ANNALES

UNIVERSITATIS SCIENTIARUM BUDAPESTINENSIS DE ROLANDO EÖTVÖS NOMINATAE

SECTIO MATHEMATICA

TOMUS LIX.

2016

ANNALES

Universitatis Scientiarum Budapestinensis de Rolando Eötvös nominatae

SECTIO CLASSICA
incepit anno MCMXXIV
SECTIO COMPUTATORICA
incepit anno MCMLXXVIII
SECTIO GEOGRAPHICA
incepit anno MCMLXVI
SECTIO GEOLOGICA
incepit anno MCMLVII
SECTIO GEOPHYSICA ET METEOROLOGICA
incepit anno MCMLXXV
SECTIO HISTORICA
incepit anno MCMLVII
SECTIO IURIDICA
incepit anno MCMLIX
SECTIO LINGUISTICA
incepit anno MCMLXX
SECTIO MATHEMATICA
incepit anno MCMLVIII
SECTIO PAEDAGOGICA ET PSYCHOLOGICA
incepit anno MCMLXX
SECTIO PHILOLOGICA
incepit anno MCMLVII
SECTIO PHILOLOGICA HUNGARICA
incepit anno MCMLXX
SECTIO PHILOLOGICA MODERNA
incepit anno MCMLXX
SECTIO PHILOSOPHICA ET SOCIOLOGICA
incepit anno MCMLXII

ANNALES UNIV. SCI. BUDAPEST., **59** (2016), 3–15

ANDRÁS HAJNAL
(1931–2016)

By
PÉTER KOMJÁTH

Courtesy by Rényi Institute

András Hajnal was born in Budapest, on May 13, 1931. Having survived WWII and the Holocaust, he entered the Budapest Scientific University in 1948, where he graduated in 1953. Then he worked towards his Candidate degree (now considered equivalent to PhD) at the Szeged university, under the supervision of László Kalmár. He obtained the degree in 1957, his thesis introduced the concept of relative constructibility.

It was in 1956 that he first met with Paul Erdős, who visited Géza Fodor, another set theorist in Szeged. Erdős introduced him to the topic of infinitary combinatorics and this started a long cooperation and friendship. Their collaboration in the late fifties and sixties introduced many of the main definitions and theorems of combinatorial set theory and when the set theory conference at the UCLA in 1967 was announced Erdős and Hajnal decided to write up the major problems arising from their research. The resulting manuscript ([40]) containing 82 problems was circulated among set theorists and many of the later stars of set started their carreer by solving some of them: J. E. Baumgartner, C. C. Chang, K. J. Devlin, Gy. Elekes, J. Folkman, F. Galvin, Gy. Hoffmann, K. Kunen, J. Larson, R. Laver, A. Máté, J. B. Paris, K. Prikry, and, above all, Saharon Shelah, who only sent the proofs to

Hajnal, when he solved two or three of the problems, in order to save stamps. A few years later, a follow-up paper was written ([74]), describing the progress.

In 1956, Hajnal joined the Department of Analysis of the Eötvös University. In 1962, he received the Doctor of Sciences degree of the Hungarian Academy of Sciences. In 1970 he transferred to the Mathematical Institute of the Hungarian Academy of Sciences, keeping a part time position at the Eötvös University where he usually tought courses on set theory and real analysis. His notes of his courses on set theory were later rewritten into the very successful textbook with P. Hamburger ([105], in English: [155]).

At the institute he formed a very successful research group in set theory and set theoretical topology: István Juhász, Péter Hamburger, Attila Máté, János Gerlits, Zsigmond Nagy, Zoltán Szentmiklóssy, Lajos Soukup. He was also instructive in the activities in finite combinatorics: beyond the publication of some results (see below), with Vera Sós he started the famous Thursday Seminar, which was instrumental in making the institute one of the world centers in combinatorics.

He was elected corresponding member (1976), member (1982) of the Hungarian Academy of Sciences. In 2012, he was elected fellow of the American Mathematical Society.

Hajnal was the director of the institute between 1983–1988 and 1988–1993. Between 1980 and 1990 he was the general secretary, then between 1990 and 1994 the president of the János Bolyai Mathematical Society.

In 1994, he became a professor at the Rutgers University, New Jersey. Between 1994 and 2004 he was the director of the DIMACS research institute. After that, he stayed at Rutgers until his retirement, then returned to Hungary.

He died on July 31, 2016, from complications arising from lung cancer.

In what follows we give a very short overview of Hajnal's scientific work, concentrating only on some of the highlights.

An important part of the work is an almost complete description of partition relations for cardinals. If $\kappa, \lambda_\xi (\xi < \mu)$ are cardinals, $2 \le r < \omega$, then

$$\kappa \to (\lambda_\xi)^r_\mu$$

denotes the following statement: if f is a coloring of the r-tuples of κ with μ colors, then for some $\xi < \mu$ there is a ξ-colored homogeneous set of cardinality λ_ξ. The main problem is to determine for which cardinals $\kappa, \lambda_\xi (\xi < \mu)$ and r does this relation hold. Erdős started this work with Rado, then they continued with Hajnal in the long paper [25] and finally presented in the book [110].

Erdős, Hajnal, and Rado introduced the so-called square bracket partition relations. One question they asked was if $\aleph_1 \not\to [\aleph_1]^2_{\aleph_1}$ held, i.e., if there exists a function F on the pairs of ω_1 with $\aleph_1$ values so that if A is an uncountable subset of ω_1 then F assumes all values on the pairs of A. The existence of such a function can easily be deduced from the Continuum Hypothesis, but the question if it exists without any hypothesis, remained a major open problem for several years. That was eventually proved by Todorcevic.

Erdős and Rado proved that there are triangle-free graphs with arbitrarily large (infinite) chromatic number. Erdős conjectured that, as in the finite case, this holds if one wants to omit C_4, the circuit of length 4, as well. To his utmost surprise, Hajnal showed this false: if the chromatic number of a graph is uncountable, then X must contain a C_4. From the proof they subtracted the notion of coloring number, if X is a graph, then its coloring number, $\mathrm{Col}(X)$ is the least cardinal μ such that the vertex set of X has a well ordering with each vertex having $< \mu$ edges going down. In [27] Erdős and Hajnal introduced this concept and proved several results which connected it with the chromatic number. Even today a large part of the results on the chromatic number of infinite graphs use this notion.

One of the long standing problems in infinitary combinatorics was the following conjecture of Ruziewicz. If $\kappa < \lambda$ are infinite cardinals, $f\colon \lambda \to \mathcal{P}(\lambda)$ is a function (set mapping) such that $|f(x)| < \kappa$ ($x \in \lambda$), then there is a free set of cardinality λ. (A set $X \subseteq \lambda$ is free if $x \notin f(y)$ for $x \neq y \in X$.) This was eventually proved by Hajnal in [12].

One of the most striking results in modern set theory was Silver's 1975 result in which he proved that if λ is a singular cardinal with uncountable cofinality, the GCH holds for all cardinals below λ, then it holds for λ. In particular, if $2^{\aleph_\alpha} = \aleph_{\alpha+1}$ for $\alpha < \omega_1$, then $2^{\aleph_{\omega_1}} = \aleph_{\omega_1+1}$. Similarly, if $\gamma < \omega_1$ and $2^{\aleph_\alpha} \leq \aleph_{\alpha+\gamma}$ for $\alpha < \omega_1$, then $2^{\aleph_{\omega_1}} \leq \aleph_{\omega_1+\gamma}$, and analogous bounds can be obtained if $2^{\aleph_\alpha} \leq \aleph_{\alpha+\alpha}$, etc. Hajnal and Galvin proved that if we only assume that $2^{\aleph_\alpha} < \aleph_{\omega_1}$ for $\alpha < \omega_1$, i.e., $\aleph_{\omega_1}$ is strong limit, then

$$2^{\aleph_{\omega_1}} < \aleph_{(2^{\aleph_1})^+}$$

holds [88]. Hajnal was strongly interested if the analogous result holds for $\aleph_\omega$. (Silver's result consistently fails for $\aleph_\omega$.) This was eventually proved using a very clever argument by Shelah. Hajnal liked it so much that included into his set theory textbook ([105]).

We mention some of Hajnal's work in finite combinatorics. In an early paper [4] with János Surányi they proved that if a finite graph does not contain induced circuits of length ≥ 4, in modern terminology, if it is *chordal*, then its vertex set can be covered by cliques, $V = \bigcup\{C_i : i \in I\}$ and vertices can be chosen $v_i \in C_i$ from each of the cliques, such that $\{v_i : i \in I\}$ is independent. This was one of the early results that led to the notion of perfect graphs and Berge's famous conjectures on them.

Erdős raised the following interesting conjecture. If X is a finite graph with all degrees less than r, then obviously the chromatic number is at most r, i.e., the vertex set can be decomposed into the union of r independent sets: $V = V_1 \cup \cdots \cup V_r$. Erdős conjectured that this can be done with independent sets as equal as possible, that is, $\big||V_i| - |V_j|\big| \leq 1$ for $1 \leq i, j \leq r$. This was then proved by an intricate argument by Hajnal and Szemerédi in [46].

With his friend, István Juhász, Hajnal published several papers on applications of set theoretic results and methods to problems in general topology, especially on connections between various cardinal invariants of topological spaces. For example, they prove that if X is a Hausdorff space and $|X| > 2^{2^\kappa}$ then X contains a discrete subpace of cardinality κ^+ ([31]). If, however, X is a strong limit singular cardinal, then X contains a discrete subspace of cardinality $|X|$ ([41]).

Publications of András Hajnal

[1] A. Hajnal: On a consistency theorem connected with the generalized continuum hypothesis, *Zeitschrift f. Math. Logik. u. Grundlagen d. Math.*, **2**(1956), 131–136.

[2] A. Hajnal, L. Kalmár: Remarks on the Gödel axiom system of set theory, I–II, *Matematikai Lapok*, **1**(1956), 26–42, 218–229 (in Hungarian).

[3] A. Hajnal, L. Kalmár: An elementary combinatorial theorem with an application to axiomatic set theory, *Publ. Math.*, **4**(1955–56), 431–449.

[4] A. Hajnal, J. Surányi: Über die Auflösung von Gräfen in vollständinge Teilgraphen, *Annales Univ. Sci. Budapest.*, **1**(1958), 113–121.

[5] P. Erdős, A. Hajnal: On the structure of set mappings, *Acta Math. Acad. Sci. Hung.*, **9**(1958), 111–130.

[6] A. Hajnal: On the work in set theory of John von Neumann, *Mat. Lapok*, **10**(1959), 5–11 (in Hungarian).

[7] P. Erdős, G. Fodor, A. Hajnal: On the structure of inner set mappings, *Acta Sci. Math.*, **20**(1959), 81–90.

[8] P. Erdős, A. Hajnal: Some remarks on set theory VIII, *Michigan Math. Journal*, **7**(1960), 187–191.

[9] P. Erdős, A. Hajnal: Some remarks on set theory VII, *Acta Sci. Math.*, **21**(1960), 154–163.

[10] P. Erdős, A. Hajnal: On the topological product of discrete λ-compact spaces, *General Topology and its relations to Modern Analysis and Algebra*, Proceedings of Symposium in Prague, September 1961, 148–151.

[11] A. Hajnal: Some results and problems in set theory, *Acta Math. Sci. Hung.*, **11**(1960), 277–298.

[12] A. Hajnal: Proof of a conjecture of S. Ruziewicz, *Fund. Math.*, **50**(1961), 123–128.

[13] A. Hajnal: On a consistency theorem connected with the generalized continuum problem, *Acta Math. Acad. Sci. Hung.*, **12**(1961), 321–376.

[14] P. Erdős, A. Hajnal: On a property of families of sets, *Acta Math. Acad. Sci. Hung.*, 87–123.

[15] J. Czipszer, P. Erdős, A. Hajnal: Some extremal problems on infinite graphs, *Publ. Math. Inst. Hung. Acad. Sci.*, **7**(1962), 441–457.

[16] P. Erdős, A. Hajnal: Some remarks concerning our paper "On the structure of set mappings", *Acta Math. Sci. Hung.*, **13**(1962), 223–226.

[17] P. Erdős, A. Hajnal: On a classification of denumerable order types and an application to partition calculus, *Fund. Math.*, **51**(1962), 117–129.

[18] K. Corrádi, A. Hajnal: On the maximal number of independent circuits in a graph, *Acta Math. Acad. Sci. Hung.*, **12**(1963), 423–439.

[19] P. Erdős, A. Hajnal: On complete topological subgraphs of certain graphs, *Annales Univ. Sci. Budapest.*, **7**(1964), 143–149.

[20] P. Erdős, A. Hajnal: Some remarks on set theory IX, *Michigan Math. Journal*, **11**(1964), 107–127.

[21] A. Hajnal: Remarks on a theorem of W. P. Hanf, *Fund. Math.*, **54**(1964), 109–113.

[22] P. Erdős, A. Hajnal, J. W. Moon: A problem in graph theory, *Amer. Math. Monthly*, **71**(1964), 1107–1110.

[23] A. Hajnal: On the topological product of discrete spaces, *Notices of the Amer. Math. Soc.*, **11**(1964), 578.

[24] A. Hajnal: A theorem on k-saturated graphs, *Canadian Math. Journ.*, **17**(1965), 720–724.

[25] P. Erdős, A. Hajnal, R. Rado: Partition relations for cardinal numbers, *Acta Math. Acad. Sci. Hung.*, **16**(1965), 93–196.

[26] P. Erdős, A. Hajnal: On a problem of B. Jónsson, *Bulletin de l'Académie Polonaise des Sciences*, **14**(1966), 19–23.

[27] P. Erdős, A. Hajnal: On chromatic number of graphs and set systems, *Acta Math. Acad. Sci. Hung.*, **17**(1966), 61–99.

[28] P. Erdős, A. Hajnal, E. C. Milner: On the complete subgraphs of graphs defined by systems of sets, *Acta Math. Acad. Sci. Hung.*, **17**(1966), 159–229.

[29] P. Erdős, A. Hajnal: On decompositions of graphs, *Acta Math. Acad. Sci. Hung.*, **18**(1967), 359–377.

[30] A. Hajnal, I. Juhász: Some results in set theoretical topology, *Dokl. Akad. Nauk SSSR*, **172**(1967), 541–542, *Soviet Math. Dokl.*, **8**(1967), 141–143.

[31] A. Hajnal, I. Juhász: On discrete subspaces of topological spaces, *Indag. Math.*, **29**(1967), 343–356.
[32] P. Erdős, A. Hajnal: On the chromatic number of infinite graphs, In: *Graph Theory Symposium, Tihany, Hungary*, 1966, 83–98.
[33] A. Hajnal, G. Petruska: Remarks on Darboux functions, *Acta Math. Acad. Sci. Hung.*, **20**(1969), 13–20.
[34] P. Erdős, A. Hajnal, E. C. Milner: On sets of almost disjoint subsets of a set, *Math. Acad. Sci. Hung.*, **19**(1968), 209–218.
[35] A. Hajnal, I. Juhász: Some remarks on a property of topological cardinal functions, *Acta Math. Acad. Sci. Hung.*, **20**(1969), 25–37.
[36] G. Fodor, A. Hajnal: On regressive functions and α-complete ideals, *Bull. de l'Academie Polonaise de Sciences*, **XV**(1967), 427–432.
[37] P. Erdős, A. Hajnal: On chromatic graphs, *Mat. Lapok*, **18**(1967), 1–4 (in Hungarian).
[38] A. Hajnal: On the history and current status of axiomatic research on the continuum hypothesis and the axiom of choice, *Mat. Lapok*, **17**(1966), 253–260 (in Hungarian).
[39] A. Hajnal, I. Juhász: On hereditarily α-Lindelöf and hereditarily α-separable spaces, *Annales Univ. Sci. Budapest.*, **11**(1968), 115–124.
[40] P. Erdős, A. Hajnal: Unsolved problems in set theory, in: *Proc. Symp. Pure Math*, **XIII**, Providence, R. I., 1971, 17–48.
[41] A. Hajnal, I. Juhász: Discrete subspaces of topological spaces II, *Indag. Math.*, **31**(1969), 18–30.
[42] P. Erdős, A. Hajnal, E. C. Milner: A problem on well ordered sets, *Acta Math. Acad. Sci. Hung.*, **20**(1969), 323–329.
[43] P. Erdős, A. Hajnal: On a problem in combinatorics, *Mat. Lapok*, **19**(1968), 345–348 (in Hungarian).
[44] A. Hajnal: On some combinatorial problems involving large cardinals, *Fund. Math.*, **69**(1970), 39–53.
[45] A. Hajnal: Ulam matrices for inaccessible cardinals, *Bull. Acad. Pol. des Sci.*, **17**(1969), 683–688.
[46] A. Hajnal, E. Szemerédi: Proof of a conjecture of Erdős, in: *Combinatorial Theory and its Applications*, Balatonfüred, 1969, 601–623.
[47] Erdős P., Hajnal A.: Set mappings and polarized partitions, in: *Combinatorial Theory and its Applications*, Balatonfüred, 1969, 327–363.
[48] Erdős P., Hajnal A.: Some results and problems for certain polarized partitions, *Acta Math. Acad. Sci. Hung.*, **21**(1970), 369–392.
[49] A. Hajnal: Report on the Miklós Schweitzer Memorial Contest in 1968, *Mat. Lapok*, **20**(1969), 145–171 (in Hungarian).
[50] A. Hajnal, E. C. Milner: Some theorems on scattered order types, *Periodica Math. Hung.*, **1**(1971), 55–63.
[51] P. Erdős, A. Hajnal, E. C. Milner: Polarized partition relations for ordinal numbers, in: *Studies in Pure Mathematics*, Acad. Press, 1971, 63–87.

[52] P. Erdős, A. Hajnal: Ordinary partition relations for ordinal numbers, *Periodica Math. Hung.*, **1**(1971), 171–185.

[53] P. Erdős, A. Hajnal, E. C. Milner: Partition relations for η_α and for $\aleph_\alpha$-saturated models, in: *Theory of Sets and Topology*, Berlin, 1972, 95–108.

[54] A. Hajnal: A negative partition relation, *Proc. Nat. Acad. Sci.*, **68**(1971), 142–144.

[55] P. Erdős, A. Hajnal: Problems and results in finite and infinite combinatorial analysis, *Annals of the New York Academy of Sciences*, **175**(1970), 115–124.

[56] A. Hajnal, I. Juhász: Discrete subspaces and de Groot's conjecture about the number of open subsets, *Proc. International Symp. on Topology and its Applications*, Belgrade, 1969, 179.

[57] A. Hajnal, E. C. Milner, E. Szemerédi: A cure for telephone disease, *Canadian Math. Bull.*, **15**(1972), 447–450.

[58] A. Hajnal, I. Juhász: On disjoint representations of ultrafilters, Theory of sets and topology, Berlin, 1972, 215–219.

[59] A. Hajnal, I. Juhász: On some consequences of Martin's axiom, *Indag. Math.*, **33**(1971), 457–463.

[60] J. E. Baumgartner, A. Hajnal: A proof (involving Martin's axiom) of a partition relation, *Fund. Math.*, **78**(1973), 193–203.

[61] P. Erdős, E. Fried, A. Hajnal, E. C. Milner: Some remarks on simple tournaments, *Algebra Universalis*, **2**(1972), 238–245.

[62] A. Hajnal, I. Juhász: On discrete subspaces of product spaces, *General Topology and its Applications*, **2**(1972), 11–16.

[63] A. Hajnal, I. Juhász: Two consistency results in topology, *Bull. Amer. Math. Soc.*, **73**(1972), 711.

[64] A. Hajnal, I. Juhász: A consistency result concerning hereditarily α-separable spaces, *Indag. Math.*, **35**(1973), 301–307.

[65] A. Hajnal, I. Juhász: A consistency result concerning hereditarily α-Lindelöf spaces, *Acta Math. Acad. Sci. Hung.*, **24**(1973), 307–312.

[66] P. Erdős, A. Hajnal, B. Rothchild: On chromatic number of graphs and set-systems Cambridge Summer School in Mathematical Logic (Cambridge, 1971), *Lecture Notes in Math.*, **337**, Springer, Berlin, 1973, 531–538.

[67] A. Hajnal, B. Rothschild: A generalization of the Erdős-Ko-Rado theorem on finite set systems, *J. Combinatorial Theory*, (A), **15**(1973), 359–362.

[68] P. Erdős, A. Hajnal, E. C. Milner: Simple one-point extensions of tournaments, *Mathematika*, **19**(1972), 57–62.

[69] P. Erdős, A. Hajnal: On Ramsey like theorems, Problems and results, in:*Proc. Oxford Conf. Comb. Math. Inst., 1972*, Inst. Math. Appl. Southend-on-Sea, 1972, 123–140.

[70] P. Erdős, A. Hajnal, S. Shelah: On some general properties of chromatic numbers, *Topics in topology*, (Proc. Colloq., Keszthely, 1972), Colloq. Math. Soc. János Bolyai, **8**, North-Holland, Amsterdam, 1974, 243–255.

[71] P. Erdős, A. Hajnal, A. Máté: Chain conditions on set mappings and free sets, *Acta Sci. Math.*, **34**(1973), 69–79.

[72] A. Hajnal, A. Kertész: Some new algebraic equivalents of the axim of choice, *Publ. Math.*, **19**(1972), 339–340.
[73] A. Hajnal, I. Juhász: On square-compact cardinals, *Period. Math. Hungar.*, **3**(1973), 285–288.
[74] P. Erdős, A. Hajnal: Unsolved and solved problems in set theory, *Proceedings of the Tarski Symposium*, (Proc. Sympos. Pure Math., **XXV**, Univ. California, Berkeley, Calif., 1971, Amer. Math. Soc., Providence, R.I., 1974, 269–287.
[75] A. Hajnal: On the set theory work of Paul Erdős, *Mat. Lapok*, **22**(1967), 197–208 (in Hungarian).
[76] A. Hajnal: Results and independence results in set-theoretical topology, in: *Proceedings of the International Congress of Mathematicians*, Vancouver, B.C., 1974, Canad. Math. Congress, Montréal, Que., 1975. II, 61–62.
[77] A. Hajnal: Weak partition relations, in: *Infinite and finite sets*, Colloq., Keszthely, 1973 dedicated to P. Erdős on his 60th birthday, Colloq. Math. Soc. János Bolyai, **10**, North-Holland, Amsterdam, 1975. II, 817–836.
[78] J. E. Baumgartner, A. Hajnal, A. Máté: Weak saturation properties of ideals, in: *Infinite and finite sets*, Colloq., Keszthely, 1973, dedicated to P. Erdős on his 60th birthday, Colloq. Math. Soc. János Bolyai, **10**, North-Holland, Amsterdam, 1975, I, 137–158.
[79] P. Erdős, F. Galvin, A. Hajnal: On set systems having large chromatic number and not containing prescribed subsytems, in: *Infinite and finite sets*, Colloq., Keszthely, 1973, dedicated to P. Erdős on his 60th birthday, Colloq. Math. Soc. János Bolyai, **10**, North-Holland, Amsterdam, 1975, II, 425–513.
[80] A. Hajnal, A. Máté: Set mappings, partitions, and chromatic numbers, Logic Colloquium '73, Bristol, 1973, Studies in Logic and the Foundations of Mathematics, **80**, North-Holland, Amsterdam, 1975. 347–379.
[81] A. Hajnal, I. Juhász: On hereditarily α-Lindelöf and α-separable spaces II, Collection of articles dedicated to Andrzej Mostowski on the occasion of his sixtieth birthday, II, *Fund. Math.* **81**(1973/74), 147–158.
[82] A. Hajnal, I. Juhász: On the number of open sets, *Ann. Univ. Sci. Budapest. Eötvös Sect. Math.*, **16**(1973), 99–102.
[83] P. Erdős, A. Hajnal: Some remarks on set theory XI, Collection of articles dedicated to Andrzej Mostowski on the occasion of his sixtieth birthday, III., *Fund. Math.*, **81**(1974), 261–265.
[84] A. Hajnal, I. Juhász: On the products of weakly Lindelöf spaces, *Proc. Amer. Math. Soc.*, **48**(1975), 454–456.
[85] A. Hajnal, I. Juhász: On first countable non-Lindelöf S-spaces, in: *Infinite and finite sets*, Colloq., Keszthely, 1973; dedicated to P. Erdős on his 60th birthday, Colloq. Math. Soc. János Bolyai, **10**, North-Holland, Amsterdam, 1975. II, 837–852.
[86] P. Erdős, A. Hajnal, L. Pósa: Strong embeddings of graphs into colored graphs, in: *Infinite and finite sets*, Colloq., Keszthely, 1973; dedicated to P. Erdős on his

60th birthday, Colloq. Math. Soc. János Bolyai, **10**, North-Holland, Amsterdam, 1975, I, 585–595.

[87] A. Hajnal, I. Juhász: Remarks on the cardinality of compact spaces and their Lindelöf subspaces, *Proc. Amer. Math. Soc.*, **59**(1976), 146–148.

[88] F. Galvin, A. Hajnal: Inequalities for cardinal powers, *Ann. Math.*, **101**(1975), 491–498.

[89] A. Hajnal, I. Juhász: On spaces in which every small subspace is metrizable, *Bull. Acad. Polon. Sci. Sci. Math. Astronom. Phys.*, **24**(1976), 727–731.

[90] P. Erdős, A. Hajnal: Embedding theorems for graphs establishing negative partition relations, *Period. Math. Hungar.*, **9**(1978), 205–230.

[91] W. W. Comfort, A. Hajnal, I. Juhász: Compactness-like properties of generalized weak products, Topology, Proc. Ninth Annual Spring Conf., Memphis State Univ., Memphis, Tenn., 1975, Lecture Notes in Pure and Appl. Math., **24**, Dekker, New York, 1976., 185–188.

[92] A. Hajnal, I. Juhász: A separable normal topological group need not be Lindelöf, *General Topology and Appl.*, **6**(1976), 199–205.

[93] P. Erdős, A. Hajnal, E. C. Milner: On set systems having paradoxical covering properties, *Acta Math. Acad. Sci. Hungar.* **31**(1978), 89–124.

[94] P. Erdős, A. Hajnal: On spanned subgraphs of graphs, *Contributions to graph theory and its applications*, Internat. Colloq., Oberhof, 1977, Tech. Hochschule Ilmenau, Ilmenau, 1977, 80–96.

[95] J. Gerlits, A. Hajnal: On the tightness of product spaces, in: *Logic Colloquium* '77, Proc. Conf., Wrocław, 1977, Stud. Logic Foundations Math., **96**, North-Holland, Amsterdam-New York, 1978, 123–133.

[96] G. Grätzer, A. Hajnal, D. Kelly: Chain conditions in free products of lattices with infinitary operations, *Pacific J. Math.*, **83**(1979), 107–115.

[97] A. Hajnal, L. Lovász: An algorithm to prevent propagation of certain diseases at minimum cost, Interfaces between computer science and operations research, *MC TRACT*, **99**(1978), 106–108.

[98] G. Elekes, P. Erdős, A. Hajnal: On some partition properties of families of sets, *Studia Sci. Math. Hungar.*, **13**(1978), 151–155.

[99] A. Hajnal, I. Juhász: Weakly separated subspaces and networks, Logic Colloquium '78, Mons, 1978, Stud. Logic Foundations Math., **97**, North-Holland, Amsterdam-New York, 1979, 235–245.

[100] A. Hajnal, I. Juhász: Lindelöf spaces à la Shelah, Topology, Proc. Fourth Colloq., Budapest, 1978, Colloq. Math. Soc. János Bolyai, **23**, North-Holland, Amsterdam-New York, 1980, II, 555–567.

[101] A. Hajnal, I. Juhász: Having a small weight is determined by the small subspaces, *Proc. Amer. Math. Soc.*, **79**(1980), 657–658.

[102] A. Hajnal, I. Juhász: When is a Pixley-Roy hyperspace ccc? *Topology Appl.*, **13**(1982), 33–41.

[103] P. Erdős, A. Hajnal, E. Szemerédi: On almost bipartite large chromatic graphs. *Theory and practice of combinatorics*, North-Holland Math. Stud., **60**, North-Holland, Amsterdam, 1982, 117–123.
[104] A. Hajnal, Z. Szentmiklóssy: On the cardinality of quasi-hereditarily Lindelöf spaces, *General topology and its relations to modern analysis and algebra, V*, Prague, 1981, *Sigma Ser. Pure Math.*, **3**, Heldermann, Berlin, 1983. 248–253.
[105] A. Hajnal, P. Hamburger: *Set Theory*, university textbook, Tankönyvkiadó, 1983 (in Hungarian).
[106] A. Hajnal, Zs. Nagy: Ramsey games, *Trans. Amer. Math. Soc.*, **284**(1984), 815–827.
[107] P. Erdős, A. Hajnal, V. T. Sós, E. Szemerédi: More results on Ramsey-Turán type problems, *Combinatorica*, **3**(1983), 69–81.
[108] A. Hajnal, V. T. Sós: Paul Erdős is seventy, *J. Graph Theory*, **7**(1983), 391–393.
[109] A. Hajnal, P. Komjáth: What must and what need not be contained in a graph of uncountable chromatic number? *Combinatorica*, **4**(1984), 47–53.
[110] P. Erdős, A. Hajnal, A. Máté, R. Rado: *Combinatorial set theory: partition relations for cardinals*, *Studies in Logic and the Foundations of Mathematics*, **106**, North-Holland Publishing Co., Amsterdam, 1984, 347 pp.
[111] A. Hajnal, J. Pach: Monochromatic paths in infinite coloured graphs, in: *Finite and infinite sets*, I, II (Eger, 1981), Colloq. Math. Soc. János Bolyai, **37**, North-Holland, Amsterdam, 1984, 359–369.
[112] P. Erdős, A. Hajnal: Chromatic number of finite and infinite graphs and hypergraphs, in: *Special volume on ordered sets and their applications*, L'Arbresle, 1982, *Discrete Math.*, **53**(1985), 281–285.
[113] A. Hajnal, I. Juhász: Intersection properties of open sets, *Topology Appl.*, **19**(1985), 201–209.
[114] A. Hajnal: The chromatic number of the product of two $\aleph_1$-chromatic graphs can be countable, *Combinatorica*, **5**(1985), 137–139.
[115] A. Hajnal, I. Juhász, S. Shelah: Splitting strongly almost disjoint families, *Trans. Amer. Math. Soc.*, **295**(1986), 369–387.
[116] P. Erdős, Z. Füredi, A. Hajnal, P. Komjáth, V. Rödl, Á. Seress: Coloring graphs with locally few colors, *Discrete Math.*, **59**(1986), 21–34.
[117] A. Hajnal, N. Sauer: Complete subgraphs of infinite multipartite graphs and antichains in partially ordered sets, *Discrete Math.*, **59**(1986), 61–67.
[118] A. Hajnal,A. Kanamori, S. Shelah: Regressive partition relations for infinite cardinals, *Trans. Amer. Math. Soc.*, **299**(1987), 145–154.
[119] A. Hajnal, P. Komjáth: Some higher-gap examples in combinatorial set theory, *Ann. Pure Appl. Logic*, **33**(1987), 283–296.
[120] J. E. Baumgartner, A. Hajnal: A remark on partition relations for infinite ordinals with an application to finite combinatorics, in: *Logic and combinatorics (Arcata, Calif., 1985)*, Contemp. Math., **65**, Amer. Math. Soc., Providence, RI, 1987, 157–167.

[121] A. Hajnal, I. Juhász, S. Soukup: On saturated almost disjoint families, *Comment. Math. Univ. Carolin.*, **28**(1987), 629–633.

[122] A. Hajnal, P. Komjáth: Embedding graphs into colored graphs, *Trans. Amer. Math. Soc.*, **307**(1988), 395–409.

[123] A. Hajnal, P. Komjáth, L. Soukup, I Szalkai: Decompositions of edge colored infinite complete graphs, *Combinatorics (Eger, 1987)*, Colloq. Math. Soc. János Bolyai, **52**, North-Holland, Amsterdam, 1988, 277–280.

[124] P. Erdős, A. Hajnal: On the number of distinct induced subgraphs of a graph, Graph theory and combinatorics (Cambridge, 1988), *Discrete Math.*, **75**(1989), 145–154.

[125] P. Erdős, A. Hajnal: Ramsey-type theorems, Combinatorics and complexity (Chicago, IL, 1987), *Discrete Appl. Math.*, **25**(1989), 37–52.

[126] A. Hajnal, I. Juhász, W. Weiss: Partitioning the pairs and triples of topological spaces, *Topology Appl.*, **35**(1990), 177–184.

[127] A. Hajnal: A remark on the homogeneity of infinite permutation groups, *Bull. London Math. Soc.*, **22**(1990), 529–532.

[128] A. Hajnal, Z. Nagy, L. Soukup: On the number of certain subgraphs of graphs without large cliques and independent subsets, *A tribute to Paul Erdős*, Cambridge Univ. Press, Cambridge, 1990. 223–248.

[129] N. Alon, A. Hajnal: Ramsey graphs contain many distinct induced subgraphs, *Graphs Combin.*, **7**(1991), 1–6.

[130] P. Erdős, A. Hajnal, Z. Tuza: Local constraints ensuring small representing sets, *J. Combin. Theory*, (A), **58**(1991), 78–84.

[131] A. Hajnal: Embedding finite graphs into graphs colored with infinitely many colors, *Israel J. Math.*, **73**(1991), 309–319.

[132] A. Hajnal, E. C. Milner: On k-independent subsets of a closure, *Studia Sci. Math. Hungar.*, **26**(1991), 467–470.

[133] A. Hajnal, P. Komjáth: Corrigendum to: "Embedding graphs into colored graphs", *Trans. Amer. Math. Soc.*, **332**(1992), 475.

[134] J. Gerlits, A. Hajnal, Z. Szentmiklóssy: On the cardinality of certain Hausdorff spaces, Topological, algebraical and combinatorial structures, Frolík's memorial volume, *Discrete Math.*, **108**(1992), 31–35.

[135] G. Elekes, A. Hajnal, P. Komjáth: Partition theorems for the power set, in: *Sets, graphs and numbers (Budapest, 1991)*, Colloq. Math. Soc. János Bolyai, **60**, North-Holland, Amsterdam, 1992, 211–217.

[136] A. Hajnal, W. Maass, P. Pudlák, M. Szegedy, G. Turán: Threshold circuits of bounded depth, *J. Comput. System Sci.*, **46**(1993), 129–154.

[137] P. Erdős, A. Hajnal, M. Simonovits, V. T. Sós, E. Szemerédi: Turán-Ramsey theorems and simple asymptotically extremal structures, *Combinatorica*, **13**(1993), 31–56.

[138] A. Hajnal, N. Sauer: Cut-sets in infinite graphs and partial orders, *Discrete Math.*, **117**(1993), 113–125.

[139] A. Hajnal, I. Juhász, Z. Szentmiklóssy: Compact ccc spaces of prescribed density via hypergraphs, *Combinatorics, Paul Erdős is eighty*, Bolyai Soc. Math. Stud., János Bolyai Math. Soc., Budapest, 1993, 1, 239–252.

[140] J. E. Baumgartner, A. Hajnal, S. Todorčević: Extensions of the Erdős-Rado theorem, in: *Finite and infinite combinatorics in sets and logic (Banff, AB, 1991)*, NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., **411**, Kluwer Acad. Publ., Dordrecht, 1993. 1–17.

[141] P. Erdős, A. Hajnal, J. A. Larson: Ordinal partition behavior of finite powers of cardinals, in: *Finite and infinite combinatorics in sets and logic (Banff, AB, 1991)*, NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., **411**, Kluwer Acad. Publ., Dordrecht, 1993, 97–115.

[142] A. Hajnal: True embedding partition relations, in: Finite and infinite combinatorics in sets and logic (Banff, AB, 1991), NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., **411**, Kluwer Acad. Publ., Dordrecht, 1993. 135–151.

[143] P. Erdős, A. Hajnal, M. Simonovits, V. T. Sós, E. Szemerédi: Turán-Ramsey theorems and K_p-independence numbers, *Combin. Probab. Comput.*, **3**(1994), 297–325.

[144] A. Hajnal, Á. Korányi, J. Dixmier, M. Duflo: On the work of L. Pukánszky, *Mat. Lapok (N.S.)*, **4**(1994), 71–81 (in Hungarian).

[145] A. Hajnal: Infinite combinatorics. in: *Handbook of combinatorics*, Elsevier, Amsterdam, 1995, 2085–2116.

[146] F. Galvin, A. Hajnal, P. Komjáth: Edge decompositions of graphs with no large independent sets, in: Duro Kurepa memorial volume, *Publ. Inst. Math. (Beograd) (N.S.)*, **57(71)**(1995), 71–80.

[147] R. Aharoni, A. Hajnal, E. C. Milner: Interval covers of a linearly ordered set, in: *Set theory (Boise, ID, 1992–1994)*, Contemp. Math., **192**, Amer. Math. Soc., Providence, RI, 1996. 1–13.

[148] A. Hajnal: Paul Erdős' set theory, in: *The mathematics of Paul Erdős*, *Algorithms Combin.*, **14**, Springer, Berlin, 1997, II, 352–393.

[149] P. Erdős, A. Hajnal, M. Simonovits, T. V. Sós, E. Szemerédi: Turán-Ramsey theorems and K_p-independence numbers in: *Combinatorics, geometry and probability*, Cambridge Univ. Press, Cambridge, 1997, 253–281.

[150] A. Hajnal, I. Juhász, Z. Szentmiklóssy: On ccc Boolean algebras and partial orders, *Comment. Math. Univ. Carolin.*, **38**(1997), 537–544.

[151] P. Erdős, A. Hajnal, J. Pach: On a metric generalization of Ramsey's theorem, *Israel J. Math.*, **102**(1997), 283–295.

[152] A. Hajnal, P Komjáth: A strongly non-Ramsey order type, *Combinatorica*, **17**(1997), 363–367.

[153] J. Dixmier, M. Duflo, A. Hajnal, R. Kadison, Á. Korányi, J. Rosenberg, M Vergne: Lajos Pukánszky (1928–1996), *Notices Amer. Math. Soc.*, **45**(1998), 492–499.

[154] G. Grätzer, A. Hajnal: On isotone maps on a countable lattice, *Algebra Universalis*, **41**(1999), 85–86.

[155] A. Hajnal, P. Hamburger: *Set theory*, Translated from the 1983 Hungarian original by Attila Máté, London Mathematical Society Student Texts, **48**, Cambridge University Press, Cambridge, 1999. viii+316 pp.

[156] A. Hajnal, I. Juhász, S. Shelah: Strongly almost disjoint families, revisited, *Fund. Math.*, **163**(2000), 13–23.

[157] P. Erdős, A. Hajnal, J. Pach: A Ramsey-type theorem for bipartite graphs, *Geombinatorics*, **10**(2000), 64–68.

[158] J. E. Baumgartner, A. Hajnal: Polarized partition relations, *J. Symbolic Logic*, **66**(2001), 811–821.

[159] M. Foreman, A. Hajnal: A partition relation for successors of large cardinals, *Math. Ann.*, **325**(2003), 583–623.

[160] A. Hajnal, P. Komjáth: Some remarks on the simultaneous chromatic number, Paul Erdős and his mathematics (Budapest, 1999), *Combinatorica*, **23**(2003), 89–104.

[161] A. Hajnal: In memory of László Kalmár, *Acta Cybernet.*, **18**(2007), 7–8.

[162] A. Hajnal: Rainbow Ramsey theorems for colorings establishing negative partition relations, *Fund. Math.*, **198**(2008), 255–262.

[163] A. Hajnal, P. Komjáth: Obligatory subsystems of triple systems, *Acta Math. Hungar.*, **119**(2008), 1–13.

[164] A. Hajnal: Gyuri Elekes and set theory, *Ann. Univ. Sci. Budapest. Eötvös Sect. Math.*, **52**(2009), 19–23

[165] A. Hajnal, J. A. Larson: Partition relations, in: *Handbook of set theory*, Springer, Dordrecht, 2010. 129–213.

[166] A. Hajnal: My early encounters with Szemerédi, in: *An irregular mind*, Bolyai Soc. Math. Stud., **21**, János Bolyai Math. Soc., Budapest, 2010. 755–758.

[167] A. Hajnal, I. Juhász, L. Soukup, Z. Szentmiklóssy: Conflict free colorings of (strongly) almost disjoint set-systems, *Acta Math. Hungar.*, **131**(2011), 230–274.

ANNALES UNIV. SCI. BUDAPEST., **59** (2016), 17–23

THE MATHEMATICAL WORK OF MÁTYÁS BOGNÁR (1927–2015)

By
GÁBOR MOUSSONG

The long active career of Mátyás Bognár in mathematical research spans more than five decades from the 1950's well into the twenty-first century. Throughout this time he taught at Eötvös Loránd University. He was with the Department of Geometry until 1985, and with the Department of Analysis from 1986 until his retirement in 1997.

Most of Bognár's work was done in geometric topology. In the early 1950's he was a student of György Hajós at the Department of Geometry. It was Hajós who presented him with embeddability questions of locally compact Abelian groups into Euclidean spaces. The complete solution of this problem, which was achieved by Bognár by the late 1950's, brought up many related questions in various branches of geometric topology. Some of these were necessary as background material for the theory of embeddings of topological groups, while some others were needed as clarification of the underlying theory. These themes, among which notable are linking theory in Euclidean spaces, the theory of generalized manifolds, and the theory of categories with involutions, are present in most of Bognár's later work. We give an overview of Bognár's results in each of these subjects.

Embeddings of topological groups

An n-dimensional torus clearly admits a topological embedding into $(n+1)$-dimensional Euclidean space. This does not generalize to all compact n-dimensional abelian topological groups as demonstrated in the case $n = 1$ by the example of a solenoid (the limit of an inverse sequence of circles where the bonding maps are covering homomorphisms) which does not embed into the plane. It was shown by Kodaira and Abe in 1940 that if the underlying topological space of an n-dimensional compact connected separable Abelian topological group G embeds into $\mathbb{R}^{n+1}$, then G is a torus.

Bognár set out to generalize this result to the locally compact case. First he proved that every n-dimensional, locally compact, second countable, Abelian topological group can be embedded into $\mathbb{R}^{n+2}$. This theorem was the main result of his 1957 CSc dissertation, of which no copy exists today. The proof later appeared in print in [4]. Bognár also proved a much harder non-embeddability theorem, analogous to the Kodaira–Abe result above, in the locally compact setting. This theorem says that for an n-dimensional, locally compact, second countable, connected, Abelian topological group there can be only two possibilities. Either it is locally connected, in which case it is known to be the direct product of a (possibly) lower dimensional torus and a vector group (which of course embeds into $\mathbb{R}^{n+1}$), or else it contains an embedded copy of an n-solenoid (which simply is the direct product of an $(n-1)$-cube and a solenoid). Then he proved that an n-solenoid does not embed into $\mathbb{R}^{n+1}$. The proof of this non-embeddability result involves some essential ideas about linking curves in $\mathbb{R}^{n+1}$ for some $(n-1)$-dimensional homology classes of embedded figures, and introduces the concept of absolute linkedness of pseudomanifolds with boundary. Here "absolutely linked" means that some interior curves are necessarily nontrivially linked with the boundary no matter how the figure is embedded in Euclidean space with codimension one. For an illustrative example, a Möbius band is absolutely linked, and this fact immediately implies that no closed nonorientable surface can be embedded into $\mathbb{R}^3$. The idea of absolute linkedness, which is the key to Bognár's non-embeddability theorem, gave significant impetus for his later related work.

Bognár apparently planned to publish a complete and fully detailed account of his work in this direction in a multi-part series of articles, but only the first installment [3] materialized at the time. Complete proofs appeared later in [28], [29], and in [34], [35], [36], intertwined with later work in which generalized cells and generalized manifolds were in focus. In [38] he extended his theorems to the case of uniform embeddings.

Linking theory

Nontrivial linking coefficients between $(n-2)$-dimensional homology classes of a compact set in $\mathbb{R}^n$ and certain Jordan curves play a determining role in the proof of Bognár's non-embeddability theorem. This motivated his work on the theory of linking of disjoint compact sets in Euclidean space. Of the homology theory he was working with he only assumed fulfilment of the

Eilenberg–Steenrod axioms. In order to specify a theory of linking, which by Bognár's definition is a simultaneous bilinear assignment of linking coefficients to pairs of homology classes, one for each of two disjoint compact sets in $\mathbb{R}^n$, he proposed a single additional axiom, namely, naturality with respect to inclusions. He proved the following especially pleasing theorem which Bognár himself considered the most significant among all of his mathematical work. Given an arbitrary homology theory (in the Eilenberg–Steenrod sense) of compact sets in $\mathbb{R}^n$, and given a pair of geometrically linked round spheres S, T of dimensions k and l with $k + l = n - 1$, any arbitrarily prescribed bilinear map from $\widetilde{H}_k(S) \times \widetilde{H}_l(T)$ to the coefficient domain extends in a unique fashion to a theory of linking (in this particular pair of dimensions) that satisfies Bognár's naturality axiom. Further, there is a rather natural requirement that connects linking theories in various different dimensions. If this is also assumed as an axiom, then an analogous theorem holds, simultaneously in all dimensions, relative to a single input in dimensions $k = l = 0$.

This pair of theorems (that is, existence and uniqueness of linking theory) with detailed proofs were among the central results presented in Bognár's legendary four-volume, 1200-page DSc dissertation in 1969. Announcement of these results appeared in [10], and a detailed exposition with proofs was published in [26]. Some applications to separation questions and to existence questions of certain Jordan curves were given in [21] and [27]. Finally, the whole theory with background, detailed definitions, proofs and applications, was collected and presented in the form of a book [33].

Generalized manifolds

Investigations of topological embeddings (e.g., in [3]) brought up questions about general separation properties and linking properties of certain subspaces of Euclidean space. For the apparatus of homology theory to be applied, these subspaces had to display some properties typically characteristic of codimension one submanifolds with boundary. Initially Bognár used the classical concept of pseudomanifolds as the primary setting of this study. Proofs of some results announced in [3], including statements about absolute linkedness properties of pseudomanifolds, finally appeared in print in the series of papers [34], [35], [36].

In later investigations Bognár proposed his own concept of generalized manifolds (the so-called k-manifolds) in which only those local separation properties are postulated which play an essential role in their main applications, namely, when they are used in non-embeddability proofs. Some results in this direction were announced in [11], and the theory was exposed in detail in [14] and [32]. A note on a related question appeared in [39].

Categories with involution

An essential ingredient in the proofs of Bognár's non-embeddability results of topological groups was the construction of certain Jordan curves. In order to carry out these constructions some standard manipulation with continuous paths in a topological space was necessary. Bognár extracted the abstract formalism from these manipulations, and observed that his results hold in a more general setting, namely, when formulated in the language of categories equipped with a preferred involution which operates on morphisms. The idea of working in such categories originated from MacLane, and Bognár coined the name "i-category" for categories with involutions. The main example of an i-category is the category of paths in a topological space where the involution assigns to each path its reverse. Bognár worked out these ideas in full detail in the articles [12], [13], and pointed out interesting links with some standard concepts of algebraic topology in [18], [19], [37].

Miscellaneous topics

Apart from his main areas of research, Bognár addressed several mathematical problems and published results in diverse areas of mathematics. Most of the time the topics were within topology. He wrote several papers on triangulations and various types of complexes: in [9] and in [15] he discussed examples on non-triangulable CW-complexes, in [16] and [43] he dealt with more general types of complexes, and in [31] he discussed a classical theorem of simplicial topology. General topology was the theme of Bognár's articles on curves ([20], [44]), proximity spaces ([5], [22]), metric topology ([40], [45]), and some other special questions ([41], [42], [47]).

Bognár also published papers on algebra ([1], [8]), geometry ([6], [23]), combinatorics ([25]), and measure theory ([2], [24], [29]). He wrote or co-authored several surveys on past or contemporary research in topology ([7], [17], [46]). It is amazing that even when he was about 80 years of age he still gave seminar talks and published technical papers on some of his favourite mathematical questions.

Publications of Mátyás Bognár

[1] M. Bognár, *Ein einfaches Beispiel direkt unzerlegbarer abelscher Gruppen*, Publ. Math. Debrecen **4** (3–4) (1956), 509–511.

[2] M. Bognár, *Remarks on the inaugural lecture of F. Riesz at the University of Szeged* (in Hungarian), Matematikai Lapok **9** (3–4) (1958), 232–259.

[3] M. Bognár, *n-dimensionale berandete Pseudomannigfaltigkeiten im $(n + 1)$-dimensionalen euklidischen Raume. I*, Acta Math. Hung. **10** (3–4) (1959), 363–373.

[4] M. Bognár, *Embedding of locally compact Abelian topological groups in Euclidean spaces* (in Russian), Uspekhi Mat. Nauk **15** (3) (1960), 133–136; English translation: Amer. Math. Soc. Transl. **30** (1963), 345–349.

