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# ANNALES 

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# GEOMETRY OF TREFOIL CONE - MANIFOLD* 

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(Received March 8, 2011)


#### Abstract

In this paper we prove that Trefoil knot cone manifold $\mathcal{T}(\alpha)$ with cone angle $\alpha$ is spherical for $\pi / 3<\alpha<5 \pi / 3$. We show also that its spherical volume is given by the formula $\operatorname{Vol}(\mathcal{T}(\alpha))=(3 \alpha-\pi)^{2} / 12$.


## 1. Introduction

Let $\mathcal{T}(\alpha)$ be a cone manifold whose underlying space is the threedimensional sphere $\mathcal{S}^{3}$ and singular set is Trefoil knot $\mathcal{T}$ with cone angle $\alpha$ (Fig. 1). Since $\mathcal{T}$ is a toric knot by the Thurston theorem its complement $\mathcal{T}(0)=\mathcal{S}^{3} \backslash \mathcal{T}$ in the $\mathcal{S}^{3}$ does not admit hyperbolic structure. We think this is the reason why the simplest nontrivial knot came out of attention of geometricians. However, it is well known that Trefoil knot admits geometric structure. H. Seifert and C. Weber (1935) [16] have shown that the spherical space of dodecahedron (= Poincaré homology 3-sphere) is a cyclic 5-fold covering of $\mathcal{S}^{3}$ branched over $\mathcal{T}$. Topological structure and fundamental groups of cyclic $n$-fold coverings have described by D. Rolfsen [14] and A.J. Sieradsky [18]. In spite of positive solution of the Orbifold Geometrization Conjecture given in [1] and [2] the geometrical structure of $\mathcal{T}(\alpha)$ for an arbitrary $\alpha$ is still unknown. The most progress is achieved for the case $\alpha=2 \pi / n, n \in \mathbb{N}$. In that case $\mathcal{T}(2 \pi / n)$ is a geometric orbifold, that is can be represented in the form $\mathbb{X}^{3} / \Gamma$, where $\mathbb{X}^{3}$ is one of the eight three-dimensional homogeneous geometries and $\Gamma$ is a discrete group of isometries of $\mathbb{X}^{3}$. By Dunbar [4] classification of non-hyperbolic orbifolds has a spherical structure for $n \leq 5$, Nil for $n=6$ and $\widetilde{\operatorname{PSL}}(2, \mathbb{R})$ for $n \geq 7$. Quite

[^0]surprising situation appears in the case of the Trefoil knot complement $\mathcal{T}(0)$. By P. Norbury (see Appendix A in [12]) the manifold $\mathcal{T}(0)$ admits two geometrical structures $\mathbb{H}^{2} \times \mathbb{R}$ and $\widetilde{\operatorname{PSL}}(2, \mathbb{R})$.

In the present paper we prove that the Trefoil knot cone manifold $\mathcal{T}(\alpha)$ is spherical for $\pi / 3<\alpha<5 \pi / 3$. We show also that spherical volume of $\mathcal{T}(\alpha)$ is given by the formula $\operatorname{Vol}(\mathcal{T}(\alpha))=(3 \alpha-\pi)^{2} / 12$.

We note that the existence of spherical structure on the figure-eight knot and others two-bridge knots was investigated in [5], [8] and [13].

A further development of the results of this paper is performed in [11] and [6].


Figure 1. Trefoil cone-manifold $\mathcal{T}(\alpha)$.

## 2. Preliminary

The standard model for 3-dimensional spherical geometry is the unit sphere $\mathbb{S}^{3}$ of $\mathbb{R}^{4}$ defined by

$$
\mathbb{S}^{3}=\left\{x \in \mathbb{R}^{4}:|x|=1\right\} .
$$

Let $x, y$ be points in $\mathbb{S}^{3}$ and let $\theta(x, y)$ be the Euclidean angle between $x$ and $y$. The spherical distance between $x$ and $y$ is defined to be the real number

$$
d_{S}(x, y)=\theta(x, y)
$$

Note that $0 \leq d_{S}(x, y) \leq \pi$ and $\cos d_{S}(x, y)=\langle x, y\rangle$, where $\langle x, y\rangle$ is the scalar product of $x$ and $y$. The metric space consisting of $\mathbb{S}^{3}$ together with its spherical
metric $d_{S}$ is called spherical 3 -space and will be denoted by the same symbol $\mathbb{S}^{3}$. For more details, see for instance [14].
Theorem 1 ([14], p. 41). A function $\lambda: \mathbb{R} \rightarrow \mathbb{S}^{3}$ is a geodesic line if and only if there are orthogonal vectors $x, y$ in $\mathbb{S}^{3}$ such that

$$
\begin{equation*}
\lambda(t)=(\cos t) x+(\sin t) y . \tag{1}
\end{equation*}
$$

If $x$ and $y$ are orthogonal vectors in $\mathbb{S}^{3}$ we will denote by $\lambda(x, y)$ the geodesic line in $\mathbb{S}^{3}$ determined by (1). For any $x \in \mathbb{R}^{4}, x \neq(0,0,0,0)$, we denote by $\widetilde{x}$ the vector $\frac{x}{|x|} \in \mathbb{S}^{3}$.

It is well known that the group of spherical isometries $\operatorname{Isom}\left(\mathbb{S}^{3}\right)$ is isomorphic to the orthogonal group $O(4)$.

Recall also the spherical Cosine Rules [20]. If $\alpha, \beta, \gamma$ are the angles of a spherical triangle and $a, b, c$ are the lengths of the opposite sides, then

$$
\begin{equation*}
\cos \gamma=\frac{\cos c-\cos a \cos b}{\sin a \sin b} \tag{2}
\end{equation*}
$$

and

$$
\begin{equation*}
\cos c=\frac{\cos \gamma+\cos \alpha \cos \beta}{\sin \alpha \sin \beta} \tag{3}
\end{equation*}
$$

## 3. Fundamental polyhedron

Let $X^{3}$ be one of the following spaces: hyperbolic space $\mathbb{H}^{3}$, Euclidean space $\mathbb{E}^{3}$, or spherical space $\mathbb{S}^{3}$. Let $C^{3}$ be a cone-manifold that can modeled on $X^{3}$. Suppose that the underlying space of $C^{3}$ is the 3 -sphere and $\Sigma$ is the singular set. Let $\varphi: \pi_{1}\left(C^{3}-\Sigma\right) \rightarrow \operatorname{Isom}\left(X^{3}\right)$ be a holonomy map. If $C^{3}$ is a complete $X^{3}$ orbifold, for instance compact, then $\Gamma=\varphi\left(\pi_{1}\left(C^{3}-\Sigma\right)\right)$ is a discrete subgroup of $\operatorname{Isom}\left(X^{3}\right)$ and $C^{3}=X^{3} / \Gamma$. In that case one can canonically construct a fundamental polyhedron $F_{\Gamma}$ for $\Gamma$. For instance one can use Dirichlet, Ford or Delaunay polyhedron. Pairwise identification of faces of $F_{\Gamma}$ gives $C^{3}$.

In general case $\Gamma$ is not a discrete group and has no fundamental polyhedron. However, in many cases there exist polyhedron $F$ such that pairwise identification of faces of $F$ gives $C^{3}$ (see [5], [19]). We call $F$ fundamental polyhedron. We emphasize that images of $F$ under the $\Gamma$ action do not necessary tessellate $X^{3}$. Our aim is to construct the polyhedron $F$ in the case when $X^{3}=\mathbb{S}^{3}$ and $C^{3}$ is the trefoil knot cone-manifold $\mathcal{T}(\alpha)$.

In this section we discuss the existence of $F$ (see Figure 2) and its metrical properties. To obtain $F$, we introduce points $S, N, P_{1}, \ldots, P_{6} \in \mathbb{R}^{4}$, depending on two real parameters $l, d, 0<l<4 \pi, 0<d<\frac{\pi}{3}$. These points, with a suitable choice of the parameters will belong to the unite sphere $\mathbb{S}^{3}$ and be the vertices of the polyhedron

$$
F=\bigcup_{i=1}^{6} T\left(S, N, P_{i}, P_{i-1}\right)
$$

where $T\left(S, N, P_{i}, P_{i-1}\right)$ is the spherical tetrahedron with vertices $S, N, P_{i}, P_{i-1}$, and subscripts are consider $\bmod 6$. The parameters $d$ and $l$ depend on $\alpha$ (the relations are established in Propositions 7 and 8) and admit the following geometrical sense. The spherical length of the singular set of $\mathcal{T}(\alpha)$ is $l=2 d_{S}\left(P_{3}, P_{6}\right)$ and $d=d_{S}(S, N)$.


Figure 2

Define $S, N, P_{1}, \ldots, P_{6} \in \mathbb{R}^{4}$ by

$$
\begin{aligned}
S & =(1,0,0,0) \\
N & =(\cos d, \sin d, 0,0) \\
P_{1} & =\left(\cos d \cos \frac{l}{4}, \sin d \cos \frac{l}{4}, \cos d \sin \frac{l}{4}, \sin d \sin \frac{l}{4}\right) \\
P_{2} & =\left(\cos d \cos \frac{l}{4}, \cos d \cos \frac{l}{4} \tan \frac{d}{2},-\cos d \sin \frac{l}{4}, \cos d \sin \frac{l}{4} \cot \frac{d}{2}\right)
\end{aligned}
$$

(4) $P_{3}=\left(\cos \frac{l}{4}, 0,-\sin \frac{l}{4}, 0\right)$
$P_{4}=\left(\cos d \cos \frac{l}{4}, \sin d \cos \frac{l}{4},-\cos d \sin \frac{l}{4},-\sin d \sin \frac{l}{4}\right)$
$P_{5}=\left(\cos d \cos \frac{l}{4}, \cos d \cos \frac{l}{4} \tan \frac{d}{2}, \cos d \sin \frac{l}{4},-\cos d \sin \frac{l}{4} \cot \frac{d}{2}\right)$, $P_{6}=\left(\cos \frac{l}{4}, 0, \sin \frac{l}{4}, 0\right)$.
Proposition 2. Let $0<d<\pi / 3$ and $l, 0<l<4 \pi$ be defined by

$$
\begin{equation*}
\cos \frac{l}{2}=\frac{3 \cos ^{2} d-1}{2 \cos ^{3} d} \tag{5}
\end{equation*}
$$

Then the polyhedron $F$ exists in $\mathbb{S}^{3}$ and has the following metrical properties:

$$
\begin{gather*}
d_{S}\left(P_{i}, P_{i+1}\right)=d_{S}(S, N)=d  \tag{i}\\
\sum_{i=1}^{6} \delta_{i}=2 \pi
\end{gather*}
$$

(ii)
where $\delta_{i}$ is the dihedral angle between the faces $S P_{i} P_{i-1}$ and $N P_{i} P_{i-1}$ of $F$, and $i=1,2, \ldots, 6$.

To prove Proposition 2 we need several lemmas.
Lemma 3. Let $0<d<\pi / 3$ and $l, 0<l<4 \pi$ be defined by

$$
\cos \frac{l}{2}=\frac{3 \cos ^{2} d-1}{2 \cos ^{3} d}
$$

Then there exist $t \in(0,2 \pi)$ such that

$$
\begin{aligned}
& \sin t=\frac{\cos d \sin \frac{l}{4}}{\sin \frac{d}{2}} \\
& \cos t=\frac{\cos d \cos \frac{l}{4}}{\cos \frac{d}{2}} .
\end{aligned}
$$

The proof is elementary. One need only check that under the conditions of Lemma 3 the equality

$$
\left(\frac{\cos d \sin \frac{l}{4}}{\sin \frac{d}{2}}\right)^{2}+\left(\frac{\cos d \cos \frac{l}{4}}{\cos \frac{d}{2}}\right)^{2}=1
$$

holds.
Consider the points $S^{*}=(0,0,1,0), N^{*}=(0,0, \cos d, \sin d)$ of $\mathbb{S}^{3}$.
Lemma 4. Let $0<d<\pi / 3$ and $l, 0<l<4 \pi$ be defined by

$$
\cos \frac{l}{2}=\frac{3 \cos ^{2} d-1}{2 \cos ^{3} d}
$$

Then $S, N, P_{1}, \ldots, P_{6} \in \mathbb{S}^{3}$. Moreover $P_{1}, N, P_{4} \in \lambda\left(N, N^{*}\right), P_{3}, S, P_{6} \in$ $\in \lambda\left(S, S^{*}\right)$, and $P_{2}, P_{5} \in \lambda\left(\widehat{N+S}, \widehat{N^{*}-S^{*}}\right)$, where $\widetilde{N}=\frac{N}{|N|}$.

Proof. [Proof of Lemma 4] We obviously have $N \in \lambda\left(N, N^{*}\right), S \in \lambda\left(S, S^{*}\right)$. Also we have

$$
\begin{aligned}
\widetilde{N+S} & =\left(\cos \frac{d}{2}, \sin \frac{d}{2}, 0,0\right) \\
\widetilde{N^{*}-S^{*}} & =\left(0,0,-\sin \frac{d}{2}, \cos \frac{d}{2}\right) .
\end{aligned}
$$

It is easy to see that

$$
\begin{aligned}
P_{1} & =\cos \frac{l}{4} N+\sin \frac{l}{4} N^{*} \\
P_{2} & =\cos t(\widehat{N+S})+\sin t\left(\widetilde{N^{*}-S^{*}}\right) \\
P_{3} & =\cos \frac{l}{4} S-\sin \frac{l}{4} S^{*} \\
P_{4} & =\cos \frac{l}{4} N-\sin \frac{l}{4} N^{*} \\
P_{5} & =\cos t(\widetilde{N+S})-\sin t\left(\widetilde{N^{*}-S^{*}}\right), \\
P_{6} & =\cos \frac{l}{4} S+\sin \frac{l}{4} S^{*},
\end{aligned}
$$

where $\cos t$ and $\sin t$ are the same as in Lemma 3. Using Theorem 1 and Lemma 3 we obtain our assertion.
Lemma 5. The order two isometries $L, L^{\prime}$ of $\mathbb{S}^{3}$ are determined by the orthogonal matrices

$$
\left(\begin{array}{cccc}
\cos d & \sin d & 0 & 0 \\
\sin d & -\cos d & 0 & 0 \\
0 & 0 & -\cos d & -\sin d \\
0 & 0 & -\sin d & \cos d
\end{array}\right),\left(\begin{array}{cccc}
\cos d & \sin d & 0 & 0 \\
\sin d & -\cos d & 0 & 0 \\
0 & 0 & \cos d & \sin d \\
0 & 0 & \sin d & -\cos d
\end{array}\right)
$$

respectively are symmetries of $F$. Moreover

$$
\begin{aligned}
L: S & \rightarrow N, & L^{\prime}: S \rightarrow N \\
P_{1} & \rightarrow P_{3}, & P_{1} \rightarrow P_{6} \\
P_{2} & \rightarrow P_{2}, & P_{2} \rightarrow P_{5} \\
P_{4} & \rightarrow P_{6}, & P_{3} \rightarrow P_{4} \\
P_{5} & \rightarrow P_{5} ; &
\end{aligned}
$$

This lemma can be proved by direct calculations
Lemma 6. Under the conditions of Proposition 2 we have the following:

$$
\begin{gather*}
d_{S}\left(P_{i}, P_{i-1}\right)=d_{S}(S, N)=d  \tag{i}\\
\delta_{1}=\delta_{4}=\theta_{1}=\theta_{4}=d \tag{ii}
\end{gather*}
$$

$$
\begin{equation*}
\delta_{2}=\delta_{3}=\delta_{5}=\delta_{6}=\theta_{2}=\theta_{3}=\theta_{5}=\theta_{6}=\frac{\pi-d}{2} \tag{iii}
\end{equation*}
$$

where $\delta_{i}$ is a dihedral angle at the edge $P_{i} P_{i-1}$ of $T_{i}=T\left(S, N, P_{i}, P_{i-1}\right), \theta_{i}$ is a dihedral angle at the edge $S N$ of $T_{i}$ and $i=1,2, \ldots, 6$.

Proof. [Proof of Lemma 6] Consider the statement $(i)$. Recall that for any points $x, y$ in $\mathbb{S}^{3}$ we have $\cos d_{S}(x, y)=\langle x, y\rangle$, where $\langle x, y\rangle$ is the scalar product of $x$ and $y$. Then the identities $\cos d_{S}\left(P_{i}, P_{i-1}\right)=\cos d_{S}(S, N)=\cos d, i=$ $1,2, \ldots, 6$ follow from (4) by direct calculations.
(ii). By definition, $\theta_{1}$ is a dihedral angle at the edge $S N$ of tetrahedron $T_{1}=$ $T_{1}\left(S, N, P_{1}, P_{6}\right)$. Since, by Lemma $4, P_{1} \in \lambda\left(N, N^{*}\right)$ and $P_{6} \in \lambda\left(S, S^{*}\right)$ we obtain that $\theta_{1}$ is also a dihedral angle at the edge $S N$ of the spherical tetrahedron $T\left(S, N, S^{*}, N^{*}\right)$. Two edges $S N$ and $S^{*} N^{*}$ of the tetrahedron are of the spherical length $d$. All other edges have spherical length $\pi / 2$. It is easy to calculate using (2) that $\theta_{1}=d$. Notice that $d_{S}\left(N, P_{1}\right)=d_{S}\left(S, P_{6}\right)=l / 4$ and the tetrahedron $T_{1}=T\left(S, N, P_{1}, P_{6}\right)$ is invariant under the order two symmetry interchanging the edges $S N$ and $P_{1} P_{6}$. Hence $\delta_{1}=\theta_{1}=d$. All other equalities of (ii) follow for the latter by making use of symmetries $L$ and $L^{\prime}$.
(iii). Consider the tetrahedron $T_{2}$. We note by Lemma 5 that the product $L \circ L^{\prime}$ is the order two rotation though axis $\lambda(S, N)$ interchanging points $P_{2}$ and $P_{5}$. Hence, the triangles $P_{2} S N$ and $P_{5} S N$ lie in the same plane. It implies that $\theta_{2}=$ $(\pi-d) / 2$. The tetrahedron $T_{2}$ also has the order two symmetry interchanging the edges $S N$ and $P_{1} P_{2}$. Hence $\delta_{2}=\theta_{2}=(\pi-d) / 2$. As above we obtain all other equalities of $(i i i)$ by symmetry.

Proof. [Proof of Proposition 2] The proof directly follows from Lemma 4 and Lemma 6.

From Lemma 5 follows that the dihedral angle at the edge $S P_{6}$ are equal to the dihedral angles at the edges $N P_{4}, S P_{3}$, and $N P_{1}$. We denote this angle by $\alpha$.

Proposition 7. For any $0<d<\pi / 3$ the angle $\alpha$ is given by the formula

$$
\begin{equation*}
\sin \frac{\alpha}{2}=\frac{1}{2 \cos d} \tag{6}
\end{equation*}
$$

Proof. Using (2) and (4) we have

$$
\angle S P_{6} P_{1}=\angle S P_{6} P_{5}=\pi / 2
$$

Hence from elementary spherical trigonometry follows that $\alpha=\angle P_{1} P_{6} P_{5}$. By (2) and (i) of Lemma 6 we have

$$
\begin{equation*}
\cos \alpha=\frac{\cos d_{S}\left(P_{1}, P_{5}\right)-\cos ^{2} d}{1-\cos ^{2} d} \tag{7}
\end{equation*}
$$

The straightforward calculation shows that

$$
\begin{equation*}
\cos d_{S}\left(P_{1}, P_{5}\right)=\cos d \cos \frac{l}{2} . \tag{8}
\end{equation*}
$$

From equations (7), (8) and (2) we obtain

$$
\cos \alpha=\frac{-2 \cos ^{4} d+3 \cos ^{2} d-1}{2 \cos ^{2} d\left(1-\cos ^{2} d\right)}=\frac{2 \cos ^{2} d-1}{2 \cos ^{2} d}=1-\frac{1}{2 \cos ^{2} d} .
$$

Hence

$$
\sin ^{2} \frac{\alpha}{2}=\frac{1}{4 \cos ^{2} d}
$$

and our assertion follows.
By straightforward calculations we obtain the following corollary.
Proposition 8. For any $\pi / 3<\alpha<5 \pi / 3$ we have

$$
\begin{equation*}
l=3 \alpha-\pi . \tag{9}
\end{equation*}
$$

Proof. By Proposition 7, for $\pi / 3<\alpha<5 \pi / 3$ we obtain $0<d<\pi / 3$. We can use $\alpha$ as a main parameter of our construction. From (6) we have

$$
\cos d=\frac{1}{2 \sin \frac{\alpha}{2}} .
$$

From (5) we now obtain

$$
\cos \frac{l}{2}=\left(3-4 \sin ^{2} \frac{\alpha}{2}\right) \sin \frac{\alpha}{2} .
$$

Since $\sin 3 x=\left(3-4 \sin ^{2} x\right) \sin x$ and $\cos x=\sin \left(x+\frac{\pi}{2}\right)$ we get

$$
\sin \frac{l+\pi}{2}=\sin \frac{3 \alpha}{2} .
$$

By assumption, $\frac{\pi}{3}<\alpha<\frac{5 \pi}{3}$ and $0<l<4 \pi$. Hence $l=3 \alpha-\pi$.
According to Lemma 4, $N$ belongs to the spherical line $P_{1} P_{4}$ and $S$ to the spherical line $P_{6} P_{3}$. Let $A$ and $A^{\prime}$ be rotations of $\alpha$ about these lines respectively.

By Proposition 2, Equation 3 and Lemma 5, we can conclude that $A$ and $A^{\prime}$ identify the faces of $F$ in the following way:

$$
\begin{array}{rlr}
A: & N P_{1} P_{2} & \rightarrow N P_{1} P_{6}, \\
& A^{\prime}: S P_{2} P_{5} \rightarrow S P_{6} P_{1}, \\
& \rightarrow N P_{6} P_{5}, & S P_{3} P_{4} \rightarrow N P_{4} \rightarrow S P_{1} P_{2}, \\
& S P_{4} P_{3} \rightarrow S P_{2} P_{3} .
\end{array}
$$

Proposition 9. Let $\pi / 3<\alpha<5 \pi / 3$, then the identification of faces of $F$ by rotations $A$ and $A^{\prime}$ gives the spherical trefoil knot cone-manifold $\mathcal{T}(\alpha)$.

Proof. From [10] (see also [7] and [8]) it follows that the identification of faces of $F$ by rotations $A$ and $A^{\prime}$ gives a cone-manifold with $\mathbb{S}^{3}$ as an underlying space and the trefoil knot $\Sigma$ as a singular set. The singular set $\Sigma$ is formed by four edges $N P_{1}, N P_{4}, S P_{3}$ and $S P_{6}$. By Proposition 2, the cone-manifold has spherical structure and cone-angle along $\Sigma$ is equal to $\alpha$.

