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MIXED ALMOST CONTINUITY AND MIXED δ-CONTINUITY ON GENERALIZED TOPOLOGICAL SPACES

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Abstract. For two generalized topologies σ_1 and σ_2, the notions of mixed generalized open sets $\delta(\sigma_1, \sigma_2)$ and $r(\sigma_1, \sigma_2)$ were introduced in [4]. And for generalized topologies μ, σ_1 and σ_2, we introduced the notion of mixed weakly $(\mu, \sigma_1\sigma_2)$-continuous functions and studied some basic properties in [8]. The purpose of this paper is to introduce the other mixed continuous functions (mixed $\delta(\mu, \sigma_1\sigma_2)$-continuous functions, mixed almost $(\mu, \sigma_1\sigma_2)$-continuous functions) on generalized topological spaces by using mixed generalized open sets. We also investigate properties of such the mixed continuous functions. Finally, we investigate the relations among mixed weak $(\mu, \sigma_1\sigma_2)$-continuity, mixed almost $(\mu, \sigma_1\sigma_2)$-continuity and mixed $\delta(\mu, \sigma_1\sigma_2)$-continuity on generalized topological spaces.

1. Introduction

Let X be a nonempty set and μ be a collection of subsets of X. Then μ is called a *generalized topology* (briefly GT) [1] on X iff $\emptyset \in \mu$ and $G_i \in \mu$ for $i \in I \neq \emptyset$ implies $G = \cup_{i \in I} G_i \in \mu$. We call the pair (X, μ) a *generalized topological space* (briefly GTS) on X. The elements of μ are called μ*-open* sets and the complements are called μ*-closed* sets. The generalized-closure of a subset A of X, denoted by $c_\mu A$, is the intersection of generalized closed sets including A. And the interior of A, denoted by

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$i_\mu A$, the union of generalized open sets included in A. Let μ and σ be generalized topologies on X and Y, respectively. Then a function $f\colon X \to Y$ is said to be

(1) *(μ,σ)-continuous* [1] if $G \in \sigma$ implies that $f^{-1}(G) \in \mu$;
(2) *weakly (μ,σ)-continuous* [5] if for each $x \in X$ and each σ-open set V containing $f(x)$, there exists a μ-open set U containing x such that $f(U) \subseteq c_\sigma V$;
(3) *almost (μ,σ)-continuous* [6] if if for each $x \in X$ and each σ-open set V containing $f(x)$, there exists a μ-open set U containing x such that $f(U) \subseteq i_\sigma c_\sigma(V)$.

THEOREM 1.1 ([1]). *Let (X,μ) be generalized topological space. Then*

(1) $c_\mu A = X - i_\mu(X-A)$;
(2) $i_\mu A = X - c_\mu(X-A)$.

Let μ be a GT on a nonempty set X. Let us define $\delta(\mu) \subseteq 2^X$ by $A \in$ $\in \delta(\mu)$ iff $A \subseteq X$ and, if $x \in A$, then there is a μ-closed set Q such that $x \in i_\mu Q \subseteq A$ [2]. We know that $\delta(\mu)$ is a GT such that $\delta(\mu) \subseteq \mu$. And the nonempty element of $\delta(\mu)$ coincide with the unions of $r(\mu)$-open sets.

Let σ_1 and σ_2 be two GT's on a nonempty set X and $A \subseteq X$. A is said to be *$r(\sigma_1,\sigma_2)$-open* (resp., *$r(\sigma_1,\sigma_2)$-closed*) [3,4] if $A = i_{\sigma_1}(c_{\sigma_2}A)$ (resp., $A = c_{\sigma_1} i_{\sigma_2} A$).

Let us define $\delta(\sigma_1,\sigma_2) \subseteq 2^X$ by $A \in \delta(\sigma_1,\sigma_2)$ iff $A \subseteq X$ and, if $x \in A$, then there is a σ_2-closed set Q such that $x \in i_{\sigma_1} Q \subseteq A$ [3,4].

We call an element A in $\delta(\sigma_1,\sigma_2)$ a *$\delta(\sigma_1,\sigma_2)$-open set*. A is called a *$\delta(\sigma_1,\sigma_2)$-closed set* if the complement of A is $\delta(\sigma_1,\sigma_2)$-open. Then

$$c_{\delta(\sigma_1,\sigma_2)} A = \cap\{F \subseteq X\colon A \subseteq F, X - F \in \delta(\sigma_1,\sigma_2)\};$$

$$i_{\delta(\sigma_1,\sigma_2)} A = \cup\{G \subseteq X\colon G \subseteq A, A \in \delta(\sigma_1,\sigma_2)\}.$$

THEOREM 1.2 ([4]). *Let σ_1 and σ_2 be two GT's on a nonempty set X. Then*

(1) $\delta(\sigma_1,\sigma_2)$ is a GT on X.
(2) $\delta(\sigma_1,\sigma_2) \subseteq \sigma_1$.
(3) The nonempty element of $\delta(\sigma_1,\sigma_2)$ coincide with the unions of $r(\sigma_1,\sigma_2)$-open sets.
(4) $x \in c_{\delta(\sigma_1,\sigma_2)} A$ iff $A \cap R \neq \emptyset$ for every $r(\sigma_1,\sigma_2)$-open set R containing x.

By using the mixed generalized open sets defined in [3], we introduced the notion of mixed weakly $(\mu, \sigma_1\sigma_2)$-continuous functions [8] as the following: Let μ be a GT on a nonempty set X, and let σ_1, σ_2 be two GT's on a nonempty set Y. Then a function $f\colon X \to Y$ is said to be *mixed weakly* $(\mu, \sigma_1\sigma_2)$*-continuous* at $x \in X$ if for each σ_1-open set V containing $f(x)$, there exists a μ-open set U containing x such that $f(U) \subseteq c_{\sigma_2}V$. Then f is said to be *mixed weakly* $(\mu, \sigma_1\sigma_2)$*-continuous* if it is mixed weakly $(\mu, \sigma_1\sigma_2)$-continuous at every point of X.

The purpose of this paper is to introduce the other mixed continuous functions (mixed $\delta(\mu, \sigma_1\sigma_2)$-continuous functions, mixed almost $(\mu, \sigma_1\sigma_2)$-continuous functions) on generalized topological spaces by using mixed generalized open sets. We also investigate the characterizations of such the mixed continuous functions. Finally, we investigate the relations among mixed weak $(\mu, \sigma_1\sigma_2)$-continuity, mixed almost $(\mu, \sigma_1\sigma_2)$-continuity and mixed $\delta(\mu, \sigma_1\sigma_2)$-continuity on generalized topological spaces.

2. Mixed almost-continuity

In this section, we introduce the notion of mixed almost $(\mu, \sigma_1\sigma_2)$-continuity and study properties of it. In particular, we investigate its properties by using (σ_1, σ_2)-semiopen sets, (σ_1, σ_2)-preopen sets and (σ_1, σ_2)-β'-open sets.

Definition 2.1. Let μ be a GT on a nonempty set X, and let σ_1, σ_2 be two GT's on a nonempty set Y. Then a function $f\colon X \to Y$ is said to be *mixed almost* $(\mu, \sigma_1\sigma_2)$*-continuous* at $x \in X$ if for each σ_1-open set V containing $f(x)$, there exists a μ-open set U containing x such that $f(U) \subseteq$ $\subseteq i_{\sigma_1}c_{\sigma_2}V$. Then f is said to be *mixed almost* $(\mu, \sigma_1\sigma_2)$*-continuous* if it is mixed almost $(\mu, \sigma_1\sigma_2)$-continuous at every point of X.

Remark 2.2. Let μ be a GT on a nonempty set X, and let σ_1, σ_2 be two GT's on a nonempty set Y. If $\sigma_1 = \sigma_2$, then a mixed almost $(\mu, \sigma_1\sigma_1)$-continuous function $f\colon X \to Y$ is just an almost (μ, σ_1)-continuous function.

THEOREM 2.3. *Let $f\colon X \to Y$ be a function, let μ be a GT on a nonempty set X, and let σ_1, σ_2 be two GT's on a nonempty set Y. If f is (μ, σ_1)-continuous, then it is mixed almost $(\mu, \sigma_1\sigma_2)$-continuous.*

PROOF. For any σ_1-open set U, it is obviously $U \subseteq i_{\sigma_1}c_{\sigma_2}U$. From the fact, a (μ, σ_1)-continuous function f is also mixed almost $(\mu, \sigma_1\sigma_2)$-continuous. ∎

Obviously we have the following implications.

(μ, σ_1)-conti. $\to$ mixed almost $(\mu, \sigma_1\sigma_2)$-conti. $\to$ mixed weak $(\mu, \sigma_1\sigma_2)$-conti.

In the next two examples, we can show that the converses are not be true in general.

EXAMPLE 2.4. Let $X = \{1, 2, 3\}$ and let us consider a generalized topology μ on X defined as follows

$$\mu = \{\emptyset, \{1, 3\}\}.$$

Let $Y = \{a, b, c, d\}$; let σ_1 and σ_2 on Y be two generalized topologies defined as follows

$$\sigma_1 = \{\emptyset, \{a\}, \{a, b\}\}; \ \ \sigma_2 = \{\emptyset, \{d\}\}.$$

Consider a function $f\colon X \to Y$ defined as $f(1) = a$, $f(2) = f(3) = c$. Then for nonempty σ_1-open sets $\{a\}$ and $\{a, b\}$, $c_{\sigma_2}(\{a\}) = c_{\sigma_2}(\{a, b\}) = \{a, b, c\}$ and $i_{\sigma_1}c_{\sigma_2}(\{a\}) = i_{\sigma_1}c_{\sigma_2}(\{a, b\}) = i_{\sigma_1}(\{a, b, c\}) = \{a, b\}$. Moreover, for the only nonempty μ-open set $\{1, 3\}$, $f(\{1, 3\}) = \{a, c\}$. From these facts, we know that f is mixed weakly $(\mu_1, \sigma_1\sigma_2)$-continuous but it is not mixed almost $(\mu_1, \sigma_1\sigma_2)$-continuous.

EXAMPLE 2.5. Let $X = \{1, 2, 3\}$ and a generalized topology $\mu = \{\emptyset, \{1, 2\}\}$.

Let $Y = \{a, b, c, d\}$, and consider two generalized topologies σ_1 and σ_2 as the following:

$$\sigma_1 = \{\emptyset, \{a, b\}, \{b, c\}, \{a, b, c\}, Y\};$$
$$\sigma_2 = \{\emptyset, \{a, c\}, \{b, c\}, \{a, b, c\}\}.$$

Let us consider a function $f\colon X \to Y$ defined as $f(1) = a, f(2) = b$, $f(3) = d$. Then f is obvious mixed almost $(\mu, \sigma_1\sigma_2)$-continuous but it is not (μ, σ_1)-continuous.

THEOREM 2.6. *Let $f\colon X \to Y$ be a function, let μ be a GT on a nonempty set X, and let σ_1, σ_2 be two GT's on a nonempty set Y. Then the following are equivalent:*

(1) f is mixed almost $(\mu, \sigma_1\sigma_2)$-continuous.
(2) $f^{-1}(V) \subseteq i_\mu f^{-1}(i_{\sigma_1}c_{\sigma_2}V)$ for every σ_1-open subset V in Y.
(3) $c_\mu f^{-1}(c_{\sigma_1}i_{\sigma_2}F) \subseteq f^{-1}(F)$ for every σ_1-closed subset F in Y.
(4) For every $r(\sigma_1, \sigma_2)$-closed subset F in Y, $f^{-1}(F)$ is μ-closed.
(5) For every $r(\sigma_1, \sigma_2)$-open subset V in Y, $f^{-1}(V)$ is μ-open.

PROOF. (1) $\Rightarrow$ (2) Let V be a σ_1-open set in Y and $x \in f^{-1}(V)$. By hypothesis, there exists a μ-open set U of X containing x such that $f(U) \subseteq i_{\sigma_1}c_{\sigma_2}V$. This implies $x \in i_\mu f^{-1}(i_{\sigma_1}c_{\sigma_2}V)$. Hence $f^{-1}(V) \subseteq$ $\subseteq i_\mu f^{-1}(i_{\sigma_1}c_{\sigma_2}V)$.

(2) $\Rightarrow$ (3) Let F be a σ_1-closed set in Y. From Theorem 1.1, it follows

$$\begin{aligned} f^{-1}(Y-F) \subseteq i_\mu f^{-1}(i_{\sigma_1}c_{\sigma_2}(Y-F)) &= \\ = i_\mu f^{-1}(Y - c_{\sigma_1}i_{\sigma_2}F) &= \\ = X - c_\mu(f^{-1}(c_{\sigma_1}i_{\sigma_2}F)&. \end{aligned}$$

Hence $c_\mu f^{-1}(c_{\sigma_1}i_{\sigma_2}F) \subseteq f^{-1}(F)$.

(3) $\Rightarrow$ (4) Let F be any $r(\sigma_1, \sigma_2)$-closed set of Y. Since $c_{\sigma_1}i_{\sigma_2}F = F$, by (3), $c_\mu f^{-1}(F) \subseteq f^{-1}(F)$ and so $f^{-1}(F) = c_\mu f^{-1}(F)$. It implies $f^{-1}(F)$ is μ-closed.

(4) $\Rightarrow$ (5) Obvious.

(5) $\Rightarrow$ (1) For each $x \in X$, let V be any $r(\sigma_1, \sigma_2)$-open set of Y containing $f(x)$. By (5), $x \in f^{-1}(V) = i_\mu f^{-1}(V) = i_\mu f^{-1}(i_{\sigma_1}c_{\sigma_2}V)$. Since $x \in i_\mu f^{-1}(i_{\sigma_1}c_{\sigma_2}V)$, there exists a μ-open set U containing x such that $U \subseteq f^{-1}(i_{\sigma_1}c_{\sigma_2}V)$. This implies $f(U) \subseteq i_{\sigma_1}c_{\sigma_2}V$. Hence f is mixed almost $(\mu, \sigma_1\sigma_2)$-continuous. ∎

We recall the notions of mixed generalized open sets introduced in [3,4]. Let σ_1 and σ_2 be two GT's on a nonempty set X and $A \subseteq X$. Then A is said to be (σ_1, σ_2)-*semiopen* [3] (respectively, (σ_1, σ_2)-*preopen* [3], (σ_1, σ_2)-β'-*open*) [4] if $A \subseteq c_{\sigma_2}i_{\sigma_1}A$ (respectively, $A \subseteq i_{\sigma_1}c_{\sigma_2}A$, $A \subseteq$ $\subseteq c_{\sigma_2}i_{\sigma_1}c_{\sigma_2}A$.

The complement of (σ_1,σ_2)-semiopen (respectively, (σ_1,σ_2)-preopen, (σ_1,σ_2)-β'-open) is called (σ_1,σ_2)-*semiclosed* (respectively, (σ_1,σ_2)-*preclosed*, (σ_1,σ_2)-β'-*closed*).

THEOREM 2.7. *Let $f\colon X\to Y$ be a function, let μ be a GT on a nonempty set X, and let σ_1,σ_2 be two GT's on a nonempty set Y. Then the following are equivalent:*

(1) f is mixed almost $(\mu,\sigma_1\sigma_2)$-continuous.
(2) $f^{-1}(i_{\sigma_1}c_{\sigma_2}G)\subseteq i_\mu f^{-1}(i_{\sigma_1}c_{\sigma_2}G)$ for every (σ_1,σ_2)-β'-open set G of Y.
(3) $f^{-1}(i_{\sigma_1}c_{\sigma_2}G)\subseteq i_\mu f^{-1}(i_{\sigma_1}c_{\sigma_2}G)$ for every (σ_1,σ_2)-semiopen set G of Y.
(4) $f^{-1}(G)\subseteq i_\mu f^{-1}(i_{\sigma_1}c_{\sigma_2}G)$ for every (σ_1,σ_2)-preopen set V of Y.

PROOF. $(1)\Rightarrow(2)$ Let G be any (σ_1,σ_2)-β'-open set. Since every $i_{\sigma_1}c_{\sigma_2}G$ is $r(\sigma_1,\sigma_2)$-open and σ_1-open, by Theorem 2.6 (2),

$$f^{-1}(i_{\sigma_1}c_{\sigma_2}G)\subseteq i_\mu f^{-1}(i_{\sigma_1}c_{\sigma_2}i_{\sigma_1}c_{\sigma_2}G)=i_\mu f^{-1}(i_{\sigma_1}c_{\sigma_2}G).$$

Thus $f^{-1}(i_{\sigma_1}c_{\sigma_2}G)\subseteq i_\mu f^{-1}(i_{\sigma_1}c_{\sigma_2}G)$.

$(2)\Rightarrow(3)$ Since every (σ_1,σ_2)-semiopen set is (σ_1,σ_2)-β'-open, it is obvious.

$(3)\Rightarrow(4)$ Let G be any (σ_1,σ_2)-preopen set. Then $c_{\sigma_2}G\subseteq c_{\sigma_2}i_{\sigma_1}c_{\sigma_2}G$ and so $c_{\sigma_2}G$ is (σ_1,σ_2)-semiopen. From (3) and definition of (σ_1,σ_2)-preopen sets, it follows $f^{-1}(G)\subseteq f^{-1}(i_{\sigma_1}c_{\sigma_2}G)=f^{-1}(i_{\sigma_1}c_{\sigma_2}c_{\sigma_2}G)\subseteq$ $\subseteq i_\mu f^{-1}(i_{\sigma_1}c_{\sigma_2}c_{\sigma_2}G)=i_\mu f^{-1}(i_{\sigma_1}c_{\sigma_2}G)$.

$(4)\Rightarrow(1)$ Let V be any $r(\sigma_1,\sigma_2)$-open set of Y. Then V also is (σ_1,σ_2)-preopen and so by (4), $f^{-1}(V)\subseteq i_\mu f^{-1}(i_{\sigma_1}c_{\sigma_2}V)=i_\mu f^{-1}(V)$. This implies $f^{-1}(V)$ is μ-open and f is mixed almost $(\mu,\sigma_1\sigma_2)$-continuous. ∎

LEMMA 2.8. *Let σ_1 and σ_2 be two GT's on a nonempty set X and $A\subseteq X$. Then the following things hold.*

(1) $i_{\delta(\sigma_1,\sigma_2)}A\subseteq A\subseteq c_{\delta(\sigma_1,\sigma_2)}A$.
(2) $i_{\delta(\sigma_1,\sigma_2)}A=X-c_{\delta(\sigma_1,\sigma_2)}(X-A)$.
(3) $c_{\delta(\sigma_1,\sigma_2)}A=X-i_{\delta(\sigma_1,\sigma_2)}(X-A)$.
(4) $c_{\delta(\sigma_1,\sigma_2)}A$ is σ_1-closed.
(5) $i_{\delta(\sigma_1,\sigma_2)}A$ is σ_1-open.

Proof. By Theorem 1.2, we know that $\delta(\sigma_1, \sigma_2)$ is a GT contained σ_1, and so the things are easily obtained. ∎

Lemma 2.9 ([4]). *If Q is σ_2-closed, then $i_{\sigma_1}Q$ is $r(\sigma_1, \sigma_2)$-open.*

Theorem 2.10. *Let σ_1 and σ_2 be two GT's on a nonempty set X and $A \subseteq X$. Then $x \in i_{\delta(\sigma_1,\sigma_2)}A$ iff there exists an $r(\sigma_1, \sigma_2)$-open set R containing x such that $R \subseteq A$.*

Proof. Suppose that $x \in i_{\delta(\sigma_1,\sigma_2)}A$. Then there exists a $\delta(\sigma_1, \sigma_2)$-open set V containing x. From the definition of $\delta(\sigma_1, \sigma_2)$-open set, there exists a σ_2-closed set Q such that $x \in i_{\sigma_1}Q \subseteq A$. By Lemma 2.9, $i_{\sigma_1}Q$ is an $r(\sigma_1, \sigma_2)$-open set and so we have the condition.

The converse is easily obtained because every $r(\sigma_1, \sigma_2)$-open set is $\delta(\sigma_1, \sigma_2)$-open. ∎

Corollary 2.11. *Let μ be a GT on a nonempty set X and $A \subseteq X$. Then $x \in i_{\delta(\mu)}A$ iff there exists an $r(\mu)$-open set R containing x such that $R \subseteq A$.*

Theorem 2.12. *Let $f\colon X \to Y$ be a function, let μ be a GT on a nonempty set X, and let σ_1, σ_2 be two GT's on a nonempty set Y. Then the following are equivalent:*

(1) f is mixed almost $(\mu, \sigma_1\sigma_2)$-continuous.
(2) $c_\mu f^{-1}\left(c_{\sigma_1}i_{\sigma_2}c_{\delta(\sigma_1,\sigma_2)}A\right) \subseteq f^{-1}\left(c_{\delta(\sigma_1,\sigma_2)}A\right)$ for every A of Y.
(3) $c_\mu f^{-1}\left(c_{\sigma_1}i_{\sigma_2}c_{\sigma_1}A\right) \subseteq f^{-1}\left(c_{\delta(\sigma_1,\sigma_2)}A\right)$ for every set A of Y.

Proof. (1) $\Rightarrow$ (2) For $G \subseteq Y$, $c_{\delta(\sigma_1,\sigma_2)}G$ is σ_1-closed by Lemma 2.8 (4). Now it follows $c_\mu f^{-1}\left(c_{\sigma_1}i_{\sigma_2}c_{\delta(\sigma_1,\sigma_2)}A\right) \subseteq f^{-1}\left(c_{\delta(\sigma_1,\sigma_2)}A\right)$ from Theorem 2.6 (3).

(2) $\Rightarrow$ (3) Since $c_{\sigma_1}A \subseteq c_{\delta(\sigma_1,\sigma_2)}G$, it is obvious.

(3) $\Rightarrow$ (1) Let F be any $r(\sigma_1, \sigma_2)$-closed set. Then by Theorem 1.2 (3) and Lemma 2.8 (4), F is $\delta(\sigma_1, \sigma_2)$-closed and $F = c_{\delta(\sigma_1,\sigma_2)}F = c_{\sigma_1}F$. From hypothesis, $c_\mu f^{-1}\left(c_{\sigma_1}i_{\sigma_2}F\right) \subseteq f^{-1}(F)$. Hence by Theorem 2.6 (3), f is mixed almost $(\mu, \sigma_1\sigma_2)$-continuous. ∎

Theorem 2.13. *Let $f\colon X \to Y$ be a function, let μ be a GT on a nonempty set X, and let σ_1, σ_2 be two GT's on a nonempty set Y. Then the following are equivalent:*

(1) f is mixed almost $(\mu, \sigma_1\sigma_2)$-continuous.
(2) $f(c_\mu A) \subseteq c_{\delta(\sigma_1,\sigma_2)}f(A)$ for every A of X.

(3) $f^{-1}(K)$ is μ-closed for every $\delta(\sigma_1,\sigma_2)$-closed K of Y.
(4) $f^{-1}(V)$ is μ-open for every $\delta(\sigma_1,\sigma_2)$-open V of Y.
(5) $f^{-1}\left(i_{\delta(\sigma_1,\sigma_2)}B\right) \subseteq i_\mu f^{-1}(B)$ for every B of Y.
(6) $c_\mu f^{-1}(B) \subseteq f^{-1}\left(c_{\delta(\sigma_1,\sigma_2)}B\right)$ for every B of Y.

Proof. $(1) \Rightarrow (2)$ For $A \subseteq Y$, let $y \notin c_{\delta(\sigma_1,\sigma_2)}f(A)$ where $f(x) = y$. Then by Theorem 1.2 (4), there exists an $r(\sigma_1,\sigma_2)$-open set R of y satisfying $R \cap f(A) = \emptyset$. Moreover, by Theorem 2.6 (5), $f^{-1}(R)$ is μ-open. Since $f^{-1}(R) \cap A = \emptyset$ and $f^{-1}(R)$ is μ-open, we have $x \notin c_\mu A$ and $f(x) \notin f(c_\mu A)$ for $x \in f^{-1}(R)$. Therefore $y \notin f(c_\mu A)$.

$(2) \Rightarrow (3)$ Let K be a $\delta(\sigma_1,\sigma_2)$-closed set of Y. By (2), $f\left(c_\mu f^{-1}(K)\right) \subseteq$ $\subseteq c_{\delta(\sigma_1,\sigma_2)}f\left(f^{-1}(K)\right) \subseteq c_{\delta(\sigma_1,\sigma_2)}K = K$. This implies $c_\mu f^{-1}(K) \subseteq f^{-1}(K)$ and so $f^{-1}(K)$ is μ-closed.

$(3) \Rightarrow (4)$ Obvious.

$(4) \Rightarrow (5)$ For $B \subseteq Y$, $i_{\delta(\sigma_1,\sigma_2)}B$ is a $\delta(\sigma_1,\sigma_2)$-open set. So by (4),

$$f^{-1}(i_{\delta(\sigma_1,\sigma_2)}B) = i_\mu f^{-1}(i_{\delta(\sigma_1,\sigma_2)}B) \subseteq i_\mu f^{-1}(B).$$

$(5) \Rightarrow (6)$ Obvious.

$(6) \Rightarrow (1)$ Let $B \subseteq Y$. Then by Lemma 2.8 (4), $c_{\delta_{(\sigma_1,\sigma_2)}}B$ is a σ_1-closed set and $c_{\sigma_1}i_{\sigma_2}c_{\delta(\sigma_1,\sigma_2)}B \subseteq c_{\delta(\sigma_1,\sigma_2)}B$. Thus from (6) and $c_{\delta(\sigma_1,\sigma_2)}c_{\delta(\sigma_1,\sigma_2)}B = c_{\delta(\sigma_1,\sigma_2)}B$, it follows

$$\begin{aligned} c_\mu f^{-1}\left(c_{\sigma_1}i_{\sigma_2}c_{\delta(\sigma_1,\sigma_2)}B\right) &\subseteq f^{-1}\left(c_{\delta(\sigma_1,\sigma_2)}c_{\sigma_1}i_{\sigma_2}c_{\delta(\sigma_1,\sigma_2)}B\right) \subseteq \\ &\subseteq f^{-1}\left(c_{\delta(\sigma_1,\sigma_2)}c_{\delta(\sigma_1,\sigma_2)}B\right) = \\ &= f^{-1}\left(c_{\delta(\sigma_1,\sigma_2)}B\right). \end{aligned}$$

Hence by Theorem 2.12 (2), f is mixed almost $(\mu,\sigma_1\sigma_2)$-continuous. ∎

We recall that: Let σ_1 and σ_2 be two GT's on a nonempty set X and $A \subseteq X$. Then X is said to be (σ_1,σ_2)-*regular* [8] on X if for $x \in X$ and σ_1-closed set F with $x \notin F$, there exist $U \in \sigma_1$, $V \in \sigma_2$ such that $x \in U$, $F \subseteq V$ and $U \cap V = \emptyset$.

Let σ_1 and σ_2 be two GT's on a nonempty set X. Then X is (σ_1,σ_2)-regular if and only if for $x \in X$ and a σ_1-open set U containing x, there is a σ_1-open set V containing x such that $x \in V \subseteq c_{\sigma_2}V \subseteq U$ (Theorem 3.15 in [8]).

THEOREM 2.14. *Let σ_1 and σ_2 be two GT's on a nonempty set X. If X is (σ_1, σ_2)-regular, every σ_1-open set is $\delta(\sigma_1, \sigma_2)$-open, that is, $\delta(\sigma_1, \sigma_2) = \sigma_1$.*

PROOF. Let A be a σ_1-open set in X. For each $x \in A$, from the above fact, there exists a σ_1-open set V such that $x \in V \subseteq c_{\sigma_2} V \subseteq A$. Then it implies $x \in V \subseteq i_{\sigma_1} c_{\sigma_2} V \subseteq A$. Since $c_{\sigma_2} V$ is σ_2-closed, by Lemma 2.9, $i_{\sigma_1} c_{\sigma_2} V$ is $r(\sigma_1, \sigma_2)$-open. Hence from Theorem 1.2 (3), A is $\delta(\sigma_1, \sigma_2)$-open. ∎

THEOREM 2.15. *Let $f \colon X \to Y$ be a function, let μ be a GT on a nonempty set X, and let σ_1, σ_2 be two GT's on a nonempty set Y. If Y is (σ_1, σ_2)-regular, then the following are equivalent:*

(1) f is (μ, σ_1)-continuous.
(2) f is mixed almost $(\mu, \sigma_1\sigma_2)$-continuous.

PROOF. By Theorem 2.14, every σ_1-open set is $r(\sigma_1, \sigma_2)$-open and so by Theorem 2.6, a mixed almost $(\mu, \sigma_1\sigma_2)$-continuous function is (μ, σ_1)-continuous. Finally, the theorem is proved from Theorem 2.3. ∎

THEOREM 2.16 ([8]). *Let $f \colon X \to Y$ be a function, let μ be a GT on a nonempty set X, and let σ_1, σ_2 be two GT's on a nonempty set Y. If Y is (σ_1, σ_2)-regular, then the following are equivalent:*

(1) f is (μ, σ_1)-continuous.
(2) f is mixed weakly $(\mu, \sigma_1\sigma_2)$-continuous.

COROLLARY 2.17. *Let $f \colon X \to Y$ be a function, let μ be a GT on a nonempty set X, and let σ_1, σ_2 be two GT's on a nonempty set Y. If Y is (σ_1, σ_2)-regular, then the following are equivalent:*

(1) f is (μ, σ_1)-continuous.
(2) f is mixed almost $(\mu, \sigma_1\sigma_2)$-continuous.
(3) f is mixed weakly $(\mu, \sigma_1\sigma_2)$-continuous.

3. Mixed δ-continuity

DEFINITION 3.1. Let μ be a GT on a nonempty set X, and let σ_1, σ_2 be two GT's on a nonempty set Y. Then a function $f \colon X \to Y$ is said to be *mixed $\delta(\mu, \sigma_1\sigma_2)$-continuous* at x if for each σ_1-open set V containing $f(x)$, there exists a μ-open set U containing x such that $f(i_\mu c_\mu U) \subseteq i_{\sigma_1} c_{\sigma_2} V$. And the function f is said to be *mixed* $\delta(\mu, \sigma_1\sigma_2)$-*continuous* if it is mixed $\delta(\mu, \sigma_1\sigma_2)$-continuous at every point of X.

From the above definition, obviously we know that every mixed $\delta(\mu, \sigma_1\sigma_1)$-continuous function is mixed almost $(\mu, \sigma_1\sigma_1)$-continuous. Furthermore, if $\sigma_1 = \sigma_2$, then a mixed $\delta(\mu, \sigma_1\sigma_1)$-continuous function $f\colon X \to Y$ is $\delta(\mu, \sigma_1)$-continuous.

In the next example, we show that every mixed almost $(\mu, \sigma_1\sigma_1)$-continuous function is not mixed $\delta(\mu, \sigma_1\sigma_1)$-continuous in general.

Example 3.2. Let $X = \{1, 2, 3, 4\}$ and let us consider a generalized topology μ on X defined as follows

$$\mu = \{\emptyset,\ \{1, 2\},\ \{1, 2, 3\}\}.$$

Let $Y = \{a, b, c\}$ and let σ_1 and σ_2 on Y be two generalized topologies defined as follows

$$\sigma_1 = \{\emptyset,\ \{a\},\ \{a, c\}\};\ \ \sigma_2 = \{\emptyset,\ \{c\},\ \{a, c\}\}.$$

Consider a function $f\colon X \to Y$ defined as $f(1) = f(2) = a$, $f(3) = c$, $f(4) = d$. Then f is obviously mixed almost $(\mu, \sigma_1\sigma_2)$-continuous. For $x = 1$ in X, $U_1 = \{1, 2\}$ and $U_2 = \{1, 2, 3\}$ are all μ-open sets containing 1. Note that $i_\mu c_\mu U_1 = i_\mu c_\mu U_1 = \{1, 2, 3\}$. Take a σ_1-open set $V = \{a\}$ containing $f(1) = a$; then $i_{\sigma_1} c_{\sigma_2} V = \{a\}$. Since $f(i_\mu c_\mu U_1) = f(i_\mu c_\mu U_2) = \{a, c\}$ for a σ_1-open set $V = \{a\}$ containing $f(1)$, there exists no a μ-open set U containing $x = 1$ satisfying $f(i_\mu c_\mu U) \subseteq i_{\sigma_1} c_{\sigma_2} V$. This implies that f is not mixed $\delta(\mu, \sigma_1\sigma_2)$-continuous at $x = 1$ and so f is not mixed $\delta(\mu, \sigma_1\sigma_2)$-continuous.

In the next example, we can show that there is no any relation between (μ, σ_1)-continuity and mixed $\delta(\mu, \sigma_1\sigma_2)$-continuity.

Example 3.3.

(1) Let $X = \{1, 2, 3, 4\}$ and let us consider a generalized topology μ on X defined as follows

$$\mu = \{\emptyset, \{2\}, \{2, 3\}, \{1, 2, 3\}\}.$$

Let $Y = \{a, b, c, d\}$ and let σ_1 and σ_2 on Y be two generalized topologies defined as follows

$$\sigma_1 = \{\emptyset, \{a\}, \{b\}, \{a, b\}\};\ \ \sigma_2 = \{\emptyset, \{c, d\}\}.$$

For any μ-open set U, $c_\mu U = X$ and $i_\mu c_\mu U = \{1,2,3\}$. And for any σ_1-open set V, $c_{\sigma_2} V = \{a,b\}$ and $i_{\sigma_1} c_{\sigma_2} V = i_{\sigma_1}(\{a,b\}) = \{a,b\}$.

Consider a function $f\colon X \to Y$ defined as $f(1) = f(2) = a$, $f(3) = b$, $f(4) = d$. Then $f(\{1,2,3\}) = \{a,b\}$, it implies f is mixed δ $(\mu, \sigma_1\sigma_2)$-continuous. But for a σ_1-open set $\{b\}$, $f^{-1}(\{b\}) = \{3\}$ is not σ_1-open and so f is not (μ, σ_1)-continuous.

(2) Let $X = \{1,2,3\}$ and let us consider a generalized topology μ on X defined as follows

$$\mu = \{\emptyset, \{1,2\}\}.$$

Let $Y = \{a,b,c,d\}$ and let two generalized topologies

$$\sigma_1 = \{\emptyset, \{b,d\}\}; \quad \sigma_2 = \{\emptyset, \{a,c\}, \{a,b,c\}\}.$$

Consider a function $f\colon X \to Y$ defined as $f(1) = f(2) = a$, $f(3) = b$, $f(4) = d$. Then f is mixed $\delta(\mu, \sigma_1\sigma_2)$-continuous but it is not (μ, σ_1)-continuous.

Remark 3.4. From Theorem 2.3 and Example 3.3, we have the following diagram:

$$\begin{array}{ccccc} & & (\mu,\sigma_2)\text{-continuity} & & \\ & & \downarrow & & \\ \text{mixed} & & \text{mixed almost} & & \text{mixed weak} \\ \delta(\mu,\sigma_1\sigma_2)\text{-conti.} & \to & (\mu,\sigma_1\sigma_2)\text{-conti.} & \to & (\mu,\sigma_1\sigma_2)\text{-conti.} \end{array}$$

We recall that the notion and properties induced in [2]: Let μ be a GT on a nonempty set X. Let us define $\delta(\mu) \subseteq 2^X$ by $A \in \delta(\mu)$ iff $A \subseteq X$ and, if $x \in A$, then there is a μ-closed set Q such that $x \in i_\mu Q \subseteq A$ [2]. We know that $\delta(\mu)$ is a GT such that $\delta(\mu) \subseteq \mu$. And the nonempty element of $\delta(\mu)$ coincide with the unions of $r(\mu)$-open sets.

Theorem 3.5. *Let $f\colon X \to Y$ be a function, let μ be a GT on a nonempty set X, and let σ_1, σ_2 be two GT's on a nonempty set Y. Then the following are equivalent:*

(1) f is mixed $\delta(\mu, \sigma_1\sigma_2)$-continuous at $x \in X$.
(2) $x \in i_\delta f^{-1}(i_{\sigma_1} c_{\sigma_2} V)$ for every σ_1-open subset V containing $f(x)$.
(3) $x \in i_\delta f^{-1}(V)$ for every $r(\sigma_1, \sigma_2)$-open subset V containing $f(x)$.
(4) For $r(\sigma_1, \sigma_2)$-open set V containing $f(x)$, there exists an $r(\mu)$-open set U containing x such that $f(U) \subseteq V$.

PROOF. (1) $\Rightarrow$ (2) For any σ_1-open subset V containing $f(x)$, there exists a μ-open subset U of X containing x such that $f(i_\mu c_\mu U) \subseteq i_{\sigma_1} c_{\sigma_2} V$. Since $x \in U \subseteq i_\mu c_\mu U \subseteq f^{-1}(i_{\sigma_1} c_{\sigma_2} V)$ and $i_\mu c_\mu U$ is $r(\mu)$-open. Since every $r(\mu)$-open set is δ-open, we have $x \in i_\delta f^{-1}(i_{\sigma_1} c_{\sigma_2} V)$.

(2) $\Rightarrow$ (3) For any $r(\sigma_1, \sigma_2)$-open subset V containing $f(x)$, since $V = i_{\sigma_1} c_{\sigma_2} V$, by (2), we have $x \in i_\delta f^{-1}(V)$.

(3) $\Rightarrow$ (4) Obvious.

(4) $\Rightarrow$ (1) For each $x \in X$ and for any σ_1-open subset V containing $f(x)$, $i_{\sigma_1} c_{\sigma_2} V$ is an $r(\sigma_1, \sigma_2)$-open set such that $f(x) \in V \subseteq i_{\sigma_1} c_{\sigma_2} V$. So by (4), there exists an $r(\mu)$-open set U containing x such that $f(U) \subseteq$ $\subseteq i_{\sigma_1} c_{\sigma_2} V$. Since every $r(\mu)$-open set is μ-open and $U = i_\mu c_\mu U$, f is mixed $\delta(\mu, \sigma_1\sigma_2)$-continuous at $x \in X$. ∎

THEOREM 3.6. *Let $f\colon X \to Y$ be a function, let μ be a GT on a nonempty set X, and let σ_1, σ_2 be two GT's on a nonempty set Y. Then the following are equivalent:*

(1) f is mixed $\delta(\mu, \sigma_1\sigma_2)$-continuous.
(2) For each $x \in X$ and each $r(\sigma_1, \sigma_2)$-open set V containing $f(x)$, there exists an $r(\mu)$-open set U containing x such that $f(U) \subseteq V$.
(3) $f\left(c_{\delta(\mu)} A\right) \subseteq c_{\delta(\sigma_1,\sigma_2)} f(A)$ for every $A \subseteq X$.
(4) $c_{\delta(\mu)} f^{-1}(B) \subseteq f^{-1}\left(c_{\delta(\sigma_1,\sigma_2)} B\right)$ for every $B \subseteq Y$.
(5) For every $r(\sigma_1, \sigma_2)$-closed subset F in Y, $f^{-1}(F)$ is $\delta(\mu)$-closed.
(6) For every $r(\sigma_1, \sigma_2)$-open subset V in Y, $f^{-1}(V)$ is $\delta(\mu)$-open.

PROOF. (1) $\Rightarrow$ (2) For each $x \in X$, let V be an $r(\sigma_1, \sigma_2)$-open set containing $f(x)$; then $V = i_{\sigma_1} c_{\sigma_2} V$ and V is also σ_1-open. By (1), there exists a μ-open set W containing x such that $f(i_\mu c_\mu W) \subseteq i_{\sigma_1} c_{\sigma_2} V$. Put $U = i_\mu c_\mu W$. Then U is an $r(\mu)$-open set satisfying $f(U) \subseteq V$.

(2) $\Rightarrow$ (3) Let $A \subseteq X$. Now we show that $f(x) \in c_{\delta(\sigma_1,\sigma_2)} f(A)$ for each $x \in c_{\delta(\mu)} A$. In order to use Theorem 1.2 (4), let V be an $r(\sigma_1, \sigma_2)$-open set V containing $f(x)$. Then by (2), there exists an $r(\mu)$-open set U containing x such that $f(U) \subseteq V$. From $x \in c_{\delta(\mu)} A$, it follows $U \cap A \neq \emptyset$. This implies $V \cap f(A) \neq \emptyset$ because of $f(U) \subseteq V$. Hence $f(x) \in c_{\delta(\sigma_1,\sigma_2)} f(A)$ by Theorem 1.2.

(3) $\Rightarrow$ (4) Obvious.

(4) $\Rightarrow$ (5) For each $r(\sigma_1, \sigma_2)$-closed subset V in Y, that is, $c_{\delta(\sigma_1,\sigma_2)} V = V$. Since V is $\delta(\sigma_1, \sigma_2)$-closed, from (4), it is obtained $c_{\delta(\mu)} f^{-1}(V) \subseteq f^{-1}(V)$. So $f^{-1}(V)$ is $\delta(\mu)$-closed.

(5) $\Rightarrow$ (6) Obvious.

(6) $\Rightarrow$ (1) Let $x \in X$ and V be any σ_1-open set containing $f(x)$. Then obviously $i_{\sigma_1} c_{\sigma_2} V$ is $r(\sigma_1, \sigma_2)$-open. By hypothesis, $f^{-1}(i_{\sigma_1} c_{\sigma_2} V)$ is $\delta(\mu)$-open and so there exists an $r(\mu)$-open set U containing x such that $x \in U = i_\mu c_\mu U \subseteq f^{-1}(i_{\sigma_1} c_{\sigma_2} V)$. This implies $f^{-1}(i_\mu c_\mu U) \subseteq (i_{\sigma_1} c_{\sigma_2} V)$, and hence f is mixed $\delta(\mu, \sigma_1\sigma_2)$-continuous. ∎

Let (X, μ) be a generalized topological space and $\mathcal{M}_\mu = \{M \subseteq X : M \in \mu\}$. Then X is said to be *relative G-regular* (simply, *G-regular*) [7] on $\mathcal{M}_\mu$ if for $x \in \mathcal{M}_\mu$ and μ-closed set F with $x \notin F$, there exist $U, V \in \mu$ such that $x \in U$, $F \cap \mathcal{M}_\mu \subseteq V$ and $U \cap V = \emptyset$.

THEOREM 3.7 ([7]). *Let (X, μ) be a generalized topological space. If X is G-regular, every μ-open set is δ-open.*

THEOREM 3.8. *Let $f\colon X \to Y$ be a function, let μ be a GT on a nonempty set X, and let σ_1, σ_2 be two GT's on a nonempty set Y. If X is G-regular, then the following are equivalent:*

(1) f is mixed almost $(\mu, \sigma_1\sigma_2)$-continuous.

(2) f is mixed $\delta(\mu, \sigma_1\sigma_2)$-continuous.

PROOF. It follows from Theorem 2.6 and Theorem 3.6. ∎

COROLLARY 3.9. *Let $f\colon X \to Y$ be a function, let μ be a GT on a nonempty set X, and let σ_1, σ_2 be two GT's on a nonempty set Y. If X is G-regular and if f is (μ, σ_1)-continuous, then f is mixed $\delta(\mu, \sigma_1\sigma_2)$-continuous.*

PROOF. It follows from Theorem 2.3 and Theorem 3.8. ∎

THEOREM 3.10. *Let $f\colon X \to Y$ be a function, let μ be a GT on a nonempty set X, and let σ_1, σ_2 be two GT's on a nonempty set Y. If Y is (σ_1, σ_2)-regular and X is G-regular, then the following are equivalent:*

(1) f is (μ, σ_1)-continuous.

(2) f is mixed almost $(\mu, \sigma_1\sigma_2)$-continuous.

(3) f is mixed weakly $(\mu, \sigma_1\sigma_2)$-continuous.

(4) f is mixed δ $(\mu, \sigma_1\sigma_2)$-continuous.

PROOF. It is obtained from Corollary 2.17 and Theorem 3.8. ∎

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ON Ω_S-I-CLOSED SETS AND A DECOMPOSITION OF CONTINUITY

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Abstract. In this paper, we introduce the notions of Ω_s-I-closed sets and Ω_s-I-continuous functions. We obtain a characterization of $\star$-extremally disconnected spaces. A decomposition of continuity is also established. Finally, we obtain a characterization of Hayashi-Samuels spaces.

1. Introduction

In this paper, a space always means a topological space. For a space (X,τ) and a subset A of X, we denote by $\operatorname{int}_\tau(A)$ and $\operatorname{cl}_\tau(A)$ (or simply, by $\operatorname{int}(A)$ and $\operatorname{cl}(A)$ if there is no confusion) the interior and the closure of A in (X,τ), respectively. For a set B, we denote by $\mathcal{P}(B)$ the power set of B.

A nonempty collection I of subsets on a topological space (X,τ) is called an *ideal* on X if it satisfies the following two conditions: (1) if $A\in I$ and $B\subseteq A$, then $B\in I$; (2) If $A\in I$ and $B\in I$, then $A\cup B\in I$.