[5] M. Bognár, *Bemerkungen zum Kongressvortrag „Stetigkeitsbegriff und abstrakte Mengenlehre" von F. Riesz*, General topology and its relations to modern analysis and algebra, Prague (1961), 96–105.

[6] M. Bognár, *On W. Fenchel's solution of the plank problem*, Acta Math. Hung. **12** (3–4) (1961), 269–270.

[7] M. Bognár, *The development of the theory of topological manifolds* (in Hungarian), Matematikai Lapok **18** (1–2) (1967), 37–57.

[8] M. Bognár, *On ordered categories*, Ann. Univ. Sci. Budapest, Sectio Math., **11** (1968), 59–70.

[9] M. Bognár, *On the triangulability of CW-complexes* (in Hungarian), Matematikai Lapok **21** (3–4) (1970) 259–267.

[10] M. Bognár, *Axiomatization of the theory of linking* (in Russian), Dokl. Akad. Nauk SSSR **203** (1972), 986–988; English translation: Soviet Math. Dokl. **13** (2) (1972), 477–480.

[11] M. Bognár, *On generalized manifolds*, Coll. Math. Soc. J. Bolyai **8**, Topics in topology, Keszthely (1972), 111–126.

[12] M. Bognár, *Über Kategorien mit Involution*, Acta Math. Hung. **23** (1–2) (1972), 147–174.

[13] M. Bognár, *Die i-Kategorie der stetigen Wege*, Acta Math. Hung. **24** (1–2) (1973), 155–178.

[14] M. Bognár, *Über Lageeigenschaften verallgemeinerten Mannigfaltigkeiten*, Acta Math. Hung. **24** (1–2) (1973), 179–198.

[15] M. Bognár, *A finite dimensional CW complex which cannot be triangulated*, Acta Math. Hung. **29** (1–2) (1977), 107–112.

[16] M. Bognár, *On complexes*, Ann. Univ. Sci. Budapest, Sectio Math. **21** (1977), 99–118.

[17] M. Bognár and J. Szenthe, *H. Poincaré's mathematical work and its effect* (in Hungarian), Matematikai Lapok **28** (4) (1977–1980), 269–286.

[18] M. Bognár, *i-categories and homology*, Coll. Math. Soc. J. Bolyai **23**, Topology, Budapest (1978), 163–174.

[19] M. Bognár, *Continuous paths and homology*, Acta Math. Hung. **34** (1–2) (1979), 121–127.

[20] M. Bognár, *On a special type of curves*, Proc. Int. Conf. Geom. Topology, PWN Warszawa (1980), 45–46.

[21] M. Bognár, *Some consequences of the decomposition theorem*, Coll. Math. Soc. J. Bolyai **41**, Topology and Appl., Eger (1983), 89–92.

[22] M. Bognár, *Extending compatible proximities*, Acta Math. Hung. **45** (3–4) (1985), 377–378.

[23] M. Bognár and G. Kertész, *On lineations*, Acta Math. Hung. **47** (1–2) (1986), 53–64.

[24] M. Bognár, *Notes on the inaugural lecture delivered by Frederic Riesz in 1925 as rector of Szeged University*, Acta Math. Hung. **48** (1- 2) (1986), 23–46.

[25] M. Bognár, *Walking in finite directed graphs*, Studia Sci. Math. Hung. **22** (1987), 305–313.

[26] M. Bognár, *On axiomatization of the theory of linking*, Acta Math. Hung. **49** (1–2) (1987), 3–28.

[27] M. Bognár, *On linking Jordan curves*, Acta Math. Hung. **51** (1–2) (1988), 15–22.

[28] M. Bognár, *On embedding of locally compact abelian topological groups in euclidean spaces. I*, Acta Math. Hung. **51** (3–4) (1988), 371–399.

[29] M. Bognár, *On embedding of locally compact abelian topological groups in euclidean spaces. II*, Acta Math. Hung. **52** (1–2) (1988), 101–131.

[30] M. Bognár, *On the exterior linear measure*, Acta Math. Hung. **53** (1–2) (1989), 159–168.

[31] M. Bognár, *On a theorem of Runge*, Coll. Math. Soc. J. Bolyai **55**, Topology, Pécs (1989), 33–46.

[32] M. Bognár, *Cohomological pseudomanifolds*, Acta Math. Hung. **57** (1–2) (1991), 91–109.

[33] M. Bognár, *Foundations of linking theory*, Akadémiai Kiadó, Budapest, 1992.

[34] M. Bognár, *On pseudomanifolds with boundary. I*, Acta Math. Hung. **59** (1–2) (1992), 227–244.

[35] M. Bognár, *On pseudomanifolds with boundary. II*, Acta Math. Hung. **61** (1–2) (1993), 151–163.

[36] M. Bognár, *On pseudomanifolds with boundary. III*, Acta Math. Hung. **65** (1) (1994), 69–105.

[37] M. Bognár, *i-categories, free groups, fundamental groupoids*, Bolyai Soc. Math. Studies **4**, Topology with Applications, Szekszárd (1993), 21–29.

[38] M. Bognár, *Uniform embedding of Abelian topological groups in Euclidean spaces*, Acta Math. Hung. **65** (4) (1994), 333–337.

[39] M. Bognár, *External characterization of generalized manifolds*, Studia Sci. Math. Hung. **30** (1995), 443–446.

[40] M. Bognár, *On the Moore–Menger theorem*, Acta Math. Hung. **71** (1–2) (1996), 91–101.

[41] M. Bognár, *On Peano mappings*, Acta Math. Hung. **74** (3) (1997), 221–227.

[42] M. Bognár, *On some variants of connectedness*, Acta Math. Hung. **79** (1–2) (1998), 117–122.

[43] M. Bognár, *Complexes and components*, Ann. Univ. Sci. Budapest, Sectio Math. **42** (1999), 73–82.

[44] M. Bognár, *On a theorem of Gyula Pál*, Acta Math. Hung. **98** (1–2) (2003), 79–83.

[45] M. Bognár, *On the Hahn–Mazurkiewicz theorem*, Acta Math. Hung. **102** (1–2) (2004), 37–83.

[46] M. Bognár and Á. Császár, *Topology*, Bolyai Soc. Math. Studies **14**, A Panorama of Hungarian Mathematics in the Twentieth Century (ed. by J. Horváth), Budapest (2006), 9–25.

[47] M. Bognár, *The Sorgenfrey line is non-metrizable*, Acta Math. Hung. **133** (1–2) (2011), 185–187.

ANNALES UNIV. SCI. BUDAPEST., **59** (2016), 25–44

ALMOST $\mathcal{I}$-CONTINUOUS MULTIFUNCTIONS

By
C. ARIVAZHAI AND N. RAJESH
(Received February 17, 2016)

Abstract. The aim of this paper is to introduce and study upper and lower almost $\mathcal{I}$-continuous multifunctions as a generalization of upper and lower $\mathcal{I}$-continuous multifunctions, respectively.

1. Introduction

It is well known that various types of functions play a significant role in the theory of classical point set topology. A great number of papers dealing with such functions have appeared, and a good number of them have been extended to the setting of multifunctions [2, 4, 5, 16, 17, 18, 19]. This implies that both, functions and multifunctions are important tools for studying other properties of spaces and for constructing new spaces from previously existing ones. The concept of ideals in topological spaces has been introduced and studied by Kuratowski [11] and Vaidyanathaswamy, [21]. An ideal $\mathcal{I}$ on a topological space (X,τ) is a nonempty collection of subsets of X which satisfies (i) $A \in \mathcal{I}$ and $B \subset A$ implies $B \in \mathcal{I}$ and (ii) $A \in \mathcal{I}$ and $B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$. Given a topological space (X,τ) with an ideal $\mathcal{I}$ on X and if $\mathcal{P}(X)$ is the set of all subsets of X, a set operator $(.)^\star : \mathcal{P}(X) \to \mathcal{P}(X)$, called the local function [21] of A with respect to τ and $\mathcal{I}$, is defined as follows: for $A \subset X$, $A^\star(\tau,\mathcal{I}) = \{x \in X | U \cap A \notin \mathcal{I}$ for every $U \in \tau(x)\}$, where $\tau(x) = \{U \in \tau | x \in U\}$. A Kuratowski closure operator $\mathrm{Cl}^\star(.)$ for a topology $\tau^\star(\tau,\mathcal{I})$ called the $\star$-topology, finer than τ is defined by $\mathrm{Cl}^\star(A) = A \cup A^\star(\tau,\mathcal{I})$ when there is no chance of confusion, $A^\star(\mathcal{I})$ is denoted by $A^\star$. If $\mathcal{I}$ is an ideal on X, then $(X,\tau,\mathcal{I})$ is called an ideal topological space. In 1990, Jankovic and Hamlett [9] introduced the notion of $\mathcal{I}$-open sets in topological spaces. In 1992, Abd El-Monsef et al. [1] further investigated $\mathcal{I}$-open

2000 Mathematics Subject Classification 54C05, 54C601, 54C08, 54C10

sets and $\mathcal{I}$-continuous functions. Several characterizations and properties of $\mathcal{I}$-open sets were provided in [1, 14]. Recently, Akdag [2] introduced and studied the concept of $\mathcal{I}$-continuous multifunctions in ideal topological spaces. Also in [17], the theory of almost continuity for multifunctions is unified using certain minimal conditions. In this paper, we introduce and study upper (lower) almost $\mathcal{I}$-continuous multifunctions and obtain several characterizations of upper (lower) almost $\mathcal{I}$-continuous multifunctions and basic properties of such functions.

2. Preliminaries

Throughout this paper, (X,τ) and (Y,σ) (or simply X and Y) always mean topological spaces in which no separation axioms are assumed unless explicitly stated. For a subset A of (X,τ), $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$ denote the closure of A with respect to τ and the interior of A with respect to τ, respectively. A subset S of an ideal topological space $(X,\tau,\mathcal{I})$ is $\mathcal{I}$-open [9] if $S \subset \mathrm{Int}(S^*)$. The complement of an $\mathcal{I}$-closed set is said to be an $\mathcal{I}$-open set. The $\mathcal{I}$-closure and the $\mathcal{I}$-interior, that can be defined in the same way as $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$, respectively, will be denoted by $\mathcal{I}\,\mathrm{Cl}(A)$ and $\mathcal{I}\,\mathrm{Int}(A)$, respectively. The family of all $\mathcal{I}$-open (resp. $\mathcal{I}$-closed) sets of $(X,\tau,\mathcal{I})$ is denoted by $\mathcal{I}O(X)$ (resp. $\mathcal{I}C(X)$). The family of all $\mathcal{I}$-open (resp. $\mathcal{I}$-closed) sets of $(X,\tau,\mathcal{I})$ containing a point $x \in X$ is denoted by $\mathcal{I}O(X,x)$ (resp. $\mathcal{I}C(X,x)$). A subset A is said to be regular open [20] (resp. semiopen [12], preopen [13], semi-preopen [3]) if $A = \mathrm{Int}(\mathrm{Cl}(A))$ (resp. $A \subset \mathrm{Cl}(\mathrm{Int}(A))$, $A \subset \mathrm{Int}(\mathrm{Cl}(A))$, $A \subset \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(A)))$). The complement of regular open (resp. semiopen, semi-preopen) set is called regular closed (resp. semiclosed, semi-preclosed) set. The intersection (resp. union) of all semiclosed (resp. semiopen) set containing (resp. contained in) $A \subset X$ is called the semiclosure (resp. semiinterior) of A and is denoted by $s\,\mathrm{Cl}(A)$ (resp. $s\,\mathrm{Int}(A)$). The family of all regular open (resp. regular closed, semiopen, semiclosed, preopen, semi-preopen, semi-preclosed) sets of (X,τ) is denoted by $RO(X)$ (resp. $RC(X)$, $SO(X)$, $SC(X)$, $PO(X)$, $SPO(X)$, $SPC(X)$). By a multifunction $F\colon (X,\tau) \to (Y,\sigma)$, we shall denote the upper and lower inverse of a set B of Y by $F^+(B)$ and $F^-(B)$, respectively, that is, $F^+(B) = \{x \in X : F(x) \subset B\}$ and $F^-(B) = \{x \in X : F(x) \cap B \neq \emptyset\}$. In particular, $F^-(y) = \{x \in X : y \in F(x)\}$ for each point $y \in Y$ and for each $A \subset X$, $F(A) = \bigcup_{x\in A} F(x)$. Then F is said to be surjection if $F(X) = Y$. A multifunction $F\colon (X,\tau,\mathcal{I}) \to$ $\to (Y,\sigma)$ is said to be lower $\mathcal{I}$-continuous [2] (resp. upper $\mathcal{I}$-continuous) multifunction if $F^-(V) \in \mathcal{I}O(X,\tau)$ (resp. $F^+(V) \in \mathcal{I}O(X,\tau)$) for every $V \in \sigma$. A subset N of a topological space (X,τ) is said to be $\mathcal{I}$-neighborhood of a point $x \in X$, if there exists an $\mathcal{I}$-open set V such that $x \in V \subset N$.

LEMMA 2.1. *The following statements are true:*

(1) *Let A be a subset of a space (X,τ). Then $A \in PO(X)$ if and only if* $s\,\mathrm{Cl}(A) = \mathrm{Int}(\mathrm{Cl}(A))$ [7].
(2) *A subset A of a space (X,τ) is semi-preopen if and only if* $\mathrm{Cl}(A)$ *is regular closed* [3].

DEFINITION 2.2. [4] A multifunction $F\colon (X,\tau,\mathcal{I}) \to (Y,\sigma)$ is said to be:

(1) lower weakly $\mathcal{I}$-continuous if for each $x \in X$ and each open set V of Y such that $x \in F^-(V)$, there exists $U \in \mathcal{I}O(X,x)$ such that $U \subset F^-(\mathrm{Cl}(V))$,
(2) upper weakly $\mathcal{I}$-continuous if for each $x \in X$ and each open set V of Y such that $x \in F^+(V)$, there exists $U \in \mathcal{I}O(X,x)$ such that $U \subset F^+(\mathrm{Cl}(V))$,
(3) weakly $\mathcal{I}$-continuous if it is both upper weakly $\mathcal{I}$-continuous and lower weakly $\mathcal{I}$-continuous.

3. On upper and lower almost $\mathcal{I}$-continuous multifunctions

DEFINITION 3.1. A multifunction $F\colon (X,\tau,\mathcal{I}) \to (Y,\sigma)$ is said to be:

(1) lower almost $\mathcal{I}$-continuous if for each $x \in X$ and each open set V of Y such that $x \in F^-(V)$, there exists $U \in \mathcal{I}O(X,x)$ such that $U \subset F^-(\mathrm{Int}(\mathrm{Cl}(V)))$,
(2) upper almost $\mathcal{I}$-continuous if for each $x \in X$ and each open set V of Y such that $x \in F^+(V)$, there exists $U \in \mathcal{I}O(X,x)$ such that $U \subset F^+(\mathrm{Int}(\mathrm{Cl}(V)))$,
(3) almost $\mathcal{I}$-continuous if it is both upper almost $\mathcal{I}$-continuous and lower almost $\mathcal{I}$-continuous.

It is clear that every upper (lower) $\mathcal{I}$-continuous function is upper (lower) almost $\mathcal{I}$-continuous. But the converse is not true as shown by the following example.

EXAMPLE 3.2. Let $X = \{a,b,c\}$, $\tau = \{\emptyset, X\}$, $\sigma = \{\emptyset, \{a\}, X\}$ and $\mathcal{I} = \{\emptyset, \{a\}\}$. The multifunction $F\colon (X,\tau,\mathcal{I}) \to (X,\sigma)$ defined by $F(x) = \{x\}$ for all $x \in X$ is upper almost $\mathcal{I}$-continuous but is not upper $\mathcal{I}$-continuous.

THEOREM 3.3. *The following statements are equivalent for a multifunction* $F\colon (X,\tau,\mathcal{I}) \to (Y,\sigma)$:

(1) *F is upper almost $\mathcal{I}$-continuous multifunction,*

(2) *for each* $x \in X$ *and for each open set* V *such that* $F(x) \subset V$*, there exists* $U \in \mathcal{I}O(X, x)$ *such that if* $y \in U$*, then* $F(y) \subset \mathrm{Int}(\mathrm{Cl}(V)) = s\,\mathrm{Cl}(V)$*,*
(3) *for each* $x \in X$ *and for each regular open set* G *of* Y *such that* $F(x) \subset G$*, there exists* $U \in \mathcal{I}O(X, x)$ *such that* $F(U) \subset G$*,*
(4) *for each* $x \in X$ *and for each closed set* K *such that* $x \in F^{+}(Y \setminus K)$*, there exists an* $\mathcal{I}$*-closed set* H *such that* $x \in X \setminus H$ *and* $F^{-}(\mathrm{Cl}(\mathrm{Int}(K))) \subset H$*,*
(5) $F^{+}(\mathrm{Int}(\mathrm{Cl}(V))) \in \mathcal{I}O(X)$ *for any open set* $V \subset Y$*,*
(6) $F^{-}(\mathrm{Cl}(\mathrm{Int}(K))) \in \mathcal{I}C(X)$ *for any closed set* $K \subset Y$*,*
(7) $F^{+}(G) \in \mathcal{I}O(X)$ *for any regular open set* G *of* Y*,*
(8) $F^{-}(K) \in \mathcal{I}C(X)$ *for any regular closed set* K *of* Y*,*
(9) *for each point* x *of* X *and each neighborhood* V *of* $F(x)$*,* $F^{+}(\mathrm{Int}(\mathrm{Cl}(V)))$ *is an* $\mathcal{I}$*-neighborhood of* x*,*
(10) *for each point* x *of* X *and each neighborhood* V *of* $F(x)$*, there exists an* $\mathcal{I}$*-neighborhood* U *of* x *such that* $F(U) \subset \mathrm{Int}(\mathrm{Cl}(V))$*.*

Proof. (1) ⇔ (2): The proof follows from Definition 3.1 and Lemma 2.1.

(2) ⇒ (3): Let $x \in X$ and G be a regular open set of Y such that $F(x) \subset G$. By (2), there exists $U \in \mathcal{I}O(X, x)$ such that if $y \in U$, then $F(y) \subset \mathrm{Int}(\mathrm{Cl}(G)) = G$. We obtain $F(U) \subset G$.

(3) ⇒ (2): Let $x \in X$ and V be an open set of Y such that $F(x) \subset V$. Then, $\mathrm{Int}(\mathrm{Cl}(V)) \in RO(Y)$. By (3), there exists $U \in \mathcal{I}O(X, x)$ such that $F(U) \subset \mathrm{Int}(\mathrm{Cl}(V))$.

(2) ⇒ (4): Let $x \in X$ and K be a closed set of Y such that $x \in F^{+}(Y \setminus K)$. By (2), there exists $U \in \mathcal{I}O(X, x)$ such that $F(U) \subset \mathrm{Int}(\mathrm{Cl}(Y \setminus K))$. We have $\mathrm{Int}(\mathrm{Cl}(Y \setminus K)) = Y \setminus \mathrm{Cl}(\mathrm{Int}(K))$ and $U \subset F^{+}(Y \setminus \mathrm{Cl}(\mathrm{Int}(K))) = X \setminus F^{-}(\mathrm{Cl}(\mathrm{Int}(K)))$. We obtain $F^{-}(\mathrm{Cl}(\mathrm{Int}(K))) \subset X \setminus U$. Take $H = X \setminus U$. Then, $x \in X \setminus H$ and H is $\mathcal{I}$-closed set.

(4) ⇒ (2): Let $x \in X$ and V be an open set of Y such that $F(x) \subset V$. Then $Y \setminus V$ is closed in Y and $x \in F^{+}(V) = F^{+}(Y \setminus (Y \setminus V))$. By (4), there exits an $\mathcal{I}$-closed set L such that $x \in X \setminus L$ and $F^{-}(\mathrm{Cl}(\mathrm{Int}(Y \setminus V))) \subset L$. This implies that $X \setminus L \subseteq F^{+}(\mathrm{Int}(\mathrm{Cl}(V)))$. Put $U = X \setminus L$. Then $U \in \mathcal{I}O(X)$ and if $y \in U$, then $F(y) \subset \mathrm{Int}(\mathrm{Cl}(V))$.

(1) ⇒ (5): Let V be any open set of Y and $x \in F^{+}(\mathrm{Int}(\mathrm{Cl}(V)))$. By (1), there exists $U_x \in \mathcal{I}O(X, x)$ such that $U_x \subset F^{+}(\mathrm{Int}(\mathrm{Cl}(V)))$. Therefore, we obtain $F^{+}(\mathrm{Int}(\mathrm{Cl}(V))) = \bigcup_{x \in F^{+}(\mathrm{Int}(\mathrm{Cl}(V)))} U_x$. Hence, $F^{+}(\mathrm{Int}(\mathrm{Cl}(V))) \in \mathcal{I}O(X)$.

(5) ⇒ (1): Let V be any open set of Y and $x \in F^{+}(V)$. By (5), $F^{+}(\mathrm{Int}(\mathrm{Cl}(V))) \in$ $\in \mathcal{I}O(X)$. Take $U = F^{+}(\mathrm{Int}(\mathrm{Cl}(V)))$. Then $F(U) \subset \mathrm{Int}(\mathrm{Cl}(V))$. Hence, F is upper almost $\mathcal{I}$-continuous.

(5) $\Rightarrow$ (6): Let K be any closed set of Y. Then, $Y \setminus K$ is an open set of Y. By (5), $F^+(\mathrm{Int}(\mathrm{Cl}(Y \setminus K))) \in \mathcal{I}O(X)$. Since $\mathrm{Int}(\mathrm{Cl}(Y \setminus K)) = Y \setminus \mathrm{Cl}(\mathrm{Int}(K))$, it follows that $F^+(\mathrm{Int}(\mathrm{Cl}(Y \setminus K))) = F^+(Y \setminus \mathrm{Cl}(\mathrm{Int}(K))) = X \setminus F^-(\mathrm{Cl}(\mathrm{Int}(K)))$. We obtain that $F^-(\mathrm{Cl}(\mathrm{Int}(K)))$ is $\mathcal{I}$-closed in X.

(6) $\Rightarrow$ (5): It can be obtained similarly as (5)$\Rightarrow$(6).

(5) $\Rightarrow$ (7): Let G be any regular open set of Y. By (5), $F^+(\mathrm{Int}(\mathrm{Cl}(G))) = F^+(G) \in \mathcal{I}O(X)$.

(7) $\Rightarrow$ (5): Let V be any open set of Y. Then, $\mathrm{Int}(\mathrm{Cl}(V)) \in RO(Y)$. By (7), $F^+(\mathrm{Int}(\mathrm{Cl}(V))) \in \mathcal{I}O(X)$.

(6) $\Rightarrow$ (8): It can be obtained similarly as (5)$\Rightarrow$(7).

(8) $\Rightarrow$ (6): It can be obtained similarly as (7)$\Rightarrow$(5).

(5) $\Rightarrow$ (9): Let $x \in X$ and V be a neighborhood of $F(x)$. Then there exists an open set G of Y such that $F(x) \subset G \subset V$. Then we have $x \in F^+(G) \subset F^+(V)$. Since $F^+(\mathrm{Int}(\mathrm{Cl}(G))) \in \mathcal{I}O(X)$, $F^+(\mathrm{Int}(\mathrm{Cl}(V)))$ is an $\mathcal{I}$-neighborhood of x.

(9) $\Rightarrow$ (10): Let $x \in X$ and V be a neighborhood of $F(x)$. By (9), $F^+(\mathrm{Int}(\mathrm{Cl}(V)))$ is an $\mathcal{I}$-neighborhood of x. Take $U = F^+(\mathrm{Int}(\mathrm{Cl}(V)))$. Then $F(U) \subset \mathrm{Int}(\mathrm{Cl}(V))$.

(10) $\Rightarrow$ (1): Let $x \in X$ and V be any open set of Y such that $F(x) \subset V$. Then V is a neighborhood of $F(x)$. By (10), there exists an $\mathcal{I}$-neighborhood U of x such that $F(U) \subset \mathrm{Int}(\mathrm{Cl}(V))$. Therefore, there exists $G \in \mathcal{I}O(X)$ such that $x \in G \subset U$ and hence $F(G) \subset F(U) \subset \mathrm{Int}(\mathrm{Cl}(V))$. We obtain that F is upper almost $\mathcal{I}$-continuous. ∎

Theorem 3.4. *For a multifunction $F\colon (X,\tau,\mathcal{I}) \to (Y,\sigma)$, the following statements are equivalent:*

(1) *F is lower almost $\mathcal{I}$-continuous multifunction,*
(2) *for each $x \in X$ and for each open set V such that $F(x) \cap V \neq \emptyset$, there exists $U \in \mathcal{I}O(X,x)$ such that if $y \in U$, then $F(y) \cap \mathrm{Int}(\mathrm{Cl}(V)) \neq \emptyset$,*
(3) *for each $x \in X$ and for each regular open set G of Y such that $F(x) \cap G \neq \emptyset$, there exists $U \in \mathcal{I}O(X,x)$ such that if $y \in U$, then $F(y) \cap G \neq \emptyset$,*
(4) *for each $x \in X$ and for each closed set K such that $x \in F^-(Y \setminus K)$, there exists an $\mathcal{I}$-closed set H such that $x \in X \setminus H$ and $F^+(\mathrm{Cl}(\mathrm{Int}(K))) \subset H$,*
(5) *$F^-(\mathrm{Int}(\mathrm{Cl}(V))) \in \mathcal{I}O(X)$ for any open set $V \subset Y$,*
(6) *$F^+(\mathrm{Cl}(\mathrm{Int}(K))) \in \mathcal{I}C(X)$ for any closed set $K \subset Y$,*
(7) *$F^-(G) \in \mathcal{I}O(X)$ for any regular open set G of Y,*
(8) *$F^+(K) \in \mathcal{I}C(X)$ for any regular closed set K of Y.*

Proof. We Prove only (1) $\Rightarrow$ (2), (2) $\Rightarrow$ (3), (3) $\Rightarrow$ (4). The other proofs can be obtained similarly as Theorem 3.3.

(1) ⇒ (2): Let $x \in X$ and V be an open subset of Y such that $F(x) \cap V \neq \emptyset$. Since F is lower almost $\mathcal{I}$-continuous, there exists $U \in \mathcal{I}O(X, x)$ such that $U \subset F^{-}(\mathrm{Int}(\mathrm{Cl}(V)))$. This implies that if $y \in U$, then $F(y) \cap \mathrm{Int}(\mathrm{Cl}(V)) \neq \emptyset$.

(2) ⇒ (3): Let $x \in x$ and G be a regular open subset of Y such that $F(x) \cap G \neq \emptyset$. Then $G = \mathrm{Int}(\mathrm{Cl}(G))$ is open in Y. By (2), there exits $U \in \mathcal{I}O(X, x)$ such that if $y \in U$, then $F(y) \cap \mathrm{Int}(\mathrm{Cl}(G)) \neq \emptyset$. That is, if $y \in U$, then $F(y) \cap G \neq \emptyset$.

(3) ⇒ (4): Let $x \in X$ and K be a closed subset of Y such that $x \in F^{-}(Y \setminus K)$. Then $\mathrm{Int}(\mathrm{Cl}(Y \setminus K))$ is regular open in Y such that $x \in F^{-}(\mathrm{Int}(\mathrm{Cl}(Y \setminus K)))$. Thus $F(x) \cap \mathrm{Int}(\mathrm{Cl}(Y \setminus K)) \neq \emptyset$. By (3), there exits $U \in \mathcal{I}O(X, x)$ such that if $y \in U$, then $F(y) \cap \mathrm{Int}(\mathrm{Cl}(Y \setminus K)) \neq \emptyset$. Hence $U \subset F^{-}(\mathrm{Int}(\mathrm{Cl}(Y \setminus K)))$, and so $U \subset X \setminus F^{+}(\mathrm{Cl}(\mathrm{Int}(K)))$. Set $L = X \setminus U$. Then L is a $\mathcal{I}$-closed set such that $x \in X \setminus L$ and $F^{+}(\mathrm{Cl}(\mathrm{Int}(K))) \subset L$.

(4) ⇒ (1): Let $x \in x$ and V be an open subset of Y such that $x \in F^{-}(V)$. Then $Y \setminus V$ is closed in Y such that $x \in F^{-}(Y \setminus (Y \setminus V))$. By (4), there exits an $\mathcal{I}$-closed set L such that $x \in X \setminus L$ and $F^{+}(\mathrm{Cl}(\mathrm{Int}(Y \setminus V))) \subset L$. Set $U = X \setminus L$. Thus U is $\mathcal{I}$-open in X such that $x \in U$ and $U \subset F^{-}(\mathrm{Int}(\mathrm{Cl}(V)))$. Therefore, F is lower almost $\mathcal{I}$-continuous. ∎

Definition 3.5. Let $(X, \tau, \mathcal{I})$ be an ideal topological space and let (x_α) be a net in X. It is said that the net $(x_\alpha)\mathcal{I}$-converges to x if for each $\mathcal{I}$-open set G containing x in X, there exists an index $\alpha_0 \in I$ such that $x_\alpha \in G$ for each $\alpha \geq \alpha_0$.

Theorem 3.6. *If $F : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ is a lower (upper) almost $\mathcal{I}$-continuous multifunction, then for each $x \in X$ and for each net (x_α) which $\mathcal{I}$-converges to x in X and for each open set $V \subset Y$ such that $x \in F^{-}(V)$ (resp. $x \in F^{+}(V)$), the net (x_α) is eventually in $F^{-}(\mathrm{Int}(\mathrm{Cl}(V)))$ (resp. $F^{+}(\mathrm{Int}(\mathrm{Cl}(V)))$).*

Proof. Let (x_α) be a net $\mathcal{I}$-converges to x in X and let V be any open set in Y such that $x \in F^{-}(V)$. Since F is lower almost $\mathcal{I}$-continuous multifunction, there exists an $\mathcal{I}$-open set U in X containing x such that $U \subset F^{-}(\mathrm{Int}(\mathrm{Cl}(V)))$. Since $(x_\alpha)\mathcal{I}$-converges to x, there exists an index $\alpha_0 \in J$ such that $x_\alpha \in U$ for all $\alpha \geq \alpha_0$. So we obtain that $x_\alpha \in U \subset F^{-}(\mathrm{Int}(\mathrm{Cl}(V)))$ for all $\alpha \geq \alpha_0$. Thus, the net (x_α) is eventually in $F^{-}(\mathrm{Int}(\mathrm{Cl}(V)))$.

The proof of the upper almost $\mathcal{I}$-continuity of F is similar to the above. ∎

Definition 3.7. Let (X, τ) be a topological space. The collection of all regular open sets forms a base for a topology τ^*. It is called the semiregularization. In case when $\tau = \tau^*$, the space (X, τ) is called semiregular [20].

THEOREM 3.8. *Let $F\colon (X,\tau,\mathcal{I}) \to (Y,\sigma)$ be a multifunction from a topological space (X,τ) to a semiregular topological space (Y,σ). Then F is lower almost $\mathcal{I}$-continuous multifunction if and only if F is lower $\mathcal{I}$-continuous.*

PROOF. Let $x \in X$ and let V be an open set such that $x \in F^-(V)$. Since (Y,σ) is a semiregular space, there exist regular open sets U_i for $i \in I$ such that $V = \underset{i\in I}{\cup} U_i$. We have $F^-(V) = F^-(\underset{i\in I}{\cup} U_i) = \underset{i\in I}{\cup} F^-(U_i)$. By Theorem 3.3, $F^-(U_i) \in \mathcal{I}O(X)$ for $i \in I$. We obtain $F^-(V) \in \mathcal{I}O(X)$. Hence F is lower $\mathcal{I}$-continuous. The converse is obvious. ∎

COROLLARY 3.9. *A multifunction $F\colon (X,\tau,\mathcal{I}) \to (Y,\sigma)$ is lower almost $\mathcal{I}$-continuous multifunction if and only if $F\colon (X,\tau,\mathcal{I}) \to (Y,\sigma^*)$ is lower $\mathcal{I}$-continuous.*

Suppose that (X,τ), (Y,σ) and (Z,η) are topological spaces. It is known that if $F_1\colon (X,\tau,\mathcal{I}) \to (Y,\sigma)$ and $F_2\colon (Y,\sigma) \to (Z,\eta)$ are multifunctions, then the composite multifunction $F_2 \circ F_1\colon (X,\tau) \to (Z,\eta)$ is defined by $(F_2 \circ F_1)(x) = F_2(F_1(x))$ for each $x \in X$.

THEOREM 3.10. *If $F\colon (X,\tau,\mathcal{I}) \to (Y,\sigma)$ is an upper (lower) semicontinuous multifunction and $G\colon (Y,\sigma) \to (Z,\eta)$ is an upper (lower) semicontinuous multifunction, then $G \circ F\colon (X,\tau) \to (Z,\eta)$ is an upper (lower) almost $\mathcal{I}$-continuous multifunction.*

PROOF. Let $V \subset Z$ be any regular open set. From the definition of $G \circ F$, we have $(G \circ F)^+(V) = F^+(G^+(V))$ (resp. $(G \circ F)^-(V) = F^-(G^-(V))$). Since G is upper (lower) semicontinuous multifunction, $G^+(V)$ (resp. $G^-(V)$) is an open set. Since F is upper (lower) $\mathcal{I}$-continuous multifunction, $F^+(G^+(V))$ (resp. $F^-(G^-(V))$) is an $\mathcal{I}$-open set. It shows that $G \circ F$ is an upper (resp. lower) almost $\mathcal{I}$-continuous multifunction. ∎

THEOREM 3.11. *A multifunction $F\colon (X,\tau,\mathcal{I}) \to (Y,\sigma)$ is upper almost $\mathcal{I}$-continuous if and only if $s\,\mathrm{Cl}\,F\colon (X,\tau,\mathcal{I}) \to (Y,\sigma)$ is upper almost $\mathcal{I}$-continuous, where $s\,\mathrm{Cl}\,F(x) = s\,\mathrm{Cl}(F(x))$ for each point $x \in X$.*

PROOF. Suppose that F is upper almost $\mathcal{I}$-continuous. Let V be any open set of Y such that $s\,\mathrm{Cl}\,F(x) \subset V$. Then $F(x) \subset V$ and by Theorem 3.3, there exists $U \in \mathcal{I}O(X,x)$ such that $F(U) \subset s\,\mathrm{Cl}(V)$. For each $u \in U$, $F(U) \subset s\,\mathrm{Cl}(V)$ and hence $(s\,\mathrm{Cl}\,F)^+(V) \subset \mathcal{I}\,\mathrm{Int}(s\,\mathrm{Cl}\,F)^+(s\,\mathrm{Cl}(V))$. It follows from Theorem 3.3, that $s\,\mathrm{Cl}\,F$ is upper almost $\mathcal{I}$-continuous. Conversely, suppose that $s\,\mathrm{Cl}\,F\colon (X,\tau,\mathcal{I}) \to (Y,\sigma)$ is upper almost $\mathcal{I}$-continuous. Let V be any open set of Y and $x \in F^+(V)$. Then $F(x) \subset V$ and $s\,\mathrm{Cl}\,F(x) \subset s\,\mathrm{Cl}(V)$. There exists $U \in \mathcal{I}O(X, x)$ such that $s\,\mathrm{Cl}\,F(U) \subset s\,\mathrm{Cl}(V)$. Therefore, we have $U \subset$ $\subset (s\,\mathrm{Cl}\,F)^+(s\,\mathrm{Cl}(V)) \subset F^+(s\,\mathrm{Cl}(V))$ and hence $x \in U \subset \mathcal{I}\,\mathrm{Int}(F^+(s\,\mathrm{Cl}(V)))$. Thus, we obtain $F^+(V) \subset \mathcal{I}\,\mathrm{Int}(F^+(s\,\mathrm{Cl}(V)))$ and by Theorem 3.3, F is upper almost $\mathcal{I}$-continuous. ∎

Theorem 3.12. *A multifunction $F\colon (X, \tau, \mathcal{I}) \to (Y, \sigma)$ is lower almost $\mathcal{I}$-continuous if and only if $s\,\mathrm{Cl}\,F\colon (X, \tau, \mathcal{I}) \to (Y, \sigma)$ is lower almost $\mathcal{I}$-continuous.*

Proof. Suppose that F is lower almost $\mathcal{I}$-continuous. Let $x \in X$ and V be any open set of Y such that $s\,\mathrm{Cl}(F)(x) \cap V \neq \emptyset$. Then we have $x \in (s\,\mathrm{Cl}\,F)^-(V)$ $= F^-(V)$ and $F(x) \cap V \neq \emptyset$. By Theorem 3.4, there exists $U \in \mathcal{I}O(X, x)$ such that $F(U) \cap s\,\mathrm{Cl}(V) \neq \emptyset$ for every $u \in U$. Therefore, we obtain that $(s\,\mathrm{Cl}\,F)(u) \cap s\,\mathrm{Cl}(V) \neq \emptyset$ for every $u \in U$. It follows from Theorem 3.4, that $s\,\mathrm{Cl}\,F$ is lower almost $\mathcal{I}$-continuous. Conversely, suppose that $s\,\mathrm{Cl}\,F$ is lower almost $\mathcal{I}$-continuous. Let $x \in X$ and V be any open set of Y such that $F(x) \cap V \neq \emptyset$. Then, we have $x \in F^-(V) = (s\,\mathrm{Cl}\,F)^-(V)$ and hence $(s\,\mathrm{Cl}\,F)(x) \cap V \neq \emptyset$. Since $s\,\mathrm{Cl}\,F$ is lower almost $\mathcal{I}$-continuous, by Theorem 3.4, there exists $U \in \mathcal{I}O(X, x)$ such that $(s\,\mathrm{Cl}\,F)(u) \cap s\,\mathrm{Cl}(V) \neq \emptyset$ for every $u \in U$. Therefore, we obtain that $F(u) \cap s\,\mathrm{Cl}(V) \neq \emptyset$ for every $u \in U$. It follows from Theorem 3.4, F is lower almost $\mathcal{I}$-continuous. ∎

Definition 3.13. A subset A of a topological space (X, τ) is said to be:

(1) α-regular [10] if for each $a \in A$ and any open set U containing a, there exists an open set G of X such that $a \in G \subset \mathrm{Cl}(G) \subset U$;

(2) α-paracompact [10] if every X-open cover A has an X-open refinement which covers A and is locally finite for each point of X.

Lemma 3.14. [10] *If A is an α-paracompact and α-regular set of a topological space (X, τ) and U an open neighborhood of A, then there exists an open set G of X such that $A \subset G \subset \mathrm{Cl}(G) \subset U$.*

Lemma 3.15. *If $F\colon (X, \tau, \mathcal{I}) \to (Y, \sigma)$ is a multifunction such that $F(x)$ is α-paracompact and α-regular for each $x \in X$, then we have the following*

(1) *$G^+(V) = F^+(V)$ for each open set V of Y,*

(2) *$G^-(V) = F^-(V)$ for each closed set V of Y, where G denotes $\mathrm{Cl}\,F$ or $\mathcal{I}\,\mathrm{Cl}\,F$.*

Proof. (1): Let V be any open set of Y and $x \in G^+(V)$. Then $G(x) \subset V$ and $F(x) \subset G(x) \subset V$. We have $x \in F^+(V)$ and hence $G^+(V) \subset F^+(V)$. Then we have $F(x) \subset V$ and by Lemma 3.14, there exists an open set H of V such that $F(x) \subset H \subset \mathrm{Cl}(H) \subset V$. Since $F^+(V) \subset G^+(V)$. Therefore, $G^+(V) = F^+(V)$.

(2): This follows from (1). ∎

THEOREM 3.16. *Let $F: (X, \tau, \mathcal{I}) \to (Y, \sigma)$ be a multifunction such that $F(x)$ is α-paracompact and α-regular for each $x \in X$. Then the following statements are equivalent:*

(1) *F is upper almost $\mathcal{I}$-continuous;*
(2) *$\mathcal{I}\,\mathrm{Cl}\,F$ is upper almost $\mathcal{I}$-continuous;*
(3) *$\mathrm{Cl}\,F$ is upper almost $\mathcal{I}$-continuous.*

PROOF. We put $G = \mathcal{I}\,\mathrm{Cl}\,F$ or $\mathrm{Cl}\,F$. Suppose that F is upper almost $\mathcal{I}$-continuous. Let $x \in X$ and V be any open set of Y containing $G(x)$. By Lemma 3.15, $x \in G^+(V) = F^+(V)$ and there exists $U \in \mathcal{I}O(X, x)$ such that $F(U) \subset s\,\mathrm{Cl}(V)$. Since $F(u)$ is α-paracompact and α-regular for each $u \in U$, there exists an open set H such that $F(u) \subset H \subset \mathrm{Cl}(H) \subset s\,\mathrm{Cl}(V)$ hence $G(u) \subset \mathrm{Cl}(H) \subset s\,\mathrm{Cl}(V)$ for each $u \in U$. Therefore, we obtain $G(U) \subset s\,\mathrm{Cl}(V)$. This shows that G is upper almost $\mathcal{I}$-continuous. Conversely, suppose that G is upper almost $\mathcal{I}$-continuous. Let $x \in X$ and V be any open set of Y containing $F(x)$. By Lemma 3.15, $x \in F^+(V) = G^+(V)$ and hence $G(x) \subset V$. There exists $U \in \mathcal{I}O(X, x)$ such that $G(U) \subset s\,\mathrm{Cl}(V)$. Therefore, we obtain $F(u) \subset s\,\mathrm{Cl}(V)$. This shows that F is upper almost $\mathcal{I}$-continuous. ∎

THEOREM 3.17. *Let $F: (X, \tau, \mathcal{I}) \to (Y, \sigma)$ be a multifunction such that $F(x)$ is α-paracompact and α-regular for each $x \in X$. Then the following statements are equivalent:*

(1) *F is lower almost $\mathcal{I}$-continuous;*
(2) *$\mathcal{I}\,\mathrm{Cl}\,F$ is lower almost $\mathcal{I}$-continuous;*
(3) *$\mathrm{Cl}\,F$ is lower almost $\mathcal{I}$-continuous.*

PROOF. We put $G = \mathcal{I}\,\mathrm{Cl}\,F$ or $\mathrm{Cl}\,F$. Suppose that F is lower almost $\mathcal{I}$-continuous. Let $x \in x$ and V be any open set of Y such that $G(x) \cap V \neq \emptyset$. Since V is open, $F(x) \cap V \neq \emptyset$ and there exists $U \in \mathcal{I}O(X, x)$ such that $F(u) \cap$ $\cap\, \mathrm{Int}(\mathrm{Cl}(V)) \neq \emptyset$ for each $u \in U$. Therefore, we obtain $G(u) \cap \mathrm{Int}(\mathrm{Cl}(V)) \neq \emptyset$ for each $u \in U$. This shows that G is lower almost $\mathcal{I}$-continuous. Conversely, suppose that G is lower almost $\mathcal{I}$-continuous. Let $x \in X$ and V any open set of Y such that $F(x) \cap V \neq \emptyset$. Since $F(x) \subset G(x)$, $G(x) \cap V \neq \emptyset$ and there exists $U \in \mathcal{I}O(X, x)$ such that $G(u) \cap \mathrm{Int}(\mathrm{Cl}(V)) \neq \emptyset$ for each $u \in U$. Therefore, by Theorem 3.4, F is lower almost $\mathcal{I}$-continuous. ∎

THEOREM 3.18. *For a multifunction* $F\colon (X, \tau, \mathcal{I}) \to (Y, \sigma)$ *such that* $F(x)$ *is an* α*-regular and* α*-paracompact set for each* $x \in X$*, the following are equivalent:*

(1) F *is upper weakly* $\mathcal{I}$*-continuous,*
(2) F *is upper almost* $\mathcal{I}$*-continuous,*
(3) F *is upper* $\mathcal{I}$*-continuous.*

PROOF. (1) $\Rightarrow$ (3). Suppose that F is upper weakly $\mathcal{I}$-continuous. Let $x \in X$ and G and open set of Y such that $F(x) \subset G$. Since $F(x)$ is α-regular α-paracompact, by Lemma 3.14, there exists an open set V such that $F(x) \subset V \subset \mathrm{Cl}(V) \subset G$. Since F is upper weakly $\mathcal{I}$-continuous at x and $F(x) \subset V$, there exists $U \in$ $\in \mathcal{I}O(X, x)$ such that $F(U) \subset \mathrm{Cl}(V)$ and hence $F(U) \subset \mathrm{Cl}(V) \subset G$. Therefore, F is upper $\mathcal{I}$-continuous. ∎

COROLLARY 3.19. *Let* $F\colon (X, \tau, \mathcal{I}) \to (Y, \sigma)$ *be a multifunction such that* $F(x)$ *is compact for each* $x \in X$ *and* Y *is regular. Then, the following are equivalent:*

(1) F *is upper weakly* $\mathcal{I}$*-continuous;*
(2) F *is upper almost* $\mathcal{I}$*-continuous;*
(3) F *is upper* $\mathcal{I}$*-continuous.*

LEMMA 3.20. [18] *If* A *is an* α*-regular set of* X*, then for every open set* G *which intersects* A*, there exists an open set* D *such that* $A \cap D \neq \emptyset$ *and* $\mathrm{Cl}(D) \subset G$*.*

THEOREM 3.21. *For a multifunction* $F\colon (X, \tau, \mathcal{I}) \to (Y, \sigma)$ *such that* $F(x)$ *is an* α*-regular set of* Y *for each* $x \in X$*, the following are equivalent:*

(1) F *is lower weakly* $\mathcal{I}$*-continuous,*
(2) F *is lower almost* $\mathcal{I}$*-continuous,*
(3) F *is lower* $\mathcal{I}$*-continuous.*

PROOF. (1) $\Rightarrow$ (3): Suppose that F is lower weakly $\mathcal{I}$-continuous. Let $x \in X$ and G an open set of Y such that $F(x) \cap G \neq \emptyset$. Since $F(x)$ is α-regular, by Lemma 3.20, there exists an open set D of Y such that $F(x) \cap D \neq \emptyset$ and $\mathrm{Cl}(D) \subset G$. Since F is lower weakly $\mathcal{I}$-continuous at x, there exists $U \in \mathcal{I}O(X, x)$ such that $F(u) \cap \mathrm{Cl}(D) \neq \emptyset$ for each $u \in U$. Since $\mathrm{Cl}(D) \subset G$, we have $F(u) \cap G \neq \emptyset$ for each $u \in U$. Therefore, F is lower $\mathcal{I}$-continuous. ∎

THEOREM 3.22. *Let* $F\colon (X, \tau, \mathcal{I}) \to (Y, \sigma)$ *be a multifunction such that* $F(x)$ *is closed in* Y *for each* $x \in X$ *and* Y *is normal. Then the following are equivalent:*

(1) F *is upper weakly* $\mathcal{I}$*-continuous,*
(2) F *is upper almost* $\mathcal{I}$*-continuous,*
(3) F *is upper* $\mathcal{I}$*-continuous.*

PROOF. (1) $\Rightarrow$ (3): Suppose that F is upper weakly $\mathcal{I}$-continuous. Let $x \in X$ and G an open set of Y containing $F(x)$. Since $F(x)$ is closed in Y, by the normality of Y there exists an open set V of Y such that $F(x) \subset V \subset \mathrm{Cl}(V) \subset G$. Since F is upper weakly $\mathcal{I}$-continuous, there exists $U \in \mathcal{I}O(X, x)$ such that $F(U) \subset \mathrm{Cl}(V) \subset G$. This shows that F is upper $\mathcal{I}$-continuous. ∎

DEFINITION 3.23. A topologial space (X, τ) is said to be rimcompact if each point of X has a base of neighborhoods with compact frontiers.