## 4. The main theorem

Denote by $V=\operatorname{Vol}(\mathcal{T}(\alpha))$ the spherical volume of $\mathcal{T}(\alpha)$. We find $V$ by making use the Schläfli differential formula (see [17], [9] or [14] for details):

$$
\mathrm{d} V=\frac{l_{\alpha}}{2} \mathrm{~d} \alpha,
$$

where $l_{\alpha}$ is the length of the singular set $\Sigma$ of $\mathcal{T}(\alpha)$. We note that $\Sigma$ is formed by four segments $N P_{1}, N P_{4}, S P_{3}$ and $S P_{6}$. By direct calculation from (4) we have

$$
d_{S}\left(N, P_{1}\right)=d_{S}\left(N, P_{4}\right)=d_{S}\left(S, P_{3}\right)=d_{S}\left(S, P_{6}\right)=\frac{l}{4} .
$$

Hence, $l_{\alpha}=l$ and by Proposition 8 we obtain $l_{\alpha}=3 \alpha-\pi$.
Theorem 10. The Trefoil cone-manifold $\mathcal{T}(\alpha)$ is spherical for $\pi / 3<\alpha<$ $<5 \pi / 3$. The spherical volume is given by the formula

$$
\operatorname{Vol}(\mathcal{T}(\alpha))=\frac{(3 \alpha-\pi)^{2}}{12}
$$

Proof. The first part of the statement follows from Corollary 9.
By the Schläfli formula and Proposition 8 we have

$$
\operatorname{Vol}(\mathcal{T}(\alpha))=\int \frac{l_{\alpha}}{2} d \alpha=\int \frac{3 \alpha-\pi}{2} d \alpha=\frac{(3 \alpha-\pi)^{2}}{12}+C .
$$

If $\alpha$ tends to $\pi / 3$ then, by Proposition 7, $d$ tends to 0 and by (5) the length $l$ also tends to 0 . It means that $F$ shrinks to a point and $\operatorname{Vol}(\mathcal{T}(\alpha)) \rightarrow 0$. It implies that the constant $C=0$. Finally we obtain

$$
\operatorname{Vol}(\mathcal{T}(\alpha))=\frac{(3 \alpha-\pi)^{2}}{12}
$$

## References

[1] M. Boileau, J. Porti, Geometrization of 3-orbifolds of cyclic type, Astérisque, 272 (2001), 1-208.
[2] M. Boileau, B. Leeb, J. Porti, Geometrization of 3-dimensional orbifolds, Annals of Math., 162 (2005), no. 1, 195-250.
[3] H. S. M. Coxeter, Regular honeycombs in elliptic space, Proc. London Math. Soc., 4 (1954), 471-501.
[4] W. D. Dunbar, Geometric orbifolds, Rev. Mat. Univ. Complut. Madrid, 1 (1988), 67-99.
[5] H. M. Hilden, M. T. Lozano and J. M. Montesinos-Amilibia, On a Remarkable Polyhedron Geometrising the Figure Eight Knot Cone Manifolds, J. Math. Sci. Univ. Tokyo, 2 (1995), no. 3, 501-561.
[6] A. A. Kolpakov and A. D. Mednykh, Spherical structures on torus knots and links, Siberian Mathematical Journal, Vol. 50, No. 5. (2009), 856-866.
[7] A. Mednykh, A. Rasskazov, On the structure of the canonical fundamental set for the 2-bridge link orbifolds, Universität Bielefeld, Sonderforshungsbereich 343, "Discrete Structuren in der Mathematik", Preprint, 98-062.
[8] A. Mednykh, A. Rasskazov, Volumes and degeneration of cone-structures on the figure-eight knot, Tokyo Journal of Mathematics, 29, no. 2 (2007), 445-464.
[9] J. Milnor, The Schläfli differential equality, in: Collected papers, Vol 1, Houston: Publish or Perish, 1994.
[10] J. Minkus, The branched cyclic coverings of 2 bridge knots and links, Mem. Amer. Math. Soc., 35 (1982), no. 255.
[11] E. Molnár, J. Szirmai, A. Vesnin, Projective metric realizations of conemanifolds with singularities along 2-bridge knots and links, J. Geom., 95 (2009), no. 1-2, 91-133.
[12] W. P. Neumann, Notes on Geometry and 3-Manifolds, with appendix by Paul Norbury, in: Low Dimensional Topology, Böröczky, Neumann, Stipsicz, Eds., Bolyai Society Mathematical Studies 8 (1999), 191-267.
[13] J. Porti, Spherical cone structures on 2-bridge knots and links, Kobe J. Math., 21 (2004), no. 1-2, 61-70.
[14] J. G. Ratcliffe, Foundations of hyperbolic manifolds, Graduate Texts in Mathematics, 149, Springer-Verlag, New York,(2006), 779 pages.
[15] D. Rolfsen, Knots and links, Publish or Perish Inc., Berkely Ca., 1976.
[16] H. Seifert, C. Weber, Die beiden Dodecaederräme, Math. Z., Bd 37, S. 237-253, 1933.
[17] L. Schläfli, Theorie der vielfachen Kontinuität, In: Gesammelte mathematishe Abhandlungen 1, Basel: Birkhäuser 1950.
[18] A. J. Sieradski, Combinatorial squashings, 3-manifolds, and the third homology of groups, Invent. Math., 84 (1986), 121-139.
[19] W. Thurston, The Geometry and Topology of Three-Manifolds, Princeton University, 1980.
[20] E. B. Vinberg (ed.), Geometry II, New York: Springer-Verlag, 1993.

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# ON SOME NEW POSITIONAL SMALL INDUCTIVE DIMENSIONS FOR UNIFORM SPACES* 

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#### Abstract

The paper defines new positional dimension-like functions of the type ind for uniform spaces and presents several theorem concerning the standard properties of dimension theory for these functions. Finally, some open questions concerning these functions are given.


## 1. Preliminaries

It was observed in the book of Gillman-Jerison (see [9]) that a better dimension theory can be built out, for covering dimension, if we do not consider all open sets, but only that base of them, that consist of the cozero sets (i.e., where a continuous function is not 0 ). Then many statements, originally valid for normal spaces, extend to all Tychonoff spaces. Later it was realized, by Charalambous, that the same idea can be extended much further: for all uniform spaces one can define covering dimension by (uniform) cozero sets (i.e., where a uniformly continuous function is not 0 ). Of course, this theory of dimension depends only on the system of cozero sets, not on the actual uniformity. Nevertheless, the usual setting is that of uniform spaces, these theorems are considered to belong to the theory of uniform spaces.

The paper intends to contribute to this theory. Its setting is a pair of uniform spaces, one a subspace of the other one, for which there are defined two basic types of small inductive dimension-like functions, and several theorems are proved for them. The paper follows rather closely the presentation of the paper [8] who investigated the corresponding theorems for topological spaces.

[^1]However, for topological spaces [8] contains several examples for distinctness of the defined dimension like functions, which, however, are not Tychonoff spaces, therefore these examples do not exist for the case of uniform spaces.

The set of real numbers with the natural metric is denoted by $R$ and the uniformity induced on $R$ by this metric is denoted by $\mathcal{U}_{R}$. Also, by a base of a topological space we mean an open base.

Let $(X, \mathcal{U})$ be a uniform space (see [11]). The uniformity $\mathcal{U}$ induces a topology $\tau_{\mathcal{U}}$ on $X$. More exactly we have $\tau_{\mathcal{U}}=\{U \subseteq X$ : for every $x \in U$ there exists $V \in \mathcal{U}$ such that $\{y \in X:(x, y) \in V\} \subseteq U\}$. Also, a map $f: X \rightarrow R$ is called uniformly continuous if it is uniformly continuous with respect to the uniformities $\mathcal{U}$ and $\mathcal{U}_{R}$, where $\mathcal{U}_{R}$ is the usual metric uniformity of $R$, that is for every $U_{R} \in \mathcal{U}_{R}$, there exists $U \in \mathcal{U}$ such that for every $\left(x, x^{\prime}\right) \in U$ we have $\left(f(x), f\left(x^{\prime}\right)\right) \in U_{R}$.

Let $(X, \mathcal{U})$ be a uniform space and $A$ a subset of $X$. The subset $A$ is called $\mathcal{U}$-cozero set if there exists a bounded uniformly continuous map $f: X \rightarrow R$ and an open set $V$ of $R$ such that $A=f^{-1}(V)$. The complement of a $\mathcal{U}$-cozero set is called $\mathcal{U}$-zero.

Let $(X, \mathcal{U})$ be a uniform space and $M \subseteq X$. The pair $\left(M, \mathcal{U}_{M}\right)$, where

$$
\mathcal{U}_{M}=\{(M \times M) \cap V: V \in \mathcal{U}\}
$$

is a uniform space which is called a subspace of the uniform space $(X, \mathcal{U})$. We note that a subset $B$ of $M$ is $\mathcal{U}_{M}$-cozero (respectively, $\mathcal{U}_{M}$-zero) if and only if for some $\mathcal{U}$-cozero (respectively, $\mathcal{U}$-zero) set $A$ of $X$, we have $B=A \cap M$ (see [2] and [11]).

We recall some properties of $\mathcal{U}$-cozero and $\mathcal{U}$-zero sets (see [1], [2], [3], and [4]).

1. If $\operatorname{Coz} \mathcal{U}$ is the collection of all $\mathcal{U}$-cozero sets, then this set is a base for the topology $\tau_{\mathcal{U}}$.
2. If $\mathcal{U}$ is induced by a metric, then the set $\mathrm{Coz} \mathcal{U}$ coincides with the topology of the topological space $X$, that is $\operatorname{Coz} \mathcal{U}=\tau_{\mathcal{U}}$.
3. If $\left(X, \tau_{\mathcal{U}}\right)$ is Lindelöf, then $\operatorname{Coz} \mathcal{U}$ is the collection of all cozero sets of $X$.
4. The union of a countable collection of $\mathcal{U}$-cozero sets is a $\mathcal{U}$-cozero set and the intersection of a finite collection of $\mathcal{U}$-cozero sets is a $\mathcal{U}$-cozero set.
5. For any two $\mathcal{U}$-cozero sets $U_{1}, U_{2}$ with $U_{1} \cup U_{2}=X$, there are disjoint $\mathcal{U}$-cozero sets $H_{1}, H_{2}$ such that $U_{1} \cup H_{1}=U_{2} \cup H_{2}=X$.
6. A $\mathcal{U}$-cozero set is a union of a countable collection of $\mathcal{U}$-zero sets.
7. If $F_{1}, F_{2}$ are disjoint $\mathcal{U}$-zero sets, then there are disjoint $\mathcal{U}$-cozero sets $U_{1}$, $U_{2}$ such that $F_{1} \subseteq U_{1}$ and $F_{2} \subseteq U_{2}$.
8. If $F \subseteq U$ with $F$ being a $\mathcal{U}$-zero set and $U$ being a $\mathcal{U}$-cozero set, then there are a $\mathcal{U}$-cozero set $H$ and a $\mathcal{U}$-zero set $\widetilde{H}$ such that $F \subseteq H \subseteq \widetilde{H} \subseteq U$.
9. If $F_{1}, F_{2}$ are disjoint $\mathcal{U}$-zero sets, then there are $\mathcal{U}$-cozero sets $H_{1}, H_{2}$ and disjoint $\mathcal{U}$-zero sets $\bar{H}_{1}, \bar{H}_{2}$ such that $F_{1} \subseteq H_{1} \subseteq \widetilde{H}_{1}$ and $F_{2} \subseteq H_{2} \subseteq \widetilde{H}_{2}$.

Also, if $F$ is a $\mathcal{U}$-zero set and $G$ a $\mathcal{U}$-cozero set of $X$, then the following are true (see [2] and [3]):
(i) $F \backslash G$ is a $\mathcal{U}$-zero set while $G \backslash F$ is a $\mathcal{U}$-cozero set of $X$.
(ii) If $E$ is a $\mathcal{U}_{F}$-zero set of $F$, then $E$ is a $\mathcal{U}$-zero set of $X$.
(iii) If $H$ is a $\mathcal{U}_{G}$-cozero set of $G$, then $H$ is a $\mathcal{U}$-cozero set of $X$.
(iv) If $H_{1}, H_{2}$ are disjoint $\mathcal{U}_{M}$-cozero sets of a subset $M$ of $X$, then there are disjoint $\mathcal{U}$-cozero sets $G_{1}, G_{2}$ of $X$ such that $H_{1} \subseteq G_{1}$ and $H_{2} \subseteq G_{2}$.
(v) If $x \in G$, where $G$ is a $\mathcal{U}$-cozero set, then there are a $\mathcal{U}$-cozero set $H$ and a $\mathcal{U}$-zero set $F$ with $x \in H \subseteq F \subseteq G$.

In this paper by $\omega$ we denote the set $\{0,1,2, \ldots\}$ and consider the symbols -1 and $\infty$ considering that: (1) $-1<n$, for every $n \in(\omega \cup\{\infty\})$, (2) $n<\infty$, for every $n \in(\omega \cup\{-1\}),(3)-1+n=n+(-1)=n$, for every $n \in(\omega \cup\{-1, \infty\})$, and $(4) \infty+n=n+\infty=\infty$, for every $n \in(\omega \cup\{-1, \infty\})$.

In [13], [14], [5], [6], [10], and [8] the so called relative and positional dimension-like functions are studied. In this paper we define new positional dimension-like functions for uniform spaces. Our presentation will follow the presentation in the paper [8] rather closely. In particular, in section 2 of this paper we define some new positional dimension-like functions and give some relations among them. In section 3 we consider subspace theorems and in section 4 sum theorems. In section 5 we give some results connected with the dimension-like functions of spaces. Finally, we give some open questions for these functions.

## 2. Some new positional small inductive dimension-like functions of the type $\mathcal{U}$-ind

In this section we define and study new positional small inductive dimension-like functions of the type $\mathcal{U}$-ind of uniform spaces.
Definition 2.1. Let $(X, \mathcal{U})$ be a uniform space, $Q \subseteq X$. and $\mathcal{B}$ a subset of Coz $\mathcal{U}$ (containing the empty set and $X$ ). The family $\mathcal{B}$ is said to be a $\mathcal{U}$-p-base for $Q$ in $X$ if the set $\{Q \cap U: U \in \mathcal{B}\}$ is a base for the subspace $Q$ of $\left(X, \tau_{\mathcal{U}}\right)$. The family $\mathcal{B}$ is said to be a $\mathcal{U}$-pos-base for $Q$ in $X$ if for every $x \in Q$ and a $\mathcal{U}$-cozero set $U$
with $x \in U$ there exists an element $V$ of $\mathcal{B}$ such that $x \in V \subseteq U$. The family $\mathcal{B}$ is said to be a $\mathcal{U}$-ps-base for $Q$ in $X$ if $\mathcal{B}$ is a base for the space $\left(X, \tau_{\mathcal{U}}\right)$.

Clearly, we have the implications:

$$
\mathcal{U} \text {-ps-base } \Longrightarrow \mathcal{U} \text {-pos-base } \Longrightarrow \mathcal{U} \text {-p-base }
$$

Since $\operatorname{Coz} \mathcal{U}$ is a base for the topology $\tau_{\mathcal{U}}$ of $X$, the requirement that $U$ is a $\mathcal{U}$ cozero set in the definition of $\mathcal{U}$-pos-base can be replaced by the condition that $U \in \tau_{\mathcal{U}}$. That is, this property is equivalent to the fact that $\mathcal{B}$ is an outer base for $Q$ in the topological space $\left(X, \tau_{\mathcal{U}}\right)$.
Definition 2.2. (See [2], [3], and [1]) We denote by $\mathcal{U}$-ind the dimension-like function whose domain is the class of all uniform spaces X and whose range is the set $\omega \cup\{-1, \infty\}$, where $\omega$ is the first infinite cardinal, satisfying the following conditions:
(i) $\mathcal{U}$ - ind $X=-1$ if and only if $X=\emptyset$.
(ii) $\mathcal{U}$-ind $X \leq n$, where $n \in \omega$ if whenever $x \in U$ with $U$ a $\mathcal{U}$-cozero set there are a $\mathcal{U}$-cozero set $V$ and a $\mathcal{U}$-zero set $\widetilde{V}$ with $x \in V \subseteq \widetilde{V} \subseteq U$ and $\mathcal{U}-\operatorname{ind}(\widetilde{V}-V) \leq n-1$.

For a subspace $Y$ of $X$, by $\mathcal{U}$ - ind $Y$ we mean $\mathcal{U}_{Y}$ - ind $Y$.
(iii) $\mathcal{U}$ - ind $X=\infty$ if for each $n \in \omega \cup\{-1\}, \mathcal{U}$ - ind $X \not \leq n$.
(iv) $\mathcal{U}$ - ind $X=n$ if $\mathcal{U}$ - ind $X \leq n$ and $\mathcal{U}$ - ind $X \not \leq n-1$.

Definition 2.3. We denote by $\mathcal{U}-\mathrm{p}_{0}$-ind the dimension-like function whose domain is the class of all pairs $(Q, X)$, where $Q$ is a subset of a uniform space $X$, and whose range is the set $\omega \cup\{-1, \infty\}$ satisfying the following conditions:
(i) $\mathcal{U}-\mathrm{p}_{0}-\operatorname{ind}(Q, X)=-1$ if and only if $X=\emptyset$.
(ii) $\mathcal{U}$ - $p_{0}-\operatorname{ind}(Q, X) \leq n$, where $n \in \omega$ if either $Q=\emptyset$ or there exists a $\mathcal{U}$-p-base $\mathcal{B}_{n}$ (depending on $n$ ) for $Q$ in $X$ such that for every $x \in Q \cap U$ with $U$ a $\mathcal{U}$-cozero set of $X$ there are a $V \in \mathcal{B}_{n}$ and a $\mathcal{U}$-zero set $\widetilde{V}$ of $X$ with $x \in V \subseteq \widetilde{V} \subseteq U$ and

$$
\mathcal{U}-\mathrm{p}_{0}-\operatorname{ind}(Q \cap(\widetilde{V}-V), \widetilde{V}-V) \leq n-1
$$

For a subspace $Y$ of $X$, by $\mathcal{U}-\mathrm{p}_{0}-\operatorname{ind}(Q \cap Y, Y)$ we mean $\mathcal{U}_{Y}-\mathrm{p}_{0}-\operatorname{ind}(Q \cap Y, Y)$.
(iii) $\mathcal{U}-\mathrm{p}_{0}-\operatorname{ind}(Q, X)=\infty$ if for each $n \in(\omega \cup\{-1\})$,

$$
\mathcal{U}-\mathrm{p}_{0}-\operatorname{ind}(Q, X) \not \leq n
$$

(iv) $\mathcal{U}-\mathrm{p}_{0}-\operatorname{ind}(Q, X)=n$ if

$$
\mathcal{U}-\mathrm{p}_{0}-\operatorname{ind}(Q, X) \leq n
$$

and

$$
\mathcal{U}-\mathrm{p}_{0}-\operatorname{ind}(Q, X) \not \leq n-1 .
$$

Definition 2.4. We denote by $\mathcal{U}-\mathrm{p}_{1}$-ind the dimension-like function whose domain is the class of all pairs $(Q, X)$, where $Q$ is a subset of a uniform space $X$, and whose range is the set $\omega \cup\{-1, \infty\}$ satisfying the following conditions:
(i) $\mathcal{U}-\mathrm{p}_{1}-\operatorname{ind}(Q, X)=-1$ if and only if $Q=\emptyset$.
(ii) $\mathcal{U}$-p p $_{1}-\operatorname{ind}(Q, X) \leq n$, where $n \in \omega$, if there exists a $\mathcal{U}$-p-base $\mathcal{B}_{n}$ (depending on $n$ ) for $Q$ in $X$ such that whenever $x \in Q$ and $x \in U$ with $U$ a $\mathcal{U}$-cozero set of $X$ there are a $V \in \mathcal{B}_{n}$ and a $\mathcal{U}$-zero set $\widetilde{V}$ of $X$ with $x \in V \subseteq \widetilde{V} \subseteq U$ and

$$
\mathcal{U}-\mathrm{p}_{1}-\operatorname{ind}(Q \cap(\widetilde{V}-V), X) \leq n-1
$$

(iii) $\mathcal{U}-\mathrm{p}_{1}-\operatorname{ind}(Q, X)=\infty$ if for each $n \in \omega \cup\{-1\}$,

$$
\mathcal{U}-\mathrm{p}_{1}-\operatorname{ind}(Q, X) \not \leq n .
$$

(iv) $\mathcal{U}-\mathrm{p}_{1}-\operatorname{ind}(Q, X)=n$ if

$$
\mathcal{U}-\mathrm{p}_{1}-\operatorname{ind}(Q, X) \leq n
$$

and

$$
\mathcal{U}-\mathrm{p}_{1}-\operatorname{ind}(Q, X) \not \leq n-1
$$

Definition 2.5. Let $i \in\{0,1\}$. If in Definitions 2.3 and 2.4 instead of the $\mathcal{U}$-p-base $\mathcal{B}$ we consider a $\mathcal{U}$-pos-base (respectively, a $\mathcal{U}$-ps-base), then the dimension-like function $\mathcal{U}-\mathrm{p}_{\mathrm{i}}$ - ind will be denoted by $\mathcal{U}-$ pos $_{\mathrm{i}}-$ ind (respectively, by $\mathcal{U}-\mathrm{ps}_{\mathrm{i}}$ - ind).

Remarks. (1) It is clear that all these dimension-like functions (in Definitions $2.2,2.3,2.4$, and 2.5 ) depend only on the cozero-structure $\mathcal{U}$.
(2) The requirement that $U$ is a $\mathcal{U}$-cozero set in the Definitions 2.2(ii), 2.3(ii), and 2.4(ii) can be replaced by the condition that $U \in \tau_{\mathcal{U}}$.