An *ideal topological space* [5] is a topological space (X,τ) with an ideal I on X and is denoted by (X,τ,I). For a subset $A\subseteq X$, $A^*(I)=\{x\in X\colon U\cap\cap A\notin I$ for every $U\in\tau(x)\}$ is called the *local function* [5] of A with respect to I and τ. We simply write A^* instead of $A^*(I)$ if there is no confusion. It has been proved that $\operatorname{cl}^*_I(A)=A\cup A^*$ defines a Kuratowski closure operator on X. We denote by $\tau^*(I)$ the topology generated by the closure operator $\operatorname{cl}^*_I()$. We simply write $\operatorname{cl}^*(A)$ instead of $\operatorname{cl}^*_I(A)$ if there is no confusion. One can prove that τ^* is finer than τ. An ideal topological space (X,τ,I) is said to be a *Hayashi–Samuels* space if it satisfies $X=X^*$, or equivalently, $\tau\cap I=\emptyset$.

At the end of this section, let us recall some definitions:

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DEFINITION 1.1. A subset A of an ideal topological space (X,τ,I) is said to be

(1) [1] pre-I-open if $A \subseteq \operatorname{int}(\operatorname{cl}^*(A))$;
(2) [3] semi-I-open if $A \subseteq \operatorname{cl}^*(\operatorname{int}(A))$;
(3) [3] α-I-open if $A \subseteq \operatorname{int}(\operatorname{cl}^*(\operatorname{int}(A)))$;
(4) [6] regular-I-closed if $A = (\operatorname{int}(A))^*$;
(5) [6] an A_I-set if $A = U \cap V$, where U is open and V is regular-I-closed.

DEFINITION 1.2. [8] A subset A of a space X is said to be Ω_s-closed if $\operatorname{scl}(A) \subseteq$ $\subseteq \operatorname{int}(\operatorname{cl}(U))$ whenever $A \subseteq U$ and U is semi-open in X.

DEFINITION 1.3. A function $f\colon (X,\tau,I) \to (Y,\sigma)$ is said to be:

(1) α-I-continuous [3], if for every $V \in \sigma, f^{-1}(V)$ is α-I-open in (X,τ,I)
(2) semi-I-continuous [3], if for every $V \in \sigma, f^{-1}(V)$ is semi-I-open in (X,τ,I)
(3) A_I-continuous [6], if for every $V \in \sigma, f^{-1}(V)$ is an A_I-set in (X,τ,I)
(4) pre-I-continuous[1]) if for every $V \in \sigma, f^{-1}(V)$ is pre-I-open in (X,τ,I).

2. Ω_s-I-closed sets and some properties

It was proved in [4] that a subset A of a space X is Ω_s-closed if $A \subseteq$ $\subseteq \operatorname{int}(\operatorname{cl}(U))$ whenever $A \subseteq U$ and U is semi-open in X. Every pre-open set is Ω_s-closed. It is natural to have the following definition.

DEFINITION 2.1. A subset A of an ideal topological space is said to be *Ω_s-I-closed* if $A \subseteq \operatorname{int}(\operatorname{cl}^*(U))$ whenever $A \subseteq U$ and U is semi-I-open.

EXAMPLE 2.2. Ω_s-I-closed sets and Ω_s-closed sets are independent concepts.

PROOF. Let $X = \{\langle x,y\rangle : (x-\frac{1}{2})^2 + (y-\frac{1}{2})^2 \le \frac{1}{4}\}$ with usual metric topology τ. We denote by $B(p,r)$ the closed disc with center p and radius r.

We set

$$I_1 = \mathcal{P}(X), \qquad A = \left\{ \left\langle \frac{1}{2} + \frac{1}{n}, \frac{1}{2} \right\rangle : n = 3,4,5,\dots \right\},$$

$$U = X \setminus A, \quad C = \bigcup_{i=3}^{i=\infty} B\left(\left\langle \frac{1}{2} + \frac{1}{i}, \frac{1}{2} \right\rangle, \frac{1}{2i(i+1)} \right) \quad \text{and} \quad I_2 = \mathcal{P}(C).$$

Clearly, I_1 and I_2 are two ideals on X.

(1) Since $\mathrm{cl}^*_{I_1}(\mathrm{int}(V)) = \mathrm{int}(V)$ for every subset V of X, then semi-I_1-open sets in (X, τ, I_1) coincide with open sets in (X, τ). Thus, every subset of X is Ω_s-I_1-closed. Meanwhile, every closed disc (except X itself) of X is not Ω_s-closed. So, Ω_s-I_1-closed sets need not to be Ω_s-closed.

(2) Since the subset A is nowhere dense, then U is α-open. Thus, U is Ω_s-closed. Now, we show that U is not Ω_s-I_2-closed. Clearly, $\mathrm{int}(U) = X \setminus (A \cup \cup \langle \frac{1}{2}, \frac{1}{2} \rangle)$ and $\mathrm{cl}^*_{I_2}(\mathrm{int}(U)) = U$. Thus, U is semi-I_2-open. But $\mathrm{int}(\mathrm{cl}^*_{I_2}(U)) = U \setminus \{\langle \frac{1}{2}, \frac{1}{2} \rangle\}$. So, $U \nsubseteq \mathrm{int}(\mathrm{cl}^*_{I_2}(U))$, which implies U is not Ω_s-I_2-closed. ∎

REMARK. When $I = \{\emptyset\}$, Ω_s-I-closed sets and Ω_s-closed sets are coincide in the ideal topological space (X, τ, I).

It is clear from the definitions that every pre-I-open set is a Ω_s-I-closed. But the converse need not to be true.

EXAMPLE 2.3. There exists a Ω_s-I-closed set which is not pre-I-open.

PROOF. It was pointed out in Example 2.2 that every subset of X is Ω_s-I_1-closed. But the subset $B = \{\langle x, y \rangle : (x - \frac{1}{2})^2 + (y - \frac{1}{2})^2 \leq \frac{1}{16}\}$ is clearly not pre-I_1-open. ∎

In [4], the authors showed that the intersection of an open set and a Ω_s-closed set need not to be Ω_s-closed set. Since open sets are clearly Ω_s-I-closed. So it implies that the intersection of two Ω_s-I-closed sets need not to be Ω_s-I-closed. But the following theorem is clear.

THEOREM 2.4. *The arbitrary union of Ω_s-I-closed sets is Ω_s-I-closed.*

In [2], Ekici and Noiri introduced the terms of *$\star$-extremally disconnected ideal topological spaces*. An ideal topological space is said to be $\star$-extremally disconnected if $\mathrm{cl}^*(A)$ is open for every open set A of X. We have the following theorem.

THEOREM 2.5. *For an ideal topological space (X, τ, I), the following conditions are equivalent:*

(1) *(X, τ, I) is $\star$-extremally disconnected;*
(2) *For each $x \in X$, $\{x\}$ is Ω_s-I-closed.*

PROOF. (1) $\Rightarrow$ (2) Suppose U is an arbitrary semi-I-open set containing x. Then $\{x\} \subseteq U \subseteq \mathrm{cl}^*(\mathrm{int}(U))$. Since (X, τ, I) is $\star$-extremally disconnected, we have $\mathrm{cl}^*(\mathrm{int}(U)) = \mathrm{int}(\mathrm{cl}^*(\mathrm{int}(U)))$. Clearly,

$$\{x\} \subseteq \mathrm{cl}^*(\mathrm{int}(U)) = \mathrm{int}(\mathrm{cl}^*(\mathrm{int}(U))) \subseteq \mathrm{int}(\mathrm{cl}^*(U)),$$

which implies $\{x\}$ is Ω_s-I-closed.

(2) $\Rightarrow$ (1) Suppose (X,τ,I) is not $\star$-extremally disconnected. Then there exists an open set U such that $\mathrm{cl}^*(U)$ is not open. So, there exists an $x \in \mathrm{cl}^*(U) \setminus \mathrm{int}(\mathrm{cl}^*(U))$. We set $V = \mathrm{int}(\mathrm{cl}^*(U)) \cup \{x\}$. Then V is semi-I-open and $\{x\} \subseteq V$. Clearly, $x \notin \mathrm{int}(\mathrm{cl}^*(U)) = \mathrm{int}(\mathrm{cl}^*(V))$, which implies $\{x\}$ is not Ω_s-I-closed. ∎

3. Decomposition properties of sets and mappings

In this section, we investigate some decomposition properties of sets and mappings by Ω_s-I-closed sets.

LEMMA 3.1 ([3]). *Let A be a subset of an ideal topological space (X,τ,I). Then A is α-I-open if and only if it is semi-I-open and pre-I-open.*

THEOREM 3.2. *Let A be a subset of an ideal topological space (X,τ,I). Then A is α-I-open if and only if it is semi-I-open and Ω_s-I-closed.*

PROOF. "$\Leftarrow$" Suppose A is semi-I-open and Ω_s-I-closed. Then A is a semi-I-open set contains itself. So $A \subseteq \mathrm{int}(\mathrm{cl}^*(A))$, since A is Ω_s-I-closed. This implies A is pre-I-open. Then A is α-I-open by Lemma 3.1.

"$\Rightarrow$" We only need to show that A is Ω_s-I-closed. Let U be an arbitrary semi-I-open set with $A \subseteq U$. Then $A \subseteq \mathrm{int}(\mathrm{cl}^*(\mathrm{int}(A))) \subseteq \mathrm{int}(\mathrm{cl}^*(\mathrm{int}(U))) \subseteq$ $\subseteq \mathrm{int}(\mathrm{cl}^*(U))$, which implies A is Ω_s-I-closed. ∎

It is clear that every pre-I-open set is Ω_s-I-closed. By Example 2.3, we know that a Ω_s-I-closed set need not to be pre-I-open. So, Theorem 3.2 is a slight improvement of Lemma 3.1.

LEMMA 3.3 ([5]). *Let (X,τ,I) be an ideal topological space, and let $A \subseteq X$ be a subset. If $U \in \tau$, then $U \cap A^* = U \cap (U \cap A)^* \subseteq (U \cap A)^*$.*

LEMMA 3.4. *Let A be a semi-I-open set of an ideal topological space (X,τ,I) and U an open set. Then $U \cap A$ is semi-I-open.*

PROOF. Since A is semi-I-open, then $A \subseteq \mathrm{cl}^*(\mathrm{int}(A))$. We have

$$U \cap A \subseteq U \cap \mathrm{cl}^*(\mathrm{int}(A)) = U \cap [\mathrm{int}(A) \cup (\mathrm{int}(A))^*] =$$
$$= [U \cap \mathrm{int}(A)] \cup [U \cap (\mathrm{int}(A))^*] \subseteq \mathrm{int}(U \cap A) \cup [\mathrm{int}(U \cap A)]^* =$$
$$= \mathrm{cl}^*(\mathrm{int}(U \cap A)),$$

which shows that $U \cap A$ is semi-I-open. ∎

By Definition 1.1, an subset A of an ideal topological space is regular-I-closed if $A = (\mathrm{int}(A))^*$. Then $A = (\mathrm{int}(A))^* = (\mathrm{int}(A))^* \cup \mathrm{int}(A)$, i.e., $A =$

$\mathrm{cl}^*(\mathrm{int}(A))$. This shows that every regular-I-closed set is semi-I-open. So we have the following corollary.

COROLLARY 3.5. *Every A_I-set is a semi-I-open set.*

Renuka and Sivaraj also got the same conclusion as Corollary 3.5 by a derect proof in [9].

In [6], Keskin and Noiri showed that a subset A of a Hayashi-Samuels space (X, τ, I) is open if and only if it is pre-I-open and an A_I-set. We have the following theorem, which is a slight improvement.

THEOREM 3.6. *Let A be a subset of a Hayashi-Samuels space (X, τ, I). Then A is open if and only if it is Ω_s-I-closed and an A_I-set.*

PROOF. "⇒" Clearly.

"⇐" Suppose A is Ω_s-I-closed and an A_I-set. Then A is α-I-open by Corollary 3.5 and Theorem 3.2. So, A is both α-I-open and an A_I-set, which implies A is open. ∎

DEFINITION 3.7. A function $f\colon (X, \tau, I) \to (Y, \sigma)$ is said to be Ω_s-I-continuous if for every $V \in \sigma, f^{-1}(V)$ is Ω_s-I-closed in (X, τ, I).

The following two theorems can be easily established:

THEOREM 3.8. *Let (X, τ, I) be an ideal topological space and (Y, σ) a topological space. For a function $f\colon (X, \tau, I) \to (Y, \sigma)$, the following conditions are equivalent:*

(1) *f is α-I-continuous;*
(2) *f is semi-I-continuous and Ω_s-I-continuous.*

THEOREM 3.9. *Let (X, τ, I) be a Hayashi-Samuels space and (Y, σ) a topological space. For a function $f\colon (X, \tau, I) \to (Y, \sigma)$, the following conditions are equivalent:*

(1) *f is continuous;*
(2) *f is A_I-continuous and Ω_s-I-continuous.*

EXAMPLE 3.10. A Ω_s-I-continuous function need not to be pre-I-continuous.

PROOF. Let $X = \left\{(x, y)\colon (x - \frac{1}{2})^2 + (y - \frac{1}{2})^2 \leq \frac{1}{4}\right\}$ with usual metric topology τ. Let $I = \mathcal{P}(X)$ be a ideal on X. So, (X, τ, I) is the same ideal topological topological space as (X, τ, I_1) in Example 2.2. We set $B = \{(x, y)\colon (x - \frac{1}{2})^2 + (y - \frac{1}{2})^2 \leq \frac{1}{16}\}$. Let $Y = \{a, b\}$, $\sigma = \{\emptyset, a, Y\}$. B is defined the same as in Example 2.3. Define $f\colon (X, \tau, I) \to (Y, \sigma)$ as: $f(x) = a$, when $x \in B$; $f(x) = b$, when $x \in X \setminus B$. By Example 2.2 and Example 2.3, we know that B is a Ω_s-I-closed set without being pre-I-open. So, clearly, f is a Ω_s-I-continuous function whicn is not pre-I-continuous. ∎

4. A characterization of Hayashi-Samuels spaces

We know that if A is regular-I-closed, then $A = (\mathrm{int}(A))^* = \mathrm{cl}^*(\mathrm{int}(A))$. But, conversely, if $A = \mathrm{cl}^*(\mathrm{int}(A))$, A need need not to be regular-I-closed. Actually, the subset U in Example 2.2 is a subset which satisfies $\mathrm{cl}^*_{I_2}(\mathrm{int}(U)) = U$ without being regular-I_2-closed. We end this paper by the following theorem.

THEOREM 4.1. *For an ideal topological space (X, τ, I), the following conditions are equivalent:*

(1) *Every subset A which satisfies $A = \mathrm{cl}^*(\mathrm{int}(A))$ is regular-I-closed;*
(2) *(X, τ, I) is a Hayashi–Samuels space.*

PROOF. (1) $\Rightarrow$ (2) Suppose (X, τ, I) is not a Hayashi–Samuels space. Then there exists an open set $U \in I$. Clearly, $U^* = \emptyset$ and $\mathrm{cl}^*(\mathrm{int}(U)) = U$. But $(\mathrm{int}(U))^* = \emptyset \neq U$, i.e., U is not regular-I-closed.

(2) $\Rightarrow$ (1) Suppose (X, τ, I) is a Hayashi–Samuels space and A is a subset satisfies $A = \mathrm{cl}^*(\mathrm{int}(A))$. Clearly, for each open set U, we have $U^* = U$. Thus, $A = \mathrm{cl}^*(\mathrm{int}(A)) = \mathrm{int}(A) \cup (\mathrm{int}(A))^* = (\mathrm{int}(A))^*$. This implies A is regular-I-closed. ∎

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ZOLTÁN SEBESTYÉN IS 70 YEARS OLD

By
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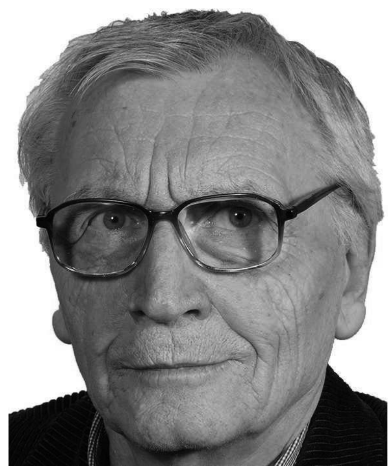

Zoltán Sebestyén celebrated his 70th birthday on the 29th of December 2013. This volume, from colleagues and students, is to pay tribute for his contributions as scientist, teacher, and friend.

Zoltán Sebestyén was born on 29 December 1943 in Sömjénmihályfa. As a high school student he won a gold medal in the International Mathematical Olympiad in 1962. He graduated from the Eötvös Loránd University in 1967 with a Diploma in Mathematics. He defended his PhD in 1971 and he became a Candidate of Mathematical Sciences under the supervision of Béla Szőkefalvi-Nagy in Szeged. Since 1967, Zoli has been working at the Department of Applied Analysis and Computational Mathematics (formerly Analysis II, and then Applied Analysis) of ELTE. Shortly after he had become a doctor of the Hungarian Academy of Sciences in 1984 he was elected (in 1985) head of that Department which was being led by Zoli for more than twenty years, until 2006. At present he is professor emeritus at Institute of Mathematics, Faculty of Sciences, ELTE. He is a member of the János Bolyai Mathematical Society and the AMS.

Zoli has played an important role in the teaching of functional analysis. He has been teaching Professor of this course (and many others) for decades. He had two students (Dénes Petz and Zoltán Magyar) who received the degree of Candidate of Sciences and five who obtained PhD degree (Vilmos Komornik, Vilmos Prokaj, Máté Matolcsi, Bálint Farkas, and Zsigmond Tarcsay). Currently he has two PhD students (Balázs Takács and Tamás Titkos).

He has made fundamental contributions to functional analysis containing relevant results on operator moment problems, operator extensions, and representation theory of C^*-algebras. One of his most famous results is that the axioms of C^*-algebras are not independent. Recently, Zoli has begun to investigate an interesting and fruitful topic, a generalized Lebesgue decomposition theory. Quite recently Zoli was awarded the Béla Szőkefalvi-Nagy Medal.

Besides of being an outstanding mathematician it is a well known fact that Zoli is a first-class amateur football player as well. No doubt, it would be worth introducing, along the lines of the Erdős Number, the concept of Sebestyén Number, describing the "football game distance" of a mathematician and Zoli.

Sebestyén's 70th anniversary conference was held at the Institute of Mathematics, Faculty of Sciences, ELTE on the 10th of January 2014. The invited speakers were not only the experts of the field of functional analysis, they were also friends or former students of Zoli. Speakers came from Finland, France, the Netherlands, Poland, Romania, and colleagues from all leading Hungarian universities. This volume contains papers from the speakers of the conference.

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WIENER'S LEMMA AND THE JACOBS–DE LEEUW–GLICKSBERG DECOMPOSITION*

By
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Dedicated to Prof. Zoltán Sebestyén on the occasion of his 70th birthday with gratitude for having taught me Hilbert spaces.

Abstract. A result of J. A. Goldstein [8] extends Wiener's lemma from harmonic analysis about continuous and atomic measures to Hilbert space contractions. In this short note we discuss the relation between that lemma of Wiener and the Jacobs–de Leeuw–Glicksberg decomposition from operator theory. By using this decomposition we prove Goldstein's result in a way that is closer to the elementary proof of Wiener's lemma, and in a slightly stronger form at that. The presented proof appears to be new and worthy of mentioning.

A classical result of Norbert Wiener characterizes continuous Borel measures μ on the unit circle S^1 via their Fourier coefficients

$$\hat{\mu}(n) := \int_{S^1} z^n \,\mathrm{d}\mu(z) \quad (n \in \mathbb{Z}).$$

More precisely we have:

THEOREM 1 (Wiener's lemma). *Let μ be a complex Borel measure on the unit circle S^1. Then*

$$\lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |\hat{\mu}(n)|^2 = \sum_{\lambda\in S^1} |\mu\{\lambda\}|^2 .$$

Moreover, we have

$$\lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |\hat{\mu}(n+k)|^2 = \sum_{\lambda\in S^1} |\mu\{\lambda\}|^2$$

where the limit is uniform in $k \in \mathbb{Z}$.

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The proof is elementary, in fact a simple application of Fubini's theorem and Lebesgue's dominated convergence theorem. Yet the result plays a fundamental role in the harmonic analysis and spectral theory of unitary operators, hence also in ergodic theory. Most notably by using Wiener's lemma one can prove a special case of a general operator theoretic result, the Jacobs–de Leeuw–Glicksberg decomposition, to be presented below. So it is not surprising that Wiener's lemma has a purely operator theoretic content related to this decomposition. The discussion of this intrinsic connection is the subject of this short note. The next result was discovered by J. A. Goldstein[1] [8], who named it *generalized Wiener theorem* and gave a proof based on the ergodic theorem (we also refer to [1], [2], [7], [9] for more background information on this approach in the time-continuous case, and for the history of the lemma). We give a different, new proof which is based on the Jacobs–Glicksberg–de Leeuw decomposition and lies nearer to the proof of the original lemma. Our proof immediately yields slightly more information about the convergence: It is uniform along weak orbits of the adjoint operator (however, with some careful modification of Goldstein's proof, this can be obtained also from his approach).

THEOREM 2 (Abstract Wiener lemma). *Let T be a contraction on a Hilbert space H, and for $\lambda \in \mathbb{C}$ denote by P_λ the orthogonal projection onto $\ker(\lambda I - T)$. Then for every $x, y \in H$*

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} |(T^n x \mid y)|^2 = \sum_{\lambda\in S^1} |(P_\lambda x \mid P_\lambda y)|^2 = \sum_{\lambda\in S^1} |(P_\lambda x \mid y)|^2 .$$

Moreover,

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \left|(T^n x \mid T^{*k} y)\right|^2 = \sum_{\lambda\in S^1} |(P_\lambda x \mid P_\lambda y)|^2 = \sum_{\lambda\in S^1} |(P_\lambda x \mid y)|^2 .$$

uniformly for $k = 0, 1, 2, \dots$.

The proof can be based on Wiener's lemma in its original version and on the spectral theorem for unitary operators, or on the ergodic theorem and tensor product constructions, see Goldstein [8]. However, we choose to take a different route relying on the previously mentioned splitting due to Jacobs [10], de Leeuw, Glicksberg [3, 4]. And at the end of this note we sketch how their result (in this special case) can be proved by using Wiener's lemma, thus indicating the alternative proof via the spectral theorem.

THEOREM 3 (Jacobs, de Leeuw, Glicksberg). *Let T be a contraction on a Hilbert space. Then one has the following orthogonal decomposition into closed T and T^*-invariant subspaces:*

$$H = H_{\mathrm{s}} \oplus H_{\mathrm{r}},$$

[1] In the previous version of the present note this reference was not known to the author and he thanks Catalin Badea and Markus Haase for calling his attention to the article [8]. This helped also to trace some other relevant references in the time-continuous case.

where H_{s} *is the closed linear hull of eigenvectors to unimodular eigenvalues, i.e.,*

$$H_{\mathrm{s}} = \overline{\mathrm{lin}}\{x\colon \text{there is } \lambda \in S^1 \text{ with } Tx = \lambda x\}$$

and

$$H_{\mathrm{r}} = \{x\colon T^{n_k}x \to 0 \text{ as } k \to \infty \text{ weakly for a some subsequence } (n_k)_{k\in\mathbb{N}} \subseteq \mathbb{N}\}.$$

The operator T *is unitary on* H_{s} *with strongly compact orbits. For every* $x \in H$ *one has that* $x \in H_{\mathrm{r}}$ *if and only if*

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \left|(T^n x \mid y)\right| = 0 \quad \text{for each } y \in H.$$

The subspace H_{s} is called the *structured* (or reversible) part while the usual terminology for H_{r} is the *random* (or almost weakly stable) part. The orthogonal projection P onto H_{s} is called the *Jacobs–de Leeuw–Glicksberg projection*. The theorem is actually far more general in its original version than the case of cyclic contraction semigroups on a Hilbert spaces as presented here. It provides, for example, an analogous decomposition with respect to weakly compact Abelian or amenable semigroups of linear operators on an arbitrary Banach space, see the original papers mentioned above, or consult [5, Ch. 16].

For the proof of Theorem 2 we need some more preparation. The following result is well-known and of extreme importance in ergodic theory.

LEMMA 4 (Koopman, von Neumann [12]). *The following assertions are equivalent for a bounded positive sequence* $(c_n)_{n\in\mathbb{N}}$*:*

(i) $\displaystyle\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} c_n = 0.$

(ii) $\displaystyle\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} c_n^2 = 0.$

(iii) $c_n \to 0$ *in density as* $n \to \infty$.

Another important fact is that the elements of the random part H_{r} enjoy even stronger stability properties:

LEMMA 5 (Jones, Lin [11]). *For* $x \in H$ *one has* $x \in H_{\mathrm{r}}$ *if and only if*

$$\sup_{\substack{y\in H\\ \|y\|\le 1}} \frac{1}{N}\sum_{n=1}^{N} \left|(T^n x \mid y)\right| \to 0 \quad \text{as } N \to 0.$$

See also [5, Prop. 8.18 and Prop. 9.17] for a proof. We also need the next, probably well-known, auxiliary result:

LEMMA 6. *Let* T *be a contraction on a Hilbert space* H*. Then the following assertions hold:*

a) *If* $\lambda \in S^1$ *and* $Ty = \lambda y$, *then* $T^*y - \overline{\lambda}y = 0$.
b) *The eigenvectors to different unimodular eigenvalues are orthogonal.*
c) *For* $\lambda \in S^1$ *we have* $T^* P_\lambda = P_\lambda T^*$.

PROOF. Let $\mu,\ \lambda$ be unimodular eigenvalues with eigenvectors $x,\ y \in H$, respectively.

a) We prove that $T^*y - \overline{\lambda}y = 0$. Indeed,

$$\left\|T^*y - \overline{\lambda}y\right\|^2 = \|T^*y\|^2 + \|y\|^2 - (T^*y \mid \overline{\lambda}y) - (\overline{\lambda}y \mid T^*y) \le \\ \le \|y\|^2 + \|y\|^2 - \|y\|^2 - \|y\|^2 = 0.$$

b) Now

$$(\mu - \lambda)(x \mid y) = (Tx \mid y) - (x \mid T^*y) = (Tx \mid y) - (Tx \mid y) = 0,$$

and the assertion is proved.

c) Let $\lambda \in S^1$ and let $x,\ y \in H$. Then

$$(P_\lambda T^* x \mid y) = (T^* x \mid P_\lambda y) = (x \mid TP_\lambda y) = \overline{\lambda}(x \mid P_\lambda y) = \overline{\lambda}(P_\lambda x \mid y) = (T^* P_\lambda x \mid y)$$

by part a). ∎

For each $\lambda \in S^1$ let P_λ denote the orthogonal projection onto $\ker(\lambda I - T)$. As a consequence of the preceding lemma the Jacobs–de Leeuw–Glicksberg projection has the form

$$P = \sum_{\lambda \in S^1} P_\lambda,$$

where the sum is orthogonal and converges strongly.

PROOF OF THEOREM 2. Consider the Jacobs–de Leeuw–Glicksberg projection P onto H_{s}. By Theorem 3 we have

$$\frac{1}{N}\sum_{j=1}^{N}\left|(T^n x \mid y) - (T^n Px \mid Py)\right| = \frac{1}{N}\sum_{n=1}^{N}\left|(T^n(x - Px) \mid y - Py)\right| \to 0 \tag{1}$$

as $N \to \infty$. For $\lambda,\ \mu \in S^1$

$$|(P_\lambda x \mid P_\lambda y)(P_\mu y \mid P_\mu x)| \le \|P_\lambda x\|\,\|P_\lambda y\|\,\|P_\mu x\|\,\|P_\mu y\| \le \\ \le \frac{1}{4}\left(\|P_\lambda x\|^2 + \|P_\lambda y\|^2\right)\left(\|P_\mu x\|^2 + \|P_\mu y\|^2\right),$$

hence in the following we can apply Fubini's theorem and the dominated convergence theorem freely. A short calculation yields (use Lemma 6.b))

$$
\begin{aligned}
|(T^nPx \mid Py)|^2 &= \left|\left(\sum_{\lambda\in S^1} \lambda^n P_\lambda x \,\Big|\, \sum_{\mu\in S^1} P_\mu y\right)\right|^2 = \\
&= \left|\sum_{\lambda\in S^1}\sum_{\mu\in S^1} \lambda^n (P_\lambda x \mid P_\mu y)\right|^2 = \left|\sum_{\lambda\in S^1} \lambda^n (P_\lambda x \mid P_\lambda y)\right|^2 = \\
&= \sum_{\lambda\in S^1} \lambda^n (P_\lambda x \mid P_\lambda y)\cdot \sum_{\mu\in S^1} \mu^{-n}(P_\mu y \mid P_\mu x) = \\
&= \sum_{\lambda,\mu\in S^1} \lambda^n\mu^{-n}(P_\lambda x \mid P_\lambda y)(P_\mu y \mid P_\mu x).
\end{aligned}
\tag{2}
$$

We thus obtain

$$
\begin{aligned}
&\frac{1}{N}\sum_{n=1}^{N} |(T^nPx \mid Py)|^2 = \\
&\quad = \sum_{\lambda\in S^1} |(P_\lambda x \mid P_\lambda y)|^2 + \sum_{\substack{\lambda,\mu\in S^1\\ \lambda\neq\mu}} \frac{1}{N}\tfrac{(\lambda\overline{\mu})^{N+1}-1}{\lambda\overline{\mu}-1}(P_\lambda x \mid P_\lambda y)(P_\mu y \mid P_\mu x) \to \\
&\quad \to \sum_{\lambda\in S^1} |(P_\lambda x \mid P_\lambda y)|^2 =: d,
\end{aligned}
$$

by dominated convergence. Since $((T^nx \mid y))_{n\in\mathbb{N}}$ and $((T^nPx \mid Py))_{n\in\mathbb{N}}$ are bounded sequences, we have

$$
\begin{aligned}
&\limsup_{N\to\infty}\left|\frac{1}{N}\sum_{n=1}^{N}|(T^nx \mid y)|^2 - d\right| = \\
&\quad = \limsup_{N\to\infty}\left|\frac{1}{N}\sum_{n=1}^{N}\Big(|(T^nx \mid y)|^2 - |(T^nPx \mid Py)|^2\Big) + \frac{1}{N}\sum_{n=1}^{N}|(T^nPx \mid Py)|^2 - d\right| \le \\
&\quad \le \limsup_{N\to\infty}\left(\frac{1}{N}\sum_{n=1}^{N}\Big||(T^nx \mid y)|^2 - |(T^nPx \mid Py)|^2\Big| + \left|\frac{1}{N}\sum_{n=1}^{N}|(T^nPx \mid Py)|^2 - d\right|\right) \le \\
&\quad \le \limsup_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}|(T^nx \mid y) - (T^nPx \mid Py)|\cdot\Big(|(T^nx \mid y)| + |(T^nPx \mid Py)|\Big) + \\
&\qquad + \limsup_{N\to\infty}\left|\frac{1}{N}\sum_{n=1}^{N}|(T^nPx \mid Py)|^2 - d\right| = 0.
\end{aligned}
$$

Replacing y by $T^{*k}y$ in (2) yields

$$\frac{1}{N}\sum_{n=1}^{N}\left|(T^nPx \mid PT^{*k}y)\right|^2 =$$

$$= \sum_{\lambda\in S^1}\left|(P_\lambda x \mid P_\lambda T^{*k}y)\right|^2 + \sum_{\substack{\lambda,\mu\in S^1\\ \lambda\neq\mu}} \frac{1}{N}\frac{(\lambda\overline{\mu})^{N+1}-1}{\lambda\overline{\mu}-1}(P_\lambda x \mid P_\lambda T^{*k}y)(P_\mu T^{*k}y \mid P_\mu x) =$$

$$= \sum_{\lambda\in S^1}\left|(P_\lambda x \mid P_\lambda T^{*k}y)\right|^2 + \sum_{\substack{\lambda,\mu\in S^1\\ \lambda\neq\mu}} \frac{1}{N}\frac{(\lambda\overline{\mu})^{N+1}-1}{\lambda\overline{\mu}-1}\mu^{-k}\lambda^k(P_\lambda x \mid P_\lambda y)(P_\mu y \mid P_\mu x).$$

From this we obtain by dominated convergence

$$\frac{1}{N}\sum_{n=1}^{N}\left|(T^nPx \mid PT^{*k}y)\right|^2 \to \sum_{\lambda\in S^1}\left|(P_\lambda x \mid P_\lambda T^{*k}y)\right|^2 = \sum_{\lambda\in S^1}\left|\lambda^k(P_\lambda x \mid P_\lambda y)\right|^2 = d$$

uniformly for $k = 0, 1, 2, \ldots$ as $N \to \infty$. Since, by Theorem 5, the convergence in (1) is uniform for $\|y\| \le 1$, the second assertion follows, too. ∎

Next, we indicate how Wiener's lemma is a special case of the previous operator-theoretic result. This is well known (see e.g., [1], [8]), and we present this example for the sake of completeness. Consider a complex Borel measure μ on the unit disc $B = \{z \in \mathbb{C}\colon |z| \le 1\}$, and the Hilbert space $H = \mathrm{L}^2(B, |\mu|)$. The multiplication operator T defined by $(Tf)(z) = zf(z)$ is a contraction on H, and it is unitary if μ is supported on the unit circle S^1. By the Radon–Nikodym theorem $\mathrm{d}\mu = f\mathrm{d}\,|\mu|$ for some $f \in \mathrm{L}^1(B, |\mu|)$ with $|f| = 1$. The orthogonal projection P_λ onto the eigenspace $\ker(\lambda I - T)$ is the multiplication operator by the characteristic function $1_{\{\lambda\}}$. Moreover, we have

$$\int_B z^{n+k}\,\mathrm{d}\mu(z) = \int_B z^n z^k f(z)\,\mathrm{d}\,|\mu|\,(z) = \left(T^n f \,\middle|\, T^{*k}1\right).$$

Since

$$(P_\lambda f \mid 1) = \int_B 1_{\{\lambda\}} f(z)\,\mathrm{d}\,|\mu|\,(z) = \int_B 1_{\{\lambda\}}\,\mathrm{d}\mu(z) = \mu\{\lambda\},$$

we obtain by Theorem 2 the following corollary:

COROLLARY 7. *Let μ be a complex Borel measure on the unit disc $B := \{z\colon |z| \le 1\}$. Then*

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\left|\int_B z^{n+k}\,\mathrm{d}\mu(z)\right|^2 = \sum_{\lambda\in S^1}|\mu\{\lambda\}|^2$$

uniformly for $k = 0, 1, 2, \ldots$. If μ is a complex Borel measure on S^1, then

$$\lim_{N\to\infty}\frac{1}{2N+1}\sum_{n=-N}^{N}|\hat{\mu}(n+k)|^2 = \sum_{\lambda\in S^1}|\mu\{\lambda\}|^2$$

uniformly for $k = 0, 1, 2, \dots.$

If μ is a complex Borel measure on S^1, then the operator T is unitary. By applying the preceding corollary to T and T^* and by taking the average of the two results we obtain Wiener's lemma.

Next, we indicate briefly how the Jacobs–de Leeuw–Glicksberg theorem can be obtained with the help of Wiener's lemma. Let T be a unitary operator on a Hilbert space H. By the spectral theorem (see, e.g., [5, Ch. 18] for details) for each $x,\ y \in H$ there is a Borel measure $\mu_{x,y}$ on S^1 such that

$$(T^n x \mid y) = \int_{S^1} z^n \,\mathrm{d}\mu_{x,y}(z) = \hat{\mu}_{x,y}(n) \quad \text{for each } n \in \mathbb{Z},$$

and the action of T on the cyclic subspace $[T, x]$ is unitarily equivalent to the multiplication operator by the identity on $\mathrm{L}^2(S^1, \mu_x)$. The measures $\mu_x := \mu_{x,x}$ are positive and $\mu_{x,y}$ is absolutely continuous with respect to μ_x and μ_y.

Given a contraction T on a Hilbert space H we consider the Szőkefalvi-Nagy–Foiaş decomposition, i.e. we orthogonally decompose H into two T and T^*-invariant closed subspaces

$$H = H_\mathrm{n} \oplus H_\mathrm{u}$$

such that T restricts to an unitary operator on H_u, and there is no T, T^*-invariant subspace in H_n on which T would act unitarily, see [13]. By a result of Foguel [6] T is weakly stable on H_n, i.e., for each $x \in H_\mathrm{n}, y \in H$ one has $(T^n x \mid y) \to 0$ as $n \to \infty$.

Moreover, by means of the spectral theorem, we can orthogonally decompose the space H_u as

$$H_\mathrm{u} = H_\mathrm{d} \oplus H_\mathrm{c}$$

with

$$H_\mathrm{d} = \{x\colon \mu_x \text{ is a discrete measure}\}$$

and

$$H_\mathrm{c} = \{x\colon \mu_x \text{ is a continuous measure}\},$$

both subspaces being T and T^*-invariant. Finally, if we set

$$H_1 := H_\mathrm{d} \quad \text{and} \quad H_0 := H_\mathrm{n} \oplus H_\mathrm{c},$$

then $H = H_0 \oplus H_1$ is an orthogonal decomposition. We claim that this is indeed the Jacobs–de Leeuw–Glicksberg decomposition, i.e., has the properties as stated in Theorem 3. (The arguments given here are standard, and we refer to [5, Ch. 18] for details concerning the spectral theorem.) By the spectral theorem H_d is the closed linear hull of eigenvectors to unimodular eigenvalues, so that $H_1 = H_\mathrm{s}$ as in Theorem 3. Let P_u be the orthogonal projection onto H_{u}, and let $P_{\mathrm{n}} := I - P_{\mathrm{u}}$ be the complementary projection. For every $x,\ y \in H$ we have by Foguel's aforementioned result

$$\limsup_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} |(T^n x \mid y)|^2 = \limsup_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} |(T^n P_{\mathrm{u}} x \mid P_{\mathrm{u}} y) + (T^n P_{\mathrm{n}} x \mid P_{\mathrm{n}} y)|^2 =$$

$$= \limsup_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \Big(|(T^n P_{\mathrm{u}} x \mid P_{\mathrm{u}} y)|^2 + 2\Re (T^n P_{\mathrm{n}} x \mid P_{\mathrm{n}} y)\overline{(T^n P_{\mathrm{u}} x \mid P_{\mathrm{u}} y)} +$$

$$+ |(T^n P_{\mathrm{n}} x \mid P_{\mathrm{n}} y)|^2 \Big) =$$

$$= \limsup_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} |(T^n P_{\mathrm{u}} x \mid P_{\mathrm{u}} y)|^2 .$$

Since

$$(T^n P_{\mathrm{u}} x \mid P_{\mathrm{u}} y) = \int_{S^1} z^n \,\mathrm{d}\mu_{P_{\mathrm{u}}x, P_{\mathrm{u}}y}(z) = \hat{\mu}_{P_{\mathrm{u}}x, P_{\mathrm{u}}y}(n),$$

by Wiener's Lemma 1 we obtain

$$\limsup_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} |(T^n x \mid y)|^2 = \limsup_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \hat{\mu}_{P_{\mathrm{u}}x, P_{\mathrm{u}}y}(n)^2 = 0$$

if and only if $\mu_{P_{\mathrm{u}}x, P_{\mathrm{u}}y}$ is a continuous measure. If this limit is 0 for each y, in particular for $y = x$, then we obtain $P_{\mathrm{u}} x \in H_{\mathrm{c}}$, i.e., $x \in H_0$. On the other hand, if $x \in H_0$, then $P_{\mathrm{u}} x \in H_{\mathrm{c}}$, so that $\mu_{P_{\mathrm{u}}x}$ is continuous, and by absolute continuity so is $\mu_{P_{\mathrm{u}}x, P_{\mathrm{u}}y}$ for each $y \in H$. This implies that

$$\limsup_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} |(T^n x \mid y)|^2 = 0,$$

so that by the Koopman–von Neumann Lemma 4 also

$$\limsup_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} |(T^n x \mid y)| = 0,$$

follows. Whence we conclude $H_0 = H_{\mathrm{r}}$ as in Theorem 3.

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ON OPERATOR POLYNOMIALS WITH NONNEGATIVE COEFFICIENTS IN A BANACH LATTICE

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Dedicated to our friend, Professor Zoltán Sebestyén on the occasion of his 70th anniversary

Abstract. We study spectral properties of polynomials $P(\cdot)$ of the form $P(\lambda) = \lambda^m I_E - A(\lambda)$, where $A(\lambda) = \lambda^l A_l + \ldots + \lambda A_1 + A_0$, the coefficients A_j are nonnegative linear operators in a (complex) Banach lattice $E, A_l \neq 0$, I_E denotes the identity operator in E, l is a nonnegative integer and m is a positive integer. In Section 3 the space is $E = C(K)$, the Banach lattice of all complex-valued continuous functions on a compact Hausdorff space K, and the main result proves that two sets connected with a strictly positive eigenvector of the operator $P(\rho)$ $(\rho > 0)$ are a group under pointwise multiplication in $C(K)$, and a subgroup of the circle group, respectively. Section 4 studies the general case of nonnegative linear operators in a Banach lattice E under the assumption that $A(1)$ is irreducible and has a compact power. The main results show that either the point spectrum of the polynomial $P(\cdot)$ contains $\mathbb{C} \setminus \{0\}$, or there is a positive integer k with the property that for every $\rho > 0$ such that the spectral radius of $A(\rho)$ is ρ^m, the point spectrum of $P(\cdot)$ on the circle of radius ρ is the set $\{\rho e^{2\pi i n/k} \colon n = 0, 1, \ldots, k-1\}$. In the second case the geometric and algebraic multiplicities in these points are studied.

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1. Introduction

We consider polynomials $P(\cdot)$ of the form

$$P(\lambda) = \lambda^m I_E - A(\lambda), \quad A(\lambda) = \lambda^l A_l + \ldots + \lambda A_1 + A_0,$$

where the coefficients A_j are nonnegative linear operators in a (complex) Banach lattice $E, A_l \neq 0$, I_E denotes the identity operator in E, l is nonnegative integer and m is a positive integer.

Monic operator- (or matrix-) polynomials (i.e. $l < m$) with nonnegative coefficients were considered in [6], [13], [17] and [21]. The main tool to study such polynomials is (as in the case of arbitrary coefficients) the companion operator, which is here a positive operator and the spectral theory of (cone) positive operators can be applied.

We are also interested in the case $l \geq m$, when linearization (in λ) does not help much. In modeling queueing problems and comparison theorems for differential equations the case $m = 1$ and entrywise nonnegative matrix coefficients appear at several places in the literature; see for example [3], [4], [9], [10], [12]; in that case, special procedures (like iteration, factorization, operator roots) can be applied to solve some problems, see [7], [14].

There are many results on and applications of such operator polynomials with positive definite and also with arbitrary coefficients, see the introduction of [14]. An instance for $0 < m < l$ with arbitrary coefficients is implicitly contained in [16, Corollary 23.5].

Here we consider the general case, where the coefficients are nonnegative operators in Banach lattices, and the polynomial $P(\cdot)$ is not monic. We follow the usual steps in the spectral theory of nonnegative operators in Banach lattices, namely, first we prove our results for the Banach lattice $C(K)$ of the continuous functions on a compact space and then use the well-known connection between the main ideals in a Banach lattice and AM-spaces to prove our main results.

The paper is organized as follows. In Section 2 we recall a few results for later use. In Section 3 we consider the problems for the Banach lattice $C(K)$ of the continuous functions on a compact space. The main results are contained in Theorem 3.1. In Section 4 we assume that the coefficients A_j are nonnegative linear operators in a Banach lattice E such that their sum is irreducible and has a compact power. If the spectrum of $P(\cdot)$ is not the whole complex plane, then there is a positive integer k with the property that for every $\rho > 0$ such that the spectral radius of $A(\rho)$ is ρ^m, the eigenvalues of $P(\cdot)$ on the circles of radius ρ are distributed as the roots of unity of order k. We study the geometric and algebraic multiplicities of these eigenvalues, and end the paper with a conjecture.

Concerning the used concepts for the theory of nonnegative operators in Banach lattices we refer to the monographs of H. H. Schaefer [23], P. Meyer-Nieberg [18] and A. C. Zaanen [25], concerning the theory of operator polynomials we refer to the monographs of A. S. Markus [16] and L. Rodman [22].

To distinguish the polynomial and its value at a point we use $P(\cdot)$ to denote the polynomial and $P(\lambda)$ to denote its value at λ, respectively.

2. Preliminaries

For a (complex) Banach lattice E the Banach algebra $\mathcal{L}(E)$ of all bounded linear operators in E is an ordered Banach algebra with the normal cone $\mathcal{L}_+(E)$ of all nonnegative operators in E.

In this section we recall without proofs results of [8, Sections 3 and 6] and [14, Section 1.1], where some of the results were proved under more general assumptions. We will use the following notation:

0_E denotes the zero element in E,

I_E denotes the identity operator in E,

$\Sigma(P(\cdot))$ denotes the spectrum of $P(\cdot)$, which is defined as the set $\{\lambda \in \mathbb{C}\colon P(\lambda)$ is not invertible in $\mathcal{L}(E)\}$,

$\Sigma_{\text{point}}(P(\cdot))$ denotes the point spectrum of $P(\cdot)$, which is defined as the set $\{\lambda \in \mathbb{C}\colon P(\lambda)$ is not injective$\}$. We call the elements in this set eigenvalues of $P(\cdot)$.

$\Pi_\rho = \{\lambda \in \mathbb{C}\colon |\lambda| = \rho\}$ for $\rho \geq 0$.