THEOREM 3.24. *If (Y, σ) is a rimcompact space and $F\colon (X, \tau, \mathcal{I}) \to (Y, \sigma)$ is a compact valued multifunction with the closed graph, then the following are equivalent:*

(1) *F is upper weakly $\mathcal{I}$-continuous;*
(2) *F is upper almost $\mathcal{I}$-continuous;*
(3) *F is upper $\mathcal{I}$-continuous.*

PROOF. Suppose that F is upper weakly α-continuous. Let $x \in X$ and V be any open set of Y containing $F(x)$. Since Y is rimcompact, for each $z \in F(x)$. Since Y is rimcompact, for each $z \in F(x)$ there exists an open set $W(z)$ such that $z \in W(z) \subset V$ and the frontier $Fr(W(z))$ is compact. The family $\{W(z)\colon z \in F(x)\}$ is a cover of $F(x)$ by open sets of Y. Since $F(x)$ is compact, there exists a finite number of points, say, $z_1, z_2, ..z_n$ in $F(x)$ such that $F(x) \subset \cup\{W(z_j)\colon 1 \leq j \leq n\}$. Let $W = \cup\{W(z_j)\colon 1 \leq j \leq n\}$, then we have $Fr(W)$ is compact, $F(x) \subset W \subset V$ and $F(x) \cap Fr(W) = F(x) \cap \mathrm{Cl}(W) \cap$ $\cap \mathrm{Cl}(Y \backslash W) \subset F(x) \cap Y \backslash W = \emptyset$. For each $y \in Fr(W)$, $(x, y) \in X \times Y \backslash G(F)$. Since $G(F)$ is closed, there exist open sets $U(y) \subset X$ and $V(y) \subset Y$ containing x and y, respectively, such that $F(U(y)) \cap V(y) = \emptyset$. The family $\{V(y)\colon y \in Fr(W)\}$ is a cover of $Fr(W)$ by open sets of Y. Since $Fr(W)$ is compact, there exists a finite subset K of $Fr(W)$ such that $Fr(W) \subset \cup\{V(y)\colon y \in K\}$. Since F is upper weakly $\mathcal{I}$-continuous, there exits $U_0 \in \mathcal{I}O(X, x)$ such that $F(U_0) \subset \mathrm{Cl}(W)$. Put $U = U_0 \cap (\cap\{U(y)\colon y \in K\})$. Then we obtain $U \in \mathcal{I}O(X, x)$, $F(U) \subset \mathrm{Cl}(W)$ and $F(U) \cap Fr(W) = \emptyset$. Therefore, we obtain $F(U) \subset W \subset V$. This shows that F is upper $\mathcal{I}$-continuous. ∎

COROLLARY 3.25. *If (Y, σ) is a rimcompact space and $f\colon (X, \tau, \mathcal{I}) \to (Y, \sigma)$ is an almost $\mathcal{I}$-continuous function with closed graph, then f is $\mathcal{I}$-continuous.*

For a multifunction $F\colon (X, \tau) \to (Y, \sigma)$, the graph multifunction $G_F\colon X \Rightarrow$ $\Rightarrow X \times Y$ is defined as follows: $G_F(x) = \{x\} \times F(x)$ for every $x \in X$.

LEMMA 3.26. *For a multifunction $F\colon (X, \tau) \to (Y, \sigma)$, the following hold:*

(1) $G^{+}F(A \times B) = A \cap F^{+}(B)$,
(2) $G^{-}F(A \times B) = A \cap F^{-}(B)$
for any subsets $A \subset X$ *and* $B \subset Y$ [16].

THEOREM 3.27. *Let* $F : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ *be a multifunction such that* $F(x)$ *is compact for each* $x \in X$. *Then* F *is upper almost* $\mathcal{I}$*-continuous if and only if* $G_F : X \to X \times Y$ *is upper almost* $\mathcal{I}$*-continuous.*

PROOF. Suppose that $G_F : X \to X \times Y$ is upper almost $\mathcal{I}$-continuous. Let $x \in X$ and V be any open set of Y containing $F(x)$. Since $X \times V$ is open in $X \times Y$ and $G_F(x) \subset X \times V$, there exists $U \in \mathcal{I}O(X, x)$ such that $G_F(U) \subset \subset \operatorname{Int}(\operatorname{Cl}(X \times V)) = X \times \operatorname{Int}(\operatorname{Cl}(V))$. By Lemma 3.26, we have $U \subset G_F^{+}(X \times \times \operatorname{Int}(\operatorname{Cl}(V))) = F^{+}(\operatorname{Int}(\operatorname{Cl}(V)))$ and $F(U) \subset \operatorname{Int}(\operatorname{Cl}(V))$. This shows that F is upper almost $\mathcal{I}$-continuous. Conversely, suppose that $F : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ is upper almost $\mathcal{I}$-continuous. Let $x \in X$ and W be any open set of $X \times Y$ containing $G_F(x)$. For each $y \in F(x)$, there exist open sets $U(y) \subset X$ and $V(y) \subset Y$ such that $(x, y) \in U(y) \times V(y) \subset W$. The family of $\{V(y) : y \in \in F(x)\}$ is an open cover of $F(x)$. Since $F(x)$ is compact, it follows that there exists a finite number of points, say $y_1, y_2, y_3, \ldots, y_n$ in $F(x)$ such that $F(x) \subset \cup\{V(y_i) : i = 1, 2, \ldots, n\}$. Take $U = \cap\{U(y_i) : i = 1, 2, \ldots, n\}$ and $V = \cup\{V(y_i) : i = 1, 2, \ldots, n\}$. Then U and V are open sets in X and Y, respectively, and $\{x\} \times F(x) \subset U \times V \subset W$. Since F is upper almost $\mathcal{I}$-continuous, there exists $U_0 \in \mathcal{I}O(X, x)$ such that $F(U_0) \subset \operatorname{Int}(\operatorname{Cl}(V))$. By Lemma 3.26, we have $U \cap U_0 \subset U \cap F^{+}(\operatorname{Int}(\operatorname{Cl}(V))) = G_F^{+}(U \times \operatorname{Int}(\operatorname{Cl}(V))) \subset G_F^{+}(\operatorname{Int}(\operatorname{Cl}(U \times \times V))) \subset G_F^{+}(\operatorname{Int}(\operatorname{Cl}(W)))$. Therefore, we obtain $U \cap U_0 \in \mathcal{I}O(X, x)$ and $G_F(U \cap U_0) \subset \operatorname{Int}(\operatorname{Cl}(W))$. This shows that G_F is upper almost $\mathcal{I}$-continuous. ∎

THEOREM 3.28. *A multifunction* $F : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ *is lower almost* $\mathcal{I}$*-continuous if and only if* $G_F : X \to X \times Y$ *is lower almost* $\mathcal{I}$*-continuous.*

PROOF. Suppose that F is lower almost $\mathcal{I}$-continuous. Let $x \in X$ and W be any open set of $X \times Y$ such that $x \in G_F^{-}(W)$. Since $W \cap (\{x\} \times F(x)) \neq \emptyset$, there exists $y \in F(x)$ such that $(x, y) \in W$ and hence $(x, y) \in U \times V \subset W$ for some open sets U and V of X and Y, respectively. Since $F(x) \cap V \neq \emptyset$, there exists $G \in \mathcal{I}O(X, x)$ such that $G \subset F^{-}(\operatorname{Int}(\operatorname{Cl}(V)))$. By Lemma 3.26, $U \cap G \subset U \cap F^{-}(\operatorname{Int}(\operatorname{Cl}(V))) = G_F^{-}(U \times \operatorname{Int}(\operatorname{Cl}(V))) \subset G_F^{-}(\operatorname{Int}(\operatorname{Cl}(W)))$. Furthermore, $x \in U \cap G \in \mathcal{I}O(X)$ and hence G_F is lower almost $\mathcal{I}$-continuous. Conversely, suppose that G_F is lower almost $\mathcal{I}$-continuous. Let $x \in X$ and V be any open set of Y such that $x \in F^{-}(V)$. Then $X \times V$ is open in $X \times Y$ and $G_F(x) \cap (X \times V) = (\{x\} \times F(x)) \cap (X \times V) = \{x\} \times (F(x) \cap V) \neq \emptyset$. Since G_F is lower almost $\mathcal{I}$-continuous, there exists an $\mathcal{I}$-open set U containing x such that

$U \subset G_F^-(\mathrm{Int}(\mathrm{Cl}(X \times V)))$. Since $G_F^-(\mathrm{Int}(\mathrm{Cl}(X \times V))) = G_F^-(X \times \mathrm{Int}(\mathrm{Cl}(V)))$, by Lemma 3.26, we have $U \subset F^-(\mathrm{Int}(\mathrm{Cl}(V)))$. This shows that F is lower almost $\mathcal{I}$-continuous. ∎

DEFINITION 3.29. Let $F\colon (X, \tau, \mathcal{I}) \to (Y, \sigma)$ be a multifunction. The multigraph $G(F)$ is said to be $\mathcal{I}$-closed graph in $X \times Y$ if for each $(x, y) \in X \times Y \setminus G(F)$, there exist $\mathcal{I}$-open set U and an open set V containing x and y, respectively, such that $(U \times V) \cap G(F) = \emptyset$.

THEOREM 3.30. *Let $F\colon (X, \tau) \to (Y, \sigma)$ be an upper almost $\mathcal{I}$-continuous and punctually α-paracompact multifunction into a Hausdorff space (Y, σ). Then the multigraph $G(F)$ of F is an $\mathcal{I}$-closed graph in $X \times Y$.*

PROOF. Suppose that $(x_0, y_0) \notin G(F)$. Then $y_0 \notin F(x_0)$. Since (Y, σ) is a Hausdorff space, then for each $y \in F(x_0)$ there exist open sets $V(y)$ and $W(y)$ containing y and y_0 respectively such that $V(y) \cap W(y) = \emptyset$. The family $\{V(y)\colon y \in F(x_0)\}$ is an open cover of $F(x_0)$ which is α-paracompact. Thus, it has a locally finite open refinement $\Phi = \{U_\beta\colon \beta \in I\}$ which covers $F(x_0)$. Let W_0 be an open neighborhood of y_0 such that W_0 intersects only finitely many members $U_{\beta_1}, U_{\beta_2}, \ldots, U_{\beta_n}$ of Φ. Choose $y_1, y_2, \ldots, y_n$ in $F(x_0)$ such that $U_{\beta_i} \subset V(y_i)$ for each $i = 1, 2, \ldots, n$ and set $W = W_0 \cap (\bigcap_{i=1}^{n} W(y_i))$. Then W is an open neighborhood of y_0 with $W \cap (\bigcup_{\beta \in I} U_\beta) = \emptyset$, which implies that $W \cap \mathrm{Int}(\mathrm{Cl}(\bigcup_{\beta \in I} U_\beta)) = \emptyset$. By the upper almost $\mathcal{I}$ continuity of F, there exists $U \in \mathcal{I}O(X, x_0)$ such that $F(U) \subset \mathrm{Int}(\mathrm{Cl}(\bigcup_{\beta \in I} U_\beta))$. It follows that $(U \times W) \cap G(F) = \emptyset$. Therefore, the graph $G(F)$ is an $\mathcal{I}$-closed graph in $X \times Y$. ∎

Let $\{X_\alpha\colon \alpha \in \nabla\}$ and $\{Y_\alpha\colon \alpha \in \nabla\}$ be any two families of topological spaces with same index set ∇. For each $\alpha \in \nabla$, let $F_\alpha\colon X_\alpha \to Y_\alpha$ be a multifunction. The product space $\prod\{X_\alpha\colon \alpha \in \nabla\}$ will be denoted by $\prod X_\alpha$ and the product multifunction $\prod F_\alpha\colon \prod X_\alpha \to \prod Y_\alpha$, defined by $F(x) = \prod\{F_\alpha(x_\alpha)\colon \alpha \in \nabla\}$ for each $x = \{x_\alpha\} \in \prod X_\alpha$, is simply denoted by $F\colon \prod X_\alpha \to Y_\alpha$.

THEOREM 3.31. *Let $F_\alpha\colon (X, \tau, \mathcal{I}) \to (Y, \sigma)_\alpha$ be a multifunction for each $\alpha \in \nabla$ and $F\colon X \to \prod Y_\alpha$ a multifunction defined by $F(x) = \prod\{F_\alpha(x)\colon \alpha \in \nabla\}$ for each $x \in X$. If F is upper almost $\mathcal{I}$-continuous (resp. lower almost $\mathcal{I}$-continuous), then F_α is upper almost $\mathcal{I}$-continuous (resp. lower almost $\mathcal{I}$-continuous) for each $\alpha \in \nabla$.*

Proof. Let $x \in X$, $\alpha \in \nabla$ and V_α any regular open set of Y_α containing $F_\alpha(x)$. Then $P_\alpha^{-1}(V_\alpha) = V_\alpha \times \prod\{Y_\beta : \beta \in \nabla \text{ and } \beta \neq \alpha\}$ is a regular open set of $\prod Y_\alpha$ containing $F(x)$, where P_α is the natural projection of $\prod Y_\alpha$ onto Y_α. Since F is upper almost $\mathcal{I}$-continuous, there exists $U \in \mathcal{I}O(X, x)$ such that $F(U) \subset \subset p_\alpha^{-1}(V_\alpha)$. Therefore, we obtain $F_\alpha(U) \subset P_\alpha(F(U)) \subset P_\alpha(P_\alpha^{-1}(V_\alpha)) = V_\alpha$. This shows that $F_\alpha : (X, \tau, \mathcal{I}) \to (Y, \sigma)_\alpha$ is upper almost $\mathcal{I}$-continuous for each $\alpha \in \nabla$. The proof for lower almost $\mathcal{I}$-continuous is similar and is thus omitted. ∎

Theorem 3.32. *If (Y, σ) is a Hausdorff space and $F, G : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ are multifunctions such that*

(1) *$F(x)$ and $G(x)$ are compact for each $x \in X$,*
(2) *G is upper weakly $\mathcal{I}$-continuous,*
(3) *F is upper almost $\mathcal{I}$-continuous,*

then the set $A = \{x \in X : F(x) \cap G(x) \neq \emptyset\}$ is $\mathcal{I}$-closed in X.

Proof. Let $x \in X \setminus A$. Then $F(x) \cap G(x) = \emptyset$. Since $F(x)$ and $G(x)$ are compact in a Hausdorff space (Y, σ), there exist disjoint regular open sets V and W of Y such that $F(x) \subset V$, $G(x) \subset W$ and $V \cap \mathrm{Cl}(W) = \emptyset$. Since F is upper almost $\mathcal{I}$-continuous, there exists $U_1 \in \mathcal{I}O(X, x)$ such that $F(U_1) \subset V$. Since G is upper weakly $\mathcal{I}$-continuous, there exists $U_2 \in \mathcal{I}O(X, x)$ such that $G(U_2) \subset \mathrm{Cl}(W)$. Put $U = U_1 \cap U_2$, then $U \in \mathcal{I}O(X, x)$ and $F(U) \cap G(U) = \emptyset$. Therefore, we obtain $U \cap A = \emptyset$ and $x \in X \setminus \mathcal{I}\,\mathrm{Cl}(A)$. Hence A is $\mathcal{I}$-closed in X. ∎

Theorem 3.33. *If $F : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ is an upper almost $\mathcal{I}$-continuous multifunction such that $F(x)$ is α-nearly paracompact for each $x \in X$ and Y is Hausdorff, then for each $(x, y) \in X \times Y \setminus G(F)$, there exist $U \in \mathcal{I}O(X, x)$ and an open set V containing y such that $(U \times \mathrm{Cl}(V)) \cap G(F) = \emptyset$.*

Proof. Let $(x, y) \in X \times Y \setminus G(F)$, then $y \in Y \setminus F(x)$. Since Y is Hausdorff, for each $a \in F(X)$ there exist open sets $V(a)$ and $W(a)$ containing a and y, respectively, such that $V(a) \cap W(a) = \emptyset$, hence $\mathrm{Int}(\mathrm{Cl}(V(a))) \cap W(a) = \emptyset$. The family $V = \{\mathrm{Int}(\mathrm{Cl}(V(a))) : a \in F(x)\}$ is a cover of $F(x)$ by regular open sets of Y and $F(x)$ is α-nearly paracompact. There exists a locally finite open refinement $H = \{H_\alpha : \alpha \in \nabla\}$ of V such that $F(x) \subset \cup\{H_\alpha : \alpha \in \nabla\}$. Since H is locally finite, there exists an open neighborhood W_0 of Y and a finite subset ∇_0 of ∇ such that $W_0 \cap H_\alpha = \emptyset$ for every $\alpha \in \nabla \setminus \nabla_0$. For each $\alpha \in \nabla_0$, there exists $a(\alpha) \in F(x)$ such that $H_\alpha \subset V(a(\alpha))$. Now, put $W = W_0 \cap (\cap\{W(a(\alpha)) : \alpha \in \nabla_0\})$ and $H = \cup\{H_\alpha : \alpha \in \nabla\}$. Then W is an open neighborhood of y, H is open in Y and $W \cap H = \emptyset$. Therefore, we obtain $F(x) \subset H$ and $\mathrm{Cl}(W) \cap H = \emptyset$ an hence $F(x) \subset Y \setminus \mathrm{Cl}(W)$. Since W is open, $Y \setminus \mathrm{Cl}(W)$ is regular open in Y. Since F is upper almost $\mathcal{I}$-continuous, there exists $U \in \mathcal{I}O(X, x)$ such that $F(U) \subset Y \setminus \mathrm{Cl}(W)$, hence $F(U) \cap \mathrm{Cl}(W) = \emptyset$. Therefore, we obtain $(U \times \mathrm{Cl}(V)) \cap G(F) = \emptyset$. ∎

Corollary 3.34. *If $F\colon (X, \tau, \mathcal{I}) \to (Y, \sigma)$ is an upper almost $\mathcal{I}$-continuous multifunction such that $F(x)$ is compact for each $x \in X$ and Y is Hausdorff, then for each $(x, y) \in X \times Y \setminus G(F)$, there exist $U \in \mathcal{I}O(X, x)$ and an open set V containing y such that $(U \times \mathrm{Cl}(V)) \cap G(F) = \emptyset$.*

Corollary 3.35. *If $f\colon (X, \tau, \mathcal{I}) \to (Y, \sigma)$ is an $\mathcal{I}$-continuous function into a Hausdorff space Y, then $G(f)$ is $\mathcal{I}$-closed.*

Theorem 3.36. *Suppose that (X, τ) and (X_α, τ_α) are topological spaces, where $\alpha \in J$. Let $F\colon X \to \prod_{\alpha \in J} X_\alpha$ be a multifunction from X to the product space $\prod_{\alpha \in J} X_\alpha$ and let $P_\alpha\colon \prod_{\alpha \in J} X_\alpha \to X_\alpha$ be the projection for each $\alpha \in J$. If F is upper (lower) almost $\mathcal{I}$-continuous multifunction, then $P_\alpha \circ F$ is upper (resp. lower) almost $\mathcal{I}$-continuous multifunction for each $\alpha \in J$.*

Proof. Take any $\alpha_0 \in J$. Let V_{α_0} be an open set in $(X_{\alpha_0}, \tau_{\alpha_0})$. Then

$$(P_{\alpha_0} \circ F)^+(\mathrm{Int}(\mathrm{Cl}(V_{\alpha_0}))) = F^+(P^+_{\alpha_0}(\mathrm{Int}(\mathrm{Cl}(V_{\alpha_0})))) = \\ = F^+(\mathrm{Int}(\mathrm{Cl}(V_{\alpha_0}))) \times \prod_{\alpha \neq \alpha_0} X_\alpha$$

(resp.

$$(P_{\alpha_0} \circ F)^-(\mathrm{Int}(\mathrm{Cl}(V_{\alpha_0}))) = F^-(P^-_{\alpha_0}(\mathrm{Int}(\mathrm{Cl}(V_{\alpha_0})))) = \\ = F^-(\mathrm{Int}(\mathrm{Cl}(V_{\alpha_0})) \times \prod_{\alpha \neq \alpha_0} X_\alpha).)$$

Since F is upper (resp. lower) almost $\mathcal{I}$-continuous multifunction and since $\mathrm{Int}(\mathrm{Cl}(V_{\alpha_0})) \times \prod_{\alpha \neq \alpha_0} X_\alpha$ is a regular open set, it follows that $F^+(\mathrm{Int}(\mathrm{Cl}(V_{\alpha_0})) \times \times \prod_{\alpha \neq \alpha_0} X_\alpha)$ (resp. $F^-(\mathrm{Int}(\mathrm{Cl}(V_{\alpha_0})) \times \prod_{\alpha \neq \alpha_0} X_\alpha)$) is $\mathcal{I}$-open in (X, τ). It shows that $P_{\alpha_0} \circ F$ is upper (lower) almost $\mathcal{I}$-continuous multifunction. Hence, we obtain that $P_\alpha \circ F$ is upper (lower) almost $\mathcal{I}$-continuous multifunction for each $\alpha \in J$. ∎

Theorem 3.37. *Suppose that for each $\alpha \in J$, $(X_\alpha, \tau_\alpha), (Y_\alpha, \sigma_\alpha)$ are topological spaces. Let $F_\alpha\colon X_\alpha \to Y_\alpha$ be a multifunction for each $\alpha \in J$ and* *let* $F\colon \prod_{\alpha\in J} X_\alpha \to \prod_{\alpha\in J} Y_\alpha$ *be defined by* $F((x_\alpha)) = \prod_{\alpha\in J} F_\alpha(x_\alpha)$ *from the product space* $\prod_{\alpha\in J} X_\alpha$ *to the product space* $\prod_{\alpha\in J} Y_\alpha$. *If* F *is upper (lower) almost* $\mathcal{I}$*-continuous multifunction, then each* F_α *is upper (resp. lower) almost* $\mathcal{I}$*-continuous multifunction for each* $\alpha \in J$.

Proof. Let $V_\alpha \subseteq Y_\alpha$ be an open set. Then $\mathrm{Int}(\mathrm{Cl}(V_\alpha)) \times \prod_{\alpha\neq\beta} Y_\beta$ is a regular open set. Since F is upper (lower) almost $\mathcal{I}$-continuous multifunction, it follows that $F^+(\mathrm{Int}(\mathrm{Cl}(V_\alpha)) \times \prod_{\alpha\neq\beta} Y_\beta) = F_\alpha^+(\mathrm{Int}(\mathrm{Cl}(V_\alpha))) \times \prod_{\alpha\neq\beta} X_\beta$ (resp. $F^-(\mathrm{Int}(\mathrm{Cl}(V_\alpha)) \times \times \prod_{\alpha\neq\beta} Y_\beta) = F_\alpha^-(\mathrm{Int}(\mathrm{Cl}(V_\alpha))) \times \prod_{\alpha\neq\beta} X_\beta$) is an $\mathcal{I}$-open set. Consequently, we obtain that $F_\alpha^+(\mathrm{Int}(\mathrm{Cl}(V_\alpha)))$ (resp. $F_\alpha^-(\mathrm{Int}(\mathrm{Cl}(V_\alpha)))$) is an $\mathcal{I}$-open set. Thus, we show that F_α is upper (resp. lower) almost $\mathcal{I}$-continuous multifunction. ∎

Theorem 3.38. *Suppose that* (X,τ), (Y,σ), (Z,η) *are topological spaces and* $F_1\colon (X,\tau,\mathcal{I}) \to (Y,\sigma)$, $F_2\colon (X,\tau) \to (Z,\eta)$ *are multifunctions. Let* $F_1 \times \times F_2\colon (X,\tau,\mathcal{I}) \to (Y,\sigma)\times Z$ *be a multifunction which is defined by* $(F_1 \times \times F_2)(x) = F_1(x)\times F_2(x)$ *for each* $x \in X$. *If* $F_1 \times F_2$ *is upper (lower) almost* $\mathcal{I}$*-continuous multifunction, then* F_1 *and* F_2 *are upper (resp. lower) almost* $\mathcal{I}$*-continuous multifunctions.*

Proof. Let $x \in X$ and let $K \subset Y$, $H \subset Z$ be open sets such that $x \in F_1^+(K)$ and $x \in F_2^+(H)$. Then we obtain that $F_1(x) \subset K$ and $F_2(x) \subset H$ and so $F_1(x)\times F_2(x) = (F_1\times F_2)(x) \subset K\times H$. We have $x \in (F_1\times F_2)^+(K\times H)$.Since $F_1 \times F_2$ is upper almost $\mathcal{I}$-continuous multifunction, there exists an $\mathcal{I}$-open set U containing x such that $U \subset (F_1\times F_2)^+(\mathrm{Int}(\mathrm{Cl}(K\times H)))$. We obtain that $U \subset F_1^+(\mathrm{Int}(\mathrm{Cl}(K)))$ and $U \subset F_2^+(\mathrm{Int}(\mathrm{Cl}(H)))$. Thus, we obtain that F_1 and F_2 are upper almost $\mathcal{I}$-continuous multifunctions. The proof of the lower almost $\mathcal{I}$ continuity of F_1 and F_2 is similar to the above. ∎

Lemma 3.39. [1] *Let* A *and* B *be subsets of a topological space* (X,τ). *Then*
(1) *If* $A \in \mathcal{I}O(X)$ *and* $B \in \tau$, *then* $A\cap B \in \mathcal{I}O(B)$;
(2) *If* $A \in \mathcal{I}O(B)$ *and* $B \in \mathcal{I}O(X)$, *then* $A \in \mathcal{I}O(X)$.

Lemma 3.40. *If* $F\colon (X,\tau,\mathcal{I}) \to (Y,\sigma)$ *is an upper almost* $\mathcal{I}$*-continuous (lower almost* $\mathcal{I}$*-continuous) multifunction and* $U \in \tau$, *then* $F_{|U}\colon (U,\tau_U) \Rightarrow (Y,\sigma)$ *is upper almost* $\mathcal{I}$*-continuous (lower almost* $\mathcal{I}$*-continuous).*

Proof. Suppose that V is an open subset of Y. Let $x \in U$ and let $x \in (F_{|U})^-(V)$. Since F is lower almost $\mathcal{I}$-continuous multifunction, there exists an $\mathcal{I}$-open set G such that $x \in G \subset F^-(\mathrm{Int}(\mathrm{Cl}(V)))$. By Lemma 3.39, we obtain that

$x \in G \cap U \in \mathcal{I}O(U)$ and $G \cap U \subset (F_{|U})^-(\text{Int}(\text{Cl}(V)))$. Hence $F_{|U}$ is lower almost $\mathcal{I}$-continuous. The proof of the upper almost $\mathcal{I}$-continuity of $F_{|U}$ is similar to the above. ∎

THEOREM 3.41. *Let $\{U_\alpha : \alpha \in \Lambda\}$ be an open cover of a space (X, τ). Then a multifunction $F: (X, \tau, \mathcal{I}) \to (Y, \sigma)$ is upper almost $\mathcal{I}$-continuous (resp. lower almost $\mathcal{I}$-continuous) if and only if the restriction $F_{|U_\alpha}: (U_\alpha, \tau_\alpha) \to (Y, \sigma)$ is upper almost $\mathcal{I}$-continuous (resp. lower almost $\mathcal{I}$-continuous) for each $\alpha \in \Lambda$.*

PROOF. We prove only the case for F upper almost $\mathcal{I}$-continuous, the proof for F lower almost $\mathcal{I}$-continuous being analogous. Let $\alpha \in \Lambda$ and V be any open set of Y. Since F is upper almost $\mathcal{I}$-continuous, $F^+(\text{Int}(\text{Cl}(V)))$ is $\mathcal{I}$-open in X. By Lemma 3.39, $(F_{|U_\alpha})^+(\text{Int}(\text{Cl}(V))) = F^+(\text{Int}(\text{Cl}(V))) \cap U_\alpha$ is $\mathcal{I}$-open in U_α and hence $F_{|U_\alpha}$ is upper almost $\mathcal{I}$-continuous. Conversely, let V be any open set of Y. Since $F_{|U_\alpha}$ is upper almost $\mathcal{I}$-continuous for each $\alpha \in \Lambda$, $(F_{|U_\alpha})^+(\text{Int}(\text{Cl}(V))) = F^+(\text{Int}(\text{Cl}(V))) \cap U_\alpha$ is $\mathcal{I}$-open in U_λ. By Lemma 3.39, $(F_{|U_\alpha})^+(\text{Int}(\text{Cl}(V)))$ is $\mathcal{I}$-open in X for each $\alpha \in \Lambda$. We obtain that $F^+(\text{Int}(\text{Cl}(V))) = \bigcup_{\alpha \in \Lambda} (F_{|U_\alpha})^+(\text{Int}(\text{Cl}(V)))$ is $\mathcal{I}$-open in X. Hence F is upper almost $\mathcal{I}$-continuous. ∎

Recall that a multifunction $F: (X, \tau) \to (Y, \sigma)$ is said to be punctually connected if for each $x \in X$, $F(x)$ is connected.

DEFINITION 3.42. An ideal topological space $(X, \tau, \mathcal{I})$ is called $\mathcal{I}$-connected provided that X is not the union of two nonempty disjoint $\mathcal{I}$-open sets.

THEOREM 3.43. *Let F be a multifunction from an $\mathcal{I}$-connected topological space $(X, \tau, \mathcal{I})$ onto a topological space (Y, σ) such that F is punctually connected. If F is an upper almost $\mathcal{I}$-continuous multifunction, then Y is a connected space.*

PROOF. Let $F: (X, \tau, \mathcal{I}) \to (Y, \sigma)$ be an upper almost $\mathcal{I}$-continuous multifunction from an $\mathcal{I}$ connected topological space X onto a topological space Y. Suppose that Y is not connected and let $Y = H \cup K$ be a partition of Y. Then both H and K are open and closed subsets of Y. Since F is upper almost $\mathcal{I}$-continuous, $F^+(H)$ and $F^+(K)$ are $\mathcal{I}$-open subsets of X. In view of the fact that $F^+(H), F^+(K)$ are disjoint and F is punctually connected, $X = F^+(H) \cup F^+(K)$ is a partition of X. This is contrary to the $\mathcal{I}$-connectedness of X. Hence, it is obtained that Y is a connected space. ∎

Recall that a multifunction $F: (X, \tau, \mathcal{I}) \to (Y, \sigma)$ is said to be punctually closed if for each $x \in X, F(x)$ is closed.

THEOREM 3.44. *Let F be an upper almost $\mathcal{I}$-continuous punctually closed multifunction and G be an upper almost continuous punctually closed multifunction from a space (X, τ) to a normal space (Y, σ). Then the set $K = \{x \in X : F(x) \cap \cap G(x) \neq \emptyset\}$ is $\mathcal{I}$-closed in X.*

PROOF. Let $x \in X \setminus K$. Then $F(x) \cap G(x) = \emptyset$. Since F and G are punctually closed multifunctions and Y is a normal space, there exists disjoint open sets U and V containing $F(x)$ and $G(x)$, respectively. Since F and G are upper almost $\mathcal{I}$-continuous and upper almost continuous, respectively the sets $F^+(\mathrm{Int}(\mathrm{Cl}(U)))$ and $G^+(\mathrm{Int}(\mathrm{Cl}(V)))$ are $\mathcal{I}$-open and open sets, respectively containing x. Let $H = F^+(\mathrm{Int}(\mathrm{Cl}(U))) \cap G^+(\mathrm{Int}(\mathrm{Cl}(V)))$. Then H is an $\mathcal{I}$-open set containing x and $H \cap K = \emptyset$. Hence, K is $\mathcal{I}$ closed in X. ∎

DEFINITION 3.45. An ideal topological space $(X, \tau, \mathcal{I})$ is said to be $\mathcal{I}$-T_2 [14] if for each pair of distinct points x and y in X, there exist disjoint $\mathcal{I}$-open sets U and V in X such that $x \in U$ and $y \in V$.

THEOREM 3.46. *Let $F : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ be an upper almost $\mathcal{I}$-continuous multifunction and punctually closed from a topological space (X, τ) to a normal topological space (Y, σ) and let $F(x) \cap F(y) = \emptyset$ for each distinct pair $x, y \in X$. Then X is an $\mathcal{I}$-T_2 space.*

PROOF. Let x and y be any two distinct points in X. Then we have $F(x) \cap \cap F(y) = \emptyset$. Since (Y, σ) is a normal space, it follows that there exist disjoints open sets U and V containing $F(x)$ and $F(y)$, respectively. Thus $F^+(\mathrm{Int}(\mathrm{Cl}(U)))$ and $F^+(\mathrm{Int}(\mathrm{Cl}(V)))$ are disjoint $\mathcal{I}$-open sets containing x and y, respectively. Thus, it is obtained that (X, τ) is $\mathcal{I}$-T_2. ∎

In the following $(D, >)$ is a directed set, (F_λ) is a net of multifunction $F_\lambda : (X, \tau) \to (Y, \sigma)$ for every $\lambda \in D$ and F is a multifunction from X into Y.

DEFINITION 3.47. Let $(F_\lambda)_{\lambda \in D}$ be a net of multifunctions from X to Y. A multifunction $F^\star : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ is defined as follows: for each $x \in \in X$, $F^\star(x) = \{y \in Y :$ for each open neighborhood V of y and each $\mu \in \in D$, there exists $\lambda \in D$ such that $\lambda > \mu$ and $V \cap F_\lambda(x) \neq \emptyset\}$ is called the upper topological limit of the net $(F_\lambda)_{\lambda \in D}$ [5].

DEFINITION 3.48. A net $(F_\lambda)_{\lambda \in D}$ is said to be equally upper almost $\mathcal{I}$-continuous at $x_0 \in X$ if for every open set V containing $F_\lambda(x_0)$, there exists an $\mathcal{I}$-open set U containing x_0 such that $F_\lambda(U) \subset \mathrm{Int}(\mathrm{Cl}(V_\lambda))$ for all $\lambda \in D$.

THEOREM 3.49. *Let $(F_\lambda)_{\lambda \in D}$ be a net of multifunctions from an ideal topological space $(X, \tau, \mathcal{I})$ into a compact space (Y, σ). If the following are satisfied:*

(1) $\cup\{F_\mu(x)\colon \mu > \lambda\}$ *is closed in* Y *for each* $\lambda \in D$ *and each* $x \in X$*;*
(2) $(F_\lambda)_{\lambda\in D}$ *is equally upper almost* $\mathcal{I}$*-continuous on* X*, then* $F^\star$ *is upper almost* $\mathcal{I}$*-continuous on* X*, then* $F^\star$ *is upper almost* $\mathcal{I}$*-continuous on* X*.*

Proof. We have $F^\star(x) = \cap\{(\cup\{F_\mu(x)\colon \mu > \lambda\})\colon \lambda \in D\}$. Since the net $(\cup\{F_\mu(x)\colon \mu > \lambda\})_{\lambda\in D}$ is a family of closed sets having the finite intersection property and Y is compact, $F^\star(x) \neq \emptyset$ for each $x \in X$. Now, let $x_0 \in X$ and let V be a proper open subset of Y such that $F^\star(x_0) \subset V$. Since $F^\star(x_0) \cap (Y \setminus V) = \emptyset$, $F^\star(x_0) \neq \emptyset$ and $Y \setminus V \neq \emptyset$, $\cap\{(\cup\{F_\mu(x_0)\colon \mu > \lambda\})\colon \lambda \in D\} \cap (Y \setminus V) = \emptyset$ and hence $\cap\{(\cup\{F_\mu(x_0) \cap (Y \setminus V)\colon \mu > \lambda\})\colon \lambda \in D\} = \emptyset$. Since Y is compact and the family $\{(\cup\{F_\mu(x_0) \cap (Y \setminus V)\colon \mu > \lambda\})\colon \lambda \in D\}$ is a family of closed sets with the empty intersection, there exists $\lambda \in D$ such that $F_\mu(x_0) \cap (Y \setminus V) = \emptyset$ for each $\mu \in D$ with $\mu > \lambda$. Since the net $(F_\lambda)_{\lambda\in D}$ is equally upper almost $\mathcal{I}$-continuous on X, there exists an $\mathcal{I}$-open set U containing x_0 such that $F_\mu(U) \subset \mathrm{Int}(\mathrm{Cl}(V))$ for each $\mu > \lambda$, that is, $F_\mu(x) \cap (Y \setminus \mathrm{Int}(\mathrm{Cl}(V))) = \emptyset$ for each $x \in U$. Then we have $\cup\{F_\mu(x) \cap (Y \setminus \mathrm{Int}(\mathrm{Cl}(V)))\colon \mu > \lambda\} = \emptyset$ and hence $\cap\{\cup\{F_\mu(x)\colon \mu > \lambda\}\colon \lambda \in D\} \cap (Y \setminus \mathrm{Int}(\mathrm{Cl}(V))) = \emptyset$. This implies that $F^*(U) \subset \mathrm{Int}(\mathrm{Cl}(V))$. If $V = Y$, then it is clear that for each $\mathcal{I}$-open set U containing x_0 we have $F^\star(U) \subset \mathrm{Int}(\mathrm{Cl}(V))$. Hence $F^\star$ is upper almost $\mathcal{I}$-continuous at x_0. Since x_0 is arbitrary, the proof completes. ∎

References

[1] M. E. Abd El-Monsef, E. F. Lashien and A. A. Nasef, On $\mathcal{I}$-open sets and $\mathcal{I}$-continuous functions, *Kyungpook Math. J.*, **32**(1) (1992), 21-30.
[2] M. Akdag, On upper and lower $\mathcal{I}$-continuous multifunctions, *Far East J. math. Sci.*, **25**(1) (2007), 49–57.
[3] D. Andrijevic, Semi-preopen sets, *Mat. Vesnik*, **38** (1986), 24–32.
[4] C. Arivazhai and N. Rajesh, On upper and lower weakly $\mathcal{I}$-continuous multifunctions (submitted).
[5] T. Banzaru, On the upper semicontinuity of the upper topological limite for multifunction nets, *Semin. Mat. Fiz. Inst. Politehn. Timisoara*, (1983), 59–64.
[6] N. Bourbaki, *General Topology, Part I*, Addison Wesley, Reading, Mass 1996.
[7] D. S. Jankovic, A note on mappings of extremally disconnected spaces, *Acta Math. Hungar.*, **46** (1985), 83Ű-92.
[8] D. Jankovic and T. R. Hamlett, Compatible extension of ideals, *Bull. U. M. I.*, **7** (1992), 453–465.
[9] D. Jankovic and T. R. Hamlett, New Toplogies From Old Via Ideals, *Amer. Math. Monthly*, **97**(4) (1990), 295–310.

[10] I. Kovacevic, Subsets and paracompactness, *Univ. u. Novom Sadu, Zb. Rad. Prirod. Mat. Fac. Ser. Mat.*, **14** (1984), 79–87.

[11] K. Kuratowski, Topology, *Academic Press, New York*, 1966.

[12] N. Levine, Semiopen sets and semicontinuity in topological spaces, *Amer. Math. Monthly*, **70** (1963), 36–41.

[13] A. S. Mashhour, M. E. Abd El-Monsef and S. N. El-Deeb, On precontinuous and weak precontinuous mappings, *Proc. Phys. Soc. Egypt*, **53** (1982), 47–53.

[14] R. A. Mahmoud and A. A. Nasef, Regularity and Normality via ideals, *Bull. Malaysian Math. Sc. Soc.*, **24** (2001), 129–136.

[15] R. L. Newcomb, Topologies which are compact modulo an ideal, Ph.D. Thesis, University of California, USA(1967).

[16] T. Noiri and V. Popa, Almost weakly continuous multifunctions, *Demonstratio Math.*, **26** (1993), 363–380.

[17] T. Noiri and V. Popa, A unified theory of almost continuity for multifunctions, *Sci. Stud. Res. Ser. Math. Inform.*, **20**(1) (2010), 185–214.

[18] V. Popa, A note on weakly and almost continuous multifunctions, *Univ, u Novom Sadu, Zb. Rad. Prirod-Mat. Fak. Ser. Mat.*, **21** (1991), 31–38.

[19] V. Popa, Weakly continuous multifunction, *Boll. Un. Mat. Ital.*, (5) 15-A (1978), 379–388.

[20] M. Stone, Applications of the theory of boolean rings to general topology, *Trans. Amer. Math. Soc.*, **41** (1937), 374–381.

[21] R. Vaidyanathaswamy, The localisation theory in set topology, *Proc. Indian Acad. Sci.*, **20** (1945), 51–61.

C. Arivazhai
Department of Mathematics
Rajah Serfoji Govt. College
Thanjavur-613005
Tamilnadu, India

N. Rajesh
Department of Mathematics
Rajah Serfoji Govt. College
Thanjavur-613005
Tamilnadu, India
nrajesh_topology@yahoo.co.in

ANNALES UNIV. SCI. BUDAPEST., **59** (2016), 45–54

(w, k)-CONTINUITY AND WEAK (w, k)-CONTINUITY IN WEAK STRUCTURE SPACES DUE TO CSÁSZÁR

By
AHMAD AL-OMARI AND TAKASHI NOIRI
(Received March 17, 2016)

Abstract. We obtain some relations between a generalized topology [2] and a weak structure [3] on a nonempty set. And also, we define and investigate the notions of (w, k)-continuity and weak (w, k)-continuity in weak structure spaces due to Császár [3].

1. Introduction and preliminaries

Császár [2] introduced a generalized structure called generalized topology. Also, Császár [3] has introduced a new notion of structures called a weak structure which is weaker than both a generalized topology [2] and a minimal structure [4, 5]. Let X be a nonempty set and $w \subseteq \mathcal{P}(X)$, where $\mathcal{P}(X)$ is the power set of X. Then w is called a weak structure (briefly WS) (resp. minimal structure) on X if $\emptyset \in w$ (resp. $\emptyset, X \in w$). Each member of w is said to be w-open and the complement of a w-open set is said to be w-closed. Let w be a weak structure on X and $A \subseteq X$. Császár [3] defined (as in the general case) $i_w(A)$ as the union of all w-open subsets of A (e.g. $\emptyset$) and $c_w(A)$ as the intersection of all w-closed sets containing A (e.g. X). Recently, Al-Omari and Noiri [1] has introduced a new notions of weak (μ, w)-continuity and weak (w, μ)-continuity between a weak structure space [3] and a generalized topological space [2].

We call a class $\mu \subseteq \mathcal{P}(X)$ a generalized topology [2] (briefly, GT) if $\phi \in \mu$ and the arbitrary union of elements of μ belongs to μ. A set X with a GT μ on it is called a generalized topological space (briefly, GTS) and is denoted by (X, μ). In this paper, we obtain some relations between a generalized topology [2] and a weak structure [3] on a nonempty set. And also, we define and investigate the

---

2000 Mathematics Subject Classification 54A05, 54C10

notions of (w, k)-continuity and weak (w, k)-continuity in weak structure spaces due to Császár [3].

The following lemmas are useful in the sequel:

LEMMA 1.1 ([3]). *Let w be a WS on X and A, B subsets of X, then the following properties hold:*

(1) $i_w(A) \subseteq A \subseteq c_w(A)$.
(2) If $A \subseteq B$ *implies that* $i_w(A) \subseteq i_w(B)$ *and* $c_w(A) \subseteq c_w(B)$.
(3) $i_w(i_w(A)) = i_w(A)$ *and* $c_w(c_w(A)) = c_w(A)$.
(4) $i_w(X - A) = X - c_w(A)$ *and* $c_w(X - A) = X - i_w(A)$.

LEMMA 1.2 ([3]). *Let w be a WS on X and A a subset of X, then the following properties hold:*

(1) $x \in i_w(A)$ *if and only if there is* $W \in w$ *such that* $x \in W \subseteq A$.
(2) $x \in c_w(A)$ *if and only if* $W \cap A \neq \emptyset$ *whenever* $x \in W \in w$.
(3) If $A \in w$, *then* $A = i_w(A)$ *and if A is w-closed, then* $A = c_w(A)$.

REMARK 1.3. If w is a WS on X, then

(1) $i_w(\emptyset) = \emptyset$ and $c_w(X) = X$.
(2) $i_w(X)$ is the union of all w-open sets in X.
(3) $c_w(\emptyset)$ is the intersection of all w-closed sets in X.

2. Some properties of weak structures

THEOREM 2.1. *For a WS space (X, w), the following properties are equivalent:*

(1) $w = \mu$ *i.e. w is a generalized topology in the sense of Császár;*
(2) $i_w(A)$ *is w-open for every subset A of X;*
(3) $c_w(A)$ *is w-closed for every subset A of X.*

PROOF. (1) ⇒ (2). Since $i_w(A) = \cup\{U \in w\colon U \subseteq A\}$ and w is a generalized topology, we have $i_w(A) \in w$.

(2) ⇔ (3). This follows easily from Lemma 1.1 (4).

(2) ⇒ (1). Suppose that $A_\alpha \in w$ for each $\alpha \in \Delta$. Let $V = \cup_{\alpha\in\Delta} A_\alpha$. By Lemma 1.1 we have $i_w(V) \subseteq V$. Let $x \in V$, then there exists $\alpha_0 \in \Delta$ such that $x \in A_{\alpha_0} \in w$. Therefore, $x \in A_{\alpha_0} = i_w(A_{\alpha_0}) \subseteq i_w(V)$. Thus we obtain $V \subseteq i_w(V)$ and hence $i_w(V) = V$. By (2), we have $V = \cup_{\alpha\in\Delta} A_\alpha \in w$ and hence w is a generalized topology. ∎

THEOREM 2.2. *Let w be a WS on X and $w^* = \{A \subset X : A = i_w(A)\}$. Then, the following properties hold:*

(1) w^* is a GT containing w;
(2) w is a GT if and only if $w = w^*$.

PROOF. (1). If $A \in w$, $i_w(A) = A$ and hence $A \in w^*$. Therefore, w^* contains w. Let $A_\alpha \in w^*$ for each $\alpha \in \Delta$. Then $A_\alpha = i_w(A_\alpha) \subseteq i_w(\cup A_\alpha)$ for each $\alpha \in \Delta$. Hence $\cup A_\alpha \subseteq i_w(\cup A_\alpha)$ and $\cup A_\alpha = i_w(\cup A_\alpha)$. Therefore, $\cup A_\alpha \in w^*$. And w^* is a GT.

(2). Let w be a GT and $A \in w^*$. Then $A = i_w(A)$. $i_w(A) = \cup\{V \in w : V \subseteq \subseteq A\} \in w$ and hence $A \in w$. By (1) $w \subseteq w^*$ and $w = w^*$. Conversely, Let $w = w^*$. Then by (1) w^* is a GT and hence w is a GT. ∎

REMARK 2.3. Let w be a WS on X. Then $i_w(A) \in w$ for every $A \subseteq X$ if and only if $w = w^*$.

3. (w, k)-continuous functions

DEFINITION 3.1. Let w be a WS on X and k be a WS on Y. A function $f : (X, w) \to (Y, k)$ is said to be (w, k)-continuous if for each $x \in X$ and each $V \in k$ containing $f(x)$, there exists $U \in w$ containing x such that $f(U) \subseteq V$.

DEFINITION 3.2 ([2]). A function $f : (X, \mu) \to (Y, \lambda)$ is said to be (μ, λ)-continuous if $f^{-1}(V) \in \mu$ for every $V \in \lambda$.

THEOREM 3.3. *For a function $f : (X, w) \to (Y, k)$, the following properties are equivalent:*

(1) f is (w, k)-continuous;
(2) for each $x \in X$ and each $V \in k$ containing $f(x)$, there exists $U \in w^*$ containing x such that $f(U) \subseteq V$;
(3) for each $x \in X$ and each $V \in k^*$ containing $f(x)$, there exists $U \in w^*$ containing x such that $f(U) \subseteq V$;
(4) $f : (X, w^*) \to (Y, k^*)$ is (w^*, k^*)-continuous (in the sense of Császár [2]).

PROOF. (1) ⇒ (2). Let $x \in X$ and $V \in k$ containing $f(x)$. Since f is (w, k)-continuous, there exists $U \in w \subseteq w^*$ containing x such that $f(U) \subseteq V$.

(2) ⇒ (3). Let $x \in X$ and $V \in k^*$ containing $f(x)$. Since $V \in k^*$, then $V = i_k(V)$ and $f(x) \in i_k(V)$. By Lemma 1.2, there exists $W \in k$ such

that $f(x) \in W \subseteq V$. By (2), there exists $U \in w^*$ containing x such that $f(U) \subseteq W \subseteq V$.

(3) $\Rightarrow$ (4). Suppose that $V \in k^*$. Let $x \in f^{-1}(V)$, then $f(x) \in V = i_k(V)$. By Lemma 1.2, there exists $W \in k \subseteq k^*$ such that $f(x) \in W \subseteq V$. Then by (3) there exists $U_x \in w^*$ containing x such that $f(U_x) \subseteq W \subseteq V$. Hence $x \in U_x \subseteq f^{-1}(V)$ and hence $f^{-1}(V) = \cup\{U_x : x \in f^{-1}(V)\} \in w^*$. Therefore, $f : (X, w^*) \to (Y, k^*)$ is (w^*, k^*)-continuous.

(4) $\Rightarrow$ (1). Let $x \in X$ and each $V \in k \subseteq k^*$ containing $f(x)$, then by (4) $x \in f^{-1}(V) \in w^*$. Thus $f^{-1}(V) = i_w(f^{-1}(V))$ and there exists $U \in w$ containing x such that $x \in U \subseteq f^{-1}(V)$; hence $f(U) \subseteq V$. Therefore, f is (w, k)-continuous. ∎

Theorem 3.4. *For a function $f : (X, w) \to (Y, k)$, the following properties are equivalent:*

(1) f is (w, k)-continuous;
(2) $f^{-1}(B) = i_w(f^{-1}(B))$ for every k-open set B in Y;
(3) $f(c_w(A)) \subseteq c_k(f(A))$ for every subset A in X;
(4) $c_w(f^{-1}(B)) \subseteq f^{-1}(c_k(B))$ for every subset B in Y;
(5) $f^{-1}(i_k(B)) \subseteq i_w(f^{-1}(B))$ for every subset B in Y;
(6) $c_w(f^{-1}(K)) = f^{-1}(K)$ for every k-closed set K in Y.