Proposition 2.1. Let $(X, \mathcal{U})$ be a uniform space. Then, for every subset $Q$ of $X$ we have

$$
\mathcal{U}-\operatorname{ind} Q \leq \mathcal{U}-\mathrm{p}_{\mathrm{i}}-\operatorname{ind}(Q, X), i \in\{0,1\}
$$

Proof. We prove that

$$
\begin{equation*}
\mathcal{U}-\text { ind } Q \leq \mathcal{U}-\mathrm{p}_{1}-\operatorname{ind}(Q, X) \tag{1}
\end{equation*}
$$

The proof of the other inequality is similar. Let

$$
\mathcal{U}-\mathrm{p}_{1}-\operatorname{ind}(Q, X)=n \in(\omega \cup\{-1, \infty\})
$$

The relation (1) is clear if $n=-1$ or $n=\infty$. Suppose that $n \in \omega$ and that (1) is true for every pair $\left(Q^{Y}, Y\right)$ with $\mathcal{U}$ - $\mathrm{p}_{1}-\operatorname{ind}\left(Q^{Y}, Y\right)<n$. Since $\mathcal{U}$ - $\mathrm{p}_{1}-\operatorname{ind}(Q, X)=$
$n$, there exists a $\mathcal{U}$-p-base $\mathcal{B}_{n}$ for $Q$ in $X$ such that for every $x \in Q$ and $x \in U$ with $U$ a $\mathcal{U}$-cozero set there are a $V \in \mathcal{B}_{n}$ and a $\mathcal{U}$-zero set $\widetilde{V}$ of $X$ with

$$
x \in V \subseteq \widetilde{V} \subseteq U
$$

and

$$
\mathcal{U}-\mathrm{p}_{1}-\operatorname{ind}(Q \cap(\widetilde{V}-V), X) \leq n-1 .
$$

To prove that $\mathcal{U}$-ind $Q \leq n$ it suffices to show that for every $x \in Q$ and $x \in U^{Q}$ with $U^{Q}$ a $\mathcal{U}$-cozero subset of $Q$ there are a $\mathcal{U}_{Q}$-zero set $V^{Q}$ of $Q$ and a $\mathcal{U}_{Q}$-cozero set $\widetilde{V}^{Q}$ of $Q$ with $x \in V^{Q} \subseteq \widetilde{V}^{Q} \subseteq U^{Q}$ and

$$
\mathcal{U}-\operatorname{ind}\left(\widetilde{V}^{Q}-V^{Q}\right) \leq n-1
$$

Let $x \in Q$ and $x \in U^{Q}$ with $U^{Q}$ a $\mathcal{U}$-cozero set of $Q$. Then, there is a $\mathcal{U}$ cozero set $U$ of $X$ such that $U^{Q}=U \cap Q$. Clearly, $x \in U$. Thus, there exist a $V \in \mathcal{B}_{n}$ and a $\mathcal{U}$-zero set $\widetilde{V}$ of $X$ with $x \in V \subseteq \widetilde{V} \subseteq U$ and

$$
\mathcal{U}-\mathrm{p}_{1}-\operatorname{ind}(Q \cap(\widetilde{V}-V), X) \leq n-1
$$

We consider the subsets

$$
V^{Q}=V \cap Q
$$

and

$$
\widetilde{V}^{Q}=\widetilde{V} \cap Q
$$

of $Q$. Clearly, the above sets $V^{Q}$ and $\bar{V}^{Q}$ are $\mathcal{U}_{Q}$-cozero and $\mathcal{U}_{Q}$-zero sets of $X$, respectively,

$$
x \in V^{Q} \subseteq \widetilde{V}^{Q} \subseteq U^{Q}
$$

and

$$
\widetilde{V}^{Q}-V^{Q}=(\widetilde{V}-V) \cap Q .
$$

By induction, we have

$$
\begin{aligned}
\mathcal{U}-\operatorname{ind}\left(\widetilde{V}^{Q}-V^{Q}\right) & =\mathcal{U}-\operatorname{ind}((\widetilde{V}-V) \cap Q) \\
& \leq \mathcal{U}-\mathrm{p}_{1}-\operatorname{ind}(Q \cap(\widetilde{V}-V), X) \leq n-1 .
\end{aligned}
$$

Thus, $\mathcal{U}$ - $\operatorname{ind}(Q) \leq n$.
Proposition 2.2. Let $(X, \mathcal{U})$ be a uniform space. Then, for every subset $Q$ of $X$ we have

$$
\mathcal{U}-\mathrm{p}_{\mathrm{i}}-\operatorname{ind}(Q, X) \leq \mathcal{U}-\operatorname{pos}_{\mathrm{i}}-\operatorname{ind}(Q, X) \leq \mathcal{U}-\mathrm{ps}_{\mathrm{i}}-\operatorname{ind}(Q, X), i \in\{0,1\}
$$

Proof. We prove by induction only the inequality

$$
\begin{equation*}
\mathcal{U}-\mathrm{p}_{1}-\operatorname{ind}(Q, X) \leq \mathcal{U}-\operatorname{pos}_{1}-\operatorname{ind}(Q, X), \tag{2}
\end{equation*}
$$

where $\mathcal{U}-\operatorname{pos}_{1}-\operatorname{ind}(Q, X)=n \in(\omega \cup\{-1, \infty\})$. The proofs of all other cases are similar. The relation (2) is clear if $n=-1$ or $n=\infty$. Let $n \in \omega$. Then, there exists a $\mathcal{U}$-pos-base $\mathcal{B}_{n}$ for $Q$ in $X$ such that for every $x \in Q$ and $x \in U$ with $U$ a $\mathcal{U}$-cozero set of $X$ there are a $V \in \mathcal{B}_{n}$ and a $\mathcal{U}$-zero set $\widetilde{V}$ with $x \in V \subseteq \widetilde{V} \subseteq U$ and

$$
\mathcal{U}-\operatorname{pos}_{1}-\operatorname{ind}(Q \cap(\widetilde{V}-V), X) \leq n-1 .
$$

By induction, we have

$$
\mathcal{U}-\mathrm{p}_{1}-\operatorname{ind}(Q \cap(\widetilde{V}-V), X) \leq \mathcal{U}-\operatorname{pos}_{1}-\operatorname{ind}(Q \cap(\widetilde{V}-V), X) \leq n-1
$$

Since $\mathcal{B}_{n}$ is also a $\mathcal{U}$-p-base for $Q$ in $X$, we have

$$
\mathcal{U}-\mathrm{p}_{1}-\operatorname{ind}(Q, X) \leq n .
$$

Proposition 2.3. Let $(X, \mathcal{U})$ be a uniform space. Then, for every subset $Q$ of $X$ we have

$$
\mathcal{U}-\mathrm{ps}_{0}-\operatorname{ind}(Q, X) \leq \mathcal{U}-\operatorname{ind} X
$$

Proof. We prove the inequality by induction. The relation is clear if $n=-1$ or $n=\infty$. Let $\mathcal{U}$-ind $X=n \in \omega$. We prove that $\mathcal{U}-\mathrm{ps}_{0}-\operatorname{ind}(Q, X) \leq n$. If $Q=\emptyset$, then the proof is clear. Suppose that $Q \neq \emptyset$. We set $\mathcal{B}_{n}=\operatorname{Coz} \mathcal{U}$. Then, $\mathcal{B}_{n}$ is a $\mathcal{U}$-ps-base. Let $x \in Q$ and $x \in U$ with $U$ a $\mathcal{U}$-cozero set of $X$. Then, there are a $\mathcal{U}$-cozero set $V$ of $X$ and a $\mathcal{U}$-zero set $\bar{V}$ of $X$ such that $x \in V \subseteq \widetilde{V} \subseteq U$ and

$$
\mathcal{U}-\operatorname{ind}(\widetilde{V}-V) \leq n-1
$$

Also, by induction, we have

$$
\mathcal{U}-\operatorname{ps}_{0}-\operatorname{ind}(Q \cap(\bar{V}-V), \bar{V}-V) \leq \mathcal{U}-\operatorname{ind}(\bar{V}-V) \leq n-1
$$

Thus,

$$
\mathcal{U}-\mathrm{ps}_{0}-\operatorname{ind}(Q, X) \leq n .
$$

Examples. (1) Let $R^{n}$ be the $n$-dimensional Euclidean space and $d$ the usual metric on $R^{n}$. Then, the sets of the form

$$
V(A, \varepsilon)=\{(x, y): d(x, y)=0, \text { or } x \notin A, y \notin A \text { and } d(x, y)<\varepsilon\},
$$

where $A$ is a finite subset of $R^{n}$ and $\varepsilon$ is a positive real number, form a base for a uniformity $\mathcal{E}$ on $R^{n}$. It is known that $\mathcal{E}$ - ind $R^{n}=0$ (see [3]).

Let $Q$ be a non empty subset of $R^{n}$. Then, by Proposition 6 of [3], we have $\mathcal{E}$ - ind $Q \leq \mathcal{E}$ - ind $R^{n}=0$ and, therefore, since $Q \neq \emptyset, \mathcal{E}$ - ind $Q=0$. Thus, by Propositions 2.2 and 2.3, we have

$$
\begin{aligned}
\mathcal{E}-\operatorname{ind} Q & =\mathcal{E}-\mathrm{p}_{0}-\operatorname{ind}\left(Q, R^{n}\right) \\
& =\mathcal{E}-\operatorname{pos}_{0}-\operatorname{ind}\left(Q, R^{n}\right) \\
& =\mathcal{E}-\operatorname{ps}_{0}-\operatorname{ind}\left(Q, R^{n}\right) \\
& =\mathcal{E}-\operatorname{ind} R^{n}=0 .
\end{aligned}
$$

(2) Let $\mathcal{U}$ be the uniformity on the 2-dimensional Euclidean space $R^{2}$ generated by sets of the form

$$
V(A, \varepsilon)=\{(x, y): x=y, \text { or } x \notin A, y \notin A \text { and } d(x, y)<\varepsilon\}
$$

where $d$ is the usual metric on $R^{2}, A$ is a finite subset of $R^{2}-\{a\}, a$ a fixed point of $R^{2}$, and $\varepsilon$ a positive real number. Then, $\mathcal{U}-\operatorname{ind}\left(R^{2}\right)=1$ and $\mathcal{U}$ - ind $\left(R^{2}-\{a\}\right)=0$ (see [3]).

Let $Q$ be a non empty subset of $R^{2}-\{a\}$. Then, by Propositions 2.2 and 2.3 and Proposition 6 of [3], we have

$$
\begin{aligned}
\mathcal{U}-\mathrm{p}_{0}-\operatorname{ind} Q & =\mathcal{U}-\mathrm{p}_{0}-\operatorname{ind}\left(Q, R^{2}-\{a\}\right) \\
& =\mathcal{U}-\operatorname{pos}_{0}-\operatorname{ind}\left(Q, R^{2}-\{a\}\right) \\
& =\mathcal{U}-\operatorname{ps}_{0}-\operatorname{ind}\left(Q, R^{2}-\{a\}\right) \\
& =\mathcal{U}-\operatorname{ind}\left(R^{2}-\{a\}\right) \\
& =0<\mathcal{U}-\operatorname{ind} R^{2}=1
\end{aligned}
$$

(3) Let $\mathcal{U}$ be the uniformity of the Example (2). Then, we have $\mathcal{U}$-ind $\{a\}=0$ (see [3]). Also, by Definition 2.4, we have

$$
\begin{aligned}
\mathcal{U}-\operatorname{ind}\{a\}=0 & =\mathcal{U}-\mathrm{p}_{1}-\operatorname{ind}\left(\{a\}, R^{2}\right) \\
& =\mathcal{U}-\operatorname{pos}_{1}-\operatorname{ind}\left(\{a\}, R^{2}\right) \\
& =\mathcal{U}-\operatorname{ps}_{1}-\operatorname{ind}\left(\{a\}, R^{2}\right) \\
& <\mathcal{U}-\operatorname{ind} R^{2}=1
\end{aligned}
$$

Proposition 2.4. Let $(X, \mathcal{U})$ be a uniform space. Then, we have

$$
\mathcal{U}-\mathrm{ps}_{0}-\operatorname{ind}(X, X)=\mathcal{U}-\operatorname{ind}(X)
$$

Proof. The inequality

$$
\mathcal{U}-\operatorname{ind}(X) \leq \mathcal{U}-\operatorname{ps}_{0}-\operatorname{ind}(X, X)
$$

is clear. Also, by Proposition 2.3, we have

$$
\mathcal{U}-\mathrm{ps}_{0}-\operatorname{ind}(X, X) \leq \mathcal{U}-\operatorname{ind}(X) .
$$

Thus,

$$
\mathcal{U}-\operatorname{ps}_{0}-\operatorname{ind}(X, X)=\mathcal{U}-\operatorname{ind}(X) .
$$

Propositions 2.1, 2.2, 2.3, and 2.4 imply the following proposition.
Proposition 2.5. For every uniform space $X$ we have

$$
\mathcal{U}-\mathrm{p}_{0}-\operatorname{ind}(\mathrm{X}, \mathrm{X})=\mathcal{U}-\operatorname{pos}_{0}-\operatorname{ind}(\mathrm{X}, \mathrm{X})=\mathcal{U}-\mathrm{ps}_{0}-\operatorname{ind}(\mathrm{X}, \mathrm{X})=\mathcal{U}-\operatorname{ind}(\mathrm{X}) .
$$

Remark. The relations between the considered positional dimension-like functions of the type $\mathcal{U}$-ind are summarized in the following diagram, where " $\rightarrow$ " means " $\leq$ ".


## Questions.

1. Let $i \in\{0,1\}$. Find a uniform space $(X, \mathcal{U})$ and a subset $Q$ of $X$ such that all the dimension-like functions $\mathcal{U}$ - $\operatorname{ind}(Q), \mathcal{U}-\mathrm{p}_{\mathrm{i}}-\operatorname{ind}(Q, X), \mathcal{U}$ - $\operatorname{pos}_{\mathrm{i}}-\operatorname{ind}(Q, X)$, and $\mathcal{U}-\mathrm{ps}_{\mathrm{i}}-\operatorname{ind}(Q, X)$ to be different.
2. In [1] M. Charalambous gave an example of a uniform space $\left(Q_{n}, \mathcal{M}^{Q_{n}}\right)$ such that

$$
\mathcal{M}^{Q_{n}-\operatorname{ind}}\left(Q_{n}\right)=n .
$$

Find subsets $K$ of $Q_{n}$ such that the defined dimensions of this paper to be between of 0 and $n$ ?
3. In [1] M. Charalambous gave an example of a uniform space $\left(S, \mathcal{M}^{S}\right)$ such that

$$
\mathcal{M}^{S}-\operatorname{ind}(S)=2
$$

Find subsets $K$ of $S$ such that the defined dimensions of this paper to be 1 ?

## 3. Subspace theorems

Proposition 3.1. Let $i \in\{0,1\}$ and $Q, K$ be two subsets of a uniform space $(X, \mathcal{U})$ with $K \subseteq Q$. Then,
(a) $\mathcal{U}-\mathrm{p}_{\mathrm{i}}-\operatorname{ind}(K, X) \leq \mathcal{U}-\mathrm{p}_{\mathrm{i}}-\operatorname{ind}(Q, X)$,
(b) $\mathcal{U}$ - $\operatorname{pos}_{\mathrm{i}}-\operatorname{ind}(K, X) \leq \mathcal{U}$ - posi $_{\mathrm{i}}-\operatorname{ind}(Q, X)$, and
(c) $\mathcal{U}-\mathrm{ps}_{\mathrm{i}}-\operatorname{ind}(K, X) \leq \mathcal{U}-\mathrm{ps}_{\mathrm{i}}-\operatorname{ind}(Q, X)$.

Proof. We prove the inequality

$$
\begin{equation*}
\mathcal{U}-\mathrm{p}_{1}-\operatorname{ind}(K, X) \leq \mathcal{U}-\mathrm{p}_{1}-\operatorname{ind}(Q, X) \tag{3}
\end{equation*}
$$

The proofs of all other inequalities are similar.
Let $\mathcal{U}-\mathrm{p}_{1}-\operatorname{ind}(Q, X)=n \in(\omega \cup\{-1, \infty\})$. The relation (3) is clear if $n=-1$ or $n=\infty$. Let $n \in \omega$ and suppose that (3) is true for any $K \subseteq Q \subseteq X$ with $\mathcal{U}$-p $p_{1}-\operatorname{ind}(Q, X)<n$. There exists a $\mathcal{U}$-p-base $\mathcal{B}_{n}$ for $Q$ in $X$ such that for every $x \in Q$ and $x \in U$ with $U$ a $\mathcal{U}$-cozero set of $X$ there exist a $V \in \mathcal{B}_{n}$ and a $\mathcal{U}$-zero set $\widetilde{V}$ of $X$ with $x \in V \subseteq \widetilde{V} \subseteq U$ and

$$
\mathcal{U}-\mathrm{p}_{1}-\operatorname{ind}(Q \cap(\widetilde{V}-V), X) \leq n-1
$$

For the $\mathcal{U}$-p-base $B_{n}$ of $X$ we have: if $x \in K \subseteq Q$ and $x \in U$ with $U$ a $\mathcal{U}$-cozero set of $X$, then there exist a $V \in \mathcal{B}_{n}$ and a $\mathcal{U}$-zero set $\widetilde{V}$ of $X$ with

$$
\mathcal{U}-\mathrm{p}_{0}-\operatorname{ind}(Q \cap(\widetilde{V}-V), X) \leq n-1 .
$$

By assumption, we have

$$
\begin{array}{r}
\mathcal{U}-\mathrm{p}_{0}-\operatorname{ind}(K \cap(\widetilde{V}-V), X) \\
\leq \mathcal{U}-\mathrm{p}_{0}-\operatorname{ind}(Q \cap(\widetilde{V}-V), X) \\
\leq n-1
\end{array}
$$

Also, the set $\mathcal{B}_{n}$ is a $\mathcal{U}$-p-base for $K$ in $X$. Thus,

$$
\mathcal{U}-\mathrm{p}_{1}-\operatorname{ind}(K, X) \leq n
$$

Proposition 3.2. Let $i \in\{0,1\}$, $Y$ be a subspace of a uniform space $(X, \mathcal{U})$, and $Q \subseteq Y$. Then,
(a) $\mathcal{U}$ - $\mathrm{p}_{\mathrm{i}}-\operatorname{ind}(Q, Y) \leq \mathcal{U}-\mathrm{p}_{\mathrm{i}}-\operatorname{ind}(Q, X)$,
(b) $\mathcal{U}$-posi $-\operatorname{ind}(Q, Y) \leq \mathcal{U}$-posi $-\operatorname{ind}(Q, X)$, and
(c) $\mathcal{U}-$ ps $_{i}-\operatorname{ind}(Q, Y) \leq \mathcal{U}-\operatorname{ps}_{i}-\operatorname{ind}(Q, X)$.

Proof. We prove the inequality

$$
\begin{equation*}
\mathcal{U}-p s_{1}-\operatorname{ind}(Q, Y) \leq \mathcal{U}-p s_{1}-\operatorname{ind}(Q, X) \tag{4}
\end{equation*}
$$

The proofs of all other inequalities are similar. Let

$$
\mathcal{U}-\mathrm{ps}_{1}-\operatorname{ind}(Q, X)=n \in \omega(\cup\{-1, \infty\}) .
$$

The relation (4) is clear if $n=-1$ or $n=\infty$. Let $n \in \omega$ and suppose that (4) is true for any $Q \subseteq Y \subseteq X$ with $\mathcal{U}$-ps $1-\operatorname{ind}(Q, X)<n$. There exists a $\mathcal{U}$-ps-base $\mathcal{B}_{n}$ for $Q$ in $X$ such that for every $x \in Q$ and $x \in U$ with $U \mathcal{U}$-cozero set of $X$ there exists a $V \in \mathcal{B}_{n}$ and a $\mathcal{U}$-zero set set $\widetilde{V}$ of $X$ such that $x \in V \subseteq \widetilde{V} \subseteq U$ and

$$
\mathcal{U}-\mathrm{ps}_{1}-\operatorname{ind}(Q \cap(\widetilde{V}-V), X) \leq n-1 .
$$

We must prove that

$$
\mathcal{U}-\mathrm{ps}_{1}-\operatorname{ind}(Q, Y) \leq n .
$$

We consider the set $\mathcal{B}_{Y}=\left\{U \cap Y: U \in \mathcal{B}_{n}\right\}$. This set is a $\mathcal{U}_{Y}$-p-base of $Y$. Let $x \in Q$ and $U^{Y}$ be a $\mathcal{U}$-cozero set of $Y$ with $x \in U^{Y}$. Then, there exists a $\mathcal{U}$-cozero set $U$ of $X$ with $U^{Y}=U \cap Y$. Clearly, $x \in U$. Thus, there exist a $V \in \mathcal{B}_{n}$ and a $\mathcal{U}$-zero set $\widetilde{V}$ of $X$ with $x \in V \subseteq \widetilde{V} \subseteq U$ and

$$
\mathcal{U}-\mathrm{ps}_{1}-\operatorname{ind}(Q \cap(\widetilde{V}-V), X) \leq n-1 .
$$

We consider the sets $V^{Y}=V \cap Y$ and $\widetilde{V}^{Y}=\widetilde{V} \cap Y$. Then, $V^{Y} \in \mathcal{B}_{Y}$ and $\widetilde{V}^{Y}$ is a $\mathcal{U}_{Y}$-closed set of $Y$. Also, we have $\widetilde{V}^{Y}-V^{Y}=Y \cap(\widetilde{V}-V)$. Thus,

$$
\begin{aligned}
& \mathcal{U}-\mathrm{ps}_{1}-\operatorname{ind}\left(Q \cap\left(\widetilde{V}^{Y}-V^{Y}\right), Y\right)=\mathcal{U}-\mathrm{ps}_{1}-\operatorname{ind}(Q \cap Y \cap(\widetilde{V}-V), Y) \\
& \leq \mathcal{U}-\mathrm{ps}_{1}-\operatorname{ind}(Q \cap(\widetilde{V}-V), X) \leq n-1 .
\end{aligned}
$$

Therefore, since the set $\mathcal{B}_{Y}$ is a $\mathcal{U}$-ps-base for $Q$ in $Y$, we have $\mathcal{U}$ - $\mathrm{ps}_{1}-\operatorname{ind}(Q, Y) \leq n$.

Proposition 3.3. Let $\mathcal{U}, \mathcal{V}$ be two uniformities on a set $X$ such that induce the same topology on $X$ and $\operatorname{Coz} \mathcal{U} \subseteq \operatorname{Coz} \mathcal{V}$. If $i \in\{0,1\}$ and $Q$ is a subset of $X$, then,
(a) $\mathcal{V}$ - $\mathrm{p}_{\mathrm{i}}-\operatorname{ind}(Q, X) \leq \mathcal{U}-\mathrm{p}_{\mathrm{i}}-\operatorname{ind}(Q, X)$,
(b) $\mathcal{V}$ - $\operatorname{posi}_{i}-\operatorname{ind}(Q, X) \leq \mathcal{U}$-posi $-\operatorname{ind}(Q, X)$, and
(c) $\mathcal{V}-\mathrm{ps}_{\mathrm{i}}-\operatorname{ind}(Q, X) \leq \mathcal{U}-\mathrm{ps}_{\mathrm{i}}-\operatorname{ind}(Q, X)$.

Proof. We prove the inequality

$$
\begin{equation*}
\mathcal{V}-\mathrm{p}_{0}-\operatorname{ind}(Q, X) \leq \mathcal{U}-\mathrm{p}_{0}-\operatorname{ind}(Q, X) \tag{5}
\end{equation*}
$$

The proofs of all other inequalities are similar. Let

$$
\mathcal{U}-\mathrm{p}_{0}-\operatorname{ind}(Q, X)=n \in(\omega \cup\{-1, \infty\})
$$

The relation (5) is clear if $n=-1$ or $n=\infty$. Let $n \in \omega$ and suppose that (5) is true for any $Q \subseteq X$ with $\mathcal{U}$ - $\mathrm{p}_{0}$ - ind $(Q, X)<n$. There exists a $\mathcal{U}$-p-base $\mathcal{B}_{n}$ for $Q$ in $X$ such that for every $x \in Q$ and $x \in U$ with $U$ a $\mathcal{U}$-cozero set of $X$ there exist a $V \in \mathcal{B}_{n}$ and a $\mathcal{U}$-zero set $\widetilde{V}$ of $X$ with $x \in V \subseteq \widetilde{V} \subseteq U$ and

$$
\mathcal{U}-\mathrm{p}_{0}-\operatorname{ind}(Q \cap(\widetilde{V}-V), \widetilde{V}-V) \leq n-1
$$

Since $\operatorname{Coz} \mathcal{U} \subseteq \operatorname{Coz} \mathcal{V}$, we have that $\mathcal{B}_{n}$ is a $\mathcal{V}$-p-base for $Q$ in $X$. Let $x \in Q$ and $U$ be a $\mathcal{U}$-cozero set of $X$. Then, there exist a $V \in \mathcal{B}_{n}$ and a $\mathcal{U}$-zero set $\widetilde{V}$ of $x$ with $x \in V \subseteq \widetilde{V} \subseteq U$ and

$$
\mathcal{U}-\mathrm{p}_{0}-\operatorname{ind}(Q \cap(\widetilde{V}-V), \widetilde{V}-V) \leq n-1
$$

Since $\widetilde{V}$ is a $\mathcal{U}$-zero set, the set $X-\widetilde{V}$ is a $\mathcal{U}$-cozero set. Thus, this set is a $\mathcal{V}$-cozero set and, therefore, $\widetilde{V}$ is a $\mathcal{V}$-zero set. Moreover, by assumption, we have

$$
\begin{aligned}
& \mathcal{V}-\mathrm{p}_{0}-\operatorname{ind}(Q \cap(\widetilde{V}-V), \widetilde{V}-V) \\
& \leq \mathcal{U}-\mathrm{p}_{0}-\operatorname{ind}(Q \cap(\widetilde{V}-V), \widetilde{V}-V) \\
& \leq n-1
\end{aligned}
$$

Thus,

$$
\mathcal{V}-\mathrm{p}_{0}-\operatorname{ind}(Q, X) \leq n
$$

## 4. Sum theorems

Proposition 4.1. Let $(X, \mathcal{U})$ be a uniform space. Then for every two subsets $Q_{1}$ and $Q_{2}$ of $X$ we have:

$$
\begin{equation*}
\mathcal{U}-\operatorname{pos}_{0}-\operatorname{ind}\left(Q_{1} \cup Q_{2}, X\right) \leq \mathcal{U}-\operatorname{pos}_{0}-\operatorname{ind}\left(Q_{1}, X\right)+\mathcal{U}-\operatorname{pos}_{0}-\operatorname{ind}\left(Q_{2}, X\right) \tag{6}
\end{equation*}
$$

Proof. We prove the relation (6) by induction on $n$, where

$$
n=\mathcal{U}-\operatorname{pos}_{0}-\operatorname{ind}\left(Q_{1}, X\right)+\mathcal{U}-\operatorname{pos}_{0}-\operatorname{ind}\left(Q_{2}, X\right)
$$

If $n=-1$, then

$$
\mathcal{U}-\operatorname{pos}_{0}-\operatorname{ind}\left(Q_{1}, X\right)=\mathcal{U}-\operatorname{pos}_{0}-\operatorname{ind}\left(Q_{2}, X\right)=-1
$$

which means that $Q_{1} \cup Q_{2}=X=\emptyset$ and, therefore, (6) is true. Also, if $Q_{1}=\emptyset$ or $Q_{2}=\emptyset$, then the relation (6) is true.