Proposition 2.1. *Let*

$$A(\lambda) = \lambda^l A_l + \ldots + \lambda A_1 + A_0$$

and $A_j \in \mathcal{L}_+(E), j = 0, 1, \ldots, l$. Then the function

$$\operatorname{spr}_A\colon [0, \infty[\to [0, \infty[, \quad \rho \mapsto \text{spectral radius of } A(\rho)$$

is geometrically convex, i.e., it satifies the functional inequality

$$\operatorname{spr}_A\left(\rho_1{}^{\mu}\rho_2{}^{1-\mu}\right) \leq \operatorname{spr}_A(\rho_1)^{\mu}\operatorname{spr}_A(\rho_2)^{1-\mu}, \quad \rho_1, \rho_2 \in]0,\infty[,\ \mu \in [0,1].$$

For a proof see [8, Proposition 3.2] or [14, p. 14, Proposition 1.14].

Recall that a function $s\colon]0,\infty[\to [0,\infty[$ is geometrically convex if and only if $]-\infty,\infty[\to [-\infty,\infty[,\ \tau \mapsto \ln(s(e^{\tau}))$ is convex. The following lemma holds for arbitrary geometrically convex functions and for arbitrary real m, its assertions are consequences of corresponding properties of convex functions.
In the following $\operatorname{spr}_A{}^{(k)}$ denotes the k-th derivative of the function spr_A.

LEMMA 2.2. *Let* $0 < \rho_1 < \rho_2$.

1.) *Assume that* $\operatorname{spr}_A(\rho_j) = \rho_j{}^m$ *($j = 1,2$), and there exists a* $\hat\rho \in]\rho_1,\rho_2[$ *such that* $\operatorname{spr}_A(\hat\rho) = \hat\rho^m$*, then* $\operatorname{spr}_A(\rho) = \rho^m$ *for all* $\rho \in]\rho_1,\rho_2[$.
2.) *Assume that* $\operatorname{spr}_A(\rho_1) = \rho_1{}^m$ *, that* spr_A *is differentiable in* $]\rho_1,\rho_2[$ *and there exists a* $\hat\rho \in]\rho_1,\rho_2[$ *such that* $\operatorname{spr}_A(\hat\rho) = \hat\rho^m$ *and* $\operatorname{spr}_A{}^{(1)}(\hat\rho) = m\hat\rho^{m-1}$*. Then* $\operatorname{spr}_A(\rho) = \rho^m$ *for all* $\rho \in]\rho_1,\hat\rho[$*; a similar assertion holds for* ρ_2 *instead of* ρ_1.
3.) *Assume that* spr_A *is twice differentiable in* $]\rho_1,\rho_2[$ *and there exists a* $\hat\rho \in]\rho_1,\rho_2[$ *such that* $\operatorname{spr}_A(\hat\rho) = \hat\rho^m$, $\operatorname{spr}_A{}^{(1)}(\hat\rho) = m\hat\rho^{m-1}$ *and* $\operatorname{spr}_A{}^{(2)}(\hat\rho) = m(m-1)\hat\rho^{m-2}$*. Then* $\operatorname{spr}_A(\rho) = \rho^m$ *for all* $\rho \in]\rho_1,\rho_2[$.

PROPOSITION 2.3. *Let* $0 < \rho_1 < \rho_2$*. Assume that* $\operatorname{spr}_A(\rho) < \rho^m$ *for all* $\rho \in]\rho_1,\rho_2[$*, then*

$$\Sigma(P(\cdot)) \cap \{\lambda \in \mathbb{C}\colon \rho_1 < |\lambda| < \rho_2\} = \emptyset;$$

here, as before, $P(\lambda) = \lambda^m I_E - A(\lambda)$.

For a proof see [8, Proposition 3.4] or [14, Proposition 1.7].

PROPOSITION 2.4. *Assume that there exist a* $\hat\rho \in]0,\infty[$ *such that* $\operatorname{spr}_A(\hat\rho) < \hat\rho^m$ *and a* $\rho \in]0,\infty[$ *such that* $\operatorname{spr}_A(\rho) = \rho^m$*. If* ρ *and* $\lambda \in \Pi_\rho$ *are poles of* $P^{-1}(\cdot)$*, then the order of the pole* λ *of* $P^{-1}(\cdot)$ *is less than or equal to the order of the pole* ρ *of* $P^{-1}(\cdot)$*. Here* $P^{-1}(\lambda)$ *denotes the inverse of* $P(\lambda)$ *for* $\lambda \notin \Sigma(P(\cdot))$*; again,* $P(\lambda) = \lambda^m I_E - A(\lambda)$.

For a proof see [8, Theorem 6.1].

REMARK. Recall that if λ is a pole of $P^{-1}(\cdot)$, then the order of the pole $P^{-1}(\cdot)$ is equal to the maximum of the lengths of the (nontrivial) Jordan chains of $P(\cdot)$ at λ.

3. $E = C(K)$

The results of this section for the special Banach lattice $C(K)$ are not only interesting in themselves, but also they are used in the following section to prove results for arbitrary Banach lattices. This is a common procedure relying on the fact that "any general Banach lattice E is in a very intimate way connected to certain families of AL- and AM-spaces canonically associated with E"; [20, p. 240].

Let K denote a nonempty compact Hausdorff space, and let $C(K)$ denote the Banach lattice of all complex continuous functions on K with the maximum norm. In this section we generalize the results in [18, Proposition 4.1.7] and [23, Proposition V.4.2]; in [14, Theorem 4.23] the case of matrix coefficients was considered, this corresponds to a finite set K. We use the following notation:

- $1_K(s) = 1$ for all $s \in K$,
- $0_K(s) = 0$ for all $s \in K$,
- δ_s is the functional on $C(K)$ defined by $\langle \delta_s, f \rangle := f(s)$,
- $A_j{}'\delta_s$ is the functional obtained by applying the dual of A_j to δ_s,
- $\mu_{A_j{}'\delta_s}$ is the representing Borel measure of the functional $A_j{}'\delta_s$,
- $\mathfrak{B}$ is the algebra of the Borel sets of K,
- supp (μ) is the support of the Borel measure μ.

The main result of this section is the following theorem:

THEOREM 3.1. *For all $\rho > 0$ and all $u \gg 0_K$ such that $P(\rho)u = 0_K$ the set*

$$G = \left\{ \frac{x}{u} : x \in C(K) \text{ such that } P(\rho\omega)x = 0_K \text{ for some } \omega \in \Pi_1 \text{ and } |x| = u \right\}$$

is a group under pointwise multiplication, and the set

$$\Gamma = \{\omega \in \Pi_1 : P(\rho\omega)x = 0_K \text{ for some } x \in C(K) \text{ and } |x| = u\}$$

is a subgroup of the circle group Π_1. Both groups do not depend on ρ and u.

REMARK. If $\rho > 0$, then $u \gg 0_K$ and $P(\rho)u = 0_K$ imply that $\rho^m = \mathrm{spr}(A(\rho))$ is an eigenvalue of the dual operator $A(\rho)'$, since $\mathrm{spr}(A(\rho))$ is a distinguished eigenvalue of $(A(\rho))'$, see [18, Theorem 4.13], [23, Proposition II.8.10].

To prepare the proof, we isolate three lemmata.

LEMMA 3.2. *Let* $P(1)1_K = 0_K$, $\omega \in \Pi_1$ *and* $x \in C(K)$ *such that*

$$P(\omega)x = 0_K, \quad |x| = 1_K.$$

Then for all $j \in \{0, 1, \ldots, l\}$, $s \in K$ *and* $t \in \operatorname{supp}(\mu_{A_j{}'\delta_s})$

$$\omega^j x(t) = \omega^m x(s)$$

holds.

PROOF. Let $s \in K$. Then

$$(\omega^m x)(s) = (A(\omega)x)(s) = \langle \delta_s, A(\omega)x \rangle =$$
$$= \sum_{j=0}^{l} \omega^j \langle A_j' \delta_s, x \rangle = \sum_{j=0}^{l} \omega^j \int_K x(t) \mu_{A_j{}'\delta_s}(dt).$$

Define the measure space $(\hat{K}, \hat{\mathfrak{B}}, \hat{\mu})$ as the disjoint union of the measure spaces $(K_j, \mathfrak{B}_\mathrm{j}, \mu_{A_j{}'\delta_s})$ where $K_j = K$ and $\mathfrak{B}_\mathrm{j} = \mathfrak{B}$, and define $\hat{x} \colon \hat{K} \to \mathbb{C}$ such that $\hat{x}(t) = \omega^j x(t)$ for t $\in K_j$. Then the first equation continues as follows:

$$= \int_{\hat{K}} \hat{x}(t) \hat{\mu}(dt),$$

and this equation is equivalent to

$$\int_{\hat{K}} \omega^{-m} x(s)^{-1} \hat{x}(t) \hat{\mu}(dt) = 1.$$

The measures $\mu_{A_j{}'\delta_s}$ are nonnegative, since the operators A_j are nonnegative, $j \in \{0, 1, \ldots, l\}$, and from $A(1)1_K = 1_K$ we get that $\hat{\mu}$ is a probability measure. Now $\frac{\hat{x}}{\omega^m x(s)}$ is unimodular, therefore $\frac{\hat{x}(t)}{\omega^m x(s)} = 1$ for $t \in$ supp $(\hat{\mu})$, and this means that

$$\omega^j x(t) = \omega^m x(s).$$

for all $j \in \{0, 1, \ldots, l\}$, $s \in K$ and $t \in \operatorname{supp}(\mu_{A_j{}'\delta_s})$. ∎

LEMMA 3.3. *Let* $P(1)1_K = 0_K$. *Then*

1.) *If* x *and* y *are unimodular eigenfunctions of* $P(\cdot)$ *to unimodular eigenvalues* ω *and* ϕ, *respectively, then* $\omega\phi$ *is a unimodular eigenvalue with corresponding unimodular eigenfunction* xy *of* $P(\cdot)$.

2.) *The set G of all unimodular eigenfunctions of $P(\cdot)$ for unimodular eigenvalues of $P(\cdot)$ is a group under pointwise multiplication, and the set*

$$\{\omega \in \Pi_1 : P(\omega)x = 0_K \text{ for some } x \in C(K) \text{ such that } |x| = 1_K\}$$

is a subgroup of the circle group Π_1.

Proof. Let x and y be unimodular eigenfunctions of $P(\cdot)$ to unimodular eigenvalues ω and ϕ, respectively. By Lemma 3.2, for all $j \in \{0, 1, \ldots, l\}$, $s \in K$ and $t \in$supp $(\mu_{A_j{}'\delta_s})$ we have

$$\omega^j x(t) = \omega^m x(s) \quad \text{and} \quad \phi^j y(t) = \phi^m y(s).$$

Using the probability measure $\hat{\mu}$ defined in the proof of Lemma 3.2 we get for all $s \in K$

$$(\omega\phi)^m(xy)(s) = \int\limits_{\hat{K}} \omega^m x(s)\phi^m y(s)\hat{\mu}(dt) = \sum_{j=0}^{l} \int\limits_{K} \omega^j x(t)\phi^j y(t)\mu_{A_j{}'\delta_s}(dt) =$$

$$= \sum_{j=0}^{l} \langle A_j{}'\delta_s, \omega^j x\phi^j y\rangle = \langle \delta_s, A(\omega\phi)\{xy\}\rangle = (A(\omega\phi)\{xy\})\,(s).$$

Therefore $P(\omega\phi)\{xy\} = 0_K$. This proves 1.), and 2.) is now evident. ∎

Lemma 3.4. *Let $\rho > 0$ and $u \in C(K)$ such that*

$$P(\rho)u = 0_K \quad \text{and} \quad u \gg 0_K.$$

Then

1.) *If x is an eigenfunction of $P(\cdot)$ corresponding to an eigenvalue $\rho\omega$ with $\omega \in \Pi_1$ and $|x| = u$ then for all $j \in \{0, 1, \ldots, l\}$, $s \in K$ and $t \in \mathrm{supp}(\mu_{A_j{}'\delta_s})$*

$$\omega^j \frac{x}{u}(t) = \omega^m \frac{x}{u}(s).$$

2.) *The set*

$$G = \left\{\frac{x}{u} : x \in C(K),\ P(\rho\omega)x = 0_K \text{ for some } \omega \in \Pi_1,\ |x| = u\right\}$$

is a group under pointwise multiplication, and the set

$$\{\omega \in \Pi_1 : P(\rho\omega)x = 0_K \text{ for some } x \in C(K) \text{ such that } |x| = u\}$$

is a subgroup of the circle group Π_1.

PROOF. Let M_u denote the operator of multiplcation by u in $C(K)$. M_u is a lattice isomorphism with inverse $M_{\frac{1_K}{u}}$. We define $\tilde{A}_j = \rho^{j-m} {M_u}^{-1} A_j M_u$ for $j = 0, 1, \ldots, l$, and

$$\tilde{P}(\lambda) = \lambda^m I_E - (\lambda^l \tilde{A}_l + \ldots + \lambda \tilde{A}_1 + \tilde{A}_0).$$

Then

$$P(\rho\lambda) = \rho^m M_u \tilde{P}(\lambda) {M_u}^{-1} \text{ for all complex } \lambda.$$

Therefore for all $x \in C(K)$, complex ρ and ω

$$P(\rho\omega)x = 0_K \quad \Longleftrightarrow \quad \tilde{P}(\rho\omega){M_u}^{-1}x = 0_K$$

holds. All assertions follow from Lemmata 3.2 and 3.3 and the following observation: Let T be a (continuous) nonnegative operator in $C(K)$ and $s \in K$, then

$$\text{supp}\,(\mu_{T'\delta_s}) = \text{supp}\left(\mu_{\left({M_u}^{-1}TM_u\right)'\delta_s}\right).$$

To see this equality take any $f \in C(K)$ such that $\text{supp}(f) \subset K \setminus \text{supp}(\mu_{T'\delta_s})$. Then

$$\int_K f(t)\mu_{\left({M_u}^{-1}TM_u\right)'\delta_s}(dt) = \langle \delta_s, {M_u}^{-1}TM_u f\rangle = \left\langle \delta_s, \frac{1_K}{u}TM_u f\right\rangle =$$

$$= \frac{1}{u(s)}\langle \delta_s, TM_u f\rangle = \frac{1}{u(s)}\int_K (uf)(t)\mu_{T'\delta_s}(dt) = 0.$$

For the last equality note that $\text{supp}(f) = \text{supp}(uf)$ since $u \gg 0_K$. Therefore the inclusion

$$\text{supp}(\mu_{T'\delta_s}) \supset \text{supp}\left(\mu_{\left({M_u}^{-1}TM_u\right)'\delta_s}\right)$$

holds. The converse inclusion follows by symmetry. ∎

We will now give a proof of Theorem 3.1.

PROOF. From Lemma 3.4 it follows that G and Γ are groups. We will now prove that G does not depend on ρ and u.

For $i = 1, 2$ let $\rho_i > 0, u_i \in C(K)$ such that $P(\rho_i)u_i = 0_K$ and $u_i \gg 0_K$, and define

$$G_i = \left\{ \frac{x}{u_i} : x \in C(K),\ P(\rho_i\omega)x = 0_K \text{ for some } \omega \in \Pi_1,\ |x| = u_i \right\}.$$

We have to prove that $G_1 = G_2$. Let $\frac{x_1}{u_1} \in G_1$ and $\omega \in \Pi_1$ such that $P(\rho_1\omega)x_1 = 0_K$. Define $x_2 = \frac{u_2}{u_1}x_1$. Then for all $s \in K$

$$\{A(\rho_2\omega)x_2\}(s) = \langle \delta_s, A(\rho_2\omega)x_2\rangle =$$
$$= \sum_{j=0}^{l} (\rho_2\omega)^j \int_K \left(\frac{u_2}{u_1}x_1\right)(t)\mu_{A_j{}'\delta_s}(dt) =$$
$$= \frac{x_2}{u_2}(s) \sum_{j=0}^{l} \rho_2^j \int_K \omega^j \frac{x_1}{u_1}(t)\frac{u_2}{x_2}(s)u_2(t)\mu_{A_j{}'\delta_s}(dt).$$

From Lemma 3.4.1.) follows for all $j \in \{0,1,\dots,l\}$, $s \in K$ and $t \in \operatorname{supp}(\mu_{A_j{}'\delta_s})$

$$\omega^m \frac{x_2}{u_2}(s) = \omega^m \frac{x_1}{u_1}(s) = \omega^j \frac{x_1}{u_1}(t).$$

Therefore we get

$$\{A(\rho_2\omega)x_2\}(s) = \omega^m \frac{x_2}{u_2}(s) \sum_{j=0}^{l} \rho_2^j \int_K u_2(t)\mu_{A_j{}'\delta_s}(dt) =$$
$$= \omega^m \frac{x_2}{u_2}(s)\langle \delta_s, A(\rho_2)u_2\rangle = \omega^m \frac{x_2}{u_2}(s)\rho_2^m u_2(s) = ((\rho_2\omega)^m x_2)(s)$$

for all $s \in K$, which is equivalent to $P(\rho_2\omega)x_2 = 0_K$. Therefore $\frac{x_1}{u_1} = \frac{x_2}{u_2} \in$ $\in G_2$. We have proved $G_1 \subset G_2$, the converse inclusion follows similarly.

Now we will prove that Γ does not depend on ρ and u. For $i = 1,2$ let $\rho_i > 0$ and $u_i \gg 0_K$ such that $P(\rho_i)u_i = 0_K$, and define

$$\Gamma_i = \{\omega \in \Pi_1 \colon P(\rho_i\omega)x = 0_K \text{ for some } x \in C(K) \text{ and } |x| = u_i\}.$$

Let $\omega \in \Gamma_1$ and $x_1 \in C(K)$ such that $P(\rho_1\omega)x_1 = 0_K$ and $|x_1| = u_1$. Then $\frac{x_1}{u_1} \in G_1 = G_2$. Therefore $\frac{x_1}{u_1} = \frac{x_2}{u_2}$ for some $x_2 \in C(K)$. From $x_2 = \frac{u_2}{u_1}x_1$ we obtain as above $A(\rho_2\omega)x_2 = (\rho_2\omega)^m x_2$, thus $\omega \in \Gamma_2$. This proves the inclusion $\Gamma_1 \subset \Gamma_2$, the converse inclusion follows similarly. ∎

4. The irreducible case

In this section we always assume that for $j = 0, 1, \ldots, l$ the coefficients A_j are nonnegative linear operators in a (complex) Banach lattice E.

LEMMA 4.1.

1.) If $0 < \rho < \tau$, *then*

$$0 \leq \frac{\rho^l}{\tau^l} A(\tau) \leq A(\rho) \leq A(\tau) \leq \frac{\tau^l}{\rho^l} A(\rho).$$

Therefore

$$A(\rho_0) \text{ is irreducible for one } \rho_0 > 0.$$
$$\Updownarrow$$
$$A(\rho) \text{ is irreducible for all } \rho > 0.$$

2.) For all $\lambda \in \mathbb{C}$ *and all* $x \in E$

$$|A(\lambda)x| \leq A(|\lambda|)\,|x|$$

holds, therefore

$$A(\rho_0) \text{ has a compact power for one } \rho_0 > 0.$$
$$\Updownarrow$$
$$A(\lambda) \text{ has a compact power for all } \lambda \in \mathbb{C}.$$

3.) If $\lambda \neq 0$ *and* $A(\lambda)$ *has a compact power, then* $A(\lambda)$ *is a Riesz operator and* $P(\lambda)$ *is a Fredholm operator with index* 0.

PROOF. The inequality in 1.) is clear. For the second assertion note that for S and $T \in \mathcal{L}_+(E)$ such that $S \leq T$ and S is irreducible, it follows that T is irreducible. Also the inequality in 2.) is clear. From [2, Theorem] and the inequality in 1.) the second assertion in 2.) follows for all positive λ, and using the inequality in 2.) [1, Theorem 2.34] proves that the assertion follows for all λ. For a proof of 3.) see [5, Chapter 3]. ∎

A warning: In this section $P(\rho)'$ will denote the dual operator of the operator $P(\rho)$, and $P^{(k)}(\rho)$ will denote the k-th derivative of the operator polynomial $P(\cdot)$ at ρ. Similarly, $u^{(k)}(\rho)$ will denote the k-th derivative of the vector valued function u at ρ.

LEMMA 4.2. *Let $\omega \in \Pi_1$ and $\rho > 0$. Then the following assertions hold:*

1.) If there exists a $u' \in E'_+$ such that $P(\rho)'u' = 0_{E'}$ and $N_{\mathrm{abs}}(u') := \{x \in E\colon \langle u', |x|\rangle = 0\} = \{0_E\}$, then $x \in E$ and $P(\rho\omega)x = 0_E$ imply $P(\rho)|x| = 0_E$.

Assume, in addition, that $A(\rho)$ is irreducible.

2.) Then $N_{\mathrm{abs}}(u') = \{x \in E\colon \langle u', |x|\rangle = 0\} = \{0_E\}$ for all $u' \in E'_+$ such that $u' \neq 0_{E'}$ and $P(\rho)'u' = 0_{E'}$.

Assume, in addition, $\operatorname{spr} A(\rho) = \rho^m$. *Then*

3.) If $\Sigma_{\mathrm{point}}(P(\cdot)) \cap \Pi_\rho$ is non-empty, then $\rho \in \Sigma_{\mathrm{point}}(P(\cdot))$ and $\ker(P(\rho)) = span\{u\}$ for some quasiinterior element $u \in E_+$,

4.) dim ker $P(\rho\omega) \leq 1$.

PROOF. 1.) We obtain $\rho^m|x| = |A(\rho)x| \leq A(\rho)|x|$. Therefore $P(\rho)|x| \leq 0_E$ and $0 = \langle P(\rho)'u', |x|\rangle = \langle u', P(\rho)|x|\rangle \leq 0$. Thus $\langle u', P(\rho)|x|\rangle = 0$, and then $P(\rho)|x| = 0_E$.

2.) Since $N_{\mathrm{abs}}(u')$ is an $A(\rho)$-invariant ideal in E, $u' \neq 0_{E'}$ and $A(\rho)$ is irreducible, $N_{\mathrm{abs}}(u')$ is trivial.

3.) The first assertion follows directly from 1.). Now $\ker(P(\rho)) = \ker(\rho^m I_E - A(\rho))$, $A(\rho)$ is irreducible, $\operatorname{spr} A(\rho) = \rho^m$ and $\operatorname{spr} A(\rho)u' = A(\rho)'u'$. Therefore the second assertion follows from [23, V.5.2 Theorem] or [18, 4.2.13].

4.) Let u be as in 3.). For $j = 0, 1, \dots, l$ the principal ideal E_u is an AM space and is invariant under A_j, since $0_E \leq A_j u \leq \rho^{-j}A(\rho)u \leq \rho^{m-j}u$. From 1.) it follows that $\ker(P(\rho)) \subset E_u$. By [23, Theorem II.7.4], there exist a nonempty, compact Hausdorff space K and a lattice isomorphism $T\colon E_u \to C(K)$ such that $Tu = 1_K$. Fix $s \in K$. For $i = 1, 2$ let $x_i \in \ker(P(\rho\omega))$ and set $f_i = Tx_i$. Then $x = f_1(s)x_2 - f_2(s)x_1 \in \ker(P(\rho\omega))$. By 1.) and 2.) we obtain $|x| = \alpha u$ for some nonnegative α, and this implies $|Tx| = T|x| = \alpha 1_K$. Therefore $\alpha = |Tx|(s) = |f_1(s)f_2(s) - f_2(s)f_1(s)| = 0$. Thus $|x| = 0_E$ and hence $x = 0_E$, i.e. x_1 and x_2 are linearly dependent. ∎

The next theorem generalizes parts of the results in [14, Theorem 4.23], where matrix polynomials were considered, and of results in [18, 4.2.13], [23, V.5.2], and [24, Theorem 137.3], where the case $l = 0$ and $m = 1$ is considered.

THEOREM 4.3. *Let $A(1)$ be irreducible, and define*

$$\Delta_P=\{\rho>0:\ \operatorname{spr} A(\rho)=\rho^m, P(\rho)'u'=0_{E'} \text{ for some non-zero } u'\in E'_+\}.$$

Let $\rho \in \Delta_P$. If the set

$$\Gamma = \{\omega \in \Pi_1 : \rho\omega \in \Sigma_{\text{point}}(P(\cdot))\}$$

is non-empty, then it is a subgroup of the circle group Π_1, and does not depend on $\rho \in \Delta_P$.

PROOF. Let $\omega_1, \omega_2 \in \Gamma$. For $i = 1, 2$ let $\rho_i \in \Delta_P$ be such that $\rho_i\omega_i \in \in \Sigma_{\text{point}}(P(\cdot))$, and let $x_i \in \ker(P(\rho_i))$, $x_i \neq 0_E$.

By Lemma 4.2.3 there exist quasiinterior elements $u_i \in E_+$ such that $P(\rho_i)u_i = 0_E$ and $|x_i| = \alpha_i u_i$ for some positive α_i; we can and will assume that $\alpha_i = 1$. Note that $x_i \in E_{u_i}$.

For $i = 1, 2$ and $j \in \{0, 1, \dots, l\}$ the principal ideal E_{u_i} is A_j invariant, since $0_E \leq A_j u_i \leq \rho^{-j}A(\rho)u_i = \rho^{m-j}u_i$. Let $A_{j,i}$ denote the operator in $\mathcal{L}(E_{u_i})$ induced by A_i, and let $A_i(\cdot)$ and $P_i(\cdot)$ denote the corresponding operator polynomials with the coefficients $A_{j,i}$.

By [23, Theorem II.7.4] there exist compact Hausdorff spaces K_i and lattice isomorphisms $T_i : E_{u_i} \to C(K_i)$ such that $T_i u_i = 1_{K_i}$. Then $T_i P_i(\rho_i) T_i^{-1} 1_{K_i} = 0_{K_i}$.

For $i = 1, 2$ set

$$\Gamma_i = \{\omega \in \Pi_1 : T_i P_i(\rho_i\omega) T_i{}^{-1} f_i = 0_{K_i} \text{ for some } f_i \in C(K_i), |f_i| = 1_{K_i}\}.$$

By Theorem 3.1, these sets are subgroups of Π_1 and are independent of $\rho \in \Delta_P$. They are contained in Γ, since $P(\rho_i\omega)T_i{}^{-1}f_i = P_i(\rho_i\omega)T_i{}^{-1}f_i = 0_{E_{u_i}} = 0_E$. Note that $\omega_i \in \Gamma_i$.

By [23, Theorem and Corollary III.4.1], there exists a homeomorphism $h_{21} : K_1 \to K_2$ with inverse h_{12}. The composition operator $C_{21} : C(K_1) \to C(K_2), f \mapsto f \circ h_{12}$ is a lattice isomorphism.

Set $L_{21} = T_2{}^{-1} C_{21} T_1 : E_{u_1} \to E_{u_2}$. L_{21} is a lattice isomorphism, and for $j \in \{0.1 \dots l\}$ we have $A_{j,2}L_{21} = L_{21}A_{j,1}$, since $A_{j,2}$ and $A_{j,1}$ are induced by A_j.

Now

$$\begin{aligned} T_2P_2(\rho_1\omega_1)T_2{}^{-1}C_{21}f_1 &= T_2P_2(\rho_1\omega_1)L_{21}T_1{}^{-1}f_1 = \\ = T_2L_{21}P_1(\rho_1\omega_1)T_1{}^{-1}f_1 &= C_{21}T_1P_1(\rho_1\omega_1)T_1{}^{-1}f_1 = 0_{E_{u_2}} \end{aligned}$$

holds and $\rho_1 \in \Delta_P$, therefore $\omega_1 \in \Gamma_2$. We proved that $\Gamma_1 \subset \Gamma_2$, the reverse inclusion follows by symmetry. Therefore $\omega_1\omega_2 \in \Gamma_1 = \Gamma_2 \subset \Gamma$. ∎

LEMMA 4.4. *Assume that $A(1)$ is irreducible and has a compact power. Then for all $\rho > 0$ such that $\operatorname{spr}(A(\rho)) = \rho^m$*

1.) $\rho \in \Sigma_{\text{point}}(P(\cdot))$.
2.) $\Delta_P = \{\rho > 0\colon \operatorname{spr}(A(\rho)) = \rho^m\}$*; for the definition of* Δ_P *see Theorem 4.3.*
3.) *each* $\lambda \in \Sigma(P(\cdot)) \cap \Pi_\rho$ *is a pole of* $P^{-1}(\cdot)$ *and an eigenvalue of* $P(\cdot)$ *of geometric multiplicity 1, i.e.* $\dim(\ker(P(\lambda))) = 1$.

PROOF. By Lemma 4.1.3.) for each $\lambda \neq 0$ the operator $P(\lambda)$ is a Fredholm operator with index 0, and any $\lambda \in \Sigma(P(\cdot)) \setminus \{0\}$ is an eigenvalue of $P(\cdot)$. By [18, Lemma 4.2.11], $\rho^m = \operatorname{spr}(A(\rho))$ is an eigenvalue of $A(\rho)$. Therefore $\rho \in \Sigma(P(\cdot))$. By the same Lemma we obtain that $\rho^m = \operatorname{spr}(A(\rho))$ is an eigenvalue of the dual operator of $A(\rho)$ with a positive eigenfunctional, therefore 2.) holds. Assertion 3.) follows from [11, Corollary XI.8.4] and from Lemma 4.2.4.). ∎

REMARK. Under the assumptions of the lemma above for all positive ρ the spectral radius $\operatorname{spr}(A(\rho))$ of the operator $A(\rho)$ is an algebraic simple eigenvalue of the operator $A(\rho)$. By analytic perturbation theory (see [15, Chapters II, VII]), the function

$$\operatorname{spr}_A\colon]0,\infty[\to [0,\infty[, \quad \rho \mapsto \operatorname{spr}(A(\rho))$$

is real-analytic, and together with Lemma 2.2.1 we obtain that

- $\Delta_P =]0,\infty[$ or
- Δ_P contains at most two positive reals

holds. Simple (scalar) examples show that Δ_P can be empty, for all cases explicit examples can be found in [7]. In the monic case (i.e. $l < m$) there is exactly one positive real in Δ_P, which is the spectral radius of $P(\cdot)$ or, equivalently, of its companion operator and $\Sigma(P(\cdot)) \cap \Pi_\rho$ is the peripheral spectrum of $P(\cdot)$.

The following theorem is [14, Proposition 4.2.7] for the matrix case, and for $m = 1$ and $l = 0$ it is the so called Jentzsch-Perron Theorem, see [18, Corollary 4.2.12], [23, Theorem V.5.2], [24, Theorem 138.2] and [25, Theorem 44.8].

We will use some results about Fredholm operators and analytic Fredholm valued functions which can be found in [11, Chap. XI].

THEOREM 4.5. *Assume that $A(1)$ is irreducible and has a compact power. Then either*

1.) $\Sigma_{\text{point}}(P(\cdot)) \cap \Pi_\rho = \Pi_\rho$ *for some (and then all)* $\rho > 0$, *or*
2.) *there exists a* $k \in \{1, 2, \ldots\}$ *such that*

$$\Sigma_{\text{point}}(P(\cdot)) \cap \Pi_\rho = \left\{ \rho e^{\frac{2\pi i}{k} n} : n = 0, 1, \ldots, k-1 \right\},$$

for all $\rho > 0$ *such that* $\operatorname{spr}(A(\rho)) = \rho^m$.

PROOF. Let $\lambda \in \mathbb{C} \setminus \{0\}$. By Lemma 4.1.3.) $A(\lambda)$ is a Riesz operator, therefore $P(\lambda) = \lambda^m I_E - A(\lambda)$ is a Fredholm operator with index 0. From [11, Theorem XI.8.2] follows that only the following two (exclusive) cases can occur:

(i) $P(\lambda)$ is not invertible for all $\lambda \in \mathbb{C} \setminus \{0\}$,
(ii) there exists a (at most countable) subset Σ of $\mathbb{C} \setminus \{0\}$ which has no accumulation point in $\mathbb{C} \setminus \{0\}$ such that $P(\lambda)$ is invertible for all $\lambda \in \mathbb{C} \setminus \{\Sigma \cup \{0\}\}$.

In the first case we obtain assertion 1.). In the second case we obtain from Lemma 4.4 that $\Sigma_{\text{point}}(P(\cdot)) \cap \Pi_\rho$ is a non-empty finite set for all $\rho > 0$ such that $\operatorname{spr}(A(\rho)) = \rho^m$. Thus assertion 2.) follows from Theorem 4.3. ∎

REMARK. For the matrix case in [14, Theorem 4.23] the number k is characterized as the so-called "index of phase irreduciblity" of an infinite graph; which is in the case that $l = 0$ and $m = 1$ up to a sign equal to the usual index of irreducibility of entrywise nonnegative matrices, see [19].

THEOREM 4.6. *Assume that $A(1)$ is irreducible and has a compact power. Let $\lambda \in \Sigma_{\text{point}}(P(\cdot)) \cap \Pi_\rho$ for some $\rho > 0$ such that $\operatorname{spr}(A(\rho)) = \rho^m$. Then the following assertions hold:*

1.) *If* $\operatorname{spr}_A{}^{(1)}(\rho) \neq m\rho^{m-1}$, *then λ is an eigenvalue of $P(\cdot)$ of algebraic multiplicity 1.*
2.) *If* $\operatorname{spr}_A{}^{(1)}(\rho) = m\rho^{m-1}$ *and* $\operatorname{spr}_A(\hat{\rho}) \neq \hat{\rho}^m$ *for at least one positive* $\hat{\rho}$, *then ρ is an eigenvalue of $P(\cdot)$ of geometric multiplicity 1 and of algebraic multiplicity 2.*

PROOF. We mentioned above that under the assumption of the theorem it follows from the analytic perturbation theory that the function $\mathrm{spr}_A\colon]0,\infty[\to [0,\infty[$ is real-analytic, therefore its derivative ${\mathrm{spr}_A}^{(1)}(\cdot)$ exists on the positive half-line. Moreover, from [15, Section II.2.1] follows that there exists an analytic function

$$u\colon]0,\infty[\to E\setminus\{0_E\}$$

such that

$$(\mathrm{spr}_A(\tau)I_E - A(\tau))u(\tau) = 0_E \quad \text{for all} \quad \tau\in]0,\infty[.$$

Then

$$({\mathrm{spr}_A}^{(1)}(\tau)I_E - A^{(1)}(\tau))u(\tau) + (\mathrm{spr}_A(\tau)I_E - A(\tau))u^{(1)}(\tau) = 0_E$$

for all $\tau\in]0,\infty[$.

PROOF OF ASSERTION 1.). First we will prove that for $\rho > 0$ such that $\mathrm{spr}(A(\rho)) = \mathrm{spr}_A(\rho) = \rho^m$, this ρ is an eigenvalue of $P(\cdot)$ of algebraic multiplicity 1, i.e., we shall prove that there does not exist a (nontrivial) Jordan chain of $P(\cdot)$ at ρ of length 2.

ρ is an eigenvalue of $P(\cdot)$, see Lemma 4.4.1.). Let $u_0\in\ker(P(\rho))$, $u_0\neq 0_E$. By Lemma 4.2.3.), u_0 is a nonzero multiple of $u(\rho)$; we can and will assume w.l.o.g. that $u_0 = u(\rho)$.

Assume that there exists an $u_1\in E$ such that $P^{(1)}(\rho)u_0 + P(\rho)u_1 = 0_E$, i. e.

$$\left(m\rho^{m-1}I_E - A^{(1)}(\rho)\right)u_0 + (\rho^m I_E - A(\rho))\,u_1 = 0_E.$$

Subtracting the equations in the last two formulas (for $\tau=\rho$) we obtain

$$0_E = \left(m\rho^{m-1}I_E - {\mathrm{spr}_A}^{(1)}(\rho)\right)u_0 + (\rho^m - A(\rho))\left(u_1 - u^{(1)}(\rho)\right).$$

Since ${\mathrm{spr}_A}^{(1)}(\rho)\neq m\rho^{m-1}$, it follows that

$$u_0\in \mathrm{Ran}(\mathrm{spr}(A(\rho))I_E - A(\rho))\cap\ker(\mathrm{spr}(A(\rho))I_E - A(\rho)).$$

$A(\rho)$ is irreducible and has a compact power. By [18, Corollary 4.2.14], then $\mathrm{spr}(A(\rho))$ is an eigenvalue of $A(\rho)$ of algebraic multiplicity 1. Therefore $\mathrm{spr}(A(\rho))$ is a pole of order 1 of the resolvent of $A(\rho)$, which implies that the intersection in the last formula is trivial. Now assertion 1.) follows from Proposition 2.4, since $\mathrm{spr}_A(\rho)=\rho^m$ and ${\mathrm{spr}_A}^{(1)}(\rho)\neq m\rho^{m-1}$ imply that there exist a $\hat\rho>0$ such that $\mathrm{spr}_A(\hat\rho)) < \hat\rho^m$.

PROOF OF ASSERTION 2.). The geometric convexity of spr_A and the assumptions in assertion 2.) imply that $\mathrm{spr}_A(\tau) > \tau^m$ for all $\tau \in]0,\infty[$ such that $\tau \neq \rho$, and

$$\mathrm{spr}_A^{(2)}(\rho) \neq m(m-1)\rho^{m-2},$$

see Lemma 2.2.3. Let $u_0 \in \ker(P(\rho))$, $u_0 \neq 0_E$. Using the function $u(\cdot)$ from the beginning of the proof, we assume that $u_0 = u(\rho)$. If we set $u_1 = u^{(1)}(\rho)$, it follows immediately that $\{u_0, u_1\}$ is a Jordan chain of $P(\cdot)$ at ρ.

Assume that $\{u_0, u_1, u_2\}$ is a Jordan chain of $P(\cdot)$ at ρ of length 3. Similarly as in the proof of assertion 1.), we obtain

$$\left(\frac{1}{2}m(m-1)\rho^{m-2} - \mathrm{spr}_A^{(2)}(\rho)\right) u_0 = P(\rho)\left(u_2 - u^{(2)}(\rho)\right).$$

This implies that

$$\begin{aligned} u_0 &\in \mathrm{Ran}(P(\rho)) \cap \ker(P(\rho)) = \\ &= \mathrm{Ran}(\mathrm{spr}(A(\rho))I_E - A(\rho)) \cap \ker(\mathrm{spr}(A(\rho))I_E - A(\rho)). \end{aligned}$$

Again we have obtained a contradiction to $u_0 \neq 0_E$. Therefore ρ is an eigenvalue of $P(\cdot)$ of geometric multiplicity 1 and of algebraic multiplicity 2. ∎

We conclude the paper with the following conjecture.

CONJECTURE. Under the assumption of assertion 2.) all

$$\lambda \in \Sigma_{\mathrm{point}}(P(\cdot)) \cap \Pi_\rho$$

are eigenvalues of $P(\cdot)$ of geometric multiplicity 1 and of algebraic multiplicity 2.

For the matrix case this conjecture is proved in [14, Prop. 4.27].

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FACTORIZATION, MAJORIZATION, AND DOMINATION FOR LINEAR RELATIONS

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To our friend Zoltán Sebestyén on the occasion of his 70th birthday

Abstract. Let $\mathfrak{H}_A$, $\mathfrak{H}_B$, and $\mathfrak{H}$ be Hilbert spaces. Let A be a linear relation from $\mathfrak{H}$ to $\mathfrak{H}_A$ and let B be a linear relation from $\mathfrak{H}$ to $\mathfrak{H}_B$. If there exists an operator $Z \in \mathbf{B}(\mathfrak{H}_B, \mathfrak{H}_A)$ such that $ZB \subset \subset A$, then B is said to dominate A. This notion plays a major role in the theory of Lebesgue type decompositions of linear relations and operators. There is a strong connection to the majorization and factorization in the well-known lemma of Douglas, when put in the context of linear relations. In this note some aspects of the lemma of Douglas are discussed in the context of linear relations and the connections with the notion of domination will be treated.

1. Introduction

Let A and B be a pair of linear relations with their domains of definition in the same Hilbert space $\mathfrak{H}$ and their ranges in the Hilbert spaces $\mathfrak{H}_A$ and $\mathfrak{H}_B$, respectively. The relation B is said to dominate the relation A if there exists a bounded linear operator Z from $\mathfrak{H}_B$ to $\mathfrak{H}_A$ such that $ZB \subset A$. Domination is preserved when the closures of A and B are considered. In the particular case that A and B are, not necessarily densily defined, operators this is equivalent to $\operatorname{dom} A \subset \operatorname{dom} B$ and the existence of a constant $c \geq 0$ such that $\|Af\| \leq c\|Bf\|$ holds for all $f \in \operatorname{dom} A$. The notion of domination, which is familiar from measure theory, plays an important role in the theory of Lebesgue type decompositions. This notion and its role in Lebesgue type decompositions for a pair of bounded operators go back to Ando [1]; it has a similar position when decomposing a nonnegative form with respect to an another nonnegative form, see [11], or when decomposing an unbounded operator or a linear relation [12, 13, 14], where some further history and references can be found.

In the present paper it will be shown that domination is closely related to the following well-known lemma of R. G. Douglas [6] when that lemma is put in the context of unbounded linear operators or, more generally, linear relations.

LEMMA 1.1 (Douglas). *Let $A, B \in \mathbf{B}(\mathfrak{H}, \mathfrak{K})$, the bounded everywhere defined linear operators from a Hilbert space $\mathfrak{H}$ to a Hilbert space $\mathfrak{K}$. Then the following statements are equivalent:*

(i) $\operatorname{ran} A \subset \operatorname{ran} B$;
(ii) $A = BW$ *for some bounded linear operator* $W \in \mathbf{B}(\mathfrak{H})$;
(iii) $AA^* \leq \lambda BB^*$ *for some* $\lambda \geq 0$.

If the equivalent conditions (i)–(iii) hold, then there is a unique operator W such that

(a) $\|W\|^2 = \inf \{\mu \colon AA^* \leq \mu BB^*\}$;
(b) $\ker A = \ker W$;
(c) $\operatorname{ran} W \subset \overline{\operatorname{ran}}\, B^*$.

In the literature one can find a statement which is equivalent to the three items in Lemma 1.1, namely

(iv) $AA^* = BMB^*$, where $M \in \mathbf{B}(\mathfrak{H})$ is nonnegative and $\|M\| \leq \lambda$.

One may take $\operatorname{ran} M \subset \overline{\operatorname{ran}}\, B^*$. In addition to the results in the above lemma Douglas indicated some further results for the case when A and B are densely defined closed linear operators; see [6]. Various extensions of these basic results by Douglas can be found in the literature; see, for instance, [4, 7, 8]. The factorization aspect of the Douglas lemma was recently put in the context of linear relations by D. Popovici and Z. Sebestyén [16]; see also some refinements by A. Sandovici and Z. Sebestyén [17]. For the majorization aspect of the Douglas lemma, see [3].

The contents of the present paper are now briefly explained. For closed linear operators or relations A and B the following equivalence will be established in Theorem 3.4:

$$A \subset BW \quad \Leftrightarrow \quad AA^* \leq c^2 BB^*,$$

where W is a bounded linear operator and $c \geq 0$, in fact $\|W\| \leq c$. This result characterizes majorization in terms of a simple factorization type inclusion. Domination for a pair of closed linear operators or relations can be characterized in a similar way:

$$ZB \subset A \quad \Leftrightarrow \quad A^*A \leq c^2 B^*B,$$

see Theorem 4.4. Some consequences of these results will be explored in Section 3 and Section 4. In particular, a characterization of the equalities $A = BW$ and $ZB = A$ is given. For bounded linear operators the factorization $A = BW$ in the original Douglas lemma can be directly connected to the notion of domination for linear relations by means of the following observation:

$$A = BW \quad \Leftrightarrow \quad WA^{-1} \subset B^{-1},$$

see Lemma 5.1. This last equivalence, when combined with the two earlier equivalences, provides a simple proof for the characterization of the ordering of nonnegative selfadjoint relations in terms of resolvents; see Theorem 5.2. For the convenience of the reader some results concerning closed nonnegative forms and associated linear relations will be recalled in Section 2.

2. Preliminaries

Let H be a linear relation from a Hilbert space $\mathfrak{H}$ to a Hilbert space $\mathfrak{K}$; i.e., H is a linear subspace of the product $\mathfrak{H} \times \mathfrak{K}$. The domain, range, kernel, and multivalued part of H are denoted by $\operatorname{dom} H$, $\operatorname{ran} H$, $\ker H$, and $\operatorname{mul} H$. The formal inverse H^{-1} of H is a relation from $\mathfrak{K}$ to $\mathfrak{H}$, defined by $H^{-1} = \{\{f',f\} : \{f,f'\} \in H\}$, so that $\operatorname{dom} H^{-1} = \operatorname{ran} H$, $\operatorname{ran} H^{-1} = \operatorname{dom} H$, $\ker H^{-1} = \operatorname{mul} H$, and $\operatorname{mul} H^{-1} = \ker H$. For $\mathfrak{L} \subset \mathfrak{H}$ the set $H(\mathfrak{L})$ is a subset of $\mathfrak{K}$ defined by

$$H(\mathfrak{L}) = \{h' : \{h, h'\} \in H \text{ for some } h \in \mathfrak{L}\}.$$

In particular, $H(\{0\}) = \operatorname{mul} H$.