Proof. (1) $\Rightarrow$ (2). Let $B \in k$ and $x \in f^{-1}(B)$. Then $f(x) \in B$. There exists $U \in w$ containing x such that $f(U) \subseteq B$. Thus $x \in U \subseteq f^{-1}(B)$. This implies that $x \in i_w(f^{-1}(B))$. This shows that $f^{-1}(B) \subseteq i_w(f^{-1}(B))$. By Lemma 1.1, $f^{-1}(B) = i_w(f^{-1}(B))$.

(2) $\Rightarrow$ (3). Let A be any subset of X. Let $x \in c_w(A)$ and $B \in k$ containing $f(x)$. Then $x \in f^{-1}(B) = i_w(f^{-1}(B))$. There exists $U \in w$ such that $x \in U \subseteq f^{-1}(B)$. Since $x \in c_w(A)$, then $U \cap A \neq \phi$ and $\phi \neq f(U \cap A) \subseteq f(U) \cap f(A) \subseteq B \cap f(A)$. Since $B \in k$ containing $f(x)$, $f(x) \in c_k(f(A))$ and hence $f(c_w(A)) \subseteq c_k(f(A))$.

(3) $\Rightarrow$ (4). Let B be any subset of Y. Then we have $f(c_w(f^{-1}(B))) \subseteq c_k(f(f^{-1}(B))) \subseteq c_k(B)$. Therefore, we obtain $c_w(f^{-1}(B)) \subseteq f^{-1}(c_k(B))$.

(4) $\Rightarrow$ (5). Let B be any subset of Y. Then we have $X - i_w(f^{-1}(B)) = c_w(f^{-1}(Y - B)) \subseteq f^{-1}(c_k(Y - B)) = f^{-1}(Y - i_k(B)) = X - f^{-1}(i_k(B))$. Therefore, we obtain $f^{-1}(i_k(B)) \subseteq i_w(f^{-1}(B))$.

(5) $\Rightarrow$ (6). Let K be any k-closed set of Y. By (5), we have $X - f^{-1}(K) = f^{-1}(i_k(Y - K)) \subseteq i_w(f^{-1}(Y - K)) = i_w(X - f^{-1}(K)) = X - c_w(f^{-1}(K))$. Therefore, we have $c_w(f^{-1}(K)) \subseteq f^{-1}(K) \subseteq c_w(f^{-1}(K))$. Thus, we obtain $c_w(f^{-1}(K)) = f^{-1}(K)$.

(6) $\Rightarrow$ (1). Let $x \in X$ and $V \in k$ containing $f(x)$. By (6), we have $X - f^{-1}(V) = f^{-1}(Y-V) = c_w(f^{-1}(Y-V)) = c_w(X-f^{-1}(V)) = X-i_w(f^{-1}(V))$. Hence we have $x \in f^{-1}(V) = i_w(f^{-1}(V))$. Therefore, there exists $U \in w$ such that $x \in U \subseteq f^{-1}(V)$. Thus $x \in U \in w$ and $f(U) \subseteq V$. This shows that f is (w,k)-continuous. ∎

Definition 3.5. A WS space (X,w) is said to be w-T_2 if for any pair of distinct points x and y of X, there exist disjoint w-open sets U and V of X containing x and y, respectively.

Theorem 3.6. *If $f\colon (X,w) \to (Y,k)$ is a (w,k)-continuous injection and (Y,k) is k-T_2, then (X,w) is w-T_2.*

Proof. Let x, y be any distinct points of X. Then $f(x) \neq f(y)$. Since (Y,k) is k-T_2, there exist disjoint sets U and V in k containing $f(x)$ and $f(y)$, respectively. Since f is (w,k)-continuous, there exist G and $H \in w$ containing x and y, respectively, such that $f(G) \subseteq U$ and $f(H) \subseteq V$. This implies that $G \cap H = \phi$. Hence (X,w) is w-T_2. ∎

Definition 3.7. A WS space (X,w) is said to be w-compact if every cover of X by sets of w has a finite subcover.

A subset K of X is said to be w-compact if every cover of K by subsets of w has a finite subcover.

Theorem 3.8. *If $f\colon (X,w) \to (Y,k)$ is a (w,k)-continuous function and H is a w-compact set of X, then $f(H)$ is k-compact.*

Proof. Let $\{V_i\colon i \in I\}$ be any cover of $f(H)$ by sets of k. For each $x \in H$, there exists $i(x) \in I$ such that $f(x) \in V_{i(x)}$. Since f is (w,k)-continuous, there exists $U(x) \in w$ containing x such that $f(U(x)) \subseteq V_{i(x)}$. The family $\{U(x)\colon x \in H\}$ is a cover of H by sets of w. Since H is w-compact, there exists a finite number of points, say $x_1, x_2, \ldots, x_n$ in H such that $H \subseteq \cup\{U(x_j)\colon x_j \in H, 1 \leq j \leq n\}$. Therefore, we obtain $f(H) \subseteq \cup\{f(U(x_j))\colon x_j \in H, 1 \leq j \leq n\} \subseteq \cup\{V_{i(x_j)}\colon x_j \in H, 1 \leq j \leq n\}$. This shows that $f(H)$ is k-compact. ∎

Definition 3.9. A function $f\colon (X,w) \to (Y,k)$ is said to have a strongly (w,k)-closed graph (resp. (w,k)-closed graph) if for each $(x,y) \in (X \times Y) - G(f)$, there exist $U \in w$ containing x and $V \in k$ containing y such that $[U \times c_k(V)] \cap G(f) = \phi$ (resp. $[U \times V] \cap G(f) = \phi$).

LEMMA 3.10. *A function $f\colon (X,w) \to (Y,k)$ has a strongly (w,k)-closed graph (resp. (w,k)-closed graph) if and only if for each $(x,y) \in (X \times Y) - G(f)$, there exist $U \in w$ containing x and $V \in k$ containing y such that $f(U) \cap c_k(V) = \phi$ (resp. $f(U) \cap V = \phi$).*

THEOREM 3.11. *If $f\colon (X,w) \to (Y,k)$ is a (w,k)-continuous function and (Y,k) is k-T_2, then $G(f)$ is strongly (w,k)-closed.*

PROOF. Suppose that $(x,y) \in (X \times Y) - G(f)$. Then $y \neq f(x)$. Since Y is k-T_2, there exist disjoint sets V and $W \in k$ containing y and $f(x)$, respectively. By Lemma 1.2 we have $c_k(V) \cap W = \phi$. Since f is (w,k)-continuous, there exists $U \in w$ containing x such that $f(U) \subseteq W$. This implies that $f(U) \cap c_k(V) = \phi$ and by Lemma 3.10 $G(f)$ is strongly (w,k)-closed. ∎

THEOREM 3.12. *If $f\colon (X,w) \to (Y,k)$ is a surjective function with a strongly (w,k)-closed, then (Y,k) is k-T_2.*

PROOF. Let y_1 and y_2 be any distinct points of Y. Then there exists $x_1 \in X$ such that $f(x_1) = y_1$. Then we have $(x_1, y_2) \in (X \times Y) - G(f)$. Since $G(f)$ is strongly (w,k)-closed, there exist $U \in w$ containing x_1 and $V \in k$ containing y_2 such that $f(U) \cap c_k(V) = \phi$. Therefore, we have $y_1 = f(x_1) \in f(U) \subseteq Y - c_k(V)$. By Lemma 1.2, there exists $H \in k$ such that $y_1 \in H$ and $H \cap V = \phi$. Moreover, we have $y_2 \in V$ and $V \in k$. This shows that (Y,k) is k-T_2. ∎

THEOREM 3.13. *If $f\colon (X,w) \to (Y,k)$ is an injective (w,k)-continuous function with a (w,k)-closed graph, then (X,w) is w-T_2.*

PROOF. Let x and y be any distinct points of X. Then, since f is injective, we have $f(x) \neq f(y)$. Then we have $(x, f(y)) \in (X \times Y) - G(f)$. Since $G(f)$ is (w,k)-closed, by Lemma 3.10 there exist $U \in w$ containing x and $V \in k$ containing $f(y)$ such that $f(U) \cap V = \phi$. Since f is (w,k)-continuous, there exists $U_0 \in w$ containing y such that $f(U_0) \subseteq V$. Therefore, $f(U) \cap f(U_0) = \phi$ and $U \cap U_0 = \phi$. This shows that (X,w) is w-T_2. ∎

4. Weakly (w,k)-continuous functions

DEFINITION 4.1. Let w be a WS on X and k be a WS on Y. A function $f\colon (X,w) \to (Y,k)$ is said to be weakly (w,k)-continuous at $x \in X$ if for each $V \in k$ containing $f(x)$, there exists $U \in w$ containing x such that $f(U) \subseteq c_k(V)$.

A function $f\colon (X,w) \to (Y,k)$ is said to be weakly (w,k)-continuous if it has the property at each point $x \in X$.

THEOREM 4.2. *A function $f\colon (X,w) \to (Y,k)$ is weakly (w,k)-continuous at x if and only if for each k-open set V containing $f(x)$, $x \in i_w(f^{-1}(c_k(V)))$.*

PROOF. Let f be weakly (w,k)-continuous at x and V a k-open set containing $f(x)$. Then, there exists a w-open set U containing x such that $f(U) \subseteq c_k(V)$. Then we have $x \in U \subseteq f^{-1}(c_k(V))$ and hence $x \in i_w(f^{-1}(c_k(V)))$.

Conversely, let V be a k-open set containing $f(x)$. Then, we have $x \in$ $\in i_w(f^{-1}(c_k(V)))$. There exists a w-open set U such that $x \in U$ and $U \subseteq$ $\subseteq f^{-1}(c_k(V))$; hence $f(U) \subseteq c_k(V)$. This shows that f is weakly (w,k)-continuous at x. ∎

THEOREM 4.3. *A function $f\colon (X,w) \to (Y,k)$ is weakly (w,k)-continuous if and only if $f^{-1}(V) \subseteq i_w(f^{-1}(c_k(V)))$ for every k-open set V of Y.*

PROOF. Suppose that f is weakly (w,k)-continuous. Let $x \in f^{-1}(V)$. Then $f(x) \in V$. Since f is weakly (w,k)-continuous at x, by Theorem 4.2 we have $x \in i_w(f^{-1}(c_k(V)))$ and hence $f^{-1}(V) \subseteq i_w(f^{-1}(c_k(V)))$.

Conversely, let x be any point of X and $V \in w$ containing $f(x)$. Then, we have $x \in f^{-1}(V) \subseteq i_w(f^{-1}(c_k(V)))$. By Theorem 4.2, f is weakly (w,k)-continuous. ∎

DEFINITION 4.4. A function $f\colon (X,w) \to (Y,k)$ is said to satisfy the interiority condition if $i_w(f^{-1}(c_k(V))) \subseteq f^{-1}(V)$ for each k-open set V of Y.

THEOREM 4.5. *If a function $f\colon (X,w) \to (Y,k)$ is weakly (w,k)-continuous and satisfies the interiority condition, then f is (w,k)-continuous.*

PROOF. Let V be any k-open set. Since f is weakly (w,k)-continuous, by Theorem 4.3 $f^{-1}(V) \subseteq i_w(f^{-1}(c_k(V)))$. By the interiority condition of f, we have $i_w(f^{-1}(c_k(V))) \subseteq f^{-1}(V)$. Therefore, we obtain $f^{-1}(V) \subseteq i_w(f^{-1}(c_k(V))) \subseteq$ $\subseteq i_w(f^{-1}(V)) \subseteq f^{-1}(V)$. Thus $f^{-1}(V) = i_w(f^{-1}(V))$. Hence by Theorem 3.4 f is (w,k)-continuous. ∎

THEOREM 4.6. *If $f\colon (X,w) \to (Y,k)$ is an injective weakly (w,k)-continuous function with a strongly (w,k)-closed graph, then (X,w) is w-T_2.*

PROOF. Let x and y be any distinct points of X. Then, since f is injective, we have $f(x) \neq f(y)$. Then we have $(x, f(y)) \in (X \times Y) - G(f)$. Since $G(f)$ is strongly (w,k)-closed, by Lemma 3.10 there exist $U \in w$ containing

x and $V \in k$ containing $f(y)$ such that $f(U) \cap c_k(V) = \phi$. Since f is weakly (w, k)-continuous, there exists $H \in w$ containing y such that $f(H) \subseteq c_k(V)$. Therefore, we have $f(H) \cap f(U) = \phi$. Clearly, we obtain $U \cap H = \phi$. This shows that (X, w) is w-T_2. ∎

Definition 4.7. A WS space (X, w) is said to be w-Urysohn if for any pair of distinct points x and y of X, there exist w-open sets U and V of X containing x and y, respectively, such that $c_w(U) \cap c_w(V) = \phi$.

Theorem 4.8. *Let (X, w) be a WS space. If for each pair of distinct points x_1 and x_2 in X there exists a function $f\colon (X, w) \to (Y, k)$ such that*

(1) (Y, k) is k-Urysohn,
(2) $f(x_1) \neq f(x_2)$,
(3) f is weakly (w, k)-continuous at x_1 and x_2,

then (X, w) is w-T_2.

Proof. Let x_1 and x_2 be any distinct points of X. Then, by the hypothesis there exists a function $f\colon (X, w) \to (Y, k)$ which satisfies the conditions (1), (2) and (3). Let $y_i = f(x_i)$ for $i = 1, 2$. Then $y_1 \neq y_2$. Since (Y, k) is k-Urysohn, there exists a k-open set V_i in Y containing y_i for $i = 1, 2$ such that $c_k(V_1) \cap c_k(V_2) = \phi$. Since f is weakly (w, k)-continuous at x_1 and x_2, for $i = 1, 2$ there exists $U_i \in w$ containing x_i such that $f(U_i) \subseteq c_k(V_i)$. Hence we obtain $U_1 \cap U_2 = \phi$. Hence (X, w) is w-T_2. ∎

Theorem 4.9. *If $f\colon (X, w) \to (Y, k)$ is a weakly (w, k)-continuous function and (Y, k) is k-Urysohn, then $G(f)$ is strongly (w, k)-closed*

Proof. Suppose that $(x, y) \in (X \times Y) - G(f)$. Then $y \neq f(x)$. Since Y is k-Urysohn, there exist k-open sets V and W in Y containing y and $f(x)$, respectively, such that $c_k(V) \cap c_k(W) = \phi$. Since f is weakly (w, k)-continuous, there exists a w-open set U containing x such that $f(U) \subseteq c_k(W)$. This implies that $f(U) \cap c_k(V) = \phi$ and by Lemma 3.10, $G(f)$ is strongly (w, k)-closed. ∎

Definition 4.10. Let A be a subset of a WS space (X, w). A point $x \in X$ is called a w_θ-adherent point of A if $c_w(U) \cap A \neq \phi$ for every w-open set U containing x.

The set of all w_θ-adherent points of A is called the w_θ-closure of A and is denoted by $c_{w_\theta}(A)$. If $A = c_{w_\theta}(A)$, then A is said to be w_θ-closed. The complement of a w_θ-closed set is said to be w_θ-open.

LEMMA 4.11. *Let A be a subset of a WS space (X, w). If A is w-open in (X, w), then $c_w(A) = c_{w_\theta}(A)$.*

PROOF. In general, it hold that $c_w(A) \subseteq c_{w_\theta}(A)$. Suppose that $x \notin c_w(A)$. Then by Lemma 1.2 there exists $U \in w$ containing x such that $A \cap U = \phi$; hence $A \cap c_w(U) = \phi$ since A is w-open. This shows that $x \notin c_{w_\theta}(A)$. Therefore, we have $c_w(A) = c_{w_\theta}(A)$. ∎

THEOREM 4.12. *For a function $f : (X, w) \to (Y, k)$, the following are equivalent:*

(1) f is weakly (w, k)-continuous;
(2) $f(c_w(A)) \subseteq c_{k_\theta}(f(A))$ for every subset A of X;
(3) $c_w(f^{-1}(B)) \subseteq f^{-1}(c_{k_\theta}(B))$ for every subset B of Y.

PROOF. (1) ⇒ (2). Let A be any subset of X. Suppose that $x \in c_w(A)$ and G is any k-open set containing $f(x)$. Since f is weakly (w, k)-continuous, there exists a w-open set U containing x such that $f(U) \subseteq c_k(G)$. Since $x \in c_w(A)$, by Lemma 1.2 we have $U \cap A \neq \phi$. It follows that $\phi \neq f(U) \cap f(A) \subseteq c_k(G) \cap f(A)$. Hence $c_k(G) \cap f(A) \neq \phi$ and $f(x) \in c_{k_\theta}(f(A))$.

(2) ⇒ (3). Let B be any subset of Y. Then $f(c_w(f^{-1}(B))) \subseteq$ $\subseteq c_{k_\theta}(f(f^{-1}(B))) \subseteq c_{k_\theta}(B)$ and hence $c_w(f^{-1}(B)) \subseteq f^{-1}(c_{k_\theta}(B))$.

(3) ⇒ (1). Let x be any point of X and V be any k-open set containing $f(x)$. Since $c_k(V) \cap (Y - c_k(V) = \phi$, clearly $f(x) \notin c_{k_\theta}(Y - c_k(V))$ and hence $x \notin f^{-1}(c_{k_\theta}(Y - c_k(V)))$. By (3), $x \notin c_w(f^{-1}(Y - c_k(V)))$. By Lemma 1.2, there exists a w-open set U containing x such that $U \cap f^{-1}(Y - c_k(V)) = \phi$; hence $f(U) \cap (Y - c_k(V)) = \phi$. This shows that $f(U) \subseteq c_k(V)$. Therefore, f is weakly (w, k)-continuous. ∎

DEFINITION 4.13. A subset K of a non-empty set X with a WS w is said to be w-closed relative to (X, w) if for any cover $\{V_i : i \in I\}$ of K by w-open sets of X, there exists a finite subset I_0 of I such that $K \subseteq \cup\{c_w(V_i) : i \in I_0\}$. If X is w-closed relative to (X, w), then (X, w) is said to be w-closed.

THEOREM 4.14. *If $f : (X, w) \to (Y, k)$ is a weakly (w, k)-continuous function and K is w-compact in (X, w), then $f(K)$ is k-closed relative to (Y, k).*

PROOF. Let K be w-compact in (X, w). Let $\{V_i : i \in I\}$ be any cover of $f(K)$ by k-open sets of (Y, k). For each $x \in K$, there exists $i(x) \in I$ such that $f(x) \in V_{i(x)}$. Since f is weakly (w, k)-continuous, there exists a w-open set U_x containing x such that $f(U_x) \subseteq c_k(V_{i(x)})$. The family $\{U_x : x \in K\}$ is a cover of K by w-open sets of X. Since K is w-compact in (X, w), there exist a finite number of points, say, $x_1, x_2, \ldots, x_n$ in K such that $K \subseteq \cup\{U_{x_j} : x_j \in K, 1 \leq j \leq n\}$. Therefore, we obtain

$$\begin{aligned} f(K) &\subseteq \cup\{f(U_{x_j})\colon x_j \in K, 1 \leq j \leq n\} \\ &\subseteq \cup\{c_k(V_{i(x_j)})\colon x_j \in K, 1 \leq j \leq n\}. \end{aligned}$$

This shows that $f(K)$ is k-closed relative to (Y, k). ∎

COROLLARY 4.15. *If $f\colon (X, w) \to (Y, k)$ is a weakly (w, k)-continuous surjection and (X, w) is w-compact, then (Y, k) is k-closed.*

References

[1] A. AL-OMARI and T. NOIRI, Weak continuity between WSS and GTS due to Császár, *Malays. J. Math. Sci.*, **7**(2) (2013), 297–313.
[2] Á. CSÁSZÁR, Generalized topology, generalized continuity, *Acta Math. Hungar.*, **96** (2002), 351–357.
[3] Á. CSÁSZÁR, Weak structures, *Acta Math. Hungar.*, **131** (2011), 193–195.
[4] V. POPA and T. NOIRI, On M-continuous functions, *Anal. Univ. "Dunarea de Jos" Galati, Ser. Mat. Fiz. Mec. Teor., Fasc. II*, **18(23)** (2000), 31–41.
[5] H. MAKI, J. UMEHARA and T. NOIRI, Every topological space is pre-$T_{\frac{1}{2}}$, *Mem. Fac. Sci. Kochi. Univ. Ser. A Math.*, **17** (1996), 33–42.

Ahmad Al-Omari
Al al-Bayt University
Faculty of Sciences
Department of Mathematics
P.O. Box 130095
Mafraq 25113, Jordan
omarimutah1@yahoo.com

Takashi Noiri
2949-1 Shiokita-cho, Hinagu
Yatsushiro-shi, Kumamoto-ken
869-5142 Japan
t.noiri@nifty.com

KENMOTSU MANIFOLD WITH SOME CURVATURE CONDITIONS

By
KANAK KANTI BAISHYA AND PARTHA ROY CHOWDHURY
(Received May 9, 2016)

Abstract. Recently the present authors have introduced the notion of *generalized quasi-conformal curvature tensor W which bridges Conformal curvature tensor, Concircular curvature tensor, Projective curvature tensor and Conharmonic curvature tensor*. The present paper attempts to study generalized *quasi-conformal like* recurrent Kenmotsu manifolds and generalized *quasi-conformal like* ϕ-recurrent Kenmotsu manifolds.

1. Introduction

Recently, in tune with Yano and Sawaki [13], Baishya et. al. [11] have introduced and studied generalized quasi-conformal curvature tensor in the context of $N(k,\mu)$-manifold. The generalized quasi-conformal curvature tensor is defined for $(2n+1)$-dimensional manifold as

$$\begin{aligned} W(X,Y)Z = \frac{2n-1}{2n+1}\left[(1+2na-b)-\{1+2n(a+b)\}c\right]C(X,Y)Z+ \\ +[1-b+2na]E(X,Y)Z+2n(b-a)P(X,Y)Z+ \\ +\frac{2n-1}{2n+1}(c-1)\{1+2n(a+b)\}\hat{C}(X,Y)Z \end{aligned} \tag{1.1}$$

where C, E, P and $\hat{C}$ stand for Conformal, Concircular, Projective and Conharmonic curvature tensor respectively and the scalar triples (a,b,c) are real constants. The beauty of such curvature tensor lies in the fact that it has the flavour of Riemann curvature tensor R if the scalar triple $(a,b,c)\equiv(0,0,0)$, Conformal curvature tensor C [15] if $(a,b,c)\equiv\left(-\frac{1}{2n-1},-\frac{1}{2n-1},1\right)$, Conharmonic curvature tensor $\hat{C}$ [22] if $(a,b,c)\equiv\left(-\frac{1}{2n-1},-\frac{1}{2n-1},0\right)$, Concircular

2000 Mathematics Subject Classification 2010, 53C15, 53C25

curvature tensor E ([14], p. 84) if $(a,b,c) \equiv (0,0,1)$, Projective curvature tensor P ([14], p. 84) if $(a,b,c) \equiv \left(-\frac{1}{2n},0,0\right)$ and m-Projective curvature tensor H [8], if $(a,b,c) \equiv \left(-\frac{1}{4n},-\frac{1}{4n},0\right)$. The equation (1.1) can also be written as

$$\begin{aligned} W(X,Y)Z &= R(X,Y)Z + a\left[S(Y,Z)X - S(X,Z)Y\right] + \\ &+ b\left[g(Y,Z)QX - g(X,Z)QY\right] - \\ &- \frac{cr}{2n+1}\left(\frac{1}{2n} + a + b\right)\left[g(Y,Z)X - g(X,Z)Y\right]. \end{aligned} \tag{1.2}$$

Hereafter in this paper, generalized quasi-conformal curvature tensor W will be called quasi-conformal like curvature tensor in order to avoid reiteration of the word "generalized".

The study of Riemann symmetric manifolds began with the work of Cartan [6] in 1926. Thereafter in 1950, Walker [4] introduced the notion of locally recurrent Riemann manifold. A Riemann manifold is said to be locally recurrent [4] if there exists a non-zero 1-form A such that

$$(\nabla_X R)(Y,Z)U = A(X)R(Y,Z)U \tag{1.3}$$

where ∇ is the Levi–Civita connection. If the 1-form A vanishes identically, then the manifold is said to be a locally symmetric.

As a weaker version of local recurrence, Dubey [17] introduced the notion of generalized recurrent manifolds. A Riemannian manifold $(M^{2n+1},g), n>1$ is called generalized recurrent ([17], [19]) if its curvature tensor R satisfies the condition

$$(\nabla_X R)(Y,Z)U = A(X)R(Y,Z)U + B(X)G(Y,Z)U \tag{1.4}$$

for all vector fields X, Y, Z, U, where A and B are two non-zero 1-forms defined as

$$A(X) = g(X,\rho),\ B(X) = g(X,\sigma),\ G(Y,Z)U = \left[g(Z,U)Y - g(Y,U)Z\right], \tag{1.5}$$

where ρ, σ are vector fields associated to the 1-forms A and B, respectively. The generalized recurrent manifolds are studied by De and Guha [19], Özgür [7], Singh and Khan [9] and the references therein. In analogy with the definition given by Maralabhavi and Rathnamma [21], a Riemannian manifold (M^{2n+1},g), $(n>1)$, is called a generalized quasi conformal like recurrent if W admits the relation

$$(\nabla_X W)(Y,Z)U = A(X)W(Y,Z)U + B(X)G(Y,Z)U \tag{1.6}$$

where A and B are non-zero 1-forms defined in (1.3). In particular, if A and B vanishes identically, then such a manifold will be called quasi conformal like symmetric manifold. Again, Kenmotsu manifold for which $A \neq 0$ but $B = 0$ will be called quasi conformal like recurrent manifold.

In 1977, Takahashi [18] introduced the notion of local ϕ-symmetry on a Sasakian manifold. Generalizing the notion of local ϕ-symmetry of Takahashi [18], Basari and Murathan [1] introduced the notion of generalized ϕ-recurrent Kenmotsu manifolds. In analogy with the definition given by Basari and Murathan [1], a Riemannian manifold (M^{2n+1}, g), $(n > 1)$, is called a generalized quasi conformal like ϕ-recurrent if its quasi conformal like curvature tensor W of type $(1, 3)$ admits the condition

$$\phi^2 (\nabla_X W)(Y, Z, U, V) = A(X)W(Y, Z, U, V) + B(X)G(Y, Z)U \tag{1.7}$$

where A and B are non-zero 1-forms defined in (1.3). In particular, if A and B vanishes identically, then such a Kenmotsu manifold will be called quasi conformal like ϕ-symmetric ([2], [18]). Again, the Kenmotsu manifold for which $A \neq 0$ but $B = 0$ is said to be quasi conformal like ϕ-recurrent ([3] and reference there in).

Our work is structured as follows. Section 2 is a very brief review of Kenmotsu manifolds. In section 3, we investigate generalized quasi conformal like recurrent Kenmotsu manifold and obtain some interesting results and it is found that every generalized conformal recurrent Kenmotsu manifold is necessarily conformal recurrent. Section 4 is concerned with generalized quasi conformal like ϕ-recurrent Kenmotsu manifold. Among others it is observed that such a manifold is Einstein provided $(1 - a + 2na) \neq 0$.

2. Kenmotsu manifolds

Let M be a $(2n + 1)$-dimensional connected differentiable manifold of class C^∞-covered by a system of coordinate neighborhoods(U, x^h) in which there are given a tensor field ϕ of type $(1, 1)$, a cotravariant vector field ξ and a 1-form η such that

$$\phi^2 X = -X + \eta(X)\xi, \tag{2.1}$$

$$\eta(\xi) = 1, \qquad \phi \cdot \xi = 0, \quad \eta(\phi X) = 0, \tag{2.2}$$

for any vector field X on M. Then the structure (ϕ, ξ, η) is called contact structure and the manifold M^{2n+1} equipped with such structure is said to be an almost contact manifold, if there is given a Riemannian compatible metric g such that

$$g(\phi X, Y) = -g(X, \phi Y), \qquad g(X, \xi) = \eta(X), \tag{2.3}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{2.4}$$

for all vector fields X and Y, then we say M is an almost contact metric manifold. An almost contact metric manifold M is called a Kenmotsu manifold if it satisfies [12],

$$(\nabla_X \phi) Y = -g(X, \phi Y)\, \xi - \eta(Y)\phi(X), \tag{2.5}$$

for all vector fields X and Y, where ∇ is a Levi–Civita connection of the Riemannian metric. From the above it follows that

$$\nabla_X \xi = X - \eta(X)\xi, \tag{2.6}$$

$$(\nabla_X \eta)Y = g(X, Y) - \eta(X)\eta(Y), \tag{2.7}$$

In a Kenmotsu manifold the following relations hold ([20], [12])

$$R(X, Y)\xi = \eta(X)\, Y - \eta(Y)X, \tag{2.8}$$

$$R(\xi, X)Y = \eta(Y)X - g(X, Y)\xi, \tag{2.9}$$

$$\eta\,(R(X, Y)Z) = \eta(Y)g(X, Z) - \eta(X)\, g(Y, Z), \tag{2.10}$$

$$S\,(X, \xi) = -2n\eta(X), \tag{2.11}$$

for any vector fields X, Y. In the sequel we shall use the following Lemmas.

Lemma 2.1. *Let $M^{2n+1}(\phi, \xi, \eta, g)$ be a Kenmotsu manifold. Then for any vector fields $X, Y,\ Z$ the following relation holds:*

$$(\nabla_X R)(Y, Z)\xi = g(X, Y)Z - g(X, Z)Y - R(Y, Z)X \tag{2.12}$$

Proof. By virtue of (2.6), (2.7) and (2.8) we can easily get (2.12). ∎

Lemma 2.2. *In a Riemannian manifold(M^{2n+1}, g) the following relation holds:*

$$g\,((\nabla_X R)(Y, Z)U, V) = -g\,((\nabla_X R)(Y, Z)V, U) \tag{2.13}$$

for all vector fields $X, Y, Z, U, V \in \chi(M)$.

Proof. Since the proof of Lemma 2.2 follows by a routine calculation, we shall omit it. ∎

Lemma 2.3. *In a Riemannian manifold(M^{2n+1}, g) the following relation holds:*

$$(\nabla_X S)(Y, \xi) = -2ng(X, Y) - S(X, Y). \tag{2.14}$$

Proof. Lemma 2.3 is the direct consequence of (2.12). ∎

3. Generalized quasi conformal like recurrent Kenmotsu manifolds

Let $(M^{2n+1}; g), (n > 1)$, be a generalized quasi conformal like recurrent Kenmotsu manifold. Then in consequence of (1.2), (1.6) and Bianchi's 2nd identity, we have

$$\begin{aligned}
&a\Big[(\nabla_X S)(Z,U)g(Y,V) - (\nabla_X S)(Y,U)g(Z,V) + (\nabla_Y S)(X,U)g(Z,V) - \\
&- (\nabla_Y S)(Z,U)g(X,V) + (\nabla_Z S)(Y,U)g(X,V) - (\nabla_Z S)(X,U)g(Y,V)\Big] + \\
&+b\Big[(\nabla_X S)(Y,V)g(Z,U) - (\nabla_X S)(Z,V)g(Y,U) + (\nabla_Y S)(Z,V)g(X,U) - \\
&- (\nabla_Y S)(X,V)g(Z,U) + (\nabla_Z S)(X,V)g(Y,U) - (\nabla_Z S)(Y,V)g(X,U)\Big] = \\
&= \frac{c}{2n+1}\left(\frac{1}{2n} + a + b\right)\Big[dr(X)\{g(Z,U)g(Y,V) - g(Y,U)g(Z,V)\} + \\
&+dr(Y)\{g(X,U)g(Z,V) - g(Z,U)g(X,V)\} + \\
&+dr(Z)\{g(Y,U)g(X,V) - g(X,U)g(Y,V)\}\Big] + \\
&+A(X)W(Y,Z,U,V) + A(Y)W(Z,X,U,V) + \\
&+A(Z)W(X,Y,U,V) + B(X)\{g(Z,U)g(Y,V) - g(Y,U)g(Z,V)\} + \\
&+B(Y)[g(X,U)g(Z,V) - g(Z,U)g(X,V)] + \\
&+B(Z)\{g(Y,U)g(X,V) - g(X,U)g(Y,V)\}. \qquad (3.1)
\end{aligned}$$

Next, from (1.2) one can easily bring out the followings

$$\begin{aligned}
&\sum_{i=1}^{2n+1} W(e_i, Z, U, e_i) = \\
&= (1 - b + 2na)\, S(Z,U) + r\left[b - \frac{c\{1 + 2n(a+b)\}}{2n+1}\right] g(Z,U),
\end{aligned} \qquad (3.2)$$

$$\sum_{i=1}^{2n+1} W(e_i, e_i, e_i, e_i) = r(1-c)[1 + 2n(a-b)], \qquad (3.3)$$

$$\sum_{i=1}^{2n+1} W(X, Y, e_i, e_i) = 0 \qquad (3.4)$$

and

$$\sum_{i=1}^{2n+1} W\left(e_i,\ X,\ e_i,\ Y\right) = \tag{3.5}$$
$$= -\left(1+2nb-a\right) S(X,Y) + r\left[\frac{c\left(1+2n(a+b)\right)}{2n+1} - a\right].$$

Contracting over Y and V in (3.1) and then using (3.2), we obtain

$$[(2n-1)\,a-b][(\nabla_X S)\,(Z,U)-(\nabla_Z S)\,(X,U)] =$$
$$= W(Z,X,U,\rho) + (1-b+2na)\Big[A(X)S(Z,U)-A(Z)S(X,U)\Big] +$$
$$+ (2n-1)[B(X)g(Z,U)-B(Z)g(X,U)] + \left[\frac{(2n-1)c}{2n+1}\left(\frac{1}{2n}+a+b\right)-\frac{b}{2}\right]\times$$
$$\times\Big[dr(X)g(Z,U)-dr(Z)g(X,U)\Big]. \tag{3.6}$$

Again, contracting Z over U in (3.6) and then using (3.2) and (3.3), we get

$$\frac{(2n-1)}{2}\Big[(2n+1)(a+b)-2c\{1+2n(a+b)\}\Big]dr(X) =$$
$$= r\left[1+(2n-1)(a+b)-\frac{c(2n-1)}{(2n+1)}\{1+2n(a+b)\}\right]A(X)- \tag{3.7}$$
$$-[2+(2n-1)(a+b)]A(QX)+2n(2n-1)B(X).$$

Replacing $a=-\frac{1}{2n-1}$, $b=-\frac{1}{2n-1}$ and $c=1$, $B(X)=0$ for all X, in (3.7) one can state the following:

THEOREM 3.1. *Every generalized quasi conformal like recurrent Kenmotsu manifold is necessarily conformally recurrent.*

PROPOSITION 3.2. *Let $(M^{2n+1};g)$, $(n>1)$, be an Einstein Kenmotsu manifold. Then every generalized quasi conformal like recurrent manifold M is necessarily generalized recurrent.*

Next, we suppose that the scalar curvature r is constant, then the equation (3.7) reduces to

$$rA(X) = \lambda A\left(QX\right) - \mu B(X), \tag{3.8}$$

where

$$\lambda = \frac{(2n+1)[2+(2n-1)(a+b)]}{\Big[(2n+1)+(4n^2-1)(a+b)-(2n-1)c\{1+2n(a+b)\}\Big]}$$

and

$$\mu = \frac{2n(4n^2 - 1)}{[(2n+1) + (4n^2-1)(a+b) - (2n-1)c\{1 + 2n(a+b)\}]}$$

THEOREM 3.3. *The 1-forms of a generalized quasi conformal like recurrent manifold are connected by the equation (3.8).*

Again, setting $Y = V = e_i$ in (3.1) and taking summation over i, $1 \leq i \leq 2n+$ $+1$, we get

$$\begin{aligned}(1 - b + 2na)\Big[(\nabla_X S)(Z,U) - A(X)S(Z,U)\Big] = \\ = r\left[b - \frac{c}{2n+1}\{1 + 2n(a+b)\}\right][A(X)g(Z,U) - dr(X)g(Z,U)] + \\ +2nB(X)g(Z,U).\end{aligned} \tag{3.9}$$

Using $dr = 0$ in (3.9) we obtain for $(1 - b + 2na) \neq 0$ that

$$(\nabla_X S)(Z,U) = A(X)S(Z,U) + \lambda(X)g(Z,U), \tag{3.10}$$

where

$$\lambda(X) = r\left[b - c\frac{\{1 + 2n(a+b)\}}{2n+1}\right] \times \left[A(X) + \frac{2n}{(1 - b + 2na)}B(X)\right].$$

From the above, we can easily bring out the followings:

PROPOSITION 3.4. *Let (M^{2n+1}, g), $n(\geq 1)$,be a Kenmotsu manifold is quasi conformal like symmetric. Then such a manifold is either conformally or conharmonic flat or Ricci symmetric.*

PROOF. For a symmetric quasi conformal like curvature tensor, one can easily bring out from (1.2) that

$$(1 - b + 2na)(\nabla_X S)(Z,U) = 0.$$

This completes the proof. ∎

THEOREM 3.5. *A generalized quasi-conformal like recurrent Kenmotsu manifold is a generalized Ricci-recurrent provided $(1 - b + 2na) \neq 0$.*

PROPOSITION 3.6. *A generalized quasi-conformal like symmetric Kenmotsu manifold is Ricci-symmetric.*

THEOREM 3.7. *A generalized quasi-conformal like recurrent Kenmotsu manifold is an Einstein space provided it is Ricci-symmetric and $(1 - b + 2na) \neq 0$.*

REMARK 3.8. The same result also true for quasi-conformal like recurrent Kenmotsu manifold.

Again, in view of (1.2) and (1.4) we obtain

$$\begin{aligned} A(X)W(Y,Z,\xi,V)+B(X)[g(Y,V)\eta(Z)-g(Z,V)\eta(Y)] = \\ =(\nabla_X R)(Y,Z,\xi,V)+a\Big[(\nabla_X S)(Z,\xi)g(Y,V)-(\nabla_X S)(Y,\xi)g(Z,V)\Big]+ \\ +b\Big[(\nabla_X S)(Y,V)\eta(Z)-(\nabla_X S)(Z,V)\eta(Y)\Big]- \\ -\frac{dr(X)c}{2n+1}\left(\frac{1}{2n}+a+b\right)[g(Y,V)\eta(Z)-g(Z,V)\eta(Y)] \end{aligned} \tag{3.11}$$

which yields from (2.12) and (2.14) for $Z=\xi$, that

$$\begin{aligned} 2nB(X)+\left[br-2n(1-b+2an)-\frac{cr}{2n+1}\{1+2n(a+b)\}\right]A(X) = \\ = dr(X)\left[b-\frac{c}{2n+1}\{1+2n(a+b)\}\right]. \end{aligned} \tag{3.12}$$

From, the foregoing equation one can state the following:

THEOREM 3.9. *Let (M^{2n+1},g), $(n>1)$, be a generalized quasi-conformal like recurrent Kenmotsu manifold. Then the 1-forms are connected by*

$$\left[2n(1-b+2an)-br+\frac{cr\{1+2n(a+b)\}}{2n+1}\right]A(X)=2nB(X). \tag{3.13}$$

for any vector field X, where $dr(X)$ denotes the covariant derivative of the scalar curvature r with respect to the vector field X.

4. Generalized quasi conformal like ϕ-recurrent Kenmotsu manifolds

Let (M^{2n+1},g), $(n>1)$, be a generalized quasi-conformal like ϕ-recurrent Kenmotsu manifold. Then in view of (1.7) and (2.1), we have

$$\begin{aligned} -g\left((\nabla_X W)(Y,Z)U,V\right)+\eta((\nabla_X W)(Y,Z)U)\eta(V) = \\ = A(X)g\left(W(Y,Z)U,V\right)+B(X)g\left(G(Y,Z)U,V\right). \end{aligned} \tag{4.1}$$

By virtue of (1.2), (4.1) becomes

$$
\begin{aligned}
&-(\nabla_X R)(Y,Z,U,V)- \\
&-a\Big[(\nabla_X S)(Z,U)g(Y,V)-(\nabla_X S)(Y,U)g(Z,V)\Big]- \\
&-b\Big[(\nabla_X S)(Y,V)g(Z,U)-(\nabla_X S)(Z,V)g(Y,U)\Big]+ \\
&+\frac{dr(X)c}{2n+1}\left(\frac{1}{2n}+a+b\right)[g(Z,U)g(Y,V)-g(Y,U)g(Z,V)]+ \\
&+\eta((\nabla_X R)(Y,Z)U)\eta(V)+ \\
&+a\Big[(\nabla_X S)(Z,U)\eta(V)\eta(Y)-(\nabla_X S)(Y,U)\eta(Z)\eta(V)\Big]+ \\
&+b\Big[(\nabla_X S)(Y,\xi)g(Z,U)\eta(V)-(\nabla_X S)(Z,\xi)g(Y,U)\eta(V)\Big]- \\
&-\frac{dr(X)c}{2n+1}\left(\frac{1}{2n}+a+b\right)\Big[g(Z,U)\eta(V)\eta(Y)-g(Y,U)\eta(Z)\eta(V)\Big]= \\
&=A(X)\Big[R(Y,Z,U,V)+a\{S(Z,U)g(Y'',V)-S(Y,U)g(Z,V)\}+ \\
&+b\{g(Z,U)S(Y,V)-g(Y,U)S(Z,V)\}- \\
&-\frac{cr}{2n+1}\left(\frac{1}{2n}+a+b\right)\{g(Z,U)g(Y,V)-g(Y,U)g(Z,V)\}]+ \\
&+B(X)\Big[g(Z,U)g(Y,V)-g(Y,U)g(Z,V)\Big],
\end{aligned}
\tag{4.2}
$$

which yields for $U=\xi$ and on a suitable contraction that

$$
\begin{aligned}
&(1-a+2na)\Big[2ng(X,Z)+S(X,Z)\Big]+ \\
&+\left[\frac{c\{1+2n(a+b)\}}{2n+1}-b\right]dr(X)\eta(Z)= \\
&=A(X)\left[br-2n(1-b+2na)-\frac{cr\,[1+2n(a+b)]}{2n+1}\right]\eta(Z)+ \\
&+2nB(X)\eta(Z).
\end{aligned}
\tag{4.3}
$$

THEOREM 4.1. *Let (M^{2n+1},g) be a quasi conformal like ϕ-symmetric Kenmotsu manifold. Then the manifold is Einstein provided $(1-a+2na)\neq 0$.*

Again, using (2.11) in (4.3) we get

$$
\begin{aligned}
&A(X)\left[br-2n(1-b+2na)-\frac{cr\,[1+2n(a+b)]}{2n+1}\right]+2nB(X)= \\
&=\left[\frac{c\{1+2n(a+b)\}}{2n+1}-b\right]dr(X).
\end{aligned}
\tag{4.4}
$$

PROPOSITION 4.2. *Let (M^{2n+1}, g) be a quasi conformal like ϕ-symmetric Kenmotsu manifold. Then the scalar curvature is constant.*

In view of (4.4) and $dr(X) = 0$, we have

$$S(X,Z) = -2ng(X,Z), \text{ provided } (1 - a + 2na) \neq 0. \tag{4.5}$$

This leads to the following results:

THEOREM 4.3. *A generalized quasi conformal like ϕ-recurrent Kenmotsu manifold (M^{2n+1}, g) is an Einstein manifold provided $(1 - a + 2na) \neq 0$.*

THEOREM 4.4. *Let (M^{2n+1}, g) be a generalized quasi conformal like ϕ-recurrent Kenmotsu manifold. Then the 1-forms are related by*

$$A(X)\left[br - 2n(1 - b + 2na) - \frac{cr\,[1 + 2n(a+b)]}{2n+1}\right] = -2nB(X). \tag{4.6}$$

ACKNOWLEDGEMENT. First author would like to thank UGC, ERO-Kolkata, for their financial support.

References

[1] A. BASARI and C. MURATHAN, On generalized ϕ-recurrent Kenmotsu manifolds, *Fen Derg.*, **3**(1) (2008), 91–97.

[2] A. A. SHAIKH and K.K. BAISHYA, On ϕ-symmetric LP-Sasakianmanifolds, *Yokohama Math. J.*, **52** (2005), 97–112.

[3] A. A. SHAIKH, T. BASU and K. K. BAISHYA, On the existence of locally ϕ-recurrent LP-Sasakian manifolds, *Bull. Allahabad Math. Soc.*, **24**(2) (2009), 281–295.

[4] A. G. WALKER, On Ruses spaces of recurrent curvature. *Proc. London Math. Soc.*, **52** (1950), 36–64.

[5] B. CHEN and K. YANO, Hypersurfaces of a conformally space, *Tensor N. S.*, **26** (1972), 315–321.

[6] E. CARTAN, Sur une classe remarquable d'espaces de Riemannian, *Bull. Soc.Math. France*, **54** (1926), 214–264.

[7] C. ÖZGÜR, On generalized recurrent Kenmotsu manifolds, *World Applied Sci. J.*, **2**(1) (2007), 9–33.

[8] G. P. POKHARIYAL and R. S. MISHRA, Curvature tensors' and their relativistics significance I, *Yokohama Mathematical Journal*, vol. **18**, 105–108, 1970.

[9] H. SINGH and Q. KHAN, On generalized recurrent Riemannian manifolds, *Publ. Math. Debrecen*, **56** (1–2)(2000), 87–95.

[10] K. Amur and Y. B. Maralabhavi, On quasi-conformally flat spaces, *Tensor (N.S.)*, **145** (31) (1977), 194–198.
[11] K. K. Baishya and P. R. Chowdhury, On generalized quasi-conformal $N(k, \mu)$-manifolds, *Commun. Korean Math. Soc.*, **31** (2016), 1, 163–176.
[12] K. Kenmotsu, A class of almost contact Riemannian manifolds, *Tohoku Mathematical Journal*, **24**(1) (1972), 93–103.
[13] K. Yano and S. Sawaki, Riemannian manifolds admitting a conformal transformation group, *J. Diff. Geom.*, **2** (1968), 161–184.
[14] K. Yano and S. Bochner, Curvature and Betti numbers, *Annals of Mathematics Studies* **32**, Princeton University Press, 1953.
[15] L. P. Eisenhart, *Riemannian Geometry*, Princeton University Press, 1949.
[16] Q. Khan, On generalized recurrent Sasakian manifold, *Kyungpook Math. J.*, **44** (2004), 167–172.
[17] R. S. D. Dubey, Generalized recurrent spaces, *Indian J. Pure Appl. Math.*, **10**(12) (1979), 1508–1513.
[18] T. Takahashi, Sasakian ϕ-symmetric spaces, *Tohoku Math. J.*, **29** (1977), 91–113.
[19] U. C. De and N. Guha, On generalized recurrent manifolds, *J. Net. Acad. Math. India*, **9** (1991), 85–92.
[20] Venkatesha, C. S. Bagewadi, On Pseudo Projective ϕ-recurrent Kenmotsu manifolds, *Soochow. J. Math.*, **32** (2006), 1–7.
[21] Y. B. Maralabhavi and M. Rathnamma, Generalized recurrent and concircular recurrent manifolds, *Indian J. Pure and Appl. Math.*, **30**(11) (1999), 1167–1171.
[22] Y. Ishii, On conharmonic transformations, *Tensor (N.S.)*, **7** (1957), 73–80.

Kanak Kanti Baishya
Department Of Mathematics
Kurseong College
Dowhill Road, Kurseong
Darjeeling-734203
West Bengal, India
kanakkanti.kc@gmail.com

Partha Roy Chowdhury
Department Of Mathematics
Shaktigarh Bidyapith(H.S), Siliguri
Darjeeling-734005
West Bengal, India
partha.raychowdhury81@gmail.com

ANNALES UNIV. SCI. BUDAPEST., **59** (2016), 67–73

EXTREMALLY DISCONNECTED SPACES VIA HEREDITARY CLASSES

By
S. ABINAYA, C. CARPINTERO, N. RAJESH AND E. ROSAS
(Received May 30, 2016)

Abstract. In this paper, we introduce and study a notion called extremally disconnected space via hereditary classes. Many characterizations of this space are obtained.