Suppose that for any uniform space $X$ and its subsets $Q_{1}, Q_{2}$ the relation (6) is true if

$$
\mathcal{U}-\operatorname{pos}_{0}-\operatorname{ind}\left(Q_{1}, X\right)+\mathcal{U}-\operatorname{pos}_{0}-\operatorname{ind}\left(Q_{2}, X\right)<n
$$

where $n \in \omega$. We shall prove (6) for the case

$$
\mathcal{U}-\operatorname{-oss}_{0}-\operatorname{ind}\left(Q_{1}, X\right)+\mathcal{U}-\operatorname{pos}_{0}-\operatorname{ind}\left(Q_{2}, X\right)=n
$$

Let

$$
\mathcal{U}-\operatorname{pos}_{0}-\operatorname{ind}\left(Q_{1}, X\right)=n_{1} \text { and } \mathcal{U}-\operatorname{pos}_{0}-\operatorname{ind}\left(Q_{2}, X\right)=n_{2}
$$

where $n_{1}, n_{2} \in(\omega \cup\{-1\})$. If one of the elements $n_{1}, n_{2}$ is equal to -1 , then the other is also equal to -1 and, therefore, $n=-1$ is not a natural number.

Hence, we can suppose that $n_{1}, n_{2} \in \omega$ and $Q_{1}, Q_{2} \neq \emptyset$.
There exists a $\mathcal{U}$-pos-base $\mathcal{B}_{1}$ for $Q_{1}$ in $X$ such that for every $x \in Q_{1}$ and $x \in U$ with $U$ a $\mathcal{U}$-cozero set of $X$ there exist a $V_{1} \in \mathcal{B}_{1}$ and a $\mathcal{U}$-zero set $\widetilde{V}_{1}$ of $X$ with $x \in V_{1} \subseteq \widetilde{V}_{1} \subseteq U$ and

$$
\mathcal{U}-\operatorname{pos}_{0}-\operatorname{ind}\left(Q_{1} \cap\left(\widetilde{V}_{1}-V_{1}\right), \widetilde{V}_{1}-V_{1}\right) \leq n_{1}-1
$$

Also, there exists a $\mathcal{U}$-pos-base $\mathcal{B}_{2}$ for $Q_{2}$ in $X$ such that for every $x \in Q_{2}$ and $x \in U$ with $U$ a $\mathcal{U}$-cozero set of $X$ there exist a $V_{2} \in \mathcal{B}_{2}$ and a $\mathcal{U}$-zero set $\bar{V}_{2}$ of $X$ with $x \in V_{2} \subseteq \widetilde{V}_{2} \subseteq U$ and

$$
\mathcal{U}-\operatorname{pos}_{0}-\operatorname{ind}\left(Q_{2} \cap\left(\widetilde{V}_{2}-V_{2}\right), \widetilde{V}_{2}-V_{2}\right) \leq n_{2}-1
$$

The set $\mathcal{B}=\mathcal{B}_{1} \cup \mathcal{B}_{2}$ is a $\mathcal{U}$-pos-base for $Q_{1} \cup Q_{2}$ in $X$. Let $x \in Q_{1} \cup Q_{2}$ and $x \in U$ with $U$ a $\mathcal{U}$-cozero set of $X$. Without loss of generality we suppose that $x \in Q_{1}$. Then, there exist a $V_{1} \in \mathcal{B}_{1}$ and a $\mathcal{U}$-zero set $\widetilde{V}_{1}$ of $X$ with $x \in V_{1} \subseteq$ $\subseteq \widetilde{V}_{1} \subseteq U$ and

$$
\mathcal{U}-\operatorname{pos}_{0}-\operatorname{ind}\left(Q_{1} \cap\left(\widetilde{V}_{1}-V_{1}\right), \widetilde{V}_{1}-V_{1}\right) \leq n_{1}-1
$$

Also, by Propositions 3.2 and 3.1, we have

$$
\begin{aligned}
& \mathcal{U}-\operatorname{pos}_{0}-\operatorname{ind}\left(Q_{2} \cap\left(\widetilde{V}_{1}-V_{1}\right), \widetilde{V}_{1}-V_{1}\right) \\
& \leq \mathcal{U}-\operatorname{pos}_{0}-\operatorname{ind}\left(Q_{2} \cap\left(\widetilde{V}_{1}-V_{1}\right), X\right) \\
& \leq \mathcal{U}-\operatorname{pos}_{0}-\operatorname{ind}\left(Q_{2}, X\right) \leq n_{2}
\end{aligned}
$$

and, therefore, by inductive assumption,

$$
\begin{aligned}
& \mathcal{U}-\operatorname{pos}_{0}-\operatorname{ind}\left(\left(Q_{1} \cup Q_{2}\right) \cap\left(\widetilde{V}_{1}-V_{1}\right), \widetilde{V}_{1}-V_{1}\right)= \\
& \mathcal{U}-\operatorname{-os}_{0}-\operatorname{ind}\left(\left(Q_{1} \cap\left(\widetilde{V}_{1}-V_{1}\right)\right) \cup\left(Q_{2} \cap\left(\widetilde{V}_{1}-V_{1}\right)\right), \widetilde{V}_{1}-V_{1}\right) \leq \\
& \mathcal{U}-\operatorname{pos}_{0}-\operatorname{ind}\left(Q_{1} \cap\left(\widetilde{V}_{1}-V_{1}\right), \widetilde{V}_{1}-V_{1}\right)+ \\
& \mathcal{U}-\operatorname{pos}_{0}-\operatorname{ind}\left(Q_{2} \cap\left(\widetilde{V}_{1}-V_{1}\right), \widetilde{V}_{1}-V_{1}\right) \leq n_{1}+n_{2}-1=n-1
\end{aligned}
$$

Thus,

$$
\mathcal{U}-\operatorname{pos}_{0}-\operatorname{ind}\left(Q_{1} \cup Q_{2}, X\right) \leq n
$$

Proposition 4.2. Let $(X, \mathcal{U})$ be a uniform space. Then for every two subsets $Q_{1}$ and $Q_{2}$ of $X$ we have:

$$
\begin{align*}
& \mathcal{U}-\operatorname{pos}_{1}-\operatorname{ind}\left(Q_{1} \cup Q_{2}, X\right) \leq \\
& \mathcal{U}-\operatorname{pos}_{1}-\operatorname{ind}\left(Q_{1}, X\right)+\mathcal{U}-\operatorname{pos}_{1}-\operatorname{ind}\left(Q_{2}, X\right)+1 \tag{7}
\end{align*}
$$

and

$$
\begin{align*}
& \mathcal{U}-\mathrm{ps}_{1}-\operatorname{ind}\left(Q_{1} \cup Q_{2}, X\right) \leq \\
& \mathcal{U}-\mathrm{ps}_{1}-\operatorname{ind}\left(Q_{1}, X\right)+\mathcal{U}-\operatorname{ps}_{1}-\operatorname{ind}\left(Q_{2}, X\right)+1 \tag{8}
\end{align*}
$$

Proof. We prove relation (8) by induction on $n$, where

$$
n=\mathcal{U}-\mathrm{ps}_{1}-\operatorname{ind}\left(Q_{1}, X\right)+\mathcal{U}-\mathrm{ps}_{1}-\operatorname{ind}\left(Q_{2}, X\right)
$$

If $n=-1$, then $\mathcal{U}-\mathrm{ps}_{1}-\operatorname{ind}\left(Q_{1}, X\right)=\mathcal{U}-\operatorname{ps}_{1}-\operatorname{ind}\left(Q_{2}, X\right)=-1$ which means that $Q_{1} \cup Q_{2}=\emptyset$ and, therefore, (8) is true.

Suppose that for any uniform space $X$ and its subsets $Q_{1}, Q_{2}$ the relation (8) is true if

$$
\mathcal{U}-\mathrm{ps}_{1}-\operatorname{ind}\left(Q_{1}, X\right)+\mathcal{U}-\mathrm{ps}_{1}-\operatorname{ind}\left(Q_{2}, X\right)<n
$$

where $n$ is a natural number. We shall prove (8) for the case

$$
\mathcal{U}-\mathrm{ps}_{1}-\operatorname{ind}\left(Q_{1}, X\right)+\mathcal{U}-\operatorname{ps}_{1}-\operatorname{ind}\left(Q_{2}, X\right)=n
$$

Let

$$
\mathcal{U}-\mathrm{ps}_{1}-\operatorname{ind}\left(Q_{1}, X\right)=n_{1} \text { and } \mathcal{U}-\mathrm{ps}_{1}-\operatorname{ind}\left(Q_{2}, X\right)=n_{2}
$$

where $n_{1}, n_{2} \in(\omega \cup\{-1\})$. If $n_{1}=-1$ or $n_{2}=-1$, then $Q_{1}=\emptyset$ or $Q_{2}=\emptyset$, respectively and the relation (8) is true.

There exists a $\mathcal{U}$-ps-base $\mathcal{B}_{1}$ for $Q_{1}$ in $X$ such that for every $x \in Q_{1}$ and $x \in U$ with $U$ a $\mathcal{U}$-cozero set of $X$ there exist a $V_{1} \in \mathcal{B}_{1}$ and a $\mathcal{U}$-zero set $V$ of $X$ such that $x \in V_{1} \subseteq \widetilde{V}_{1} \subseteq U$ and

$$
\mathcal{U}-\operatorname{ps}_{1}-\operatorname{ind}\left(Q_{1} \cap\left(\widetilde{V}_{1}-V_{1}\right), X\right) \leq n_{1}-1
$$

Also, there exists a $\mathcal{U}$-ps-base $\mathcal{B}_{2}$ for $Q_{2}$ in $X$ such that for every $x \in Q_{2}$ and $x \in U$ with $U$ a $\mathcal{U}$-cozero set of $X$ there exist a $V_{2} \in \mathcal{B}_{2}$ and a $\mathcal{U}$-zero set $\widetilde{V}$ of $X$ such that $x \in V_{2} \subseteq \widetilde{V}_{2} \subseteq U$ and

$$
\mathcal{U}-\mathrm{ps}_{1}-\operatorname{ind}\left(Q_{2} \cap\left(\widetilde{V}_{2}-V_{2}\right), X\right) \leq n_{2}-1
$$

The set $\mathcal{B}=\mathcal{B}_{1} \cup \mathcal{B}_{2}$ is a $\mathcal{U}$-ps-base for $Q_{1} \cup Q_{2}$ in $X$. Let $x \in Q_{1} \cup Q_{2}$ and $x \in U$ with $U$ a $\mathcal{U}$-cozero set of $X$. Without loss of generality we suppose that $x \in Q_{1}$. Then, there exist a $V_{1} \in \mathcal{B}_{1}$ and a $\mathcal{U}$-zero set $\widetilde{V}_{1}$ of $X$ with $x \in V_{1} \subseteq$ $\subseteq \widetilde{V}_{1} \subseteq U$ and

$$
\mathcal{U}-\mathrm{ps}_{1}-\operatorname{ind}\left(Q_{1} \cap\left(\widetilde{V}_{1}-V_{1}\right), X\right) \leq n_{1}-1
$$

and, by Proposition 3.1,

$$
\mathcal{U}-\mathrm{ps}_{1}-\operatorname{ind}\left(Q_{2} \cap\left(\widetilde{V}_{1}-V_{1}\right), X\right) \leq \mathcal{U}-\operatorname{ps}_{1}-\operatorname{ind}\left(Q_{2}, X\right)=n_{2}
$$

By inductive assumption we have

$$
\begin{array}{r}
\mathcal{U}-\mathrm{ps}_{1}-\operatorname{ind}\left(\left(Q_{1} \cup Q_{2}\right) \cap\left(\widetilde{V}_{1}-V\right), X\right)= \\
\mathcal{U}-\operatorname{ps}_{1}-\operatorname{ind}\left(\left(Q_{1} \cap\left(\widetilde{V}_{1}-V_{1}\right)\right) \cup\left(Q_{2} \cap\left(\widetilde{V}_{1}-V_{1}\right)\right), X\right) \leq \\
\mathcal{U}-\operatorname{ps}_{1}-\operatorname{ind}\left(Q_{1} \cap\left(\widetilde{V}_{1}-V_{1}\right), X\right)+ \\
\mathcal{U}-\mathrm{ps}_{1}-\operatorname{ind}\left(Q_{2} \cap\left(\widetilde{V}_{1}-V_{1}\right), X\right)+1 \leq n_{1}-1+n_{2}+1=n
\end{array}
$$

Thus, $\mathcal{U}$ - $\mathrm{ps}_{1}-\operatorname{ind}\left(Q_{1} \cup Q_{2}, X\right) \leq n+1$.
The proof of the relation (7) is similar.
Question. Is the sum theorems (Propositions 4.1 and 4.2) true for the positional dimension-like functions of the type $\mathcal{U}$-ind that are not mentioned in Propositions 4.1 and 4.2?

## 5. Some other results

DEFINITION 5.1. Let $(X, \mathcal{U})$ and $(Y, \mathcal{V})$ be two uniform spaces and $f: X \rightarrow Y$.
The map $f$ is called $(\mathcal{U}, \mathcal{V})$-cozero map if $f^{-1}(U) \in \operatorname{Coz} \mathcal{U}$ for every $U \in$ $\in \operatorname{Coz} \mathcal{V}$. It is clear that every $(\mathcal{U}, \mathcal{V})$-cozero map from a uniform space $(X, \mathcal{U})$ to a uniform space $(Y, \mathcal{V})$ is a continuous map from the topological space $\left(X, \tau_{\mathcal{U}}\right)$ to the topological space $\left(Y, \tau_{\mathcal{V}}\right)$.

The map $f: X \rightarrow Y$ is called $(\mathcal{U}, \mathcal{V})$-cozero-set preserving map if $f(U) \in$ $\in \operatorname{Coz} \mathcal{V}$, for every $U \in \operatorname{Coz} \mathcal{U}$.

Also, the map $f$ is called $(\mathcal{U}, \mathcal{V})$-isomorphism if the map $f$ is $1-1$, onto, $(\mathcal{U}, \mathcal{V})$-cozero map, and $(\mathcal{U}, \mathcal{V})$-cozero-set preserving map.

Proposition 5.1. Let $(X, \mathcal{U})$ and $(Y, \mathcal{V})$ be two uniform spaces and $Q \subseteq X$. If the map $f: X \rightarrow Y$ is a $(\mathcal{U}, \mathcal{V})$-isoomorphism, then

$$
\begin{equation*}
\mathcal{U}-\mathrm{p}_{1}-\operatorname{ind}(Q, X) \leq \mathcal{V}-\mathrm{p}_{1}-\operatorname{ind}(f(Q), Y) \tag{9}
\end{equation*}
$$

Proof. We prove the relation (9) by induction on the element

$$
\mathcal{V}-\mathrm{p}_{1}-\operatorname{ind}(f(Q), Y) \in(\omega \cup\{-1, \infty\})
$$

This relation is true if

$$
\mathcal{V}-\mathrm{p}_{1}-\operatorname{ind}(f(Q), Y)=-1
$$

or

$$
\mathcal{V}-\mathrm{p}_{1}-\operatorname{ind}(f(Q), Y)=\infty
$$

Suppose that the relation (9) is true if $\mathcal{V}-\mathrm{p}_{1}-\operatorname{ind}(f(Q), Y)<n \in \omega$ and prove it in the case where $\mathcal{V}$ - $\mathrm{p}_{1}-\operatorname{ind}(f(Q), Y)=n$.

There exists a $\mathcal{V}$-p-base $\mathcal{B}_{n}$ for $f(Q)$ in $Y$ such that for every $y \in f(Q)$ and $y \in U$ with $U$ a $\mathcal{U}$-cozero set of $Y$ there exist a $V \in \mathcal{B}$ and a $\mathcal{V}$-zero set $\widetilde{V}$ of $Y$ with $y \in V \subseteq \widetilde{V} \subseteq U$ and

$$
\mathcal{V}-\mathrm{p}_{1}-\operatorname{ind}(f(Q) \cap(\widetilde{V}-V), Y) \leq n-1
$$

We consider the set

$$
\left\{f^{-1}(W): W \in \mathcal{B}_{n}\right\}
$$

We observe that this set is a $\mathcal{U}$-p-base for $Q$ in $X$. Now, let $x \in Q$ and $x \in W$ with $W$ a $\mathcal{U}$-cozero set of $X$. Then, $y=f(x) \in f(Q)$ and since $f$ is $(\mathcal{U}, \mathcal{V})$-cozero-set preserving map, the set $f(W)$ is a $\mathcal{V}$-cozero set of $Y$. Also, $y \in f(W)$. Thus, there exist a $V \in \mathcal{B}_{n}$ and a $\mathcal{V}$-zero set $\widetilde{V}$ of $Y$ with $y \in V \subseteq \widetilde{V} \subseteq f(W)$ and

$$
\mathcal{V}-\mathrm{p}_{1}-\operatorname{ind}(f(Q) \cap(\widetilde{V}-V), Y) \leq n-1 .
$$

Since the map $f$ is a $(\mathcal{U}, \mathcal{V})$-cozero map, the set $f^{-1}(\widetilde{V})$ is a $\mathcal{U}$-zero set of $X$ and the set $f^{-1}(V)$ is a $\mathcal{U}$-cozero set of $X$. Also, we have $x \in f^{-1}(V) \subseteq f^{-1}(\widetilde{V}) \subseteq$ $\subseteq f^{-1}(f(W))=W$ and

$$
\mathcal{U}-p_{1}-\operatorname{ind}\left(Q \cap\left(f^{-1}(\widetilde{V})-f^{-1}(V)\right), X\right)<n .
$$

Indeed, we have $f\left(f^{-1}(\widetilde{V})-f^{-1}(V)\right) \subseteq \widetilde{V}-V$. By inductive assumption and Proposition 3.1, we have

$$
\begin{aligned}
& \mathcal{U}-\mathrm{p}_{1}-\operatorname{ind}\left(Q \cap\left(f^{-1}(\widetilde{V})-f^{-1}(V)\right), X\right) \\
& \leq \mathcal{V}-\mathrm{p}_{1}-\operatorname{ind}\left(f\left(Q \cap\left(f^{-1}(\widetilde{V})-f^{-1}(V)\right), Y\right) \leq\right. \\
& \mathcal{V}-\mathrm{p}_{1}-\operatorname{ind}\left(f(Q) \cap f\left(f^{-1}(\widetilde{V})-f^{-1}(V)\right), Y\right) \leq \\
& \mathcal{V}-\mathrm{p}_{1}-\operatorname{ind}(f(Q) \cap(\widetilde{V}-V), Y)<n .
\end{aligned}
$$

Thus,

$$
\mathcal{U}-\mathrm{p}_{1}-\operatorname{ind}\left(Q \cap\left(f^{-1}(\widetilde{V})-f^{-1}(V)\right), X\right)<n
$$

and, therefore, $\mathcal{U}-\mathrm{p}_{1}-\operatorname{ind}(Q, X) \leq n$.
Similarly, we have the following proposition.
Proposition 5.2. Let $(X, \mathcal{U})$ and $(Y, \mathcal{V})$ be two uniform spaces, $f: X \rightarrow Y$ a $(\mathcal{U}, \mathcal{V})$-isomorphism, and $Q \subseteq X$. If the restriction $\left.f\right|_{Q}$ of the map $f$ to $Q$ is a $\left(\mathcal{U}_{Q}, \mathcal{V}_{f(Q)}\right)$-isomorphism, then

$$
\mathcal{U}-\operatorname{pos}_{1}-\operatorname{ind}(Q, X) \leq \mathcal{U}-\operatorname{pos}_{1}-\operatorname{ind}(f(Q), Y)
$$

Acknowledgements. I am grateful to the referee for a number of helpful suggestions for improvement in the article.

## References

[1] M. G. Charalambous, Two new inductive dimension functions for topological spaces, Annales Univ. Sci. Budapest., Sectio Math., 17 (1974), 21-28.
[2] M. G. Charalambous, A new covering dimension function for uniform spaces, J. London Math. Soc. (2), 11 (1975), 137-143.
[3] M. G. Charalambous, Inductive dimension theory for uniform spaces, Annales Univ. Sci. Budapest., Sectio Math., 18 (1975), 15-25.
[4] M. G. Charalambous, Dimension theory for $\sigma$-frames, J. London Math. Soc. (2) 8 (1974), 149-160.
[5] A. Chigogidze, Inductive dimensions for completely regular spaces, Comment. Math. Univ. Carolinae, 18 (1977), no. 4, 623-637.
[6] A. Chigogidze, Relative dimensions. (Russian) General topology, 67-117, 132, Moskov. Gos. Univ., Moscow, 1985.
[7] R. Engelking, General Topology, Revised and completed edition, Heldermann Verlag, Berlin, 1989.
[8] D. N. Georgiou, S. D. Iliadis, and A. C. Megaritis, On positional dimensionlike functions, Topology Proceedings, 33 (2009), 285-296.
[9] Leonard Gillman, Meyer Jerison, Rings of continuous functions, D. Van Nostrand Co., Inc., Princeton, 1960. ix +300.
[10] S. D. Iliadis, Universal spaces and mappings, North-Holland Mathematics Studies, 198. Elsevier Science B. V., Amsterdam, 2005. xvi + 559.
[11] J. R. Isbell, Uniform spaces, Amer. Math. Soc. (1964).
[12] K. Kuratowski and A. Mostowski, Set Theory, PWN and North-Holland, 1976.
[13] V. V. Tкachuk, On the dimension of subspaces, Moscow Univ. Math. Bull., 36 (1981), no.2, 25-29.
[14] V. V. Tкachuк, On the relative small inductive dimension, Moscow Univ. Math. Bull., 37 (1982), no.5, 25-29.
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# ON PROPERTIES OF GENERALIZED NEIGHBOURHOOD SYSTEMS 

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#### Abstract

Neighbourhood structures are particular cases of generalized neighbourhood systems. Let $X \neq \emptyset$ be a set and $N(X)$ be the set of all neighbourhood structures on $X$, partially ordered as follows: $\psi \leq \varphi$ for $\psi, \varphi \in N(X)$ iff $\psi(x) \leq \varphi(x)$ for each $x \in X$. Then $N(X)$ is a complete sublattice of the $G N(X)$ which denotes the set of all strongly generalized neighbourhood systems on $X$ partially ordered as above. We investigate some properties of $G N(X)$. In addition we discuss the product of generalized neighbourhood systems and present some new results concerning $g n$-continuity related to this product.


## 1. Introduction

In 1914, Hausdorff [6] defined toplogical spaces in terms of a system of neighbourhoods at each point. Csaszar [1] continued to study this approach under the name of neighbourhood spaces, with various conditions on the systems of neighbourhoods at each point. Recently, the properties of neighbourhood spaces have investigated by using neighbourhood $p$-stacks instead of neighbourhood filters in [7,9] and Richmond and Slapal [11] continued to study these concepts by using neighbourhood rasters which is a subclass of neighbourhood $p$-stacks. The concept of generalized neighbourhood systems which is a strict generalization of neighbourhood structures recalled below was given by Csaszar [2]. Let $X \neq \emptyset$ then a map $\psi: X \rightarrow \exp (\exp X)$ satisfying $x \in V$ for $V \in \psi(x), x \in X$ is called a generalized neighbourhood system (briefly GNS) on $X$. In this paper, $G N(X)$ denotes the set of all strongly generalized neighbourhood structures on $X$ partially ordered as follows: $\psi \leq \varphi$ for $\psi, \varphi \in G N(X)$ iff $\psi(x) \subseteq \varphi(x)$ for
each $x \in X$. We investigate the properties of $G N(X)$ and show that $N(X)$ is a complete sublattice of $G N(X)$. In addition we describe the product of GNSs' and give some new results concerning $g n$-continuity related to this product.