Let H_1 and H_2 be relations from a Hilbert space $\mathfrak{H}$ to a Hilbert space $\mathfrak{K}$. Then H_1 is a *restriction* of H_2 and H_2 is an *extension* of H_1 if $H_1 \subset H_2$.

Proposition 2.1. *Let H_1 and H_2 be relations from a Hilbert space $\mathfrak{H}$ to a Hilbert space $\mathfrak{K}$ and assume that $H_1 \subset H_2$. Then the following statements are equivalent:*

(i) $\operatorname{dom} H_1 = \operatorname{dom} H_2$;
(ii) $H_2 = H_1 \,\widehat{+}\, (\{0\} \times \operatorname{mul} H_2)$.

Moreover, the following statements are equivalent:

(iii) $\operatorname{ran} H_1 = \operatorname{ran} H_2$;
(iv) $H_2 = H_1 \,\widehat{+}(\ker H_2 \times \{0\})$.

Proof. By symmetry it suffices to show the equivalence between (i) and (ii).

(i) $\Rightarrow$ (ii) It suffices to show that $H_2 \subset H_1 \,\widehat{+}(\{0\} \times \operatorname{mul} H_2)$. Let $\{h, h'\} \in H_2$. Since $h \in \operatorname{dom} H_2 \subset \operatorname{dom} H_1$, there exists an element $k' \in \mathfrak{K}$ such that $\{h, k'\} \in H_1$. Hence, with $\varphi' = h' - k'$, it follows that

$$\{h, h'\} = \{h, k'\} + \{0, \varphi'\},$$

and thus $\{0, \varphi'\} \in H_2$ or $\varphi' \in \operatorname{mul} H_2$. Hence (ii) follows.

(ii) $\Rightarrow$ (i) This implication is trivial. ∎

The useful result in the following corollary can be found in [2].

Corollary 2.2. *Let H_1 and H_2 be relations from a Hilbert space $\mathfrak{H}$ to a Hilbert space $\mathfrak{K}$ and assume that $H_1 \subset H_2$. Then the following statements are equivalent:*

(i) $H_1 = H_2$;
(ii) $\operatorname{dom} H_1 = \operatorname{dom} H_2$ *and* $\operatorname{mul} H_1 = \operatorname{mul} H_2$;
(iii) $\operatorname{ran} H_1 = \operatorname{ran} H_2$ *and* $\ker H_1 = \ker H_2$.

Corollary 2.3. *Let H_1 and H_2 be relations from a Hilbert space $\mathfrak{H}$ to a Hilbert space $\mathfrak{K}$ and assume that $H_1 \subset H_2$. Then*

(i) $\operatorname{dom} H_1 = \mathfrak{H}$, $\operatorname{mul} H_2 = \{0\} \Rightarrow H_1 = H_2$;
(ii) $\operatorname{ran} H_1 = \mathfrak{K}$, $\ker H_2 = \{0\} \Rightarrow H_1 = H_2$.

The sum of two linear relations H_1 and H_2 from $\mathfrak{H}$ to $\mathfrak{K}$ is a linear relation defined by

$$H_1 + H_2 = \{\{f, f' + f''\} : \{f, f'\} \in H_1, \{f, f''\} \in H_2\},$$

while their componentwise sum is a linear relation defined by

$$H_1 \,\widehat{+}\, H_2 = \{\{f + g, f' + g'\} : \{f, f'\} \in H_1, \{g, g'\} \in H_2\}.$$

Let H_1 be a relation from a Hilbert space $\mathfrak{H}$ to a Hilbert space $\mathfrak{M}$ and let H_2 be a relation from a Hilbert space $\mathfrak{M}$ to a Hilbert space $\mathfrak{K}$. The product H_2H_1 is a linear relation from $\mathfrak{H}$ to $\mathfrak{K}$ defined by

$$H_2H_1 = \{\{f, f'\} : \{f, \varphi\} \in H_1, \{\varphi, f'\} \in H_2 \text{ for some } \varphi \in \mathfrak{M}\}. \tag{2.1}$$

Observe, that

$$\ker(H_2H_1) = H_1^{-1}(\ker H_2) = \{f \in \mathfrak{H} : \{f, \varphi\} \in H_1 \text{ for some } \varphi \in \ker H_2\}, \tag{2.2}$$

and

$$\operatorname{mul} H_2H_1 = H_2(\operatorname{mul} H_1) = \{f' \in \mathfrak{K} : \{\varphi, f'\} \in H_2 \text{ for some } \varphi \in \operatorname{mul} H_1\}. \tag{2.3}$$

In particular, $\ker H_1 \subset \ker H_1H_2$ and $\operatorname{mul} H_2 \subset \operatorname{mul} H_2H_1$. The following identities are also easy to check:

$$HH^{-1} = I_{\operatorname{ran} H} \,\widehat{+}\, (\{0\} \times \operatorname{mul} H) \quad \text{and} \quad H^{-1}H = I_{\operatorname{dom} H} \,\widehat{+}\, (\{0\} \times \ker H) \tag{2.4}$$

with both sums direct. Hence, in particular,

$$\operatorname{mul} H = \{0\} \Rightarrow HH^{-1} = I_{\operatorname{ran} H}; \quad \ker H = \{0\} \Rightarrow H^{-1}H = I_{\operatorname{dom} H}. \tag{2.5}$$

The closure of a linear relation H from $\mathfrak{H}$ to $\mathfrak{K}$ is the closure of the linear subspace in $\mathfrak{H} \times \mathfrak{K}$, when the product is provided with the product topology. The closure of an operator need not be an operator; if it is then one speaks of a closable operator. The relation H is called closed when it is closed as a subspace of $\mathfrak{H} \times \mathfrak{K}$. In this case both $\ker H \subset \mathfrak{H}$ and $\operatorname{mul} H \subset \mathfrak{K}$ are closed subspaces.

Let H be a closed linear relation from $\mathfrak{H}$ to $\mathfrak{K}$. Then $H_{\mathrm{mul}} = \{0\} \times \operatorname{mul} H$ is a closed linear relation and $H_{\mathrm{s}} = H \,\widehat{\ominus}\, H_{\mathrm{mul}}$, so that $\operatorname{dom} H_{\mathrm{s}} = \operatorname{dom} H$ is dense in $\overline{\operatorname{dom}} H = \mathfrak{H} \ominus \operatorname{mul} H^*$, while $\operatorname{ran} H_{\mathrm{s}} \subset \overline{\operatorname{dom}} H^* = \mathfrak{K} \ominus \operatorname{mul} H$. The operator part H_{s} and H_{mul} lead to the componentwise orthogonal decomposition

$$H = H_{\mathrm{s}} \,\widehat{\oplus}\, H_{\mathrm{mul}}. \tag{2.6}$$

The adjoint relation H^* from $\mathfrak{K}$ to $\mathfrak{H}$ is defined by $H^* = JH^{\perp} = (JH)^{\perp}$, where $J\{f, f'\} = \{f', -f\}$. The adjoint is automatically a closed linear relation and the closure of H is given by H^{**}. The operator part $(H^*)_{\mathrm{s}}$ is densely defined in $\overline{\operatorname{dom}} H^* = \mathfrak{H} \ominus \operatorname{mul} H^{**}$ and maps into $\overline{\operatorname{dom}} H = \overline{\operatorname{dom}} H^{**} = \mathfrak{H} \ominus \operatorname{mul} H^*$. When H is closed the operator parts H_{s} and $(H^*)_{\mathrm{s}}$ are connected by

$$(H_{\mathrm{s}})^{\times} = (H^*)_{\mathrm{s}}, \tag{2.7}$$

where $(H_{\mathrm{s}})^{\times}$ denotes the adjoint of H_{s} in the sense of the smaller spaces $\overline{\operatorname{dom}} H$ and $\overline{\operatorname{dom}} H^*$.

Let H_1 be a relation from a Hilbert space $\mathfrak{H}$ to a Hilbert space $\mathfrak{M}$ and let H_2 be a relation from a Hilbert space $\mathfrak{M}$ to a Hilbert space $\mathfrak{K}$. The product satisfies

$$H_1^* H_2^* \subset (H_2 H_1)^*. \tag{2.8}$$

Moreover, if $H_2 \in \mathbf{B}(\mathfrak{M}, \mathfrak{K})$ then there is actually equality

$$H_1^* H_2^* = (H_2 H_1)^*, \tag{2.9}$$

see [10, Lemma 2.4], so that, in particular

$$H_2 H_1^{**} \subset (H_2 H_1)^{**}.$$

Assume that the relations H_1 and H_2 are closed. In general the product $H_2 H_1$ is not closed. However, if for instance $H_1 \in \mathbf{B}(\mathfrak{H}, \mathfrak{M})$, then the product $H_2 H_1$ is closed.

A linear relation H in a Hilbert space $\mathfrak{H}$ is symmetric if $H \subset H^*$ and selfadjoint if $H = H^*$. If the relation H is selfadjoint then $H_{\rm s}$ is a selfadjoint operator in $\overline{\operatorname{dom}}\, H = \mathfrak{H} \ominus \operatorname{mul} H$. A linear relation H in a Hilbert space $\mathfrak{H}$ is nonnegative if $(h', h) \geq 0$ for all $\{h, h'\} \in H$. In particular a nonnegative relation is symmetric.

An important special case of a nonnegative selfadjoint relation appears when one considers relations of the form $T^* T$ where T is a closed linear relation from a Hilbert space $\mathfrak{H}$ to a Hilbert space $\mathfrak{K}$; cf. [18].

LEMMA 2.4. *Let T be a closed relation from a Hilbert space $\mathfrak{H}$ to a Hilbert space $\mathfrak{K}$. Then the product $T^* T$ is a nonnegative selfadjoint relation in $\mathfrak{H}$. Furthermore,*

$$T^* T = T^* T_{\rm s} = (T_{\rm s})^* T_{\rm s}, \tag{2.10}$$

so that in particular

$$\ker(T^* T) = \ker T = \ker T_{\rm s}, \quad \operatorname{mul}(T^* T) = \operatorname{mul} T^* = \operatorname{mul}(T_{\rm s})^*. \tag{2.11}$$

The operator part of $T^* T$ can be rewritten as

$$(T^* T)_{\rm s} = (T^*)_{\rm s} T_{\rm s} = (T_{\rm s})^{\times} T_{\rm s}. \tag{2.12}$$

PROOF. It is clear from the definition that $T^* T$ is a nonnegative selfadjoint relation in $\mathfrak{H}$. In fact $T^* T$ is selfadjoint since $\operatorname{ran}(T^* T + I) = \mathfrak{H}$, which follows from $\mathfrak{H}^2 = T \,\widehat{\oplus}\, T^{\perp} = T \,\widehat{\oplus}\, J T^*$.

Next, let P be the orthogonal projection from $\mathfrak{K}$ onto $\overline{\operatorname{dom}}\, T^*$, so that $PT = T_{\rm s} \subset T$. Since $\operatorname{dom} T^* \subset \overline{\operatorname{dom}}\, T^* = \operatorname{ran} P$ it also follows that $T^* \subset T^* P$. Hence

$$T^* T \subset T^* P T \subset T^* T, \tag{2.13}$$

so that the inclusions are both equalities. From $PT = T_{\rm s}$ one obtains that $(T_{\rm s})^* = (PT)^* = T^* P$, so that (2.13) leads to (2.10). Since $T_{\rm s}$ is an operator, (2.11) is immediate from (2.10). Furthermore, (2.12) is clear from (2.7). ∎

LEMMA 2.5. *Let H be a nonnegative selfadjoint relation in a Hilbert space $\mathfrak{H}$. Then there exists a unique nonnegative selfadjoint relation K in $\mathfrak{H}$, denoted by $K = H^{\frac{1}{2}}$, such that $K^2 = H$. Moreover, $H^{\frac{1}{2}}$ has the representation*

$$H^{\frac{1}{2}} = H_{\rm s}^{\frac{1}{2}} \,\widehat{\oplus}\, H_{\rm mul}. \tag{2.14}$$

PROOF. It is clear that K defined by the right hand side of (2.14) is a nonnegative selfadjoint relation with $\operatorname{mul} K = \operatorname{mul} H$. To see that $K^2 = H$, let $\{f, f'\} \in K^2$. Then $\{f, \varphi\} \in K$ and $\{\varphi, f'\} \in K$. Clearly,

$$\varphi = H_s^{\frac{1}{2}} f + \alpha, \quad f' = H_s^{\frac{1}{2}} \varphi + \beta,$$

with $\alpha \in \operatorname{mul} H$ and $\beta \in \operatorname{mul} H$. Since $\varphi \in \operatorname{dom} H_s^{\frac{1}{2}}$ it follows that $\alpha = 0$ and $f' = H_s f + \beta$, so that $\{f, f'\} \in H$. It follows that $K^2 \subset H$, and since $K^2 = K^* K$ is selfadjoint, it follows that $K^2 = H$.

In order to show uniqueness, let K be a nonnegative selfadjoint relation such that $K^2 = H$. Then

$$\operatorname{mul} K = \operatorname{mul} H.$$

To see this, first observe that by (2.3) $\operatorname{mul} K \subset \operatorname{mul} K^2 = \operatorname{mul} H$. For the reverse inclusion, let $\{0, \psi\} \in H = K^2$. Then $\{0, \varphi\} \in K$ and $\{\varphi, \psi\} \in K$. Since K is selfadjoint it follows that $\varphi = 0$, so that $\{0, \psi\} \in K$ and $\operatorname{mul} H \subset \operatorname{mul} K$. This implies that $K = K_s \oplus H_{\mathrm{mul}}$, where K_s is a nonnegative selfadjoint operator. It will now be shown that $H_s = (K_{\mathrm{s}})^2$, and since the square root of a nonnegative selfadjoint operator is uniquely determined it follows that $K_{\mathrm{s}} = H_{\mathrm{s}}^{1/2}$.

To see that $H_{\mathrm{s}} = K_{\mathrm{s}}^2$, let $\{f, f'\} \in H_{\mathrm{s}}$. Then $\{f, f'\} \in H = K^2$ and $f' \perp \operatorname{mul} H$. Now $\{f, \varphi\} \in K$ and $\{\varphi, f'\} \in K$ for some $\varphi \in \overline{\operatorname{dom} K} = \overline{\operatorname{dom} H}$. Hence $\{f, \varphi\} \in K_{\mathrm{s}}$ and $\{\varphi, f'\} \in K_{\mathrm{s}}$, so that $\{f, f'\} \in K_{\mathrm{s}}^2$. Thus $H_{\mathrm{s}} \subset (K_{\mathrm{s}})^2$. For the converse inclusion, let $\{f, f'\} \in (K_{\mathrm{s}})^2$. Then $\{f, \varphi\} \in K_{\mathrm{s}} \subset K$, $\{\varphi, f'\} \in K_{\mathrm{s}} \subset K$, so that $\{f, f'\} \in K^2 = H$. Since $f' \perp \operatorname{mul} H$, it follows that $\{f, f'\} \in H_{\mathrm{s}}$. ∎

Let H be a nonnegative selfadjoint relation. Since Lemma 2.5 implies that $\operatorname{mul} H^{\frac{1}{2}} = \operatorname{mul} H$, it follows that

$$(H^{\frac{1}{2}})_{\mathrm{s}} = (H_{\mathrm{s}})^{\frac{1}{2}},$$

so that the notation $H_{\mathrm{s}}^{\frac{1}{2}}$ is unambiguous. Furthermore it is clear that

$$\operatorname{dom} H \subset \operatorname{dom} H^{\frac{1}{2}} \subset \overline{\operatorname{dom}} H^{\frac{1}{2}} = \overline{\operatorname{dom}} H. \tag{2.15}$$

Therefore the following statements are equivalent:

$$\operatorname{dom} H \text{ closed}; \quad \operatorname{dom} H^{\frac{1}{2}} \text{ closed}; \quad \operatorname{dom} H = \operatorname{dom} H^{\frac{1}{2}}. \tag{2.16}$$

Let H be a nonnegative selfadjoint relation. Then for each $x > 0$,

$$\operatorname{dom} (H + x)^{1/2} = \operatorname{dom} H^{1/2}, \tag{2.17}$$

and, moreover,

$$\left\| (H_s + x)^{1/2} h \right\|^2 = \left\| (H^{1/2})_s h \right\|^2 + x \left\| h \right\|^2, \quad h \in \operatorname{dom} H^{1/2}. \tag{2.18}$$

It is clear that the identity holds for $h \in \operatorname{dom} H$ and since $\operatorname{dom} H$ is a core for $H^{1/2}$ it holds for $h \in \operatorname{dom} H^{1/2}$.

There is a natural ordering for nonnegative selfadjoint relations in a Hilbert space $\mathfrak{H}$; it is inspired by the corresponding situation for selfadjoint operators $H_1, H_1 \in \mathbf{B}(\mathfrak{H})$.

Two nonnegative selfadjoint relations H_1 and H_2 are said to satisfy the inequality $H_1 \leq H_2$ if

$$\operatorname{dom} H_{2s}^{\frac{1}{2}} \subset \operatorname{dom} H_{1s}^{\frac{1}{2}}, \quad \left\| H_{1s}^{\frac{1}{2}} h \right\| \leq \left\| H_{2s}^{\frac{1}{2}} h \right\|, \quad h \in \operatorname{dom} H_2^{\frac{1}{2}}. \tag{2.19}$$

It follows from (2.17) and (2.18) that $H_1 \leq H_2$ if and only if $H_1 + x \leq H_2 + x$ for some (and hence for all) $x > 0$.

A sesquilinear form (or form for short) $\mathfrak{t}[\cdot,\cdot]$ in a Hilbert space $\mathfrak{H}$ is a mapping from $\mathfrak{D} \times \mathfrak{D}$ to $\mathbb{C}$ where $\mathfrak{D}$ is a (not necessarily densely defined) linear subspace of $\mathfrak{H}$, such that it is linear in the first entry and anti-linear in the second entry. The domain $\operatorname{dom} \mathfrak{t}$ is defined by $\operatorname{dom} \mathfrak{t} = \mathfrak{D}$. The corresponding quadratic form $\mathfrak{t}[\cdot]$ is defined by $\mathfrak{t}[\varphi] = \mathfrak{t}[\varphi, \varphi]$, $\varphi \in \operatorname{dom} \mathfrak{t}$. A sesquilinear form $\mathfrak{t}$ is said to be nonnegative if

$$\mathfrak{t}[\varphi] \geq 0, \quad \varphi \in \operatorname{dom} \mathfrak{t}.$$

The nonnegative form $\mathfrak{t}$ in $\mathfrak{H}$ is said to be *closed* if for any sequence (φ_n) in $\operatorname{dom} \mathfrak{t}$ one has

$$\varphi_n \to \varphi, \quad \mathfrak{t}[\varphi_n - \varphi_m] \to 0, \quad \Rightarrow \quad \varphi \in \operatorname{dom} \mathfrak{t}, \quad \mathfrak{t}[\varphi_n - \varphi] \to 0. \tag{2.20}$$

The inequality $\mathfrak{t}_1 \leq \mathfrak{t}_2$ for forms $\mathfrak{t}_1$ and $\mathfrak{t}_2$ is defined by

$$\operatorname{dom} \mathfrak{t}_2 \subset \operatorname{dom} \mathfrak{t}_1, \quad \mathfrak{t}_1[h] \leq \mathfrak{t}_2[h], \quad h \in \operatorname{dom} \mathfrak{t}_2. \tag{2.21}$$

In particular, $\mathfrak{t}_2 \subset \mathfrak{t}_1$ implies $\mathfrak{t}_1 \leq \mathfrak{t}_2$.

The theory of nonnegative forms can be found in [15]. The representation theorem gives a connection between nonnegative selfadjoint relations and nonnegative closed forms; see [9, 15].

THEOREM 2.6 (representation theorem). *Let $\mathfrak{t}$ be a closed nonnegative form in the Hilbert space $\mathfrak{H}$. Then there exists a nonnegative selfadjoint relation H in $\mathfrak{H}$ such that*

(i) $\operatorname{dom} H \subset \operatorname{dom} \mathfrak{t}$ *and*

$$\mathfrak{t}[\varphi, \psi] = (\varphi', \psi) \tag{2.22}$$

for every $\{\varphi, \varphi'\} \in H$ and $\psi \in \operatorname{dom} \mathfrak{t}$;

(ii) $\operatorname{dom} H$ *is a core for* $\mathfrak{t}$ *and* $\operatorname{mul} H = (\operatorname{dom} \mathfrak{t})^{\perp}$*;*

(iii) *if $\varphi \in \operatorname{dom} \mathfrak{t}$, $\omega \in \mathfrak{H}$, and*

$$\mathfrak{t}[\varphi, \psi] = (\omega, \psi) \tag{2.23}$$

holds for every ψ in a core of $\mathfrak{t}$, then $\{\varphi, \omega\} \in H$.

The nonnegative selfadjoint relation H is uniquely determined by (i).

The following result is a direct consequence of the representation theorem.

PROPOSITION 2.7. *Let T be a closed linear relation from a Hilbert space $\mathfrak{H}$ to a Hilbert space $\mathfrak{K}$. The nonnegative selfadjoint relation T^*T in the Hilbert space $\mathfrak{H}$ corresponds to the closed nonnegative form*

$$\mathfrak{t}[h, k] = (T_s h, T_s k)_{\mathfrak{K}}, \quad h, k \in \operatorname{dom} \mathfrak{t} = \operatorname{dom} T_s = \operatorname{dom} T, \tag{2.24}$$

and, in particular,

$$\mathfrak{t}[h,k] = \big((T_{\mathrm{s}})^{\times} T_{\mathrm{s}} h, k\big)_{\mathfrak{K}}, \quad h \in \operatorname{dom} T^*T, \quad k \in \operatorname{dom} \mathfrak{t}. \tag{2.25}$$

PROOF. Since T_{s} is a closed linear operator, it follows that the form in (2.24) is closed. Clearly, if in (2.24) one assumes that $h \in \operatorname{dom} T^*T = \operatorname{dom}(T_{\mathrm{s}})^{\times} T_{\mathrm{s}}$, see (2.12), then (2.25) follows. The result is now obtained from Theorem 2.6. ∎

Proposition 2.7 combined with (2.12) in Lemma 2.4 yields the so-called second representation theorem for closed forms.

COROLLARY 2.8. *Let $\mathfrak{t}$ be a closed nonnegative form in the Hilbert space $\mathfrak{H}$ and let H be the corresponding nonnegative selfadjoint relation H in $\mathfrak{H}$ as in Theorem 2.6. Then*

$$\operatorname{dom} \mathfrak{t} = \operatorname{dom} H_{\mathrm{s}}^{\frac{1}{2}} \quad \textit{and} \quad \mathfrak{t}[\varphi,\psi] = \left(H_{\mathrm{s}}^{\frac{1}{2}}\varphi, H_{\mathrm{s}}^{\frac{1}{2}}\psi\right), \quad \varphi,\psi \in \operatorname{dom} \mathfrak{t}. \tag{2.26}$$

A subset of $\operatorname{dom} \mathfrak{t} = \operatorname{dom} H_{\mathrm{s}}^{\frac{1}{2}}$ *is a core of the form* $\mathfrak{t}$ *if and only if it is a core of the operator* $H_{\mathrm{s}}^{\frac{1}{2}}$. *In particular,* $\operatorname{dom} H$ *is a core of* $H^{\frac{1}{2}}$.

As a straightforward consequence of the representation theorem one can state the following result which connects inequalities between nonnegative selfadjoint relations with inequalities between the corresponding nonnegative closed forms.

COROLLARY 2.9. *Let $\mathfrak{t}_1$ and $\mathfrak{t}_2$ be closed nonnegative forms and let H_1 and H_2 be the corresponding nonnegative selfadjoint relations. Then*

$$\mathfrak{t}_1 \le \mathfrak{t}_2 \quad \textit{if and only if} \quad H_1 \le H_2. \tag{2.27}$$

COROLLARY 2.10. *Let $\mathfrak{H}$, $\mathfrak{H}_1$, and $\mathfrak{H}_2$ be Hilbert spaces. Let T_1 be a closed linear relation from $\mathfrak{H}$ into $\mathfrak{K}_1$ and let T_2 be a closed linear relation from $\mathfrak{H}$ into $\mathfrak{K}_2$. Then $T_1^*T_1 \le T_2^*T_2$ if and only if*

$$\operatorname{dom} T_2 \subset \operatorname{dom} T_1 \quad \textit{and} \quad \|(T_1)_{\mathrm{s}}h\|_{\mathfrak{K}_1} \le \|(T_2)_{\mathrm{s}}h\|_{\mathfrak{K}_2}, \quad h \in \operatorname{dom} T_2.$$

PROOF. Let $\mathfrak{t}_1$ and $\mathfrak{t}_2$ be the closed nonnegative forms in the Hilbert space $\mathfrak{H}$ induced by $T_1^*T_1$ and $T_2^*T_2$. Hence by Corollary 2.9 one has $T_1^*T_1 \le T_2^*T_2$ if and only if $\mathfrak{t}_1 \le \mathfrak{t}_2$. By Proposition 2.7 $\mathfrak{t}_1 \le \mathfrak{t}_2$ if and only if

$$\operatorname{dom} T_2 \subset \operatorname{dom} T_1, \quad ((T_1)_{\mathrm{s}}h, (T_1)_{\mathrm{s}}h)_{\mathfrak{K}_1} \le ((T_2)_{\mathrm{s}}h, (T_2)_{\mathrm{s}}h)_{\mathfrak{K}_2}, \quad h \in \operatorname{dom} T_2. \quad ∎$$

3. The lemma of Douglas in the context of linear relations

In this section the lemma of Douglas, see Introduction, will be discussed in the context of linear relations. The first result to be presented is about range inclusion and factorization. It goes back to D. Popovici and Z. Sebestyén [16], who stated it actually in the context of linear spaces. Some refinements can be found in [17].

PROPOSITION 3.1. *Let $\mathfrak{H}_A$, $\mathfrak{H}_B$, and $\mathfrak{H}$ be Hilbert spaces, let A be a linear relation from $\mathfrak{H}_A$ to $\mathfrak{H}$, and let B be a linear relation from $\mathfrak{H}_B$ to $\mathfrak{H}$. Then* $\operatorname{ran} A \subset \operatorname{ran} B$ *if and only if there exists a linear relation W from $\mathfrak{H}_A$ to $\mathfrak{H}_B$ such that $A \subset BW$.*

PROOF. ($\Rightarrow$) Let the linear relation W from $\mathfrak{H}_A$ to $\mathfrak{H}_B$ be defined by the product

$$W = B^{-1}A.$$

Let $\{f,f'\} \in A$. Then $f' \in \operatorname{ran} A$, so that $f' \in \operatorname{ran} B$ and there exists $\varphi \in \mathfrak{H}_B$ such that $\{\varphi,f'\} \in B$ or $\{f',\varphi\} \in B^{-1}$. Hence $\{f,\varphi\} \in W$ and $\{f,f'\} \in BW$.

($\Leftarrow$) Let $f' \in \operatorname{ran} A$, then for some $f \in \mathfrak{H}_A$ one has $\{f,f'\} \in A$. Hence there is $\varphi \in \mathfrak{H}_B$ such that $\{f,\varphi\} \in W$ and $\{\varphi,f'\} \in B$. This implies that $f' \in \operatorname{ran} B$. ∎

For the next result, see [17, Proposition 2]; for completeness a short proof is presented.

PROPOSITION 3.2. *Let $\mathfrak{H}_A$, $\mathfrak{H}_B$, and $\mathfrak{H}$ be Hilbert spaces, let A be a linear relation from $\mathfrak{H}_A$ to $\mathfrak{H}$, and let B be a linear relation from $\mathfrak{H}_B$ to $\mathfrak{H}$. Then there exists a linear relation W from $\mathfrak{H}_A$ to $\mathfrak{H}_B$ such that $A = BW$ if and only if*

$$\operatorname{ran} A \subset \operatorname{ran} B \quad \textit{and} \quad \operatorname{mul} B \subset \operatorname{mul} A.$$

PROOF. ($\Rightarrow$) It follows from (2.3) that $\operatorname{mul} B \subset \operatorname{mul} BW = \operatorname{mul} A$ while $\operatorname{ran} A \subset \operatorname{ran} B$ holds by Proposition 3.1.

($\Leftarrow$) As in the proof of Proposition 3.1 consider $W = B^{-1}A$ which satisfies $A \subset$ $\subset BW$. In view of (2.4) one can write

$$BW = BB^{-1}A = \left(I_{\operatorname{ran} B} \,\widehat{+}\, (\{0\} \times \operatorname{mul} B)\right) A. \tag{3.1}$$

Since $\operatorname{ran} A \subset \operatorname{ran} B$, it is clear from (3.1) that $\operatorname{dom} BW = \operatorname{dom} A$ and since $\operatorname{mul} B \subset$ $\subset \operatorname{mul} A$ one also concludes from (3.1) that $\operatorname{mul} BW = \operatorname{mul} A$. Therefore, the equality $BW = A$ holds by Corollary 2.2. ∎

Observe that if W is a linear relation from $\mathfrak{H}_A$ to $\mathfrak{H}_B$, then the inclusion $A \subset BW$ shows that

$$\operatorname{dom} A \subset \operatorname{dom} W \quad \text{and} \quad \operatorname{ran} A \subset \operatorname{ran} B.$$

Furthermore, if W is an operator, then the inclusion $A \subset BW$ is equivalent to:

$$\operatorname{dom} A \subset \operatorname{dom} W \quad \text{and} \quad \{Wf,f'\} \in B \quad \text{for all} \quad \{f,f'\} \in A,$$

so that in particular W takes $\operatorname{dom} A$ into $\operatorname{dom} B$. Hence when the relation W is a bounded operator then it may be assumed that $W \in \mathbf{B}(\overline{\operatorname{dom} A}, \overline{\operatorname{dom} B})$. In this case the zero continuation W_c of W to $(\operatorname{dom} A)^{\perp}$ satisfies $A \subset BW \subset BW_c$ and $\|W_c\| = \|W\|$, so that without loss of generality it may be assumed that $W \in \mathbf{B}(\mathfrak{H}_A, \mathfrak{H}_B)$.

LEMMA 3.3. *Let $A \subset BW$ for some $W \in \mathbf{B}(\mathfrak{H}_A, \mathfrak{H}_B)$. Then*

$$W^*B^* \subset A^* \quad \textit{and} \quad A^{**} \subset B^{**}W.$$

PROOF. Clearly $A \subset BW$ implies via (2.8) that

$$W^*B^* \subset (BW)^* \subset A^*.$$

This inclusion combined with $W^* \in \mathbf{B}(\mathfrak{H}_B, \mathfrak{H}_A)$ and (2.9) in turn gives rise to

$$A^{**} \subset (W^* B^*)^* = B^{**} W^{**} = B^{**} W. \qquad \blacksquare$$

The main result in this section concerns factorization and majorization. If A and B are closed linear relations, then the case that $W \in \mathbf{B}(\mathfrak{H}_A, \mathfrak{H}_B)$ can be characterized as follows; see also [3, 6].

THEOREM 3.4. *Let $\mathfrak{H}_A$, $\mathfrak{H}_B$, and $\mathfrak{H}$ be Hilbert spaces, let A be a closed linear relation from $\mathfrak{H}_A$ to $\mathfrak{H}$, and let B be a closed linear relation from $\mathfrak{H}_B$ to $\mathfrak{H}$. Then there exists an operator $W \in \mathbf{B}(\mathfrak{H}_A, \mathfrak{H}_B)$ (or equivalently an operator $W \in \mathbf{B}(\overline{\mathrm{dom}}\, A, \overline{\mathrm{dom}}\, B)$) such that*

$$A \subset BW, \tag{3.2}$$

if and only if there exists $c \geq 0$ such that

$$AA^* \leq c^2 BB^*. \tag{3.3}$$

One can take $\|W\| \leq c$.

PROOF. ($\Rightarrow$) Let $A \subset BW$ with $W \in \mathbf{B}(\overline{\mathrm{dom}}\, A, \overline{\mathrm{dom}}\, B)$. By considering the zero continuation of W, again denoted by W, it may be assumed that $W \in \mathbf{B}(\mathfrak{H}_A, \mathfrak{H}_B)$. Then

$$W^* B^* \subset A^*, \tag{3.4}$$

cf. Lemma 3.3. In particular this implies that $\mathrm{dom}\, B^* \subset \mathrm{dom}\, A^*$. Now let $\{f, f'\} \in \in (B^*)_{\mathrm{s}} \subset B^*$. Then it follows from (3.4) that

$$\{f, W^* f'\} \in A^*.$$

Hence there is an element $\chi \in \mathrm{mul}\, A^*$ such that

$$W^* (B^*)_{\mathrm{s}} f = (A^*)_{\mathrm{s}} f + \chi.$$

Observe that

$$\|(A^*)_{\mathrm{s}} f\|^2 \leq \|(A^*)_{\mathrm{s}} f\|^2 + \|\chi\|^2 = \|W^* (B^*)_{\mathrm{s}} f\|^2 \leq \|W\|^2 \|(B^*)_{\mathrm{s}} f\|^2.$$

Together with $\mathrm{dom}\, B^* \subset \mathrm{dom}\, A^*$ this inequality proves (3.3); see Corollary 2.10.

($\Leftarrow$) Assume that (3.3) holds, in other words, assume that there exists $c \geq 0$ such that

$$\mathrm{dom}\, B^* \subset \mathrm{dom}\, A^*, \quad c\, \|(B^*)_{\mathrm{s}} f\| \geq \|(A^*)_{\mathrm{s}} f\|, \quad f \in \mathrm{dom}\, B^*. \tag{3.5}$$

Consider A_{s} as a densely defined operator from $\overline{\mathrm{dom}}\, A$ to $(\mathrm{mul}\, A)^{\perp}$ and B_{s} as a densely defined operator from $\overline{\mathrm{dom}}\, B$ to $(\mathrm{mul}\, B)^{\perp}$. Then the assumption (3.5) is equivalent to

$$\mathrm{dom}\, B^* \subset \mathrm{dom}\, A^*, \quad c\|(B_{\mathrm{s}})^{\times} f\| \geq \|(A_{\mathrm{s}})^{\times} f\|, \quad f \in \mathrm{dom}\, B^*, \tag{3.6}$$

where the adjoints $(A_{\mathrm{s}})^{\times}$ and $(B_{\mathrm{s}})^{\times}$ are with respect to these smaller spaces; see (2.7). Define the linear relation D by

$$D = \left\{ \{(B_s)^{\times} f, (A_{\mathrm{s}})^{\times} f\} : f \in \mathrm{dom}\, B^* \right\}.$$

Then by (3.6) D is a bounded operator from $\overline{\mathrm{dom}}\,B$ to $\overline{\mathrm{dom}}\,A$ with $\|D\| \leq c$. It has a unique extension, again denoted by D, from $\overline{\mathrm{dom}\,B}$ to $\overline{\mathrm{dom}\,A}$ with $\|D\| \leq c$, such that

$$D(B_{\mathrm{s}})^{\times} \subset (A_{\mathrm{s}})^{\times},$$

or taking adjoints, using (2.9),

$$A_{\mathrm{s}} = (A_{\mathrm{s}})^{\times\times} \subset (D(B_{\mathrm{s}})^{\times})^{\times} = (B_{\mathrm{s}})^{\times\times} D^{\times} = B_{\mathrm{s}} W_0, \tag{3.7}$$

where $W_0 = D^{\times}$ is a bounded linear operator from $\overline{\mathrm{dom}\,A}$ to $\overline{\mathrm{dom}\,B}$, with $\|W_0\| = \|D\| \leq c$. Observe that the inclusion $\mathrm{dom}\,B^* \subset \mathrm{dom}\,A^*$ implies that

$$\mathrm{mul}\,A \subset \mathrm{mul}\,B. \tag{3.8}$$

Now let $\{f,f'\} \in A$, so that $f' = A_{\mathrm{s}}f + \varphi$ with $\varphi \in \mathrm{mul}\,A$. By (3.8) one has $\varphi \in \mathrm{mul}\,B$. By (3.7) the inclusion $\{f, A_{\mathrm{s}}f\} \in A_{\mathrm{s}}$ implies that

$$\{f, W_0 f\} \in W_0, \quad \{W_0 f, A_{\mathrm{s}} f\} \in B_{\mathrm{s}},$$

and, hence

$$\{W_0 f, A_{\mathrm{s}} f + \varphi\} \in B.$$

Therefore one concludes that $\{f,f'\} \in BW_0$, i.e., $A \subset BW_0$ holds with $W_0 \in \mathbf{B}(\overline{\mathrm{dom}\,A}, \overline{\mathrm{dom}\,B})$. Finally, let W be the zero continuation of W_0 to $(\mathrm{dom}\,A)^{\perp}$. Then $W \in \mathbf{B}(\mathfrak{H}_A, \mathfrak{H}_B)$ with $\|W\| = \|W_0\|$ and, moreover, the inclusion $A \subset BW$ is satisfied. ∎

In particular, the equivalences $AA^* \leq BB^* \Leftrightarrow W^*B^* \subset A^* \Leftrightarrow A \subset BW$ with $\|W\| \leq 1$ can be found in [3, Proposition 2.2, Remark 2.3]. For densely defined operators A and B the implication $AA^* \leq BB^* \Rightarrow A \subset BW$, $\|W\| \leq 1$, can be found in [6, Theorem 2].

The following two corollaries are variations on the theme of Theorem 3.4.

Corollary 3.5. *Let A and B be closed linear relations as in Theorem 3.4 and, in addition, let $T \in \mathbf{B}(\mathfrak{K}, \mathfrak{H}_A)$ with $\mathfrak{K}$ a Hilbert space. Then*

$$AA^* \leq c^2 BB^* \quad \Rightarrow \quad AT\overline{T^*A^*} \leq c^2 \|T\|^2 BB^*,$$

where $c \geq 0$. In particular,

$$BT\overline{T^*B^*} \leq \|T\|^2 BB^*$$

holds for every $T \in \mathbf{B}(\mathfrak{K}, \mathfrak{H}_A)$.

Proof. Assume that $AA^* \leq c^2 BB^*$, which by Theorem 3.4 is equivalent to the inclusion $A \subset BW$. Therefore it follows that

$$AT \subset BWT.$$

Observe that AT is closed and that WT is bounded. Hence again by Theorem 3.4 one obtains

$$AT(AT)^* \leq \|WT\|^2 BB^*$$

Now observe that (2.9) shows that

$$(AT)^* = \left((T^*A^*)^*\right)^* = \overline{T^*A^*}$$

Hence this leads to

$$A\overline{TT^*A^*} \leq c_T^2 BB^*,$$

where one can take $c_T = \|WT\| \leq c\|T\|$. The last statement follows from the first one with the choices $A = B$ and $c = 1$. ∎

COROLLARY 3.6. *Let A and B be closed linear relations as in Theorem 3.4 and let T be a linear relation from the Hilbert space $\mathfrak{H}$ to the Hilbert space $\mathfrak{K}$. Then*

$$AA^* \leq c^2 BB^* \quad \Rightarrow \quad \overline{TA}(TA)^* \leq c^2 \overline{TB}(TB)^*, \tag{3.9}$$

where $c \geq 0$. In particular, if $T \in \mathbf{B}(\mathfrak{H}, \mathfrak{K})$ then

$$AA^* \leq c^2 BB^* \quad \Rightarrow \quad \overline{TAA^*T^*} \leq c^2 \overline{TBB^*T^*}.$$

PROOF. Assume that $AA^* \leq c^2 BB^*$. Then by Theorem 3.4 $A \subset BW$ for some $W \in \mathbf{B}(\mathfrak{H}_A, \mathfrak{H}_B)$ with $\|W\| \leq c$. Hence it follows that

$$TA \subset T(BW) = (TB)W \subset \overline{TB}W. \tag{3.10}$$

Due to (2.9) the following identity holds

$$\overline{TB}W = \left(W^* (TB)^*\right)^*,$$

which implies that the relation $\overline{TB}W$ is closed. Therefore one concludes from (3.10) that

$$AA^* \leq c^2 BB^* \quad \Rightarrow \quad \overline{TA} \subset \overline{TB}W.$$

By Theorem 3.4 this implication can be rewritten as the implication stated in (3.9). If $T \in \mathbf{B}(\mathfrak{H}, \mathfrak{K})$ the last statement is obtained by applying (2.9) to (3.9). ∎

The occurrence of the equality $A = BW$ in Theorem 3.4 can be characterized as follows.

PROPOSITION 3.7. *Let $\mathfrak{H}_A$, $\mathfrak{H}_B$, and $\mathfrak{H}$ be Hilbert spaces, let A be a closed linear relation from $\mathfrak{H}_A$ to $\mathfrak{H}$, and let B be a closed linear relation from $\mathfrak{H}_B$ to $\mathfrak{H}$. Then there exists a bounded (not necessarily closed) operator W from $\operatorname{dom} A$ into $\overline{\operatorname{dom}} B$ such that*

$$A = BW, \tag{3.11}$$

if and only if the following conditions are satisfied

(i) *the inequality* (3.3) *holds for some $c \geq 0$;*
(ii) $\operatorname{mul} A = \operatorname{mul} B$.

PROOF. ($\Rightarrow$) If $A = BW$ holds for some bounded operator W from $\operatorname{dom} A$ into $\overline{\operatorname{dom}} B$, then clearly $A \subset BW^{**}$ and here $W^{**} \in \mathbf{B}(\overline{\operatorname{dom}} A, \overline{\operatorname{dom}} B)$, since $\operatorname{dom} A \subset \operatorname{dom} W$. Now the inequality (3.3) is obtained from Theorem 3.4. Since W is an operator, one obtains $\operatorname{mul} A = \operatorname{mul} BW = \operatorname{mul} B$; see (2.3).

($\Leftarrow$) The inequality (3.3) implies the existence of $W_0 \in \mathbf{B}(\overline{\operatorname{dom}} A, \overline{\operatorname{dom}} B)$ such that $A \subset BW_0$ by Theorem 3.4. Then $\operatorname{dom} A \subset \operatorname{dom} W_0$ and the restriction $W := W_0 \upharpoonright \operatorname{dom} A$ is a bounded operator such that $A \subset BW$ and $\operatorname{dom} BW = \operatorname{dom} A$. The second assumption implies that $\operatorname{mul} BW = \operatorname{mul} B = \operatorname{mul} A$ and hence the equality $A = BW$ follows from Corollary 2.2. ∎

The following result concerns the alternative formulation of the Douglas lemma which is known in the literature, but now in the context of relations. The domain condition is a sufficient condition.

PROPOSITION 3.8. *Let $\mathfrak{H}_A$, $\mathfrak{H}_B$, and $\mathfrak{H}$ be Hilbert spaces, let A be a closed linear relation from $\mathfrak{H}_A$ to $\mathfrak{H}$, and let B be a closed linear relation from $\mathfrak{H}_B$ to $\mathfrak{H}$. Assume that* $\operatorname{dom} A^* = \operatorname{dom} B^*$. *Then the following statements are equivalent:*

(i) $AA^* \leq c^2 BB^*$ *for some* $c \geq 0$*;*
(ii) $AA^* = BMB^*$ *for some* $0 \leq M \in \mathbf{B}(\mathfrak{H}_B)$ *with* $\|M\| \leq c^2$.

PROOF. (i) $\Rightarrow$ (ii) By Theorem 3.4 it follows that $A \subset BW$, and that $W^*B^* \subset A^*$. Let Q be the orthogonal projection onto $(\operatorname{mul} A^*)^\perp$. Then clearly

$$QW^*B^* \subset QA^*,$$

where QA^* is an operator. The assumption $\operatorname{dom} A^* = \operatorname{dom} B^*$ implies that actually equality holds

$$QW^*B^* = QA^*.$$

Therefore one obtains via $AA^* = AQA^*$, see Lemma 2.4, that

$$AA^* = AQA^* \subset BWQA^* = BWQW^*B^* = (BWQ)\,(QW^*B^*)\,,$$

where the relation BWQ is closed and

$$(QW^*B^*)^* = B\,(QW^*)^* = BWQ.$$

Hence the term $(BWQ)(QW^*B^*)$ is selfadjoint and equality prevails:

$$AA^* = BWQW^*B^* = BMB^* \quad \text{with} \quad M = WQW^*.$$

Note that $\|M\| \leq \|W\|^2 \leq c^2$.

(ii) $\Rightarrow$ (i) Since $M \geq 0$ is bounded one can rewrite (ii) in the form

$$AA^* = BMB^* = BM^{1/2}M^{1/2}B^* \subset \left(BM^{1/2}\right)\overline{M^{1/2}B^*}. \tag{3.12}$$

Observe that by (2.9)

$$\overline{M^{1/2}B^*} = \left(M^{1/2}B^*\right)^{**} = \left(BM^{1/2}\right)^*.$$

This equality and the fact that $BM^{1/2}$ is closed together show that both sides in (3.12) are selfadjoint; see Lemma 2.4. Thus there is actually equality in (3.12):

$$AA^* = \left(BM^{1/2}\right)\overline{M^{1/2}B^*}.$$

Now Corollary 3.5 implies that

$$AA^* = BM^{1/2}\overline{M^{1/2}B^*} \leq \|M\| BB^*,$$

so that (3.3) follows with $c^2 = \|M\|$. ∎

4. Domination of linear relations

The following notions and terminology are strongly influenced by the theory of Lebesgue type decompositions of linear relations and forms, cf. [11], [12], [19]. In fact in these papers the notion of domination is used for (mostly closable) operators. However domination can be defined also in the context of linear relations as follows.