1. introduction and Preliminaries

In 2002, Császár [1] introduced the notions of generalized topology and generalized continuity. A nonempty family $\mathcal{H}$ of subsets of X is said to be a hereditary class [2], if $A \in \mathcal{H}$ and $B \subset A$, then $B \in \mathcal{H}$. A generalized topological space (X, μ) with a hereditary class $\mathcal{H}$ is called hereditary generalized topological space and is denoted by $(X, \mu, \mathcal{H})$. Given a generalized topological space (X, μ) with a hereditary class $\mathcal{H}$, for each $A \subset X$, $A^\star(\mathcal{H}, \mu) = \{x \in X : A \cap V \notin \mathcal{H}$ for every $V \in \mu$, such that $x \in V\}$ [2]. If $c^\star(A) = A \cup A^\star(\mathcal{H}, \mu)$ for every subset A of X, then $\mu^\star = \{A \subset X : X \backslash A = c^\star(X \backslash A)\}$ is a GT, $\mu^\star$ is finer than μ [2]. A subset A of $(X, \mu, \mathcal{H})$ is said to be $\mathcal{H}$-open (resp. σ-$\mathcal{H}$-open. π-$\mathcal{H}$-open, α-$\mathcal{H}$-open, β-$\mathcal{H}$-open, strong β-$\mathcal{H}$-open) if $A \subset i(A^\star)$ (resp. $A \subset c^\star i(A)$, $A \subset ic^\star(A)$, $A \subset ic^\star i(A)$, $A \subset cic^\star(A)$, $A \subset c^\star ic^\star(A)$). The collection of $\mathcal{H}$-open (resp. σ-$\mathcal{H}$-open. π-$\mathcal{H}$-open, α-$\mathcal{H}$-open, β-$\mathcal{H}$-open) subsets of $(X, \mu, \mathcal{H})$ is denoted by $\mathcal{H}(\mu)$ (resp. $\sigma(\mathcal{H}, \mu)$, $\pi(\mathcal{H}, \mu)$, $\alpha(\mathcal{H}, \mu)$, $\beta(\mathcal{H}, \mu)$, $s\beta(\mathcal{H}, \mu)$). A subset A of (X, μ) is said to be μ-semiopen (resp. μ-preopen, μ-α-open, μ-β-open, μ-regular open) if $A \subset ci(A)$ (resp. $A \subset ic(A)$, $A \subset ici(A)$, $A \subset cic(A)$, $A = ic(A)$). The collection of μ-semiopen (resp. μ-preopen, μ-α-open, μ-β-open, μ-regular open) subsets of (X, μ) is denoted by $\sigma(\mu)$ (resp. $\pi(\mu)$, $\alpha(\mu)$, $\beta(\mu)$, $r(\mu)$). In this paper, we introduce and study the notion called extremally disconnected

2010 Mathematics Subject Classification 54A05, 54A10, 54C08, 54C10

spaces via hereditary classes. Many characterizations of this type of space are obtained. A generalized topology is said to be a quasi topology if it is closed under finite intersection [3].

The following lemmas will be useful in the sequel and we use some of the results without mentioning it, when the context is clear.

Lemma 1.1. *If A is a subset of $(X, \mu, \mathcal{H})$ such that $A \subset A^\star$, then* $A^\star = c(A^\star) = c^\star(A) = c(A)$.

Lemma 1.2. *Let (X, μ) be a quasi topology with a hereditary class $\mathcal{H}$ and A, B be subsets of X. If $U \in \mu$, then $U \cap A^\star \subset (U \cap A)^\star$.*

2. Properties of $\mathcal{H}$-extremally disconnected spaces

Definition 2.1. A hereditary generalized topological space $(X, \mu, \mathcal{H})$ is said to be $\mathcal{H}$-extremally disconnected if $c^\star(A) \in \mu$ for each $A \in \mu$.

Example 2.2. Let $X = \{a, b, c, d\}$, $\mu = \{\emptyset, \{a\}, \{a, b\}, \{b, c\}, \{a, b, c\}, \{b, c, d\}, X\}$ and $\mathcal{H} = \{\emptyset, \{b\}, \{c\}, \{b, c\}\}$. It follows that $\mathcal{H}$ is extremally disconnected.

Example 2.3. Let $X = \{a, b, c, d\}$, $\mu = \{\emptyset, \{a\}, \{b, c\}, \{a, b, c\}, X\}$ and $\mathcal{H} = \{\emptyset, \{c\}\}$. Take $A = \{a\}$, then $A^\star = \{a, d\}$. It follows that $\mathcal{H}$ is not extremally disconnected, because $c^\star(A) \notin \mu$.

Definition 2.4. A subset A of a hereditary generalized topological space $(X, \mu, \mathcal{H})$ is said to be weak regular-$\mathcal{H}$-closed if $A = c^\star(i(A))$. The family of weak regular-$\mathcal{H}$-closed subsets of $(X, \mu, \mathcal{H})$ is denoted by $rc(\mathcal{H}, \mu)$.

Example 2.5. In the Example 2.2, $A = \{a, b, d\}$, $i(A) = \{a\}$ and $c^\star(i(A)) = \{a, d\}$, in consequence, A is not A weak regular-$\mathcal{H}$-closed. In the same Example, the set $B = \{a, b\}$, $i(B) = B$ and $c^\star(i(B)) = \{a, b\} = B$. Follows that B is weak regular-$\mathcal{H}$-closed

Theorem 2.6. *For a hereditary generalized topological space $(X, \mu, \mathcal{H})$, the following are equivalent:*

(1) $(X, \mu, \mathcal{H})$ is $\mathcal{H}$-extremally disconnected,
(2) $\sigma(\mathcal{H}, \mu) \subset \pi(\mathcal{H}, \mu)$,
(3) $rc(\mathcal{H}, \mu) \subset \mu$.

PROOF. (1) → (2): Let $A \in \sigma(\mathcal{H}, \mu)$. Then $A \subset c^\star(i(A))$ and by (1) $c^\star(i(A)) \in \in \mu$. Therefore, we have $A \subset c^\star(i(A)) = i(c^\star(i(A))) \subset i(c^\star((A))$. This shows that $A \in \pi(\mathcal{H}, \mu)$.

(2) → (3): Let $A \in rc(\mathcal{H}, \mu)$. Then $A = c^\star(i(A))$ and hence $A \in \sigma(\mathcal{H}, \mu)$. By (2), $A \in \pi(\mathcal{H}, \mu)$ and $A \subset i(c^\star(A)$. Moreover, A is $\star$-closed and $A \subset \subset i(c^\star(A) = i(A)$. Therefore, we obtain $A \in \mu$.

(3) → (1): For $A \in \mu$, we show that $c^\star(A) \in rc(\mathcal{H}, \mu)$. Since $i(c^\star(A)) \subset \subset c^\star(A)$, we have $(i(c^\star(A)))^\star \subset (c^\star(A))^\star = (A \cup A^\star)^\star = A^\star \cup (A^\star)^\star \subset \subset A^\star \cup A^\star = A^\star \subset c^\star(A)$ and hence $(i(c^\star(A)))^\star \subset c^\star(A)$. So, we have $c^\star(i(c^\star(A))) = i(c^\star(A)) \cup (i(c^\star(A)))^\star \subset c^\star(A)$ and hence $c^\star(i(c^\star(A))) \subset \subset c^\star(A)$. On the other hand, since $A \in \mu$, $A \in \pi(\mathcal{H}, \mu)$ and hence we have $A \subset i(c^\star(A))$. Then, we have $c^\star(A) \subset c^\star(i(c^\star(A)))$. Hence, we have $c^\star(A) = c^\star(i(c^\star(A)))$. This shows that $c^\star(A) \in rc(\mathcal{H}, \mu)$. Furthermore, since $rc(\mathcal{H}, \mu) \subset \mu$, we have $c^\star(A) \in \mu$. This shows that $(X, \mu, \mathcal{H})$ is $\mathcal{H}$-extremally disconnected. ∎

THEOREM 2.7. *Let (X, μ) be a quasi topology with a hereditary class $\mathcal{H}$. If $A \cap B = \emptyset$ for every $A, B \in \mu$, then $A \cap c^\star(B) = \emptyset$.*

PROOF. Since $A \cap B = \emptyset$,

$$A \cap c^\star(B) \subset A \cap (B \cup B^\star) =$$

$$= (A \cap B) \cup (A \cap B^\star) \subset (A \cap B) \cup (A \cap B)^\star = c^\star(A \cap B).$$

On the other hand, since $\emptyset^\star = \emptyset$ and $c^\star(\emptyset) = \emptyset$, we have $A \cap c^\star(B) \subset c^\star(A \cap \cap B) = \emptyset$. Thus, we obtain that $A \cap c^\star(B) = \emptyset$. ∎

LEMMA 2.8. *Let $(X, \mu, \mathcal{H})$ be a $\mathcal{H}$-extremally disconnected space. If $A \cap B = \emptyset$ for every $A, B \in \mu$, then $c^\star(A) \cap c^\star(B) = \emptyset$.*

PROOF. The proof follows from Theorem 2.7. ∎

LEMMA 2.9. *Let $(X, \mu, \mathcal{H})$ be a $\mathcal{H}$-extremally disconnected space. If $c^\star(A) \cap \cap c^\star(B) = \emptyset$ for any subsets A and B of X, then $A \cap B = \emptyset$.*

PROOF. Since $A \subset c^\star(A)$ and $B \subset c^\star(B)$, $A \cap B \subset c^\star(A) \cap c^\star(B) = \emptyset$. Then $A \cap B = \emptyset$. ∎

EXAMPLE 2.10. Let $X = \{a, b, c, d\}$, $\mu = \{\emptyset, \{a\}, \{b, c\}, \{a, b, c\}\}$ and $\mathcal{H} = \{\emptyset\}$. $(X, \mu, \mathcal{H})$ is a $\mathcal{H}$-extremally disconnected space. Take $A = \{b\}$ and $B = \{c\}$. $A \cap B = \emptyset$ but $c^\star(A) \cap c^\star(B) \neq \emptyset$.

THEOREM 2.11. *Let $(X, \mu, \mathcal{H})$ be a $\mathcal{H}$-extremally disconnected space. Then $A \cap B = \emptyset$ if and only if $c^\star(A) \cap c^\star(B) = \emptyset$ for $A, B \in \mu$.*

PROOF. The proof is clear. ∎

THEOREM 2.12. *Let $(X, \mu, \mathcal{H})$ be a $\mathcal{H}$-extremally disconnected space and A a subset of X. Then the following properties hold:*

(1) $A \in \sigma(\mathcal{H}, \mu)$ if and only if $A \in \mathcal{H}(\mu)$,
(2) $A \in \pi(\mathcal{H}, \mu)$ if and only if $A \in s\beta(\mathcal{H}, (\mu))$,
(3) If $A \in \mathcal{H}(\mu)$, then $A \in \pi(\mu)$.

PROOF. (1). Sufficient condition is clear. On the other hand, let $A \in \sigma(\mathcal{H}, \mu)$. Then, we have $A \subset c^\star(i(A))$. Since $(X, \mu, \mathcal{H})$ is a $\mathcal{H}$-extremally disconnected space, for $i(A) \in \mu$, we have $c^\star(i(A) \in \mu$. Therefore, we have $A \subset c^\star(i(A)) \subset \subset i(c^\star(i(A)))$ and hence $A \in \mathcal{H}(\mu)$.

(2) Necessary condition is obvious. On the other hand, let $A \in sI(X, \mu)$ and hence $A \subset c^\star(i(c^\star(A)))$. Since $(X, \mu, \mathcal{H})$ is a hereditary generalized topological space, for $i(c^\star(A)) \in \mu$, we have $c^\star(i(c^\star(A))) \in \mu$. So, we have

$$A \subset c^\star(i(c^\star(A))) \subset i(c^\star(i(c^\star(A)))),$$

that is, $A \subset i(c^\star(i(c^\star(A))))$. Besides, since $i(c^\star(A)) \subset c^\star(A)$, we have $c^\star(i(c^\star(A))) \subset c^\star(c^\star(A)) = c^\star(A)$ and hence $i(c^\star(i(c^\star(A)))) \subset i(c^\star(A))$. Consequently, we have $A \subset i(c^\star(A))$ and hence $A \in \pi(\mathcal{H}, \mu)$.

(3). Let $A \in A(\mathcal{H}, \mu)$, then we have $A \subset c(i(c^\star(A)))$. Since $(X, \mu, \mathcal{H})$ is a hereditary generalized topological space, for $i(c^\star(A)) \in \mu$, we have $c(i(c^\star(A))) \in \mu$. So, we have $A \subset c(i(c^\star(A))) \subset i(c(i(c^\star(A)))) \subset \subset i(c(c^\star(A))) \subset i(c(A \cap A^\star)) = i(c(A) \cap c(A^\star)) \subset i(c(A))$. Therefore, $A \subset i(c(A))$ and hence $A \in \pi(\mu)$. ∎

EXAMPLE 2.13. In the Example 2.2, $A = \{a, b, d\}$. $A \in \pi(\mu)$ but $A \notin \mathcal{H}(\mu)$.

THEOREM 2.14. *For a hereditary generalized topological space $(X, \mu, \mathcal{H})$, the following are equivalent:*

(1) $(X, \mu, \mathcal{H})$ is $\mathcal{H}$-extremally disconnected.
(2) $\mathcal{H}(r) = \mu \cap \mathcal{F}$, where $\mathcal{F}$ is the family of all $\star$-$\mathcal{H}$-closed subsets of X.

PROOF. (1) $\Rightarrow$ (2): Suppose $(X, \mu, \mathcal{H})$ is $\mathcal{H}$-extremally disconnected. Then $\mu \cap \mathcal{F} \subset \mathcal{H}(r)$. clearly, every $\mathcal{H}(r)$-open set is open and every $\mathcal{H}(r)$-open set is $\star$-$\mathcal{H}$-closed, and so $\mathcal{H}(r) \subset \mu \cap \mathcal{F}$ which proves (2).

(2) $\Rightarrow$ (1) Suppose (2) holds. Let $G \in \mu$. Then $i(c^\star(G))$ is a $\mathcal{H}(r)$-open set and so it is $\star$-$\mathcal{H}$-closed by hypothesis. Hence $c^\star(G) \subset c^\star ic^\star(G)$ and

so $c^\star(G) = ic^\star(G)$ which implies that $c^\star(G) \in \mu$. Therefore, $(X, \mu, \mathcal{H})$ is $\mathcal{H}$-extremally disconnected. ∎

Theorem 2.15. *For a hereditary generalized topological space $(X, \mu, \mathcal{H})$, the following are equivalent:*

(1) $(X, \mu, \mathcal{H})$ is $\mathcal{H}$-extremally disconnected.
(2) $c^\star(A) \in \mathcal{H}(r)$ for every $A \in \beta(\mathcal{H}, \mu)$.
(3) $c^\star(A) \in \mathcal{H}(r)$ for every $A \in \pi(\mathcal{H}, \mu)$.
(4) $c^\star(A) \in \mathcal{H}(r)$ for every $A \in \alpha(\mathcal{H}, \mu)$.
(5) $c^\star(A) \in \mathcal{H}(r)$ for every $A \in \mu$.
(6) $c^\star(A) \in \mathcal{H}(r)$ for every $A \in \sigma(\mathcal{H}, \mu)$.
(7) $c^\star(A) \in \mathcal{H}(r)$ for every $A \in \delta(\mathcal{H}, \mu)$.
(8) $c^\star(A) \in \mathcal{H}(r)$ for every $A \in \mathcal{H}(r)$.

Proof. (1) ⇒ (2): Suppose $(X, \mu, \mathcal{H})$ is $\mathcal{H}$-extremally disconnected. If $G \in$ $\in s\beta(\mathcal{H}, \mu)$, then $G \subset c^\star(i(c^\star(G)))$ and so $c^\star(G) \subset c^\star(i(c^\star(G)))$. By hypothesis, it follows that $c^\star(G) \subset i(c^\star(i(c^\star(G))) = i(c^\star(G))$ and so $c^\star(G) = i(c^\star(G)) = i(c^\star(c^\star(G)))$. Therefore $c^\star(G) \in \mathcal{H}(r)$.

(2) ⇒ (3),(3) ⇒ (4),(4) ⇒ (5),(5) ⇒ (6), and (6) ⇒ (7), are clear. Also (2) ⇒ (6), and (6) ⇒ (4) are clear.

(2) ⇒ (1): Clearly, $\mu \cap \mathcal{F} \subset \mathcal{H}(r)$. Suppose $G \in \mathcal{H}(r)$. Then $G = i(c^\star(G))$ and $i(c^\star(G)) \in \mathcal{H}(r)$. By hypothesis, $c^\star(i(c^\star(G))) \in \mathcal{H}(r)$. Therefore, $c^\star(i(c^\star(G))) = i(c^\star(c^\star(i(c^\star(G))))) = i(c^\star(G)) = G$, which implies that $G \in \mathcal{H}$. Hence $\mathcal{H}(r) = \mu \cap \mathcal{F}$; consequently $(X, \mu, \mathcal{H})$ is $\mathcal{H}$-extremally disconnected. ∎

Theorem 2.16. *For a hereditary generalized topological space $(X, \mu, \mathcal{H})$, the following are equivalent:*

(1) $(X, \mu, \mathcal{H})$ is $\mathcal{H}$-extremally disconnected.
(2) $i^\star(A)$ is μ-closed for every μ-closed subset A of X.
(3) $c^\star i(A) \subset ic^\star(A)$ for every subset A of X.
(4) $c^\star(A) \in \mu$ for every $A \in s\beta(\mathcal{H}, \mu)$.
(5) $\beta(\mathcal{H}, \mu) \subset \pi(\mathcal{H}, \mu)$.
(6) $\alpha(\mathcal{H}, \mu) = \sigma(\mathcal{H}, \mu)$.

Proof. (1) ⇒ (2): Let A be a μ-closed subset of X. Then $X \backslash A \in \mu$. By (1), $c^\star(X \backslash A) = X \backslash i^\star(A) \in \mu$. Hence $i^\star(A)$ is μ-closed.

(2) ⇒ (3): Let A be any subset of X. Then $X \backslash i(A)$ is μ-closed in X and by (2), $i^\star(X \backslash i(A))$ is μ-closed in X. Therefore, $c^\star i(A) \in \mu$ and hence $c^\star i(A) \subset ic^\star(A)$.

(3) $\Rightarrow$ (4): Let $A \in \beta(\mathcal{H},\mu)$. Then $c^\star(A) \in \sigma(\mathcal{H},\mu)$. By Theorem 2.14, $c^\star(A) \in \pi(\mathcal{H},\mu)$. Thus, $c^\star(A) \subset ic^\star(A)$; hence $c^\star(A) \in \mu$.

(4) $\Rightarrow$ (5): Let $A \in \beta(\mathcal{H},\mu)$. By(4), $c^\star(A) = ic^\star(A)$. Hence $A \subset c^\star(A) = = ic^\star(A)$; hence $A \in \pi(\mathcal{H},\mu)$.

(5) $\Rightarrow$ (6): Let $A \in \sigma(\mathcal{H},\mu)$. Since $\sigma(\mathcal{H},\mu) \subset \beta(\mathcal{H},\mu)$. By (5), $A \in \pi(\mathcal{H},\mu)$. Since $A \in \sigma(\mathcal{H},\mu)$ and $A \in \pi(\mathcal{H},\mu)$, $A \in \alpha(\mathcal{H},\mu)$.

(6) $\Rightarrow$ (1): Let $A \in \mu$. Then $c^\star(A) \in \sigma(\mathcal{H},\mu)$, by (6), we have $c^\star(A) \in \in \alpha(\mathcal{H},\mu)$. Hence $c^\star(A) \subset ic^\star ic^\star(A) = ic^\star(A)$; hence $c^\star(A) \subset ic^\star(A)$. Hence $c^\star(A) \in \mu$ and hence $(X,\mu,\mathcal{H})$ is $\mathcal{H}$-extremally disconnected. ∎

Theorem 2.17. *For a hereditary generalized topological space $(X,\mu,\mathcal{H})$, the following are equivalent:*

(1) $(X,\mu,\mathcal{H})$ is $\mathcal{H}$-extremally disconnected.
(2) Every $\mathcal{H}(r)$-open subset of X is $\star$-$\mathcal{H}$-closed in X.
(3) Every $\mathcal{H}(r)$-closed subset of X is $\star$-$\mathcal{H}$-open in X.

Proof. (1) $\Leftrightarrow$ (2): Let $A \in \mathcal{H}(r)$. Then $ic^\star(A)$. Since $A \in \mu$, $c^\star \in \mu$. Hence $A = ic^\star(A) = c^\star(A)$; hence A is $\star$-$\mathcal{H}$-closed in X. Conversely, let $A \in \mu$. Since $ic^\star(A) \in \mathcal{H}(r)$, it is $\star$-$\mathcal{H}$-closed. This implies that $c^\star(A) \subset c^\star ic^\star(A) = ic^\star(A)$ since $A \subset ic^\star(A)$. Hence $c^\star \in \mu$; hence X is $\mathcal{H}$-extremally disconnected.

(2) $\Leftrightarrow$ (3): Obvious. ∎

References

[1] A. Császár, Generalized topology, generalized continuity, *Acta Math. Hungar.*, **96** (4) (2002), 351–357.

[2] A. Császár, Modifications of generalized topologies via hereditary casses, *Acta Math. Hungar.*, **115** (2007), 29–36.

[3] A. Császár, Remarks on quasi-topologies, *Acta Math. Hungar.*, **119** (1) (2008), 197–200.

[4] R. Ramesh and R. Mariapppan, Generalized open sets in hereditary generalized topological spaces, *J. Math. Compute. Sci.*, **5** (2) (2015), 149–159.

S. Abinaya
Department of Mathematics
Star Lion College of Engg.
Thanjavur-613005
Tamilnadu, India
abinaya4120@gmail.com

C. Carpintero
Department of Mathematics
Universidad De Oriente
Cumaná, Venezuela
carpintero.carlos@gmail.com

N. Rajesh
Department of Mathematics
Rajah Serfoji Govt. College
Thanjavur-613005
Tamilnadu, India
nrajesh_topology@yahoo.co.in

E. Rosas
Department of Mathematics
Universidad De Oriente
Cumana
Venezuela
ennisrafael@gmail.com

ANNALES UNIV. SCI. BUDAPEST., **59** (2016), 75–80

ON TWO VARIABLES FUNCTIONS OF ϕ-BOUNDED VARIATION IN THE MEAN

By
K. N. DARJI AND R. G. VYAS
(Received July 31, 2016)

Abstract. Here, we have extended the Riesz type result for the class $\phi BVM([0, 2\pi]^2)$ of two variables functions of ϕ-bounded variation in the sense of L_ϕ-norm.

1. Introduction

While studying the convergence of Fourier series, Jordan introduced the class $BV([0, 2\pi])$ of functions of bounded variation over $[0, 2\pi]$. The concept of bounded variation is generalized in many ways and many different notions of generalized bounded variations are introduced. While investigating the convergence of Fourier series in the $L^1([0, 2\pi])$-norm, in 1996 F. Móricz and A. H. Siddiqi [6] introduced the class $BVM([0, 2\pi])$ of functions of bounded variation in the mean. The concept of bounded variation in the mean was generalized by R. E. Castillo [2] in 2005 and it was to the class $BV^{(p)}M([0, 2\pi])$ of functions of p-bounded variation in the mean. Finally in 2013, R. E. Castillo, N. Merentes and E. Trousselot [3] generalized the class $BV^{(p)}M([0, 2\pi])$ by introducing the class $\phi BVM([0, 2\pi])$ of functions of ϕ-bounded variation in the sense of $L_\phi([0, 2\pi])$-norm. Here, extending the class $\phi BVM([0, 2\pi])$ to the class $\phi BVM([0, 2\pi]^2)$ of two variables functions of ϕ-bounded variation in the mean, we have obtained Riesz type result for this extended class.

We need following notations and definitions.

In the sequel $\mathbb{T} = [0, 2\pi)$ and a function $\phi : [0, \infty) \to [0, \infty)$ is continuous, convex, and satisfies $\phi(0) = 0$, $\phi(x) > 0$ for $x > 0$, and $\phi(x) \to \infty$ as $x \to \infty$.

2000 Mathematics Subject Classification 26A45, 26B30, 26A16, 26A24

We say $f \in L_\phi(\overline{\mathbb{T}})$ if

$$\int_{\overline{\mathbb{T}}} \phi(|f(x)|)\, dx < \infty.$$

DEFINITION 1.1. Let $f \in L_\phi(\overline{\mathbb{T}})$ be 2π-periodic. We say $f \in \phi BVM(\overline{\mathbb{T}})$ (that is, f is a function of ϕ-bounded variation in the mean over $\overline{\mathbb{T}}$) if

$$V_\phi^m(f, \overline{\mathbb{T}}) = \sup_{I^1} \left\{ \sum_i \int_{\overline{\mathbb{T}}} \phi\left(\frac{|f(I_{ix})|}{|I_{ix}|} \right) |I_{ix}|\, dx \right\} < \infty,$$

where I^1 is a finite collection of non-overlapping subintervals $\{[x_i, x_{i+1}]\}$ in $\overline{\mathbb{T}}$, $\{I_{ix}\} = \{[x+x_i, x+x_{i+1}]\}$, $f(I_{ix}) = f(x+x_{i+1}) - f(x+x_i)$ and $|I_{ix}| = |x_{i+1} - x_i|$.

Note that, for $\phi(x) = x^p$ $(p \geq 1)$ one gets the class $BV^{(p)}M(\overline{\mathbb{T}})$ (see, [2, Defination 2.1, p. 62]) and for $\phi(x) = x$ one gets the class $BVM(\overline{\mathbb{T}})$ (see, [6, p. 19]).

R. E. Castillo, N. Merentes and E. Trousselot [3] proved the following result.

THEOREM A: Let $f \in \phi BVM(\overline{\mathbb{T}})$ be such that f' is continuous on $\overline{\mathbb{T}}$. Then $f' \in L_\phi(\overline{\mathbb{T}})$ and

$$V_\phi^m(f, \overline{\mathbb{T}}) = 2\pi \int_{\overline{\mathbb{T}}} \phi(|f'(x)|)\, dx.$$

For a function f of two variables, where f is 2π-periodic in each variables, the notion of ϕ-bounded variation in the sense of L_ϕ-norm is defined as follows.

We say $f \in L_\phi(\overline{\mathbb{T}}^2)$ if

$$\iint_{\overline{\mathbb{T}}^2} \phi(|f(x, y)|)\, dx\, dy < \infty.$$

Consider a function f on $\mathbb{R}^k$. For $k = 1$ and $I = [a, b]$, define $f(I) = f(b) - f(a)$. For $k = 2$, $I = [a, b]$ and $J = [c, d]$, define

$$f(I \times J) = f(I, d) - f(I, c) = f(b, d) - f(a, d) - f(b, c) + f(a, c).$$

DEFINITION 1.2. Let $f \in L_\phi(\overline{\mathbb{T}}^2)$. We say $f \in \phi BVM(\overline{\mathbb{T}}^2)$ (that is, f is a function of ϕ-bounded variation in the mean over $\overline{\mathbb{T}}^2$) if each of

$$V_\phi^m(f, \overline{\mathbb{T}}^2) = \sup_{I^1, I^2} \left\{ \sum_i \sum_j \iint_{\overline{\mathbb{T}}^2} \phi\left(\frac{|f(I_{ix} \times I_{jy})|}{|I_{ix}|\,|I_{jy}|} \right) |I_{ix}|\,|I_{jy}|\, dx\, dy \right\},$$

$$V_\phi^m(f(., s), \overline{\mathbb{T}}) = \sup_{I^1} \left\{ \sum_i \int_{\overline{\mathbb{T}}} \phi\left(\frac{|f(I_{ix} \times s)|}{|I_{ix}|} \right) |I_{ix}|\, dx \right\},\ s \in \overline{\mathbb{T}},$$

and

$$V_{\phi}^{m}(f(t,.),\overline{\mathbb{T}}) = \sup_{I^2}\left\{\sum_{j}\int_{\overline{\mathbb{T}}}\phi\left(\frac{|f(t\times I_{jy})|}{|I_{jy}|}\right)|I_{jy}|\,dy\right\},\ t\in\overline{\mathbb{T}},$$

are finite, where I^1 and I^2 are finite collections of non-overlapping subintervals $\{[x_i, x_{i+1}]\}$ and $\{[y_j, y_{j+1}]\}$ respectively in $\overline{\mathbb{T}}$, $\{I_{ix}\} = \{[x+x_i, x+x_{i+1}]\}$, $\{I_{jy}\} = \{[y+y_j, y+y_{j+1}]\}$, $|I_{ix}| = |x_{i+1}-x_i|$ and $|I_{jy}| = |y_{j+1}-y_j|$.

Note that, for $\phi(x) = x^p$ $(p \geq 1)$ one gets the class $BV^{(p)}M(\overline{\mathbb{T}}^2)$ (see, [4, Defination 1.2, p. 62]) and for $\phi(x) = x$ one gets the class $BVM(\overline{\mathbb{T}}^2)$.

We now extend Riesz type result (Theorem A) for the class $\phi BVM(\overline{\mathbb{T}}^2)$ of two variables functions of ϕ-bounded variation in the mean as follows.

2. Statement of the result

THEOREM 2.1. *Let $f \in \phi BVM(\overline{\mathbb{T}}^2) \cap C^2(\overline{\mathbb{T}}^2)$. Then $f_x(.,s) \in L_\phi(\overline{\mathbb{T}})$, $f_y(t,.) \in L_\phi(\overline{\mathbb{T}})$, $f_{yx} \in L_\phi(\overline{\mathbb{T}}^2)$ and*

$$\begin{aligned} V_{\phi}^{m}(f,\overline{\mathbb{T}}^2) + V_{\phi}^{m}(f(.,s),\overline{\mathbb{T}}) + V_{\phi}^{m}(f(t,.),\overline{\mathbb{T}}) = \\ = 4\pi^2 \iint_{\overline{\mathbb{T}}^2}\phi(|f_{yx}(x,y)|)\,dx\,dy + 2\pi\int_{\overline{\mathbb{T}}}\phi(|f_x(.,s)|)\,dx + \\ + 2\pi\int_{\overline{\mathbb{T}}}\phi(|f_y(t,.)|)\,dy. \end{aligned}$$

3. Proof of the result

PROOF OF THEOREM 2.1. In view of [3, Theorem, p. 133], we have $f_x(.,s) \in L_\phi(\overline{\mathbb{T}})$, $f_y(t,.) \in L_\phi(\overline{\mathbb{T}})$,

$$V_{\phi}^{m}(f(.,s),\overline{\mathbb{T}}) = 2\pi\int_0^{2\pi}\phi(|f_x(.,s)|)\,dx \tag{1}$$

and

$$V_\phi^m(f(t,.),\overline{\mathbb{T}}) = 2\pi \int_0^{2\pi} \phi(|f_y(t,.)|)\, dy. \tag{2}$$

Now, by the rectangular mean value theorem [5, Proposition 3.11, p. 93], for any $(x,y) \in \overline{\mathbb{T}}^2$ there exists $(\epsilon_i^1, \epsilon_j^2) \in (x+x_i, x+x_{i+1}) \times (y+y_j, y+y_{j+1})$ such that

$$|A_1 - B_1 - C_1 + D_1| = |f_{xy}(\epsilon_i^1, \epsilon_j^2)||x_{i+1} - x_i||y_{j+1} - y_j|,$$

where

$A_1 = f(x+x_{i+1}, y+y_{j+1}),\ B_1 = f(x+x_i, y+y_{j+1}),\ C_1 = f(x+x_{i+1}, y+y_j)$ and $D_1 = f(x+x_i, y+y_j)$.

Therefore,

$$\phi\left(\frac{|A_1 - B_1 - C_1 + D_1|}{|x_{i+1} - x_i||y_{j+1} - y_j|}\right)|x_{i+1} - x_i||y_{j+1} - y_j| = \\ = \phi(|f_{xy}(\epsilon_i^1, \epsilon_j^2)|)|x_{i+1} - x_i||y_{j+1} - y_j|.$$

Integrating the above equality over $\overline{\mathbb{T}}^2$ and then summing it for $i = 0$ to $n-1$ and $j = 0$ to $r-1$, we get

$$4\pi^2 \sum_{i=0}^{n-1}\sum_{j=0}^{r-1} \phi(|f_{xy}(\epsilon_i^1, \epsilon_j^2)|)\ (x_{i+1} - x_i)(y_{j+1} - y_j) = \\ = \sum_{i=0}^{n-1}\sum_{j=0}^{r-1} \iint_{\overline{\mathbb{T}}^2} \phi\left(\frac{|A_1 - B_1 - C_1 + D_1|}{|x_{i+1} - x_i||y_{j+1} - y_j|}\right)\ |x_{i+1} - x_i||y_{j+1} - y_j|\, dx\, dy \leq \\ \leq V_\phi^m(f, \overline{\mathbb{T}}^2).$$

Therefore,

$$4\pi^2 \iint_{\overline{\mathbb{T}}^2} \phi(|f_{xy}(x,y)|)\, dx\, dy \leq V_\phi^m(f, \overline{\mathbb{T}}^2) < \infty. \tag{3}$$

Thus, $f_{xy} \in L_\phi(\overline{\mathbb{T}}^2)$.

On the other hand, by Jensen inequality,

$$\phi\left(\frac{|A_1 - B_1 - C_1 + D_1|}{|x_{i+1} - x_i||y_{j+1} - y_j|}\right) |x_{i+1} - x_i||y_{j+1} - y_j| =$$

$$= \phi\left(\frac{\left|\int_{y+y_j}^{y+y_{j+1}} \int_{x+x_i}^{x+x_{i+1}} f_{uv}(u, v)\, du\, dv\right|}{\int_{x_i}^{x_{i+1}} \int_{y_j}^{y_{j+1}} du\, dv}\right) |x_{i+1} - x_i||y_{j+1} - y_j| \leq$$

$$\leq \phi\left(\frac{\int_{y+y_j}^{y+y_{j+1}} \int_{x+x_i}^{x+x_{i+1}} |f_{uv}(u, v)|\, du\, dv}{\int_{x_i}^{x_{i+1}} \int_{y_j}^{y_{j+1}} du\, dv}\right) |x_{i+1} - x_i||y_{j+1} - y_j| \leq$$

$$\leq \left(\frac{\int_{y+y_j}^{y+y_{j+1}} \int_{x+x_i}^{x+x_{i+1}} \phi(|f_{uv}(u, v)|)\, du\, dv}{\int_{x_i}^{x_{i+1}} \int_{y_j}^{y_{j+1}} du\, dv}\right) (x_{i+1} - x_i)(y_{j+1} - y_j) =$$

$$= \int_{y+y_j}^{y+y_{j+1}} \int_{x+x_i}^{x+x_{i+1}} \phi(|f_{uv}(u, v)|)\, du\, dv.$$

Integrating the above inequality over $\overline{\overline{\mathbb{T}}}^2$ and then summing it for $i = 0$ to $n - 1$ and $j = 0$ to $r - 1$, we get

$$V_\phi^m(f, \overline{\overline{\mathbb{T}}}^2) \leq 4\pi^2 \int_0^{2\pi} \int_0^{2\pi} \phi(|f_{yx}(x, y)|)\, dx\, dy. \tag{4}$$

From (3) and (4), we have

$$V_\phi^m(f, \overline{\overline{\mathbb{T}}}^2) = 4\pi^2 \int_0^{2\pi} \int_0^{2\pi} \phi(|f_{yx}(x, y)|)\, dx\, dy. \tag{5}$$

Hence, the theorem follows from (1), (2) and (5). ∎

References

[1] C. R. Adams and J. A. Clarkson, Properties of functions $f(x, y)$ of bounded variation, *Trans. Amer. Math. Soc.*, **36** (1934), 711–730.

[2] R. E. Castillo, Bounded p-variation in the mean, *East-West J. Math.*, **7**, (1) (2005), 61–67.

[3] R. E. Castillo, N. Merentes and E. Trousselot, The Nemytskii operator on bounded ϕ-variation in the mean spaces, *Proyecciones Journal of Mathematics*, **32**, (2) (2013), 119–142.

[4] K. N. Darji and R. G. Vyas, On two variables functions of bounded p-variation in the mean, *J. Indian Math. Soc.*, **81**, (1–2) (2014), 61–66.

[5] S. R. Ghorpade and B. V. Limaye, *A course in multivariable calculus and analysis*, Springer, 2009.

[6] F. Móricz and A. Siddiqi, A quatified version of the Dirichlet-Jordan test in L^1-norm, *Rend. Circ. Mat. Palermo*, **45**, (1996), 19–24.

K. N. Darji
Department of Science and Humanities
Tatva Institute of Technological Studies
Modasa, Aravalli, Gujarat, India
darjikiranmsu@gmail.com

R. G. Vyas
Department of Mathematics
Faculty of Science
The Maharaja Sayajirao University of Baroda, Vadodara, Gujarat, India
drrgvyas@yahoo.com

ANNALES UNIV. SCI. BUDAPEST., **59** (2016), 81–93

THE IDEMPOTENT SUM NUMBER AND n-THIN UNITAL RINGS

By
PETER DANCHEV AND EBRAHIM NASIBI
(Received October 1, 2016)

Abstract. We define the concepts of *idempotent sum number* for unital rings and *n-thin* unital rings, where $n \in \mathbb{N}$. We also investigate their fundamental properties, some of which are in contrast to certain principal facts that are well-known for the so-called n-good rings.

1. Introduction and Basic Definitions

Throughout the present paper, all rings into consideration will be associative with identity element 1; such rings will be hereafter called *unital*. As usual, for such a ring R, $U(R)$ denotes the group of units of R, $J(R)$ denotes the Jacobson radical of R, $P(R)$ denotes the prime radical of R, $\mathrm{Nil}(R)$ denotes the set of all nilpotent elements of R, and $\mathrm{Id}(R)$ denotes the set of all idempotents of R. The most part of our terminology and notations are standard and will precisely follow these from [2] and [16]. Those that are not well-known will be stated explicitly below.

Motivated by the idea in [10] for the unit sum number, but related to idempotents (see [12], where things are slightly different), one can state the following new notions:

Definition 1. Let $n \geq 2$ be a natural number. A ring R is said to have the *idempotent n-sum property* if every element $r \in R \setminus \{0\}$ is the sum of exactly n non-zero idempotents, that is, $r = e_1 + \cdots + e_n$, where $e_1, \ldots, e_n \in \mathrm{Id}(R) \setminus \{0\}$.

2010 Mathematics Subject Classification 16S34, 16U60

For instance, in $\mathbb{Z}_2 = \{0, 1\}$ we have $1 = 1 + 1 + 1$, while in $\mathbb{Z}_2 \times \mathbb{Z}_2$ we have $(1, 1) = (1, 0) + (0, 1) = (1, 1) + (1, 1) + (1, 1)$. In that aspect, in any ring R it could be that $2 = 1 + 1 = 1 + (1 - e) + e$ for some existing non-trivial idempotent $e \in R$, etc.

The following observation is crucial: If R has the idempotent n-sum property for some $n > 1$, then it possesses the idempotent k-sum property for each $k \geq n$.

In fact, to prove this, given $0 \neq x \in R$. If $n > 1$, then we write $x = e_1 + \cdots + e_n$ with $e_1, \ldots, e_n \in \mathrm{Id}(R) \setminus \{0\}$. Hence $x - (e_1 + \cdots + e_{n-1}) = e_n \neq 0$ and thus $x - (e_1 + \cdots + e_{n-1}) = f_1 + \cdots + f_n$ for some non-zero idempotents $f_1, \ldots, f_n$ which gives that $x = e_1 + \cdots + e_{n-1} + f_1 + \cdots + f_n$ is a sum of $k = 2n - 1 > n$ idempotents. Repeating the same procedure, we can obtain such a representation of x for an arbitrary large k (depending on n).

It is immediate that all Boolean rings R, that are rings in which any element is an idempotent, will have the idempotent 2-sum property when they are decomposable and 3-sum property when they are indecomposable, i.e., are isomorphic to $\mathbb{Z}_2$. In fact, each non-trivial idempotent e can be written as a sum of the two non-zero idempotents 1 and $e - 1$. In this light, in any ring the existence of a non-trivial idempotent e yields that $1 = e + (1 - e)$. Moreover, if R is a commutative ring of characteristic 2, then the sum of two different non-trivial idempotents e_1 and e_2 is again an idempotent because $(e_1 + e_2)^2 = e_1^2 + e_2^2 = e_1 + e_2$. Also, 0 can be expressed as $0 = e + (1 - e) + 1 = 1 + 1$, where $e \in \mathrm{Id}(R)$. Same will hold for an arbitrary ring R, provided that e_1 and e_2 are mutually orthogonal non-trivial idempotents.

Definition 2. Suppose R is a ring and n is a positive integer. We will say that *the idempotent sum number* of R is n, and shall write $\mathrm{isn}(R) = n$ for short, if n is the minimal natural for which R has the idempotent n-sum property.

If there is an element of R which cannot be written as a sum of idempotents, we will write $\mathrm{isn}(R) = \infty$, whereas if every element of R is a sum of idempotents but R does not have the idempotent n-sum property for any natural n, we will write $\mathrm{isn}(R) = \omega$.

Note that Boolean rings R, i.e., rings whose elements are idempotents, have $\mathrm{isn}(R)$ equal to either 2 or 3 which depends on the fact whether R is decomposable or not.

Definition 3. Let $n \in \mathbb{N}$. A ring R is called *n-thin* if every element in R possesses the idempotent n-sum property. When this is fulfilled for any n, R is just said to be *thin*.

It is self-evident that Boolean rings are either 2-thin or 3-thin. Likewise, each n-thin ring is k-thin whenever $k \geq n$.

A concrete application of the above terminology and definitions is given in [7] for abelian groups with a variation of thin endomorphism rings.

On the other hand, imitating [1], a ring is called *clean* if each element is the sum of a unit and an idempotent. Likewise, in [14] were defined the so-called *n-good* rings that are rings for which every element is the sum of n-units. Generalizing these both notions, in [15] was stated and studied the concept of *n-clean rings* whose elements are sums of n-units and an idempotent. We shall discuss these concepts in the next sections.

2. Some Examples and Preliminaries

EXAMPLE 2.1. $\mathrm{isn}(\mathbb{Z}) = \mathrm{isn}(\mathbb{Q}) = \infty$.

PROOF. Straightforward. ∎

EXAMPLE 2.2. Suppose that $\mathbb{Z}_p$ is the simple p element field. If $p \neq 2$, then $\mathrm{isn}(\mathbb{Z}_p) = \omega$. If $p = 2$, then $\mathrm{isn}(\mathbb{Z}_2) = 3$.

PROOF. Each non-zero element x of $\mathbb{Z}_p \cong \mathbb{Z}/p\mathbb{Z}$ can be expressed as $x = t + (p)$, where $1 \leq t \leq p - 1$. Since t is the sum of idempotents 1 written t-times and 1 is written $(p + 1)$-times as the sum of itself, the assertion follows. ∎

As usual, $M_n(R)$ designates the full $n \times n$ matrix ring over a ring R.

EXAMPLE 2.3. Let R be a ring and $n \geq 2$. Then $\mathrm{isn}(M_n(R)) > 2$.

PROOF. Utilizing Lemma 3 in [12], we deduce that in $M_n(R)$ there exists an element which cannot be expressed as the sum of two idempotents. This forces that $\mathrm{isn}(M_n(R)) > 2$, as asserted. ∎

Let $R[[X]]$ be the ring of all formal power series and $R[X]$ the polynomial ring over a commutative ring R. The following holds.

EXAMPLE 2.4. For any ring R, $\mathrm{isn}(R[[X]]) = \mathrm{isn}(R[X]) = \infty \geq \mathrm{isn}(R)$.

PROOF. We know that $\mathrm{Id}\, R[[X]] = \mathrm{Id}(R)$. But it is plainly verified that there is no element from $R[[X]]$ which can be expressed as a ring-theoretic sum of idempotents from R. We can process similarly for $R[X]$, which gives the assertion. ∎

EXAMPLE 2.5. Let $R = J_p$ be the ring of p-adic integers. Then $\operatorname{isn}(R/J(R)) \leq$ $\leq \omega < \operatorname{isn}(R) = \infty$.

PROOF. It is well known that $J(R) = pR$ and that $R/J(R) \cong \mathbb{Z}_p$, whence by Example 2.2, we infer that $\operatorname{isn}(R/J(R)) \leq \omega$. However, R being a domain easily implies that $\operatorname{Id}(R) = \{0, 1\}$ and hence $\operatorname{isn}(R) = \infty$, thus sustaining the claim. ∎

The following simple observations are useful:

LEMMA 2.6. *If R is a commutative ring of prime* $\operatorname{char}(R) = p$ *and* $\operatorname{isn}(R) < \infty$, *then* $N(R) = \{0\}$.

PROOF. Letting $x \in N(R)$ and since $\operatorname{isn}(R)$ is either finite or ω, we express x as the finite sum $x = \sum_i e_i$ with all $e_i \in \operatorname{Id}(R)$. Since $x^{p^t} = 0$ for some $t \in \mathbb{N}$, we obtain $0 = x^{p^n} = (\sum_i e_i)^{p^n} = \sum_i e_i = x$. This substantiates our claim. ∎

When $p = 2$, we can say even much more by offering the following:

PROPOSITION 2.7. *If R is a commutative ring of* $\operatorname{char}(R) = 2$ *and* $\operatorname{isn}(R) < \infty$, *then R is Boolean.*

PROOF. Since $\operatorname{isn}(R)$ is either finite or ω, each element of R is a finite sum of idempotents. But it is readily checked that this sum is also an idempotent, so that R is Boolean and thus the assertion directly follows. ∎

As a valuable consequence, we derive:

COROLLARY 2.8. *Suppose R is a finite ring such that $U(R) = 1$. Then* $\operatorname{isn}(R) < \infty$ *if, and only if, R is Boolean.*

PROOF. Since $-1 = 1$, we observe that R has characteristic 2. It now follows from [8] or [5] that R is commutative, and hence we apply Proposition 2.7 to get the wanted claim. ∎

LEMMA 2.9. *Let R be a ring in which* 2 *is invertible and R is n-thin. Then R is $2n$-good.*

PROOF. For every $e \in \operatorname{Id}(R)$ one may observe that $e = \frac{2e-1}{2} + \frac{1}{2}$ is the sum of two units and so the assertion follows routinely. ∎

3. Main Results

We first give some properties of n-thin rings, i.e., of rings that have the idempotent n-sum property.

PROPOSITION 3.1. *Suppose R is a ring. Then the following statements are true:*

(1) If K is an ideal of the ring R which has the idempotent n-sum property, then R/K has the idempotent n-sum property with $\mathrm{isn}(R/K) \leq \mathrm{isn}(R)$.
(2) If R has the idempotent n-sum property, then every homomorphic image of R has the idempotent n-sum property.
(3) The direct product $R = \prod_{i \in I} R_i$ of rings $\{R_i\}_{i \in I}$ has the idempotent n-sum property if, and only if, each R_i has the idempotent n-sum property.

PROOF. To prove (1), let R have the idempotent n-sum property and let K be an ideal of the ring R. Also, let $\overline{x} = x + K \in \frac{R}{K}$. We have $0 \neq x = e_1 + e_2 + \cdots + e_n$, where $e_1, e_2, \ldots, e_n \in \mathrm{Id}(R) \setminus \{0\}$. So, $\overline{x} = \overline{e_1} + \overline{e_2} + \cdots + \overline{e_n}$, where $\overline{e_i} = e_i + K$; $1 \leq i \leq n$. But since $\overline{e_1}, \overline{e_2}, \ldots, \overline{e_n} \in \mathrm{Id}\left(\frac{R}{K}\right)$, it follows that $\frac{R}{K}$ has the idempotent l-sum property for some $l \leq n$, because $\overline{e_j} = 0$ may occur for some index $j \in [1, n]$. Thus, by the observation quoted above, R/K has the idempotent n-sum property. The second part is immediate.

Item (2) follows directly from item (1).

One direction of (3) follows directly from (2). Conversely, let R_i has the idempotent n-sum property for each $i \in I$. Set $0 \neq x = (x_1, x_2, \ldots, x_i, \ldots) \in$ $\in \prod_{i \in I} R_i$. There exist $e_{1_i}, e_{2_i}, \ldots, e_{n_i} \in \mathrm{Id}(R_i) \setminus \{0\}$ such that $x_i = e_{1_i} + e_{2_i} +$ $+ \cdots + e_{i_i} + \cdots + e_{n_i}$, provided $x_i \neq 0$ for $i \in I$. Hence

$$x = (e_{1_1}, e_{1_2}, \ldots, e_{1_i}, \ldots) + (e_{2_1}, e_{2_2}, \ldots, e_{2_i}, \ldots) + \ldots$$
$$\cdots + (e_{n_1}, e_{n_2}, \ldots, e_{n_i}, \ldots).$$

One can easily prove that $(e_{j_1}, e_{j_2}, \ldots, e_{j_i}, \ldots) \in \mathrm{Id}(\prod_{i \in I} R_i)$ for every j, $1 \leq j \leq n$. Therefore, $\prod_{i \in I} R_i$ has the idempotent n-sum property. If some $x_i = 0$, the situation is analogous, because 0 is an idempotent. ∎

REMARK 1. If now R is a boolean ring, then any non-zero idempotent e can be written as $e = (e - 1) + 1 = e + 1 + 1$, where certainly $2 = 0$.

On the other side, it was proved in [12, Theorem 2] that if R is a commutative ring having the idempotent 2-sum property, then $R/P(R)$ has the idempotent 2-sum property. However, this is a simple consequence of the above issue (1) when $n = 2$.