## 2. Preliminaries

Suppose $\mathcal{H}$ is be a collection of subsets of a nonempty set $X$. Consider the following conditions on $\mathcal{H}$.
(a) $A \in \mathcal{H}, A \subseteq B$ implies $B \in \mathcal{H}$. ( $\mathcal{H}$ is a stack)
(b) $A_{1}, A_{2} \in \mathcal{H}$ implies $A_{1} \cap A_{2} \neq \emptyset$. $(\mathcal{H}$ has the pairwise intersection property.)
(b') $A_{1}, A_{2}, \ldots, A_{n} \in \mathcal{H}$ implies $A_{1} \cap A_{2} \cap \ldots \cap A_{n} \neq \emptyset .(\mathcal{H}$ has the finite intersection property.)
( $\mathrm{b}^{\prime \prime}$ ) $\cap \mathcal{H} \neq \emptyset$. ( $\mathcal{H}$ is an $m$-family [8].)
If $\mathcal{H}$ satisfies (a) and (b), it is called a p-stack in [7]. If $\mathcal{H}$ satisfies (a) and $\left(\mathrm{b}^{\prime}\right)$ it is called a raster in [11]. Clearly every raster is a $p$-stack and every filter is a raster, but not conversely. For every point $x \in X \neq \emptyset, \dot{x}$ denotes the filter of all supersets of $\{x\}$. A neighbourhood space, in the sense of [7] (resp. [11]), is a pair $(X, \nu)$ where $\nu: X \rightarrow \exp (\exp X)$ is a map such that $\nu(x) \subseteq \dot{x}$ is a $p$-stack (resp. raster) for each $x \in X$. Then $\nu$ is called a neighbourhood structure on $X$. Clearly, a neighbourhood space is defined to be pretopological if $\nu(x)$ is a filter for all $x \in X$.

Let $X \neq \emptyset$ and $\psi$ be a GNS on $X$. Then the pair $(X, \psi)$ is called a $g n$-space [8]. Now consider the following conditions on $\psi$.
(a) $\psi(x) \neq \emptyset$ for all $x \in X$.
(b) $U, V \in \psi(x)$ implies $U \cap V \in \psi(x)$ for all $x \in X$.
(c) $\psi(x)$ is a stack for all $x \in X$.
(d) for each $x \in X$ and $V \in \psi(x)$, there is a set $O$ satisfying $x \in O \subset V$, and $y \in O$ implies the existence of a set $U \in \psi(y)$ with $U \subset O$.
(e) $X \in \psi(x)$ for all $x \in X$.

If $\psi$ satisfies (a) and (b), it is called a weak neighbourhood system in [10]. If $\psi$ satisfies (c), it is called ascending [4]. If $\psi$ satisfies (c) and (d), it is called complete [12]. If $\psi$ satisfies (e), we shall say that it is strongly. If $\psi$ satisfies (a), then $\psi(x)$ is an $m$-family for each $x \in X$. Clearly a neighbourhood structure $\nu$ is an ascending, strongly GNS.

Now we recall some concepts and notations defined in [2]. The collection of all GNSs' on $X$ is denoted by $\Psi(X)$. For $\psi \in \Psi(X)$ and $A \subset$ $\subset X$, the interior and closure of $A$ on $\psi$ (denoted by $\imath_{\psi} A, \gamma_{\psi} A$, respectively) are defined as $\imath_{\psi} A=\{x \in A$ : there exists $V \in \psi(x)$ such that $V \subset A\}$ and $\gamma_{\psi} A=\{x \in X: V \cap A \neq \emptyset$ for all $V \in \psi(x)\}$, respectively. A generalized topology (briefly GT) on $X$ is a subset $\mu$ of the power set $\exp X$ such that $\emptyset \in \mu$ and every union of some elements of $\mu$ belongs to $\mu$. The elements of $\mu$ are called $\mu$-open sets and their complements are called $\mu$-closed sets. Let $A \subset X$, the $\mu$-interior of $A$ (denoted by $i_{\mu} A$ ) is the union of all $\mu$-open sets contained in $A$ and the $\mu$-closure of $A$ (denoted by $c_{\mu} A$ ) is the intersection of all $\mu$-closed sets containing $A$. If $X \in \mu$, then $\mu$ is said to be a strongly generalized topology [3] on $X$. If $\mu$ and $v$ are generalized topologies on $X$ and $Y$, respectively, then a mapping $f: X \rightarrow Y$ is said to be $(\mu, v)$-continuous [2] if $f^{-1}(v) \subseteq \mu$. If $\psi \in \Psi(X)$, then $\mu=\mu_{\psi}$ is defined as the collection of all subsets $M \subset X$ such that $x \in M$ implies the existence of a set $V \in \psi(x)$ satisfying $V \subset M$. Also it is shown that $\mu_{\psi}$ is a GT on $X$, generated by the GNS $\psi$ and anyone can write $i_{\psi}$ for $i_{\mu_{\psi}}$ and $c_{\psi}$ for $c_{\mu_{\psi}}$. In addition, for an arbitrary subfamily $\mu$ of $\exp X$ (thus $\mu$ need not to be a GT) consider $\psi(x)=\psi_{\mu}(x)=\{M \in \mu: x \in M\}$ for each $x \in X$, then $\psi_{\mu}$ is a GNS on $X$. Also $\Psi_{\mu}(X)$ is defined as the set of all $\psi \in \Psi(X)$ satisfying $V \in \mu$ for $V \in \psi(x), x \in X$.

Lemma 2.1. (a) If $\psi \in \Psi_{\mu}(X)$ for the $G T \mu=\mu_{\psi}$ on $X$, then $\imath_{\psi}=i_{\psi}$ and $\gamma_{\psi}=c_{\psi}$.
(b) If $\mu$ is a GT on $X$ and $\psi=\psi_{\mu}$, then $\mu_{\psi}=\mu$.

In addition, for a GNS $\psi$ on $X$ and $A \subset X$, two more operators (denoted by $I_{\psi}^{*}$ and $\mathrm{cl}_{\psi}^{*}$ ) are defined by K. Min [8] as $I_{\psi}^{*} A=\{x \in A: A \in \psi(x)\}$ and $\mathrm{cl}_{\psi}^{*} A=\{x \in X: X-A \notin \psi(x)\}$. If $\psi$ and $\varphi$ are GNSs' on $X$ and $Y$, respectively, then a mapping $f: X \rightarrow Y$ is said to be $(\psi, \varphi)$-continuous [2] if for each $x \in X$ and $V \in \varphi(f(x))$, there is $U \in \psi(x)$ such that $f(U) \subset V$, $g n$-continuous [8] if $f^{-1}(\varphi(f(x))) \subseteq \psi(x)$ for each $x \in X$ and $g n$-open [8] if $f(\psi(x)) \subseteq \varphi(f(x))$ for each $x \in X$.

## 3. The Lattice $G N(X)$

$G N(X)$ denotes the set of all strongly generalized neighbourhood structures on $X$, partially ordered as follows: $\psi \leq \varphi$ for $\psi, \varphi \in G N(X)$ iff $\psi(x) \subseteq \varphi(x)$ for each $x \in X$ (in which case $\psi$ is coarser than $\varphi$ and $\varphi$ is finer than $\psi$ ). For $\psi, \varphi \in G N(X)$, we denote the meet and join as $\psi \wedge \varphi=\psi \cap \varphi$ and $\psi \vee \varphi=\psi \cup \varphi$,
respectively, such that

$$
\begin{aligned}
& (\psi \wedge \varphi)(x)=\{V: V \in \psi(x) \text { and } V \in \varphi(x)\} \\
& (\psi \vee \varphi)(x)=\{V: V \in \psi(x) \text { or } V \in \varphi(x)\}
\end{aligned}
$$

for each $x \in X$. Therefore, for an arbitrary subfamily $\mathcal{A}=\left\{\psi_{k}: k \in K\right\} \subseteq$ $\subseteq G N(X)$, we shall define $\wedge=\inf _{G N(x)} \mathcal{A}$ and $\vee=\sup _{G N(X)} \mathcal{A}$ such that

$$
\wedge(x)=\cap\left\{\psi_{k}(x): k \in K\right\} \text { and } \vee(x)=\cup\left\{\psi_{k}(x): k \in K\right\}
$$

for all $x \in X$, respectively. Clearly $G N(X)$ is a complete lattice, with $\psi_{\sup }(x)=$ $\{V: x \in V\}$ for all $x \in X$ and $\psi_{\mathrm{inf}}(x)=\{X\}$ for all $x \in X$ as greatest and least elements.

Lemma 3.1. If $\mathcal{A}=\left\{\psi_{k}: k \in K\right\} \subseteq G N(X), \wedge=\inf _{G N(x)} \mathcal{A}$ and $\vee=$ $\sup _{G N(X)} \mathcal{A}$, then the following statements are valid.
(a) If each $\psi_{k}$ is ascending, so is $\inf _{G N(x)} \mathcal{A}$ and $\sup _{G N(X)} \mathcal{A}$.
(b) If each $\psi_{k}$ is complete, so is $\sup _{G N(X)} \mathcal{A}$;
(c) If each $\psi_{k}$ is a weak $G N S$, so is $\inf _{G N(X)} \mathcal{A}$.

Corollary 3.2. $N(X)$ is a complete sublattice of $G N(X)$, with $\psi_{\text {sup }}(x)=\dot{x}$ for all $x \in X$ and $\psi_{\inf }(x)=\{X\}$ for all $x \in X$, as greatest and least elements.

Proof. It is straightforward from Lemma 3.1 (a).
The following result is a consequence of the definitions.
Proposition 3.3. Let $\mathcal{A}=\left\{\psi_{k}: k \in K\right\} \subseteq G N(X), \wedge=\inf _{G N(x)} \mathcal{A}, \vee=$ $\sup _{G N(X)} \mathcal{A}$ and $A \subset X$. Then we have
(a) $\imath_{\vee} A=\cup_{k \in K} \imath_{\psi_{k}} A$ and $\gamma_{\vee} A \subseteq \cup_{k \in K} \gamma_{\psi_{k}} A$,
(b) $\imath_{\wedge} A \subseteq \cap_{k \in K} \psi_{\psi_{k}} A$ and $\cap_{k \in K} \gamma_{\psi_{k}} A \subseteq \gamma_{\wedge} A$,
(c) $I_{\vee}^{*} A=\cup_{k \in K} I_{\psi_{k}}^{*} A$ and $\mathrm{cl}_{\vee}^{*} A \subseteq \cup_{k \in K} \mathrm{cl}_{\psi_{k}}^{*} A$
(d) $I_{\wedge}^{*} A=\cap_{k \in K} I_{\psi_{k}}^{*} A$ and $\cap_{k \in K} \operatorname{cl}_{\psi_{k}}^{*} A \subseteq \operatorname{cl}_{\wedge}^{*} A$.

Proposition 3.4. Let $\mathcal{A}=\left\{\psi_{k}: k \in K\right\} \subseteq G N(X), \wedge=\inf _{G N(x)} \mathcal{A}$ and $\vee=$ $\sup _{G N(X)} \mathcal{A}$. Then $\mu_{\wedge} \subseteq \cup_{k \in K} \mu_{\psi_{k}} \subseteq \mu_{\vee}$.

Proof. If $A \in \mu_{\wedge}$, then for each $x \in A$ there exists $V \in \wedge(x)$ such that $V \subset A$. Thus $A \in \cup_{k \in K} \mu_{\psi_{k}}$ since $V \in \psi_{k}(x)$ for all $k \in K$. If $A \in \mu_{\psi_{k}}$ for some $k \in K$, then there exists $V_{k} \in \psi_{k}(x)$ such that $V_{k} \subset A$ for each $x \in A$. Hence $A \in \mu_{\vee}$.

## 4. Initial and final structures for GNSs’

Let $X$ be a set, $\left\{\left(X_{k}, \psi_{k}\right): k \in K\right\}$ be collection of $g n$-spaces and $\left\{f_{k}: k \in\right.$ $\in K\}$ a corresponding collection of functions $f_{k}: X \rightarrow X_{k}$. Then let us define the
mapping $\vartheta: X \rightarrow \exp (\exp X)$ such that

$$
\vartheta(x)=\left\{f_{k}^{-1}(V): k \in K \text { and } V \in \psi_{k}\left(f_{k}(x)\right)\right\}
$$

for each $x \in X$. Clearly $\vartheta$ is a GNS on $X$. We call $\vartheta$ the initial generalized neighbourhood structure on $X$ for the family $\left(f_{k}\right)_{k \in K}$.

Clearly $\vartheta$ is the coarsest GNS on $X$ for which the mappings $f_{k}$ are $g n$ continuous. Then the following result is clear.

Proposition 4.1. Let $X$ be a set, $\left\{\left(X_{k}, \psi_{k}\right): k \in K\right\}$ be a collection of strongly $g n$-spaces and $\left\{f_{k}: k \in K\right\}$ be a corresponding collection of functions $f_{k}: X \rightarrow$ $X_{k}$. The initial generalized neighbourhood structure on $X$ exists and is specified by $\vartheta(x)=\sup _{G N(X)}\left\{f_{k}^{-1}\left(\psi_{k}\left(f_{k}(x)\right)\right): k \in K\right\}$ for all $x \in X$.

The concept of convergence in $g n$-spaces was given in [8] by using $m$ families. Let $\psi$ be a GNS and $\mathcal{H}$ be an $m$-family on $X$. Then $\mathcal{H}$ converges to $x \in X$ if $\mathcal{H}$ is finer than $\psi(x)$ i.e. $\psi(x) \subset \mathcal{H}$. So we can give the following result.

Proposition 4.2. Let $\vartheta$ be the initial generalized neighbourhood structure on $X$ induced by the collection of strongly gn-spaces $\left\{\left(X_{k}, \psi_{k}\right): k \in K\right\}$ and functions $\left\{f_{k}: k \in K\right\}$, where $f_{k}: X \rightarrow X_{k}$. For an $m$-family $\mathcal{H}$ on $X$,
(a) if $\mathcal{H}$ converges to $x \in X$, then $f_{k}(\mathcal{H})$ converges to $f_{k}(x)$ for all $k \in K$.
(b) if $\left\{f_{k}: k \in K\right\}$ is a collection of injective functions and $f_{k}(\mathcal{H})$ converges to $f_{k}(x)$ for all $k \in K$, then $\mathcal{H}$ converges to $x \in X$.

Proof. (a) For all $k \in K$, we have $f_{k}^{-1}\left(\psi_{k}\left(f_{k}(x)\right)\right) \subset \vartheta(x)$ since $f_{k}$ is $g n$ continuous. Thus $\psi_{k}\left(f_{k}(x)\right) \subset f_{k}(\vartheta(x)) \subset f_{k}(\mathcal{H})$.
(b) If $V \in \vartheta(x)$, then $V \in f_{k}^{-1}\left(\psi_{k}\left(f_{k}(x)\right)\right)$ for some $k \in K$. Thus $f_{k}(V) \in$ $\in f_{k}\left(f_{k}^{-1}\left(\psi_{k}\left(f_{k}(x)\right)\right)\right) \subset \psi_{k}\left(f_{k}(x)\right)$. Therefore we have $f_{k}(V) \in f_{k}(\mathcal{H})$ by hypothesis. Hence $V \in \mathcal{H}$.

Let $X$ be a set, $\left\{\left(X_{k}, \psi_{k}\right): k \in K\right\}$ be a collection of strongly $g n$-spaces, $\left\{h_{k}: k \in K\right\}$ be a corresponding collection of functions $h_{k}: X_{k} \rightarrow X$ and $Y=\cup$ $\cup_{k \in K} h_{k}\left(X_{k}\right)$. We shall define the mapping $\Phi: X \rightarrow \exp (\exp X)$ as;
(a) if $x \in Y, \Phi(x)=\left\{V \subseteq X: h_{k}^{-1}(V) \in \psi_{k}(z)\right.$, for all $z \in h_{k}^{-1}(x)$ and for all $k \in K$ such that $\left.x \in h_{k}\left(X_{k}\right)\right\}$;
(b) if $x \notin Y, \Phi(x)=\{V \subseteq X: x \in V\}$.

Clearly $\Phi$ is the finest strongly GNS on $X$, for which the mappings $f_{k}$ are $g n$-continuous. Then we shall say that $\Phi$ is the final generalized neighbourhood structure on $X$ for the family $\left(f_{k}\right)_{k \in K}$.
Proposition 4.3. Let $X$ be a set and $\left\{\left(X_{k}, \psi_{k}\right): k \in K\right\}$ be a collection of strongly gn-spaces. $\vartheta$ and $\Phi$ are the initial and final generalized neighbourhood structures
on $X$ for the functions $\left\{f_{k}: X \rightarrow X_{k}\right\}_{k \in K}$ and $\left\{h_{k}: X_{k} \rightarrow X\right\}_{k \in K}$, respectively. Then the following statements are valid.
(a) If $\left\{f_{k}: k \in K\right\}$ is a collection of injective functions and each $\psi_{k}$ is an ascending (resp. complete) GNS, so is $\vartheta$.
(c) If each $\psi_{k}$ is an ascending (resp. weak) GNS, so is $\Phi$.

Proof. (a) Let $U \in \vartheta(x)$ and $U \subset V$. Then $U \in f_{k}^{-1}\left(\psi_{k}\left(f_{k}(x)\right)\right)$ for some $k \in K$. Thus $f_{k}(U) \in \psi_{k}\left(f_{k}(x)\right)$, and so $f_{k}(V) \in \psi_{k}\left(f_{k}(x)\right)$. Therefore $V \in$ $\in f_{k}^{-1}\left(\psi_{k}\left(f_{k}(x)\right)\right) \subset \vartheta(x)$. It can be easily seen that $\vartheta$ is complete.
(b) Let $U \in \Phi(x)$ and $U \subset V$. If $x \in Y$, then $h_{k}^{-1}(U) \in \psi_{k}(z)$ for all $z \in$ $\in h_{k}^{-1}(x)$ and for all $k \in K$ such that $x \in h_{k}\left(X_{k}\right)$. Thus $h_{k}^{-1}(V) \in \psi_{k}(z)$ for all $z \in h_{k}^{-1}(x)$ and for all $k \in K$ such that $x \in h_{k}\left(X_{k}\right)$ since each $\psi_{k}$ is ascending. Therefore $V \in \Phi(x)$. If $x \notin Y$, then clearly $V \in \Phi(x)$.

Now let $U, V \in \Phi(x)$. If $x \in Y$, then $h_{k}^{-1}(U), h_{k}^{-1}(V) \in \psi_{k}(z)$ and so $h_{k}^{-1}(U) \cap h_{k}^{-1}(V)$ for all $z \in h_{k}^{-1}(x)$ and for all $k \in K$ such that $x \in h_{k}\left(X_{k}\right)$. Thus $U \cap V \in \Phi(x)$. If $x \notin Y$, then clearly $U \cap V \in \Phi(x)$.

## 5. Product of GNSs' as an application

In [5], Császár show that how the definition of the product of topologies can be modified in order to define the product of GT's. Let $\mu_{k}$ be a GT on $X_{k}$ and $M_{\mu_{k}}$ is the union of all elements of $\mu_{k}$ for $k \in K$. By $\mu=P_{k \in K} \mu_{k}$, Császár denoted the all unions of some elements of the sets of the form; $M=\prod_{k \in K} M_{k}$ where $M_{k} \in \mu_{k}$ and, with the exception of a finite number of indices $k, M_{k}=M_{\mu_{k}} . \mu$ is called as the product of the GT's $\mu_{k}$. In this section, we discuss this concept on generalized neighbourhood systems.

Let $K \neq \emptyset$ be an index set, $X_{k} \neq \emptyset$ for $k \in K$ and $X=\prod_{k \in K} X_{k}$ is the Cartesian product of the sets $X_{k}$. Then consider the initial generalized neighbourhood structure $\vartheta$ on $X$, induced by the collection of strongly $g n$-spaces $\left\{\left(X_{k}, \psi_{k}\right): k \in K\right\}$ and the projections $\left\{p_{k}: k \in K\right\}$, where $p_{k}: X \rightarrow X_{k}$. If we denote the set of all finite intersections of $\vartheta(x)$ by $\vartheta_{\cap}(x)$, then $\vartheta_{\cap}(x)$ is all sets of the form $V=\prod_{k \in K} V_{k}$, where $V_{k} \in \psi_{k}\left(x_{k}\right)$ for each $k \in K$ and $V_{k} \neq X_{k}$ only finitely many times, for each $x \in X$. Clearly $\vartheta_{\cap}$ is a finer GNS than $\vartheta$ on $X$.

Then we can give the following result by Lemma 2.1 (b).
Proposition 5.1. If $\mu_{k}$ is a strongly GT on $X_{k}$ and $\psi_{k}=\psi_{\mu_{k}}$ for each $k \in K$, then $\mu_{\vartheta_{\cap}}=P_{k \in K} \mu_{k}$.

In the particular case, when $\mu_{k}=\tau_{k}$ is a topology on $X_{k}$ and $\psi^{k}=\psi_{\tau_{k}}$ for $k \in K, \mu_{\vartheta_{\cap}}$ coincides with the product topology of the factors $\tau_{k}$.
Proposition 5.2. The projection $p_{k}:\left(X, \vartheta_{\cap}\right) \rightarrow\left(X_{k}, \psi_{k}\right), p_{k}(x)=x_{k}$ is $g n$ continuous and gn-open for all $k \in K$.

Proof. $p_{k}$ is $g n$-continuous since $\vartheta_{\cap}$ is finer than $\vartheta$. Now let $x \in X$ and $V \in$ $\in \vartheta_{\cap}(x)$. Then $V=\prod_{k \in K} V_{k}$ where $V_{k} \in \psi^{k}\left(x_{k}\right)$ for each $k \in K$ and $V_{k} \neq X_{k}$ only finitely many times. Thus we have $p_{k}(V)=V_{k} \in \psi_{k}\left(x_{k}\right)$. Hence $p_{k}$ is $g n$-open.

Clearly $g n$-continuity of $p_{k}$ implies $\left(\psi, \psi_{k}\right)$-continuity of $p_{k}$. So $p_{k}$ is $\left(\mu_{\psi}, \mu_{\psi_{k}}\right)$-continuous by Proposition 2.1 of [2]. Then we obtain Proposition 2.7 of [5] as a corollary by Proposition 5.1 and Lemma 2.1 (b).
Corollary 5.3. Let $\mu_{k}$ be a strongly GT on $X_{k}$ and $\psi_{k}=\psi_{\mu_{k}}$ for $k \in K$. Then the projection $p_{k}$ is $\left(P_{k \in K} \mu_{k}, \mu_{k}\right)$-continuous.

Proposition 5.4. Let $A_{k} \subset X_{k}$ for $k \in K$ and $A=\prod_{k \in K} A_{k}$. Then
(a) $\imath_{\vartheta_{\cap}} A \subset \prod_{k \in K} \imath_{\psi_{k}} A_{k}$.

If there exists a finite subset $J \subset K$ such that $A_{k}=X_{k}$ for $k \in(K-J)$, then
(b) $\imath_{\vartheta_{\cap}} A=\prod_{k \in K} \imath_{\psi_{k}} A_{k}$,
(c) $I_{\vartheta_{\cap}}^{*} A=\prod_{k \in K} I_{\psi_{k}}^{*} A_{k}$.