DEFINITION 4.1. Let $\mathfrak{H}_A$, $\mathfrak{H}_B$, and $\mathfrak{H}$ be Hilbert spaces, let A be a linear relation from $\mathfrak{H}$ to $\mathfrak{H}_A$, and let B be a linear relation from $\mathfrak{H}$ to $\mathfrak{H}_B$. Then B dominates A if there exists an operator $Z \in \mathbf{B}(\mathfrak{H}_B, \mathfrak{H}_A)$ such that

$$ZB \subset A. \tag{4.1}$$

Note that the inclusion $ZB \subset A$ in (4.1) means that

$$\{\{f, Zf'\} : \{f,f'\} \in B\} \subset A. \tag{4.2}$$

This shows that $\operatorname{dom} B \subset \operatorname{dom} A$ and that $\ker B \subset \ker A$. Furthermore,

$$\operatorname{mul} ZB = Z(\operatorname{mul} B) \subset \operatorname{mul} A.$$

It follows from the definition that Z takes $\operatorname{ran} B$ into $\operatorname{ran} A$; the boundedness implies that Z takes $\overline{\operatorname{ran}}\, B$ into $\overline{\operatorname{ran}}\, A$. Hence one can assume that $(\operatorname{ran} B)^\perp \subset \ker Z$, in which case Z is uniquely determined. Domination is transitive: if $Z_1 B \subset A$ and $Z_2 C \subset B$ then

$$Z_1(Z_2 C) \subset Z_1 B \subset A,$$

so that $(Z_1 Z_2)C \subset A$.

Let A and B be relations in a Hilbert space $\mathfrak{H}$ which satisfy $B \subset A$. Then clearly B dominates A (with $Z = 1$). In particular, since $A \subset A^{**}$, it follows that A dominates A^{**}.

In the particular case when A and B in the above definition are linear operators it is possible to give an equivalent characterization of domination.

LEMMA 4.2. *Let $\mathfrak{H}_A$, $\mathfrak{H}_B$, and $\mathfrak{H}$ be Hilbert spaces, let A be a linear operator from $\mathfrak{H}$ to $\mathfrak{H}_A$, and let B be a linear operator from $\mathfrak{H}$ to $\mathfrak{H}_B$. Then B dominates A if and only if there exists $c \geq 0$ such that*

$$\operatorname{dom} B \subset \operatorname{dom} A \quad \text{and} \quad \|Af\| \leq c\|Bf\|, \quad f \in \operatorname{dom} B. \tag{4.3}$$

PROOF. Assume that B dominates A. Then (4.1) shows that $\operatorname{dom} B \subset \operatorname{dom} A$ and that for all $f \in \operatorname{dom} B$ one has $ZBf = Af$, which leads to

$$\|Af\| \leq \|Z\|\|Bf\|, \quad f \in \operatorname{dom} B.$$

The desired result follows from this with $c = \|Z\|$.

Conversely, assume that (4.3) holds. Define an operator Z_0 from $\operatorname{ran} B$ to $\operatorname{ran} A$ by $Z_0 Bf = Af, f \in \operatorname{dom} B$. It follows from (4.3) that the operator Z_0 is well defined and bounded with $\|Z_0\| \leq c$. Thus Z_0 can be continued to a bounded operator from $\overline{\operatorname{ran}}\, B$ to $\overline{\operatorname{ran}}\, A$ with the same norm. Let Z be the extension of $\operatorname{clos} Z_0$ obtained by defining Z to be 0 on $(\operatorname{ran} B)^\perp$. Then clearly $Z : \mathfrak{H}_B \to \mathfrak{H}_A$ is bounded and $ZB \subset A$ holds. ∎

A weaker version of Lemma 4.2 with densely defined operators on a Banach space appears in [8, Theorem 2.8]; see also [4, 7].

LEMMA 4.3. *Let the relation B dominate the relation A as in* (4.1), *then*

$$A^* \subset B^* Z^*, \tag{4.4}$$

and, consequently

$$ZB^{**} \subset A^{**}. \tag{4.5}$$

In other words, B^{**} dominates A^{**} with the same operator Z. In particular, if B dominates A then the following inclusions are valid

$$\operatorname{dom} B \subset \operatorname{dom} A, \quad \operatorname{ran} A^* \subset \operatorname{ran} B^*, \quad \textit{and} \quad \operatorname{dom} B^{**} \subset \operatorname{dom} A^{**}.$$

PROOF. It follows from (4.1) and (2.9) that

$$A^* \subset (ZB)^* = B^* Z^*.$$

Now taking adjoints again yields

$$Z^{**} B^{**} \subset (B^* Z^*)^* \subset A^{**},$$

and this proves (4.5). The remaining statements are clear from (4.4) and (4.5) . ∎

So far domination has been defined for linear relations which are not necessarily closed. Due to Lemma 4.3 domination of closed linear relations can be characterized in terms of majorization.

THEOREM 4.4. *Let $\mathfrak{H}_A$, $\mathfrak{H}_B$, and $\mathfrak{H}$ be Hilbert spaces, let A be a closed linear relation from $\mathfrak{H}$ to $\mathfrak{H}_A$, and let B be a closed linear relation from $\mathfrak{H}$ to $\mathfrak{H}_B$. Then there exists an operator $Z \in \mathbf{B}(\mathfrak{H}_B, \mathfrak{H}_A)$ such that*

$$ZB \subset A \tag{4.6}$$

if and only if there exists $c \geq 0$ such that

$$A^* A \leq c^2 B^* B. \tag{4.7}$$

One can take $\|Z\| \leq c$.

PROOF. Since A and B are assumed to be closed the inclusions (4.6) and (4.4) are equivalent. Hence the result follows from Theorem 3.4. ∎

PROPOSITION 4.5. *Let $\mathfrak{H}_A$, $\mathfrak{H}_B$, and $\mathfrak{H}$ be Hilbert spaces, let A be a closed linear relation from $\mathfrak{H}$ to $\mathfrak{H}_A$, and let B be a closed linear relation from $\mathfrak{H}$ to $\mathfrak{H}_B$. Then there exists an operator $Z \in \mathbf{B}(\mathfrak{H}_B, \mathfrak{H}_A)$ such that*

$$A = ZB \tag{4.8}$$

if and only if the following three conditions are satisfied:

(i) *the inequality* (4.7) *holds for some $c \geq 0$;*
(ii) $\operatorname{dom} A = \operatorname{dom} B$;
(iii) $\dim(\operatorname{mul} A) \leq \dim(\operatorname{mul} B)$.

PROOF. ($\Rightarrow$) Property (i) follows directly from Theorem 4.4. Since $\operatorname{dom} Z = \mathfrak{H}_B$, the equality (4.8) implies (ii). Finally, it follows from (2.3) and (4.8) that $\operatorname{mul} A = \operatorname{mul} ZB = Z(\operatorname{mul} B)$, i.e. Z maps $\operatorname{mul} B$ onto $\operatorname{mul} A$, and hence (iii) holds.

($\Leftarrow$) Decompose A and B via their operator parts:

$$A = A_{\mathrm{s}} \,\widehat{\oplus}\, A_{\mathrm{mul}}, \quad B = B_{\mathrm{s}} \,\widehat{\oplus}\, B_{\mathrm{mul}}.$$

By Lemma 2.4 the condition (4.7) is equivalent to $(A_{\mathrm{s}})^* A_{\mathrm{s}} \le c^2 (B_{\mathrm{s}})^* B_{\mathrm{s}}$, $c \ge 0$. Now by Theorem 4.4 there exists $Z_0 \in \mathbf{B}(\mathfrak{H}_B \ominus \mathrm{mul}\, B, \mathfrak{H}_A \ominus \mathrm{mul}\, A)$ such that

$$Z_0 B_{\mathrm{s}} \subset A_{\mathrm{s}}.$$

By the condition (ii) $\mathrm{dom}\, A_{\mathrm{s}} = \mathrm{dom}\, B_{\mathrm{s}}$ and hence, in fact, the equality $Z_0 B_{\mathrm{s}} = A_{\mathrm{s}}$ prevails. Moreover, the condition (iii) guarantees the existence of a surjective operator $Z_m \in \mathbf{B}(\mathrm{mul}\, B, \mathrm{mul}\, A)$. Finally, by taking $Z = Z_0 \oplus Z_m$ one gets the desired identity $ZB = A$. ∎

Finally note that the result in Proposition 3.8 has a counterpart in the setting of Theorem 4.4. Let $\mathfrak{H}_A$, $\mathfrak{H}_B$, and $\mathfrak{H}$ be Hilbert spaces, let A be a closed linear relation from $\mathfrak{H}$ to $\mathfrak{H}_A$, and let B be a closed linear relation from $\mathfrak{H}$ to $\mathfrak{H}_B$. If $\mathrm{dom}\, A = \mathrm{dom}\, B$, then the following statements are equivalent:

(i) $A^*A \le c^2 B^*B$, $c \ge 0$;
(ii) $A^*A = B^*MB$ for some $0 \le M \in \mathbf{B}(\mathfrak{H}_B)$ with $\|M\| \le c^2$.

5. Majorization and domination

There is a direct connection between the majorization of bounded operators as in the original Douglas lemma and the notion of domination of linear relations as in Definition 4.1.

Lemma 5.1. *Let $\mathfrak{H}_A$, $\mathfrak{H}_B$, and $\mathfrak{H}$ be Hilbert spaces, let $A \in \mathbf{B}(\mathfrak{H}_A, \mathfrak{H})$, $B \in \mathbf{B}(\mathfrak{H}_B, \mathfrak{H})$, and $W \in \mathbf{B}(\mathfrak{H}_A, \mathfrak{H}_B)$. Then*

$$A = BW \quad \Leftrightarrow \quad WA^{-1} \subset B^{-1}. \tag{5.1}$$

Proof. First observe that if $H \in \mathbf{B}(\mathfrak{H}_1, \mathfrak{H}_2)$, then

$$HH^{-1} = I_{\mathrm{ran}\, H} \subset I_{\mathfrak{H}_2}, \quad H^{-1}H = I_{\mathfrak{H}_1} \widehat{+} \big(\{0\} \times \ker H\big) \supset I_{\mathfrak{H}_1}, \tag{5.2}$$

as is clear from (2.4) and (2.5).

($\Rightarrow$) Assume that $A = BW$. Then by (5.2) it follows that

$$WA^{-1} \subset B^{-1}BWA^{-1} = B^{-1}AA^{-1} \subset B^{-1}.$$

($\Leftarrow$) Assume that $WA^{-1} \subset B^{-1}$. Then by (5.2) it follows that

$$BW \subset BWA^{-1}A \subset BB^{-1}A \subset A,$$

so that $BW \subset A$. Actually equality $BW = A$ prevails here, since both BW and A are everywhere defined operators. ∎

In other words, the lemma expresses the fact that when A and B are bounded operators, then B majorizes A in the sense of $AA^* \le \lambda BB^*$ (cf. Lemma 1.1) if and only if the relation A^{-1} dominates the relation B^{-1} in the sense of Definition 4.1

The connection in Lemma 5.1 is useful as it yields a particularly simple proof for the characterization of the ordering of nonnegative selfadjoint relations as in (2.19). For earlier treatments of the ordering, see [5, 9].

THEOREM 5.2. *Let H_1 and H_2 be nonnegative selfadjoint relations in a Hilbert space $\mathfrak{H}$. Then the following statements are equivalent:*

(i) $H_1 \leq H_2$;
(ii) $(H_1+x)^{-1} \geq (H_2+x)^{-1}$ *for some and hence for every* $x>0$;
(iii) $H_1^{-1} \geq H_2^{-1}$.

PROOF. (i) $\Leftrightarrow$ (ii) Recall that $H_1 \leq H_2$ if and only if for some (and hence for all) $x>0$

$$H_1+x \leq H_2+x,$$

and note that for $x>0$ the inverses $(H_1+x)^{-1}$ and $(H_2+x)^{-1}$ belong to $\mathbf{B}(\mathfrak{H})$. By Theorem 4.4 $H_1+x \leq H_2+x$ is equivalent to the existence of $Z \in \mathbf{B}(\mathfrak{H})$ such that

$$Z(H_2+x)^{1/2} \subset (H_1+x)^{1/2}, \quad \|Z\| \leq 1; \tag{5.3}$$

cf. Corollary 2.10. Now an application of Lemma 5.1 shows that (5.3) is equivalent to

$$(H_2+x)^{-1/2} = (H_1+x)^{-1/2}Z. \tag{5.4}$$

Finally, by Lemma 1.1 (or Theorem 3.4) (5.4) is equivalent to

$$(H_2+x)^{-1/2}(H_2+x)^{-1/2} \leq (H_1+x)^{-1/2}(H_1+x)^{-1/2},$$

since $\|Z\| \leq 1$.

(ii) $\Leftrightarrow$ (iii) Let H be a nonnegative selfadjoint relation. Then clearly also H^{-1} is a nonnegative selfadjoint relation and it is connected to H via

$$(H+x)^{-1} = \frac{1}{x} - \frac{1}{x^2}\left(H^{-1}+\frac{1}{x}\right)^{-1}, \tag{5.5}$$

where $x>0$. Hence for a pair of nonnegative selfadjoint relations H_1 and H_2 one obtains for each $x>0$:

$$(H_2+x)^{-1} - (H_1+x)^{-1} = \frac{1}{x^2}\left[\left(H_1^{-1}+\frac{1}{x}\right)^{-1} - \left(H_2^{-1}+\frac{1}{x}\right)^{-1}\right].$$

Now the equivalence is obtained from (i) $\Leftrightarrow$ (ii). ∎

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OPTIMAL EXPANSIONS IN NONINTEGER BASES II

By
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Dedicated to Zoltán Sebestyén on his 70th birthday

Abstract. Decimal and the more general integer base expansions have an obvious *each-step optimal approximation* property. It was investigated by Dajani, de Vries et al. whether this property remains valid for expansions in noninteger bases $q > 1$ with respect to the alphabet $\{0, 1, \dots, m\}$ with $m = \lceil q \rceil - 1$. The purpose of this note is to see what happens without the last assumption.

We fix a *real base* $q > 1$ and a *finite alphabet* $A = \{0, 1, \dots, m\}$, $m \geq 1$. By an *expansion* of a real number x we mean a sequence of integers $c_i \in A$ satisfying the equality

$$x = \sum_{i=1}^{\infty} \frac{c_i}{q^i}.$$

Following Rényi's seminal work [11] hundreds of papers were devoted to various aspects of such expansions. We cite some of them at the end of this paper that are related to the number of expansions. Many more are cited in the review paper [9].

Let us denote by $J_{m,q}$ the set of numbers x having at least one expansion. The inequalities

$$0 = \sum_{i=1}^{\infty} \frac{0}{q^i} \leq x \leq \sum_{i=1}^{\infty} \frac{m}{q^i} = \frac{m}{q-1}$$

show that

$$J_{m,q} \subset \left[0, \frac{m}{q-1}\right].$$

The inclusion is strict if $q > m+1$. Indeed, since

$$c_1 = 0 \Longrightarrow x \le \sum_{i=2}^{\infty} \frac{m}{q^i} = \frac{m}{q(q-1)}$$

and

$$c_1 \ge 1 \Longrightarrow x \ge \frac{1}{q},$$

$J_{m,q}$ contains the endpoints of the interval

$$\left[\frac{m}{q(q-1)}, \frac{1}{q}\right]$$

but not its interior. (There are interior points because of our assumption $q > m+1$.)

On the other hand, the inclusion becomes an equality if $q \le m+1$:

Proposition 1 (Rényi [11]). *Given $x \in \left[0, \frac{m}{q-1}\right]$, let $(b_i) \in A^{\infty}$ be the lexicographically largest sequence satisfying the inequality*

$$\sum_{i=1}^{\infty} \frac{b_i}{q^i} \le x.$$

If $q \le m+1$, then this is in fact an expansion of x:

$$\sum_{i=1}^{\infty} \frac{b_i}{q^i} = x.$$

See, e.g., [3] or [9] for simple proofs of this theorem and its variants.

Remark. The sequence (b_i) may be constructed by the *greedy* algorithm as follows. Let b_1 be the largest integer in A satisfying $\frac{b_1}{q} \le x$. If $n \ge$ ≥ 2 and $b_1, \ldots, b_{n-1}$ have already been defined, then let b_n be the largest integer in A satisfying

$$\frac{b_1}{q} + \ldots + \frac{b_n}{q^n} \le x.$$

Let us give some examples.

EXAMPLES.

- For $q = m+1$ we have $J_{m,q} = [0,1]$, and we recover the familiar integer base expansions.

  It is well-known that if $x \in (0,1)$ has the form $x = \frac{k}{q^n}$ with some integers $k, n \geq 0$, then x has exactly two expansions: a *finite* one ending with 0^∞ (this is the greedy expansion) and and *infinite* one ending with m^∞. Otherwise the expansion of x is unique.
- (Erdős, Horváth, Joó [6].) If $A = \{0,1\}$ and $q = \frac{1+\sqrt{5}}{2} \approx 1.618$ is the Golden ratio, then $x = 1$ has infinitely many expansions:
  -- a periodic expansion

$$1 = \sum_{k=1}^{\infty} \frac{1}{q^{2k-1}} = \sum_{i=1}^{\infty} \frac{a_i}{q^i} \quad \text{with} \quad (a_i) = (10)^\infty;$$

  -- for each $n = 1, 2, \ldots$, there is exactly one expansion (c_i) first differing from (a_i) at the nth place:

$$(c_i) = \begin{cases} (10)^{k-1}01^\infty & \text{if } n = 2k-1, \\ (10)^{k-1}110^\infty & \text{if } n = 2k. \end{cases}$$

  The greedy expansion is the last one with $n = 2$:

$$1 = \frac{1}{q} + \frac{1}{q^2}.$$

- (Erdős, Joó, K. [7].) If $A = \{0,1\}$ and $q < \frac{1+\sqrt{5}}{2}$, then each interior point of $J_{m,q} = \left[0, \frac{1}{q-1}\right]$ has a continuum of distinct expansions.

We are interested here by the existence of expansions providing the best approximation at each step:

DEFINITION. An expansion

$$x = \sum_{i=1}^{\infty} \frac{c_i}{q^i}$$

is *optimal* if for each finite sequence $d_1 \dots d_k \in A^k$ satisfying the inequality

$$\sum_{i=1}^{k} \frac{d_i}{q^i} \le x,$$

we have

$$\sum_{i=1}^{k} \frac{d_i}{q^i} \le \sum_{i=1}^{k} \frac{c_i}{q^i}.$$

In other words, for each finite sequence $d_1 \dots d_k \in A^k$ we have

$$\text{either} \quad \sum_{i=1}^{k} \frac{d_i}{q^i} > x \quad \text{or} \quad \sum_{i=1}^{k} \frac{d_i}{q^i} \le \sum_{i=1}^{k} \frac{c_i}{q^i}.$$

Of course, it is sufficient to check the definition for finite sequences $d_1 \dots d_k \in A^k$ differing from $c_1 \dots c_k$.

It is well-known that in the familiar integer base case every $x \in [0,1]$ has an optimal expansion. (We will reprove this later.) The following simple result shows the relevance of Rényi's greedy expansions in the general case:

Lemma 2. *If $q \le m+1$, and a number has an optimal expansion, then it is its greedy expansion.*

Proof. Let (c_i) be an expansion of x. If it is not the greedy expansion of x, then for some $k \ge 1$ we have $c_k + 1 \in A$ and

$$\frac{c_1}{q} + \dots + \frac{c_{k-1}}{q^{k-1}} + \frac{c_k + 1}{q^k} \le x.$$

Hence

$$\frac{c_1}{q} + \dots + \frac{c_k}{q^k} < \frac{c_1}{q} + \dots + \frac{c_{k-1}}{q^{k-1}} + \frac{c_k + 1}{q^k} \le x$$

so that the expansion (c_i) is not optimal. ∎

The converse is false in general:

Example (Dajani, de Vries, K., Loreti [3].). Let $m = 1$, $A = \{0, 1\}$ and $q < (1+\sqrt{5})/2$. Then the greedy expansion of

$$x := \frac{1}{q^2} + \frac{1}{q^3}$$

starts with 100 because

$$\frac{1}{q} < x, \quad \frac{1}{q} + \frac{1}{q^2} > x \quad \text{and} \quad \frac{1}{q} + \frac{1}{q^3} > x.$$

This expansion is not optimal, because

$$\frac{1}{q} + \frac{0}{q^2} + \frac{0}{q^3} < \frac{0}{q} + \frac{1}{q^2} + \frac{1}{q^3} (= x).$$

In order to clarify the general situation we denote by $q_{m,n,p}$ the positive solution of the equation

$$1 = \frac{m}{q} + \ldots + \frac{m}{q^n} + \frac{p}{q^{n+1}}$$

for all integers $m \geq p \geq 1$ and $n \geq 1$. We have the following properties:

- the map $(m, n, p) \mapsto q_{m,n,p}$ is strictly increasing if we consider the lexicographic order for the triplets (m, n, p);
- for each fixed m we have $m < q_{m,n,p} < m + 1$ and $(m, n, p) \to$ $\to m + 1$ as $n \to \infty$.

The smallest such number is the Golden ratio: $q_{1,1,1} = \frac{1+\sqrt{5}}{2}$.

Now we can describe the general picture:

THEOREM 3. *Consider the expansions in some base $q > 1$ with respect to an alphabet $A = \{0, 1, \ldots, m\}$, $m \geq 1$.*

(a) If $q \leq m$, then every $x \in J_{m,q}$ has an optimal expansion $\iff$ q is integer.
(b) If $m < q < m + 1$, then every $x \in J_{m,q}$ has an optimal expansion $\iff$ $q = q_{m,n,p}$ for some n, p.
(c) If $q \geq m + 1$, then every $x \in J_{m,q}$ has an optimal expansion.

The case (b) was proved before in [3]. For the reader's convenience we give a short direct proof for this case, too.

PROOF OF THE THEOREM. We distinguish several cases.

The case $q > m + 1$. We apply the second version of the definition of optimality. Let $d_1 \ldots d_k \in A^k$ be different from $c_1 \ldots c_k$, and let j be the first index such that $d_j \neq c_j$.

If $d_j > c_j$, then

$$\sum_{i=1}^{k} \frac{d_i}{q^i} \geq \left(\sum_{i=1}^{j-1} \frac{c_i}{q^i} \right) + \frac{c_j + 1}{q^j} = \left(\sum_{i=1}^{j} \frac{c_i}{q^i} \right) + \sum_{i=j+1}^{\infty} \frac{q-1}{q^i} >$$

$$> \left(\sum_{i=1}^{j} \frac{c_i}{q^i} \right) + \sum_{i=j+1}^{\infty} \frac{m}{q^i} \geq \sum_{i=1}^{\infty} \frac{c_i}{q^i} = x.$$

If $d_j < c_j$, then

$$\sum_{i=1}^{k} \frac{d_i}{q^i} \leq \left(\sum_{i=1}^{j-1} \frac{c_i}{q^i} \right) + \frac{c_j - 1}{q^j} + \sum_{i=j+1}^{k} \frac{m}{q^i} <$$

$$< \left(\sum_{i=1}^{j-1} \frac{c_i}{q^i} \right) + \frac{c_j - 1}{q^j} + \sum_{i=j+1}^{\infty} \frac{q-1}{q^i} = \sum_{i=1}^{j} \frac{c_i}{q^i} \leq \sum_{i=1}^{k} \frac{c_i}{q^i}.$$

The case of integer bases $1 < q \leq m + 1$. Since every $x \in J_{m,q}$ has a greedy expansion

$$x = \sum_{i=1}^{\infty} \frac{b_i}{q^i}$$

by Rényi's theorem, it is sufficient to show that these expansions are optimal. More precisely, we have to show that if $c_1 \dots c_k \in A^k$ and

$$\sum_{i=1}^{k} \frac{c_i}{q^i} \leq \sum_{i=1}^{\infty} \frac{b_i}{q^i},$$

then

$$\sum_{i=1}^{k} \frac{c_i}{q^i} \leq \sum_{i=1}^{k} \frac{b_i}{q^i}.$$

We may assume here that $c_1 \dots c_k$ is different from $b_1 \dots b_k$.

By the definition of the greedy algorithm there exists a first index j such that $c_j < b_j$.

In the classical case $q = m + 1$ we have simply

$$\sum_{i=1}^{k} \frac{c_i}{q^i} \leq \left(\sum_{i=1}^{j-1} \frac{b_i}{q^i} \right) + \frac{b_j - 1}{q^j} + \sum_{i=j+1}^{k} \frac{q-1}{q^i} < \sum_{i=1}^{j} \frac{b_i}{q^i} \leq \sum_{i=1}^{k} \frac{b_i}{q^i}.$$

If $q < m+1$, then we may assume without loss of generality that $c_1 \dots c_k$ is lexicographically maximal among all sequences $d_1 \dots d_k \in A^k$ satisfying

$$\sum_{i=1}^{k} \frac{d_i}{q^i} = \sum_{i=1}^{k} \frac{c_i}{q^i}.$$

Then $c_1 \dots c_k$ contains no block ab with $a < m$ and $b \geq q$: for otherwise we could change a to $a+1$ and b to $b-q$, contradicting the lexicographic maximality.

Since $q < m+1$, $c_j < m$ implies that $c_{j+1} \leq q-1 < m$ if $j < k$; continuing by induction we get $c_i \leq q-1$ for all $j < i \leq k$. Therefore

$$\sum_{i=1}^{k} \frac{c_i}{q^i} \leq \left(\sum_{i=1}^{j-1} \frac{b_i}{q^i}\right) + \frac{b_j - 1}{q^j} + \sum_{i=j+1}^{k} \frac{q-1}{q^i} < \sum_{i=1}^{j} \frac{b_i}{q^i} \leq \sum_{i=1}^{k} \frac{b_i}{q^i}$$

as required.

The case of non-integer bases $q < m$. We show that

$$x := \frac{[q]+1}{q^2} = \frac{0}{q} + \frac{[q]+1}{q^2}$$

has no optimal expansion.

Since

$$\frac{1}{q} < x < \frac{2}{q},$$

the greedy expansion of x starts with $\frac{1}{q} + \frac{b}{q^2}$ for some $b \in A$. This is not equal to x, because the equality

$$\frac{1}{q} + \frac{b}{q^2} = \frac{[q]+1}{q^2}$$

would imply that $q = [q]+1-b$ is integer. We conclude by applying Lemma 2.

The case where $m < q < m+1$ is different from the numbers $q_{m,n,p}$. It follows from our assumption on q that

$$(1) \quad \frac{m}{q^2} + \dots + \frac{m}{q^{n+1}} + \frac{p-1}{q^{n+2}} < \frac{1}{q} < \frac{m}{q^2} + \dots + \frac{m}{q^{n+1}} + \frac{p}{q^{n+2}} =: x$$

for some n and p.

For the proof we use the definition and basic properties of the numbers $q_{m,n,p}$, given before the statement of Theorem 3.

Let (n,p) be the lexicographically smallest pair of positive integers such that $q < q_{m,n,p}$. Then the second inequality of (1) is satisfied.

If $p \geq 2$, then we have $q_{m,n,p-1} \leq q$ by the minimality of (n,p), and we cannot have equality because of our assumption on q in the present case. This implies the first inequality in (1).

If $p = 1$ and $n \geq 2$, then $q_{m,n-1,m} < q$ by the minimality of (n,p) and by our assumption on q, and the first inequality in (1) follows again.

Finally, if $p = 1$ and $n = 1$, then the first inequality of (1) is equivalent to $q > m$, and this is satisfied by our assumption.

It follows from (1) that

$$\frac{1}{q} < x < \frac{1}{q} + \frac{1}{q^{n+2}}$$

and therefore that

$$x < \frac{1}{q} + \frac{1}{q^k}, \quad k = 1, \dots, n+2.$$

These relations imply that the greedy expansion (b_i) of

$$x := \frac{m}{q^2} + \dots + \frac{m}{q^{n+1}} + \frac{p}{q^{n+2}}$$

begins with 10^{n+1}. Since

$$\frac{b_1}{q} + \dots + \frac{b_{n+2}}{q^{n+2}} = \frac{1}{q} < \frac{m}{q^2} + \dots + \frac{m}{q^{n+1}} + \frac{p}{q^{n+2}} = x,$$

the greedy expansion of x is not optimal. In view of Lemma 2 we conclude that x has no optimal expansion.

The case where $q = q_{m,n,p}$ for some n,p. Since $q < m+1$, every $x \in J_{m,q}$ has a greedy expansion

$$x = \sum_{i=1}^{\infty} \frac{b_i}{q^i}$$

by Rényi's theorem. We have to show that if $c_1 \dots c_k \in A^k$ and

$$\sum_{i=1}^{k} \frac{c_i}{q^i} \leq \sum_{i=1}^{\infty} \frac{b_i}{q^i},$$

then

$$\sum_{i=1}^{k} \frac{c_i}{q^i} \leq \sum_{i=1}^{k} \frac{b_i}{q^i}.$$

We may assume again without loss of generality that $c_1 \dots c_k$ is lexicographically maximal among all sequences $d_1 \dots d_k \in A^k$ satisfying

$$\sum_{i=1}^{k} \frac{d_i}{q^i} = \sum_{i=1}^{k} \frac{c_i}{q^i}.$$

Then $c_1 \dots c_k$ contains no block am^nb with $a < m$ and $b \geq p$: for otherwise we could change a to $a + 1$, m to 0 (n times) and b to $b - p$, contradicting the lexicographic maximality.

The case $c_1 \dots c_k = b_1 \dots b_k$ is obvious. If $c_1 \dots c_k \neq b_1 \dots b_k$, then by the definition of the greedy algorithm there exists a first index j such that $c_j < b_j$. Now we have

$$\begin{aligned}\sum_{i=1}^{k} \frac{c_i}{q^i} < \left(\sum_{i=1}^{j-1} \frac{b_i}{q^i}\right) + \frac{b_j - 1}{q^j} + \\ + \frac{1}{q^j} \sum_{\ell=0}^{\infty} \left(\frac{1}{q^{n+1}}\right)^{\ell} \left(\frac{m}{q} + \dots + \frac{m}{q^n} + \frac{p-1}{q^{n+1}}\right) = \\ = \sum_{i=1}^{j} \frac{b_i}{q^i} \leq \sum_{i=1}^{k} \frac{b_i}{q^i}.\end{aligned}$$

∎

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TWO CHARACTERIZATIONS OF UNITARY-ANTIUNITARY SIMILARITY TRANSFORMATIONS OF POSITIVE DEFINITE OPERATORS ON A FINITE DIMENSIONAL HILBERT SPACE*

By
LAJOS MOLNÁR

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Dedicated to Professor Zoltán Sebestyén on the occasion of his seventieth birthday

Abstract. In this paper we characterize the unitary-antiunitary similarity transformations on the set of all positive definite operators on a finite dimensional Hilbert space by means of two particular invariance properties. The first one concerns the preservation of a general relative entropy like quantity while the second one is related to the preservation of a measure of difference between the arithmetic mean as the maximal symmetric operator mean and any other given symmetric operator mean.

1. Introduction and statement of the results

Let us begin with a general remark. As the title and the abstract suggest the results of this paper are about transformations on certain structures of linear operators acting on a finite dimensional Hilbert space. Of course, the problems and results could be formulated in the context of matrices but, on the one hand we prefer treating operators to matrices, on the other hand we believe our present approach may help someone to extend these investigations to other settings, e.g., to that of certain classes of von Neumann algebras.

So, in what follows let H be any finite dimensional complex Hilbert space with $\dim H > 1$. Denote by $B(H)$ the algebra of all linear operators on H. An

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element A of $B(H)$ is called positive semidefinite if $\langle Ax, x\rangle \geq 0$ holds for all $x \in H$. The set of all positive semidefinite operators in $B(H)$ is denoted by $B(H)_+$. The invertible elements of $B(H)_+$ are called positive definite and $B(H)_+^{-1}$ stands for their collection. Denote by Tr the usual trace functional on $B(H)$.

Our aim in this paper is to give two characterizations of the unitary-antiunitary similarity transformations on $B(H)_+^{-1}$. These maps are of particular importance, they originate from the algebra *-automorphisms and the algebra *-antiautomorphisms of $B(H)$ (which, anyway, are just the so-called Jordan *-automorphisms of that algebra). Those transformations appear and play fundamental roles in very many applications of the theory of operator algebras.

Our first characterization is in a way connected with quantum information theory. One of the most important concepts there is that of the relative entropy. In fact, there is not just one but several notions of quantum relative entropy. Here we recall the one named after Belavkin and Staszewski which is defined by the formula

$$S_{BS}(A||B) = -\operatorname{Tr} A \log(A^{-1/2}BA^{-1/2}), \quad A, B \in B(H)_+^{-1}. \tag{1}$$

Motivated by Wigner's famous theorem on the structure of quantum mechanical symmetry transformations (i.e., bijections of the space of all pure states preserving transition probability), in Theorem 5 in the paper [5] we determined the structure of all bijective maps ϕ on $B(H)_+^{-1}$ which leave the Belavkin-Staszewski relative entropy invariant, i.e., which satisfy

$$S_{BS}(\phi(A)||\phi(B)) = S_{BS}(A||B), \quad A, B \in B(H)_+^{-1}. \tag{2}$$

It turned out that in spite of the high nonlinearity reflected in the definition (1) of the quantity $S_{BS}(\cdot,\cdot)$, the transformations ϕ which leave it invariant are unitary-antiunitary similarity transformations. To be honest in [5, Theorem 5] the map ϕ was originally defined on the class of all nonsingular density operators (i.e., elements of $B(H)_+^{-1}$ with unit trace which represent mixed quantum states), but in the very first step of the proof we extended it to the whole space $B(H)_+^{-1}$ keeping its bijectivity and preserver property (2). In Theorem 1 below we extend this result to a much more general setting, namely, we consider bijective maps on $B(H)_+^{-1}$ which preserve a quantity similar to (1) but the function $-\log$ is replaced by any nonconstant real valued operator monotone decreasing function f on $]0,\infty[$ having the property $\lim_{t\to\infty} f(t)/t = 0$. The precise formulation of the result reads as follows.

THEOREM 1. *Let* $f\colon\ ]0,\infty[\ \to \mathbb{R}$ *be a nonconstant operator monotone decreasing function such that* $\lim_{t\to\infty} f(t)/t = 0$. *The bijective map* $\phi\colon B(H)_+^{-1} \to \to B(H)_+^{-1}$ *satisfies*

$$\operatorname{Tr}\phi(A)f\left(\phi(A)^{-1/2}\phi(B)\phi(A)^{-1/2}\right) = \operatorname{Tr} Af\left(A^{-1/2}BA^{-1/2}\right),\ A,B \in B(H)_+^{-1}$$

if and only if there is a unitary or antiunitary operator U *on* H *such that we have*

$$\phi(A) = UAU^*, \quad A \in B(H)_+^{-1}.$$

We now turn to our second characterization of unitary-antiunitary similarity transformations on $B(H)_+^{-1}$.

In the recent paper [6] we have discussed transformations on structures of positive semidefinite operators which preserve a given norm of a given operator mean. Our second theorem in this paper is also connected to means, hence we briefly collect the necessary preliminaries on the Kubo-Ando theory of operator means. Following the fundamental paper [2], a binary operation σ on $B(H)_+$ is called a connection if it satisfies the following conditions. For any operators $A, B, C, D \in B(H)_+$ and sequences $(A_n), (B_n)$ in $B(H)_+$ we have

(O1) if $A \le C$ and $B \le D$ then $A\sigma B \le C\sigma D$;
(O2) $C(A\sigma B)C \le (CAC)\sigma(CBC)$;
(O3) if $A_n \downarrow A$ and $B_n \downarrow B$ then $A_n\sigma B_n \downarrow A\sigma B$,

where the arrow $\downarrow$ refers to monotone decreasing convergence in the strong operator topology. (We remark that in this definition H should be an infinite dimensional Hilbert space in order to finally obtain the Kubo-Ando theory of operator means which is independent of the underlying Hilbert space.) A connection σ is called a mean if it is normalized in the sense that for the identity operator I on H we have $I\sigma I = I$. Clearly, for any connection (or mean) σ on $B(H)_+$, its so-called transpose σ' defined by

$$A\sigma' B = B\sigma A, \quad A, B \in B(H)_+$$

is a connection (or mean) again and σ is called symmetric if $\sigma = \sigma'$. The most simple means are the weighted arithmetic means (which are just the fixed convex combinations); $A\sigma B = \lambda A + (1-\lambda)B$ with some given $\lambda \in [0,1]$.

One of the most important results in the Kubo-Ando theory says that there is an affine order-isomorphism between the class of all connections σ on $B(H)_+$ and the class of all nonnegative real valued operator monotone increasing functions f on $]0,\infty[$, see Theorem 3.2 in [2]. In fact, as seen in the proof of that theorem, if σ is a connection then the operator monotone function f associated with it is $f(t) = I\sigma(tI)$, $t > 0$. Conversely, if f is a nonnegative real valued operator monotone increasing function on $]0,\infty[$ then the connection with which it is associated satisfies

$$A\sigma B = A^{1/2} f\left(A^{-1/2} B A^{-1/2}\right) A^{1/2} \tag{3}$$

for all invertible elements A, B of $B(H)_+$. We remark that the property (O3) implies that the above formula extends also to the case where $A \in B(H)_+$ is invertible but $B \in B(H)_+$ is arbitrary (to be correct, in that case the quantity $f(0)$ should be defined; we set $f(0) = \lim_{t \to 0} f(t)$). The case where $f(t) = \sqrt{t}$, $t > 0$ is especially important. The corresponding mean is called the geometric mean of positive semidefinite operators which has very many applications in different areas of science. By Corollary 4.2 in [2], if σ is a connection with associated operator monotone function f, then $t \mapsto tf(1/t)$, $t > 0$ is the operator monotone function on $]0, \infty[$ associated with the transpose σ' of σ. Therefore, for symmetric means we have that the corresponding operator monotone function f satisfies $f(t) = tf(1/t)$ for all $t > 0$.

By Theorem 4.5 in [2] the arithmetic mean $A \nabla B = (A + B)/2$, $A, B \in B(H)_+$ is the maximal symmetric mean. That is, for any symmetric mean σ we have

$$A\sigma B \leq A \nabla B \tag{4}$$

for all $A, B \in B(H)_+$. One can use the trace-norm (which, for positive semidefinite operators, simply equals the trace) to measure the gap between the two sides of the inequality in (4). In our second result we show the hopefully interesting fact that the bijective transformations on $B(H)_+$ which preserve this quantity for any given symmetric mean (satisfying some mild assumptions) are exactly the unitary-antiunitary similarity transformations on $B(H)_+$.

THEOREM 2. *Let σ be a symmetric operator mean and $f\colon\ ]0, \infty[\ \to [0, \infty[$ the nonnegative scalar valued operator monotone function associated with σ. Assume $f(0) := \lim_{t \to 0} f(t) = 0$ and $\lim_{t \to \infty} f(t) = \infty$. A bijective map $\phi\colon B(H)_+ \to B(H)_+$ satisfies*

$$\operatorname{Tr}(\phi(A) \nabla \phi(B) - \phi(A) \sigma \phi(B)) = \operatorname{Tr}(A \nabla B - A \sigma B), \quad A, B \in B(H)_+ \tag{5}$$

if and only if there is a unitary or antiunitary operator U on H such that

$$\phi(A) = UAU^*, \quad A \in B(H)_+.$$

Observe that many means, among them the geometric mean, satisfy the requirements of the theorem. Furthermore, we remark that applying very simple modifications in the proof one check easily that the same result holds true also in the case where ϕ acts on $B(H)_+^{-1}$, not on $B(H)_+$.

2. Proofs

In this section we present the proofs of our results. Our strategy is the following. We present characterizations of the usual order $\leq$ between self-adjoint operators in terms of the quantities what our transformations preserve. This will imply that they are order automorphisms. Then we can apply our former results on the structures of those automorphisms and then finish the proof rather easily.

The proof of Theorem 1 is based on the following characterization of the order $\leq$ on $B(H)_+^{-1}$. Recall that for any two self-adjoint operators A, B in $B(H)$ we write $A \leq B$ if and only if $\langle Ax, x\rangle \leq \langle Bx, x\rangle$ holds for all $x \in H$.

LEMMA 3. *Let f be as in Theorem 1. For any $A, B \in B(H)_+^{-1}$ we have $A \leq B$ if and only if*

$$\operatorname{Tr} Xf(X^{-1/2}AX^{-1/2}) \geq \operatorname{Tr} Xf(X^{-1/2}BX^{-1/2})$$

holds for all $X \in B(H)_+^{-1}$.

PROOF. First, let g be any real valued continuous function on the interval $]0, \infty[$. We claim that for any invertible operator $A \in B(H)$ we have

$$Ag(A^*A) = g(AA^*)A. \tag{6}$$

Indeed, since $|A|$ clearly commutes with $g(|A|^2)$, we infer $|A|g(|A|^2) = g(|A|^2)|A|$. Let U be the unitary operator in the polar decomposition of A, i.e., $A = U|A|$. We have $g(UXU^*) = Ug(X)U^*$ for any self-adjoint operator $X \in \in B(H)$ (in fact, this follows from the fact that g can be uniformly approximated by polynomials on every compact subinterval of $]0, \infty[$). Then we deduce

$$Ag(A^*A) = U|A|g\left(|A|^2\right) = Ug\left(|A|^2\right)|A| = g\left(U|A|^2U^*\right)U|A| = g\left(AA^*\right)A.$$

Now, let g be defined by $g(t) = f(1/t)$, $t > 0$. Using (6) we compute

$$\begin{aligned}
\operatorname{Tr} Af\left(A^{-1/2}BA^{-1/2}\right) &= \operatorname{Tr} Ag\left(A^{1/2}B^{-1}A^{1/2}\right) = \\
&= \operatorname{Tr} Ag\left(\left(A^{1/2}B^{-1/2}\right)\left(A^{1/2}B^{-1/2}\right)^*\right) = \\
&= \operatorname{Tr} B^{1/2}A^{1/2}g\left(\left(A^{1/2}B^{-1/2}\right)\left(A^{1/2}B^{-1/2}\right)^*\right)A^{1/2}B^{-1/2} = \\
&= \operatorname{Tr} B^{1/2}A^{1/2}A^{1/2}B^{-1/2}g\left(\left(A^{1/2}B^{-1/2}\right)^*\left(A^{1/2}B^{-1/2}\right)\right) = \\
&= \operatorname{Tr} BB^{-1/2}AB^{-1/2}g\left(B^{-1/2}AB^{-1/2}\right).
\end{aligned} \tag{7}$$

Denote h the function defined by $h(t) = tg(t)$, $t > 0$ and let $h(0) = 0$. Since $\lim_{t\to\infty} f(t)/t = 0$, we have $\lim_{t\to 0} h(t) = 0$ and hence h is a continuous function on $[0,\infty[$ implying that $X \longmapsto h(X)$ is continuous on $B(H)_+$.

Now, select any $A, B \in B(H)_+^{-1}$. If $A \le B$, then by the operator monotone decreasing property of f it follows that

$$\operatorname{Tr} Xf\left(X^{-1/2}AX^{-1/2}\right) \ge \operatorname{Tr} Xf\left(X^{-1/2}BX^{-1/2}\right) \tag{8}$$

holds for all $X \in B(H)_+^{-1}$. Conversely, assume that the equality in (8) is valid for all $X \in B(H)_+^{-1}$. By the definition of h, (7) and (8) we obtain that

$$\operatorname{Tr} Ah\left(A^{-1/2}XA^{-1/2}\right) = \operatorname{Tr} AA^{-1/2}XA^{-1/2}g\left(A^{-1/2}XA^{-1/2}\right) \ge$$
$$\ge \operatorname{Tr} BB^{-1/2}XB^{-1/2}g\left(B^{-1/2}XB^{-1/2}\right) = \operatorname{Tr} Bh\left(B^{-1/2}XB^{-1/2}\right)$$

holds for all $X \in B(H)_+^{-1}$. Let X converge to an arbitrary rank-one projection P on H. Then we have

$$\operatorname{Tr} Ah\left(A^{-1/2}PA^{-1/2}\right) \ge \operatorname{Tr} Bh\left(B^{-1/2}PB^{-1/2}\right). \tag{9}$$

Pick a unit vector $x \in H$ from the range of P. Obviously, $P = x \otimes x$ holds where the operator $x \otimes x$ is defined by $(x \otimes x)z = \langle z, x\rangle x$, $z \in H$. We compute

$$A^{-1/2}PA^{-1/2} = \left\|A^{-1/2}x\right\|^2 \left(A^{-1/2}x \big/ \|A^{-1/2}x\|\right) \otimes \left(A^{-1/2}x \big/ \|A^{-1/2}x\|\right)$$

and hence

$$A^{1/2}h\left(A^{-1/2}PA^{-1/2}\right)A^{1/2} =$$
$$= A^{1/2}h\left(\left\|A^{-1/2}x\right\|^2\right)\left(A^{-1/2}x \big/ \|A^{-1/2}x\|\right) \otimes \left(A^{-1/2}x/\left\|A^{-1/2}x\right\|\right)A^{1/2} =$$
$$= h\left(\left\|A^{-1/2}x\right\|^2\right)\left(x/\left\|A^{-1/2}x\right\|\right) \otimes \left(x/\left\|A^{-1/2}x\right\|\right) = g\left(\left\|A^{-1/2}x\right\|^2\right) x \otimes x.$$

Consequently, by (9) we have

$$g\left(\left\|A^{-1/2}x\right\|^2\right) \ge g\left(\left\|B^{-1/2}x\right\|^2\right).$$

Since f is a nonconstant operator monotone function, it is strictly monotone (according to a famous theorem of Löwner, operator monotone functions on any open interval have analytic continuations onto the upper half plane). We deduce that g is strictly increasing which implies that $\|A^{-1/2}x\|^2 \ge \|B^{-1/2}x\|^2$, i.e., $\langle A^{-1}x, x\rangle \ge \langle B^{-1}x, x\rangle$. The rank-one projection P was arbitrary which means that this inequality holds for every unit vector $x \in H$. Therefore, $A^{-1} \geq B^{-1}$ which is equivalent to $A \leq B$. This completes the proof of the lemma. ∎

Now we are in a position to prove our first theorem.