Corollary 3.2. *Suppose that R is a ring. Then*

(i) If R_i has the idempotent n_i-sum property for each $i \in [1, m]$, then $\prod_{i=1}^{m} R_i$ has the idempotent k-sum property where $k = \max\{n_i : i \in [1, m]\}$.

(ii) Let R have the idempotent n-sum property and let e be a central idempotent in R. Then eRe has the idempotent n-sum property. In particular, if R is n-thin, then eRe is n-thin.

Proof. (i) For each $i \in [1, m]$ we infer by the listed above observation that the system R_i has the idempotent k-sum property, where $k = \max\{n_i : i \in [1, m]\}$. Hence, with the aid of Proposition 3.1 (3), the product $\prod_{i=1}^{m} R_i$ has too the idempotent k-sum property.

(ii) Since e is central, it follows that $eRe = eR$ is a homomorphic image of R. Hence the result follows directly from Proposition 3.1 (2). ∎

This can be extended like this:

Theorem 3.3. *Let e be a central idempotent element of a ring R. Then R has the idempotent n-sum property if, and only if, eRe and $(1-e)R(1-e)$ both have the idempotent n-sum property. Thus $\mathrm{isn}(R) = \max\{\mathrm{isn}(eRe), \mathrm{isn}((1-e)R(1-e))\}$.*

In particular, R is n-thin if, and only if, both eRe and $(1-e)R(1-e)$ are n-thin.

Proof. We use $\overline{e}$ to denote $1-e$. Since $R = eRe \oplus \overline{e}R\overline{e} \cong \begin{pmatrix} eRe & 0 \\ 0 & \overline{e}R\overline{e} \end{pmatrix} = R'$, for every $A \in R'$ we formally write $A = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$, where a, b belong to eRe and $\overline{e}R\overline{e}$, respectively. By our hypothesis $a = e_1 + \cdots + e_n$, $b = f_1 + \cdots + f_n$, where $e_i, f_i \in \mathrm{Id}(R)$ are non-zero for i, $1 \le i \le n$. So

$$A = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = \begin{pmatrix} e_1 & 0 \\ 0 & f_1 \end{pmatrix} + \cdots + \begin{pmatrix} e_n & 0 \\ 0 & f_n \end{pmatrix}.$$

But $\begin{pmatrix} e_i & 0 \\ 0 & f_i \end{pmatrix} \in \mathrm{Id}(R')$ are non-zero for any i with $1 \le i \le n$. It now follows that R has the idempotent n-sum property.

Conversely, given $x \in R$, we write $x = er$ for some $r \in R$. Thus $r = e_1 + \cdots + e_n$ for $e_1, \dots, e_n \in \mathrm{Id}(R)$, and hence $x = e(e_1 + \cdots + e_n) = ee_1 + \cdots + ee_n$. Since it is obviously checked that $ee_1, \dots, ee_n$ are idempotents, we are done. ∎

Inductively, one can deduce the following:

Corollary 3.4. *Let $e_1, \dots, e_k$ be orthogonal central idempotents with $e_1 + \cdots + e_k = 1$. Then R has the idempotent n-sum property if, and only if, $e_i R e_i$ has the idempotent n-sum property for each index $1 \leq i \leq k$.*

In particular, R is n-thin if, and only if, $e_i R e_i$ is n-thin for every index $1 \leq i \leq k$.

Proof. One direction follows easily from Theorem 3.3 by induction.

Conversely, let R have the idempotent n-sum property and let $e_1, \dots, e_k$ be orthogonal central idempotents with $e_1 + \cdots + e_k = 1$. Thus, since $R = e_1 R e_1 \oplus \cdots \oplus e_k R e_k$, it plainly follows in view of Proposition 3.1 (3) that $e_i R e_i$ has the idempotent n-sum property for each $i \in [1, k]$. ∎

Theorem 3.5. *Let R be a ring and let $\mathrm{diag}(a_1, \dots, a_n)$ be the $n \times n$ diagonal matrix with a_i in each entry on the main diagonal; $1 \leq i \leq n$. Then $\mathrm{diag}(a_1, \dots, a_n)$ has the idempotent n-sum property in $\mathbb{M}_n(R)$ if, and only if, $a_1, \dots, a_n \in R \setminus \{0\}$ has the idempotent n-sum property in R.*

Proof. For the "if" part, write $a_i = e_{i1} + \cdots + e_{in}$, where $e_{ij} \in \mathrm{Id}(R)$ are non-zero for every i, j; $1 \leq i, j \leq n$. So,

$$diag(a_1, \dots, a_n) = diag(e_{11}, \dots, e_{1n}) + \cdots + diag(e_{n1}, \dots, e_{nn}).$$

But it is readily verified that $diag(e_{i1}, \dots, e_{in})$ is an idempotent for every i, where $1 \leq i \leq n$, as required.

As for the "and only if" part, it follows referring to Proposition 3.1 (2). ∎

The following technicality is well-known (see [13]), but we state it only for the sake of completeness and for the reader's convenience.

Lemma 3.6. *Let R be a commutative ring and let $f = \sum_{i=0}^{n} a_i x^i \in R[x]$ be an idempotent element. Then a_0 is an idempotent in R and a_i is a nilpotent in R for each i with $0 \leq i \leq n$.*

Proof. It is easy to check that a_0 is an idempotent. Now, to show the second part, it is enough to verify that for each prime ideal P of R we have that every $a_i \in P$; $i \in [1, n]$. Define $\varphi\colon R[x] \to (\frac{R}{P})[x]$ by $\varphi(\sum_{i=0}^{n} a_i x^i) = \sum_{i=0}^{n} (a_i + P) x^i$. As $\varphi(f)\varphi(f) = \varphi(f^2) = \varphi(f)$, so we have $deg(\varphi(f)) + deg(\varphi(f)) = deg(\varphi(f))$. Therefore, $deg(\varphi(f)) = 0$. Hence $a_1 + P = \cdots = a_n + P = P$, as required. ∎

Theorem 3.7. *If R is a commutative ring with the idempotent n-sum property, then $R[x]$ does not have the idempotent n-sum property.*

Proof. Let $x \in \mathrm{Id}(R[X])$ with $x = e_1 + \cdots + e_n$, where $e_i \in \mathrm{Id}(R[x])$ for every i, $1 \leq i \leq n$. Since $\mathrm{Id}(R) = \mathrm{Id}(R[x])$, we deduce that $x - (e_1 + \cdots + e_{n-1})$ is an idempotent in R. Hence, by Lemma 3.6, the element 1 should be a nilpotent, which is an obvious contradiction. ∎

Proposition 3.8. *Let R be a ring. Then R has the idempotent 2-sum property if, and only if, for every $x \in R$ there exist $e_1, e_2 \in \mathrm{Id}(R)$ such that $e_1 x = e_1 e_2$ and $(e_1 - 1)(x - 1) = (e_1 - 1)e_2$.*

Proof. If R has the idempotent 2-sum property and $x \in R$, then $x = f_1 + f_2$, where $f_1, f_2 \in \mathrm{Id}(R)$. Letting $e_1 = 1 - f_1$, we have $e_1 x = e_1(f_1 + f_2) = e_1 f_2$ and $(e_1 - 1)(x - 1) = (e_1 - 1)f_2$.

Conversely, let for every $x \in R$ there exist $e_1, e_2 \in \mathrm{Id}(R)$ such that $e_1 x = e_1 e_2$ and $(e_1 - 1)(x - 1) = (e_1 - 1)e_2$. As $e_1 e_2 - e_2 = (e_1 - 1)(x - 1) = e_1 x - e_1 - x + 1$, so $x = e_2 + (1 - e_1)$. Therefore, x has the idempotent 2-sum property, as expected. ∎

A Morita context, denoted by $(A, B, M, N, \psi, \varphi)$, consists of two rings A, B, two bimodules ${}_A N_B, {}_B M_A$ and a pair of bimodule homomorphisms $\psi \colon N \otimes_B M \longrightarrow A$ and $\varphi \colon M \otimes_A N \longrightarrow B$ which satisfy the following associativity: $\psi(v, w)v' = v\varphi(w, v')$ and $\varphi(w, v)w' = w\psi(v, w')$. These conditions will insure that the set T of generalized matrices

$$\begin{pmatrix} a & n \\ m & b \end{pmatrix}; a \in A, b \in B, m \in M, n \in N$$

will form a ring, called *the ring of the Morita context* or shortly *the Morita ring*. Many authors studied Morita rings – for instance, the interested reader can see [3], [4] and [11], respectively.

Proposition 3.9. *Let $T = \begin{pmatrix} A & N \\ M & B \end{pmatrix}$ be the Morita ring with $\varphi, \psi = 0$. If both A and B have the idempotent n-sum property, then $T/J(T)$ has the idempotent n-sum property.*

Proof. As A and B have the idempotent n-sum property, so by Proposition 3.1 (1) the quotients $A/J(A)$ and $B/J(B)$ have the idempotent n-sum property. Hence, adapting Proposition 3.1 (3), $[A/J(A)] \oplus [B/J(B)]$ has the idempotent n-sum property too. Now, since $T/J(T) \cong [A/J(A)] \oplus [B/J(B)]$, we deduce that $T/J(T)$ has the idempotent n-sum property as well. ∎

PROPOSITION 3.10. *Let* $C = \begin{pmatrix} A & N \\ M & B \end{pmatrix}$ *be the Morita ring with* $\varphi, \psi = 0$. *If* C *has the idempotent* n*-sum property, then* A *and* B *have the idempotent* n*-sum property.*

PROOF. Setting $I = \begin{pmatrix} 0 & N \\ M & B \end{pmatrix}$ and $J = \begin{pmatrix} A & N \\ M & 0 \end{pmatrix}$, we derive that I and J are ideals of C and that $C/I \cong A$ and $C/J \cong B$. Therefore, both A and B have the idempotent n-sum property by Proposition 3.1 (1). ∎

COROLLARY 3.11. *(1) Let* R, S *be two rings, and let* M *be an* (R, S)*-bimodule. Suppose also that* $E = \begin{pmatrix} R & M \\ 0 & S \end{pmatrix}$ *is the formal triangular matrix ring. If* E *has the idempotent* n*-sum property, then* R *and* S *have both the idempotent* n*-sum property.*

(2) Let, for each integer $n \geq 2$, *the ring* $\mathbb{T}_n(R)$ *of all* $n \times n$ *lower (resp., upper) triangular matrices over* R *has the idempotent* n*-sum property. Then* R *has the idempotent* n*-sum property.*

PROOF. Since formal triangular matrix rings are special cases of the Morita rings with zero morphisms, item (1) follows by Proposition 3.10. Also, point (2) follows immediately from point (1). ∎

The next two assertions are similar to Corollary 3.11.

THEOREM 3.12. *If* A *and* B *are rings,* $M = {}_BM_A$ *is a bimodule and the formal triangular matrix ring* $T = \begin{pmatrix} A & 0 \\ M & B \end{pmatrix}$ *has the idempotent* n*-sum property, then both* A *and* B *have the idempotent* n*-sum property.*

PROOF. Let $T = \begin{pmatrix} A & 0 \\ M & B \end{pmatrix}$ have the idempotent n-sum property. So, for every $a \in A$, $b \in B$ and $m \in M$, one can write $\begin{pmatrix} a & 0 \\ m & b \end{pmatrix} = \begin{pmatrix} e_1 & 0 \\ f_1 & g_1 \end{pmatrix} + \cdots + \begin{pmatrix} e_n & 0 \\ f_n & g_n \end{pmatrix}$, where $\begin{pmatrix} e_i & 0 \\ f_i & g_i \end{pmatrix} \in \mathrm{Id}(T)$ are non-zero for every i, $1 \leq i \leq n$. Thus $a = e_1 + \cdots + e_n$ and $b = g_1 + \cdots + g_n$. It is easy to check that $e_i \in \mathrm{Id}(A)$ and $g_i \in \mathrm{Id}(B)$, $\forall\, i \in [1, n]$. Therefore, A and B have the idempotent n-sum property, as claimed. ∎

As an immediate consequence, we yield (compare with Corollary 3.11 (2)):

COROLLARY 3.13. *Let, for each integer $n \geq 2$, the ring $\mathbb{T}_n(R)$ of all $n \times n$ lower (resp., upper) triangular matrices over R be n-thin. Then R is n-thin.*

The following result gives a partial converse of Theorem 3.12.

THEOREM 3.14. *If A and B are rings that have the idempotent n-sum property and $M = {}_B M_A$ is a bimodule, then the formal triangular matrix ring $T = \begin{pmatrix} A & 0 \\ M & B \end{pmatrix}$ has the idempotent $(n+2)$-sum property, provided* $\mathrm{char}(A) = 2$.

PROOF. For every $t = \begin{pmatrix} a & 0 \\ m & b \end{pmatrix} \in T$, we have $a = e_1 + \cdots + e_n$ and $b = f_1 + \cdots + f_n$, where $e_i \in \mathrm{Id}(A)$ and $f_i \in \mathrm{Id}(B)$ for each i with $1 \leq i \leq n$. Hence, one may write

$$t = \begin{pmatrix} e_1 & 0 \\ 0 & f_1 \end{pmatrix} + \cdots + \begin{pmatrix} e_n & 0 \\ 0 & f_n \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ m & 0 \end{pmatrix} + \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix},$$

which manifestly shows that T possesses the idempotent $(n+2)$-sum property, as asserted. ∎

For more examples of rings that have the idempotent n-sum property, in what follows we consider the amalgamated duplication of a ring R. Let R be a commutative ring and I an ideal of R. As in [9], the amalgamated duplication of the ring R along the ideal I is defined to be the subring $R \bowtie I = \{(r, r+i) : r \in R, i \in I\}$ of the direct product $R \times R$.

PROPOSITION 3.15. *Let R be a commutative ring and I an ideal of R. If $R \bowtie I$ has the idempotent n-sum property, then R has the idempotent n-sum property.*

In particular, if $R \bowtie I$ is n-thin, then R is n-thin.

PROOF. Suppose that $(r, r+i) \in R \bowtie I$ is non-zero with $r \neq 0$. Thus $(r, r+i) = (r_1, r_1+i_1) + \cdots + (r_n, r_n+i_n)$, where $(r_1, r_1+i_1), \ldots, (r_n, r_n+i_n) \in \mathrm{Id}(R \bowtie I)$ are non-zero. But one can infer that

$$\mathrm{Id}(R \bowtie I) = (R \bowtie I) \cap (\mathrm{Id}(R) \times \mathrm{Id}(R)).$$

Hence $r_1, \ldots, r_n \in \mathrm{Id}(R)$ and $r = r_1 + \cdots + r_n$, as required. ∎

REMARK 2. As $(R \bowtie I)/((0) \times I) \cong R$, so Proposition 3.15 follows from Proposition 3.1 (2). Therefore, we have a different proof for Proposition 3.15. Further, if $I = R$, then $R \bowtie I \cong R \times R$. Thus, in this case, the converse of Proposition 3.15 is true.

In the next statement we also have a construction which shows that the converse of Proposition 3.15 could be true in some non-trivial situations.

Proposition 3.16. *Let R be a commutative ring and I an ideal of R. If there exists an ideal J of R such that $I \cap J = 0$ and $R = I + J$, then $R \bowtie I$ has the idempotent n-sum property if, and only if, R has the idempotent n-sum property.*

Proof. With the aid of the Chinese Remainder Theorem, we have that $R \cong (R/I) \times (R/J)$. As $R \bowtie I \cong (R/I) \times (R/J) \times (R/J) \cong R \times (R/J)$, the result follows from Proposition 3.1 (3). ∎

Let R be a ring and let ${}_RM_R$ be an R-R-bimodule. The trivial extension of R by M is the ring $T(R, M) = R \oplus M$ with the usual addition and the following multiplication

$$(r_1, m_1)(r_2, m_2) = (r_1 r_2, r_1 m_2 + m_1 r_2).$$

The trivial extension of R by a bimodule ${}_RM_R$ is isomorphic to the ring of all matrix $T = \begin{pmatrix} r & m \\ 0 & r \end{pmatrix}$, where $r \in R$ and $m \in M$ and the usual matrix operations are used. So, if the trivial extension of R has the idempotent n-sum property, then R also has the idempotent n-sum property.

Let R be a ring and let ${}_RV_R$ be an R-R-bimodule which is a general ring (possibly with no unity) in which $(vw)r = v(wr)$, $(vr)w = v(rw)$ and $(rv)w = r(vw)$ hold for all $v, w \in V$ and $r \in R$. Then the ideal-extension $I(R; V)$ of R by V is defined to be the additive abelian group $I(R; V) = R \oplus V$ with the following multiplication

$$(r, v)(s, w) = (rs, rw + vs + vw).$$

So, we are now ready to proceed by proving

Proposition 3.17. *Let R and V be as above. Then the following statements hold:*

(1) If the ideal-extension $I(R; V)$ has the idempotent n-sum property, then R has the idempotent n-sum property.
(2) The ideal-extension $I(R; V)$ has the idempotent n-sum property if R has the idempotent n-sum property and, for any $v \in V$, there exists $e_n \in \mathrm{Id}(R)$ such that $e_n v + v e_n + v^2 = v$.

Proof. Let $E = I(R; V)$. To show the validity of (1), let $x \in R$. Then $(x, 0) \in E$. Thus there exist $(e_1, e_1'), \ldots, (e_n, e_n') \in \mathrm{Id}(E)$ such that $(x, 0) = (e_1, e_1') + \cdots + (e_n, e_n')$. But $e_1, \ldots, e_n \in \mathrm{Id}(R)$, so $x = e_1 + \cdots + e_n$, as required.

To obtain (2), let $w = (a, v) \in E$. Then $a = e_1 + \cdots + e_n$, where $e_i \in \mathrm{Id}(R)$ for every i, $1 \leq i \leq n$. But by our hypothesis $w = (e_1, 0) + \cdots + (e_{n-1}, 0) + (e_n, v)$. But $(e_n, v)^2 = (e_n, v)(e_n, v) = (e_n^2, e_n v + v e_n + v^2) = (e_n, v)$, which shows that $I(R; V)$ has the idempotent n-sum property, too. ∎

4. Left-Open Problems

In closing, we shall state some unsettled questions that still elude us.

Recall that a ring R is called *n-clean* (see, e.g., [15]) if each its element can be expressed as the sum of n units and an idempotent. So, the following naturally occurs:

Problem 1. Characterize those rings (we call them *co-n-clean*) for which every element is the sum of n idempotents and a unit.

Evidently, 1-clean = co-1-clean = clean. However, for any $n \geq 2$, the situation seems to be more complicated.

Problem 2. If R is an n-thin ring such that $2 \in U(R)$, does it follow that it is n-good?

Notice just that we proved in Lemma 2.9 that R has to be $2n$-good.

Problem 3. Suppose p is a prime integer. Does it follow that R has the idempotent p-sum property if, and only if, each element of R satisfies the equality $x^{p+1} = x$?

Notice that when $p = 2$ this is precisely Theorem 1 from [12]; compare also with [6].

Acknowledgment. The authors are very indebted to the referee for the invaluable comments and the competent insightful thoughts on the paper.

References

[1] D. D. Anderson, V. P. Camillo, Commutative rings whose elements are a sum of a unit and an idempotent, *Commun. Algebra*, **30** (2002), 3327–3336.

[2] M. Atiyah, I. Macdonald, *Introduction to Commutative Algebra*, Addison-Wesley (1969).
[3] M. C. Cabezos, M. G. Lozano, M. S. Molina, Exchange Morita rings, *Commun. Algebra*, **29** (2001), 907–925.
[4] H. Chen, Morita contexts with many units, *Commun. Algebra*, **30** (2002), 1499–1512.
[5] R. Coleman, Some properties of finite rings, arXiv: 1302.3192v1 [math. RA], 13 Feb. 2013.
[6] P. Danchev, A new characterization of $J(n)$-rings, preprint (2016).
[7] P. Danchev, B. Goldsmith, On projectively fully transitive Abelian p-groups, *Results Math.*, **63** (2013), 1109–1130.
[8] P. V. Danchev and T. Y. Lam, Rings with unipotent units, *Publ. Math. Debrecen*, **88** (2016), 449–466.
[9] M. D'Anna and M. Fontana, An amalgamated duplication of a ring along an ideal: the basic properties, *J. Algebra Appl.*, **6** (2007), 443–459.
[10] B. Goldsmith, S. Pabst and A. Scott, Unit sum numbers of rings and modules, *Quart. J. Math. Oxford*, **49** (1998), 331–344.
[11] A. Haghany, Hopficity and co-Hopficity for Morita contexts, *Commun. Algebra*, **27** (1999), 477–492.
[12] Y. Hirano and H. Tominaga, Rings in which every element is a sum of two idempotents, *Bull. Austral. Math. Soc.*, **37** (1988), 161–164.
[13] H. Matsumura, *Commutative Algebra*, Benjamin-Cummings Publ. and Co. (1980).
[14] P. Vámos, 2-good rings, *Quart. J. Math. Oxford*, **56** (2005), 417–430.
[15] G. Xiao and W. Tong, n-clean rings, *Algebra Colloq.*, **13** (2006), 599–606.
[16] O. Zariski and P. Samuel, *Commutative Algebra I*, Van Nostrand (1965).

Peter Danchev
University of Plovdiv
4000 Plovdiv, Bulgaria
pvdanchev@yahoo.com

Ebrahim Nasibi
University of Shahreza
P.O. Box: 86149-56841
Shahreza, Iran
ebrahimnasibi@yahoo.com

ANNALES UNIV. SCI. BUDAPEST., **59** (2016), 95–100

UNIFORM CONVERGENCE FOR CONVEXIFICATION OF DOMINATED POINTWISE CONVERGENT CONTINUOUS FUNCTIONS

By
ZOLTÁN KÁNNAI
(Received October 19, 2016)

ABSTRACT. In this paper we give a more general form and a compendious proof for Arzelá's dominated convergence theorem. It also implies a uniformly convergent convexification of pointwise convergent dominated continuous functions, by which the harder representations can be eliminated from the proof of Krein's weak compactness theorem.

1. Introduction

The Lebesgue dominated convergence theorem of the measure theory implies that the Riemann integral of a bounded sequence of continuous functions over the interval [0, 1] pointwise converging to zero, also converges to zero. The validity of this result is independent of measure theory, on the other hand, this result together with only elementary functional analysis, can generate measure theory itself [2]. The result was also known before the appearance of measure theory [1][5], but the original proof was very complicated. For this reason the involved result, when presented in teaching, is generally obtained from measure theory. Later, Eberlein [3] gave an elementary, but still relatively complicated proof, and there were other simpler proofs but burdened with complicated concepts, like measure theory (e.g. [4]). In this paper we give a short and elementary proof for the involved theorem. Furthermore we strenghen it in the following way: a bounded sequence of continuous functions defined on a compact topological space K, pointwise converging to zero, has a suitable convexification converging also uniformly to zero on K, thus, e.g., the original sequence converges weakly to zero in $C(K)$. On the other hand, our proof for the theorem on keeping the dominated pointwise convergence, is also a new station among the aforementioned attempts for simpler proofs. However, the convexificated uniform convergence is even more suitable for wider applications. For example, the usual proof of the Krein-Smulian theorem [6] beyond the simple tools of the functional analysis, uses heavy embedding theorems and the Riesz' representation theorem with the whole apparatus of measure theory. Our main result, however, reduces the cited proof to a form in which we need abstract tools only, namely the Hahn-Banach separation theorem and Alaoglu's theorem, without a direct application of Jordan decomposition for linear functionals, or Riesz' representation, or any statement of measure theory.

2. Positive functionals keep the dominated pointwise convergence

Let K be a compact topological space, denote by $C(K)$ the Banach space of real valued continuous functions over K. The operator "upper envelope" (resp. "lower envelope") is denoted by the symbol $\vee$ (resp. $\wedge$). Fix a positive linear functional $\Phi\colon C(K) \to \mathbb{R}$. For every function $f\colon K \to \mathbb{R}_+$, define

$$\overline{\Phi}(f) := \sup_{\substack{g \in C(K) \\ 0 \le g \le f}} \Phi g.$$

$\overline{\Phi}$ is obviously monotone, i.e. $\overline{\Phi}(f_1) \ge \overline{\Phi}(f_2)$ if $f_1 \ge f_2$, moreover $\overline{\Phi}$ is an extension of Φ.

We note that $\overline{\Phi}$ is not subadditive. This fact gives importance to the next lemma.

LEMMA 1. *Let* $f_1, f_2\colon K \to \mathbb{R}_+$, $f_1 \ge f_2$ *and* $g, h \in C(K)$, $0 \le h \le f_1$, $0 \le g \le f_2$. *Then*

$$\overline{\Phi}(f_2) - \Phi(g \wedge h) \le \left(\overline{\Phi}(f_1) - \Phi h\right) + \left(\overline{\Phi}(f_2) - \Phi g\right).$$

PROOF. In view of the identity $g + h = (g \vee h) + (g \wedge h)$ we obtain

$$\Phi g + \Phi h = \Phi(g \vee h) + \Phi(g \wedge h),$$

hence $g \vee h \le f_1$ implies

$$\Phi g + \Phi h \le \overline{\Phi}(f_1) + \Phi(g \wedge h),$$

whence obviously

$$\overline{\Phi}(f_2) - \Phi(g \wedge h) \le \left(\overline{\Phi}(f_1) - \Phi h\right) + \left(\overline{\Phi}(f_2) - \Phi g\right). \qquad \blacksquare$$

LEMMA 2. *Take a sequence of functions* $f_n\colon K \to \mathbb{R}_+$ *with* $f_1 \leq \alpha < +\infty$ *and* $f_n(x)$ *decreasingly tending to* 0 *for every* $x \in K$. *Then* $\overline{\Phi}(f_n) \to 0$.

PROOF. Fix a number $\varepsilon > 0$. For every integer n choose a function $g_n \in C(K)$ such that $0 \leq g_n \leq f_n$ and

$$\Phi g_n > \overline{\Phi}(f_n) - \frac{\varepsilon}{2^{n+1}}.$$

Put $h_n := \bigwedge_{k=1}^{n} g_k$. Prove by induction that

$$\overline{\Phi}(f_n) - \Phi h_n \leq \sum_{k=1}^{n} \frac{\varepsilon}{2^{k+1}}.$$

The case $n = 1$ is obvious since $h_1 = g_1$. Supposing the statement is true for n, from the previous lemma we get

$$\begin{aligned}\overline{\Phi}(f_{n+1}) - \Phi h_{n+1} &= \overline{\Phi}(f_{n+1}) - \Phi(g_{n+1} \wedge h_n) \leq \\ &\leq \left(\overline{\Phi}(f_n) - \Phi h_n\right) + \left(\overline{\Phi}(f_{n+1}) - \Phi g_{n+1}\right) \leq \\ &\leq \sum_{k=1}^{n} \frac{\varepsilon}{2^{k+1}} + \frac{\varepsilon}{2^{n+2}} = \sum_{k=1}^{n+1} \frac{\varepsilon}{2^{k+1}},\end{aligned}$$

which completes the induction. Now the sequence $(h_n) \subseteq C(K)$ tends pointwise decreasingly to 0, so by Dini's theorem we obtain that h_n uniformly tends to 0 on K, hence by continuity of Φ we get $\Phi h_n \to 0$. Thus, there is an index N such that $0 \leq \Phi h_n \leq \frac{\varepsilon}{2}$ for every $n \geq N$, consequently $0 \leq \overline{\Phi}(f_n) \leq \Phi h_n + \sum_{k=1}^{n} \frac{\varepsilon}{2^{k+1}} \leq \frac{\varepsilon}{2} + \sum_{k=1}^{n} \frac{\varepsilon}{2^{k+1}} < \varepsilon$. ∎

THEOREM 3. *Let* $(g_n) \subseteq C(K)$ *with* $0 \leq g_n \leq \alpha < +\infty$ *for every* $n \in \mathbb{N}$, *pointwise converging to* 0. *Then* $\Phi g_n \to 0$.

PROOF. The sequence of functions $f_n := \bigvee_{k=n}^{\infty} g_k$ fulfills the assumptions of the previous lemma. Thus,

$$0 \leq \Phi g_n \leq \overline{\Phi}(f_n) \to 0.$$ ∎

This theorem yields, for example, that a dominated sequence $(g_n) \subseteq \subseteq C[0,1]$ tending pointwise to 0 has also $\int_0^1 g_n \to 0$, without any application of measure theory, or other complicated ideas and methods.

3. A convexification admitting uniform convergence

Let K be a compact topological space.

Lemma 4. *Let $(f_n) \subseteq C(K)$ with $0 \le f_n \le \alpha < +\infty$ for every $n \in \mathbb{N}$, pointwise converging to 0. Then for arbitrary $\varepsilon > 0$, there is a convex combination g of the functions $f_1, f_2, \ldots$ such that $g \le \varepsilon$ on the whole K.*

Proof. Suppose, by contradiction, that there is an $\varepsilon > 0$ such that the convex subsets

$$M := \operatorname{co}(f_1, f_2, \ldots) \qquad \text{and} \qquad \varepsilon + C(K)_- := \{f \in C(K) \colon f \le \varepsilon\}$$

of $C(K)$ are disjoint. Since the interior of $\varepsilon + C(K)_-$ is nonempty, by the Hahn-Banach separation theorem we obtain a nonzero functional $\Phi \in (C(K))^*$ such that

$$\sup_{\varepsilon + C(K)_-} \Phi \le \inf_M \Phi.$$

This implies that the functional Φ is positive. Then by the latest theorem, $\Phi f_n \to 0$, on the other hand

$$\Phi f_n \ge \inf_M \Phi \ge \sup_{\varepsilon + C(K)_-} \Phi \ge \Phi(\varepsilon \cdot \mathbf{1}) > 0$$

for every n, since Φ is positive and nonvoid. This is a contradiction. ∎

Definition 5. Let (f_n) be a sequence from a set X. A sequence $(g_n) \subseteq X$ is called a *convexification* of the sequence (f_n) if

$$g_n \in \operatorname{co}(f_n, f_{n+1}, \ldots)$$

for every $n \in \mathbb{N}$.

Theorem 6. *Let $(f_n) \subseteq C(K)$ with $|f_n| \le \alpha < +\infty$ for every $n \in \mathbb{N}$, pointwise converging to 0. Then there is a convexification (g_n) of the sequence (f_n) converging uniformly to 0 on K.*

Proof. Apply the previous lemma for $\varepsilon = \frac{1}{n}$ and the sequence $|f_n|, |f_{n+1}|, \ldots$. ∎

Remark 7. From the previous theorem immediately follows that an arbitrary functional $\Phi \in (C(K))^*$ has the property of dominated 0-convergence on $C(K)$, without the Jordan-type decomposition.

4. An application: Krein's theorem with elementary tools

The analogue of the following idea is due to Whitley [6].

Let X be a real Banach space and $K \subseteq X$ be a separable weakly compact set. Denote $J\colon X \to X^{**}$ the usual canonical embedding.

Theorem 8. $\overline{\mathrm{co}}(K)$ *is weakly compact (the closure is taken in norm).*

Proof. By Alaoglu's theorem, the unit ball $\mathcal{B}_{(C(K,w^*))^*}$ of the Banach space $(C(K, w^*))^*$ is w^*-compact. The restriction operator $R\colon X^* \to C(K)$, $\varphi \mapsto \varphi_{|K}$ is continuous, so

$$M := \left\{\Phi R\colon \Phi \in \mathcal{B}_{(C(K))^*}\right\} \subseteq X^{**}$$

is also convex and w^*-compact, hence $\overline{\mathrm{co}}\,(J(K)) \subseteq M$. Now it is enough to prove that every $\Phi R \in M$ $\left(\Phi \in \mathcal{B}_{(C(K))^*}\right)$ belonging to the w^*-closure of $\overline{\mathrm{co}}\,(J(K))$, is an element of $J\,(X)$. Suppose it is not; then by Hahn-Banach theorem we obtain a functional $A \in B_{X^{***}}$ such that $AJ = 0$ and $\alpha := A\,(\Phi R) > 0$. Let (x_n) be a norm dense sequence in K. By Goldstine's theorem, for every integer n there is a functional $\varphi_n \in \mathcal{B}_{X^*}$ such that

$$|\Phi R\varphi_n - \alpha|, |\varphi_n x_1|, |\varphi_n x_2|, \ldots, |\varphi_n x_n| \le \frac{1}{n}.$$

Then the sequence (φ_n) tends to 0 at every x_k, thus, $(R\varphi_n)$ being bounded tends to 0 pointwise on the whole K. Hence by the Remark 7 after Theorem 6 we get that $\Phi R\varphi_n \to 0$. On the other hand, by the choice, $\Phi R\varphi_n$ obviously tends to α, which is a contradiction. ∎

References

[1] C. Arzelà, Sulla integrazione per serie, *Rendinconti Accad.Lincei Roma*, **1** (1885), 532–537, 566–569.
[2] P. J. Daniell, A general form of integral, *Annals of Mathematics*, **19** (1918), 279–294.
[3] W. F. Eberlein, Notes on integration I: the underlying convergence theorem, *Communications on Pure and Appl. Math.*, **10** (1957), 357–360.
[4] J. W. Lewin, A Truly Elementary Approach to the Bounded Convergence Theorem, *The American Mathematical Monthly*, **93** (1986), 395–397.
[5] W. F. Osgood, Non-uniform convergence and the integration of series term by term, *Amer. Journal of Math.*, **19** (1897), 155–190.
[6] R. Whitley, The Krein-Smulian Theorem, *Proceedings of the American Mathematical Society*, **97** (1986), 376–377.

Zoltán Kánnai
Department of Mathematics
Corvinus University Budapest

ANNALES UNIV. SCI. BUDAPEST., **59** (2016), 101–112

CURVES ON SOME CLASSES OF KENMOTSU MANIFOLDS

By
AVIJIT SARKAR AND AMIT SIL
(Received November 8, 2016)

Abstract. The object of the present paper is to study slant curves on β-Kenmotsu manifolds and biharmonic almost contact curves on Kenmotsu space forms of dimension three with respect to semisymmetric metric connection. Biharmonic curves on β-Kenmotsu generalized Sasakian space forms have also been considered. An example is given.

1. Introduction

The concept of curves of constant slopes is a beautiful notion of classical differential geometry. Such a curve of constant slope is also called cylindrical helix. This is a curve in the Euclidean space E^3 for which the tangent vector field has a constant angle with a fixed direction, called the axis. Analogous to the above concept, there are notions of slant curves and in particular Legendre curves in contact structure geometry. The study of Legendre curves was introduced by Baikoussis and Blair [3]. The almost contact version of Legendre curve was introduced by Inoguchi and Lee [12]. An almost contact curve on an almost contact metric manifold is a curve whose tangent vector field is orthogonal to the Reeb vector field. If the almost contact metric manifold is contact metric manifold then the curve is known as Legendre curve. If a curve makes a constant angle with the Reeb vector field, then the curve is called slant curve. A Legendre curve is a particular case of slant curve. Slant curves and Legendre curves have been studied by several authors [5], [10], [14], [16], [17], [18].

In 1972, K. Kenmotsu introduced a new class of almost contact manifolds which are known as Kenmotsu manifolds. Tanno [22] has classified the connected

---

2000 Mathematics Subject Classification 53C15, 53D25

almost contact Riemannian manifolds whose automorphism groups have the maximum dimensions into the following three classes:

1. homogeneous normal contact Riemannian manifolds with constant ϕ -holomorphic sectional curvature;

2. global Riemannian products of a line or circle and a Kählerian space form;

3. warped product spaces $L \times_f F$, where L is a line and F is a Kählerian manifold.

Kenmotsu manifolds belongs to the third type of the above classification.

On the other hand Kenmotsu manifolds can be viewed as a particular case of trans-Sasakian manifold of type (α, β). A trans-Sasakian manifolds is called β-Kenmotsu manifold if $\alpha = 0$ [7]. If $\alpha = 0, \quad \beta = 1$, the manifold is known as Kenmotsu manifold. A Kenmotsu manifold with constant ϕ-sectional curvature is known as Kenmotsu space form [2]. P. Alegre and A. Carriazo introduced the notion of trans-Sasakian generalized Sasakian Space forms [1]. U. C .De and A. Sarkar studied quasi-Sasakian generalized Sasakian space forms [6]. Motivated by these works, in this paper we are interested to study Kenmotsu generalized Sasakian space forms. Actually we study biharmonic almost contact curve on such manifolds. In 1924, Friedmann and Schouten [9], introduced the notion of semisymmetric linear connection in a differentiable manifold. Hayden [11] introduced a metric connection with a non-zero torsion on a Riemannian manifold in 1932. In 1970, K. Yano [23] studied semisymmetric metric connection systematically on a Riemannian manifold. The study of semisymmetric metric connection in the area of almost contact manifolds was introduced by A. Sharfuddin [21]. Semisymmetric metric connection on almost contact manifolds has been studied by several authors [8], [15], [19]. The geometric interpretation of semisymmetric metric connection can be found in the book [20], page 143. The present paper is organized as follows:

After the introduction we give required preliminaries in Section 2. In Section 3 we study slant curves in three-dimensional β-Kenmotsu manifolds with semi symmetric metric connection. In Section 4 we consider biharmonic almost contact curves on Kenmotsu space forms. Section 5 is devoted to investigate biharmonic almost contact curves on three-dimensional β-Kenmotsu generalized Sasakian space forms. The last Section contains example.

2. Preliminaries

Let M be an almost contact metric manifold with an almost contact metric structure (ϕ, ξ, η, g) i.e., ϕ is a 1-1 tensor field, ξ is a unit vector field, η is a 1-form and g is a compatible Riemannian metric such that [4]

$$\phi^2 X = -X + \eta(X)\xi, \qquad \eta(\xi) = 1, \qquad \phi\xi = 0, \qquad \eta(\phi) = 0 \tag{2.1}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y) \tag{2.2}$$

$$g(X, \phi Y) = -g(\phi X, Y), \qquad g(X, \xi) = \eta(X) \tag{2.3}$$

for all $X, Y \in \chi(M)$. The fundamentai 2-form Φ of the manifold is defined by

$$\Phi(X, Y) = g(X, \phi Y) \tag{2.4}$$

for $X, Y \in \chi(M)$

An almost contact metric manifold is normal if

$$[\phi, \phi](X, Y) + 2d\eta(X, Y)\xi = 0$$

For a β-Kenmotsu manifold

$$(\nabla_X \phi)Y = \beta(g(\phi X, Y)\xi - \eta(Y)\phi X) \tag{2.5}$$

From (2.5),

$$\nabla_X \xi = \beta(X - \eta(X)\xi) \tag{2.6}$$

Putting $\beta = 1$, we get for Kenmotsu manifolds

$$\nabla_X \xi = X - \eta(X)\xi \tag{2.7}$$

The semi symmetric metric connection $\tilde{\nabla}$ and the Levi-Civita connection ∇ on an almost contact metric manifold are related by

$$\tilde{\nabla}_X Y = \nabla_X Y + \eta(Y)X - g(X, Y)\xi \tag{2.8}$$

for all vector fields X, Y on M.

The torsion tensor of a semi symmetric metric connection on an almost contact metric manifold is given by

$$\tilde{T}(X, Y) = \eta(Y)X - \eta(X)Y \tag{2.9}$$

A curve γ on M is called Frenet curve with respect to semi-symmetric metric connections if it satisfies

$$\tilde{\nabla}_T T = \tilde{k} N \tag{2.10}$$

$$\tilde{\nabla}_T N = -\tilde{k} T + \tilde{\tau} B \tag{2.11}$$

$$\tilde{\nabla}_T B = -\tilde{\tau} N \tag{2.12}$$

where $\tilde{k}, \tilde{\tau}$ are the curvature and torsion of the curve with respect to semi symmetric metric connection, $\{T, N, B\}$ is an orthonormal frame with $\dot{\gamma} = T$.

Actually the above formulas are generalization of Serret-Frenet formulas for curves with respect to Levi-Civita connection. We shall denote the curvature and torsion of the curve with respect to Levi-Civita connection ∇ by k and τ.

A unit speed curve γ on an almost contact metric manifold is called an almost contact curve if $\eta(\dot{\gamma}) = 0$.

3. Slant curves on β-Kenmotsu manifolds with respect to semisymmetric metric connection

Definition 3.1. A unit speed curve γ is called a slant curve if it satisfies $\eta(\dot{\gamma}) = \cos\theta$, where θ is a constant.

In this section, we study slant curves on β-Kenmotsu manifolds with respect to semisymmetric metric connection and prove the following:

Theorem 3.1. *If a three-dimensional β-Kenmotsu manifold admits a proper slant curve whose curvature is zero with respect to semisymmetric metric connection, then the manifold is cosymplectic.*

Proof. Let us consider a slant curve γ on a β-Kenmotsu manifold with semisymmetric connection.

Here $\dot{\gamma}(s) = T(s)$ is given by

$$\cos\theta(s) = g(T(s), \xi) \tag{3.1}$$

Where θ is the constant slant angle. By covariant differentiation with respect to $\tilde{\nabla}$ we get from (3.1)

$$-\sin\theta.\theta' = -g(\tilde{\nabla}_T T, \xi) - g(T, \tilde{\nabla}_T \xi) \tag{3.2}$$

Now using (2.6) and (2.10) in (3.2) we get,

$$\begin{aligned} -\sin\theta.\theta' &= -g(\tilde{k}N,\xi) - g(T,\beta(T-\eta(T)\xi)) = \\ &= -\tilde{k}\eta(N) - \beta g(T,T) + \eta(T)\eta(T)\beta = \\ &= -\tilde{k}\eta(N) - \beta + \cos^2\theta.\beta \end{aligned} \tag{3.3}$$

If θ = constant, then from (3.3) we get, $\tilde{k}\eta(N) = -\beta\sin^2\theta$.

Consider a proper slant curve with zero curvature with respect to semisymmetric metric connection. Then from above we get $\beta = 0$.

This completes the proof. ∎

4. Biharmonic almost contact curves on three-dimensional Kenmotsu space forms with respect to semisymmetric metric connection

Definition 4.1. An almost contact curve γ on a three-dimensional β-Kenmotsu manifold is called biharmonic with respect to semi-symmetric metric connection $\tilde{\nabla}$ if it satisfies [5] [12]

$$\tilde{\nabla}^3_T T + \tilde{\nabla}_T \tilde{T}(\tilde{\nabla}_T T, T)T + \tilde{R}(\tilde{\nabla}_T T, T)T = 0 \tag{4.1}$$

where $\dot{\gamma} = T$, $\tilde{T}$ is the torsion of semi symmetric connection and $\tilde{R}$ is the curvature of the semisymmetric metric connection.

Definition 4.2. If the curvature tensor of a differentiable manifold satisfies [2]

$$\begin{aligned} R(X,Y)Z = \frac{c-3}{4}&[g(Y,Z)X - g(X,Z)Y]+ \\ &+ \frac{c+1}{4}[(\eta(X)Y - \eta(Y)X)\eta(Z)+ \\ &+ (g(X,Z)\eta(Y) - g(Y,Z)\eta(X))\xi + g(\phi Y,Z)\phi X - \\ &- g(\phi X,Z)\phi Y - 2g(\phi X,Y)\phi Z] \end{aligned} \tag{4.2}$$

Where c is the sectional curvature on the manifold and $\nabla_X\xi = X - \eta(X)\xi$, then the manifold will be called Kenmotsu space forms.

Theorem 4.1. *The curvature of a non-geodesic almost contact curve with respect to semisymmetric metric connection on a Kenmotsu space form is a constant and the torsion of the curve is given by* $\tilde{\tau} = 2\int kds$, *where k is the curvature of that curve with respect to Levi-Civita connection and s is the arc length parameter.*

Proof. Let $\tilde{R}$ and R be the curvature tensor of a three-dimensional quasi-Sasakian manifold with respect to semi-symmetric metric connection and Levi-Civita connection respectively. Then the relation between $\tilde{R}$ and R is given by [23]

$$\begin{aligned}\tilde{R}(X,Y)Z &= R(X,Y)Z - L(Y,Z)X + L(X,Z)Y+ \\ &+ 2g(\nabla_Y X, Z)\xi - 2g(\nabla_X Y, Z)\xi + \eta(Z)([X,Y])+ \\ &+ \eta(X)g(Y,Z)\xi + \eta(Y)g(X,Z)\xi\end{aligned} \tag{4.3}$$

where

$$L(Y,Z) = (\nabla_Y \eta)Z - \eta(Y)\eta(Z) + g(Y,Z) \tag{4.4}$$

Now using $(\nabla_X \eta)Y = g(\phi X, \phi Y)$ in (4.4) we get,

$$L(Y,Z) = g(\phi Y, \phi Z) - \eta(Y)\eta(Z) + g(Y,Z) \tag{4.5}$$

Using (4.5) in (4.3) we get,

$$\begin{aligned}\tilde{R}(X,Y)Z &= R(X,Y)Z - g(\phi Y, \phi Z)X + g(\phi X, \phi Z)Y+ \\ &+ \eta(Z)(\eta(Y)X - \eta(X)Y) - (g(Y,Z)X - g(X,Z)Y)+ \\ &+ 2(g(\nabla_Y X, Z) - g(\nabla_X Y, Z))\xi + (\eta(X)g(Y,Z)\xi + \eta(Y)g(X,Z)\xi)+ \\ &+ \eta(Z)([X,Y])\end{aligned} \tag{4.6}$$

For semisymmetric metric connection, we have $\tilde{\nabla}_T \tilde{T}(\tilde{\nabla}_T T, T)T = 0$.

Using this fact and the Serret-Frenet formula, from (4.1) we get

$$\tilde{\nabla}^3_T T + \tilde{k}\tilde{R}(N,T)T = 0 \tag{4.7}$$

Since we have considered Frenet frame as T, ϕT, ξ, where $\phi T = -N$, so for an almost contact curve we get $\eta(T) = 0$, $\eta(N) = 0$. Using this fact and putting $X = N$, $Y = T$, $Z = T$ in (4.6) we get,

$$\tilde{R}(N,T)T = R(N,T)T - 2N + 2[g(\nabla_T N, T) - g(\nabla_N T, T)]\xi \tag{4.8}$$

Now putting $X = N$, $Y = T$, $Z = T$ in (4.2) we get,

$$R(N,T)T = \frac{c-3}{4}N - 3\frac{c+1}{4}N \tag{4.9}$$

From (4.8) and (4.9) after some simplification and setting $\xi = B$ we get,

$$\tilde{R}(N,T)T = \frac{N}{4}(-2c-6) - 2N - 2kB \tag{4.10}$$

Again by Serret-Frenet formula we get,

$$\tilde{\nabla}^3_T T = -3\tilde{k}\tilde{k}'T + (\tilde{k''} - \tilde{k^3} - \tilde{k}\tilde{\tau^2})N + (2\tilde{\tau}\tilde{k}' + \tilde{k}\tilde{\tau}')B \tag{4.11}$$

From (4.7), (4.10) and (4.11) we get,

$$(-3\tilde{k}\tilde{k'})T + (\tilde{k''} - \tilde{k^3} - \tilde{k}\tilde{\tau^2} - 2\tilde{k}c - 6\tilde{k} - 2\tilde{k})N + (2\tilde{\tau}\tilde{k'} + \tilde{k}\tilde{\tau'} - 2k\tilde{k})B = 0$$

So we have,

$$-3\tilde{k}\tilde{k'} = 0 \tag{4.12}$$

$$\tilde{k''} - \tilde{k^3} - \tilde{k}\tilde{\tau^2} - 2\tilde{k}c - 6\tilde{k} - 2\tilde{k} = 0 \tag{4.13}$$

$$2\tilde{\tau}\tilde{k'} + \tilde{k}\tilde{\tau'} - 2k\tilde{k} = 0 \tag{4.14}$$

If $\tilde{k} \neq 0$, then $\tilde{k}$ = constant.

Now from (4.14) we get

$$\tilde{k}\tilde{\tau}' - 2k\tilde{k} = 0,$$

i.e., $\tilde{\tau} = 2\int kds$.

This completes the proof. ∎

5. Biharmonic almost contact curves on three-dimensional β-Kenmotsu generalized Sasakian space forms with respect to semisymmetric metric connection

Definition 5.1. If the curvature tensor of a differentiable manifold satisfies

$$\begin{aligned} R(X,Y)Z = {} & f_1[g(Y,Z)X - g(X,Z)Y]+ \\ & + f_2[g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X + 2g(X,\phi Y)\phi Z]+ \\ & + f_3[(\eta(X)Y - \eta(Y)X)\eta(Z)+ \\ & + (g(X,Z)\eta(Y) - g(Y,Z)\eta(X))\xi] \end{aligned} \tag{5.1}$$

where f_1, f_2 and f_3 are three functions on the manifold and $\nabla_X \xi =$ $= \beta(X - \eta(X)\xi)$, then the manifold will be called β-Kenmotsu generalized Sasakian space forms.

To give the definition, we have followed the papers [1] and [6].