Proof. (a) If $x \in \imath_{\vartheta_{\cap}} A$, then there exists $V \in \vartheta_{\cap}(x)$ such that $V=\prod_{k \in K} V_{k} \subset A$ where $V_{k} \in \psi_{k}\left(x_{k}\right)$ for each $k \in K$ and $V_{k} \neq X_{k}$ only finitely many times. Thus $p_{k}(x)=x_{k} \in p_{k}(V)=V_{k} \subset p_{k}(A)=A_{k}$ for each $k \in K$. Hence $x_{k} \in \imath_{\psi_{k}} A_{k}$ for each $k \in K$.
(b) If $x \in \prod_{k \in K} \imath_{\psi_{k}} A_{k}$, then $p_{k}(x)=x_{k} \in \imath_{\psi_{k}} A_{k}$ for each $k \in K$. Thus there exists $V_{k} \in \psi_{k}\left(x_{k}\right)$ such that $V_{k} \subset A_{k}$. Now let $V_{k}=X_{k}$ for $k \in(K-J)$. Then we have $V=\prod_{k \in K} V_{k} \in \vartheta_{\cap}(x)$ and $V \subset A$. Hence $x \in \imath_{\vartheta_{\cap}} A$.
(c) For eack $k \in K, p_{k}\left(I_{\vartheta_{\cap}}^{*} A\right) \subset I_{\psi_{k}}^{*} p_{k}(A)=I_{\psi_{k}}^{*} A_{k}$ by Theorem 5.7 of [8] since $p_{k}$ is $g n$-open. Then $I_{\vartheta_{\cap}}^{*} A \subset \cap_{k \in J} p_{k}^{-1}\left(I_{\psi_{k}}^{*} A_{k}\right)=\prod_{k \in K} I_{\psi_{k}}^{*} A_{k}$. Conversely; let $x \in \prod_{k \in K} I_{\psi_{k}}^{*} A_{k}$, then for $p_{k}(x)=x_{k}$ we have $x_{k} \in I_{\psi_{k}}^{*} A_{k}$ for each $k \in K$. Thus $A_{k} \in \psi_{k}\left(x_{k}\right)$ for each $k \in K$ and $A_{k}=X_{k}$ for $k \in K-J$. Therefore $A \in \vartheta_{\cap}(x)$ and this implies that $x \in I_{\vartheta_{\cap}}^{*} A$.
Proposition 5.5. If $A_{k} \subset X_{k}$ for $k \in K$ and $A=\prod_{k \in K} A_{k}$, then $\gamma_{\vartheta_{\cap}} A=$ $\prod_{k \in K} \gamma_{\psi_{k}} A_{k}$.

Proof. Let $x \in \gamma_{\vartheta_{\cap}} A$ and $V_{j} \in \psi_{j}\left(x_{j}\right)$ for a fixed index $j \in K$ and $V_{k}=X_{k}$ for $k \in K-\{j\}$. Then clearly $V \in \vartheta_{\cap}(x)$ and we have $V \cap A \neq \emptyset$. Therefore $p_{j}(V \cap A) \neq \emptyset$ and this implies that $V_{j} \cap A_{j} \neq \emptyset$. Thus $x_{j} \in \gamma_{\psi_{j}} A_{j}$ for each $j \in K$.

Hence $x \in \prod_{k \in K} \gamma_{\psi_{k}} A_{k}$. Conversely let $x \in \prod_{k \in K} \gamma_{\psi_{k}} A_{k}$ and an arbitrary $V \in$ $\in \vartheta_{\cap}(x)$. Then $V=\prod_{k \in K} V_{k}$ where $V_{k} \in \psi_{k}\left(x_{k}\right)$ for each $k \in K$ and $V_{k} \neq X_{k}$ only finitely many times. Thus $p_{k}(x)=x_{k} \in p_{k}(V)=V_{k}$ and so we have $V_{k} \cap$ $\cap A_{k} \neq \emptyset$ for each $k \in K$. Therefore $\prod_{k \in K} V_{k} \cap \prod_{k \in K} A_{k} \neq \emptyset$. Hence $x \in \gamma_{\vartheta_{\cap}} A$.

The following example shows that a similar equality need not to be valid for the closure operator in the sense of Min.

Example 5.6. Let $X=\{1,2\}$ and denote the GNSs'; $\psi$ as $\psi(1)=\{X,\{1\}\}$, $\psi(2)=\{X\}$, and $\phi$ as $\phi(1)=\{X\}, \phi(2)=\{X,\{2\}\}$ on $X$. Now consider the GNS $\vartheta_{\cap}$ on $X \times X$ induced by $\psi$ and $\phi$, then $\vartheta_{\cap}((1,1))=\{X \times X,\{1\} \times X\}$, $\vartheta_{\cap}((1,2))=\{X \times X, X \times\{2\},\{1\} \times X,\{1\} \times\{2\}\}, \vartheta_{\cap}((2,1))=\{X \times$ $\times X\}, \vartheta_{\cap}((2,2))=\{X \times X, X \times\{2\}\}$. For $A=\{1\}$, we have $\mathrm{cl}_{\psi}^{*} A=X$ and $\mathrm{cl}_{\phi}^{*} A=\{1\}$, so cll $_{\psi}^{*} A \times \mathrm{cl}_{\phi}^{*} A=X \times\{1\}$. However $\mathrm{cl}_{\vartheta_{\cap}}^{*}(A \times A)=X \times X$.

The following result is clear by Lemma 2.1 (a), Proposition 5.4 and 5.5.
Corollary 5.7. Let $\mu_{k}$ be a strongly $G T$ on $X_{k}, \psi_{k}=\psi_{\mu_{k}}$ and $A_{k} \subset X_{k}$ for $k \in K$. If $\eta=P_{k \in K} \mu_{k}$ and $A=\prod_{k \in K} A_{k}$, then we have
(a) $i_{\eta} A \subset \prod_{k \in K} i_{\mu_{k}} A_{k}$.
(b) $i_{\eta} A=\prod_{k \in K} i_{\mu_{k}} A_{k}$ for a finite index set $K$.
(c) $c_{\eta} A=\prod_{k \in K} c_{\mu_{k}} A_{k}$.

Lemma 5.8. If each $\psi_{k}$ is a weak GNS, so is $\vartheta_{\cap}$.
Proof. Let $U, V \in \psi(x)$. Then $U=\prod_{k \in K} U_{k}$ where $U_{k} \in \psi_{k}\left(x_{k}\right)$ for each $k \in$ $\in K$ and $U_{k} \neq X_{k}$ for a finite subset $I \subset K$ and $V=\prod_{k \in K} V_{k}$ where $V_{k} \in \psi_{k}\left(x_{k}\right)$ for each $k \in K$ and $V_{k} \neq X_{k}$ for a finite subset $J \subset K$. Thus $U \cap V=\prod_{k \in K}\left(U_{k} \cap\right.$ $\cap V_{k}$ ) where $U_{k} \cap V_{k} \in \psi_{k}\left(x_{k}\right)$ for each $k \in K$ and $U_{k} \cap V_{k} \neq X_{k}$ for the finite subset $I \cup J \subset K$. Hence $U \cap V \in \psi(x)$.

Therefore we can give the following result.
Theorem 5.9. If each $\psi_{k}$ is a weak GNS, then $\vartheta_{\cap}$ is the coarsest GNS on the product set $X$ for which the projections $p_{k}$ are $g n$-continuous.

Proof. Let $\phi$ be a weak GNS on the product set $X$ for which the projections $p_{k}$ are $g n$-continuous and $\phi(x) \subset \vartheta_{\cap}(x)$ for each $x \in X$. If $V \in \vartheta_{\cap}(x)$, then $V=\prod_{k \in K} V_{k}=\cap_{k \in J} p_{k}^{-1}\left(V_{k}\right)$ since $V_{k} \in \psi_{k}\left(x_{k}\right)$ for each $k \in K$ and $V_{k} \neq X_{k}$ for a finite subset $J \subset K$. On the other hand, $p_{k}^{-1}\left(V_{k}\right) \in \phi(x)$ for each $k \in K$. Thus $V \in \phi(x)$. Hence $\phi(x)=\psi(x)$ for each $x \in X$.
Theorem 5.10. Let $(Z, \phi)$ be a strongly gn-space and $f:(Z, \phi) \rightarrow\left(X, \vartheta_{\cap}\right)$, $f(z)=\left(x_{k}\right)_{k \in K}$ be a mapping. If $\phi$ is a weak GNS, then $f$ is gn-continuous iff $f_{k}=p_{k} \circ f$ is $g n$-continuous for each $k \in K$.

Proof. If $f$ is $g n$-continuous, then clearly $f_{k}=p_{k} \circ f$ is $g n$-continuous. Conversely, let $a \in Z$ and $V \in \vartheta_{\cap}(f(a))$ for $f(a)=\left(f_{k}(a)\right)_{k \in K}$. Then $V=\prod_{k \in K} V_{k}$ where $V_{k} \in \psi_{k}\left(f_{k}(a)\right)$ for each $k \in K$ and $V_{k} \neq X_{k}$ for a finite subset $J \subset$ $\subset K$. Thus $f_{k}^{-1}\left(V_{k}\right) \in \phi(a)$ for each $k \in K$ and $f_{k}^{-1}\left(V_{k}\right)=Z$ for $k \in$ $\in K-J$. Therefore $f^{-1}(V)=f^{-1}\left(\prod_{k \in K} V_{k}\right)=f^{-1}\left(\cap_{k \in K} p_{k}^{-1}\left(V_{k}\right)\right)=\cap$ $\cap_{k \in J} f^{-1}\left(p_{k}^{-1}\left(V_{k}\right)\right)=\cap_{k \in J}\left(p_{k} \circ f\right)^{-1}\left(V_{k}\right)=\cap_{k \in J} f_{k}^{-1}\left(V_{k}\right)$. Hence $f^{-1}(V) \in$ $\in \phi(a)$.

Corollary 5.11. Let $\left(Z_{k}\right)_{k \in K}$ be a family of nonempty sets and $\phi$ denotes the initial generalized neighbourhood structure on $Z=\prod_{k \in K} Z_{k}$, induced by the collection of strongly gn-spaces $\left\{\left(Z_{k}, \phi_{k}\right): k \in K\right\}$ and the projections $\left\{p_{k}: k \in\right.$ $\in K\}$, where $p_{k}: Z \rightarrow Z_{k}$. If $\left(f_{k}\right)_{k \in K}$ be a collection of mappings from $Z_{k}$ into $X_{k}$ and $\left(\phi_{k}\right)_{k \in K}$ be a collection of weak GNSs', then the product mapping

$$
\begin{aligned}
f:\left(Z, \phi_{\cap}\right) & \rightarrow\left(X, \vartheta_{\cap}\right) \\
z=\left(z_{k}\right)_{k \in K} & \rightarrow f(z)=\left(f_{k}\left(z_{k}\right)\right)_{k \in K}
\end{aligned}
$$

is $g n$-continuous iff $f_{k}$ is $g n$-continuous for each $k \in K$.
Proof. Let $a=\left(a_{k}\right)_{k \in K} \in X$ be a fixed point and $B^{j}=Z_{j} \times \prod_{k \in K-J}\left\{a_{k}\right\}$ for each $j \in K$. Now for each $k \in K$ define $\phi^{a_{k}}:\left\{a_{k}\right\} \rightarrow \exp \left(\exp \left\{a_{k}\right\}\right), \phi^{a_{k}}\left(a_{k}\right)=$ $\left\{\left\{a_{k}\right\}\right\}$. Therefore $\phi_{j}^{a}=\phi_{j} \times P_{k \in K-\{j\}} \phi^{a_{k}}$ is a weak GNS for each $j \in K$. Now consider the mappings $h^{j}:\left(Z_{j}, \phi_{j}\right) \rightarrow\left(B^{j}, \phi_{j}^{a}\right), h^{j}\left(z_{j}\right)=\left\{z_{j}\right\} \times \prod_{k \in K-J}\left\{a_{k}\right\}$ and the restriction $f_{/ B^{j}}$ of $f$ to $B^{j}$ for $j \in K . h^{j}$ is $g n$-continuous since the mappings $\left(h^{j}\right)_{k}:\left(Z_{j}, \phi_{j}\right) \rightarrow\left(Z_{k}, \phi_{k}\right)$ such that $\left(h^{j}\right)_{k}\left(z_{j}\right)=z_{j}$ for $k=j$ and $\left(h^{j}\right)_{k}\left(z_{j}\right)=a_{k}$ for $k \neq j$ are $g n$-continuous for each $k \in K$. In addition, let $z \in B^{j}$ and $V \in$ $\in \vartheta_{\cap}\left(f_{\mid B^{i}}(z)\right)$, then $f_{\mid B^{j}}^{-1}(V)=f^{-1}(V) \cap B^{j}$. We have $f^{-1}(V) \in \phi_{\cap}(z)$ since $f$ is $g n$-continuous. Therefore $f^{-1}(V)=\prod_{k \in K} U_{k}$ where $U_{k} \in \phi_{k}\left(z_{k}\right)$ for $k=j$ and $U_{k} \in \phi_{k}\left(a_{k}\right)$ for $k \neq j$ and $U_{k} \neq Z_{k}$ only finitely many times. Thus $f^{-1}(V) \cap$ $\cap B^{j}=U_{j} \times \prod_{k \in K-J}\left\{a_{k}\right\}$, so $f_{/ B^{j}}^{-1}(V) \in \phi_{j}^{a}(z)$. This implies the $g n$-continuity of $f_{/ B^{j}}$. Hence $f_{j}=p_{j} \circ f_{\mid B^{j}} \circ h^{j}$ is $g n$-continuous for each $j \in K$. Conversely; $g n$ continuity of the mappings $h_{k}=f_{k} \circ p_{k}$ implies the $g n$-continuity of the product mapping $f$ by Theorem 5.10.

Anyone can obtain similar results for $\left(\psi, \psi^{\prime}\right)$-continuity.

## References

[1] Á. CsÁszÁr, General Topology, Akadémiai Kiadó, Budapest, 1978.
[2] Á. CsÁszÁr, Generalized topology, generalized continuity, Acta Math. Hungar., 96 (4) (2002), 351-357.
[3] Á. CsÁszÁr, Extremally disconnected generalized topologies, Annales Univ. Sci. Budapest., 47 (2004), 91-96.
[4] Á. CsÁszár, On generalized neighbourhood systems, Acta Math. Hungar., 121 (4) (2008), 394-400.
[5] Á. CsÁszÁr, Product of generalized topologies, Acta Math. Hungar., 123 (1-2) (2009), 127-132.
[6] F. Hausdorff, Grundzüge der Mengenlehre, Verlag von Velt\&Co. (Leipzig 1914).
[7] D. C. Kent and W. K. Min, Neighborhood spaces, Internat. J. Math. \& Math. Sci., 32 (2002), 387-399.
[8] W. K. Min, Some results on generalized topological spaces and systems, Acta Math. Hungar., 108 (1-2) (2005), 171-181.
[9] W. K. Min, p-Stacks on supratopological spaces, Commun. Korean Math. Soc. 21 (2006), 749-758.
[10] W. K. Min, On weak neighborhood systems and spaces, Acta Math. Hungar., 121 (3) (2008), 283-292.
[11] T. Richmond and J. Šlapal, Neighborhood spaces and convergence, Topology Proceedings, 35 (2010), 165-175.
[12] R. Shen, Complete generalized neighborhood systems, Acta Math. Hungar., 129 (1-2) (2010), 160-165.

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## GENERALIZED ABSOLUTE CONVERGENCE OF FOURIER SERIES

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#### Abstract

Here, sufficiency conditions are obtain for the convergence of the Fourier series of the form $\sum_{k \in Z} \varpi_{|k|}\left(\varphi\left(\left|\hat{f}\left(n_{k}\right)\right|\right)\right)$, where $\hat{f}\left(n_{k}\right)$ are Fourier coefficients of $f,\left\{\varpi_{n}\right\}$ is a certain sequence of positive numbers, $\varphi(u)(u \geq 0)$ is an increasing concave function and $\left\{n_{k}\right\}_{k=1}^{\infty}$ is an increasing sequence of natural numbers with $n_{-k}=-n_{k}$ for all $k$.


## 1. Introduction

Let $f$ be a $2 \pi$-periodic real function in $L^{1}[0,2 \pi]$ and

$$
\begin{equation*}
f(x) \sim \frac{1}{2} a_{0}+\sum_{k=1}^{\infty}\left(a_{k} \cos k x+b_{k} \sin k x\right) \equiv \sum_{k \in Z} \hat{f}(k) e^{i k x} \tag{1}
\end{equation*}
$$

be the Fourier series of $f$, wherein $a_{n}, b_{n}$ are Fourier coefficients of $f$ and $\hat{f}(k)=$ $\frac{a_{|k|}-i b_{|k|} \operatorname{sgn}(k)}{2},(k \in \mathbb{Z})$.

Generalizing the concept of $\beta$-absolute convergence of Fourier series [4], for $f \in L^{p}([0,2 \pi])(1<p \leq 2)$ L. Leindler [2] obtained sufficiency condition for the convergence of the series

$$
\begin{equation*}
\sum_{k \in Z} \varpi_{|k|}(\varphi(|\hat{f}(k)|)) \tag{2}
\end{equation*}
$$

where $\varphi(u)(u \geq 0, \varphi(0)=0)$ is an increasing and concave function, $\varpi_{0}=0$ and $\left\{\varpi_{n}\right\}_{n=1}^{\infty}$ is a certain sequence of positive numbers. For $\varpi_{n}=n^{0}, \forall n$ and $\varphi(x)=x^{\beta}(0<\beta \leq 1)$, one gets $\beta$-absolute convergence of Fourier series.

Here, we have generalized non-lacunary analogue of Ogata result [3, Theorem 1] and also generalize the results [5, Theorem 1 and Theorem 2] by obtaining certain sufficiency conditions for the convergence of series

$$
\begin{equation*}
\sum_{k \in Z} \varpi_{|k|}\left(\varphi\left(\left|\hat{f}\left(n_{k}\right)\right|\right)\right) \tag{3}
\end{equation*}
$$

where $\left\{n_{k}\right\}_{k=1}^{\infty}$ is an increasing sequence of natural numbers with $n_{-k}=-n_{k}$ for all $k$. Here, for $n_{k}=k$, for all $k$, (3) reduces to (2).

Definition 1.1. A sequence $\gamma:=\left\{\gamma_{n}\right\}$ of positive terms is quasi $\beta$-power monotone increasing (decreasing) if there exists a constant $K:=K(\beta, \gamma) \geq 1$ such that

$$
\begin{equation*}
K n_{\beta} \gamma_{n} \geq m_{\beta} \gamma_{m} \quad\left(n_{\beta} \gamma_{n} \leq K m_{\beta} \gamma_{m}\right) \tag{4}
\end{equation*}
$$

holds for any $n \geq m, m=1,2, \ldots$.
Definition 1.2. Given an interval I, a sequence of non-decreasing positive real numbers $\Lambda=\left\{\lambda_{m}\right\}(m=1,2, \ldots)$ such that $\sum_{m} \frac{1}{\lambda_{m}}$ diverges and nonnegative convex function $\phi(x)$ defined on $[0, \infty)$ such that $\frac{\phi(x)}{x} \rightarrow 0$ as $x \rightarrow 0$. We say that $\mathrm{f} \in \phi \Lambda B V(I)$ (that is f is a function of $\phi \Lambda$-bounded variation over (I)) if

$$
V_{\Lambda_{\phi}}(f, I)=\sup _{\left\{I_{m}\right\}}\left\{V_{\Lambda_{\phi}}\left(\left\{I_{m}\right\}, f, I\right)\right\}<\infty
$$

where

$$
V_{\Lambda_{\phi}}\left(\left\{I_{m}\right\}, f, I\right)=\sum_{m}\left(\frac{\phi\left(\left|f\left(b_{m}\right)-f\left(a_{m}\right)\right|\right)}{\lambda_{m}}\right)
$$

and $\left\{I_{m}\right\}$ is a sequence of non-overlapping subintervals $I_{m}=\left[a_{m}, b_{m}\right] \subset I=$ $[a, b]$.

Here, $\phi$ is said to have property $\Delta_{2}$ if there is a constant $d(d \geq 2)$ so that $\phi(2 x) \leq d \phi(x)$ for all $x \geq 0$.

In the above definition, for $\phi(x)=x^{p}(1 \leq p<\infty)$ one gets the class $\Lambda B V^{(p)}(I)$ and for $\lambda_{m} \equiv 1$, for all $m$, one gets the class $\phi B V$.

To formulate the theorems we need following notations. Let the quadraticintegral modulus of continuity of $f$ over $[0,2 \pi]$ of higher differences of order $l \geq 1$ with a weight function $\alpha$ is defined as

$$
\omega_{l}^{(2)}(\delta, f, \alpha)=\sup _{0 \leq h \leq \delta}\left(\int_{0}^{2 \pi}\left|\triangle_{h}^{l} f(x)\right|^{2} \alpha(x) d x\right)^{1 / 2}
$$

where

$$
\triangle_{h}^{l} f(x)=\sum_{i=0}^{l}(-1)^{i}\binom{l}{i} f(x+(l-i) h)
$$

In the above notation, for $\alpha(x)=1$, omit writing $\alpha$, we get $\omega_{l}^{(2)}(\delta, f)$. Similarly, for $l=1$ we omit writing $l$. Let

$$
R_{n+1}^{(2)}(f, \alpha):=\left(\int_{0}^{2 \pi}\left|f(x)-S_{n}(x)\right|^{2} \alpha(x) d x\right)^{1 / 2}
$$

where

$$
S_{n}(x):=\sum_{|k| \leq n} \hat{f}(k) e^{i k x}
$$

THEOREM 1.3. Let $\left\{\alpha_{k}\right\}_{k \in \mathbb{N}}$ be a sequence of non-negative functions satisfying $\int_{0}^{2 \pi} \alpha_{k}(x) d x=1$ and $\lim _{M \rightarrow \infty} \sum_{m=M}^{\infty}\left|\hat{\alpha_{k}}(m)\right|^{2} \sigma(m)=0$, uniformly in $k, \varphi(u)$ $(u \geq 0, \varphi(0)=0)$ be an increasing and concave function and $\varpi:=\left\{\varpi_{n}\right\}$ be a quasi $\eta$-power-monotone decreasing sequence of positove numbers with some negative $\eta$. Here $\hat{\alpha}_{k}(m)=0$ whenever $\sigma(m)=\infty$. If

$$
\sum_{k=1}^{\infty} \varpi_{k} \varphi\left(\frac{\omega_{l}^{(2)}\left(\frac{\pi}{n_{k}}, f, \alpha_{k}\right)}{\sqrt{k}}\right)<\infty
$$

and

$$
\sum_{k=1}^{\infty} \varpi_{k} \varphi\left(\frac{1}{\sqrt{k} n_{k}^{l}}\right)<\infty
$$

or

$$
\sum_{k=1}^{\infty} \varpi_{k} \varphi\left(\frac{R_{n_{k}}^{(2)}\left(f, \alpha_{k}\right)}{\sqrt{k}}\right)<\infty
$$

then (3) holds.
Theorem 1.4. Let $\varpi$ and $\varphi$ be as in Theorem 1.1. Iff $\in \Lambda B V^{(p)}([0,2 \pi]), 1 \leq$ $\leq p<2 r, 1<r<\infty$ and

$$
\sum_{k=1}^{\infty} \varpi_{k} \varphi\left(\left(\frac{\left(\omega^{((2-p) s+p)}\left(\frac{1}{n_{k}}, f\right)\right)^{2-p / r}}{k\left(\sum_{j=1}^{n_{k}}\left(\frac{1}{\lambda_{j}}\right)\right)^{1 / r}}\right)^{1 / 2}\right)<\infty
$$

where $\frac{1}{r}+\frac{1}{s}=1$, then (3) holds.
In the above theorem, for $\omega_{n}=1$, for all $n$, and $\varphi(u)=u^{\beta}$ one gets result [5, Theorem 1, for $0<\beta \leq 1$ ] as a particular case.