PROOF OF THEOREM 1. The necessity part of the statement is easy. As for sufficiency, it follows from the characterization of the order $\leq$ given in the above lemma that ϕ is an order automorphism of $B(H)_+^{-1}$. The structure of such maps is known. By Theorem 1 in [5] there is an invertible either linear or conjugate linear operator T on H such that $\phi(A) = TAT^*$, $A \in B(H)_+^{-1}$. In addition to that we have the preserver property of ϕ, i.e.,

$$\operatorname{Tr} TAT^*f\left((TAT^*)^{-1/2}(TBT^*)(TAT^*)^{-1/2}\right) = \operatorname{Tr} Af\left(A^{-1/2}BA^{-1/2}\right)$$

holds for all $A, B \in B(H)_+^{-1}$. Let $t > 0$ be such that $f(t) \neq 0$. Plugging $B = tA$, we have $\operatorname{Tr} TAT^*f(tI) = \operatorname{Tr} Af(tI)$. Since $f(tI) = f(t)I$, we infer that $\operatorname{Tr} TAT^* = \operatorname{Tr} A$ and hence $\operatorname{Tr} AT^*T = \operatorname{Tr} A$ holds for all $A \in B(H)_+^{-1}$. This clearly implies $T^*T = I$, consequently T is a unitary or antiunitary operator on H. The proof is complete. ∎

Before turning to the proof of our second theorem we present some more preliminaries which will be needed.

From the famous Löwner theory of operator monotone functions it is well-known that all such functions have a certain integral representation. In fact, the formula

$$f(t) = \int_{[0,\infty]} \frac{t(1+s)}{t+s} dm(s), \quad t > 0 \tag{10}$$

gives an affine order-isomorphism from the class of all positive Radon measures m on the extended interval $[0, \infty]$ onto the set of all nonnegative scalar valued operator monotone increasing functions f on $]0, \infty[$, see Lemma 3.1 in [2]. We note that here $f(0) := \lim_{t\to 0} f(t) = m(\{0\})$ and $\lim_{t\to\infty} f(t)/t = m(\{\infty\})$.

Next, let us recall the concept of the strength of a positive semidefinite operator along any rank-one projection. Following [1], p. 329, for any $A \in B(H)_+$ and rank-one projection P on H we define the numerical quantity $\lambda(A, P)$ by

$$\lambda(A, P) = \sup\{t \geq 0 \colon tP \leq A\}$$

and call it the strength of A along P. The function $P \longmapsto \lambda(A, P)$ is said to be the strength function of A. By Theorem 1 in [1] we know that for any $A, B \in$ $\in B(H)_+$ we have $A \leq B$ if and only if $\lambda(A, P) \leq \lambda(B, P)$ holds for all rank-one projections P on H.

The proof of Theorem 2 is again based on a characterization of the usual order $\leq$ on $B(H)_+$. This is given in the next lemma. Denote $d(A,B) = \mathrm{Tr}(A\nabla B - A\sigma B)$, $A,B \in B(H)_+$.

LEMMA 4. *Let σ, f be as in Theorem 2. For any $A,B \in B(H)_+$ we have $A \leq B$ if and only if the set $\{d(A,X) - d(B,X) \colon X \in B(H)_+\}$ is bounded from below.*

PROOF. Assume $A \leq B$, then by the monotonicity of operator means (O1) we obtain

$$d(A,X) - d(B,X) = \mathrm{Tr}\left((A-B)/2\right) - \mathrm{Tr}\left(A\sigma X - B\sigma X\right) \geq \mathrm{Tr}\left((A-B)/2\right)$$

for any $X \in B(H)_+$. This gives us the necessity part of the statement. As for the converse, assume that the set $\{d(A,X) - d(B,X) \colon X \in B(H)_+\}$ is bounded from below. This means that

$$\mathrm{Tr}\left(B\sigma X - A\sigma X\right) \geq c$$

holds with a given constant c for arbitrary $X \in B(H)_+$. By Lemma 2.6 in [4] we have the formula

$$A\sigma P = f\left(\lambda\left(A,P\right)\right)P \tag{11}$$

for every $A \in B(H)_+$ and rank-one projection P on H. Since the operator means are clearly homogeneous (see (3) and then use the continuity property (O3)), we obtain

$$A\sigma(tP) = t\left(\left(1/t\right)A\right)\sigma P = tf\left(\left(1/t\right)\lambda\left(A,P\right)\right)P$$

for all $t > 0$. Therefore, letting t range over the set of all positive real numbers and P range over the set of all rank-one projections on H, it follows that the set of all quantities

$$tf\left(\left(1/t\right)\lambda\left(B,P\right)\right) - tf\left(\left(1/t\right)\lambda\left(A,P\right)\right)$$

is bounded from below by the number c given above. We fix P and claim that $\lambda(A,P) \leq \lambda(B,P)$. Obviously, we need to check this only in the case where $\lambda(A,P)$ is positive. Assuming that, $\lambda(B,P)$ is also positive. Indeed, by the symmetricity of σ we have $f(t) = tf(1/t)$ for all $t > 0$ and hence, if $\lambda(B,P)$ were zero, we would get that

$$-tf\left(\left(1/t\right)\lambda\left(A,P\right)\right) = -\lambda\left(A,P\right)f\left(t/\lambda\left(A,P\right)\right), \quad t > 0$$

is bounded from below meaning that f is bounded from above, a contradiction. Hence, we have $\lambda(B,P) > 0$. Let us introduce the new variable $s = t/\lambda(A,P)$. Then denoting $\gamma = \lambda(A,P)/\lambda(B,P)$, simple calculation using the symmetricity of f again yields

$$tf\left(\left(1/t\right)\lambda\left(B,P\right)\right) - tf\left(\left(1/t\right)\lambda\left(A,P\right)\right) = \lambda\left(A,P\right)\left(f\left(\gamma s\right)/\gamma - f\left(s\right)\right)$$

and hence we deduce that the function

$$t \longmapsto f(\gamma t)/\gamma - f(t), \quad t > 0$$

is bounded from below. We show that this implies $\gamma \leq 1$. To see this, we use the integral representation (10) of f (observe that in our present case $m(\{0\}) = m(\{\infty\}) = 0$) and compute

$$f(\gamma t) - \gamma f(t) = \int_{]0,\infty[} \frac{\gamma t(1+s)}{\gamma t + s} - \frac{\gamma t(1+s)}{t+s} dm(s) =$$

$$= \gamma(1-\gamma) \int_{]0,\infty[} \frac{t^2(1+s)}{(\gamma t+s)(t+s)} dm(s).$$

Assuming $\gamma > 1$, the quantity $\gamma(1-\gamma)$ is negative and it follows that

$$\int_{]0,\infty[} \frac{t^2(1+s)}{(\gamma t+s)(t+s)} dm(s) = \int_{]0,\infty[} \frac{1+s}{(\gamma + s/t)(1+s/t)} dm(s)$$

as a function of $t > 0$ is bounded from above. However, for any fixed s, the values under the sign of integral increase as t increases to ∞. Therefore, applying Beppo Levi theorem we deduce that

$$\lim_{t\to\infty} \int_{]0,\infty[} \frac{1+s}{(\gamma + s/t)(1+s/t)} dm(s) = \int_{]0,\infty[} \lim_{t\to\infty} \frac{1+s}{(\gamma + s/t)(1+s/t)} dm(s) =$$

$$= (1/\gamma) \int_{]0,\infty[} 1 + s dm(s) =$$

$$= (1/\gamma) \int_{]0,\infty[} \lim_{t\to\infty} \frac{t(1+s)}{t+s} dm(s) =$$

$$= (1/\gamma) \lim_{t\to\infty} \int_{]0,\infty[} \frac{t(1+s)}{t+s} dm(s) =$$

$$= (1/\gamma) \lim_{t\to\infty} f(t).$$

Hence we obtain that $\lim_{t\to\infty} f(t)$ is finite which is a contradiction. It follows that we necessarily have $\gamma \leq 1$ implying that $\lambda(A,P) \leq \lambda(B,P)$. Since this holds for every rank-one projection P on H, we obtain that $A \leq B$. ∎

Having proven the above characterization, the proof of our second theorem is simple. We shall use the fact, usually referred to as the transfer property, that for any invertible linear or conjugate linear operator $T \in B(H)$ and $A, B \in$ $\in B(H)_+$ we have $T(A\sigma B)T^* = (TAT^*)\sigma(TBT^*)$. This follows quite easily from the inequality (O2).

PROOF OF THEOREM 2. The necessity part of the statement is apparent. As for the sufficiency, it follows from Lemma 4 that ϕ is an order isomorphism of $B(H)_+$. The structure of those transformations is also known. By Theorem 1 in [3], it follows that we have an invertible either linear or conjugate linear operator T on H such that

$$\phi(A) = TAT^*, \quad A \in B(H)_+.$$

Plugging $A = I$ into (5) and using the transfer property mentioned above we deduce

$$\begin{aligned} \operatorname{Tr} T\left((I+B)/2 - f(B)\right) T^* &= \operatorname{Tr} T\left(I \nabla B - I\sigma B\right) T^* = \\ &= \operatorname{Tr}\left(\phi(I)\nabla\phi(B) - \phi(I)\sigma\phi(B)\right) = \\ &= \operatorname{Tr}\left(I\nabla B - I\sigma B\right) = \operatorname{Tr}\left((I+B)/2 - f(B)\right). \end{aligned}$$

Substituting any projection P on H into the place of B, by $f(P) = P$ we obtain from the last equalities that

$$\operatorname{Tr}(I-P)T^*T = \operatorname{Tr} T(I-P)T^* = \operatorname{Tr}(I-P).$$

Since $I - P$ ranges over the set of all projections on H we infer from this that $T^*T = I$, i.e., T is either a unitary or an antiunitary operator on H. The proof is complete. ∎

To conclude the paper we make a few remarks. Throughout we have assumed that H is at least two-dimensional. If $\dim H = 1$, then the treated problems reduce to certain functional equations that can possibly be solved with related methods. As for the first theorem, we mention that in the case where $f(1) \neq$ $\neq 0$ (which is not satisfied by the motivating example $f = -\log$) we immediately obtain that the only solution of the corresponding functional equation is the identity function. Otherwise, we need to consider an implicit functional equation and the situation is similar regarding the second theorem, too. Concerning Theorem 2 we further note that dual to the case of the arithmetic mean, the harmonic mean is known to be the minimal symmetric operator mean. So, it would be a natural question to investigate a similar problem for the harmonic mean in the place of the arithmetic mean. That question seems to be challenging. Finally, as already referred to it in the introduction, one might consider similar problems in more general operator algebras, e.g., in von Neumann algebras carrying scalar valued traces.

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ON THE SUM BETWEEN A CLOSABLE OPERATOR T AND A T-BOUNDED OPERATOR

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Abstract. We provide several perturbation theorems regarding closable operators on a real or complex Hilbert space. In particular we extend some classical results due to Hess–Kato, Kato–Rellich and Wüst. Our approach involves ranges of matrix operators of the form $\begin{pmatrix} I & A \\ -B & I \end{pmatrix}$.

1. Introduction

In the present paper we develop a perturbation theory of closable operators between Hilbert spaces. Operators we consider are (unless it is otherwise indicated) not necessarily densely defined linear transformations and the Hilbert spaces are allowed to be either real or complex. The domain, kernel and range of an operator A is denoted by $\operatorname{dom} A$, $\ker A$ and $\operatorname{ran} A$, respectively. The scalar product of each Hilbert space we encounter is denoted by the same symbol $(\cdot\,|\,\cdot)$ in the hope that we do not cause any confusion. As usual, A^* stands for the adjoint operator of a densely defined operator A. A not necessarily densely defined operator S is said to be symmetric if it satisfies

$$(Sx\,|\,y) = (x\,|\,Sy), \qquad x, y \in \operatorname{dom} S,$$

and skew-symmetric if

$$(Sx\,|\,y) = -(x\,|\,Sy), \qquad x, y \in \operatorname{dom} S.$$

For a densely defined S the above relations mean that $S \subset S^*$ and $S \subset -S^*$, respectively. S is called selfadjoint (resp., skew-adjoint) if $S = S^*$ (resp., $S = -S^*$).

If S and T are arbitrary operators with $\operatorname{dom} S \subseteq \operatorname{dom} T$ and there exist $a, b \geq 0$ satisfying

$$\|Sx\|^2 \leq a\,\|Tx\|^2 + b\|x\|^2, \qquad x \in \operatorname{dom} T, \tag{1.1}$$

2010 Mathematics Subject Classification Primary 47A55, 47B25 then S is called T-bounded. The T-bound of S in that case is defined to be the infimum of all nonnegative numbers a for which a $b \geq 0$ exists such that (1.1) satisfies. The notion of T-boundedness is a useful tool of the classical perturbation theory. For example, it appears as a basic condition of the Hess–Kato [3], Kato–Rellich [10] and Wüst [15] perturbation theorems. The main goal of this note is to provide similar perturbation results under weaker conditions (see Theorem 3.3, Corollary 3.7 and Corollary 3.9 below). Our results involve ranges of 2-by-2 matrix operators of the form $\begin{pmatrix} I & A \\ -B & I \end{pmatrix}$. In our most recent works [8, 9] the conditions provided by these matrices are replaced by similar conditions involving the operators $I+AB$ and $I+BA$.

2. Closability of operators

We start our discussion with an easy but useful result which in fact is a part of [7, Lemma 3.1]. We present also its proof for the sake of the reader.

LEMMA 2.1. *Let S and T be (possibly unbounded) linear operators between the Hilbert spaces $\mathscr{H}$ and $\mathscr{K}$, resp., $\mathscr{K}$ and $\mathscr{H}$. If*

(a) $\operatorname{ran} S = \mathscr{K}$,
(b) $\overline{\operatorname{ran} T} = \mathscr{H}$,
(c) $(Sh \,|\, k) = (h \,|\, Tk)$ *for all* $h \in \operatorname{dom} S, k \in \operatorname{dom} T$,

then T is automatically densely defined and $T^* = S$.

PROOF. In order to prove that T is densely defined, let $v \in \operatorname{dom} T^{\perp}$. Consider $h \in \operatorname{dom} S$ such that $Sh = v$. As for any $k \in \operatorname{dom} T$

$$0 = (v \,|\, k) = (Sh \,|\, k) = (h \,|\, Tk),$$

we deduce that $h \in \operatorname{ran} T^{\perp} = \{0\}$. Hence $v = Sh = 0$, as required. It is obvious by (c) that $S \subset T^*$. For the converse inclusion we should show that $\operatorname{dom} T^* \subseteq \operatorname{dom} S$. With this aim let us take $k \in \operatorname{dom} T^*$ and $u \in \operatorname{dom} S$ such that $Su = T^*k$. As $S \subset T^*$, we have $T^*k = T^*u$ and hence

$$k = k - u + u \in \ker T^* + \operatorname{dom} S \subseteq \operatorname{dom} S.$$

Here, for the last equality we used the fact that $\ker T^* = \operatorname{ran} T^{\perp} = \{0\}$. ∎

THEOREM 2.2. *Let $\mathscr{H}_1$ and $\mathscr{H}_2$ be Hilbert spaces and consider the (not necessarily densely defined) linear operators $A, C\colon \mathscr{H}_1 \to \mathscr{H}_2$, $B, D\colon \mathscr{H}_2 \to \mathscr{H}_1$. If*

(a) $\operatorname{ran} \begin{pmatrix} I & A \\ -B & I \end{pmatrix} = \mathscr{H}_2 \times \mathscr{H}_1$,

(b) $\overline{\operatorname{ran} \begin{pmatrix} I & -C \\ D & I \end{pmatrix}} = \mathscr{H}_2 \times \mathscr{H}_1$,

(c) $(Ax_1 \,|\, x_4) - (x_1 \,|\, Dx_4) = (Bx_2 \,|\, x_3) - (x_2 \,|\, Cx_3) (= 0), x_1 \in \operatorname{dom} A, x_2 \in \operatorname{dom} B, x_3 \in$ $\in \operatorname{dom} C, x_4 \in \operatorname{dom} D$,

then C and D are densely defined such that $D^* = A$ and $C^* = B$.

PROOF. We may use Lemma 2.1 for the Hilbert spaces $\mathscr{H} = \mathscr{K} = \mathscr{H}_2 \times \mathscr{H}_1$ and the operators

$$S = \begin{pmatrix} I & A \\ -B & I \end{pmatrix}, \qquad T = \begin{pmatrix} I & -C \\ D & I \end{pmatrix}.$$

We firstly observe that

$$\begin{aligned} \left(\begin{pmatrix} I & A \\ -B & I \end{pmatrix} \begin{pmatrix} x_2 \\ x_1 \end{pmatrix} \,\middle|\, \begin{pmatrix} x_4 \\ x_3 \end{pmatrix} \right) &= (x_2 + Ax_1 \,|\, x_4) + (-Bx_2 + x_1 \,|\, x_3) = \\ &= (x_2 \,|\, x_4) + (Ax_1 \,|\, x_4) - (Bx_2 \,|\, x_3) + (x_1 \,|\, x_3) \end{aligned}$$

and that

$$\begin{aligned} \left(\begin{pmatrix} x_2 \\ x_1 \end{pmatrix} \,\middle|\, \begin{pmatrix} I & -C \\ D & I \end{pmatrix} \begin{pmatrix} x_4 \\ x_3 \end{pmatrix} \right) &= (x_2 \,|\, x_4 - Cx_3) + (x_1 \,|\, Dx_4 + x_3) = \\ &= (x_2 \,|\, x_4) + (x_1 \,|\, Dx_4) - (x_2 \,|\, Cx_3) + (x_1 \,|\, x_3). \end{aligned}$$

The two quantities are equal if and only if (c) holds true. We deduce therefore by Lemma 2.1 that

$$\overline{\operatorname{dom} \begin{pmatrix} I & -C \\ D & I \end{pmatrix}} = \mathscr{H}_2 \times \mathscr{H}_1 \qquad \text{and} \qquad \begin{pmatrix} I & -C \\ D & I \end{pmatrix}^* = \begin{pmatrix} I & A \\ -B & I \end{pmatrix}.$$

As $\operatorname{dom} T = \operatorname{dom} D \times \operatorname{dom} C$, we conclude that C and D are densely defined operators. In addition, since

$$\begin{pmatrix} I & -C \\ D & I \end{pmatrix}^* = \begin{pmatrix} I & D^* \\ -C^* & I \end{pmatrix},$$

it follows that $D^* = A$ and $C^* = B$. ∎

REMARK 2.3. We mention here the elementary fact that for given two operators S and T, $\begin{pmatrix} I & S \\ -T & I \end{pmatrix}$ has full (resp., dense) range if and only if $\begin{pmatrix} I & -S \\ T & I \end{pmatrix}$ has full (resp., dense) range. The proof is left to the reader.

THEOREM 2.4. *Let A and B be (possibly unbounded) linear operators between the Hilbert spaces $\mathscr{H}$ and $\mathscr{K}$, resp., $\mathscr{K}$ and $\mathscr{H}$. The following assertions are equivalent:*

(i) B is a densely defined closable operator such that $B^* = A$;

(ii) A is a densely defined closed operator such that

$$(Ax \,|\, y) = (x \,|\, By), \qquad x \in \operatorname{dom} A, y \in \operatorname{dom} B,$$

and that $\overline{\operatorname{ran} \begin{pmatrix} I & A \\ -B & I \end{pmatrix}} = \mathscr{K} \times \mathscr{H}$.

PROOF. We firstly observe that $\begin{pmatrix} I & A \\ -A^* & I \end{pmatrix}$ is surjective under assumption (ii). Indeed, as A is densely defined and closed, we have by von Neumann's formulae

$$\begin{aligned}\mathscr{K} \times \mathscr{H} &= \{(y, -A^*y) \,|\, y \in \operatorname{dom} A^*\} \oplus \{(Ax, x) \,|\, x \in \operatorname{dom} A\} = \\ &= \{(y + Ax, x - A^*y) \,|\, x \in \operatorname{dom} A, y \in \operatorname{dom} A^*\} = \operatorname{ran} \begin{pmatrix} I & A \\ -A^* & I \end{pmatrix}.\end{aligned}$$

Furthermore, by Theorem 2.2 we conclude that

$$\overline{\operatorname{dom} B} = \mathscr{H} \qquad \text{and} \qquad B^* = A.$$

Hence (ii) implies (i). Conversely, assume that A, B fulfill (i). Obviously, A is closed (being the adjoint of B) and densely defined (as B is closable). Our only claim is therefore to show that $\overline{\operatorname{ran} \begin{pmatrix} I & B^* \\ -B & I \end{pmatrix}} = \mathscr{K} \times \mathscr{H}$ holds for any closable operator between $\mathscr{K}$ and $\mathscr{H}$. Equivalently,

$$\ker \begin{pmatrix} I & -B^* \\ B^{**} & I \end{pmatrix} = \{0\}.$$

Indeed, for if $x \in \operatorname{dom} B^*, y \in \operatorname{dom} B^{**}$ then

$$\begin{aligned}\left\| \begin{pmatrix} I & -B^* \\ B^{**} & I \end{pmatrix} \begin{pmatrix} y \\ x \end{pmatrix} \right\|^2 &= \|y - B^*x\|^2 + \|B^{**}y + x\|^2 = \\ &= \|y\|^2 + \|B^*x\|^2 + \|x\|^2 + \|B^{**}y\|^2 = \\ &-(B^*x \,|\, y) - (y \,|\, B^*x) + (B^{**}y \,|\, x) + (x \,|\, B^{**}y) = \\ &= \|y\|^2 + \|B^*x\|^2 + \|x\|^2 + \|B^{**}y\|^2 \geq \\ &\geq \left\| \begin{pmatrix} y \\ x \end{pmatrix} \right\|^2.\end{aligned}$$

(Here we used the well known identity $B^* = (B^{**})^*$). Hence (i) implies (ii). ∎

REMARK 2.5. The surjectivity of the matrix operator $\begin{pmatrix} I & A \\ -A^* & I \end{pmatrix}$ (for a given densely defined operator A) has been observed in [7, Propostion 2.3 (b)] to be equivalent with the property of A to be closed. In [7, Propostion 2.3 (a)] it was shown, with a different proof, that $\begin{pmatrix} I & A \\ -A^* & I \end{pmatrix}$ has dense range even without the assumption on the closability of A.

REMARK 2.6. In the previous result we obtain the same conclusion if we replace I by αI for a certain/for any $\alpha \in \mathbb{R} \setminus \{0\}$.

REMARK 2.7. We also mention, without proof, that for given densely defined linear operators A, B between the Hilbert spaces $\mathscr{H}$ and $\mathscr{K}$, resp., $\mathscr{K}$ and $\mathscr{H}$, the conditions

(i) $\overline{\operatorname{ran} \begin{pmatrix} I & A \\ B & I \end{pmatrix}} = \mathscr{K} \times \mathscr{H}$,

(ii) $1 \notin \mathrm{Sp}_p(B^*A^*)$,
(iii) $1 \notin \mathrm{Sp}_p(A^*B^*)$,
are equivalent. This result is obtained in [7, Proposition 2.4].

Theorem 2.4 provides a useful characterization of essentially selfadjointness, cf. also [7, Corollary 4.8]:

COROLLARY 2.8. *Let S be a linear operator acting in the (real or complex) Hilbert space $\mathscr{H}$. The following statements are equivalent:*
(i) S is essentially selfadjoint;
(ii) S is densely defined symmetric such that $\overline{\operatorname{ran}\begin{pmatrix} I & S \\ -S & I \end{pmatrix}} = \mathscr{H} \times \mathscr{H}$.

PROOF. Theorem 2.4 can be applied for $A = S^{**}$ and $B = S$. ∎

3. Perturbation theorems for closable operators

This section is devoted to the perturbation theory of linear operators. More precisely, we deal with perturbations $S+T$ of closable, selfadjoint and essentially selfadjoint operators T by a T-bounded operator S and we provide some generalizations of important results due to Hess–Kato [3], Kato–Rellich [10], and Wüst [15].

THEOREM 3.1. *Let W, Z be linear operators in the Hilbert space $\mathscr{H}$ with $\operatorname{dom} Z \subseteq \subseteq \operatorname{dom} W$ and so that Z and $W|_{\operatorname{dom} Z}$ are skew-symmetric:*

$$(Zh \,|\, k) + (h \,|\, Zk) = 0,$$
$$(Wh \,|\, k) + (h \,|\, Wk) = 0,$$

for $h, k \in \operatorname{dom} Z$. Assume furthermore that W and Z fulfill each of the following conditions:
(a) Z is closable (or $W|_{\operatorname{dom} Z}$ is closable);
(b) $T := I + Z$ has dense range;
(c) $\|Wh\|^2 \leq \|Th\|^2$ for all $h \in \operatorname{dom} T = \operatorname{dom} Z$.
Then $T + W$ has dense range in $\mathscr{H}$.

PROOF. Let $k \in \operatorname{ran}(T+W)^{\perp}$, that is to say,

$$\big(k \,\big|\, (T+W)h\big) = 0 \qquad \text{for all } h \in \operatorname{dom} Z.$$

By b) we may choose a sequence $(h_n)_{n\in\mathbb{N}}$ in $\operatorname{dom} Z$ such that

$$\|k - Th_n\| = \|k - (I+Z)h_n\| \to 0, \qquad n \to \infty.$$

Here $\|T(h_n - h_m)\|^2 = \|Z(h_n - h_m)\|^2 + \|h_n - h_m\|^2$ by skew-symmetry, whence we conclude that $(h_n)_{n\in\mathbb{N}}$ and $(Zh_n)_{n\in\mathbb{N}}$ also converge:

$$h_n \to \widetilde{h}, \qquad Zh_n \to z.$$

Let us observe that for $h \in \operatorname{dom} Z$

$$\begin{aligned}\|k+Wh\|^2 &= \|k\|^2 + \|Wh\|^2 + (k\,|\,Wh) + (Wh\,|\,k) = \\ &= \|k\|^2 + \|Wh\|^2 - (k\,|\,Th) - (Th\,|\,k) = \\ &\leq \|k\|^2 + \|Th\|^2 - (k\,|\,Th) - (Th\,|\,k) = \\ &= \|k-Th\|^2.\end{aligned}$$

Hence if we take $h = h_n$ and let $n \to \infty$ we see that $\|Wh_n + k\|^2 \to 0$. In particular,

$$(W+T)h_n \to 0.$$

Next we claim that

$$\|k-(W+T)h\|^2 = \|k\|^2 + \|(Z+W)h\|^2 + \|h\|^2, \qquad h \in \operatorname{dom} Z. \tag{3.1}$$

Indeed, since $k \in \operatorname{ran}(W+T)^{\perp}$ and by skew-symmetry we have

$$\begin{aligned}\|k-(W+T)h\|^2 &= \|k\|^2 + \|(I+Z+W)h\|^2 = \\ &= \|k\|^2 + \|h\|^2 + \|(Z+W)h\|^2 + (h\,|\,(Z+W)h) + ((Z+W)h\,|\,h) = \\ &= \|k\|^2 + \|(Z+W)h\|^2 + \|h\|^2.\end{aligned}$$

Letting, as before, $h = h_n$ and passing to limit we deduce by (3.1) that

$$\|k\|^2 = \lim_{n\to\infty} \|k-(W+T)h_n\|^2 = \|k\|^2 + \|z-k\|^2 + \left\|\widetilde{h}\right\|^2,$$

so $\widetilde{h} = 0$ and $k = z$, that is,

$$h_n \to 0, \qquad Zh_n \to k.$$

Finally, we deduce by the fact that Z is closable that $k = 0$. Consequently, $\operatorname{ran}(W+T)^{\perp} = \{0\}$, as required. ∎

The key to the upcoming perturbation theorems is the next corollary (combined with Theorem 2.4):

COROLLARY 3.2. *Let A, R and B, S be linear operators between the Hilbert spaces $\mathscr{H}$ and $\mathscr{K}$, respectively $\mathscr{K}$ and $\mathscr{H}$, with $\operatorname{dom} A \subseteq \operatorname{dom} R$ and $\operatorname{dom} B \subseteq \operatorname{dom} S$ so that*

$$\begin{aligned}(Ah\,|\,k) &= (h\,|\,Bk), \qquad h \in \operatorname{dom} A, k \in \operatorname{dom} B,\\ (Rh\,|\,k) &= (h\,|\,Sk), \qquad h \in \operatorname{dom} R, k \in \operatorname{dom} S.\end{aligned}$$

Assume furthermore that A, B, R, S fulfill each of the following conditions:

(a) A, B are closable (or $R|_{\operatorname{dom} A}$ and $S|_{\operatorname{dom} B}$ are closable);

(b) $\begin{pmatrix} I & A \\ -B & I\end{pmatrix}$ has dense range in $\mathscr{H} \times \mathscr{K}$;

(c) $\|Rh\|^2 \leq \|Ah\|^2 + \|h\|^2$ for all $h \in \operatorname{dom} A$,

(d) $\|Sk\|^2 \leq \|Bk\|^2 + \|k\|^2$ for all $h \in \operatorname{dom} B$.

Then

$$\overline{\operatorname{ran}\begin{pmatrix} I & A+R \\ -(B+S) & I \end{pmatrix}} = \mathscr{K} \times \mathscr{H}.$$

PROOF. The proof is a simple application of Theorem 3.1 for

$$Z = \begin{pmatrix} 0 & A \\ -B & 0 \end{pmatrix} \qquad \text{and} \qquad W = \begin{pmatrix} 0 & R \\ -S & 0 \end{pmatrix}. \qquad ∎$$

The next result is an immediate generalization of the Hess–Kato [3] perturbation theorem:

THEOREM 3.3. *Let R and B, S be linear operators from $\mathscr{H}$ to $\mathscr{K}$, and $\mathscr{K}$ to $\mathscr{H}$, respectively, satisfying the following conditions:*

(a) $\operatorname{dom} B^* \subseteq \operatorname{dom} R$ *and* $\operatorname{dom} B \subseteq \operatorname{dom} S$*;*
(b) *B is densely defined and closable;*
(c) *$B^* + R$ is closed;*
(d) $(Rh \mid k) = (h \mid Sk)$ *for all* $h \in \operatorname{dom} B^*$ *and* $k \in \operatorname{dom} B$*;*
(e) $\|Rh\|^2 \leq \|B^*h\|^2 + \|h\|^2$ *for* $h \in \operatorname{dom} B^*$*;*
(f) $\|Sk\|^2 \leq \|Bk\|^2 + \|k\|^2$ *for* $k \in \operatorname{dom} B$*.*

Then $B + S$ is (densely defined and) closable such that

$$(B+S)^* = B^* + R$$

PROOF. Since B is densely defined it follows that

$$\overline{\operatorname{ran}\begin{pmatrix} I & B^* \\ -B & I \end{pmatrix}} = \mathscr{K} \times \mathscr{H}.$$

Moreover, $B^* + R$ is densely defined and closed by assumption, such that

$$\big((B^* + R)h \,\big|\, k\big) = \big(h \,\big|\, (B+S)k\big), \qquad h \in \operatorname{dom} B^*, k \in \operatorname{dom} B.$$

Consequently, due to the previous Corollary, $B^* + R$ and $B + S$ fulfill the conditions of Theorem 2.4: $B + S$ is closable and $(B+S)^* = B^* + R$. ∎

REMARK 3.4. A simple condition which guarantees the closedness of $B^* + R$ is that R is assumed to be B^*-bounded, by B^*-bound less than 1, see [2, §3, Theorem 4.2].

COROLLARY 3.5. *Let B, S be linear operators from $\mathscr{H}$ to $\mathscr{K}$ satisfying the following conditions:*

(a) $\operatorname{dom} B \subseteq \operatorname{dom} S$ *and* $\operatorname{dom} B^* \subseteq \operatorname{dom} S^*$*;*
(b) *B is densely defined and closable;*
(c) *$B^* + S^*$ is closed;*
(d) $\|Sh\|^2 \leq \|Bh\|^2 + \|h\|^2$ *for* $h \in \operatorname{dom} B$*;*
(e) $\|S^*k\|^2 \leq \|B^*k\|^2 + \|k\|^2$ *for* $k \in \operatorname{dom} B^*$*;*

Then $B + S$ is (densely defined and) closable such that

$$(B+S)^* = B^* + S^*.$$

PROOF. Use Theorem 3.3 for $R := S^*$. ∎

The following proposition extends a classical result due to Hess–Kato [3]; cf. also [2, §3, Theorem 4.3]:

PROPOSITION 3.6. *Let S and T be densely defined linear operators between the Hilbert spaces $\mathscr{H}$ and $\mathscr{K}$ satisfying*

(a) T is closable;
(b) $\operatorname{dom} T \subseteq \operatorname{dom} S$ and $\operatorname{dom} T^* \subseteq \operatorname{dom} S^*$;
(c) $\|Sh\|^2 \leq \|Th\|^2 + \|h\|^2$ for all $h \in \operatorname{dom} T$;
(d) $\|S^*k\|^2 \leq q\|T^*k\|^2 + \|k\|^2$ for all $k \in \operatorname{dom} T^*$ and for some $0 \leq q < 1$.

Then $B + S$ is closable and $(B + S)^* = B^* + S^*$.

PROOF. Use Corollary 3.5 and Remark 3.4. ∎

The next corollary is a generalized Wüst perturbation theorem (cf. [15]) for essentially selfadjoint operators in real or complex Hilbert spaces (cf. [14, Theorem 5.30]; see also [9]).

COROLLARY 3.7. *Let S, T be linear operators acting on a Hilbert space $\mathscr{H}$. Assume that*

(a) S is essentially selfadjoint with $\operatorname{dom} S \subseteq \operatorname{dom} T$;
(b) $\|Th\|^2 \leq \|Sh\|^2 + \|h\|^2$ for all $h \in \operatorname{dom} S$;
(c) $(Th \mid k) = (h \mid Tk)$ for all $h, k \in \operatorname{dom} S$ (that is, the restriction of T to $\operatorname{dom} S$ is symmetric).

Then $S + T$ is essentially selfadjoint.

PROOF. We show that $A := B := S$ and $R := S := T$ fulfill each of the conditions (a)-(d) of Corollary 3.2. Indeed, $\operatorname{dom} S \subseteq \operatorname{dom} T$, and, being S, T symmetric,

$$(Sh \mid k) = (h \mid Sk) \qquad (Th \mid k) = (h \mid Tk), \qquad h, k \in \operatorname{dom} S.$$

Furthermore, S is essentially selfadjoint, thus $\overline{\operatorname{ran}\begin{pmatrix} I & S \\ -S & I \end{pmatrix}} = \mathscr{H} \times \mathscr{H}$ by Corollary 2.8. Finally, $\|Th\|^2 \leq \|Sh\|^2 + \|h\|^2$ for all $h \in \operatorname{dom} T$, by assumption. Hence

$$\operatorname{ran}\begin{pmatrix} I & S+T \\ -(S+T) & I \end{pmatrix} = \mathscr{H} \times \mathscr{H}.$$

Since $S + T$ is symmetric and densely defined, Corollary 2.8 applies. ∎

An immediate consequence of the previous result is the following

COROLLARY 3.8. *Let S, T be linear operators acting on a Hilbert space $\mathscr{H}$. Assume that*

(a) S is essentially selfadjoint with $\operatorname{dom} S \subseteq \operatorname{dom} T$;
(b) $\|Th\|^2 \leq \|Sh\|^2 + \|h\|^2$ for all $h \in \operatorname{dom} S$;
(c) $(Th \mid k) = (h \mid Tk)$ for all $h, k \in \operatorname{dom} S$;
(d) $S + T$ is closed.

Then $S + T$ is selfadjoint.

We close our paper with a generalized version of the classical Kato–Rellich [10] perturbation theorem; cf. also [14, Theorem 5.28] and [11]:

COROLLARY 3.9. *Let S,T be linear operators acting on a Hilbert space $\mathscr{H}$. Assume that*

(a) S is selfadjoint with $\operatorname{dom} S \subseteq \operatorname{dom} T$;
(b) $\|Th\|^2 \leq q\|Sh\|^2 + \|h\|^2$ for all $h \in \operatorname{dom} S$ with some $0 \leq q < 1$;
(c) $(Th \mid k) = (h \mid Tk)$ for all $h,k \in \operatorname{dom} S$.

Then $S+T$ is selfadjoint.

PROOF. Apply Corollary 3.8 and Remark 3.4. ∎

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OPERATORS HAVING SELFADJOINT SQUARES

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Abstract. The main goal of this paper is to show that a (not necessarily densely defined or closed) symmetric operator A acting on a real or complex Hilbert space is selfadjoint exactly when $I + A^2$ is a full-range operator.

1. Introduction

If T is a densely defined closed operator between two Hilbert spaces, $\mathfrak{H}$ and $\mathfrak{K}$, a classical theorem due to John von Neumann [3] states that $I + T^*T$ is a selfadjoint operator with full range. As an immediate consequence of that result one obtains also that the square of a selfadjoint operator, say A, is selfadjoint as well, furthermore, that $I + A^2$ is surjective. If the underlying Hilbert space $\mathfrak{H}$ is complex, by employing the classical theory of deficiency indices, also due to von Neumann [2], we conclude that the converse of the latter statement is also true. More precisely, if A is a densely defined symmetric operator in a complex Hilbert space $\mathfrak{H}$ such that $I + A^2$ is surjective, then the original operator A must be selfadjoint. Indeed, according to the factorizations

$$A^2 + I = (A + iI)(A - iI) = (A - iI)(A + iI), \tag{1.1}$$

it is seen readily that both $A \pm iI$ must be onto, and therefore that A is selfadjoint.

Factorization (1.1) cannot be used, of course, when the underlying Hilbert space $\mathfrak{H}$ is real. Furthermore, if the symmetric operator is not densely defined then the theory of deficiency indices is unapplicable, even if $\mathfrak{H}$ is complex.

The main purpose of this note is to prove that the following characterization of selfadjointness holds, be the underlying Hilbert space real or complex: a symmetric operator A is selfadjoint if and only if $I + A^2$ is surjective. We emphasize that the symmetric operator under consideration is not assumed to be densely defined a priori. On the contrary, densely definedness is also a direct consequence of our other assumptions.

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2. Operators having selfadjoint squares

Recall that an operator A defined in a Hilbert space $\mathfrak{H}$ is said to be symmetric if

$$(Ax \mid y) = (x \mid Ay), \qquad x, y \in \operatorname{dom} A,$$

and skew-symmetric if

$$(Ax \mid y) = -(x \mid Ay), \qquad x, y \in \operatorname{dom} A.$$

If A is densely defined in addition then the symmetry (resp., skew-symmetry) of A means that $A \subseteq A^*$ (resp., $A \subseteq -A^*$). Furthermore, a densely defined operator A is said to be selfadjoint (resp., skew-adjoint) if $A = A^*$ (resp., $A = -A^*$). Note also immediately that each selfadjoint (resp., skew-adjoint) operator is closed.

Our first result is a characterization of the skew-adjointness of an operator in terms of its square:

THEOREM 2.1. *Let $\mathfrak{H}$ be real or complex Hilbert space and $A\colon \mathfrak{H} \to \mathfrak{H}$ a skew-symmetric linear operator, whose domain $\operatorname{dom} A$ is not assumed to be dense. The following statements are equivalent:*

(i) A is densely defined and skew-adjoint;
(ii) $-A^2$ is a (positive) selfadjoint operator;
(iii) $I - A^2$ is a full range operator, i.e. $\operatorname{ran}(I - A^2) = \mathfrak{H}$.

PROOF. If A is skew-adjoint, then clearly, A is closed, and the following identity

$$-A^2 = A^*A$$

shows statement (ii), thanks to von Neumann's classical theorem. By assuming (ii), the operator $I - A^2$ is positive and selfadjoint, and bounded below (by one), therefore its range is dense, and closed in $\mathfrak{H}$, that is $\operatorname{ran}(I - A^2) = \mathfrak{H}$. Assume finally that the symmetric operator $I - A^2$ is of full range. Then it is densely defined and positive selfadjoint, as we see at once. First, $\operatorname{dom}(I - A^2)$ is dense, for if y is from $\{\operatorname{dom}(I - A^2)\}^\perp = \{\operatorname{dom} A^2\}^\perp$, then one takes into account that $y = (I - A^2)z$ for some $z \in \operatorname{dom} A^2$. We have at once for each x from $\operatorname{dom}(I - A^2)$ that

$$0 = \big(x \mid (I - A^2)\,z\big) = \big((I - A^2)\,x \mid z\big).$$

Therefore, z belongs to $\{\operatorname{ran}(I - A^2)\}^\perp = \{0\}$ which implies $y = 0$, as claimed.

An other immediate consequence is that A is densely defined skew-symmetric operator, thus it fulfills the following identity:

$$A \subseteq -A^*.$$

To prove statement (i) one only has to check that $\operatorname{dom} A^* \subseteq \operatorname{dom} A$. To see this, let $y \in \operatorname{dom} A^*$ and take $z \in \operatorname{dom} A^2$ by assumption such that

$$y - A^*y = \big(I - A^2\big)\,z = (I + A)(I - A)z.$$

Since $I+A \subset I-A^*$, we infer that

$$(y-(I-A)z) \in \ker(I-A^*) = \ker(I-A)^* = \{\operatorname{ran}(I-A)\}^{\perp} \subseteq \\ \subseteq \left\{\operatorname{ran}(I-A^2)\right\}^{\perp} = \{0\}.$$

This means just that $y = (I-A)z \in \operatorname{dom} A$, as it is claimed. ∎

The main result of our paper is the following statement:

THEOREM 2.2. *Let $\mathfrak{H}$ be real or complex Hilbert space and $A\colon \mathfrak{H} \to \mathfrak{H}$ a symmetric operator whose domain is not assumed to be dense subspace in $\mathfrak{H}$. The following assertions are equivalent:*

(i) A is densely defined and selfadjoint operator;
(ii) A^2 is a positive selfadjoint operator;
(iii) $I+A^2$ is a full range operator, i.e. $\operatorname{ran}(I+A^2) = \mathfrak{H}$.

PROOF. If A is assumed to be selfadjoint, then $A^2 = A^*A$ is positive selfadjoint operator in virtue of von Neumann's classical theorem. Therefore, (i) implies (ii). Statement (ii) also clearly implies (iii) as $(I+A^2)$ is positive selfadjoint and bounded below (by one) operator whose range is dense and closed as well, therefore is the whole space $\mathfrak{H}$. It remains to prove implication (i)⇒(ii). First of all, A^2 is densely defined: for if y is from $\left\{\operatorname{dom} A^2\right\}^{\perp}$, then $y = (I+A^2)z$ for some $z \in \operatorname{dom} A^2$, thus for each x from $\operatorname{dom} A^2$

$$0 = (x \mid y) = \left(x \,\middle|\, \left(I+A^2\right)z\right) = \left(\left(I+A^2\right)x \,\middle|\, z\right)$$

holds true. This means, of course, that z is from $\left\{\operatorname{ran}(I+A^2)\right\}^{\perp} = \{0\}$, and therefore that $y = 0$, indeed. Another consequence is that $\operatorname{dom} A$ is dense as well, and thus

$$A \subseteq A^*,$$

by our assumption on the symmetricity of A.