Definition 5.2. An almost contact curve γ on a three-dimensional β-Kenmotsu manifold is called biharmonic with respect to semi-symmetric metric connection $\tilde{\nabla}$ if it satisfies [5] [12]

$$\tilde{\nabla}_T^3 T + \tilde{\nabla}_T \tilde{T}(\tilde{\nabla}_T T, T)T + \tilde{R}(\tilde{\nabla}_T T, T)T = 0 \tag{5.2}$$

where $\tilde{\gamma} = T$, $\tilde{T}$ is the torsion of semi symmetric connection and $\tilde{R}$ is the curvature of the semisymmetric metric connection.

THEOREM 5.1. *The curvature of a biharmonic almost contact curve on a three dimensional β-Kenmotsu generalized Sasakian space form is a constant and the torsion will be constant if β and ϕ -sectional curvature of the manifold are constants.*

PROOF. Let $\tilde{R}$ and R be the curvature tensor of a three-dimensional quasi-Sasakian manifold with respect to semi-symmetric metric connection and Levi-Civita connection respectively. Then the relation between $\tilde{R}$ and R is given by [23]

$$\begin{aligned}\tilde{R}(X,Y)Z &= R(X,Y)Z - L(Y,Z)X + L(X,Z)Y+ \\ &+ 2g(\nabla_Y X, Z)\xi - 2g(\nabla_X Y, Z)\xi + \eta(Z)([X,Y])+ \\ &+ \eta(X)g(Y,Z)\xi + \eta(Y)g(X,Z)\xi\end{aligned} \tag{5.3}$$

where

$$L(Y,Z) = (\nabla_Y \eta)Z - \eta(Y)\eta(Z) + g(Y,Z). \tag{5.4}$$

Now using $(\nabla_X \eta)Y = \beta g(\phi X, \phi Y)$ in (4.4) we get,

$$L(Y,Z) = \beta g(\phi Y, \phi Z) - \eta(Y)\eta(Z) + g(Y,Z) \tag{5.5}$$

Using (5.5) in (5.3) we get,

$$\begin{aligned}\tilde{R}(X,Y)Z &= R(X,Y)Z - \beta(g(\phi Y, \phi Z)X - g(\phi X, \phi Z)Y)+ \\ &+ \eta(Z)(\eta(Y)X - \eta(X)Y) - (g(Y,Z)X - g(X,Z)Y)+ \\ &+ 2(g(\nabla_Y X, Z) - g(\nabla_X Y, Z))\xi + (\eta(X)g(Y,Z)\xi + \eta(Y)g(X,Z)\xi)+ \\ &+ \eta(Z)([X,Y])\end{aligned} \tag{5.6}$$

Let us consider a biharmonic almost contact curve γ on a three-dimensional β Kenmotsu generalized Sasakian space form.

For semisymmetric metric connection, we have

$$\tilde{\nabla}_T \tilde{T}(\tilde{\nabla}_T T, T)T = 0$$

Using this fact and Serret-Frenet formula from (5.2) we get

$$\tilde{\nabla}_T^3 T + \tilde{k}\tilde{R}(N,T)T = 0 \tag{5.7}$$

Since we have considered Frenet Frame as T, ϕT, ξ, where $\phi T = -N$, so for an almost contact curve we get $\eta(T) = 0$, $\eta(N) = 0$. Using this fact and putting $X = N$, $Y = T$, $Z = T$ in (5.6) we get,

$$\tilde{R}(N,T)T = R(N,T)T - \beta N - N + 2[g(\nabla_T N, T) - g(\nabla_N T, T)]\xi \tag{5.8}$$

Now putting $X = N, Y = T, Z = T$ in (5.1) we get,

$$R(N, T)T = N(f_1 + 3f_2) \tag{5.9}$$

From (5.8) and (5.9) after some simplification and setting $\xi = B$ we get,

$$\tilde{R}(N, T)T = N(f_1 + 3f_2) - \beta N - N - 2kB \tag{5.10}$$

Again by Serret-Frenet formula we get,

$$\tilde{\nabla}^3_T T = -3\tilde{k}\tilde{k}'T + (\tilde{k}'' - \tilde{k}^3 - \tilde{k}\tilde{\tau}^2)N + (2\tilde{\tau}\tilde{k}' + \tilde{k}\tilde{\tau}')B \tag{5.11}$$

From (5.7), (5.10) and (5.11) we get,

$$(-3\tilde{k}\tilde{k}')T + (\tilde{k}'' - \tilde{k}^3 - \tilde{k}\tilde{\tau}^2 + f_1\tilde{k} + 3f_2\tilde{k} - \tilde{k}\beta - \tilde{k})N + (2\tilde{\tau}\tilde{k}' + \tilde{k}\tilde{\tau}' - 2k\tilde{k})B = 0$$

So we have,

$$-3\tilde{k}\tilde{k}' = 0 \tag{5.12}$$

$$\tilde{k}'' - \tilde{k}^3 - \tilde{k}\tilde{\tau}^2 + f_1\tilde{k} + 3f_2\tilde{k} - \tilde{k}\beta - \tilde{k} = 0 \tag{5.13}$$

$$2\tilde{\tau}\tilde{k}' + \tilde{k}\tilde{\tau}' - 2k\tilde{k} = 0 \tag{5.14}$$

If $\tilde{k} \neq 0$, then $\tilde{k}$ = constant.

Now from (5.13) we get

$$\tilde{k}(3f_2 + f_1 - \tilde{\tau}^2 - \tilde{k}^2 - \beta - 1) = 0.$$

Consider the ϕ-sectional curvature $f_1 + 3f_2$ of the manifold is constant and the function β is constant. Then we get $\tilde{\tau}$=constant.

This completes the proof. ∎

6. Example

In this section we shall give an example of a β-Kenmotsu manifold and give an example of an almost contact curve. To give the example we have followed the paper [19].

We consider the well-known three-dimensional manifold

$$M = ((x, y, z) \in \mathbb{R}^3, z \neq 0),$$

where (x, y, z) are the standard co-ordinates in $\mathbb{R}^3$. The vector fields

$$e_1 = z\frac{\partial}{\partial x}, \quad e_2 = z\frac{\partial}{\partial y}, \quad , e_3 = \frac{\partial}{\partial z}$$

are linearly independent at each point of M. Let g be the Riemannian metric defined by

$$g(e_1, e_3) = g(e_2, e_3)g(e_1, e_2) = 0,\ g(e_1, e_1) = g(e_2, e_2) = g(e_3, e_3) = 1.$$

Let η be the 1-form defined by $\eta(Z) = g(Z, e_3)$ for any $Z \in \chi(M)$. Let ϕ be the (1,1) tensor field defined by $\phi(e_1) = -e_2$, $\phi(e_2) = e_1$, $\phi(e_3) = 0$. Then, using the linearity of ϕ and g we have

$$\eta(e_3) = 1, \quad \phi^2 Z = -Z + \eta(Z)e_3, \quad g(\phi Z, \phi W) = g(Z, W) - \eta(Z)\eta(W),$$

for any $Z, W \in \chi(M)$. Thus for $e_3 = \xi$, (ϕ, ξ, η, g) defines an almost contact metric structure on M. Now by direct computations we obtain

$$[e_1, e_2] = 0, \qquad [e_2, e_3] = -e_2, \qquad [e_1, e_3] = -e_1.$$

By Koszul formula

$$\begin{array}{lll} \nabla_{e_1} e_3 = -e_2, & \nabla_{e_1} e_2 = 0, & \nabla_{e_1} e_1 = e_3 \\ \nabla_{e_2} e_3 = -e_2, & \nabla_{e_2} e_2 = e_3, & \nabla_{e_2} e_1 = 0 \\ \nabla_{e_3} e_3 = 0, & \nabla_{e_3} e_2 = 0, & \nabla_{e_3} e_1 = 0 \end{array}$$

From above we see that the manifold satisfies $\nabla_X \xi = \beta(X - \eta(X)\xi)$, with $\beta = -1$ and $e_3 = \xi$. Hence the manifold is a β-Kenmotsu manifold with $\beta = -1$.

It can be easily calculated that

$$\begin{array}{ll} \tilde{\nabla}_{e_1} e_1 = 0, & \tilde{\nabla}_{e_2} e_1 = 0, \\ \tilde{\nabla}_{e_1} e_2 = 0, & \tilde{\nabla}_{e_2} e_2 = 0. \end{array}$$

Using these results we construct an almost contact curve on this β-Kenmotsu manifold.

Consider a curve $\gamma\colon I \longrightarrow M$ defined by $\gamma(s) = \left(\sqrt{\frac{2}{3}}s, \sqrt{\frac{1}{3}}s, 1\right)$. Hence

$$\dot{\gamma}_1 = \sqrt{\frac{2}{3}}, \quad \dot{\gamma}_2 = \sqrt{\frac{1}{3}} \quad \text{and} \quad \dot{\gamma}_3 = 0, \quad \gamma(s) = (\gamma_1(s), \gamma_2(s), \gamma_3(s)).$$

Now we have $\eta(\dot{\gamma}) = g(\dot{\gamma}, e_3) = g(\dot{\gamma}_1 e_1 + \dot{\gamma}_2 e_2 + \dot{\gamma}_3 e_3, e_3) = 0$

$$\begin{aligned} g(\dot{\gamma}, \dot{\gamma}) &= g(\dot{\gamma}_1 e_1 + \dot{\gamma}_2 e_2 + \dot{\gamma}_3 e_3, \dot{\gamma}_1 e_1 + \dot{\gamma}_2 e_2 + \dot{\gamma}_3 e_3) = \\ &= \dot{\gamma_1}^2 + \dot{\gamma_2}^2 + \dot{\gamma_3}^2 = \\ &= \dot{\gamma_1}^2 + \dot{\gamma_2}^2 = \\ &= \frac{2}{3} + \frac{1}{3} \\ &= 1 \end{aligned}$$

Hence the curve is a Legendre curve. For this curve $\tilde{\nabla}_{\dot{\gamma}}\dot{\gamma} = 0$. So the curve is a geodesic with respect to semisymmetric metric connection. The curvature and torsion with respect to semisymmetric metric of this curve are zero. The curve is trivially biharmonic.

References

[1] P. Alegre, and A. Carriazo, Structures on generalized Sasakian space forms, *Differential Geom. Appl.*, **26** (2008), 656–666.

[2] K. Arslan, et. al., Ricci curvature of submanifolds in Kenmotsu space forms, *IJMMS* **29** (2002), 719–726.

[3] C. Baikoussis, and D. E. Blair, On Legendre curves in contact 3-manifolds, *Geom. Dedicata*, **49** (1994), 135–142.

[4] D. E. Blair, Riemannian geometry of contact and symplectic manifolds, *Progress in math*, vol 203, Birkhäuser, Boston 2002.

[5] J. T. Cho, and J. E. Lee, Slant curves in contact pseudo-Hermitian manifolds, *Bull, Austral. Math. Soc.*, **78** (2008), 383–396.

[6] U. D. De, and A. Sarkar, Some results in generalized Sasakian space forms, *Thai J. of Math.*, **8** (2010), 1–10.

[7] U. C. De, and A. Sarkar, On Three-dimensional trans-Sasakian manifolds, *Extracta Mathematicae*, **23** (2008), 265–277.

[8] U. C. De, and J. Sengupta, On a type of semisymmetric metric connection on an almost contact metric manifold, *Facta Universitatis* (Nis), **16** (2001), 87–96.

[9] A. Friedmann, and J. A. Schouten, Über die Geometrie der halbsymmetrischen übertragung, *Math. Z.*, **21** (1924), 211–223.

[10] S. Güvenc, and C. Özgur, On slant curves in trans-Sasakian manifolds, *Revista De La Union Mat, Argentina*, **55** (2014), 81–100.

[11] H. A. Hayden, Subspaces of spaces of space with torsion, *Proc. London Math. Soc.*, **34** (1932), 27–50.

[12] J. Inoguchi, and J. E. Lee, Affine biharmonic curves in 3-dimensional homogeneous geometries, *Mediterr, J. Math.*, **10** (2013), 571–592.

[13] K. Kenmotsu, A class of almost contact Riemannian manifolds, *Töhoku Math. J.*, **24** (1972), 93–103.

[14] C. Özgur, and S. Güvenc, On some classes of biharmonic Legendre curves in generalized Sasakian space forms, *Collect . Math.*, **65** (2014), 203–218.

[15] D. G. Prakasha, et. al., Some classes of Kenmotsu manifolds with respect to semisymmetric metric connection, *Acta Mathematica Sinica, Eng. Series*, **21** (2013), 1–12.

[16] A. Sarkar, and D. Biswas, Legendre curves in three-dimensional Heisenberg group, *Facta Univ.*, **28** (2013), 241–248.

[17] A. Sarkar, S. K. Hui, and M. Sen, A study on Legendre curves in 3-dimensional trans-Sasakian manifolds, Lobachevskii, *J. Math.*, **35** (2014), 11–18.
[18] A. Sarkar, and A. Mondal, Certain curves on some classes of three-dimensional almost contact metric manifolds, *Revista De La Union Mat. Argentina* (to appear).
[19] A. Sarkar, A. Mondal, and D. Biswas, Some curves on three-dimensional trans-Sasakian manifolds with semisymmetric metric connection, *Palest. J. Math.*, **5** (2016), 195–203.
[20] J. A. Schouten, *Ricci calculus, An introduction to tensor analysis and its geometric applications*, Springer, 1954.
[21] A. Sharfuddin, and S. E. Hussain, Semisymmetric metric connection in almost contact manifolds, *Tensor* (N.S), **30** (1976), 133–139.
[22] S. Tanno, The automorphism groups of almost contact Riemannian manifolds, *Töhoku Math. J.*, **21** (1969), 21–38.
[23] K. Yano, On semisymmetric metric connection, *Rev. Roumaine Math., Pures Appl.*, **15** (1970),1579–1586.

Avijit Sarkar
Department of Mathematics
University of Kalyani
Kalyani, Nadia, WB-741235
avjaj@yahoo.co.in

Amit Sil
Department of Mathematics
University of Kalyani
Kalyani, Nadia, WB-741235
amitsil666@gmail.com

ANNALES UNIV. SCI. BUDAPEST., **59** (2016), 113–122

MORE ON THE h-CRITICAL NUMBERS OF FINITE ABELIAN GROUPS

By
BÉLA BAJNOK
(Received November 21, 2016)

ABSTRACT. For a finite abelian group G, a nonempty subset A of G, and a positive integer h, we let hA denote the h-fold sumset of A; that is, hA is the collection of sums of h not-necessarily-distinct elements of A. Furthermore, for a positive integer s, we set $[0, s]A = \cup_{h=0}^{s} hA$. We say that A is a generating set of G if there is a positive integer s for which $[0, s]A = G$.

The h-critical number $\chi(G, h)$ of G is defined as the smallest positive integer m for which $hA = G$ holds for every m-subset A of G; similarly, $\chi(G, [0, s])$ is the smallest positive integer m for which $[0, s]A = G$ holds for every m-subset A of G. We define $\widehat{\chi}(G, h)$ as the smallest positive integer m for which $hA = G$ holds for every generating m-subset A of G; $\widehat{\chi}(G, [0, s])$ is defined similarly.

The value of $\chi(G, h)$ has been determined by this author for all G and h, and $\widehat{\chi}(G, [0, s])$ was introduced and resolved for some special cases by Klopsch and Lev. Here we determine the remaining two quantities in all cases.

1. Introduction

Let G be a finite abelian group of order $n \geq 2$, written in additive notation. When G is cyclic, we will identify it with $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$. More generally, we recall that G has a unique *type* $(n_1, \ldots, n_r)$, where r and $n_1, \ldots, n_r$ are positive integers so that $n_1 \geq 2$, n_i is a divisor of n_{i+1} for $i = 1, \ldots, r-1$, and

$$G \cong \mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_r};$$

here r is the *rank* of G and n_r is the *exponent* of G.

2000 Mathematics Subject Classification 11B75; 05D99, 11B25, 11P70, 20K01

For a nonempty subset A of G, we let ΣA be the set of all subset sums of A; that is,

$$\Sigma A = \{\Sigma_{a \in B} a \mid B \subseteq A\}$$

(with the subset sum of the empty-set defined as 0). We then let $\mathrm{cr}(G)$ denote the smallest integer m for which $\Sigma A = G$ holds for all m-subsets A of G; the smallest integer m for which $\Sigma A = G$ holds for all m-subsets A of $G \setminus \{0\}$ is denoted by $\mathrm{cr}^*(G)$.

The study of *critical numbers* originated with the 1964 paper [7] of Erdős and Heilbronn, in which they asked for $\mathrm{cr}^*(\mathbb{Z}_p)$ for prime values of p. It took nearly half a century, but now, due to the combined results of Diderrich and Mann [6], Diderrich [5], Mann and Wou [14], Dias Da Silva and Hamidoune [4], Gao and Hamidoune [10], Griggs [11], and Freeze, Gao, and Geroldinger [8, 9], we have the critical number of every group:

Theorem 1 (The combined results of authors above). *Suppose that G is an abelian group of order $n \geq 10$, and let p be the smallest prime divisor of n. Then*

$$\mathrm{cr}^*(G) = \mathrm{cr}(G) - 1 = \begin{cases} \lfloor 2\sqrt{n-2} \rfloor & \text{if } G \text{ is cyclic of order } n = p \\ & \text{or } n = pq \text{ where } q \text{ is prime and} \\ & 3 \leq p \leq q \leq p + \lfloor 2\sqrt{p-2} \rfloor + 1,^{1} \\ n/p + p - 2 & \text{otherwise.} \end{cases}$$

We note that, while it is easy to see that $\mathrm{cr}(G)$ is at least one more than $\mathrm{cr}^*(G)$, there is no obvious reason known for the fact that they differ by exactly one.

In this paper we consider some variations of the critical numbers defined above.

We recall the following definitions. For a positive integer h and a nonempty subset A of G, we let hA denote the *h-fold sumset* of A; that is, hA is the collection of sums of h not-necessarily-distinct elements of A. Additionally, for a positive integer s, we set $[0, s]A = \cup_{h=0}^{s} hA$. Recall also that, for a subset A of G, $\langle A \rangle$ is the subgroup generated by A in G; that is, $\langle A \rangle$ is the intersection of all subgroups H of G for which $A \subseteq H$. When $\langle A \rangle = G$, we say that A is a *generating set* of G. Clearly, A is a generating set of G if, and only if, there is a positive integer s for which $[0, s]A = G$.

The subject of our paper is the study of the following four quantities:

$$\chi(G, h) = \min\{m \colon A \subseteq G, |A| \geq m \Rightarrow hA = G\},$$

[1] Observe that $\lfloor 2\sqrt{n-2} \rfloor = n/p + p - 1$ in this case.

$$\chi(G,[0,s]) = \min\{m\colon A \subseteq G, |A| \geq m \Rightarrow [0,s]A = G\},$$
$$\widehat{\chi}(G,h) = \min\{m\colon A \subseteq G, \langle A\rangle = G, |A| \geq m \Rightarrow hA = G\},$$
$$\widehat{\chi}(G,[0,s]) = \min\{m\colon A \subseteq G, \langle A\rangle = G, |A| \geq m \Rightarrow [0,s]A = G\}.$$

It is easy to see that for all G and h we have $hG = G$, so all four quantities are well defined.

The value of $\chi(G,h)$ is now known for every G and h. To state the result, we let $D(n)$ denote the set of positive divisors of the order n of G; then set

$$f_d(n,h) = \left(\left\lfloor \frac{d-2}{h} \right\rfloor + 1\right) \cdot \frac{n}{d}$$

for each $d \in D(n)$, and let

$$v(n,h) = \max\left\{f_d(n,h)\colon d \in D(n)\right\}.$$

We should note that the function $v(n,h)$ has appeared elsewhere in additive combinatorics already. For example, according to the classical result of Diananda and Yap (see [3]), the maximum size of a sum-free set (that is, a set A that is disjoint from $2A$) in the cyclic group $\mathbb{Z}_n$ is given by

$$v(n,3) = \begin{cases} \left(1+\frac{1}{p}\right)\frac{n}{3} & \text{if } n \text{ has prime divisors congruent to 2 mod 3,} \\ & \text{and } p \text{ is the smallest such divisor,} \\ \left\lfloor \frac{n}{3} \right\rfloor & \text{otherwise;} \end{cases}$$

see e.g. [1] for generalizations.

Our result for $\chi(G,h)$ is then the following:

THEOREM 2 (BAJNOK; CF. [2]). *For any abelian group G of order n and for all positive integers h we have*

$$\chi(G,h) = v(n,h) + 1.$$

By Theorem 2, the size of the largest *h-incomplete subset* of any group of order n—that is, a subset whose h-fold sumset is not the entire group—equals $v(n,h)$.

Given Theorem 2, the evaluation of $\chi(G,[0,s])$ is immediate. Clearly, $\chi(G,s)$ is an upper bound for $\chi(G,[0,s])$. Suppose then that A is an s-incomplete subset of G of size $\chi(G,s) - 1$. Choose an element $a_0 \in A$, and let

$$B = A - a_0 = \{a - a_0 \mid a \in A\}.$$

Since $|A| = |B|$ and $|sA| = |sB|$, B is also an s-incomplete subset of G of size $\chi(G,s) - 1$. But $[0,s]B = sB$, and thus $[0,s]B \neq G$, which implies that $\chi(G,[0,s])$ is an upper bound for $\chi(G,s)$. Therefore:

THEOREM 3. *For any abelian group G of order n and for all positive integers s we have*

$$\chi(G,[0,s]) = \chi(G,s) = v(n,s) + 1.$$

The quantity $\widehat{\chi}(G,[0,s])$ was introduced and investigated by Klopsch and Lev in [12] (though earlier works had treated the case of elementary abelian 2-groups). The value of $\widehat{\chi}(G,[0,s])$ is not known in general. In Section 2 we provide a general lower bound for $\widehat{\chi}(G,[0,s])$, and summarize the main results in a way that allows for comparisons to our other quantities and may generate renewed interest.

Finally, we consider $\widehat{\chi}(G,h)$. In Section 3 we prove that, perhaps surprisingly, $\chi(G,h)$ and $\widehat{\chi}(G,h)$ are always equal:

THEOREM 4. *For any abelian group G of order n and for all positive integers h we have*

$$\widehat{\chi}(G,h) = \chi(G,h) = v(n,h) + 1.$$

2. On the value of $\widehat{\chi}(G,[0,s])$

It is an easy exercise to show that, if G is an abelian group of order n that is not isomorphic to $\mathbb{Z}_2$, then

$$\widehat{\chi}(G,[0,1]) = \chi(G,[0,1]) = v(n,1) + 1 = n,$$

and if it is not isomorphic to $\mathbb{Z}_2$ or $\mathbb{Z}_2^2$, then

$$\widehat{\chi}(G,[0,2]) = \chi(G,[0,2]) = v(n,2) + 1 = \lfloor n/2 \rfloor + 1.$$

For $s = 3$, the result is considerably more complicated; in particular, it depends on the structure of G and not just on the order n of G:

THEOREM 5 (KLOPSCH AND LEV; CF. [12]). *If G is a finite abelian group of order n that is not isomorphic to an elementary abelian 2-group, then*

$$\widehat{\chi}(G,[0,3]) = \begin{cases} \left(1+\frac{1}{d}\right)\cdot\frac{n}{3}+1 & \text{if } G \text{ has a subgroup whose order is congruent to 2 mod 3 that is not isomorphic to an elementary abelian 2-group, and } d \text{ is the minimum size of such a subgroup;} \\ \left\lfloor \frac{n}{3} \right\rfloor + 1 & \text{otherwise.} \end{cases}$$

(The value of $\widehat{\chi}(G,[0,s])$ for elementary abelian 2-groups had been determined earlier—see below. We should note that this result appeared in [12] via different expressions.) Theorems 3 and 5 allow for an interesting comparison; for example, we see that

$$\widehat{\chi}(G,[0,3]) = \chi(G,[0,3])$$

holds if, and only if, n is odd.

The authors of [12] warn that an expression for $\widehat{\chi}(G,[0,s])$ when $s \geq 4$ "is very difficult, if at all feasible." One thus may focus on special types of groups. It appears that only two such results are currently known:

THEOREM 6 (KLOPSCH AND LEV; CF. [12]). *Let n and s be positive integers. If $n \leq s+1$, then $\widehat{\chi}(\mathbb{Z}_n,[0,s]) = 1$; otherwise, we have*

$$\widehat{\chi}(\mathbb{Z}_n,[0,s]) = \max\{f_d(n,s) \colon d \in D(n), d \geq s+2\} + 1,$$

where $f_d(n,s)$ is as defined above.

THEOREM 7 (LEV; CF. [13]). *Let r and s be positive integers, $s \geq 2$. If $r \leq s$, then $\widehat{\chi}(\mathbb{Z}_2^r,[0,s]) = 1$; otherwise we have*

$$\widehat{\chi}(\mathbb{Z}_2^r,[0,s]) = (s+2)\cdot 2^{r-s-1} + 1.$$

While the two results appear dissimilar, the following general lower bound evaluates to the stated values of $\widehat{\chi}(G,[0,s])$ in both cases.

PROPOSITION 8. *Let G be an abelian group of order n, and let H be a subgroup of G of index $d > 1$ for which G/H is of type $(d_1,\ldots,d_t)$. For each $i = 1,\ldots,t$, let c_i be a positive integer with $c_i \leq d_i - 1$, and suppose that*

$$\Sigma_{i=1}^t \lceil (d_i-1)/c_i \rceil \geq s+1.$$

Then we have

$$\widehat{\chi}(G,[0,s]) \geq \left(1 + \Sigma_{i=1}^t c_i\right)\cdot n/d + 1.$$

Before proving Proposition 8, we deduce how it provides exact lower bounds for $\widehat{\chi}(G,[0,s])$ in both Theorems 6 and 7.

Suppose first that G is cyclic and of order n. Clearly, if $n \leq s+1$, then $\widehat{\chi}(G,[0,s]) = 1$. Assume then that $n \geq s+2$, and let d be any divisor of n with $d \geq s+2$. Let H be a subgroup of G of index d, in which case G/H is cyclic and of order d. Then, with $t = 1$ and $c = \lfloor (d-2)/s \rfloor$, we have $c \geq 1$ and $\lceil (d-1)/c \rceil \geq s+1$, so by Proposition 8, we get

$$\widehat{\chi}(G,[0,s]) \geq \left(\left\lfloor \frac{d-2}{s} \right\rfloor + 1\right)\cdot \frac{n}{d} + 1 = f_d(n,s) + 1,$$

and our claim follows.

Next, we consider $\mathbb{Z}_2^r$, the elementary abelian 2-group of rank r. The result is trivial when $r \leq s$, so assume that $s+1 \leq r$, and let t be an integer with

$$s+1 \leq t \leq r.$$

Then choosing $H = \mathbb{Z}_2^t$ and $c_i = 1$ for all $i \in \{1, \ldots, t\}$, Proposition 8 implies that

$$\widehat{\chi}(\mathbb{Z}_2^r, [0,s]) \geq (t+1) \cdot 2^{r-t} + 1;$$

in particular, we have

$$\widehat{\chi}(\mathbb{Z}_2^r, [0,s]) \geq (s+2) \cdot 2^{r-s-1} + 1,$$

as claimed.

Proof of Proposition 8. We shall prove our claim by exhibiting a subset A of G of size

$$|A| = \left(1 + \Sigma_{i=1}^t c_i\right) \cdot n/d$$

for which $\langle A \rangle = G$, but $[0,s]A \neq G$.

Let us identify G/H with

$$K = \mathbb{Z}_{d_1} \times \cdots \times \mathbb{Z}_{d_t},$$

and for $i = 1, \ldots, t$, set

$$N_i = \{1, 2, \ldots, c_i\} \subseteq \mathbb{Z}_{d_i}.$$

Now let

$$B_i = \{0\}^{i-1} \times N_i \times \{0\}^{t-i}$$

with the understanding that $\{0\}^0$ is to be ignored, and let $B_0 = \{0\}^t$.

Consider $B = \cup_{i=0}^t B_i$. We have $\langle B \rangle = K$, and

$$|B| = 1 + \Sigma_{i=1}^t c_i.$$

Furthermore, observe that when

$$\Sigma_{i=1}^t \lceil (d_i - 1)/c_i \rceil \geq s+1,$$

then $[0,s]B = sB \neq K$.

Now set $A = \pi^{-1}(B)$, where $\pi \colon G \to G/H$ is the canonical homomorphism; it is easy to see that A satisfies our requirements. ∎

3. The evaluation of $\widehat{\chi}(G, h)$

In this section we prove Theorem 4, namely, that for any abelian group G of order n and for all positive integers h, we have

$$\widehat{\chi}(G, h) = \chi(G, h) = v(n, h) + 1.$$

Note that, obviously,

$$\widehat{\chi}(G, h) \leq \chi(G, h),$$

so by Theorem 2, we have

$$\widehat{\chi}(G, h) \leq v(n, h) + 1.$$

Therefore, to establish Theorem 4, it suffices to find a subset A of G of size $v(n, h)$ for which $\langle A \rangle = G$, but $hA \neq G$.

We proceed by induction on the order n of G. The claim can be easily verified for $n = 2$; we will assume that it also holds for all groups of order at most $n - 1$.

Recall that we have set

$$v(n, h) = \max \{ f_d(n, h) \colon d \in D(n) \},$$

where $D(n)$ is the set of positive divisors of n, and

$$f_d(n, h) = \left(\left\lfloor \frac{d-2}{h} \right\rfloor + 1 \right) \cdot \frac{n}{d}.$$

We consider two cases.

Case 1. There is a $d_0 \in D(n) \setminus \{n\}$ for which $v(n, h) = f_{d_0}(n, h)$.

Let H be a subgroup of index d_0 in G. Then G/H has order $d_0 < n$, so by our inductive hypotheses, it contains a subset B of size $v(d_0, h)$ for which $\langle B \rangle = G/H$, but $hB \neq G/H$.

Let $\pi \colon G \to G/H$ denote the canonical homomorphism, and let $A = \pi^{-1}(B)$. Then $\langle A \rangle = G$, $hA \neq G$, and the size of A is $v(n, h)$, since

$$\begin{aligned} |A| = |B| \cdot |H| = v(d_0, h) \cdot \frac{n}{d_0} &\geq f_{d_0}(d_0, h) \cdot \frac{n}{d_0} = \\ = \left(\left\lfloor \frac{d_0 - 2}{h} \right\rfloor + 1 \right) \cdot \frac{n}{d_0} &= f_{d_0}(n, h) = v(n, h). \end{aligned}$$

This completes the proof of Case 1.

Case 2. For all $d \in D(n) \setminus \{n\}$,

$$f_d(n,h) \leq v(n,h) - 1.$$

We then must have

$$v(n,h) = f_n(n,h) = \left\lfloor \frac{n-2}{h} \right\rfloor + 1.$$

Let us consider first the case when n equals a prime number p; we will then identify G with the cyclic group $\mathbb{Z}_p$. Let

$$A = \left\{1, 2, \dots, \left\lfloor \frac{p-2}{h} \right\rfloor + 1\right\} \subseteq \mathbb{Z}_p.$$

Clearly, A has size $v(p,h)$, and $\langle A \rangle = \mathbb{Z}_p$. Moreover,

$$hA = \left\{h, h+1, \dots, h \cdot \left\lfloor \frac{p-2}{h} \right\rfloor + h\right\},$$

from which we see that $h-1 \notin hA$, and thus $hA \neq \mathbb{Z}_p$. Therefore, our claim holds for prime values of n.

Assume now that n is composite. We will show that for each prime divisor p of n, $p-1$ must be divisible by h.

To do so, let p be a prime divisor of n, and let

$$p - 2 = ch + r$$

for some (unique) integers c and r with

$$0 \leq r \leq h-1.$$

Note that n is composite, so $p < n$. Therefore, by assumption, we have

$$f_p(n,h) \leq v(n,h) - 1 = \left\lfloor \frac{n-2}{h} \right\rfloor.$$

Here

$$f_p(n,h) = \left(\left\lfloor \frac{p-2}{h} \right\rfloor + 1\right) \cdot \frac{n}{p} = \left(\frac{p-2-r}{h} + 1\right) \cdot \frac{n}{p} = \frac{n}{h} + \frac{h-2-r}{h} \cdot \frac{n}{p},$$

which is more than $\left\lfloor \frac{n-2}{h} \right\rfloor$, unless $r = h-1$. But then

$$p - 2 = ch + h - 1,$$

so $p-1$ is divisible by h, as claimed.

This implies that every positive divisor of n is congruent to 1 mod h; in particular, so is n, and thus

$$v(n,h) = f_n(n,h) = \left\lfloor \frac{n-2}{h} \right\rfloor + 1 = \frac{n-1}{h}.$$

We thus need to find a subset A of G of size $\frac{n-1}{h}$ for which $\langle A \rangle = G$, but $hA \neq G$.

Suppose that G has rank r and that it is of type $(n_1, \ldots, n_r)$; that is,

$$G \cong \mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_r}$$

for integers $n_1, \ldots, n_r$ for which $n_1 \geq 2$ and n_i is a divisor of n_{i+1} for each $i = 1, \ldots, r-1$. As we just proved, $n_i - 1$ is divisible by h for each $i = 1, \ldots, r$.

We construct A as follows: for each $i = 1, \ldots, r$, let

$$N_i = \left\{ 1, 2, \ldots, \frac{n_i - 1}{h} \right\} \subseteq \mathbb{Z}_{n_i}$$

and

$$A_i = \mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_{i-1}} \times N_i \times \{0\}^{r-i};$$

we then set

$$A = \cup_{i=1}^{r} A_i.$$

Then

$$|A_i| = n_1 \cdots n_{i-1} \cdot \frac{n_i - 1}{h};$$

since the r sets are pairwise disjoint, the size of A equals

$$|A| = \sum_{i=1}^{r} |A_i| = \sum_{i=1}^{r} n_1 \cdots n_{i-1} \cdot \frac{n_i - 1}{h} = \frac{n_1 \cdots n_r - 1}{h} = \frac{n-1}{h} = v(n,h).$$

Furthermore, since $n_r - 1$ is divisible by h, we have $n_r \geq h+1$, and thus

$$A \supseteq A_r \supseteq \mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_{r-1}} \times \{1\},$$

which implies that $\langle A \rangle = G$.

Finally, we show that $hA \neq G$ by verifying that $0 \notin hA$. Let $a_1, \ldots, a_h$ be (not-necessarily distinct) elements of A. For each $i = 1, \ldots, r$, there is a unique $k_i \in \{1, \ldots, r\}$ for which $a_i \in A_{k_i}$; let

$$k = \max\{k_i \mid i = 1, \ldots, r\}.$$

But then the k-th coordinate of $a_1 + \cdots + a_h$ (as an element of $\mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_r}$) is at least 1 and at most $n_k - 1$, so

$$a_1 + \cdots + a_h \neq 0.$$

This completes our proof. ∎

References

[1] B. Bajnok, On the maximum size of a (k, l)-sum-free subset of an abelian group, *Int. J. Number Theory* **5**(6) (2009), 953–971.

[2] B. Bajnok, The h-critical number of finite abelian groups, *Unif. Distrib. Theory* **10**, no.2 (2015), 93–15.

[3] P. H. Diananda and H. P. Yap, Maximal sum-free sets of elements of finite groups, *Proceedings of the Japan Academy*, **45** (1969) 1–5.

[4] J. A. Dias Da Silva and Y. o. Hamidoune, Cyclic space for Grassmann derivatives and additive theory, *Bull. London Math. Soc.*, **26** (1994) 140–146.

[5] G. T. Diderrich, An Addition Theorem for Abelian Groups of Order pq, *J. Number Theory*, **7** (1975) 33–48.

[6] G. T. Diderrich and H. B. Mann, Combinatorial Problems in Finite Abelian Groups, *A Survey of Combinatorial Theory*, J. N. Srivastava et al., ed., North-Holland (1973).

[7] P. Erdős and H. Heilbronn, On the addition of residue classes (mod p), *Acta Arith.*, **9** (1964) 149–159.

[8] M. Freeze, W. Gao and A. Geroldinger, The critical number of finite abelian groups, *J. Number Theory*, **129** (2009) 2766–2777.

[9] M. Freeze, W. Gao and A. Geroldinger, Corrigendum to "The critical number of finite abelian groups, *J. Number Theory*, **129** (2009) 2766–2777", *J. Number Theory*, **152** (2015) 205–207.

[10] W. Gao and Y. O. Hamidoune, On additive bases, *Acta Arithmetica*, **88** (1999) 233-237.

[11] J. R. Griggs, Spanning subset sums for Finite Abelian groups, *Discrete Mathematics*, **229** (2001) 89–99.

[12] B. Klopsch and V. F. Lev, Generating abelian groups by addition only, *Forum Math.*, **21** (2009) 23–41.

[13] V. F. Lev, Generating binary spaces, *J. Combin. Theory, Ser. A* **102** (2003) 94–109.

[14] H. B. Mann and Y. F. Wou, Addition theorem for the elementary abelian group of type (p, p), *Monatshefte für Math.* **102** (1986) 273–308.

Béla Bajnok

Department of Mathematics
Gettysburg College
Gettysburg, PA 17325-1486 USA
`bbajnok@gettysburg.edu`

ANNALES UNIV. SCI. BUDAPEST., **59** (2016), 123–132

ISOMETRIES OF VIRTUAL QUADRATIC SPACES

By
MÁTÉ L. JUHÁSZ
(Received December 3, 2016)

Abstract. In this article, we introduce a new object, a virtual quadratic space, and its group of isometries. They are presented as natural generalizations of quadratic spaces and orthogonal groups. It is then shown that by replacing quadratic spaces with virtual quadratic spaces, we can unify certain enumerative properties of finite fields, without distinguishing between even and odd characteristics, such as the number of non-isomorphic non-degenerate quadratic forms, and the order of groups of isometries.

1. Introduction

Quadratic forms over finite fields have different properties depending on whether the characteristic of the field is 2 or not. However, several general statements can be made without referencing the characteristic of the field. In particular, in even dimension, the number of non-degenerate quadratic forms is 2 up to isomorphism, and the number of elements in its group of isometries is a polynomial in the number of elements of the field. However, when the dimension of a quadratic space is odd, every bilinear form is degenerate in characteristic 2, and the number of isometries for a non-degenerate quadratic form is different if the characteristic is even.

Given a quadratic space (V, Q) and a subspace $F \subseteq V$, the group of isometries that fix F are in a bijection with the isometries of $(F^{\perp}, Q|_{F^{\perp}})$, provided that the associated bilinear forms on V and F are non-degenerate. In characteristic 2, this is only possible if $\dim F^{\perp}$ is even. However, we may still consider pairs $(V, F^{\perp})$ where we only assume that $\dim V$ is even, and from Theorem 3.9, this gives a natural generalization of quadratic spaces. Furthermore, the order of orthogonal groups is given by a polynomial that does not depend on the characteristic, as seen in Corollary 4.7.

2. Quadratic spaces

Let us fix some field and denote it by $\mathbb{K}$.

Definition 2.1. A *quadratic space* is a pair (V, Q) consisting of a vector space and a quadratic form on it. It has an *associated bilinear from* defined as $B(x, y) = Q(x + y) - Q(x) - Q(y)$. The *radical* of a quadratic form Q is the set of vectors v such that $Q(v + u) = Q(u)$, and it is denoted by $\operatorname{Rad} Q$. The *direct sum* of two quadratic spaces (V, Q) and (V', Q') is $(V \oplus V', Q \oplus Q')$ such that $(Q \oplus Q')(v \oplus v') = Q(v) + Q(v')$. An *isometry* is a linear map $\varphi\colon V \to V$ such that $Q(\varphi(v)) = Q(v)$ for all $v \in V$. The *group of isometries* is denoted by $\operatorname{Iso}(V)$.

Definition 2.2. Let us denote the function $u \to B(v, u)$ by v^*. A bilinear form is *non-degenerate* or *regular* if for every non-zero vector $v \in V$, the linear function v^* is non-zero. A quadratic form is *non-degenerate* if its associated bilinear form is non-degenerate.

We need a few properties of hyperbolic spaces:

Definition 2.3. A *hyperbolic space* of dimension 2 is a pair (Σ, B) is such that the bilinear form B takes the form $B(u, v) = u_1 v_2 + u_2 v_1$ in some basis. A hyperbolic space of dimension $2k$ is the orthogonal direct sum of k hyperbolic spaces of dimension 2 each.

Lemma 2.4. *Assume V is a vector space with a non-degenerate bilinear form B, and $N < V$ is a subspace with $B|_N \equiv 0$. Then there is a hyperbolic subspace $\Sigma = N \oplus \widetilde{N}$ of dimension $2 \dim N$, such that $N^\perp \cap \Sigma = N$.*

Proof. We will construct the spaces $\widetilde{N}$ and Σ recursively. Choose a vector u in N. Since B is non-degenerate, there is a vector v such that $B(u, v) = 1$. Since $B|_N \equiv 0$, clearly $v \notin N$. Furthermore, $\Sigma_0 = \langle u, v\rangle$ is such that $B|_{\Sigma_0^\perp}$ is non-degenerate, hence we may apply the construction to $V' := \Sigma_0^\perp$ and $N' := N \cap V'$, unless $N' = \{0\}$, in which case we are done. The construction will give us $\widetilde{N}'$ and Σ' of dimension $2 \dim N' = 2(\dim N - 1)$, since $N < u^\perp$ and $u \in v^\perp$, and we may choose $\Sigma := \Sigma_0 \oplus \Sigma'$, $\widetilde{N} := \langle v\rangle \oplus \widetilde{N}'$. ∎

The following lemma shows that in characteristic 2, one can not construct a non-degenerate quadratic form in odd dimensions:

Lemma 2.5. *Given a non-degenerate quadratic space (V, Q) in characteristic 2, its associated bilinear form gives (V, B) a hyperbolic space structure.*

PROOF. It can be checked that any quadratic space decomposes as the direct sum of quadratic spaces of two kinds: those of the form Ax^2 and $A(x^2 + xy + By^2)$. Since $B(u, u) = 0$ for any u in characteristic 2, the first kind may not appear in the decomposition, while the second kind gives a hyperbolic space. For details, see [2]. ∎

Since the dimension of a hyperbolic space is even, a non-degenerate quadratic space must have even dimension.

3. Virtual quadratic spaces

DEFINITION 3.1. A *virtual quadratic space* is a tuple (V, Q, U) or (V, U) for short, with U a subspace of a vector space V and Q a non-degenerate quadratic form on V. Its *dimension* is $\dim U$. An *isometry* of the virtual quadratic space is an isometry of (V, Q) that fixes $U^\perp$. The group of isometries is denoted by $\mathrm{Iso}(V, U)$.

In general, we are only interested in the quadratic subspace (U, Q), and we only use V as an aid in theorems. As such, we must establish whether such a (V, Q) exists for a quadratic space (U, Q), and whether it is unique. First let us look at the question of existence.

PROPOSITION 3.2. *Any quadratic space U can be embedded into some virtual quadratic space (V, U).*

PROOF. Let us denote $N := U \cap U^\perp$. Since $B|_N \equiv 0$, we shall embed N into a hyperbolic space. Define $V := \widetilde{N} \oplus U$ where $\widetilde{N}$ is isomorphic as a vector space to N. We shall give $\widetilde{N} \oplus N$ a hyperbolic space structure in the following way. Let us choose a basis f_i for N^* and the equivalent $\tilde{f}_i$ for $\widetilde{N}^*$, and extend the f_i to U in an arbitrary manner. Now for $\tilde{u} \oplus v \in V$, with $\tilde{u} \in \widetilde{N}$ and $v \in U$, define $\tilde{Q}(\tilde{u} \oplus v) = Q(v) + \sum \tilde{f}_i(\tilde{u}) f_i(v)$. It can be verified that this quadratic form has non-degenerate associated bilinear form. ∎

Uniqueness of V can of course not be guaranteed, but we may look for a *minimal* virtual quadratic space, and show some type of uniqueness for that. The following theorem shows what such a space looks like, and how we may characterize this minimality.

PROPOSITION 3.3. *For any virtual quadratic space (V, Q, U) with associated bilinear form B, there is a subspace $U \leq V_m \leq V$ with $B|_{V_m}$ non-degenerate,* *such that* $U^\perp \cap V_m \subseteq U$. *For such a subspace,* $\mathrm{Iso}(V,U) = \mathrm{Iso}(V_m,U)$. *A virtual quadratic space* (V,U) *is called* minimal *if* $U^\perp \subseteq U$, *and* $\dim V = \dim U + \dim(U \cap U^\perp)$ *only depends on* $Q|_U$ *for minimal virtual quadratic spaces.*

Proof. We don't actually need Q in the proof of the first statement, and may only look at B. We will proceed by constructing the same space as in Proposition 3.2. Let us fix $N := U \cap U^\perp$. Since $B|_N \equiv 0$, by Lemma 2.4 we may embed N into a subspace $\Sigma = N \oplus \widetilde{N}$ of dimension $2\dim N$, and $B|_\Sigma$ is clearly non-degenerate, giving $\Sigma \cap \Sigma^\perp = \{0\}$.

We can see that $U \cap \Sigma = N$, since $U \cap \Sigma \supset N$ by the definition of N and Σ, and if $u \in U \cap \Sigma$, then $u \perp N$, but $u \in N^\perp \cap \Sigma = N$ by the construction of Σ. For similar reasons, $U^\perp \cap \Sigma = N$.

Therefore U decomposes as orthogonal subspaces $N \oplus M$ with $M = U \cap \Sigma^\perp$. Furthermore, $M \cap M^\perp = \{0\}$, since for all $u \in M \subseteq U \setminus N$, there is a $v \in U$ such that $u \not\perp v$. Then v decomposes as $v_M \oplus v_N$ with $v_M \in M$ and $v_N \in N$, and since $u \perp v_N$, we have $u \not\perp v_M \in M$. Therefore $B|_M$ is non-degenerate. Also, V decomposes as orthogonal subspaces $M \oplus \widehat{M} \oplus \Sigma$ with $\widehat{M} = M^\perp \cap \Sigma^\perp$.

Let us define $V_m := U + \Sigma$, which is $M \oplus \Sigma$ as an orthogonal decomposition. Clearly (V_m, U) is a virtual quadratic space, since (V_m, Q) is non-degenerate as $V_m \cap V_m^\perp = \{0\}$. Since $U^\perp \cap \Sigma = N \subseteq U$, we have $U^\perp \cap (M \oplus \Sigma) \subseteq U$, and so (V_m, U) is minimal.

Clearly $\dim V_m = \dim M + \dim \Sigma = \dim U + \dim N$. Given a virtual quadratic space (V,U), the above construction gives a minimal subspace (V_m, U). Since V decomposes as orthogonal subspaces $M \oplus \widehat{M} \oplus \Sigma$, we get $U^\perp = M^\perp \cap N^\perp = (\widehat{M} \oplus \Sigma) \cap (M \oplus N \oplus \widehat{M}) = \widehat{M} \oplus N$. Then $U \subseteq U^\perp$ if and only if $\dim \widehat{M} = 0$ and $V = V_m$. Therefore the dimension formula holds for all minimal spaces.

Finally, since $U^\perp = \widehat{M} \oplus N$, $\mathrm{Iso}(V,U)$ fixes $U^\perp \supseteq \widehat{M}$, and the action restricts to $\widehat{M}^\perp$, giving $\mathrm{Iso}(V,U) = \mathrm{Iso}(\widehat{M}^\perp, U \cap \widehat{M}^\perp) = \mathrm{Iso}(V_m, U)$. ∎

This proof tells us not only that a minimal virtual quadratic space exists, it also shows us the structure it has, and we shall use the notations introduced in this proof in other propositions as well.

However, since the theorem extends only the bilinear form to V from U, and in characteristic 2 that does not define the quadratic form, the minimal virtual quadratic space (V,U) containing U is still not unique. Fortunately, this is not an issue, at least for the isometry groups, as seen from the following theorem.

Proposition 3.4. *For any non-degenerate virtual quadratic space* (V, U), *the group* $\mathrm{Iso}(V, U)$ *depends only on* $Q|_U$. *In particular, if given two non-degenerate virtual quadratic spaces* (V, Q, U) *and* (V', Q', U'), *with* $(U, Q|_U)$ *and* $(U', Q'|_{U'})$ *isomorphic as quadratic spaces, then there is a bijection between* $\mathrm{Iso}(V, U)$ *and* $\mathrm{Iso}(V', U')$.

Proof. Consider two virtual quadratic spaces (V, U) and (V', U') such that $Q|_U \cong Q'|_{U'}$. First we may assume that they are both minimal, as all such spaces have a minimal subspace with an isometry group isomorphic to the isometry group of the containing space, in which case $\dim V = \dim V'$. We may assume that $V = V'$ and $U = U'$. Then by introducing the difference $\delta Q = Q' - Q$, we get $\delta Q|_U \equiv 0$.