Theorem 1.5. Let $\varpi$ and $\varphi$ be as in Theorem 1.1. If $\phi$ satisfies $\Delta_{2}$ property, $f \in \phi \Lambda B V([0,2 \pi]), 1 \leq p<2 r ; 1<r<\infty$, and

$$
\sum_{k=1}^{\infty} \varpi_{k} \varphi\left(\left(\frac{\phi^{-1}\left(\omega^{((2-p) s+p)}\left(\frac{1}{n_{k}}, f\right)\right)^{2-p / r}}{k\left(\sum_{j=1}^{n_{k}}\left(\frac{1}{\lambda_{j}}\right)\right)^{1 / r}}\right)^{1 / 2}\right)<\infty
$$

where $\frac{1}{r}+\frac{1}{s}=1$, then (3) holds.
Theorem 1.3, with $\omega_{n}=1$, for all $n$, and $\varphi(u)=u^{\beta}$ gives the result [5, Theorem 2, for $0<\beta \leq 1$ ] as a particular case.

We need the following lemmas to prove the results.
Lemma 1.6. ([2, Lemma 2]) Let $1<p \leq 2$, $\varpi$ and $\varphi$ be as in Theorem 1.1, $m$ be an arbitrary natural number and $\left\{\alpha_{n}\right\}$ be a monotone non-increasing sequence of non-negative numbers. Then the conditions

$$
\sigma(\varpi, m):=\sum_{k=1}^{\infty} k^{\frac{1}{m}-1} \varpi_{\left[k^{1 / m}\right]} \varphi\left(k^{\frac{1-p}{p m}} \alpha_{k^{1 / m}}\right)<\infty
$$

and

$$
\sigma(\varpi):=\sum_{k=1}^{\infty} \varpi_{k} \varphi\left(k^{\frac{1-p}{p}} \alpha_{k}\right)<\infty
$$

are equivalent, wherein $[x]$ denotes the integral part of $x$.
Lemma 1.7. ([3, Lemma 1]) Put $\hat{\alpha}_{v}(m)=\frac{1}{2 \pi} \int_{0}^{2 \pi} \alpha_{v}(x) e^{-i m x} d x(v=1,2, \ldots)$. If $\lim _{M \rightarrow \infty} \sum_{m=M}^{\infty}\left|\hat{\alpha}_{k}(m)\right|^{2} \sigma(m)=0$, uniformly in $k$, and $\hat{\alpha}_{k}(m)=0$ whenever $\sigma(m)=\infty$, then for a given constant $\lambda>1$ there exists a positive integer $\mu$ which satisfies the following property; for

$$
g(x) \sim \sum_{k=-\infty}^{k=\infty} \hat{g}\left(n_{k}\right) e^{i n_{k} x}
$$

where

$$
\hat{g}\left(n_{k}\right)= \begin{cases}0, & |k|<\mu \\ \hat{f}\left(n_{k}\right), & \text { otherwise }\end{cases}
$$

we have

$$
\sum_{|k| \geq v}\left|\hat{g}\left(n_{k}\right)\right|^{2} \leq \frac{\lambda}{2^{l-1}} \omega_{l}^{(2)}\left(\frac{\pi}{n_{v}}, g, \alpha_{v}\right)^{2}
$$

and

$$
\sum_{|k| \geq v}\left|\hat{g}\left(n_{k}\right)\right|^{2} \leq \lambda R_{n_{v}}^{(2)}\left(g, \alpha_{v}\right)^{2}
$$

Lemma 1.8. ([2, Theorem 1]) Let $\varpi$ and $\varphi$ be as in Theorem 1.1.
If $\sum_{n=1}^{\infty} \varpi_{n} \varphi\left(\left\{\frac{1}{n} \sum_{k \geq n} \rho_{k}^{2}\right\}^{1 / 2}\right)<\infty$, where $\rho_{n}=\left(a_{n}^{2}+b_{n}^{2}\right)^{1 / 2}$, then

$$
\sum_{n=1}^{\infty} \varpi_{n} \varphi\left(\rho_{n}\right)<\infty
$$

Lemma 1.9. Let $1<p \leq 2$ and $\varpi$ be as in Lemma 1.4 and let $f$ and $g$ be as in Lemma 1.5.
(i) $\sum_{k=1}^{\infty} \varpi_{k} \varphi\left(k^{\frac{1-p}{p}} \omega_{l}^{(2)}\left(\frac{\pi}{n_{k}}, f, \alpha_{k}\right)\right)<\infty$ and $\sum_{k=1}^{\infty} \varpi_{k} \varphi\left(k^{\frac{1-p}{p}} n_{k}^{-l}\right)<\infty, \Rightarrow$

$$
\sum_{k=1}^{\infty} \varpi_{k} \varphi\left(k^{\frac{1-p}{p}} \omega_{l}^{(2)}\left(\frac{\pi}{n_{k}}, g, \alpha_{k}\right)\right)<\infty .
$$

(ii) $\sum_{k=1}^{\infty} \varpi_{k} \varphi\left(k^{\frac{1-p}{P}} R_{n_{k}}^{(2)}\left(f, \alpha_{k}\right)\right)<\infty \Rightarrow$

$$
\sum_{k=1}^{\infty} \varpi_{k} \varphi\left(k^{\frac{1-p}{p}} R_{n_{k}}^{(2)}\left(g, \alpha_{k}\right)\right)<\infty .
$$

Lemma 1.7 can be proved in a similar way to the corresponding Lemma [3, Lemma 3].
Proof of Theorem i.i. In order to simplify writing we shall write $k^{1 / m}$ instead of $\left[k^{1 / m}\right]$.

Let $m>-\eta+1$. From Abel rearrangement and Jensen inequality, we have

$$
\begin{gathered}
\sum_{|j|=1}^{\infty} \varpi_{|j|} \varphi\left(\left|\hat{g}\left(n_{j}\right)\right|\right)=\sum_{|j|=1}^{\infty}\left(\sum_{k=1}^{|j|^{m}} \frac{\varpi_{|j|} \varphi\left(\left|\hat{g}\left(n_{j}\right)\right|\right.}{|j|^{m}}\right) \leq \\
\leq \sum_{k=1}^{\infty}\left(\sum_{|j|=k^{1 / m}}^{\infty} \frac{\varpi_{|j| \varphi\left(\hat{g}\left(n_{j}\right) \mid\right.}}{|j|^{m}}\right) \leq \\
\leq \sum_{k=1}^{\infty}\left(\sum_{|j|=k^{1 / m}}^{\infty} \frac{\varpi_{|j|}}{|j|^{m}}\right) \varphi\left(\frac{\sum_{|j|=k^{1 / m}}^{\infty} \varpi_{|j|}|j|^{-m}\left|\hat{g}\left(n_{j}\right)\right|}{\sum_{|j|=k^{1 / m}}^{\infty} \varpi_{|j|}|j|^{-m}}\right)=: S_{1} .
\end{gathered}
$$

Since $n+\eta>1$ and the sequence $\varpi$ is quasi $\eta$-power monotone decreasing, we have

$$
\sum_{n=\mu}^{\infty} \varpi_{n} n^{-m}=\sum_{n=\mu}^{\infty} \varpi_{n} n^{\eta} n^{-m-\eta} \leq K \varpi_{\mu} \mu^{\eta} \sum_{n=\mu}^{\infty} n^{-m-\eta} \leq K_{1} \varpi_{\mu} \mu^{1-m} .
$$

Thus, from Lemmas 1.4, 1.5 and 1.7 we have

$$
\begin{gathered}
\sum_{j=-\infty}^{j=\infty} \varpi_{j} \varphi\left(\left|\hat{g}\left(n_{j}\right)\right|\right) \leq \\
\leq \sum_{k=1}^{\infty} \varpi_{k^{\frac{1}{m}}} k^{\frac{1}{m}-1} \varphi\left(\frac{\left(\sum_{|j|=k^{1 / m}}^{\infty} \varpi_{|j|}^{2}|j|^{-2 m}\right)^{1 / 2}\left(\sum_{|j|=k^{1 / m}}^{\infty}\left|\hat{g}\left(n_{j}\right)\right|^{2}\right)^{1 / 2}}{\sum_{|j|=k^{1 / m}}^{\infty} \varpi_{|j|}|j|^{-m}}\right) \leq \\
\leq \sum_{k=1}^{\infty} \varpi_{k^{\frac{1}{m}}} k^{\frac{1}{m}-1} \varphi\left(k^{\frac{-1}{2 m}}\left(\sum_{|j|=k^{1 / m}}^{\infty}\left|\hat{g}\left(n_{j}\right)\right|^{2}\right)^{1 / 2}\right)<\infty
\end{gathered}
$$

Hence, we obtain $\sum_{j=-\infty}^{j=\infty} \varpi_{j} \varphi\left(\left|\hat{g}\left(n_{j}\right)\right|\right)<\infty$, which is equivalent to

$$
\sum_{j=-\infty}^{j=\infty} \varpi_{j} \varphi\left(\left|\hat{f}\left(n_{j}\right)\right|\right)<\infty .
$$

This completes the proof of the Theorem 1.1.
Proof of Theorem i.2. Proceeding as in the proof of result [5, Theorem 1, with $n_{k}=k$, for all $k$ ], we get

$$
R_{n_{M}}=\sum_{\left|n_{k}\right| \geq n_{M}}\left|\hat{f}\left(n_{k}\right)\right|^{2}=O\left(\left(\frac{\Omega_{1 / n_{M}}}{\sum_{j=1}^{n_{M}} \frac{1}{\lambda_{j}}}\right)^{1 / r}\right),
$$

where $\Omega_{h}=\left(\omega^{(2-p) s+p}(h, f)\right)^{2 r-p}$. Therefore, we have

$$
\sum_{|k|=1}^{\infty} \omega_{|k|} \varphi\left(\left\{\frac{1}{k} R_{n_{k}}\right\}^{1 / 2}\right)=O\left(\sum_{k=1}^{\infty} \omega_{k} \varphi\left(\left(\frac{1}{k}\left(\frac{\Omega_{1 / n_{k}}}{\left.\sum_{j=1}^{n_{k} \frac{1}{\lambda_{j}}}\right)^{1 / r}}\right)^{1 / 2}\right)\right)\right.
$$

In view of $f \in \phi \Lambda B V([0,2 \pi]) \Rightarrow f$ is bounded $\Rightarrow f \in L^{2}[0,2 \pi]$, the result follows from Lemma 1.6.

Similarly, we can prove the Theorem 1.3.

## References

[1] L. LeindLer, Comments on the absolute convergence of Fourier series, Hokkaido Mathematical J., V. 30 (2001), 221-230.
[2] L. Leindler, On the generalized absolute convergence of Fourier series, Hokkaido Mathematical J., V. 30 (2001), 241-251.
[3] Naoko Ogata, On the absolute convergence of lacunary Fourier series with the generalized condition $B_{2}$, Acta Sci. Math. (Szeged), 69 (2003), 656-666.
[4] J. R. Patadia and V. M. Shah, On the absolute convergence of lacunary Fourier series, J. Indian Math. Soc., 44(1980), 267-273.
[5] R. G. Vyas, On the absolute convergence of Fourier series of functions of $\Lambda B V^{(p)}$ and $\varphi \Lambda B V$, Georgian Math. J., 14 (2007), no.4, 769-774.
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# ON GENERALIZED $\alpha$-CLOSED SETS IN ISOTONIC SPACES 

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#### Abstract

The purpose of the present paper is to introduce the concept of generalized $\alpha$-closed sets in isotonic spaces and study their fundamental properties. The generalized closed sets are then used to define generalized $\alpha$-continuous functions and investigate some of their characterizations.


## 1. Introduction

Closure spaces and (more generally) isotonic spaces have already been studied by Hausdorff [8], Day [1], Hammer [6,7], Gnilka [2,3,4], Stadler [10, 11], and Habil and Elzenati [5].

A function $\mu$ from the power set $P(X)$ of a nonempty set $X$ into itself is called a generalized closure operator (briefly GCO) on $X$ and the pair $(X, \mu)$ is said to be generalized closure space (briefly GCS). Generalized closure spaces, a strong generalization of topological spaces, have application in several branches of pure and applied mathematics, as lattice theory, logic, general topology, digital topology and convex geometry. In 1993 Maki et al [9] introduced the notion of generalized $\alpha$-closed sets in topology. For each result known in topological spaces it is interesting to find out which are the minimal assumption that allow its extension to generalized closure spaces.

As in topological spaces, there are many hereditary properties that hold in isotonic spaces, and we note that not every property which holds in topological spaces must hold in isotonic spaces. However, since every topological space is an isotonic space, we note that if a property does not hold in a topological space, it must not hold in any isotonic space either. In this paper, we introduced the notion of generalized $\alpha$-closed sets in $(X, \mu)$ and study some of its basic properties.

## 2. Preliminaries

Let $X$ be a set, $P(X)$ its power set and $\mu: P(X) \rightarrow P(X)$ be an arbitrary set-valued set-function, called a closure function. We call $\operatorname{cl}(A), A \subseteq X$, the closure of $A$, and we call the pair $(X, \mu)$ a generalized closure space. Consider the following axioms of the closure function for all $A, B, A_{\lambda} \in P(X)$ and $\lambda \in \Lambda$ :
(K0) $\operatorname{cl}(\phi)=\phi$.
(Kl) $A \subseteq B$ implies $\operatorname{cl}(A) \subseteq \operatorname{cl}(B)$ (isotonic).
(K2) $A \subseteq \operatorname{cl}(A)$ (expanding).
(K3) $\operatorname{cl}(A \cup B) \subseteq \operatorname{cl}(A) \cup \operatorname{cl}(B)$ (sub-additive).
(K4) $\operatorname{cl}(\operatorname{cl}(A))=\operatorname{cl}(A)$ (idempotent).
(K5) $\cup_{\lambda \in \Lambda} \operatorname{cl}\left(A_{\lambda}\right) \subseteq \operatorname{cl}\left(\cup_{\lambda \in \Lambda} A\right)$ (additive).
The dual of a closure function is the interior function Int: $P(X) \rightarrow P(X)$ which is defined by

$$
\operatorname{Int}(A):=X-\operatorname{cl}(X-A)
$$

Given the interior function $\operatorname{Int}: P(X) \rightarrow P(X)$, the closure function is recovered by

$$
\operatorname{cl}(A):=X-\operatorname{Int}(X-A)
$$

for all $A \in P(X)$. A set $A \in P(X)$ is closed in the generalized closure space $(X, \mu)$ if $\operatorname{cl}(A)=A$ holds. It is open if its complement $X-A$ is closed or equivalently $A=\operatorname{Int}(A)$.
Definitions 2.1. (i) The space ( $X, \mu$ ) is said to be isotonic if $\mu$ is grounded and isotonic.
(ii) The space $(X, \mu)$ is said to be a neighborhood space if $\mu$ is grounded, expansive and isotonic.
(iii) The space $(X, \mu)$ is said to be a closure space if $\mu$ is grounded, expansive, and isotonic and idempotent.
(iv) The space $(X, \mu)$ is said to be a Cech closure space if $\mu$ is grounded, expansive, isotonic and additive.
(v) A subset $A$ of $X$ is said to be closed if $\mu A=A$. It is open if its complement is closed.
(vi) The empty set and the whole space are both open and closed.

Definition 2.2 ([10]). Let $(X, \mu)$ and $(Y, \nu)$ be isotonic spaces. A function $f:(X, \mu) \rightarrow(Y, \nu)$ is said to be continuous if $\mu f^{-1}(B) \subseteq f^{-1}(\nu B)$ for all $B \in$ $\in P(Y)$.
Definition 2.3 ([10]). Let $(X, \mu)$ and ( $Y, \nu$ ) be isotonic spaces. A function $f:(X, \mu) \rightarrow(Y, \nu)$ is said to be closure preserving if $\mu f(A) \subseteq \nu f(A)$ for all $A \in P(X)$.

Theorem 2.4 ([10]). Let $(X, \mu)$ and ( $Y, \nu$ ) be isotonic spaces and let $f:(X, \mu) \rightarrow(Y, \nu)$ be a function. Then the following properties are equivalent:
(i) $f$ is continuous.
(ii) $f$ is closure preserving.
(iii) $f(A) \subseteq B$ implies $f(\mu A) \subseteq \nu B$ for all $A \in P(X)$ and $B \in P(Y)$.

Let $(X, \mu)$ and $(Y, \nu)$ be isotonic spaces and let $f:(X, \mu) \rightarrow(Y, \nu)$ be a function. If $f$ is closure preserving, then $f^{-1}(G)$ is an open subset of $(X, \mu)$ for every open subset $G$ of $(Y, \nu)$. Let $(X, \mu)$ and $(Y, \nu)$ be isotonic spaces.

A function $f:(X, \mu) \rightarrow(Y, \nu)$ is said to be closed (resp. open) if $f(F)$ is a closed (resp. open) subset of $(Y, \nu)$ whenever $F$ is a closed (resp. open) subset of $(X, \nu)$.
Lemma 2.5. Let $\left(A, \mu_{A}\right)$ be a closed subspace of $(X, \mu)$. If $G$ is an open subset of $\left(A, \mu_{A}\right)$, then $G$ is an open subset of $(X, \mu)$.
Definition 2.6. A set $A$ in $(X, \mu)$ is said to be $\alpha \mu$-open (resp. semi-open) if $A \subseteq$ $\subseteq \operatorname{Int}(\mu(\operatorname{Int}(A)))($ resp. $A \subseteq \mu(\operatorname{Int}(A)))$. The complement of an $\alpha \mu$-open (resp. semi-open) set is $\alpha \mu$-closed (resp. semi $\mu$-closed).

## 3. Generalized $\alpha$-closed sets and generalized $\alpha$-open sets

In this section, we introduce generalized $\alpha$-closed sets in isotonic spaces and study some of its fundamental properties.
Definition 3.1. Let $(X, \mu)$ be an isotonic space. A subset $A \subseteq X$ is called a generalized $\alpha$-closed set (briefly g $\alpha$-closed) set, if $\alpha \mu(A) \subseteq G$ whenever $G$ is an open subset of $(X, \mu)$ with $A \subseteq G$. A subset $A \subseteq X$ is called a generalized $\alpha$-open set, (briefly g $\alpha$-open) set, if its complement is $\mathrm{g} \alpha$-closed.
Remark 3.2. Every closed set is $g \alpha$-closed. The converse is not true as can be seen from the following example.
Example 3.3. Let $X=\{1,2\}$ and define an isotonic operator $\mu$ on $X$ by $\mu \phi=\phi$ and $\mu\{1\}=\mu\{2\}=\mu X=X$. The closed sets are $\{X, \phi\}$ and the $\mathrm{g} \alpha$-closed sets are $\{X, \phi,\{1\}\}$. Then $\{1\}$ is $\mathrm{g} \alpha$-closed but it is not closed.
Theorem 3.4. Let ( $X, \mu$ ) be an isotonic space and let $\mu$ be additive. If $A$ and $B$ are $\mathrm{g} \alpha$-closed subsets of $(X, \mu)$, then $A \cup B$ is $\mathrm{g} \alpha$-closed.

Proof. Let $G$ be an open subset of $(X, \mu)$ such that $A \cup B \subseteq G$. Then $A \subseteq G$ and $B \subseteq G$. Since $A$ and $B$ are $\mathrm{g} \alpha$-closed, $\alpha \mu A \subseteq G$ and $\alpha \mu B \subseteq G$. Consequently, $\alpha \mu(A \cup B)=\alpha \mu(A) \cup \alpha \mu(B) \subseteq G$. Therefore, $A \cup B$ is $\mathrm{g} \alpha$-closed.

The intersection of two $g \alpha$-closed sets need not be a $g \alpha$-closed set as can be seen from the following example.

Example 3.5. Let $X=\{1,2,3\}$ and define an isotonic operator $\mu$ on $X$ by $\mu \phi=\phi$ and $\mu\{1\}=\{1,2\}, \mu\{2\}=\mu\{3\}=\mu\{2,3\}=\{2,3\}$ and $\mu\{1,2\}=\mu\{1,3\}=\mu X=X$. Then $\{1,2\}$ and $\{1,3\}$ are $g \alpha$-closed but $\{1,2\} \cap$ $\cap\{1,3\}=\{1\}$ is not $\mathrm{g} \alpha$-closed.
Theorem 3.6. Let $(X, \mu)$ be an isotonic space. If $A$ is $\mathrm{g} \alpha$-closed and $F$ is closed in ( $X, \mu$ ), then $A \cap F$ is $\mathrm{g} \alpha$-closed.

Proof. Let $G$ be an open subset of $(X, \mu)$ such that $A \cap F \subseteq G$. Then $A \subseteq G \cup$ $\cup(X-F)$ and so $\alpha \mu(A) \subseteq G \cup(X-F)$. Then $\alpha \mu(A) \cap F \subseteq \bar{G}$. Since $F$ is closed, $\alpha \mu(A \cap F) \subseteq G$. Hence, $A \cap F$ is g $\alpha$-closed.
Theorem 3.7. Let $(Y, \nu)$ be a closed subspace of $(X, \mu)$. If $F$ is a g $\alpha$-closed subset of $(Y, \nu)$, then $F$ is a g $\alpha$-closed subset of $(X, \mu)$.
Proof. Let $G$ be an open subset of $(X, \mu)$ such that $F \subseteq G$. Then $F \subseteq G \cap Y$. Since $F$ is $\mathrm{g} \alpha$-closed and $G \cap Y$ is open in $(Y, \nu), \alpha \mu(F) \cap Y=\alpha \nu(F) \subseteq G$. But $Y$ is a closed subset of $(X, \mu)$ and $\alpha \mu(F) \subseteq G$. Hence, $F$ is a g $\alpha$-closed subset of $(X, \mu)$.

The following statement is obviuos:
Theorem 3.8. Let $(X, \mu)$ be an isotonic space and let $A \subseteq X$. If $A$ is both open and $\mathrm{g} \alpha$-closed, then $A$ is closed.
Theorem 3.9. Let $(X, \mu)$ be an isotonic space and let $A \subseteq X$. If $A$ is a $\alpha$-closed, then $\alpha \mu(A)-A$ has no nonempty closed subset.

Proof. Suppose that $A$ is $\mathrm{g} \alpha$-closed. Let $F$ be a closed subset of $\alpha \mu(A)-A$. Then $F \subseteq \alpha \mu(A) \cap(X-A)$ and so $A \subseteq X-F$. Consequently, $F \subseteq X-\alpha \mu(A)$. Since $F \subseteq \alpha \mu(A), F \subseteq \alpha \mu(A) \cap(X-\alpha \mu(A))=\phi$, thus $F=\phi$. Therefore, $\alpha \mu(A)-A$ contains no nonempty closed set.

The converse of the previous theorem is not true as can be seen from the following example.
Example 3.10. Let $X=\{a, b, c, d\}$ and define an isotonic operator $\mu$ on $X$ by $\mu \phi=\phi$ and $\mu\{a\}=\{a, c\}, \mu\{b\}=\{b, c\}, \mu\{c\}=\mu\{d\}=\mu\{c, d\}=\{c, d\}$ and $\mu\{a, b\}=\mu\{a, c\}=\mu\{a, d\}=\mu\{b, c\}=\mu\{b, d\}=\mu\{a, b, c\}=$ $=\mu\{a, b, d\}=\mu\{b, c, d\}=\mu\{a, c, d\}=\mu X=X$. Then $\alpha \mu\{b\}-\{b\}=\{c\}$ does not contain nonempty closed set. But $\{b\}$ is not $\mathrm{g} \alpha$-closed.
Corollary 3.11. Let $(X, \mu)$ be an isotonic space and let $A$ be a g $\alpha$-closed subset of $(X, \mu)$. Then $A$ is closed if and only if $\alpha \mu(A)-A$ is closed.

Proof. Let $A$ be a $\alpha$-closed subset of $(X, \mu)$. If $A$ is closed, then $\alpha \mu(A)-A=\phi$. But $\phi$ is always closed. Therefore, $\alpha \mu(A)-A$ is closed.