The last step is to show that $\operatorname{dom} A^* \subseteq \operatorname{dom} A$. Take $z \in \operatorname{dom} A^*$, then for some x and y from $\operatorname{dom} A^2$ we have that

$$A^*z = \left(I+A^2\right)x \qquad \text{and} \qquad -z = \left(I+A^2\right)y.$$

This means simultaneously that

$$\begin{cases} -z = A(x+Ay) - (Ax-y), \\ A^*z = A(Ax-y) + (x+Ay). \end{cases}$$

Consequently, as $Ax-y \in \operatorname{dom} A \subseteq \operatorname{dom} A^*$, we see that $(z-(Ax-y)) \in \operatorname{dom} A^*$ and

$$A^*\left(z-(Ax-y)\right) = A^*z - A^*(Ax-y) = A^*z - A(Ax-y) = x + Ay.$$

Observe also that

$$0 = -A^*z + A^*z = A^*A(Ax+y) + (x+Ay).$$

As a consequence we finally have that

$$\begin{aligned}0 &= (A^*A(Ax+y) \mid Ax+y) + (x+Ay \mid x+Ay)\\ &= \|A(Ax+y)\|^2 + \|x+Ay\|^2 .\end{aligned}$$

Therefore $x+Ay=0$, so that $z = Ax - y \in \operatorname{dom} A$, indeed. The proof is complete. ∎

Another characterization of selfadjoint and skew-adjoint operators involving the ranges of $I \pm A^2$ is given in the next corollary:

COROLLARY 2.3. *Let A be a densely defined symmetric (resp., skew-symmetric) operator in the real or complex Hilbert space $\mathfrak{H}$. Then the following are equivalent:*

(i) A is selfadjoint (resp., skew-adjoint);
(ii) $\operatorname{dom} A^* \subseteq \operatorname{ran}(I+A^2)$ and $\operatorname{ran} A^{**} \subseteq \operatorname{ran}(I+A^2)$ (resp., $\operatorname{dom} A^* \subseteq \operatorname{ran}(I-A^2)$ and $\operatorname{ran} A^{**} \subseteq \operatorname{ran}(I-A^2)$);
(iii) $\operatorname{dom} A^{**} \subseteq \operatorname{ran}(I+A^2)$ and $\operatorname{ran} A^* \subseteq \operatorname{ran}(I+A^2)$ (resp., $\operatorname{dom} A^{**} \subseteq \operatorname{ran}(I-A^2)$ and $\operatorname{ran} A^* \subseteq \operatorname{ran}(I-A^2)$).

PROOF. If A is selfadjoint (resp., skew-adjoint), then Theorem 2.2 (resp., Theorem 2.1) implies that $I+A^2$ (resp., $I-A^2$) has full range. Thus (i) implies either of (ii) and (iii). Conversely, for a densely defined closable operator T, acting between Hilbert spaces $\mathfrak{H}$ and $\mathfrak{K}$, we have the following well known identities:

$$\operatorname{dom} T^{**} + \operatorname{ran} T^* = \mathfrak{H}, \qquad \operatorname{dom} T^* + \operatorname{ran} T^{**} = \mathfrak{K}.$$

Hence, each of (ii) and (iii) implies that $\operatorname{ran}(I+A^2) = \mathfrak{H}$ (resp., $\operatorname{ran}(I-A^2) = \mathfrak{H}$). Due to Theorem 2.2 (resp., Theorem 2.1) this means that A is selfadjoint (resp., skew-adjoint). ∎

COROLLARY 2.4. *Let $(X, \mathfrak{M}, \mu)$ be a measure space and f be any real valued measurable function of X. The multiplication operator A by f on the (real or complex) Hilbert space $\mathscr{L}^2(X, \mathfrak{M}, \mu)$ with maximal domain,*

$$\operatorname{dom} A = \left\{ g \in \mathscr{L}^2(X, \mathfrak{M}, \mu) \;\middle|\; f \cdot g \in \mathscr{L}^2(X, \mathfrak{M}, \mu) \right\},$$

is selfadjoint.

PROOF. It is seen readily that A is a symmetric operator. For a given $g \in \mathscr{L}^2(X, \mathfrak{M}, \mu)$, one obtains at once that $h = \dfrac{g}{1+f^2}$ belongs to $\operatorname{dom} A^2$ so that $(I+A^2)h = g$. That means precisely that $I+A^2$ is of full range, and therefore, in account of Theorem 2.2, that A is selfadjoint. ∎

THEOREM 2.5. *Let $\mathfrak{H}$ be a real or complex Hilbert space, and $A\colon \mathfrak{H} \to \mathfrak{H}$ be a (not necessarily densely defined) positive symmetric operator. The following statements are equivalent:*

(i) A is selfadjoint;
(ii) $I+A$ is of full range, i.e. $\operatorname{ran}(I+A) = \mathfrak{H}$.

PROOF. It is clear that (i) implies (ii): $I+A$ is a closed operator which is bounded below (by one), therefore its range is simultaneously dense and closed, i.e. $\operatorname{ran}(I+A) = \mathfrak{H}$. Conversely, assume $I+A$ to be a full range operator. We start by observing that A is densely defined: for if $y \in \{\operatorname{dom} A\}^{\perp}$ then $y = (I+A)z$ for some $z \in \operatorname{dom} A$, hence

$$0 = (x \mid y) = \big(x \mid (I+A)z\big) = \big((I+A)x \mid z\big),$$

for each $x \in \operatorname{dom} A$. Therefore, $z \in \{\operatorname{ran}(I+A)\}^{\perp} = \{0\}$, i.e. $z = 0$ and then $y = 0$ as claimed.

Next, we have that

$$(I+A) \subset (I+A)^* = (I+A^*),$$

therefore $A^* = A$ precisely when $(I+A)^* = I+A$. If $y \in \operatorname{dom} A^*$ then we see that for some $z \in \operatorname{dom} A$

$$y + A^*y = (I+A)z = (I+A^*)z,$$

thus

$$(y-z) \in \ker(I+A^*) = \{\operatorname{ran}(I+A)\}^{\perp} = \{0\}.$$

Consequently, $y = z \in \operatorname{dom} A$, as it is claimed. ∎

COROLLARY 2.6. *Let $\mathfrak{H}$ and $\mathfrak{K}$ be real or complex Hilbert spaces, $T\colon \mathfrak{H} \to \mathfrak{K}$ be densely defined linear operator. Then T^*T is a positive selfadjoint if and only if $\operatorname{ran}(I+T^*T) = \mathfrak{H}$. If T is closed, then T^*T is positive selfadjoint operator on $\mathfrak{H}$.*

PROOF. We should only check that if T is closed then $\operatorname{ran}(I+T^*T) = \mathfrak{H}$. Of course, this is the case when the two closed subspaces are orthogonal complements on $\mathfrak{H} \times \mathfrak{K}$:

$$\big\{(x, Tx) \bigm| x \in \operatorname{dom} T\big\} \qquad \text{and} \qquad \big\{(-T^*z, z) \bigm| z \in \operatorname{dom} T^*\big\}.$$

Therefore, for each $y \in \mathfrak{H}$ we find $x \in \operatorname{dom} T$ and $z \in \operatorname{dom} T^*$ such that

$$y = x - T^*z \qquad \text{and} \qquad 0 = Tx + z.$$

Consequently, $-z = Tx \in \operatorname{dom} T^*$ and $-T^*z = T^*Tx$ so that

$$y = x + T^*Tx \in \operatorname{ran}(I+T^*T),$$

as desired. ∎

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LEBESGUE DECOMPOSITION VIA RIESZ ORTHOGONAL DECOMPOSITION

By
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Dedicated to Professor Zoltán Sebestyén on the occasion of his 70th birthday

Abstract. The aim of this short note is to give a simple proof of the classical Lebesgue decomposition theorem of measures via the Riesz orthogonal decomposition theorem of Hilbert spaces.

1. Introduction

J. von Neumann in [4] gave a very elegant proof of the classical Radon–Nikodym differentiation theorem, namely, he proved that the Radon–Nikodym theorem follows (relatively easily) from the Riesz representation theorem for bounded linear functionals. Our purpose in this paper is to show how the Lebesgue decomposition theorem derives from Riesz' orthogonal decomposition theorem. More precisely, if μ and ν are finite measures on a fixed measurable space $(T, \mathscr{R})$ then the μ-absolutely continuous and μ-singular parts of ν correspond to an appropriate orthogonal decomposition $\mathfrak{M} \oplus \mathfrak{M}^{\perp}$ of $\mathscr{L}^2(\nu)$.

A very similar approach can be used by discussing several general Lebesgue-type decomposition problems such as decomposing finitely additive set functions [9], positive operators on Hilbert spaces [8], nonnegative Hermitian forms [7], and representable functionals on *-algebras [10]. The main tool in the mentioned papers, just as in this note, is the Riesz orthogonal decomposition theorem.

We must also mention that Neumann's proof simultaneously proves Lebesgue's decomposition, at least after making minimal modifications, see e.g. Rudin [6]. His treatment is undoubtedly more elegant and simpler than ours. Our only claim is nothing but to point out the deep connection between Lebesgue decomposition and orthogonal decomposition. An other elementary functional analytic proof of the Lebesgue decomposition theorem is found in [11].

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Our proof of the Lebesgue decomposition theorem is based on the following easy lemma which states that the so called multivalued part of a closed linear relation is closed itself, cf. [1] or [3]. The proof is just an easy exercise, however, we present it here for the sake of completeness.

LEMMA 1.1. *Let $\mathfrak{H}$ and $\mathfrak{K}$ be (real or complex) Hilbert spaces and let L be a linear subspace of $\mathfrak{H} \times \mathfrak{K}$, that is to say, a linear relation from $\mathfrak{H}$ into $\mathfrak{K}$. Then*

$$\mathfrak{M}(L) := \left\{k \in \mathfrak{K} \mid \exists (h_n, k_n)_{n\in\mathbb{N}} \in L^{\mathbb{N}}, h_n \to 0, k_n \to k\right\} \tag{1.1}$$

is a closed subspace of $\mathfrak{K}$.

PROOF. Note first that for any $k \in \mathfrak{K}$ the assertion $k \in \mathfrak{M}(L)$ is equivalent to $(0,k) \in \overline{L}$, where $\overline{L}$ denotes the closure of L in the Cartesian product $\mathfrak{H} \times \mathfrak{K}$. Now, if $(k_n)_{n\in\mathbb{N}}$ is any sequence from $\mathfrak{M}(L)$ with limit point $k \in \mathfrak{K}$, then clearly, $(0,k_n)_{n\in\mathbb{N}}$ converges in $\overline{L}$, namely to $(0,k)$. Consequently, $(0,k) \in \overline{L}$ according to the closedness of $\overline{L}$ and hence $k \in \mathfrak{M}(L)$. ∎

2. The Lebesgue decomposition theorem

Henceforth, we fix two finite measures μ and ν on a measurable space $(T, \mathscr{R})$, where T is a non-empty set, and $\mathscr{R}$ is a σ-algebra of subsets of T. For $E \in \mathscr{R}$, the characteristic function of E is denoted by χ_E. The vector space of $\mathscr{R}$-simple functions, i.e., the linear span of characteristic functions is denoted by $\mathscr{S}$. We also associate (real) Hilbert spaces $\mathscr{L}^2(\mu)$ and $\mathscr{L}^2(\nu)$ to the measures μ and ν, respectively, which are endowed with the usual inner products, denoted by $(\cdot \mid \cdot)_\mu$ and $(\cdot \mid \cdot)_\nu$, respectively. Note that functions belonging to $\mathscr{L}^2(\mu)$ (resp., to $\mathscr{L}^2(\nu)$) are just μ-almost everywhere (resp., ν-almost everywhere) determined. For any measurable function $f \colon T \to \mathbb{R}$ and $c \in \mathbb{R}$ we define the measurable set $[f \leq c]$ by letting

$$[f \leq c] := \{x \in T \mid f(x) \leq c\}.$$

The sets $[f \geq c]$, $[f = c]$, $[f \neq c]$, etc., are defined similarly.

Recall the notions of absolute continuity and singularity: ν is said to be *absolutely continuous* with respect to μ (shortly, μ-absolutely continuous) if $\mu(E) = 0$ implies $\nu(E) = 0$ for all $E \in \mathscr{R}$; ν is called *singular* with respect to μ (shortly, μ-singular) if there exists $S \in \mathscr{R}$ such that $\mu(S) = 0$ and $\nu(T \setminus S) = 0$. The Lebesgue decomposition theorem states that ν admits a unique decomposition $\nu = \nu_a + \nu_s$, where ν_a is μ-absolutely continuous and ν_s is μ-singular. The uniqueness can be easily proved via a simple measure theoretic argument (see e.g. [2] or [6]). The essential part of the statement is in the existence of such a decomposition.

Let us consider now the following linear subspace of $\mathscr{L}^2(\nu)$:

$$\mathfrak{M} = \left\{f \in \mathscr{L}^2(\nu) \mid \exists (\phi_n)_{n\in\mathbb{N}} \in \mathscr{S}^{\mathbb{N}},\ \phi_n \to 0 \text{ in } \mathscr{L}^2(\mu),\ \phi_n \to f \text{ in } \mathscr{L}^2(\nu)\right\}.$$

Then, by choosing

$$L := \{(\phi, \phi) \mid \phi \in \mathscr{S}\} \subseteq \mathscr{L}^2(\mu) \times \mathscr{L}^2(\nu),$$

Lemma 1.1 says that $\mathfrak{M} = \mathfrak{M}(L)$ is a closed linear subspace of $\mathscr{L}^2(\nu)$. Hence, according to the classical Riesz orthogonal decomposition theorem, we can consider the orthogonal projection P of $\mathscr{L}^2(\nu)$ onto $\mathfrak{M}$. Let us introduce the following two (signed) measures:

$$\nu_a(E) := \int_E (I-P)1\, d\nu, \qquad \nu_s(E) := \int_E P1\, d\nu, \qquad E \in \mathscr{R}. \tag{2.1}$$

Clearly, $\nu = \nu_a + \nu_s$. We state that this is the Lebesgue decomposition of ν with respect to μ:

THEOREM 2.1. *Let μ and ν be finite measures on a measurable space $(T, \mathscr{R})$. Then both ν_a and ν_s from* (2.1) *are (finite) measures such that ν_a is μ-absolutely continuous, and ν_s is μ-singular.*

PROOF. We start by proving the μ-absolutely continuity of ν_a: let $E \in \mathscr{R}$ be any measurable set with $\mu(E) = 0$. Then $\chi_E \in \mathfrak{M}$ (choose $\phi_n := \chi_E$ for all integer n), and therefore

$$\nu_a(E) = \int_E (I-P)1\, d\nu = (\chi_E \mid (I-P)1)_\nu = (P\chi_E \mid (I-P)1)_\nu = 0.$$

This means that the signed measure ν_a is absolutely continuous with respect to μ. In order to prove the μ-singularity of ν_s, consider a sequence $(\phi_n)_{n\in\mathbb{N}}$ from $\mathscr{S}$ with $\phi_n \to 0$ in $\mathscr{L}^2(\mu)$, and $\phi_n \to P1$ in $\mathscr{L}^2(\nu)$. By turning to an appropriate subsequence along the classical Riesz argument [5], we may also assume that $\phi_n \to 0$ μ-a.e. This means that $P1 = 0$ μ-a.e., and therefore that ν_s and μ are singular with respect to each other:

$$\mu([P1 \neq 0]) = 0 \qquad \text{and} \qquad \nu_s([P1 = 0]) = \int_{[P1=0]} P1\, d\nu = 0. \tag{2.2}$$

It remains only to show that ν_a and ν_s are *positive* measures, i.e., $0 \le P1 \le 1$ ν-a.e. Indeed, the left side of (2.2) yields $\mu([P1 < 0]) = 0$ and hence $\chi_{[P1<0]} \in \mathfrak{M}$. That gives

$$0 \ge \int_{[P1<0]} P1\, d\nu = (P1 \mid \chi_{[P1<0]})_\nu = (1 \mid \chi_{[P1<0]})_\nu = \nu([P1 < 0]) \ge 0,$$

which means that $P1 \ge 0$ ν-a.e. A very same argument shows that $P1 \le 1$ ν-a.e. ∎

REMARK 2.2. Observe that $P1$ in fact is the characteristic function of the set $[P1 \neq 0]$: since $0 \le P1 \le 1$, it follows that $0 \le P1 - (P1)^2$, furthermore we have

$$\int_T P1 - (P1)^2\, d\nu = \int_T P1 \cdot (I-P)1\, d\nu = (P1 \mid (I-P)1)_\nu = 0,$$

which yields $P1 = \chi_{[P1 \neq 0]}$, indeed. Consequently, by letting $S := [P1 \neq 0]$ the standard form of the Lebesgue decomposition is obtained as follows:

$$\nu_a(E) = \nu(E \setminus S), \quad \nu_s(E) = \nu(E \cap S), \qquad E \in \mathscr{R},$$

c.f. [2, 32 §, Theorem C].

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POSITIVE DEFINITE OPERATOR FUNCTIONS AND SESQUILINEAR FORMS

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Dedicated to Zoltán Sebestyén on the occasion of his 70*th birthday*

Abstract. Due to the fundamental works of T. Ando, W. Szymański, F. H. Szafraniec, and many others it is well known that sesquilinear forms play an important role in dilation theory. The crucial fact is that every positive definite operator function induces a sesquilinear form in a natural way. The purpose of this survey-like paper is to apply some recent results of Z. Sebestyén, Zs. Tarcsay, and the author for such functions. While most of the results are not new, the paper's main contribution is the unified discussion from the viewpoint of sesquilinear forms.

1. Sesquilinear forms

In this preliminary section we review first some of the standard notions and notations and give a brief survey of some recent results needed throughout the paper. We focus on the decomposition and Radon–Nikodym theory of nonnnegative sesquilinear forms that we will apply on positive definite operator functions in Section 2. Our main references are [7, Section 2] and [11].

1.1. Notions, notations

Let $\mathfrak{X}$ be a complex linear space and let $\mathfrak{t}$ be a nonnegative sesquilinear form (or shortly just *form*) on it. That is, $\mathfrak{t}$ is a mapping from $\mathfrak{X} \times \mathfrak{X}$ to $\mathbb{C}$, which is linear in the first argument, antilinear in the second argument, and the corresponding quadratic form

$$\forall x \in \mathfrak{X}: \quad \mathfrak{t}[x] := \mathfrak{t}(x,x)$$

is nonnegative. A crucial fact is that a form is uniquely determined by its quadratic form due to the polarization formula

$$\forall x,y \in \mathfrak{X}: \quad \mathfrak{t}(x,y) = \frac{1}{4}\sum_{k=0}^{3} i^k \mathfrak{t}\left[x + i^k y\right].$$

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The set of forms will be denoted by $\mathcal{F}_+(\mathfrak{X})$. For $\mathfrak{t},\mathfrak{w} \in \mathcal{F}_+(\mathfrak{X})$ we write $\mathfrak{t} \le \mathfrak{w}$ if $\mathfrak{t}[x] \le \mathfrak{w}[x]$ for all $x \in \mathfrak{X}$. *Domination* means that there exists a constant c such that $\mathfrak{t} \le c \cdot \mathfrak{w}$. Using the ordering we can define singularity and almost domination. The forms $\mathfrak{t}$ and $\mathfrak{w}$ are *singular* ($\mathfrak{t} \perp \mathfrak{w}$) if for every form $\mathfrak{s}$ the inequalities $\mathfrak{s} \le \mathfrak{t}$ and $\mathfrak{s} \le \mathfrak{w}$ imply that $\mathfrak{s} = \mathfrak{o}$ (i.e., $\mathfrak{s}$ is the identically zero form). We say that $\mathfrak{t}$ is *almost dominated by* $\mathfrak{w}$ (in symbols: $\mathfrak{t} \ll_{\mathrm{ad}} \mathfrak{w}$) if there exists a monotonically nondecreasing sequence of forms $\mathfrak{t}_n$, each dominated by $\mathfrak{w}$, such that $\mathfrak{t} = \sup_{n\in\mathbb{N}} \mathfrak{t}_n$ (pointwise supremum).

Now, we define two important notions, which are motivated by classical measure theory. Let $\mathfrak{t}$ and $\mathfrak{w}$ be forms, $\mathfrak{t}$ is called *absolutely continuous with respect to* $\mathfrak{w}$ (or $\mathfrak{t}$ is $\mathfrak{w}$-absolutely continuous, in symbols: $\mathfrak{t} \ll_{\mathrm{ac}} \mathfrak{w}$), if $\mathfrak{w}[x] = 0$ implies $\mathfrak{t}[x] = 0$ for all $x \in \mathfrak{X}$. We say that $\mathfrak{t}$ is *strongly* $\mathfrak{w}$*-absolutely continuous* ($\mathfrak{t} \ll_{\mathrm{s}} \mathfrak{w}$, in symbols), if

$$\forall (x_n)_{n\in\mathbb{N}} \in \mathfrak{X}^{\mathbb{N}}: \qquad ((\mathfrak{t}[x_n - x_m] \to 0) \ \wedge \ (\mathfrak{w}[x_n] \to 0)) \implies \mathfrak{t}[x_n] \to 0.$$

Remark that this notion is called *closability* in [7]; cf. also [16] and [9]. The following theorem says that strong absolute continuity is closely related to the ordering. For the proof see [7, Theorem 3.8].

THEOREM 1.1. *Let* $\mathfrak{t}$ *and* $\mathfrak{w}$ *be forms on* $\mathfrak{X}$. *Then* $\mathfrak{t}$ *is almost dominated by* $\mathfrak{w}$ *if and only if* $\mathfrak{t}$ *is strongly* $\mathfrak{w}$*-absolutely continuous.*

It is important to mention that if $\mathfrak{t} \in \mathcal{F}_+(\mathfrak{X})$ then the square root of its quadratic form defines a seminorm on $\mathfrak{X}$. Hence the set

$$\ker \mathfrak{t} := \left\{ x \in \mathfrak{X} \;\middle|\; \mathfrak{t}[x] = 0 \right\}$$

is a linear subspace of $\mathfrak{X}$. The Hilbert space $\mathcal{H}_{\mathfrak{t}}$ denotes the completion of the inner product space $\mathfrak{X}/_{\ker \mathfrak{t}}$ equipped with the natural inner product

$$\forall x,y \in \mathfrak{X}: \quad (x + \ker \mathfrak{t} \mid y + \ker \mathfrak{t})_{\mathfrak{t}} := \mathfrak{t}(x,y).$$

Observe that $\mathfrak{t}$ is $\mathfrak{w}$-absolutely continuous if and only if the canonical embedding (which assigns the coset $x+\ker \mathfrak{t}$ to $x+\ker \mathfrak{w}$) from $\mathcal{H}_{\mathfrak{w}}$ to $\mathcal{H}_{\mathfrak{t}}$ is well-defined. Strong absolute continuity means that this embedding is a closable operator.

We close this subsection with a Radon–Nikodym-type result. This was proved independently by Zs. Tarcsay, from a different point of view. For more background we refer the reader to [19].

LEMMA 1.2. *Let* $\mathfrak{t}$ *and* $\mathfrak{w}$ *be forms on* $\mathfrak{X}$ *and assume that* $\mathfrak{t} \le c \cdot \mathfrak{w}$ *for some* $c > 0$. *Then for every* $y \in \mathfrak{X}$ *there exists a unique vector* ξ_y *in* $\mathcal{H}_{\mathfrak{w}}$ *such that*

$$\forall x \in \mathfrak{X}: \qquad \mathfrak{t}(x,y) = (x + \ker \mathfrak{w} \mid \xi_y)_{\mathfrak{w}}\,.$$

PROOF. Let y be an arbitrary but fixed element of $\mathfrak{X}$ and define the linear functional Φ_y as follows

$$\Phi_y\colon \mathfrak{X}/_{\ker \mathfrak{w}} \to \mathbb{C}; \qquad x + \ker \mathfrak{w} \mapsto (x + \ker \mathfrak{t} \mid y + \ker \mathfrak{t})_{\mathfrak{t}}.$$

According to the Cauchy-Schwarz inequality and the assumption it is clear that Φ_y is a bounded linear functional. Indeed,

$$|\Phi_y(x+\ker \mathfrak{w})|^2 \le \|x+\ker \mathfrak{t}\|_{\mathfrak{t}}^2 \cdot \|y+\ker \mathfrak{t}\|_{\mathfrak{t}}^2 \le c^2 \cdot \|x+\ker \mathfrak{w}\|_{\mathfrak{w}}^2 \cdot \|y+\ker \mathfrak{w}\|_{\mathfrak{w}}^2.$$

Consequently, due to the Riesz representation theorem there exists a unique vector ξ_y in $\mathcal{H}_{\mathfrak{w}}$ such that

$$\forall x \in \mathfrak{X}: \qquad \mathfrak{t}(x,y) = (x+\ker \mathfrak{t} \mid y+\ker \mathfrak{t})_{\mathfrak{t}} = \Phi_y(x+\ker \mathfrak{t}) = (x+\ker \mathfrak{w} \mid \xi_y)_{\mathfrak{w}}.$$

∎

THEOREM 1.3. *Let $\mathfrak{t},\mathfrak{w} \in \mathcal{F}_+(\mathfrak{X})$ be forms on $\mathfrak{X}$ and let $\mathfrak{t}$ be almost dominated by $\mathfrak{w}$. Then for every $y \in \mathfrak{X}$ there exists a sequence $(y_n)_{n\in\mathbb{N}} \in \mathfrak{X}^{\mathbb{N}}$ such that*

$$\forall x \in \mathfrak{X}: \qquad \mathfrak{t}(x,y) = \lim_{n\to+\infty} \mathfrak{w}(x,y_n).$$

PROOF. Fix an arbitrary $y \in \mathfrak{X}$. Since $\mathfrak{t}$ is almost dominated by $\mathfrak{w}$, there exists a suitable sequence $(\mathfrak{t}_n)_{n\in\mathbb{N}}$ of $\mathfrak{w}$-dominated forms and a sequence $(\xi_{y,n})_{n\in\mathbb{N}}$ of representant vectors such that

$$\lim_{n\to+\infty} \mathfrak{t}_n = \mathfrak{t} \qquad \text{and} \qquad (\forall x \in \mathfrak{X})\ (\forall n \in \mathbb{N}): \quad \mathfrak{t}_n(x,y) = (x+\ker \mathfrak{w} \mid \xi_{y,n})_{\mathfrak{w}}.$$

As $\mathfrak{t}_n \le \mathfrak{t}$, we can apply the Cauchy–Schwarz inequality on the form $\mathfrak{t}-\mathfrak{t}_n$ that gives

$$|(\mathfrak{t}-\mathfrak{t}_n)(x,y)|^2 \le (\mathfrak{t}-\mathfrak{t}_n)[x]\,(\mathfrak{t}-\mathfrak{t}_n)[y] \to 0, \qquad n \to +\infty,$$

whence we infer that

$$\mathfrak{t}(x,y) = \lim_{n\to+\infty} t_n(x,y) = \lim_{n\to+\infty} (x+\ker \mathfrak{w} \mid \xi_{y,n})_{\mathfrak{w}}.$$

Since $\mathfrak{X}/_{\ker \mathfrak{w}}$ is dense in $\mathcal{H}_{\mathfrak{w}}$ we can choose a sequence $(y_n)_{n\in\mathbb{N}} \in \mathfrak{X}^{\mathbb{N}}$ such that

$$\|\xi_{y,n} - (y_n + \ker \mathfrak{w})\|_{\mathfrak{w}} \to 0.$$

According to the Cauchy–Schwarz inequality, this implies that

$$|(x+\ker \mathfrak{w} \mid \xi_{y,n})_{\mathfrak{w}} - (x+\ker \mathfrak{w} \mid y_n + \ker \mathfrak{w})_{\mathfrak{w}}| \to 0$$

and thus

$$\forall x \in \mathfrak{X}: \qquad \mathfrak{t}(x,y) = \lim_{n\to+\infty} \mathfrak{w}(x,y_n).$$

∎

1.2. Decomposition theorems

In this subsection we recall two basic results of decomposition theory of forms. The first one is the so-called short-type decomposition, which is a decomposition of $\mathfrak{t}$ into absolutely continuous and singular parts. The key notion is the *short* of a form to a linear subspace of $\mathfrak{X}$ (for the details see [12]), which is a generalization of the well known concept of operator short [1, 6, 10].

Let $\mathfrak{t}$ and $\mathfrak{w}$ be forms on $\mathfrak{X}$, then the short of $\mathfrak{t}$ to the subspace $\ker \mathfrak{w}$ is defined by

$$\forall x \in \mathfrak{X}: \quad \mathfrak{t}_{\ker \mathfrak{w}}[x] := \inf_{y\in\ker \mathfrak{w}} \mathfrak{t}[x-y].$$

The short-type decomposition theorem is stated as follows ([12, Theorem 1.2]).

THEOREM 1.4. *Let $\mathfrak{t},\mathfrak{w} \in \mathcal{F}_+(\mathfrak{X})$ be forms on $\mathfrak{X}$. Then there exists a short-type decomposition of $\mathfrak{t}$ with respect to $\mathfrak{w}$. Namely,*

$$\mathfrak{t} = \mathfrak{t}_{\ker \mathfrak{w}} + (\mathfrak{t} - \mathfrak{t}_{\ker \mathfrak{w}}),$$

where the first summand is $\mathfrak{w}$*-absolutely continuous and the second one is* $\mathfrak{w}$*-singular. Furthermore,* $\mathfrak{t}_{\ker \mathfrak{w}}$ *is the largest element of the set*

$$\left\{\mathfrak{s} \in \mathcal{F}_+(\mathfrak{X}) \,\middle|\, (\mathfrak{s} \leq \mathfrak{t}) \ \wedge \ (\mathfrak{s} \ll_{\mathrm{ac}} \mathfrak{w})\right\}.$$

The decomposition is unique precisely when $\mathfrak{t}_{\ker \mathfrak{w}}$ *is dominated by* $\mathfrak{w}$*.*

A decomposition of $\mathfrak{t}$ into strongly $\mathfrak{w}$-absolutely continuous (or $\mathfrak{w}$-almost dominated) and $\mathfrak{w}$-singular parts is called Lebesgue-type decomposition. This is a generalization of the well-known decomposition result of T. Ando [2] (see also [20]). The existence of such a decomposition for forms was proved first by Hassi, Sebestyén, and de Snoo in [7]. In order to present their result we need to introduce the notion of parallel sum. The parallel sum $\mathfrak{t} : \mathfrak{w}$ of the forms $\mathfrak{t}$ and $\mathfrak{w}$ is determined by the formula

$$\forall x \in \mathfrak{X}: \quad (\mathfrak{t} : \mathfrak{w})[x] := \inf_{y \in \mathfrak{X}} \{\mathfrak{t}[x-y] + \mathfrak{w}[y]\}.$$

We will see that the form

$$\mathbf{D}_{\mathfrak{w}}\mathfrak{t} := \sup_{n \in \mathbb{N}} (\mathfrak{t} : n\mathfrak{w})$$

plays an important role in this paper. (For the properties of parallel addition and the operator $\mathbf{D}$ see [7, Proposition 2.3, Lemma 2.4].)

Primarily, $\mathbf{D}_{\mathfrak{w}}\mathfrak{t}$ is the so-called almost dominated part of $\mathfrak{t}$ with respect to $\mathfrak{w}$, as the following fundamental theorem states [7, Theorem 2.11] (see also [11, Theorem 2.3] and [14, Theorem 3]).

THEOREM 1.5. *Let* $\mathfrak{t}$ *and* $\mathfrak{w}$ *be forms on* $\mathfrak{X}$*. Then the decomposition*

$$\mathfrak{t} = \mathbf{D}_{\mathfrak{w}}\mathfrak{t} + (\mathfrak{t} - \mathbf{D}_{\mathfrak{w}}\mathfrak{t})$$

is a Lebesgue-type decomposition of $\mathfrak{t}$ *with respect to* $\mathfrak{w}$*. That is,* $\mathbf{D}_{\mathfrak{w}}\mathfrak{t}$ *is almost dominated by* $\mathfrak{w}$*,* $(\mathfrak{t} - \mathbf{D}_{\mathfrak{w}}\mathfrak{t})$ *is* $\mathfrak{w}$*-singular. Furthermore,* $\mathbf{D}_{\mathfrak{w}}\mathfrak{t}$ *is the largest element of the set*

$$\left\{\mathfrak{s} \in \mathcal{F}_+(\mathfrak{X}) \,\middle|\, (\mathfrak{s} \leq \mathfrak{t}) \ \wedge \ (\mathfrak{s} \ll_{\mathrm{ad}} \mathfrak{w})\right\}.$$

The decomposition is unique precisely when $\mathbf{D}_{\mathfrak{w}}\mathfrak{t}$ *is dominated by* $\mathfrak{w}$*.*

Moreover, for the almost dominated part we have the following two formulae (for the proofs see [11, Lemma 2.2, Theorem 2.3] and [11, Theorem 2.7]):

$$(\mathbf{D}_{\mathfrak{w}}\mathfrak{t})[x] = \inf\left\{\lim_{n\to+\infty} \mathfrak{t}[x - x_n] \,\middle|\, (x_n)_{n\in\mathbb{N}} \in \mathfrak{X}^{\mathbb{N}} : \ (\mathfrak{t}[x_n - x_m] \to 0) \ \wedge \ (\mathfrak{w}[x_n] \to 0)\right\}$$

and

$$(\mathbf{D}_{\mathfrak{w}}\mathfrak{t})[x] = \inf\left\{\liminf_{n\to+\infty} \mathfrak{t}[x - x_n] \,\middle|\, (x_n)_{n\in\mathbb{N}} \in \mathfrak{X}^{\mathbb{N}} : \mathfrak{w}[x_n] \to 0\right\}.$$

The first interesting observation is [23, Theorem 1.5], which implies for example for every finite measures μ and ν that the ν-absolutely continuous part of μ is absolutely continuous with respect to the μ-absolutely continuous part of ν [23, Theorem 3.5 (b)]. An analogous result regarding representable functionals can be found in [21]. We present here a proof which is simpler than the original one in [23].

THEOREM 1.6. *Let* $\mathfrak{t}$ *and* $\mathfrak{w}$ *be forms on* $\mathfrak{X}$, *and consider their Lebesgue-type decompositions with respect to each other. Then the almost dominated parts are mutually almost dominated, i.e.,*

$$\mathbf{D}_{\mathfrak{w}}\mathfrak{t} \ll_{\mathrm{ad}} \mathbf{D}_{\mathfrak{t}}\mathfrak{w} \qquad \text{and} \qquad \mathbf{D}_{\mathfrak{t}}\mathfrak{w} \ll_{\mathrm{ad}} \mathbf{D}_{\mathfrak{w}}\mathfrak{t}.$$

PROOF. Observe first that if $\mathfrak{u}_1$, $\mathfrak{u}_2$, and $\mathfrak{v}$ are forms such that $\mathfrak{u}_1 \leq \mathfrak{u}_2$ and $\mathfrak{u}_1 \ll_{\mathrm{ad}} \mathfrak{v}$, then

$$\mathfrak{u}_1 \ll_{\mathrm{ad}} \mathbf{D}_{\mathfrak{u}_2}\mathfrak{v}.$$

Indeed, if $\mathfrak{u}_1$ is almost dominated by $\mathfrak{v}$, then there exists a monotonically nondecreasing sequence of forms $(\mathfrak{u}_{1,n})_{n\in\mathbb{N}}$ such that $\sup\limits_{n\in\mathbb{N}} \mathfrak{u}_{1,n} = \mathfrak{u}$ and $\mathfrak{u}_{1,n}$ is dominated by $\mathfrak{v}$ for all $n \in \mathbb{N}$ (i.e., $\mathfrak{u}_{1,n} \leq c_n\mathfrak{v}$ for some $c_n \geq 0$). Consequently,

$$\mathfrak{u}_{1,n} = \mathbf{D}_{\mathfrak{u}_2}\mathfrak{u}_{1,n} \leq \mathbf{D}_{\mathfrak{u}_2}c_n\mathfrak{v} = c_n\mathbf{D}_{\mathfrak{u}_2}\mathfrak{v}$$

which means that $\mathfrak{u}_1 \ll_{\mathrm{ad}} \mathbf{D}_{\mathfrak{u}_2}\mathfrak{v}$. Now, apply the previous observation with $\mathfrak{u}_1 := \mathbf{D}_{\mathfrak{w}}\mathfrak{t}$, $\mathfrak{u}_2 = \mathfrak{t}$, and $\mathfrak{v} = \mathfrak{w}$. ∎

1.3. Order structure and some extremal problems

This subsection is devoted to investigating the connection between some order properties of $(\mathcal{F}_+(\mathfrak{X}), \leq)$ and the Lebesgue type decomposition.

The first natural question is whether the infimum (i.e., the greatest lower bound) $\mathfrak{t} \wedge \mathfrak{w}$ of $\mathfrak{t}$ and $\mathfrak{w}$ exists in $\mathcal{F}_+(\mathfrak{X})$. The infimum problem has a long history in the theory of Hilbert space operators. Kadison proved that the set of bounded self-adjoint operators is a so-called anti-lattice [8]. For bounded positive operators the infimum problem was proved by Moreland and Gudder provided the space is finite dimensional [5].

The general case was solved by T. Ando in [3]. He showed that the infimum of two positive operators A and B exists in the positive cone if and only if the generalized shorts (for this notion see [2]) $[B]A$ and $[A]B$ are comparable. An analogous result concerning forms was given in [22].

Recall that the infimum of $\mathfrak{t}$ and $\mathfrak{w}$ exists if there is a form denoted by $\mathfrak{t} \wedge \mathfrak{w}$, for which $\mathfrak{t}\wedge\mathfrak{w} \leq \mathfrak{t}$, $\mathfrak{t}\wedge\mathfrak{w} \leq \mathfrak{w}$, and the inequalities $\mathfrak{u} \leq \mathfrak{t}$ and $\mathfrak{u} \leq \mathfrak{w}$ imply that $\mathfrak{u} \leq \mathfrak{t}\wedge\mathfrak{w}$.

THEOREM 1.7. *Let* $\mathfrak{t}, \mathfrak{w} \in \mathcal{F}_+(\mathfrak{X})$ *be forms on* $\mathfrak{X}$. *Then the following statements are equivalent.*

(i) $\mathbf{D}_{\mathfrak{t}}\mathfrak{w} \leq \mathbf{D}_{\mathfrak{w}}\mathfrak{t}$ *or* $\mathbf{D}_{\mathfrak{w}}\mathfrak{t} \leq \mathbf{D}_{\mathfrak{t}}\mathfrak{w}$.

(ii) $\mathbf{D}_{\mathfrak{t}}\mathfrak{w} \leq \mathfrak{t}$ *or* $\mathbf{D}_{\mathfrak{w}}\mathfrak{t} \leq \mathfrak{w}$.

(iii) The infimum $\mathfrak{t} \wedge \mathfrak{w}$ *exists.*

A nonzero form $\mathfrak{t}$ is called *minimal*, if $\mathfrak{w} \leq \mathfrak{t}$ implies that $\lambda\mathfrak{t} = \mathfrak{w}$ for some $\lambda \geq 0$. Or equivalently (see [15, Theorem 5.10]), for every form $\mathfrak{w}$ there exists a $\lambda \geq 0$ such that $\mathbf{D}_{\mathfrak{t}}\mathfrak{w} = \lambda\mathfrak{t}$.

COROLLARY 1.8. *Let* $\mathfrak{t}$ *be a minimal form on* $\mathfrak{X}$. *Then for every* $\mathfrak{w} \in \mathcal{F}_+(\mathfrak{X})$ *the infimum* $\mathfrak{t} \wedge \mathfrak{w}$ *exists.*

The Lebesgue decomposition theory of forms is encountered again by examining the extremal points of the convex set $[0, \mathfrak{t}]$. Here the segment $[\mathfrak{t}_1, \mathfrak{t}_2]$ for $\mathfrak{t}_1, \mathfrak{t}_2\mathcal{F}_+$ $+(\mathfrak{X})$, $\mathfrak{t}_1 \leq \mathfrak{t}_2$ is defined to be the convex set

$$[\mathfrak{t}_1, \mathfrak{t}_2] = \left\{\mathfrak{s} \in \mathcal{F}_+(\mathfrak{X}) \;\middle|\; \mathfrak{t}_1 \leq \mathfrak{s} \leq \mathfrak{t}_2\right\}.$$

The following theorems characterize the extremal points of form segments; for the proofs and other references see [13, Theorem 11] and [15, Section 5].

THEOREM 1.9. *Let* $\mathfrak{u}$ *and* $\mathfrak{t}$ *be forms on* $\mathfrak{X}$*, such that* $\mathfrak{u} \leq \mathfrak{t}$*. The following statements are equivalent*

(i) $\mathfrak{u}$ *and* $\mathfrak{t} - \mathfrak{u}$ *are singular,*

(ii) $\mathbf{D}_{\mathfrak{u}}\mathfrak{t} = \mathfrak{u}$,

(iii) $\mathfrak{u}$ *is an extreme point of the convex set* $[\mathfrak{o}, \mathfrak{t}]$.

THEOREM 1.10. *Let* $\mathfrak{t}$ *and* $\mathfrak{w}$ *be forms on* $\mathfrak{X}$*. Then the following statements are equivalent*

(i) $\mathfrak{t}$ *is an extreme point of* $[\mathfrak{o}, \mathfrak{t} + \mathfrak{w}]$,

(ii) $\mathrm{ex}[\mathfrak{t}, \mathfrak{t} + \mathfrak{w}] \subseteq \mathrm{ex}[\mathfrak{o}, \mathfrak{t} + \mathfrak{w}]$.

Replacing $\mathfrak{w}$ *with* $\mathfrak{w} - \mathfrak{t}$ *(if* $\mathfrak{t} \leq \mathfrak{w}$*) we have*

$$\mathfrak{t} \in \mathrm{ex}[\mathfrak{o}, \mathfrak{w}] \quad \Leftrightarrow \quad \mathrm{ex}[\mathfrak{t}, \mathfrak{w}] \subseteq \mathrm{ex}[\mathfrak{o}, \mathfrak{w}].$$

2. Positive definite operator functions

In this section we carry over the previous theorems for positive definite operator functions. Szymański in [17] presented a general dilation theory governed by forms. We will see (after making some generalities) that the absolutely continuous part in Theorem 1.4 (and the almost dominated part in Theorem 1.5) is the largest dilatable part in some sense. Finally, we describe some order properties of kernels. Throughout this section we will use the notations of [7, Section 7], which is our main reference. Recall again that almost domination and strong absolute continuity (or closability) are equivalent concepts for forms.

Let S be a non-empty set, and let $\mathfrak{E}$ be a complex Banach space (with topological dual $\mathfrak{E}^*$). The dual pairing of $x \in \mathfrak{E}$ and $x^* \in \mathfrak{E}^*$ is denoted by $\langle x, x^* \rangle$. Here the mapping

$$\langle \cdot, \cdot \rangle \colon \mathfrak{E} \times \mathfrak{E}^* \to \mathbb{C}$$

is linear in its first, conjugate linear in its second variable. The Banach space of bounded linear operators from $\mathfrak{E}$ to $\mathfrak{E}^*$ will be denoted by $\mathbf{B}(\mathfrak{E}, \mathfrak{E}^*)$.

Let $\mathfrak{X}$ be the complex linear space of functions on S with values in $\mathbf{B}(\mathfrak{E}, \mathfrak{E}^*)$ with finite support. We say that the function

$$\mathsf{K} \colon S \times S \to \mathbf{B}(\mathfrak{E}, \mathfrak{E}^*)$$

is a *positive definite operator function*, or shortly a *kernel* on S if

$$\forall f \in \mathfrak{X}\colon \qquad \sum_{s,t \in S} \langle f(t), \mathsf{K}(s,t) f(s) \rangle \geq 0.$$

We associate a form with K by setting

$$\forall f, g \in \mathfrak{X}\colon \qquad \mathfrak{w}_{\mathsf{K}}(f, g) := \sum_{s,t \in S} \langle f(t), \mathsf{K}(s,t) g(s) \rangle.$$

The set of kernels will be denoted by $\mathcal{K}_+(\mathfrak{X})$. If K and L are kernels, we write $\mathsf{K} \prec \mathsf{L}$ if $\mathfrak{w}_\mathsf{K} \leq \mathfrak{w}_\mathsf{L}$.

The following lemma states that the order structures of forms and of kernels are the same. Here we give just an outline, for the complete proof see [7, Lemma 7.1]. An analogous result in context of bounded positive operators can be found in [4, (2.2) Theorem].

LEMMA 2.1. *Let* $\mathsf{K} \in \mathcal{K}_+(\mathfrak{X})$ *be a kernel on* S *with associated form* $\mathfrak{w}_\mathsf{K}$ *and let* $\mathfrak{w}$ *be a form on* $\mathfrak{X}$. *Then the following statements are equivalent*

(i) $\mathfrak{w} \leq \mathfrak{w}_\mathsf{K}$,

(ii) $\mathfrak{w} = \mathfrak{w}_\mathsf{L}$ *for a unique kernel* $\mathsf{L} \prec \mathsf{K}$.