If the characteristic is not 2, then two quadratic spaces are isomorphic if an isomorphism of the vector spaces identifies their bilinear forms. By using the notations of the previous proof, U decomposes as a direct sum of orthogonal subspaces $N \oplus M$, where $N = U \cap U^\perp$ and Σ is a hyperbolic subspace of dimension $2 \dim N$ containing N. Since the pair (V, U) is minimal, $V = M \oplus \Sigma$. By introducing the analoguous symbols for the pair (V', U'), all the components are isomorphic, hence we can choose the isomorphism that sends $V = M \oplus \Sigma$ to $V' = M' \oplus \Sigma'$ component-wise. Under this isomorphism, $Q|_V = Q'|_V$, so the theorem becomes a trivial condition.

Let us look at the characteristic 2 case, where the associated bilinear form is always hyperbolic by Lemma 2.5. Since $B|_U = B'|_{U'}$, we may fix an identification between V and V' that extends the isometric map $U \to U'$ in a way such that $B = B'$. Then the map $\delta Q(x) := Q'(x) - Q(x)$ is an additive map.

Let us choose an automorphism $\varphi \in \mathrm{Iso}_Q(V, U)$. In order to prove that φ is also in $\mathrm{Iso}_{Q'}(V, U)$, it is sufficient to show that φ preserves the function δQ, since then $Q'(\varphi(x)) = Q(\varphi(x)) + \delta Q(\varphi(x)) = Q(x) + \delta Q(x) = Q'(x)$.

Since $U^\perp$ is fixed under the automorphism φ, the scalar product functions $B(u, .)$ are preserved for all $u \in U^\perp$, and in particular, the subspace U is preserved. Equivalently, the automorphism acts trivially on the quotient vector space V/U, since for any x, assuming that $\delta x := \varphi(x) - x \notin U$, there is at least one associated $u \in U^\perp$ such that $B(u, \delta x) \neq 0$, which would contradict the preservation of the function $B(u, .)$.

Since the function δQ is additive and vanishes on U, it is well defined on the quotient additive group V/U, which is naturally isomorphic to the quotient vector space V/U. On V/U, φ acts trivially, and so δQ is preserved. ∎

These two propositions motivate the following definition:

Definition 3.5. Two virtual quadratic spaces (V, Q, U) and (V', Q', U') are *isomorphic* if $(U, Q|_U)$ and $(U', Q|_{U'})$ are isomorphic quadratic spaces.

Now we will show a relationship between the isometry groups $\operatorname{Iso}(U)$ and $\operatorname{Iso}(V, U)$. It is known (see [2]) that in a non-degenerate quadratic space, the group of isometries acts transitively on non-zero vectors of equal norm:

Theorem 3.6. *Given a quadratic space U and two subspaces V and W such that $V \cap U^\perp = W \cap U^\perp = \{0\}$ with an isometry $\sigma\colon V \to W$, it extends to an isometry of U.*

Proposition 3.7. *Given a quadratic space (U, Q), and assuming that for any non-zero $u \in U \cap U^\perp$ we have $Q(u) \neq 0$, then every isometry of U fixes $U \cap U^\perp$. Furthermore, if given an embedding of U into a minimal virtual quadratic space (V, U), there is a natural map $\operatorname{Iso}(V, U) \to \operatorname{Iso}(U)$ that is surjective.*

Proof. We will first show that Q as a map is an injection from $N := U \cap U^\perp$ to the base field. In fact, given $u, v \in N$, $Q(u - v) = Q(u) - Q(v)$ since $u \perp v$. Therefore if $Q(u) = Q(v)$, we get $Q(u - v) = 0$, which is only possible if $u = v$. Since Q is injective, and any isometry maps N to itself, it must fix each vector, thus fixing N.

Given a virtual quadratic space (V, U), any isometry of V fixes $N \subseteq U^\perp$. If (V, U) is minimal, $N^\perp = U$, hence the subspace U is preserved, and there is a restriction map $\operatorname{Iso}(V, U) \to \operatorname{Iso}(U)$. Since (Q, V) is a non-degenerate quadratic space, by Theorem 3.6, any isometry of U extends to an isometry of V. Hence the restriction map is surjective. ∎

Proposition 3.8. *The condition that for any non-zero $u \in U \cap U^\perp$, $Q(u) \neq 0$, is equivalent to $\operatorname{Rad} Q = \{0\}$. If the characteristic is not 2, this is equivalent to $U \cap U^\perp = \{0\}$.*

Proof. In fact we shall prove that $\operatorname{Rad} Q = \{u \in U \cap U^\perp \mid Q(u) = 0\}$. In one direction, if $u \in \operatorname{Rad} Q$, then $B(u, v) = Q(u + v) - Q(u) - Q(v) = 0$ and $Q(u) = Q(u + 0) = Q(0) = 0$. Now assume $Q(u) = 0$ and $B(u, v) = 0$ for all $v \in U$. Then $Q(u + v) = Q(u) + Q(v) + B(u, v) = Q(v)$, hence $u \in \operatorname{Rad} Q$.

If the characteristic is not 2, the bilinear form B defines Q, and so $Q|_{U \cap U^\perp} \equiv$ $\equiv 0$ since $B|_{U \cap U^\perp} \equiv 0$. ∎

Propositions 3.2, 3.3, 3.4, 3.7 and 3.8 may be combined into the following theorem:

THEOREM 3.9. *Consider a quadratic space* (Q_U, U). *Then it can be embedded into a virtual quadratic space* (V, Q_V, U), *and for such an embedding,* $\operatorname{Iso}(V, U)$ *depends only on* Q_U. *In fact, such a* V *can always be chosen so that* $\dim V = \dim U + \dim(U \cap U^\perp)$, *even as a subspace of some other virtual quadratic space* $(V', Q_{V'}, U)$, *which is equivalent to the condition that* $U^\perp \subseteq U$ *in* V. *Furthermore, if* $\operatorname{Rad} Q_U = \{0\}$, *the restriction map* $\operatorname{Iso}(V, U) \to \operatorname{Iso}(U)$ *is a surjective map.*

Proposition 3.7 motivates also the following definition.

DEFINITION 3.10. A virtual quadratic space (V, U) is *non-degenerate* if $\operatorname{Rad} Q = \{0\}$.

4. Finite fields

One interesting application of virtual quadratic spaces is that they provide a common language for finite fields of even and odd characteristic. First, consider the following lemma.

LEMMA 4.1. *In a prefect field* $\mathbb{K}$ *of characteristic* 2, *any non-degenerate virtual quadratic space* (V, U) *is such that* $\dim(U \cap U^\perp) \leq 1$.

PROOF. Assume that there are two linearly independent vectors $u, v \in U \cap U^\perp$ with $Q(u) \neq 0 \neq Q(v)$. Since $\mathbb{K}$ is perfect, there is an element $\lambda \in \mathbb{K}$ such that $\lambda^2 = \frac{Q(u)}{Q(v)}$. Then the vector $w := u + \lambda v$ has $Q(w) = 0$. Since (V, U) is non-degenerate, this contradicts the fact that u and v are linearly independent. ∎

Now let us recall a few simple theorems. See [2] for details.

THEOREM 4.2. *Let us fix a finite field* $\mathbb{F}$ *of odd characteristic, and choose a non-square element* $\mathbf{e} \in \mathbb{F}$. *Then every non-degenerate quadratic form is of one of the following forms, up to isomorphism:*

- $\sum_{i=1}^k x_{2i-1}x_{2i}$ *for* $n = 2k$;
- $\sum_{i=1}^k x_{2i-1}x_{2i} + x_{2k+1}^2$ *for* $n = 2k + 1$;
- $\sum_{i=1}^k x_{2i-1}x_{2i} + x_{2k+1}^2 - \mathbf{e}x_{2k+2}^2$ *for* $n = 2k + 2$.

THEOREM 4.3. *Let us fix a finite field* $\mathbb{F}$ *of characteristic* 2, *and choose an element* $\mathbf{e} \in \mathbb{F}$ *for which the polynomial* $x^2 + x + \mathbf{e}$ *has no roots. Then every quadratic form with trivial radical is of one of the following forms, up to isomorphism:*

- $\sum_{i=1}^{k} x_{2i-1}x_{2i}$ *for* $n = 2k$;
- $\sum_{i=1}^{k} x_{2i-1}x_{2i} + x_{2k+1}^2$ *for* $n = 2k + 1$;
- $\sum_{i=1}^{k} x_{2i-1}x_{2i} + x_{2k+1}^2 + x_{2k+1}x_{2k+2} + \mathbf{e}x_{2k+2}^2$ *for* $n = 2k + 2$.

The first and last have non-degenerate associated bilinear forms, but the second has not.

The first and last cases are denoted for all fields as +-type and −-type, respectively. These two theorems can be combined into the following corollary:

Corollary 4.4. *Let us fix a finite field* $\mathbb{F}$. *Then the number of non-degenerate virtual quadratic spaces of dimension* n *up to isomorphism is* 2 *if* n *is even and* 1 *if* n *is odd.*

Proof. We may assume that the virtual quadratic space is minimal. A non-degenerate virtual quadratic space is a triple (V, Q, U) such that $Q(u) = 0$ for $u \in U \cap U^{\perp}$ only if $u = 0$. If the characteristic of the field is odd, this is only possible if $V = U$, hence this is the same case as Theorem 4.2. If the characteristic is 2, V must be of even dimension by Theorem 4.3. Since all finite fields are perfect, by Lemma 4.1 and Theorem 3.9 we have $\dim V - \dim U \leq 1$ since V is minimal. Hence if $\dim U$ is even, then $V = U$.

Assume that $\dim U$ is odd and $\dim V = \dim U + 1$. Then $\operatorname{Rad} Q|_U = \{0\}$, since the virtual quadratic space is non-degenerate. Hence $Q|_U$ is of the form prescribed in 4.3, which determines the virtual quadratic space up to isomorphism. ∎

The order of orthogonal groups over finite fields is known (see [1]).

Theorem 4.5. *Consider a quadratic space* (U, Q) *with* $\operatorname{Rad} Q = \{0\}$ *over the finite field* $\mathbb{F}_q$, *and let us denote by* $\mathrm{O}^{\varepsilon}(2k, q)$ *the group* $\operatorname{Iso}(U)$ *when* $\dim U = 2k$ *and* Q *is of type* ε, *and by* $\mathrm{O}(2k+1, q)$ *the group* $\operatorname{Iso}(U)$. *Then*

$$\mathrm{O}^{\varepsilon}(2k, q) = 2q^{k^2-k}(q^k - \varepsilon)\prod_{i=1}^{k-1}(q^{2i} - 1)$$

If $2 \nmid q$,

$$\mathrm{O}(2k+1, q) = 2q^{k^2}\prod_{i=1}^{k}(q^{2i} - 1)$$

If $2 \mid q$,

$$\mathrm{O}(2k+1, q) = q^{k^2}\prod_{i=1}^{k}(q^{2i} - 1)$$

The formula for even dimension does not discriminate between even and odd characteristics, but the formula for odd dimension does. Virtual quadratic spaces give us a hint that the problem is that the space has degenerate associated bilinear form, and as it turns out, it is:

THEOREM 4.6. *Let* (V, U) *be a non-degenerate virtual quadratic space of dimension* $2k+1$ *over a field* $\mathbb{F}_q$ *of characteristic* 2. *Then*

$$|\operatorname{Iso}(V, U)| = 2q^{k^2} \prod_{i=1}^{k} (q^{2i} - 1)$$

PROOF. It is known from 3.9 that the restriction map $\operatorname{Iso}(V, U) \to \operatorname{Iso}(U)$ is a surjection, and that $\operatorname{Iso}(U) \cong \mathrm{O}(2k+1, q)$, hence we only need to show that the kernel of the restriction map is of order 2.

Assume that (V, U) is minimal, and let us take an isometry $\varphi \in \operatorname{Iso}(V, U)$ that is in the kernel, hence it fixes U. We may decompose V as the orthogonal sum $M \oplus \Sigma$ where Σ is a hyperbolic space, and U as $M \oplus N$ where $N = U \cap U^{\perp}$. Then φ fixes $M \subset U$, and thus preserves the subspace Σ.

By Lemma 4.1, $\dim N = 1$, and Σ has a basis $\{e_1, e_2\}$ with $\langle e_1 \rangle = N$, where the bilinear form takes the form $B(u, v) = u_1 v_2 + u_2 v_1$. Since (V, U) is non-degenerate, $\operatorname{Rad} Q|_U = \{0\}$, and $Q(e_1) \neq 0$, in fact we may assume $Q(e_1) = 1$ by rescaling, as the field is perfect. Since φ fixes $U^{\perp}$, which contains e_1, we only need to check the image of e_2. Let $\varphi(e_2) = \alpha e_1 + \beta e_2$ for some parameters $\alpha, \beta \in \mathbb{F}_q$.

First of all, $1 = B(e_1, e_2) = B(e_1, \varphi(e_2)) = \beta$. Then $Q(e_2) = Q(\varphi(e_2)) = = \alpha^2 + \alpha\beta + \beta^2 Q(e_2)$, which gives us $\alpha(\alpha + 1) = 0$, hence $\alpha \in \{0, 1\}$. Since either choice gives us an isometry, we have the kernel containing 2 elements. ∎

COROLLARY 4.7. *Consider a non-degenerate virtual quadratic space* (V, Q, U) *over the finite field* $\mathbb{F}_q$, *and let us denote by* $\operatorname{Ort}^{\varepsilon}(2k, q)$ *the group* $\operatorname{Iso}(V, U)$ *when* $\dim U = 2k$ *and* Q *is of type* ε, *and by* $\operatorname{Ort}(2k+1, q)$ *the group* $\operatorname{Iso}(V, U)$. *Then*

$$\operatorname{Ort}^{\varepsilon}(2k, q) = 2q^{k^2-k}(q^k - \varepsilon) \prod_{i=1}^{k-1} (q^{2i} - 1)$$

$$\operatorname{Ort}(2k+1, q) = 2q^{k^2} \prod_{i=1}^{k} (q^{2i} - 1)$$

ACKNOWLEDGEMENT. I am grateful for the help of József Pelikán for providing me with references.

References

[1] Bertram Huppert, *Endliche Gruppen I*, Springer Verlag. 1967.
[2] Yoshiyuki Kitaoka, *Arithmetic of quadratic forms*, Cambridge University Press. 1993.

Máté L. Juhász
Alfréd Rényi Institute of Mathematics
Hungarian Academy of Sciences
juhasz.mate.lehel@renyi.mta.hu

ANNALES UNIV. SCI. BUDAPEST., **59** (2016), 133–138

CONNECTIONS BETWEEN THE CARDINALITY OF SUMSETS AND DIFFERENCE SETS NEAR THE EXTREME

By
MÁRTON HORVÁTH AND IMRE Z. RUZSA
(Received January 13, 2017)

Abstract. We study connections between the cardinality of the sets $A + A$ and $A - A$, where A is a finite set in a commutative group and one of them is near to the maximal possible value. We improve earlier results of the second author [1].

1. Introduction

Let A be a finite nonempty set in a commutative group, and set $n = |A|$. The cardinality of the difference set $A-A$ is at most n^2-n+1 as the trivial differences $a-a$ all gives 0. By the commutativity of the group, we have $|A+A| \le n(n+1)/2$. The second author [1] proved some estimates to express the phenomenon that if one of these quantities is near to the maximal possible value, then the other cannot be very small. Results of this kind are conveniently expressed in terms of the difference deficit and sum deficit introduced in [1]

$$\Delta_-(A) = n^2 - n + 1 - |A - A|,$$
$$\Delta_+(A) = \frac{n(n+1)}{2} - |A + A|.$$

The aim of this note is to improve certain estimates of [1].

Second author was supported by ERC–AdG Grant No.321104 and Hungarian National Foundation for Scientific Research (OTKA), Grants No.109789, and NK104183.

2. Differences near the maximum

The second author proved the following result in [1].

THEOREM 1. *Let A be a finite set in a commutative group, $|A| = n$. We have*

$$|A+A|\left(\Delta_-(A)^2 + \frac{n^3}{6}\right) \geq \frac{n^5}{20}.$$

In particular, if $|A+A| < n^2/10$, then

$$|A+A|\Delta_-(A)^2 \geq \frac{n^5}{30}.$$

The above theorem provides no information if $|A+A| > 0.3n^2$. Our first result extends this for the whole range up to $n^2/2$.

THEOREM 2. *We have*

$$|A+A|\Delta_-(A)^2 \geq \left(\frac{n}{2} - \frac{|A+A|}{n}\right)^4 \left(\frac{n}{2} + \frac{|A+A|}{n}\right).$$

If $|A+A| < cn^2$ for some constant $c < \frac{1}{2}$, then $|A+A|\Delta_-(A)^2 > c'n^5$ with

$$c' = \frac{(1-2c)^4(1+2c)}{32}.$$

PROOF. For a positive integer k, define the function f_k as

$$f_k(n) = \min\{\Delta_-(A) \colon |A+A| \leq k, |A| = n\}$$

for $1 \leq n \leq k$. Firstly, we study this function. Let A be a finite set such that $|A+A| \leq k, |A| = n$, and $\Delta_-(A)$ is minimal under these conditions, so $\Delta_-(A) = f_k(n)$.

Denote by $r(x)$ the number of representations of any element x as a sum from A, i.e.,

$$r(x) = |\{(a_1, a_2) \colon a_1, a_2 \in A, a_1 + a_2 = x\}|.$$

Obviously,

$$\sum_x r(x) = n^2. \tag{1}$$

We define a similar function as

$$q(x) = |\{(a_1, a_2, a_3) \colon a_1, a_2, a_3 \in A, a_1 + a_2 - a_3 = x\}|.$$

Counting the number of the quadruples (a_1, a_2, a_3, a_4) such that $a_1 + a_2 = a_3 + a_4$ in different ways, we have

$$\sum_x r^2(x) = \sum_{a \in A} q(a). \tag{2}$$

The inequality of arithmetic and quadratic means gives that

$$\sum_{a \in A} q(a) = \sum_x r^2(x) \geq \frac{\left(\sum_x r(x)\right)^2}{|A + A|} \geq \frac{n^4}{k},$$

from which we conclude that there is an $a \in A$ such that $q(a) \geq \frac{n^3}{k}$. This means that there are at least $\frac{n^3}{k}$ triples (a_1, a_2, a_3) such that $a = a_1 + a_2 - a_3$.

Our goal is to give a lower bound to the increment of the difference deficit when we consider A instead of $A' = A \setminus \{a\}$. This increment is the number of "new" pairs (a_1, a_2) (that is, $a_1 = a$ or $a_2 = a$) which do not give a new difference.

We can write these differences in two different forms with an element $\tilde{a} \in A'$. The first ones are in the form $\tilde{a} - a$, when this difference is contained by the set $A' - A'$. The second ones are the differences $a - \tilde{a} \in A' - A$ (note the slight difference: elements of $A' - a$ are now old, counted in the first form). For our estimate, we count only the differences in the second form.

If $a = a_1 + a_2 - a_3$, where $a_1 \neq a$ and $a_2 \neq a$, then the differences $a - a_1 = a_2 - a_3$ and $a - a_2 = a_1 - a_3$ are in the set $A' - A$. Ruling out the cases when a is equal to a_1 or a_2, at least $n^3/k - 2n$ triples remain. Hence the number of elements that appear as a_1 or a_2 is at least $\sqrt{n^3/k - 2n}$. This gives that

$$\Delta_-(A) - \Delta_-(A \setminus \{a\}) \geq \sqrt{\frac{n^3}{k} - 2n},$$

which implies

$$f_k(n) - f_k(n-1) \geq \sqrt{\frac{n^3}{k} - 2n} = \sqrt{\frac{n}{k}}\sqrt{n^2 - 2k}. \tag{3}$$

The latter form shows that our bound is an increasing function of n.

A repeated application of this inequality gives

$$\begin{aligned} f_k(n) \geq f_k(n) - f_k(n-t-1) &\geq \sum_{i=0}^{t} \sqrt{\frac{n-i}{k}}\sqrt{(n-i)^2 - 2k} \\ &\geq (t+1)\sqrt{\frac{n-t}{k}}\sqrt{(n-t)^2 - 2k} \end{aligned}$$

for $t < n - \sqrt{2k}$.

We put

$$t = \left\lfloor \frac{n}{2} - \frac{k}{n} \right\rfloor.$$

With this choice

$$n - t \geq \frac{n}{2} + \frac{k}{n}, \ (n-t)^2 - 2k \geq \left(\frac{n}{2} - \frac{k}{n} \right)^2,$$

hence

$$f_k(n) \geq \left(\frac{n}{2} - \frac{k}{n} \right)^2 \left(\frac{n}{2} + \frac{k}{n} \right)^{1/2} k^{-1/2}.$$

By squaring, multiplying by k and putting $k = |A + A|$ we get the claim of the theorem. ∎

The constant $1/32$ could be reduced by a more careful calculation.

3. Sums near the maximum

In [1] it is shown that

$$|A - A|\Delta_+(A)^2 > n^4/9 \tag{4}$$

assuming $|A - A| < n^2/2$, and it is conjectured that the lower bound can be improved to cn^5. We prove an estimate which improves (4), though it is not as strong as the conjectured one.

Theorem 3. *We have*

$$|A - A|\Delta_+(A) > \frac{(n - \sqrt{2|A - A|})^2 (n + 2\sqrt{2|A - A|})}{3}.$$

If $|A - A| < cn^2$ *for some constant* $c < \frac{1}{2}$, *then* $|A - A|\Delta_+(A) > c'n^3$ *with*

$$c' = \frac{(1 - \sqrt{2c})^2 (1 + 2\sqrt{2c})}{3}.$$

This bound has a different form; the bound we get for Δ_+ behaves like (4) when $|A - A|$ is of order n^2, and like the conjectural improvement when $|A - A| = O(n)$.

Proof. We proceed similarly to the previous proof. We introduce

$$g_k(n) = \min\{\Delta_+(A) \colon |A - A| \leq k, \ |A| = n\}$$

for $1 \leq n \leq k$.

Instead of the function r we consider the difference-counting function

$$d(x) = |\{(a_1, a_2)\colon a_1, a_2 \in A, a_1 - a_2 = x\}|.$$

The analogs of the equations (1), (2) are true for the function d, i.e.,

$$\sum_x d(x) = n^2, \qquad \sum_x d^2(x) = \sum_{a\in A} q(a).$$

As in the previous proof, we apply the inequality of arithmetic and quadratic means to get

$$\sum_{a\in A} q(a) = \sum_x d^2(x) \geq \frac{\left(\sum_x d(x)\right)^2}{|A-A|} \geq \frac{n^4}{k},$$

so there is an $a \in A$ such that $q(a) \geq \frac{n^3}{k}$.

If $a = a_1 + a_2 - a_3$, then $a + a_3 = a_1 + a_2$, so the sum $a + a_3$ is not a new sum in $A + A$ compared to $A' + A'$ except in the trivial cases when $a = a_1$ or $a = a_2$. For an element a_3, there might be at most n distinct pairs (a_1, a_2) which implies

$$g_k(n) - g_k(n-1) \geq \Delta_+(A) - \Delta_+(A') \geq \frac{\frac{n^3}{k} - 2n}{n} = \frac{n^2}{k} - 2.$$

(Observe that this is weaker than (3); we could obtain the conjectured bound if we had a similar estimate here.)

A repeated application of this inequality gives

$$g_k(n) \geq g_k(n) - g_k(n-t-1) \geq \sum_{i=0}^{t} \left(\frac{(n-i)^2}{k} - 2\right).$$

The best choice is $t = \left\lfloor n - \sqrt{2k} \right\rfloor$. As the function is decreasing, we can estimate the sum by an integral to conclude

$$g_k(n) > \int_0^{n-\sqrt{2k}} \left(\frac{(n-x)^2}{k} - 2\right) dx = \frac{(n-\sqrt{2k})^2(n+2\sqrt{2k})}{3k}.$$

By multiplying by k and putting $k = |A - A|$ we get the claim of the theorem. ∎

In Theorem 2, $n^2/2$ was a natural boundary, while here it is not. However, the situation for $|A - A| > n^2/2$ is already satisfactorily described in [1] in the form

$$\Delta_-(A) \leq 2(\Delta_+(A)^2 + \Delta_+(A)),$$

with examples of equality.

References

[1] IMRE Z. RUZSA, Many differences, few sums, *Ann. Univ. Sci. Budapest. Eötvös Sect. Math.*, **51** (2008), 27–38.

Márton Horváth
Budapest University
of Technology and Economics
Budapest, Hungary
horvathm@math.bme.hu

Imre Z. Ruzsa
Alfréd Rényi Institute of Mathematics
Budapest, Hungary
ruzsa@renyi.hu

ANNALES UNIV. SCI. BUDAPEST., **59** (2016), 139–140

ANALYSIS OF RANDOM GRAPHS WITH METHODS OF MARTINGALE THEORY
Abstract of Ph.D. Thesis

By
ÁGNES BACKHAUSZ
SUPERVISOR: TAMÁS MÓRI
(Defended Jnue 17, 2013)

The main goal of the research was understanding the local behaviour of certain scale free random graph models and analysing some new models using the methods of martingale theory. We deal with random graph models in which a new vertex is born at each step, and it is connected to some of the old vertices randomly. In many cases old vertices with larger degree get new edges with larger probability. This is the so called preferential attachment property. We included interactions of more than two vertices in the new models. In most of the already known models the evolution of the graph is based on the degrees of the old vertices, and it is not taken into account which groups of vertices get new edges in the same step.

In Chapter 2 (based on [2, 3]) we examined scale free random graph models, where the proportion of vertices of degree d tends to a constant c_d almost surely as the number of vertices goes to infinity, and $c_d \sim Kd^{-\gamma}$ holds as $d \to \infty$. If we would like to estimate c_d or γ, but the network is too large, we have to be careful with the sampling method. For example, if we consider the proportion of vertices of degree d in the neighbourhood of a given vertex (the degree is counted in the whole network), then the almost sure limit may differ from c_d. Moreover, the exponent of the new asymptotic degree distribution is less than γ. We present sufficient conditions that imply that the new exponent is $\gamma(\alpha - 1) + 1$, where α is the exponent of the regular growth of the number of selected vertices.

The background of this phenomenon may be the following. A fixed vertex gets new neighbours more and more rarely. Hence its neighbours are typically older than a typical vertex of the whole graph. Older vertices usually have larger degree, especially because of the preferential attachment property. Therefore the degrees are typically large in the neighbourhood of a fixed vertex (or more generally, in the set of selected vertices), and the exponent of the asymptotic degree distribution decreases.

In Chapter 3 (based on [1]) we examined a vertex at a fixed position in the random recursive tree with linear weights. We may consider the third child of the root (the first vertex) for example, in spite of the fact that we do not know in which step it is born. The model is the following. At each step a new vertex is born, and the probability that it is connected to a given old vertex of degree d is proportional to $d+\beta$, where $\beta > -1$ is a parameter of the model. The question is the asymptotic behaviour of the degree of a vertex in a fixed position. We proved that the degree of a fixed vertex and the degree of a vertex in a fixed position have the same asymptotic behaviour, as expected. The structure of the limiting random variable shows an embedded urn model.

In Chapter 4 (based on [4]) we introduced and analysed a model with dynamics based on interactions of three vertices. Every vertex, edge and triangle has a nonnegative weight, which corresponds to the number of interactions of this object. At each step three vertices interact; either a new one with two old vertices, or three old vertices. These may be chosen uniformly at random, or with probabilities proportional to the weight of the edge or that of the triangle. We proved that the degree is strongly concentrated for large weights. However, there is no deterministic connection between these quantities.

The methods include tools from the theory of discrete time martingales (to keep track of the changes in the number of vertices with a given degree) and of urn processes.

References

[1] Ágnes Backhausz, Limit distribution of degrees in random family trees, *Electronic Communications in Probability*, **16** (2011), 27–37.

[2] Ágnes Backhausz, Tamás F. Móri, Local degree distribution in scale free random graphs, *Electronic Journal of Probability*, **16** (2011), 1465–1488.

[3] Ágnes Backhausz, Local degree distributions: examples and counterexamples, *Periodica Mathematica Hungarica*, **63** (2011), 153–171.

[4] Ágnes Backhausz, Tamás F. Móri, A random graph model based on 3-interactions, *Annales Univ. Sci. Budapest., Sect. Comp.*, **36** (2012), 41–52.

ANNALES UNIV. SCI. BUDAPEST., **59** (2016), 141–146

CONVERGENCE ANALYSIS OF MARKOVIAN SYSTEMS
Abstract of Ph.D. Thesis

By
BALÁZS GERENCSÉR
SUPERVISOR: GÁBOR TUSNÁDY
(Defended June 17, 2013)

The understanding of random processes that surround us and the challenge to control them has been a constant goal of humanity for ages. Some desperately want to know how stock prices change, others simply whether it will rain or not tomorrow.

Mathematical tools have been created to allow crafting models and to get a precise and deep understanding of these processes. One of the most important concepts in this area is the class of Markov processes, which is a central concept in this PhD thesis. It is a family of processes that on one hand gives a strong structure allowing the development of efficient mathematical tools. On the other hand, it still offers enough freedom to be appropriate for a wide range of applications. In a simplified way a process is called a Markov process if the state of today of the process depends on the state of yesterday, but not on the states of before.

Markov processes have a clean and well developed theory, initiated by people among the brightest mathematicians of the early years of the 20th century, like Markov, Kolmogorov or Doob. However, it is still far from being complete. Very often strong assumptions are needed for the theory to stand. In this thesis, we take on the challenge to broaden the range of processes the theory can handle. The thesis tackles this goal in two directions. First, we consider processes where the evolution is more complex than a single old state becoming a single new state during a time step. Second, we explore a case where we break the symmetry of a random walk. We now give a detailed introduction to the two problems together with the new results we have obtained.

For the first part, the most direct way to understand the target process is by looking at the motivation: genetics. In the process of inheritance, the state of two individuals of one generation - the two parents - create a new state of the next generation. This structure has an added layer of complexity compared to the classical case when a single state evolves to another single state as time passes.

The process we mention above is a special instance of the so-called *Bi-Markov processes*: one for which two current states affect a single state of the next step. The current states being X_1, X_2 and the next state being X^+ we can properly define a Bi-Markov process by declaring the transition probabilities

$$P(X^+ = i \mid X_1 = j,\ X_2 = k).$$

We want to understand the limiting behavior of such processes after a long time. This is a very challenging mathematical problem, as it turns out that very different patterns can happen depending on the actual transition probabilities. Previous research showed examples when the distribution changes periodically, and simulations suggest that even chaotic behavior can be possible. There are also examples with a critical nature, where a very slight change in the parameters of the process induces a radical change for the long term behavior. More details on the possible cases and also on open questions leading onwards can be found in the work of Tusnády, Dénes, and Komlós. Usually we aim to prove convergence to a single distribution for specific processes. There are such results for some special cases, but this is still an open question in general. Our contribution to this part is proving the convergence of the inheritance process described below to a specific stationary distribution, and develop statistics in order to use it for modeling purposes. Note that there is an additional complexity in this case, as we allow the process to stop and not continue its evolution in time (corresponding to someone without descendants).

Let us turn back to the problem of genetics we mentioned earlier. To be more specific, we model the inheritance of congenital abnormalities. A proper understanding of such processes is very beneficial from an applied point of view: it would allow better prenatal support knowing certain risks for the child to be born. The contribution is threefold. First, we introduce a novel model to represent this inheritance process. Second, we develop the mathematical background to make the model solid and sound. Third, we compare our model to well estabilished genetic rules and real world data.

The details of the model we consider is as follows. Knowing that the copying of genes might have errors, we assume each individual has a certain number of erroneous, *mutant* genes. These mutant genes pose a double threat: the individual may develop the abnormality, and in a more severe case, it may cause to lose fertility. Each of the mutant genes randomly and indepentently cause these issues. In order to get to the next generation, two parents combine their genetic information. Therefore each of the mutant genes gets inherited with probability $1/2$. The child might get additional mutant genes caused by copying errors, external radiation, etc.

We give a quick overview of the proper mathematical formulation of the model. The probability of a single gene *not* causing the disorder is denoted by Δ, and the probability of *not* inhibiting fertility is ρ. For an individual with X mutant genes, we then get

$$P(\text{no disorder}) = \Delta^X, \qquad P(\text{fertile}) = \rho^X.$$

Given the number of mutant genes X_f, X_m of the parents, for the child X^+ we get

$$X^+ = Binom(X_f, 1/2) + Binom(X_m, 1/2) + Z.$$

Here *Binom* represents a Binomial random variable (translating the fact that each mutant gene is inherited only with probability $1/2$), while Z stands for the additional mutations and follows a Poisson distribution (corresponding to rarely happening events). In the thesis, we allow Δ, ρ, and the parameter of Z to depend on the sex of the individual, but for the purpose of this summary we omit this extra layer of complexity. We assume an infinite population to avoid unnecessary technical complications. At this point, we have a proper definition describing both the evolution of the population from generation to generation and the appearance of the abnormality. Knowing the biological model, let us now turn to the mathematical challenges.

During the mathematical analysis of the genetic model, we first prove that there exists a unique stationary distribution, and the process will approach this distribution no matter how it started. Also, we develop the formulas for the overall theoretical probabilities for the disorder and infertility. Moreover, we also estabilish the joint probabilities of certain relatives having some of these symptoms (e.g., a boy and his cousin both having the abnormality). The formulas are quite intimidating, however, it allows direct computation of the quantities mentioned.

This is very useful as it allows us to drive back to the motivating application and compare it with alternative models and real world data. We perform multiple tests to validate our model. There are theoretical genetic statements with which we match our model. We demonstrated that our model can obey the genetic rules if properly parametrized. We compare our model with another reference model, the Gaussian model, where a single real number decides whether the individual is healthy, malformed, or even infertile. We show that our model can exhibit even stronger correlation between relatives than what we see for the Gaussian model. We discover that the gender-dependent parameters allow a richer expression of abnormalities when there are gender differences. Finally, we got access to the excellent database of Hungarian birth data, which allowed us to fit our model to real world data on different congenital abnormalities. We are satisifed to see a very good fit for some malformations and a reasonable fit for others.

As it is normally the case, the end of this part of the thesis is not the end of the road. Our results lead to new interesting questions. Specifically, the convergence of Bi-Markov processes still have large unexplored territories. For the applied part, further comparisons to real world data and alternative models would be beneficial, and there are slight variations of our model which would be interesting to investigate.

In the second part, we work on the more classical processes of Markov chains. Imagine a system with different states, then a Markov chain is a random walk on these states, it jumps from one state to a neighbouring one at every time step. As this is a rather abstract and general concept, it allows to model a wide range of processes. We can think of the evolution of stock prices, or how the queue length changes at a service desk, or even how the wealth of someone evolves while playing at a casino. Often Markov chains are created on purpose, given the base system of states. For example, the famous Markov chain Monte Carlo method is one of the most used concept for numerical evaluation of certain quantities based on simulation. It is also regularly used to generate a random sample from complicated distributions. For example, think of generating a random folding of a certain protein.

One of the most natural question to ask for these applications is the performance of the Markov chain. Broadly speaking, we do not want the Markov chain to wander near the inital state but to spread out quickly and approach the stationary distribution as fast as possible. This performance is quantified by the *mixing time*, the time we need to wait for the Markov chain to be near its stationary distribution. Let us also formalize this concept:

$$t_{\text{mix}}(\epsilon) = \sup_{\sigma} \min\{k \;:\; \|\sigma P^k - \pi\|_{TV} < \epsilon\},$$

where ϵ is a tolerance parameter, σ runs through all possible starting distributions, σP^k represents the distribution after k steps, and π is the stationary distribution to approach. Distance is measured using the total variation norm. In other words, we check how much we have to wait to get to the ϵ neighborhood of the stationary distribution if we start from the worst possible starting point.

Often the situation we face is that the system of states are given, but we are free to construct a Markov chain on it. The goal is to come up with the best performing one, minimizing the mixing time. There is a natural technical condition that is often assumed for the Markov chain: *reversibility*. This is a type of symmetry, it means that between any two states, transition is equally likely to happen in the two directions. Formally,

$$\pi_i P_{ij} = \pi_j P_{ji}, \qquad \forall i, j,$$

Where P_{ab} stands for the probability of jumping to b if the current state is a.

But why should we assume reversibility? It does not come at all from the problem statement, it is only a technical condition that makes the analysis simple for those Markov chains. In this thesis, we take on the challenge to drop the reversibility condition, and find the best Markov chain among all.

We work on the fundamental case where the states and their connections are forming a cycle. For reversible Markov chains it is known and easy to show that for a cycle of n nodes, the best mixing time is of the order of n^2. Our first contribution is a structural statement: we show that any Markov chain can be viewed as a reversible one with a drift added along the cycle. By a drift we mean that transitions are made more likely around the cycle in one direction, and less likely in the other one. Our main theorem shows that even in this case, we cannot substantially decrease the mixing time, it will still be of the order of n^2. The proof to settle this is quite involved, as it can be seen in the thesis. We need very different tools when the drift is dominant and when it is weak.

We then move on to more complicated structures: we add some new allowed transitions, "long range edges" to the cycle. We initially tried to find the best new transitions to introduce, but could not see any structure in the optimization results. Therefore we switched to randomly chosen new transitions, which turned out to be an idea that performs very well. When the number of new transitions is high, we get one of the famous Small World Network models. This time we keep the edge count low, allowing the cycle to remain dominant. Here we do a double analysis. For reversible Markov chains, we develop lower and upper bounds on the mixing time that are very close to each other, resulting in a very good estimate. Afterwards, we investigate how a non-reversible Markov chain can perform. We demonstrate that there can be a huge increase in speed, making it worth considering this wider class of Markov chains. Knowing that such a speedup is possible inspires future research to exploit this phenomenon for other problems and let applications benefit from this increase in performance.

Overall, the message of this part is the following. Working with non-reversible chains, we might find much faster Markov chains than if we restrict ourselves only to reversible ones. However, this is not a universal magic tool, we see cases when there is no gain.

In the end, we see that various natural questions remain unsolved for Markov processes. This thesis does critical advances for some of these questions. On one hand we give a solid mathematical background for an inheritance process together with support for modeling and for fitting to real world data. On the other hand, we explore an extension of the well understood class of reversible Markov chains, showing that non-reversible chains may give a substantial increase in perfomance depending on the situation. We contributed to the great challenge of exploring Markov processes in two areas, providing a better understanding for the corresponding two classes of processes.

ANNALES UNIV. SCI. BUDAPEST., **59** (2016), 147–148

SOME GRAPH THEORETIC ASPECTS OF FINITE GEOMETRIES
Abstract of Ph.D. Thesis

By
TAMÁS HÉGER
SUPERVISORS: ANDRÁS GÁCS AND TAMÁS SZŐNYI
(Defended 14 April 2014)

The thesis treats problems in finite geometry that either try to answer graph theoretical questions, or originate from graph theory. However, the emphasis is on the finite geometrical viewpoint. In some problems we use the polynomial method (Rédei-polynomials, the Szőnyi–Weiner Lemma, and the combinatorial Nullstellensatz with multiplicities).

In Chapter 2 we collect results on multiple blocking sets, which are closely related to three of the four main problems. We show that through any essential point of a t-fold blocking set of size $t(q+1)+k$ in $\mathrm{PG}(2,q)$ there pass at least $q+1-k-t$ distinct t-secants. If q is not a square, no general construction for small t-fold blocking sets were known if $t \geq 2$. We construct a small double blocking set in $\mathrm{PG}(2,q)$ for each $q = p^h$, where p and $h \geq 3$ are odd.

The classical problem of (k,g)-cages asks for the order of the smallest k-regular graph of girth g. For $n = 3,4,6$, $(q+1,2n)$ cages are the incidence graphs of generalized n-gons of order q. In Chapter 3 we study the regular subgraphs of these extremal graphs. We characterize all $(q+1-t)$-regular induced subgraphs of the incidence graph of $\mathrm{PG}(2,q)$, $q = p^h$, if $t \leq p$ and $t \leq \sqrt{q}/2$ (roughly), and also construct some induced and non-induced regular subgraphs for $n = 4,6$.

In Chapter 4 we show that any small semi-resolving set for $\mathrm{PG}(2,q)$ can be extended to a double blocking set by adding at most two points to it. As a corollary, we obtain a new lower bound on the size of blocking semiovals.

The upper chromatic number of $\mathrm{PG}(2,q)$, considered as a hypergraph, is proved to equal $q^2+q+2-\tau_2(\mathrm{PG}(2,q))$ in Chapter 5, provided that $q > 256$ is a square or $q = p^h$, $p \geq 29$ and $h \geq 3$ is odd. In addition, we show that the coloring reaching this bound is essentially unique.

We treat the Zarankiewicz problem in Chapter 6, focusing on the case of C_4-free bipartite graphs. We provide exact values for several sets of the parameters, and also exhibit a table of exact values for small parameters.

ON THE INTERSECTION OF GEODESIC BALLS
Abstract of Ph.D. Thesis

By
MÁRTON HORVÁTH
SUPERVISOR: BALÁZS CSIKÓS
(Defended December 19, 2013)

The motivation of this dissertation was the Kneser–Poulsen conjecture, which claims that if some congruent balls of the Euclidean space are rearranged in such a way that the distances between the centers do not increase, then the volume of the union of the balls does not increase. Though there is no complete proof of this conjecture, many partial results are known. Based on these results, the conjecture seems to be true also for non-congruent balls, and also in the spherical and hyperbolic spaces. Thus, it is natural to ask whether the conjecture can be true in Riemannian manifolds more general than the constant curvature spaces.

If the conjecture is true in a Riemannian manifold, then the volume of the union of k geodesic balls can depend only on the distances between the centers and the radii of the balls. By the inclusion-exclusion principle, an analogous claim is true for the intersection of k balls. Call the latter property the $KP_k^=$ or KP_k property depending on whether the balls are supposed to be congruent or not.

Z. I. Szabó showed that the connected simply connected and complete harmonic manifolds have the KP_2 property. We proved that the other direction also true with the weaker $KP_2^=$ property, that is, if a connected, simply connected, and complete Riemannian manifold has the $KP_2^=$ property, then it is harmonic.

B. Csikós and D. Kunszenti-Kovács showed that if a connected simply connected and complete Riemannian manifold has the KP_3 property, then it is one of the simply connected spaces of constant curvature. We proved that it is enough to suppose the $KP_3^=$ property for this claim. Thus, the original Kneser–Poulsen conjecture cannot be generalized for spaces of non-constant curvature.

ANNALES UNIV. SCI. BUDAPEST., **59** (2016), 151–153

EFFICIENT NUMERICAL METHODS FOR ELLIPTIC AND PARABOLIC PARTIAL DIFFERENTIAL EQUATIONS
Abstract of Ph.D. Thesis

By
BALÁZS KOVÁCS
SUPERVISOR: JÁNOS KARÁTSON
(Defended December 18, 2015)

The topic of our research is centered around the numerical treatment of – linear and nonlinear, elliptic and parabolic – partial differential equations. The main focus being on the development of efficient numerical solution techniques for elliptic problems using a suitable iterative method. We also pay special attention to numerical implementations and experiments.

Although, there are various ways to numerically treat elliptic partial differential equations, cf. [2, 3], our main interest lies in developing *Newton-type methods* and *conjugate gradient methods*, within a finite element framework. We use an abstract theoretical setting, usually working in a Hilbert or Banach space framework, that is, we look for the solution of the corresponding weak problem. These abstract equations enable us to apply and develop iterative methods to solve such problems in an organized way. We usually work within a finite element framework.

Our investigations are twofold. Firstly, we study problems whose numerical solution is still considered as a challenge in the literature. We show robust convergence results for convection-dominated elliptic problems; [1]. The numerical study of the nonlinear Schrödinger equation leads to the extension of the results of the variable preconditioning Newton-like method developed by Karátson and Faragó to the case of complex Hilbert spaces; [6].

Secondly, we are interested in the computational performance of numerical methods: efficiency of the presented iterative methods [8], accuracy of a sharp upper global a posteriori error estimator developed by Karátson and Korotov [9]. Based on our presented numerical experiments we suggest and investigate possible improvements and discuss practical aspects as well.

Convection-dominated elliptic equations form an important class in the modelling of stationary convection–diffusion problems, and hence are the subject of intense research. A common point is that standard finite element discretizations are inadequate for such problems, see e.g. [2]. Hence, the finite element method is replaced by some stabilized version, such as the so-called streamline diffusion finite element method. The arising linear systems are generally solved by some preconditioned conjugate gradient method. The convergence of these iterations is also influenced by the convection-dominated character. We prove that the convergence using streamline diffusion preconditioning can in fact be robust. We give bounds independently of the diffusion parameter, and illustrate our theoretical results using various numerical experiments.

In [8] we studied the computational efficiency of three iterative methods: the gradient method, the Newton method and the quasi-Newton method, using nonlinear elliptic problems. A suitable version of quasi-Newton method is proposed and shown to be the least costly in most cases.

A common and important issue in numerics is the error estimation of the numerical methods. We aim to discuss a sharp upper global a posteriori error estimator due to Karátson and Korotov [5]. In [9] we check the numerical performance, demonstrate the efficiency and accuracy of this estimator using a second order elliptic quasilinear equation. The focus is on the technical and numerical aspects and on the components of the error estimation, especially on the adequate solution of the involved auxiliary problem, which enables *sharpness* in the sense that by the investment of ”computation time” the true error can be estimated by any desired precision.

In [6] we develop a damped Newton-like method, with a stepwise variable preconditioner, for solving complex nonlinear operator equations. This is a nontrivial complex Hilbert space extension of a preconditioning iterative method developed by Karátson and Faragó [4] for real Hilbert spaces. The motivation for this extension comes from the fully discrete numerical solution of the time-dependent nonlinear Schrödinger equation. We use the Rothe-method [10] for the numerical solution of the (complex) nonlinear Schrödinger equation: first applying a time discretization, then use our method to obtain the solution of the complex nonlinear elliptic boundary value problems on each time level. We show global convergence up to second order via a damped preconditioned iterative method, where the preconditioner is obtained by spectral equivalence. The result is provided by a number of preliminary lemmas and it follows the classical ideas of the usual convergence proofs for damped quasi-Newton methods.

We combined our separately developed codes into a solver for a very general class of parabolic partial differential equation system. The same code has been used later on in the recent article [7].

References

[1] O. Axelsson, J. Karátson and B. Kovács, Robust preconditioning estimates for convection-dominated elliptic problems via a streamline Poincaré–Friedrichs inequality, *SIAM J. on Num. Anal.*, **52** (2014), 2957–2976.

[2] H. C. Elman, D. Silvester and A. Wathen, *Finite elements and fast iterative solvers: with applications in incompressible fluid dynamics*, Oxford University Press, 2014.

[3] I. Faragó and J. Karátson, *Numerical solution of nonlinear elliptic problems via preconditioning operators: Theory and applications*, Volume 11, Nova Publishers, 2002.

[4] J. Karátson and I. Faragó, Variable preconditioning via quasi-Newton methods for nonlinear problems in Hilbert space, *SIAM J. on Num. Anal.*, **41** (2003), 1242–1262.

[5] J. Karátson and S. Korotov, Sharp upper global a posteriori error estimates for nonlinear elliptic variational problems, *Appl. Math.*, **54** (2009), 297–336.

[6] J. Karátson and B. Kovács, Variable preconditioning in complex Hilbert space and its application to the nonlinear Schrödinger equation, *CAMWA*, **65** (2013), 449–459.

[7] J. Karátson and B. Kovács, A parallel numerical solution approach for nonlinear parabolic systems arising in air pollution transport problems, in: A. Bátkai, P. Csomós, I. Faragó, A. Horányi & G. Szépszó, Springer series: Mathematics in Industry: *Mathematical Problems in Meteorological Modelling*, (24) 57–70, 2016.

[8] B. Kovács, A comparison of some efficient numerical methods for a nonlinear elliptic problem, *Central European J. of Math.*, **10** (2012), 217–230.

[9] B. Kovács, On the numerical performance of a sharp a posteriori error estimator for some nonlinear elliptic problems, *Appl. Math.*, **59** (2014), 489–508.

[10] E. H. Rothe, Zweidimensionale parabolische Randwertaufgaben als Grenzfall eindimensionaler Randwertaufgaben, *Mathematische Annalen*, **102** (1930), 650–670.

INDEX

ISSN 0524-9007

Address:

MATHEMATICAL INSTITUTE
EÖTVÖS LORÁND UNIVERSITY
ELTE TTK KARI KÖNYVTÁR
MATEMATIKAI TUDOMÁNYÁGI SZAKGYŰJTEMÉNY
PÁZMÁNY PÉTER SÉTÁNY 1/c
1117 BUDAPEST, HUNGARY

Műszaki szerkesztő:

Fried Katalin PhD

A kiadásért felelős: az Eötvös Loránd Tudományegyetem rektora
A kézirat a nyomdába érkezett: 2017. március.
Készült a LaTeX szedőprogram felhasználásával
az MSZ 5601–59 és 5602–55 szabványok szerint