PROOF. Implication $(ii) \Rightarrow (i)$ follows from the definitions. To prove the converse implication define for each $s \in S$ and $x \in \mathfrak{E}$ the function

$$h_{s,x} \in \mathfrak{X}; \qquad \forall u \in S\colon \quad h_{s,x}(u) := \delta_s(u)x$$

where δ_s is the Dirac function concentrated to s. Now, define L pointwise as follows. For each $s,t \in S$

$$\forall x,y \in \mathfrak{E}\colon \qquad \langle x, \mathsf{L}(s,t)y\rangle := \mathfrak{w}(h_{t,x}, h_{s,y})$$

It follows from the nonnegativity of $\mathfrak{w}[\,\cdot\,]$ that

$$\sum_{s,t\in S} \langle f(t), \mathsf{L}(s,t)f(s)\rangle$$

is nonnegative for all $f \in \mathfrak{X}$. The only thing we need is to show that $L(s,t) \in \mathbf{B}(\mathfrak{E},\mathfrak{E}^*)$. According to the Cauchy-Schwarz inequality, we have for all $x,y \in \mathfrak{E}$ that

$$|\langle x, L(s,t)y\rangle|^2 = |\mathfrak{w}(h_{t,x}, h_{s,y})|^2 \leq \mathfrak{w}[h_{t,x}]\cdot\mathfrak{w}[h_{s,y}] \leq \mathfrak{w}_\mathsf{K}[h_{t,x}]\cdot\mathfrak{w}_\mathsf{K}[h_{s,y}] =$$
$$= \langle x, K(t,t)x\rangle \cdot \langle y, K(s,s)y\rangle \leq \|K(t,t)\|_{\mathbf{B}(\mathfrak{E},\mathfrak{E}^*)}\cdot\|K(s,s)\|_{\mathbf{B}(\mathfrak{E},\mathfrak{E}^*)}\cdot\|x\|_\mathfrak{E}^2\cdot\|y\|_\mathfrak{E}^2. \quad \blacksquare$$

We emphasize here that the preceding is the key observation of this section. Most of the results gathered below are immediate consequences of this lemma, and the theorems listed in Section 1.

Now, we can define domination, almost domination, singularity, closability, and (strong) absolute continuity of kernels via their associated forms. We say that K *is* L-*almost dominated;* L-*closable; (strongly)*-L-*absolutely continuous* if $\mathfrak{w}_\mathsf{K}$ is $\mathfrak{w}_\mathsf{L}$-almost dominated; $\mathfrak{w}_\mathsf{L}$-closable; (strongly)-$\mathfrak{w}_\mathsf{L}$-absolutely continuous, respectively. K *and* L *are singular* if $\mathfrak{w}_\mathsf{K}$ and $\mathfrak{w}_\mathsf{L}$ are singular.

Before stating the short-type and Lebesgue-type decomposition of kernels, we mention a result of W. Szymański (reduced to our less general setting). For the details we refer the reader to [17, (3.5) Theorem].

THEOREM 2.2. *Let* $\mathsf{K},\mathsf{L} \in \mathcal{K}_+(\mathfrak{X})$ *be kernels on* S *with associated forms* $\mathfrak{w}_\mathsf{K}$ *and* $\mathfrak{w}_\mathsf{L}$. *Then*

(a) K *is absolutely continuous with respect to* L *(i.e.,* $\ker\mathfrak{w}_\mathsf{L} \subseteq \ker\mathfrak{w}_\mathsf{K}$*) if and only if there exists a Hilbert space* $\mathcal{H}$ *and a linear mapping* $T\colon \mathfrak{X}/_{\ker\mathfrak{w}_\mathsf{L}} \to \mathcal{H}$ *such*

that

$$\langle y, K(s,t)x\rangle = \big(T(h_{t,y} + \ker \mathfrak{w}_{\mathsf{L}}) \,\big|\, T(h_{s,x} + \ker \mathfrak{w}_{\mathsf{L}})\big)_{\mathcal{H}},$$

(b) K *is strongly absolutely continuous with respect to* L *(i.e.,* $\mathfrak{w}_{\mathsf{K}}$ *is strongly* $\mathfrak{w}_{\mathsf{L}}$*-absolutely continuous) if and only if there exists a Hilbert space* $\mathcal{H}$ *and a closed linear mapping* $T\colon \mathfrak{X}/_{\ker \mathfrak{w}_{\mathsf{L}}} \to \mathcal{H}$ *such that*

$$\langle y, K(s,t)x\rangle = \big(T(h_{t,y} + \ker \mathfrak{w}_{\mathsf{L}}) \,\big|\, T(h_{s,x} + \ker \mathfrak{w}_{\mathsf{L}})\big)_{\mathcal{H}}.$$

The operator T is called the *dilation* of K and the auxiliary space $\mathcal{H}$ is called the dilation space.

In view of the previous theorem, the following two decomposition theorems can be stated as follows. For every pair of kernels K and L there is a maximal part of K which has a (closed) dilation with respect to L. These are straightforward consequences of Theorem 1.4 and of Theorem 1.5.

THEOREM 2.3. *Let* $\mathsf{K}, \mathsf{L} \in \mathcal{K}_+(\mathfrak{X})$ *be kernels on S. Then there exists a short-type decomposition of* K *with respect to* L*, i.e., the first summand is* L*-absolutely continuous and the second one is* L*-singular. Namely*

$$\mathsf{K} = \mathsf{K}_{\mathrm{ac},\mathsf{L}} + \mathsf{K}_{\mathrm{s},\mathsf{L}},$$

where

$$\sum_{s,t\in S} \langle f(t), \mathsf{K}_{\mathrm{ac},\mathsf{L}}(s,t) f(s)\rangle = \inf_{g\in\ker \mathfrak{w}_{\mathsf{L}}} \sum_{s,t\in S} \langle f(t) - g(t), \mathsf{K}(s,t)(f(s) - g(s))\rangle.$$

The decomposition is unique precisely when $\mathsf{K}_{\mathrm{ac},\mathsf{L}}$ *is dominated by* L.

THEOREM 2.4. *Let* $\mathsf{K}, \mathsf{L} \in \mathcal{K}_+(\mathfrak{X})$ *be kernels on S. Then the decomposition*

$$\mathsf{K} = \mathbf{D}_{\mathsf{L}}\mathsf{K} + (\mathsf{K} - \mathbf{D}_{\mathsf{L}}\mathsf{K}),$$

is a Lebesgue-type decomposition of K *with respect to* L*. That is,* $\mathbf{D}_{\mathsf{L}}\mathsf{K}$ *is strongly* L-absolutely *continuous,* $(\mathsf{K} - \mathbf{D}_{\mathsf{L}}\mathsf{K})$ *is* L-singular. *The almost dominated part* $\mathbf{D}_{\mathsf{L}}\mathsf{K}$ *is defined by*

$$\mathfrak{w}_{\mathbf{D}_{\mathsf{L}}\mathsf{K}} := \mathbf{D}_{\mathfrak{w}_{\mathsf{L}}}\mathfrak{w}_{\mathsf{K}},$$

and hence

$$\mathfrak{w}_{\mathbf{D}_{\mathsf{L}}\mathsf{K}}[f] =$$
$$= \inf\left\{ \lim_{n\to+\infty} \mathfrak{w}_{\mathsf{K}}[f - g_n] \,\middle|\, (g_n)_{n\in\mathbb{N}} \in \mathfrak{X}^{\mathbb{N}} \colon (\mathfrak{w}_{\mathsf{K}}[g_n - g_m] \to 0) \wedge (\mathfrak{w}_{\mathsf{L}}[g_n] \to 0)\right\}$$

and

$$\mathfrak{w}_{\mathbf{D}_{\mathsf{L}}\mathsf{K}}[f] = \inf\left\{ \liminf_{n\to+\infty} \mathfrak{w}_{\mathsf{K}}[f - g_n] \,\middle|\, (g_n)_{n\in\mathbb{N}} \in \mathfrak{X}^{\mathbb{N}} \colon \mathfrak{w}_{\mathsf{L}}[x_n] \to 0\right\}.$$

The decomposition is unique precisely when $\mathbf{D}_{\mathsf{L}}\mathsf{K}$ *is dominated by* L.

Due to Theorem 1.3 we have the following Radon–Nikodym-type result for kernels.

COROLLARY 2.5. *Let* $\mathsf{K}, \mathsf{L} \in \mathcal{K}_+(\mathfrak{X})$ *be kernels on* S *and assume that* K *is almost dominated by* L. *Then for every* $g \in \mathfrak{X}$ *there exists a sequence* $(g_n)_{n\in\mathbb{N}} \in \mathfrak{X}^{\mathbb{N}}$ *such that*

$$\forall f \in \mathfrak{X}: \quad \sum_{s,t\in S} \langle f(t), \mathsf{K}(s,t)g(s)\rangle = \lim_{n\to+\infty} \sum_{s,t\in S} \langle f(t), \mathsf{L}(s,t)g_n(s)\rangle.$$

The following statements are immediate consequences of Theorem 1.6 and Theorem 1.7.

COROLLARY 2.6. *Let* $\mathsf{K}, \mathsf{L} \in \mathcal{K}_+(\mathfrak{X})$ *be kernels on* S, *then* $\mathbf{D}_{\mathsf{L}}\mathsf{K}$ *is* $\mathbf{D}_{\mathsf{K}}\mathsf{L}$*-almost dominated. And by symmetry,* $\mathbf{D}_{\mathsf{K}}\mathsf{L}$ *is* $\mathbf{D}_{\mathsf{L}}\mathsf{K}$-almost *dominated.*

COROLLARY 2.7. *Let* K *and* L *be kernels on* S. *Then the infimum* $\mathsf{K} \wedge \mathsf{L}$ *of* K *and* L *exists precisely when* $\mathbf{D}_{\mathsf{K}}\mathsf{L}$ *and* $\mathbf{D}_{\mathsf{L}}\mathsf{K}$ *are comparable.*

Finally, we have the following characterizations according to Theorem 1.9 and Theorem 1.10.

COROLLARY 2.8. *Let* $\mathsf{J}, \mathsf{K} \in \mathcal{K}_+(\mathfrak{X})$ *be kernels on* S, *such that* $\mathsf{J} \prec \mathsf{K}$. *The following statements are equivalent.*

(i) J *and* $\mathsf{K} - \mathsf{J}$ *are singular.*
(ii) $\mathbf{D}_{\mathsf{J}}\mathsf{K} = \mathsf{J}$.
(iii) J *is an extreme point of the convex set* $[0, \mathsf{K}] = \{\mathsf{U} \in \mathcal{K}_+(\mathfrak{X}) \mid 0 \prec \mathsf{U} \prec \mathsf{K}\}$.

In view of Theorem 2.2 the previous corollary says that the extremal points of the convex set $[0, \mathsf{K}]$ are precisely those kernels that have closed dilation.

COROLLARY 2.9. *Let* $\mathsf{K}, \mathsf{L} \in \mathcal{K}_+(\mathfrak{X})$ *be kernels on* S. *Then the following statements are equivalent*

(i) K *is an extreme point of* $[0, \mathsf{K} + \mathsf{L}]$.
(ii) $\operatorname{ex}[\mathsf{K}, \mathsf{K} + \mathsf{L}] \subseteq \operatorname{ex}[0, \mathsf{K} + \mathsf{L}]$.

Replacing L *with* $\mathsf{L} - \mathsf{K}$ *(if* $\mathsf{K} \prec \mathsf{L}$*) we have*

$$\mathsf{K} \in \operatorname{ex}[0, \mathsf{L}] \quad \Leftrightarrow \quad \operatorname{ex}[\mathsf{K}, \mathsf{L}] \subseteq \operatorname{ex}[0, \mathsf{L}].$$

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LINEAR FUNCTIONAL EQUATIONS
Abstract of Ph.D. Thesis

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(Defended February 27, 2015)

1. Introduction

We are concerned with the linear functional equation

$$\sum_{i=1}^{n} a_i f(b_i x + c_i y) = 0 \qquad (x, y \in \mathbb{C}) \tag{1}$$

where a_i, b_i, c_i are given complex numbers, and $f\colon \mathbb{C} \to \mathbb{C}$ is the unknown function.

We shall use the following notations. Let $(G, +)$ be an Abelian group. The difference operator Δ_h is defined by

$$\Delta_h f(x) = f(x+h) - f(x) \qquad (x, h \in G)$$

for every $f\colon G \to \mathbb{C}$. A function $f\colon G \to \mathbb{C}$ is called a *generalized polynomial* if there is an n such that $\Delta_{h_1} \dots \Delta_{h_{n+1}} f(x) = 0$ for every $h_1, \dots, h_{n+1}, x \in G$. The smallest n for which f satisfies this condition is called the degree of f. We note that every polynomial $p\colon \mathbb{C} \to \mathbb{C}$ is a generalized polynomial with the same degree but the family of generalized polynomials are wider. We say that the function $f\colon G \to \mathbb{C}$ is *additive*, if f is a homomorphism of G into the additive group of $\mathbb{C}$. A function f is a generalized polynomial of degree 1 if and only if there is an additive function a such that $f - a$ is constant.

By a well-known result of L. Székelyhidi [10], under some mild conditions on the equation (see (2) below), every solution of equation (1) is a generalized polynomial. But the finer structure of the solutions has been investigated only recently. The description of the space of solutions is the main object of the dissertation.

Let $\mathbb{C}^G$ denote the linear space of all complex valued functions defined on G equipped with the product topology. By a *variety* on G we mean a translation invariant closed linear subspace of $\mathbb{C}^G$. A function is a *polynomial* if it belongs to the algebra generated by the constant functions and the additive functions. A nonzero function $m \in \mathbb{C}^G$ is called an *exponential* if m is multiplicative; that is, if $m(x+y) = m(x) \cdot m(y)$ for every $x, y \in G$. An *exponential monomial* is the product of a polynomial and an exponential, a *polynomial-exponential function* is a finite sum of exponential monomials. If a variety contains an exponential element, then we say that *spectral analysis* holds on this variety. If a variety is spanned by exponential monomials, then we say that *spectral synthesis* holds on this variety. If spectral analysis or synthesis holds in every variety on G, then we say that spectral analysis or synthesis holds on G, respectively.

The most important contribution of our results to the theory of linear functional equations is the application of spectral analysis and synthesis to some varieties related to the spaces of solutions of the equations. The idea of the algebraic point of view of the spectral analysis and synthesis on locally compact Abelian groups goes back to the pioneer work of L. Schwartz [8]. The investigation for case of the discrete Abelian groups started by M. Laczkovich, G. Székelyhidi and L. Székelyhidi [5, 6, 11]. The idea of applying spectral analysis to the varieties related to the space of solutions first appeared in [3]. The method of spectral synthesis was first used in [1] and in full generality was proved in [2].

2. Existence of nonzero solutions of linear functional equations

By the result of L. Székelyhidi [10], the following condition on the parameters implies that every solution of (1) is a generalized polynomial.

(2) *The numbers $a_1, \dots, a_n$ are nonzero, and* **there exists** *an* $1 \le i \le n$ *such that* $b_i c_j \neq b_j c_i$ *holds for any* $1 \le j \le n,\ j \neq i$.

Hereinafter, we assume this condition hence every solution is a generalized polynomial, although without this assumption there can be found some other solutions of (1). For some special case we can describe the space of solutions but for full generality it is still open.

We shall restrict our attention to the solutions defined on a subfield K of $\mathbb{C}$, more regularly on the field $\mathbb{Q}(b_1, \dots, b_n, c_1, \dots, c_n)$. This is justified in that *any* *function $f\colon K \to \mathbb{C}$ satisfying $\sum_{i=1}^{n} a_i f(b_i x + c_i y) = 0$ for every $x, y \in K$ can be extended to a solution on $\mathbb{C}$.*

The idea of applying spectral analysis to varieties on $K^* = \{x \in K\colon x \neq 0\}$ (which is an Abelian groups with the multiplication) and on $(K^*)^k$ was introduced in [3] .

For the existence of non-constant solutions it needs two ingredients. First, we use the fact that if there is a non-constant solution of (1), then there exists a nonzero additive solution, as well. The second one is following:

THEOREM 2.1. *There is a nonzero additive solution of* (1) *if and only if there exists a solution of* (1) *which is an automorphism $\phi\colon \mathbb{C} \to \mathbb{C}$ or, equivalently, an automorphism satisfying*

$$\sum_{i=1}^{n} a_i \phi(b_i) = 0 \text{ and } \sum_{i=1}^{n} a_i \phi(c_i) = 0. \tag{3}$$

Theorem 2.1 has many applications. We show a generalization of the theorem of A. Varga [12].

THEOREM 2.2. (i) *Suppose that the parameters $b_1, \dots, b_s$ are algebraic numbers and $b_{s+1}, \dots, b_n$ are algebraically independent over $\mathbb{Q}$, where $0 \leq s < n$. If the parameters $a_1, \dots, a_n$ are algebraic numbers, then*

$$\sum_{i=1}^{n} a_i f(b_i x) = 0 \tag{4}$$

has no nonzero additive solution.

(ii) *Suppose that the parameters $a_1, \dots, a_s$ are algebraic numbers and $a_{s+1}, \dots, a_n$ are algebraically independent over $\mathbb{Q}$, where $0 \leq s < n$. If the parameters $b_1, \dots, b_n$ are algebraic numbers, then* (4) *has no nonzero additive solution.*

We can generalize Theorem 2.1 for the existence of generalized polynomials of degree $k > 1$ in the following way.

THEOREM 2.3. *For every positive integer k the following are equivalent.*

(i) *There exists a generalized polynomial of degree k which is a solution of* (1).
(ii) *There exist field automorphisms $\phi_1, \dots, \phi_k$ of $\mathbb{C}$ such that $\phi_1 \cdot \ldots \cdot \phi_k$ is a solution of* (1).

(iii) *There exist field automorphisms* $\phi_1, \dots, \phi_k$ *of* $\mathbb{C}$ *such that*

$$\sum_{i=1}^{n} a_i \prod_{j \in J} \phi_j(b_i) \prod_{j' \notin J} \phi_{j'}(c_i) = 0$$

for every $J \subseteq \{1, \dots, k\}$.

3. Space of solutions of (1)

In this section we also assume that the every solution is generalized polynomial, which is true if we assume condition (2).

3.1. Algebraic inner parameters

First we deal with the additive case ($k = 1$), moreover we start with the special case when b_i and c_i are algebraic numbers. The following theorem is a direct application of spectral synthesis proved in [6].

THEOREM 3.1. *Let* $b_1, \dots, b_n, c_1, \dots, c_n$ *be algebraic numbers, and put* $K = \mathbb{Q}(b_1, \dots, b_n, c_1, \dots, c_n)$. *Then every additive solution of* (1) *defined on* K *is of the form*

$$d_1\phi_1 + \cdots + d_k\phi_k,$$

where $d_1, \dots, d_k$ *are complex numbers and* $\phi_1, \dots, \phi_k \colon K \to \mathbb{C}$ *are injective homomorphisms satisfying*

$$\sum_{i=1}^{n} a_i\phi_j(b_i) = 0 \text{ and } \sum_{i=1}^{n} a_i\phi_j(c_i) = 0 \tag{5}$$

for every $j \in \{1, \dots, k\}$.

This result can be easily generalized composing Theorems 2.3 and 3.1.

Theorem 3.1 might suggest that if there are many injective homomorphisms which are solutions of (1), then the closed linear space generated by these injective homomorphisms contains every additive solution as well. This is not true in general.

THEOREM 3.2. *Let* $K \subset \mathbb{C}$ *be a field which contains a transcendental number. Then there exist linear functional equations of the form*

$$\sum_{i=1}^{n} a_i f(b_i x + c_i y) = 0$$

such that $b_i, c_i \in K$ for every $i = 1, \dots, n$, and there exists an additive solution d on K which is not contained by the variety generated by the injective homomorphism solutions.

A function $h\colon K \to K$ is a *derivation on* K if h is additive and satisfies the Leibnitz's rule (i.e.: $h(xy) = h(x)y + xh(y)$ for every $x, y \in K$). We note that the additive solution d in Theorem 3.2 is a derivation on K. This motivates the direction of our investigation of the general case.

3.2. Differential operators on a field

Suppose that the complex numbers $t_1, \dots, t_n$ are algebraically independent over $\mathbb{Q}$. The elements of the field $\mathbb{Q}(t_1, \dots, t_n)$ are the rational functions of $t_1, \dots, t_n$ with rational coefficients. By a *differential operator on* $\mathbb{Q}(t_1, \dots, t_n)$ we mean an operator of the form

$$D = \sum c_{i_1,\dots,i_n} \cdot \frac{\partial^{i_1+\cdots+i_n}}{\partial t_1^{i_1} \cdots \partial t_n^{i_n}}, \tag{6}$$

where $\partial/\partial t_i$ are the usual partial derivatives, the sum is finite, in each term the coefficient is a complex number, and the exponents $i_1, \dots, i_n$ are nonnegative integers. If $i_1 = \dots = i_n = 0$, then by $\partial^{i_1+\cdots+i_n}/\partial t_1^{i_1} \cdots \partial t_n^{i_n}$ we mean the identity operator on $\mathbb{Q}(t_1, \dots, t_n)$. The degree of the differential operator D is the maximum of the numbers $i_1 + \dots + i_n$ such that $c_{i_1,\dots,i_n} \neq 0$.

Let K be an arbitrary finitely generated subfield of $\mathbb{C}$. Then it can be written of the form

$$K = \mathbb{Q}(t_1, \dots, t_n, \alpha)$$

where $t_1, \dots, t_n$ are algebraically independent over $\mathbb{Q}$ and α is in algebraic over $\mathbb{Q}(t_1, \dots, t_n)$. It can be proved that if there is a differential operator D on $\mathbb{Q}(t_1, \dots, t_n)$, then it is uniquely extended as a differential operator (on K).

3.3. Spectral sythesis and the space of additive solutions

THEOREM 3.3. *Suppose that the transcendence degree of the field K over $\mathbb{Q}$ is finite. Let $f\colon K \to \mathbb{C}$ be additive, and let m be an exponential on K^*. Let ϕ be an extension of m to $\mathbb{C}$ as an automorphism of $\mathbb{C}$. Then the following are equivalent.*

(i) *$f = p \cdot m$ on K^*, where p is a generalized polynomial on K^*.*
(ii) *$f = p \cdot m$ on K^*, where p is a polynomial on K^*.*
(iii) *There exists a unique differential operator D on K such that $f = \phi \circ D$ on K.*

THEOREM 3.4. *Suppose that the transcendence degree of the field K over $\mathbb{Q}$ is finite. Then spectral synthesis holds in every variety on K^* consisting of additive functions (with respect to addition).*

The proof of the Theorem 3.4 is based on relatively new result of (local) spectral synthesis on countably generated Abelian groups [4].

As an application of Theorems 3.3 and 3.4 we describe the additive solutions of the linear functional equation (1). We denote by S_k the set of solutions of degree k of (1). We can show that

$$S_1^* = \{f|_{K^*} : f \in S_1\}$$

is a variety on K^*. For $k > 1$ the analogue statement is not true, we need to extend our attention for k-additive functions.

The next theorem is our main result concerning the additive solutions of linear functional equations and it has many applications

THEOREM 3.5. *The linear space S_1 is spanned by the functions $\phi \circ D$, where ϕ and D are as above.*

3.4. Spectral synthesis and the space of solutions of higher degree

As it was mentioned before the analogues theorems of Theorem 3.3 and 3.4 can be proved for the k-additive functions on K^k and $(K^*)^k$ instead of K and K^*, respectively. Finally, we may obtain the following result:

THEOREM 3.6. *The linear space S_k is spanned by the functions $\prod_{i=1}^k \phi_i \circ D_i$, where ϕ_i are an automorphism of $\mathbb{C}$ and $\prod_{i=1}^k \phi_i$ in S_k, and D_i are differential operators on K.*

We remark that most of the cases there is no boundary for the number of terms in D_i's in general, nevertheless it is a finite expression.

3.5. The discrete Pompeiu problem

In the last section we are concerned with the discrete Pompeiu problem and its connection to linear functional equations. The problem is stemmed from the classical Pompeiu problem and from the question asked by L. Pósa.

QUESTION 3.7 (Pósa). *Suppose that the function $f : \mathbb{R}^2 \to \mathbb{R}$ has the property that the sum of the values of f at the vertices of any square of fix size is zero. Is it true that $f \equiv 0$?*

Let D be a finite set of $\mathbb{R}^2$ and let G be a transformation group on $\mathbb{R}^2$. We say that D has the *discrete Pompeiu property with respect to G* if for every function $f : \mathbb{R}^2 \to \mathbb{C}$ the equation

$$\sum_{d\in\sigma(D)} f(d) = 0 \tag{7}$$

for all $\sigma \in G$ implies $f \equiv 0$.

The answer to Pósa's question is affirmative.

THEOREM 3.8. *Let D be the vertex set of the unit square. Then D has the discrete Pompeiu property with respect to the congruences of $\mathbb{R}^2$.*

The proof of Theorem 3.8 uses spectral analysis and some results of Euclidean Ramsey theory based on the following theorem of L. E. Shader [9].

THEOREM 3.9. *For any 2-coloring of the plane all right triangles are Ramsey.*

Pósa's question can be generalized as follows:

QUESTION 3.10 (Discrete Pompeiu problem). *Let $D \subset \mathbb{R}^2$ be a finite set. Is it true that D has the discrete Pompeiu property with respect to the congruences?*

In full generality this question remains open.

We denote by Σ the similarity group of $\mathbb{R}^2$. it can be shown that the discrete Pompeiu problem with respect to Σ is equivalent to the existence of non-constant solution of a linear functional equation.

THEOREM 3.11. *Suppose that D is a nonempty finite subset of $\mathbb{C}$. Let $f\colon \mathbb{R}^2 \to \mathbb{C}$ be a function which satisfies equation* (7) *for every $\sigma \in \Sigma$. Then $f \equiv 0$.*

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DIRECTIONS AND OTHER TOPICS IN GALOIS-GEOMETRIES
Abstract of Ph.D. Thesis

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(Defended February 27, 2015)

In the thesis we study geometries over finite fields (Galois-geometries) and "geometry style" properties of finite fields. The two main ways of finite geometrical investigations are the combinatorial and the algebraic one, there are examples for both methods in the thesis. In both cases we define a point set by a combinatorial property, e.g. by its intersection numbers with certain subspaces. In the first method we examine the set using combinatorial and geometrical tools; the other way is the algebraic one. The connection between algebra and finite geometry is said to be classical (e.g. Mathieu-groups – Witt-designs). We take a point set in a geometry over a finite field and translate its "nice" combinatorial property to a "nice" algebraic structure. In the thesis we mainly use the so-called polynomial method, created by Blokhuis and Szőnyi and developed by many others: we assign a polynomial over a finite field to the point set, examine it with various tools, then we translate the algebraic information we get back to the original, geometrical language.

Throughout the summary, let p be a prime, $q = p^h$ be a prime power, and let $\mathrm{GF}(q)$ be a finite field of q elements. Π_n refers to a (combinatorially defined) projective plane of order n, and let $\mathrm{PG}(n, q)$ denote the projective space of dimension n over $\mathrm{GF}(q)$. Let $\mathrm{AG}(n, q)$ denote the affine space of dimension n over $\mathrm{GF}(q)$ that corresponds to the co-ordinate space $\mathrm{GF}(q)^n$ of rank n over $\mathrm{GF}(q)$. We can embed $\mathrm{AG}(n, q)$ into $\mathrm{PG}(n, q)$ in the usual way: $\mathrm{PG}(n, q) = \mathrm{AG}(n, q) \cup \cup H_\infty$, where H_∞ is called the *hyperplane at infinity* or the *ideal hyperplane*, its points are called *ideal points* or *directions*. Our main tool in the algebraic methods is the careful investigation of a polynomial assigned to the point set, called *Rédei-polynomial*, which is a totally reducible polynomial, where each linear factor corresponds to a point of the set.

The main part of the thesis is related to the *direction problem*. The problem, which was suggested by Rédei, has many non-geometrical applications.

We say a point d at infinity is a *direction*, *determined* by an affine point set, if there is an affine line with the ideal point d containing at least two points of the set. The investigated questions are the *number* of determined directions, and the *size* and the *structure* of sets with few determined directions. Note that in the n-dimensional space if $|U| > q^{n-1}$ then every direction is determined. In fact, already a random point set of size much less than q^{n-1} determines all the directions. In case of a set of size q^{n-1} we investigate the questions mentioned above. The examination of smaller sets leads to stability questions as well: can we extend such a set to a set of maximal size determining the same directions only. Sets of maximal size in the plane over $\mathrm{GF}(q)$, i. e. sets of cardinality q are the most studied ones. Rédei and Megyesi proved in [13] that in the plane of p prime order if the points are not collinear then a set of size p determines at least $\frac{p+3}{2}$ directions, while Lovász and Schrijver showed in [12] that a set with that many determined directions is unique (up to an affine transformation). The case of sets of maximal size in a plane of prime power order was completely characterized by Blokhuis, Ball, Brouwer, Storme and Szőnyi in [9].

In Chapter 2 we give some results on the number of determined directions by a small point set in the plane. This is based on a joint work with Szabolcs Fancsali and Péter Sziklai. As these results have already appeared in [4] and also in the Ph.D. thesis of Szabolcs Fancsali, in the thesis we just give a short summary of the topic; the method and the results. Although the case of maximal point sets (i. e. of size q) in the plane of order q is completely characterized in [9], the classification of sets of size less than q is open. A theorem of Szőnyi in [17] describes the case $q = p$ prime. Here we study the case $q = p^h$ prime power, and give some partial results on the number of determined directions by less than q points.

Let $U \subset \mathrm{AG}(n,q) \subset \mathrm{PG}(n,q)$ be a point set, $|U| < q$. Let s denote the greatest power of p such that each line ℓ of a determined direction meets U in 0 modulo s points. During the examination of the Rédei-polynomial of the set U we define a parameter t (where $s \leq t$), which is somehow an analogue of the parameter s; it will be essential in order to reach the bounds on the number of determined directions.

THEOREM 2.14. *Let $U \subset \mathrm{AG}(2,q)$ be an arbitrary set of points and let D denote the directions determined by U. We use the notation s and t defined above. Suppose that $\infty \in D$. One of the following holds:*

(i) $1 = s \leq t < q$ *and* $\dfrac{|U|-1}{t+1} + 2 \leq |D| \leq q+1$*;*

(ii) $1 < s \le t < q$ *and* $\dfrac{|U|-1}{t+1} + 2 \le |D| \le \dfrac{|U|-1}{s-1}$*;*
(iii) $1 \le s \le t = q$ *and* $D = \{\infty\}$.

In the special case $q = p$ prime we get back the result of Szőnyi. If $q > p$ the value of this result is decreased by the fact that t was defined in an algebraic way, and its geometrical meaning is not yet clear.

In Chapter 3 – based on [5] – we examine a stability question. Given a point set of size less than q^{n-1} in the n-dimensional affine space, the question is whether we can add some points to it to reach a set of maximal size (i. e. of cardinality q^{n-1}) such that the set of determined directions remains the same. Earlier results (the strongest ones are known in the case $n = 2$) contain restrictions on the size of the affine point set or on the size of the set of determined directions. The main result of the chapter is a new method we use in order to tackle the old problem. Instead of investigating the *number* of non-determined directions, we examine the *structure* of the set of non-determined directions.

Let U be a point set, $|U| = q^{n-1} - \varepsilon$. We define a polynomial $f(X_0, X_1, \ldots, X_n)$ of degree ε, which describes the deficiency of the set, i. e. its difference from the maximal cardinality. In order to reach the results we examine this polynomial. The equation $f = 0$ defines an algebraic hypersurface in the dual space $\mathrm{PG}(n, q)$. If the polynomial splits completely into linear factors then in the dual space the surface $f = 0$ is a union of ε hyperplanes. These hyperplanes correspond to exactly ε points in the original space, and by adding these points to the set we reach the maximal size. An undetermined direction refers to a hyperplane in the dual space such that the intersection of the hyperplane and the surface $f = 0$ is *totally reducible*, i. e. it splits into $(n-2)$-dimensional subspaces. (We call such a hyperplane a TRI hyperplane, where the abbreviation TRI stands for Totally Reducible Intersection.) Thus, if the surface is not totally reducible then the non-determined directions have a very restricted (strong) structure.

We have an extendability result in general dimension for $\varepsilon = 2$.

THEOREM 3.10. *Let* $n \ge 3$. *Let* $U \subset \mathrm{AG}(n, q) \subset \mathrm{PG}(n, q)$, $|U| = q^{n-1} - 2$. *Let* $D \subseteq H_\infty$ *be the set of directions determined by* U *and put* $N = H_\infty \setminus D$ *the set of non-determined directions. Then* U *can be extended to a set* $\bar{U} \supseteq U$, $|\bar{U}| = q^{n-1}$ *determining the same directions only, or the points of* N *are collinear and* $|N| \le \lfloor \frac{q+3}{2} \rfloor$, *or the points of* N *are on a (planar) conic curve.*

We show a general stability theorem in the 3-space if $\varepsilon < p$.

THEOREM 3.11. *Let* $U \subset \mathrm{AG}(3, q) \subset \mathrm{PG}(3, q)$, $|U| = q^2 - \varepsilon$, *where* $\varepsilon < p$. *Let* $D \subseteq H_\infty$ *be the set of directions determined by* U *and put* $N = H_\infty \setminus D$ *the* *set of non-determined directions. Then N is contained in a plane curve of degree $\varepsilon^4 - 2\varepsilon^3 + \varepsilon$ or U can be extended to a set $\bar{U} \supseteq U$, $|\bar{U}| = q^2$.*

We consider the case when U is extendable as the typical one, otherwise the non-determined directions are contained in a (planar) curve of low degree. Although note that there exist examples of maximal point sets of size $q^2 - 2$, $q \in \{3, 5, 7, 11\}$, not determining the points of a conic at infinity.

To reach the total strength of this theory, we would like to use an argument stating that it is a "very rare" situation that the intersection of a hyperplane and the surface is totally reducible – this difficult problem seems to be interesting for its own sake, and it is yet unsolved.

CONJECTURE 3.12. *Let $f(X_0, X_1, \ldots, X_n)$ be a homogeneous irreducible polynomial of degree $d > 2$ and let F be the hypersurface in $\mathrm{PG}(n, q)$ determined by $f = 0$. Then the number of TRI hyperplanes to F is "small" or F is a cone with a low dimensional base.*

The proof of the conjecture would imply extendability of direction sets under very general conditions.
Finally we describe an application of the result in the theory of ovoids.

COROLLARY 3.17. *Let $\mathcal{B}$ be a partial ovoid of size $q^2 - 2$ of the partial geometry $T_2^*(\mathcal{K})$, then $\mathcal{B}$ is always extendable to an ovoid.*

In Chapter 4, which is based on [3], we generalize the original direction problem such that we define *determined k-dimensional subspaces* of the hyperplane at infinity.

DEFINITION 4.1. Let $U \subset \mathrm{AG}(n, q) \subset \mathrm{PG}(n, q)$ be a point set, and k be a fixed integer, $k \leq n - 2$. We say a subspace S_k of dimension k in H_∞ is *determined* by U if there is an affine subspace T_{k+1} of dimension $k + 1$, having S_k as its hyperplane at infinity, containing at least $k + 2$ affinely independent points of U (i.e. spanning T_{k+1}).

The questions here are the analogues of that in the classical problem: that is, for a fixed k we ask for the size of the point set if it does not determine all the k-subspaces of H_∞; and for the structure of U in case of "few" determined subspaces. Note that $|U| \leq q^{n-1}$ if it does not determine all the k-subspaces at infinity. In the thesis we consider point sets of the maximal "interesting" cardinality, i. e. sets of size q^{n-1}. In Proposition 4.2 and 4.14 we give some constructions for point sets with relatively many non-determined ideal subspaces in arbitrary dimensions. We describe the hierarchy of determined subspaces of *different* dimensions for a given affine point set.

PROPOSITION 4.6. *Let $U \subset \mathrm{AG}(n,q) \subset \mathrm{PG}(n,q)$, $|U| = q^{n-1}$ and k be a fixed integer, $k \leq n-3$. If there is a subspace V_{n-2} of dimension $n-2$, $V_{n-2} \subset H_\infty$ such that all of the k-dimensional subspaces of V_{n-2} are determined by U then V_{n-2} is determined by U as well.*

Our main result here is the complete characterization of point sets of maximal size in 3 dimensions.

THEOREM 4.7. *Let $U \subset \mathrm{AG}(3,q) \subset \mathrm{PG}(3,q)$, $|U| = q^2$. Let L be the set of lines in H_∞ determined by U and put N the set of non-determined lines. Then one of the following holds:*
a) $|N| = 0$, *i.e. U determines all the lines of H_∞;*
b) $|N| = 1$ *and then there is a parallel class of affine planes such that U contains one (arbitrary) complete line in each of its planes;*
c) $|N| = 2$ *and then (i) U together with the two undetermined lines in H_∞ form a hyperbolic quadric or (ii) U contains q parallel lines (U is a cylinder);*
d) $|N| \geq 3$ *and then U contains q parallel lines (U is a cylinder).*

It means that if there are “many” (≥ 3) undetermined lines then the point set is somehow “reducible”: it must form a cone (cylinder). In case of two undetermined lines one other example – the hyperbolic quadric – occurred.

In Chapter 1 we describe a problem which seems to be purely algebraic, Vandermonde sets and super-Vandermonde sets [2]. Beyond the algebraic motivation they are also interesting from the finite geometrical point of view.

Let $S = \{x_1, \ldots, x_n\} \subseteq \mathrm{GF}(q)$ be a subset. The *k-th power sum* of the elements of S is $\pi_k(x_1, \ldots, x_n) = \sum_{i=1}^{n} x_i^k$. Let $w = w_S$ be the smallest positive integer k such that $\pi_k \neq 0$ if such a k exists, otherwise $w = \infty$.

DEFINITION 1.4. Let $1 < t < q$. We say that $T = \{y_1, \ldots, y_t\} \subseteq \mathrm{GF}(q)$ is a *Vandermonde set*, if $\pi_k = \sum_i y_i^k = 0$ for all $1 \leq k \leq t-2$.

In other words, the Vandermonde property is equivalent to $w_T \geq t-1$. If $p \mid t$, then a t-set cannot have more than $t-2$ zero power sums, so $w_T \leq t-1$, it follows from the fact that a Vandermonde determinant of distinct elements cannot be zero. So in this sense Vandermonde sets are extremal with $w = t-1$, and the name “Vandermonde” comes from here. In general a t-set cannot have more than $t-1$ zero power sums (so for a Vandermonde set $w_T = t-1$ or t holds). This consideration leads to the following definition.

DEFINITION 1.5. Let $1 < t < q$. We say that $T = \{y_1, \ldots, y_t\} \subseteq \mathrm{GF}(q)$ is a *super-Vandermonde set*, if $\pi_k = \sum_i y_i^k = 0$ for all $1 \leq k \leq t-1$.

So the super-Vandermonde property is equivalent to $w_T = t$, and the argument above shows that such a set exists only if $p \nmid t$ holds. Any additive subgroup of $\mathrm{GF}(q)$ is a Vandermonde set, and any multiplicative subgroup of $\mathrm{GF}(q)$ is a super-Vandermonde set. There also exist finite geometrical examples: many interesting point sets can be translated to Vandermonde sets in a natural way. Our main result here is the characterization of small and large super-Vandermonde sets. What does "small" and "large" mean? By removing the zero element from an additive subgroup of $\mathrm{GF}(q)$ one gets a super-Vandermonde set. The smallest and largest non-trivial additive subgroups are of cardinality p and q/p, respectively. This motivates that, for our purposes small and large will mean "of size $< p$" and "of size $> q/p$", resp.

THEOREM 1.12. *Suppose that $T \subset \mathrm{GF}(q)$ is a super-Vandermonde set of size $|T| < p$. Then T is a (transform of a) multiplicative subgroup.*

THEOREM 1.13. *Suppose that $T \subset \mathrm{GF}(q)$ is a super-Vandermonde set of size $|T| > q/p$. Then T is a (transform of a) multiplicative subgroup.*

Note that we classified the case $q = p^2$: then a super-Vandermonde set of $GF(q)$ is (a coset of) a multiplicative subgroup.

The concept of super-Vandermonde sets led to further research. Sets of size $p+1$ and $q/p-1$ are recently examined by Blokhuis in [10] and in [11].

In the last two chapters we show some connections between finite geometries and other fields in combinatorics. In Chapter 5, which is a joint work with T. Héger [6], we examine a graph theoretical question due to R. Bailey and P. Cameron. We examine resolving sets of the incidence graph of a finite projective plane. We give the metric dimension of the incidence graph, and classify the smallest resolving sets of it, using combinatorial tools.

Let $\Gamma = (V, E)$ be a simple graph, for $x, y \in V$, $d(x, y)$ denotes the distance of x and y.

DEFINITION 5.1. $S = \{s_1, \ldots, s_k\} \subset V$ is a *resolving set* in $\Gamma = (V, E)$, if the ordered distance lists $(d(x, s_1), \ldots, d(x, s_k))$ are unique for all $x \in V$. The *metric dimension of* Γ, denoted by $\mu(\Gamma)$, is the size of the smallest resolving set in it.

Equivalently, S is a resolving set in $\Gamma = (V, E)$ if and only if for all $x, y \in V$, there exists a point $z \in S$ such that $d(x, z) \neq d(y, z)$. In other words, the vertices of Γ can be distinguished by their distances from the elements of a resolving set. We say that a vertex v is *resolved* by S if its distance list with respect to S is unique. A set $A \subset V$ is resolved by S if all its elements are resolved by S. Note that the distance list is ordered, the (multi)set of distances is not sufficient.

Take a projective plane $\Pi = (\mathcal{P}, \mathcal{L})$ of order q, where $\mathcal{P}$ denotes the set of points and $\mathcal{L}$ stands for the set of lines. The incidence graph $\Gamma(\Pi)$ of Π is a bipartite graph with vertex classes $\mathcal{P}$ and $\mathcal{L}$, where $P \in \mathcal{P}$ and $\ell \in \mathcal{L}$ are adjacent in Γ if and only if P and ℓ are incident in Π. By a resolving set or the metric dimension of Π we mean that of its incidence graph.

We prove the following theorem regarding the metric dimension of a finite projective plane.

THEOREM 5.2. *The metric dimension of a projective plane of order* $q \geq 23$ *is* $4q - 4$.

We give the description of all resolving sets of a projective plane Π of size $4q - 4$ ($q \geq 23$).

The starting point of combinatorial search theory is the following problem: given a set X of n elements out of which one x is marked, what is the minimum number s of queries of the form of subsets $A_1, A_2, \ldots, A_s$ of X such that after getting to know whether x belongs to A_i for all $1 \leq i \leq s$ we are able to determine x. Since decades, the number s is known to be equal to $\lceil \log_2 n \rceil$ no matter if the ith query might depend on the answers to the previous ones (*adaptive search*) or we have to ask our queries at once (*non-adaptive search*). In Chapter 6 we address the q-analogue of the basic problem [7]. Let V denote an n-dimensional vector space over $\mathrm{GF}(q)$ and let $\mathbf{v}$ be a marked 1-dimensional subspace of V. We will be interested in determining the minimum number of queries that is needed to find $\mathbf{v}$ provided all queries are subspaces of V and the answer to a query U is YES if $\mathbf{v} \leqslant U$ and NO if $\mathbf{v} \nleqslant U$. This number will be denoted by $A(n, q)$ in the adaptive case and $M(n, q)$ in the non-adaptive case. Note that a set $\mathcal{U}$ of subspaces of V can be used as query set to determine the marked 1-space in a non-adaptive search if and only if for every pair $\mathbf{u}, \mathbf{v}$ of 1-subspaces of V there exists a subspace $U \in \mathcal{U}$ with $\mathbf{u} \leqslant U, \mathbf{v} \nleqslant U$ or $\mathbf{u} \nleqslant U, \mathbf{v} \leqslant U$. Such systems of subspaces are called *separating*. It is easy to show that $A(n, 2) = M(n, 2) = n$ for all $n \geq 2$. Thus we will mainly focus on the case when $q \geq 3$. As usual, the subspaces of an n-dimensional vector space over $\mathrm{GF}(q)$ are considered as the elements of the Desarguesian projective geometry $\mathrm{PG}(n - 1, q)$. In the case $n = 3$ we determine $A(3, q)$ for all prime powers q.

THEOREM 6.1. *Consider a projective plane* Π_q *of order* q. *Let* $A(\pi_q)$ *denote the minimum number of queries in adaptive search that is needed to determine a point of* Π_q *provided the queries can be either points or lines of* π_q. *With this notation we have* $A(\Pi_q) \leq 2q - 1$; *if* q *is a prime power, then* $A(\mathrm{PG}(2, q)) = 2q - 1$, *that is the equality* $A(3, q) = 2q - 1$ *holds.*

We also address the problem of determining $M(3, q)$. We obtain upper and lower bounds but not the exact value except if $q \geq 121$ is a square. The most important consequence of our results is the following theorem that states that the situation is completely different from that in the classical case where adaptive and non-adaptive search require the same number of queries.

THEOREM 6.2. *For* $q \geq 9$ *the inequality* $A(3,q) < M(3,q)$ *holds.*

We also address in arbitrary dimensions the general problem of giving upper and lower bounds on $A(n,q)$ and $M(n,q)$. Our main results are the following theorems.

THEOREM 6.3. *For any prime power* $q \geq 2$ *and positive integer* n *the inequalities* $\log_2 \begin{bmatrix} n \\ 1 \end{bmatrix}_q \leq A(n,q) \leq (q-1)(n-1)+1$ *hold.*

THEOREM 6.4. *There exists an absolute constant* $C > 0$ *such that for any positive integer* n *and prime power* q *the inequalities* $\frac{1}{C}q(n-1) \leq M(n,q) \leq 2q(n-1)$ *hold. Moreover, if* q *tends to infinity, then* $(1-o(1))q(n-1) \leq M(n,q)$ *holds.*

Throughout the proofs we use semi-resolving sets of finite projective planes, which is a variant of resolving sets we mentioned in the previous chapter.

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