Conversely, suppose that $\alpha \mu(A)-A$ is closed. As $A$ is g $\alpha$-closed, $\alpha \mu(A)-A=\phi$ by Theorem 3.9. Consequently, $\alpha \mu(A)=A$. Hence, $A$ is closed.

Theorem 3.12. Let $(X, \mu)$ be an isotonic space. $A$ set $A \subseteq X$ is g $\alpha$-open if and only if $F \subseteq X-\alpha \mu(X-A)$ whenever $F$ is closed and $F \subseteq A$.

Proof. Suppose that $A$ is g $\alpha$-open and let $F$ be a closed subset of $(X, \mu)$ such that $F \subseteq A$. Then $X-A \subseteq X-F$. But $X-A$ is $g \alpha$-closed and $X-F$ is open. It follows that $\alpha \mu(X-A) \subseteq X-F$ and hence $F \subseteq X-\alpha \mu(X-A)$.

Conversely, let $G$ be an open subset of $(X, \mu)$ such that $X-A \subseteq G$. Then $X-G \subseteq A$. Since $X-G$ is closed, $X-G \subseteq X-\alpha \mu(X-A)$. Consequently, $\alpha \mu(X-A) \subseteq G$. Hence, $X-A$ is $\mathrm{g} \alpha$-closed and so $A$ is $\mathrm{g} \alpha$-open.

The union of two g $\alpha$-open sets need not be a g $\alpha$-open set as we can see in Example 3.5: Put $A=\{2\}$ and $B=\{3\}$. Then $A$ and $B$ are g $\alpha$-open but $A \cup B=\{2,3\}$ is not $\mathrm{g} \alpha$-open.

Theorem 3.13. Let $(X, \mu)$ be an isotonic space. If $A$ is $\mathrm{g} \alpha$-open and $B$ is open in $(X, \mu)$, then $A \cup B$ is $g \alpha$-open.

Proof. Let $F$ be a closed subset of $(X, \mu)$ such that $F \subseteq A \cup B$. Then $X-(A \cup B) \subseteq X-F$. Hence, $(X-A) \cap(X-B) \subseteq X-F$. By Theorem 3.6., $(X-A) \cap(X-B)$ is g $\alpha$-closed. Therefore, $\alpha \mu((X-A) \cap(X-B)) \subseteq X-F$. Consequently, $F \subseteq X-\alpha \mu((X-A) \cap(X-B))=X-\alpha \mu(X-(A \cup B))$. By Theorem 3.12., $A \cup B$ is g $\alpha$-open.
Theorem 3.14. Let $(X, \mu)$ be an isotonic space. If $A$ and $B$ are $g \alpha$-open subsets of $(X, \mu)$, then $A \cap B$ is $g \alpha$-open.

Proof. Let $F$ be a closed subset of $(X, \mu)$ such that $F \subseteq A \cap B$. Then $X-(A \cap B) \subseteq X-F$. Consequently, $(X-A) \cup(X-B) \subseteq X-F$. By Theorem 3.6., $(X-A) \cup(X-B)$ is g $\alpha$-closed. Thus, $\alpha \mu((X-A) \cup(X-B)) \subseteq X-F$. Consequently, $F \subseteq X-\alpha \mu((X-A) \cup(X-B))=X-\alpha \mu(X-(A \cap B))$. By Theorem 3.12., $A \cap B$ is g $\alpha$-open.
Theorem 3.15. Let $(X, \mu)$ be an isotonic space. If $A$ is a g $\alpha$-open subset of $(X, \mu)$, then $G=X$ whenever $G$ is open and $(X-\alpha \mu(X-A)) \cup(X-A) \subseteq G$.

Proof. Suppose that $A$ is g $\alpha$-open. Let $G$ be an open subset of $(X, \mu)$ such that $(X-\alpha \mu(X-A)) \cup(X-A) \subseteq G$. Then $X-G \subseteq X-((X-\alpha \mu(X-A)) \cup(X-A))$. Therefore, $X-G \subseteq \alpha \mu(X-A) \cap A$ or, equivalently, $X-G \subseteq \alpha \mu(X-A)-(X-A)$. But $X-G$ is closed and $X-A$ is $\mathrm{g} \alpha$-closed. Thus, by Theorem 3.9., $X-G=\phi$. Consequently, $X=G$.

The converse of this proposition is not true as can be seen from Example 3.10: $\operatorname{Put} A=\{a, c, d\}$. Then $A$ is not $\mathrm{g} \alpha$-open and $(X-\alpha \mu(X-A)) \cup(X-A)=$ $=\{a, d\} \cup\{b\} \subseteq G$ gives $G=X$. But $A$ is not $g \alpha$-open.
Theorem 3.16. Let $(X, \mu)$ be an isotonic space and let $A \subseteq X$. If $A$ is a g $\alpha$ closed, then $\alpha \mu(A)-A$ is $\mathrm{g} \alpha$-open.
Proof. Suppose that $A$ is $\mathrm{g} \alpha$-open. Let $F$ be a closed subset of $(X, \mu)$ such that $F \subseteq \alpha \mu(A)-A$. By Theorem 3.9., $F=\phi$ and hence $F \subseteq X-\alpha \mu(X-(\alpha \mu(X)-A))$. By Theorem 3.12., $\alpha \mu(A)-A$ is $\mathrm{g} \alpha$-open.

The converse of this result is not true as can be seen from Example 3.10: Put $A=\{b\}$. Then $\alpha \mu\{b\}-\{b\}=\{c\}$ which is $\mathrm{g} \alpha$-open. But $\{b\}$ is not g $\alpha$-closed.
Definition 3.17. An isotonic space $(X, \mu)$ is said to be a $T_{g \alpha}$-space if every $\mathrm{g} \alpha$-closed subset of $(X, \mu)$ is closed.
Theorem 3.18. Let $(X, \mu)$ be an isotonic space. Then
(i) If $(X, \mu)$ is a $T_{g \alpha}$-space then every singleton subset of $X$ is either closed or open.
(ii) If every singleton subset of $X$ is a closed subset of $(X, \mu)$, then $(X, \mu)$ is a $T_{g \alpha}$-space.

Proof. (i) Suppose that $(X, \mu)$ is a $T_{g \alpha}$-space. Let $x \in X$ and assume that $\{x\}$ is not closed. Then $X-\{x\}$ is not open. Since $X$ is the only open set which contains $X-\{x\}$ this implies $X-\{x\}$ is $g \alpha$-closed. Since $(X, \mu)$ is a $T_{g \alpha}$-space, $X-\{x\}$ is closed or equivalently $\{x\}$ is open.
(ii) Let $A$ be a $\alpha$-closed subset of ( $X, \mu$ ). To prove: $A$ is closed. Suppose that $x \in A$. Then $\{x\} \in X-\{x\}$. Since $A$ is $g \alpha$-closed and $X-\{x\}$ is open, $\alpha \mu(A) \in X-\{x\}$,(i.e) $\{x\} \in X-\alpha \mu(A)$. Hence $x \in \alpha \mu(A)$ and thus $\alpha \mu(A) \in$ $\in A$. Therefore $A$ is closed subset of $(X, \mu)$. Hence $(X, \mu)$ is a $T_{g \alpha}$-space.

## 4. Generalized continuous functions

In this section, we introduce concept of generalized $\alpha$-continuous functions by using the notion of generalized $\alpha$-closed sets and investigate some of their characterizations.

Definition 4.1. Let $(X, \mu)$ and $(Y, \nu)$ be isotonic spaces. A function $f:(X, \mu) \rightarrow$ $\rightarrow(Y, \nu)$ is said to be generalized $\alpha$-continuous (briefly $\mathrm{g} \alpha$-continuous) if $f^{-1}(F)$ is a $\mathrm{g} \alpha$-closed subset of $(X, \mu)$ for every closed subset $F$ of $(Y, \nu)$.

Clearly, if $f:(X, \mu) \rightarrow(Y, \nu)$ is $g \alpha$-continuous, then $f^{-1}(G)$ is a $\alpha \alpha$-open subset of $(X, \mu)$ for every open subset $G$ of $(Y, \nu)$.

REmARK 4.2. Every continuous map is g $\alpha$-continuous. The converse is not true as can be seen from the following example.
Example 4.3. Let $X=\{1,2,3\}=Y$ and define an isotonic operator $\mu$ on $X$ by $\mu \phi=\phi, \mu\{1\}=\mu\{2\}=\mu\{1,2\}=\{1,2\}, \mu\{3\}=\{3\}$ and $\mu\{1,3\}=$ $=\mu\{2,3\}=\mu X=X$. Define an isotonic operator $\nu$ on $Y$ by $\nu \phi=\phi, \nu\{1\}=$ $\{1,2\}, \nu\{2\}=\{2\}, \nu\{3\}=\{3\}, \nu\{1,2\}=\{1,2\}, \nu\{2,3\}=\{2,3\}$ and $\nu\{1,3\}=\nu Y=Y$. Let $\varphi:(X, \mu) \rightarrow(Y, \nu)$ be defined by $\varphi(1)=1, \varphi(2)=3$, $\varphi(3)=2$. It easy to see that $\varphi$ is $\mathrm{g} \alpha$-continuous but not closure preserving because $\varphi(\mu\{1,3\}) \nsubseteq \mu(\varphi\{1,3\})$. Therefore, $\varphi$ is not continuous.

Regarding the restriction $f_{\mid H}$ of a map $f:(X, \mu) \rightarrow(Y, \nu)$ to a subset $H$ of $X$, we have the following:

Theorem 4.4. Let $(X, \mu),(Y, \nu)$ be isotonic space and let $\left(H, \mu_{H}\right)$ be a closed subspace of $(X, \mu)$. If $f:(X, \mu) \rightarrow(Y, \nu)$ is g $\alpha$-continuous, then the restriction $f_{\mid H}:\left(H, \mu_{H}\right) \rightarrow(Y, \nu)$ is g $\alpha$-continuous.

Proof. Let $F$ be a closed subset of $(Y, \nu)$. Then the set $M=\left(f_{\mid H}\right)^{-1}(F)=$ $f^{-1}(F) \cap H$ is a g $\alpha$-closed subset of $(X, \mu)$ by Theorem 3.6. Since $\left(f_{\mid H}\right)^{-1}(F)=$ $M$, it is sufficient to show that $M$ is a $g \alpha$-closed subset of $\left(H, \mu_{H}\right)$. Let $G$ be an open subset of $\left(H, \mu_{H}\right)$ such that $M \subseteq G$. Then $G$ is an open subset of $(X, \mu)$ by Lemma 2.5. Since $M$ is $g \alpha$-closed and $H$ is a g $\alpha$-closed subset of $(X, \mu)$, $\mu_{H}(M)=\mu M \cap H=\mu M \subseteq G$. Therefore, $\left(f_{\mid H}\right)^{-1}(F)$ is a $\alpha \alpha$-closed subset of $\left(H, \mu_{H}\right)$. Hence, $f_{\mid H}$ is $g \alpha$-continuous.

In Theorem 4.4., the assumption of closedness of $H$ cannot be removed as can be seen from the following example.

Example 4.5. Let $X=\{1,2,3\}$ and define an isotonic operator $\mu$ on $X$ by $\mu \phi=$ $\phi, \mu\{2\}=\{1,2\}$ and $\mu\{1\}=\mu\{3\}=\mu\{1,3\}=\mu\{1,2\}=\mu\{2,3\}=\mu X=$ $X$. Let $Y=\{a, b\}$ and define an isotonic operator $\nu$ on $Y$ by $\nu \phi=\phi, \nu\{a\}=\{a\}$ and $\nu\{b\}=\nu Y=Y$. Let $f:(X, \mu) \rightarrow(Y, \nu)$ be defined by $f(1)=f(3)=a$ and $f(2)=b$. Then $H=\{2,3\}$ is not a closed subset of $(X, \mu)$ and $f$ is $g \alpha$ continuous. But the restriction $f_{\mid H}$ is not $g \alpha$-continuous.

Theorem 4.6. Let $(X, \mu)$ and $(Y, \nu)$ be isotonic spaces and let $\mu$ be additive. Let $A$ and $B$ be closed subsets of $(X, \mu)$ such that $X=A \cup B$. Let $f:\left(A, \mu_{A}\right) \rightarrow(Y, \nu)$ and $g:\left(B, \mu_{B}\right) \rightarrow(Y, \nu)$ be $g \alpha$-continuous maps such that $f(x)=g(x)$ for every $x \in A \cap B$. Let $h:(X, \mu) \rightarrow(Y, \nu)$ be defined by $h(x)=f(x)$ if $x \in A$ and $h(x)=g(x)$ if $x \in B$. Then $h:(X, \mu) \rightarrow(Y, \nu)$ is $g \alpha$-continuous.

Proof. Let $F$ be a closed subset of $(Y, \nu)$. Clearly, $h^{-1}(F)=f^{-1}(F) \cup g^{-1}(F)$. Since $f:\left(A, \mu_{A}\right) \rightarrow(Y, \nu)$ and $g:\left(B, \mu_{B}\right) \rightarrow(Y, \nu)$ are $g \alpha$-continuous, $f^{-1}(F)$ and $g^{-1}(F)$ are $g \alpha$-closed subset of $\left(A, \mu_{A}\right)$ and $\left(B, \mu_{B}\right)$, respectively. As $A$ is
a closed subset of $(X, \mu), f^{-1}(F)$ is a $\alpha \alpha$-closed subset of $(X, \mu)$ by Theorem 3.6. Similarly, $g^{-1}(F)$ is a $\mathrm{g} \alpha$-closed subset of $(X, \mu)$. By Theorem 3.4., $f^{1}(F) \cup$ $\cup g^{-1}(F)$ is a $\alpha \alpha$-closed subset of $(X, \mu)$. Therefore, $h^{-1}(F)$ is a $\alpha \alpha$-closed subset of $(X, \mu)$. Hence, $h$ is $\mathrm{g} \alpha$-continuous.

The following statement is obvious:
Theorem 4.7. Let $(X, \mu),(Y, \nu)$ and $(Z, \rho)$ be isotonic spaces. If $f:(X, \mu) \rightarrow$ $\rightarrow(Y, \nu)$ is $\mathrm{g} \alpha$-continuous and $g:(Y, \nu) \rightarrow(Z, \rho)$ is closure preserving, then $g \circ f:(X, \mu) \rightarrow(Z, \rho)$ is $g \alpha$-continuous.

## References

[1] M. M. Day, Convergence, closure and neighborhood, Duke Math. J., 11 (1944), 181.
[2] S. Gnilka, On extended topologies I: Closure operators, Ann. Soc. Math. Pol., Ser. I, commented. Math., 34 (1994), 81.
[3] S. Gnilka, On extended topologies II: Compactness, quasi-metrizability, symmetry, Ann. Soc. Math. Pol., Ser. I, commented. Math., 35 (1995), 147.
[4] S. Gnilka, On continuity in extended topologies, Ann. Soc. Math. Pol., Ser. I, commented. Math., 37 (1997), 99.
[5] E. D. Habil and K. A. Elzenati, Connectedness in isotonic spaces, Turk. J. Math., 30 (2006), 247.
[6] P. C. Hammer, Extended topology: Set-valued set functions, Nieuw Arch. Wisk. III., 10 (1962), 55.
[7] P. C. Hammer, Extended topology: Continuity I, Portug. Math., 25 (1964), 77.
[8] F. Hausdorff, Gestufte Raume, Fund. Math, 25 (1935), 486.
[9] K. Maki, R. Devi and K. Balachandran, Generalized $\alpha$-closed sets in topology, Bull. Fukuoka Univ. Ed. Part III, 42 (1993), 13.
[10] B. M. R. Stadler and P. F. Stadler, Basic properties of closure spaces, J. Chem. Inf. Comput. Sci., 42 (2002), 577.
[11] B. M. R. Stadler, and P. F. Stadler, Higher separation axioms in generalized closure spaces, Commentationes MathematicaeWarszawa, Ser. I., 43 (2003), 257.

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# ON MIXED QUASI-EINSTEIN MANIFOLDS 

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#### Abstract

The object of the present paper is to introduce a new type of Riemannian manifold called mixed quasi-Einstein manifolds $M(Q E)_{n}$ and prove the existence Theorem of a mixed quasi-Einstein manifold. Some geometric properties of mixed quasiEinstein manifolds have been studied. The totally umbilical hypersurfaces of $M(Q E)_{n}$ are also studied. The existence of a mixed quasi-Einstein manifold have been proved by two non-trivial examples.


## 1. Introduction

A Riemannian manifold $\left(M^{n}, g\right), n=\operatorname{dim} M \geq 2$, is said to be an Einstein manifold if the following condition

$$
\begin{equation*}
S=\frac{r}{n} g \tag{1}
\end{equation*}
$$

holds on $M$, where $S$ and $r$ denote the Ricci tensor and the scalar curvature of $\left(M^{n}, g\right)$ respectively. According to ([1],p.432), (1) is called the Einstein metric condition. Einstein manifolds play an important role in Riemannian Geometry as well as in general theory of relativity. Also Einstein manifolds form a natural subclass of various classes of Riemannian manifolds by a curvature condition impossed on their Ricci tensor ([1],p.432-433). For instance, every Einstein manifold belongs to the class of Riemannian manifolds $\left(M^{n}, g\right)$ realizing the following relation:

$$
\begin{equation*}
S(X, Y)=a g(X, Y)+b A(X) A(Y) \tag{2}
\end{equation*}
$$

where $a, b \in \mathbb{R}$ and $A$ is a non-zero 1-form such that

$$
\begin{equation*}
g(X, U)=A(X), \tag{3}
\end{equation*}
$$

for all vector fields $X$. Moreover, different structures on Einstein manifolds have been studied by several authors. In 1993, Tamassay and Binh [26] studied weakly symmetric structures on Einstein manifolds.

A non-flat Riemannian manifold $\left(M^{n}, g\right)(n>2)$ is defined to be a quasiEinstein manifold if its Ricci tensor $S$ of type $(0,2)$ is not identically zero and satisfies the condition (2).

It is to be noted that Chaki and Maity [2] also introduced the notoin of quasiEinstein manifolds in a different way. They have taken $a, b$ are scalars and the vector field $U$ metrically equivalent to the 1 -form $A$ as a unit vector field. Such an $n$-dimensional manifold is denoted by $(Q E)_{n}$. Quasi-Einstein manifolds have been studied by several authors such as De and Ghosh [8], De and De [9] and De, Ghosh and Binh [10] and many others.

Quasi-Einstein manifolds arose during the study of exact solutions of the Einstein field equations as well as during considerations of quasi-umbilical hypersurfaces of semi-Euclidean spaces. For instance, the Robertson-Walker spacetimes are quasi-Einstein manifolds. Also, quasi-Einstein manifold can be taken as a model of the perfect fluid spacetime in general relativity [7]. So quasiEinstein manifolds have some importance in the general theory of relativity.

Quasi-Einstein manifolds have been generalized by several authors in several ways such as generalized quasi-Einstein manifolds ([3, 11, 12, 18, 22]), super quasi-Einstein manifolds ([4, 13, 20]), pseudo quasi-Einstein manifolds [23], $N(k)$-quasi-Einstein manifolds ([19, 24, 25]) and many others.

In a recent paper [21] Nagaraja generalizes the quasi-Einstein manifold as follows:

A non-flat Riemannian manifold $\left(M^{n}, g\right)(n \geq 3)$ is called mixed quasiEinstein manifold if its Ricci tensor $S$ of type $(0,2)$ is not identically zero and satisfies the condition

$$
\begin{equation*}
S(X, Y)=a g(X, Y)+b A(X) B(Y)+c B(X) A(Y) \tag{4}
\end{equation*}
$$

where $a, b$ and $c$ are smooth functions and $A$ and $B$ are non-zero 1-forms such that $g(X, U)=A(X)$ and $g(X, V)=B(X)$ for all vector fields $X$ and $U$ and $V$ being the orthogonal unit vector fields called the generator of the manifold.

From (4), it follows that

$$
\begin{equation*}
S(Y, X)=a g(Y, X)+b A(Y) B(X)+c B(Y) A(X) \tag{5}
\end{equation*}
$$

From (4) and (5), it follows that

$$
(b-c)[A(X) B(Y)-A(Y) B(X)]=0
$$

This shows that either $b=c$ or, $A(X) B(Y)=A(Y) B(X)$. Motivated by this result we give the following definition:

A non-flat Riemannian manifold $\left(M^{n}, g\right)(n \geq 3)$ is called mixed quasiEinstein manifold if its Ricci tensor $S$ of type ( 0,2 ) is not identically zero and satisfies the condition:

$$
\begin{equation*}
S(X, Y)=a g(X, Y)+b[A(X) B(Y)+A(Y) B(X)] \tag{6}
\end{equation*}
$$

where $a, b$ are scalars of which $b \neq 0$ and $A$ and $B$ are non-zero 1-forms such that

$$
g(X, U)=A(X), \quad g(X, V)=B(X), \quad g(U, V)=0
$$

where $U, V$ are unit vector fields. In such a case $A, B$ are called associated 1forms and $U, V$ are called the generators of the manifold. Such an $n$-dimensional manifold is denoted by the symbol $M(Q E)_{n}$.

If $b=0$, then the manifold becomes an Einstein manifold. If $A=B$, then the manifold reduces to a quasi-Einstein manifold. This justifies the name mixed quasi-Einstein manifold.

A Riemannian manifold of quasi-constant curvature was given by Chen and Yano [6] as a conformally flat manifold with the curvature tensor $R$ of type ( 0,4 ) satisfied the condition

$$
\begin{align*}
\dot{R}(X, Y, Z, W) & =a[g(Y, Z) g(X, W)-g(X, Z) g(Y, W)]+ \\
& +b[g(X, W) A(Y) A(Z)-g(X, Z) A(Y) A(W)+ \\
& +g(Y, Z) A(X) A(W)-g(Y, W) A(X) A(Z)] \tag{7}
\end{align*}
$$

where $\dot{R}(X, Y, Z, W)=g(R(X, Y) Z, W), R$ is the curvature tensor of type $(1,3)$, $a, b$ are scalar functions of which $b \neq 0$ and $A$ is a non-zero 1-form defined by

$$
\begin{equation*}
g(X, U)=A(X) \tag{8}
\end{equation*}
$$

for all $X$ and $U$ being a unit vector field.
It can be easily seen that if the curvature tensor $\hat{R}$ of the form (7), then the manifold is conformally flat. On the other hand, Gh. Vranceanu [27] defined the notion of almost constant curvature by the same expression (7). Later A. L. Mocanu [17] pointed out that the manifold introduced by Chen and Yano and the manifold introduced by Gh.Vranceanu are same. Hence a Riemannian manifold is said to be of quasi-constant curvature if the curvature tensor $k$ satisfied the relation (7).

A generalization of a manifold of quasi-constant curvature, called a manifold of mixed quasi-constant curvature is needed for the study of a $M(Q E)_{n}$. Such a manifold is denoted by the symbol $M(Q C)_{n}$ and is defined as follows:

A non-flat Riemannian manifold $\left(M^{n}, g\right)(n \geq 3)$ is called a manifold of mixed quasi-constant curvature if its curvature tensor $R$ of type $(0,4)$ satisfies
the condition

$$
\begin{align*}
\dot{R}(X, Y, Z, W) & =p[g(Y, Z) g(X, W)-g(X, Z) g(Y, W)] \\
& +q[g(Y, Z)\{A(X) B(W)+B(X) A(W)\} \\
& +g(X, W)\{A(Y) B(Z)+B(Y) A(Z)\} \\
& -g(X, Z)\{A(Y) B(W)+A(W) B(Y)\} \\
& -g(Y, W)\{A(X) B(Z)+A(Z) B(X)\}], \tag{9}
\end{align*}
$$

where $\dot{R}(X, Y, Z, W)=g(R(X, Y) Z, W)$ and $p, q$ are scalars and $A, B$ are nonzero 1 -forms. If the 1 -forms $A$ and $B$ are equal, then the manifold reduces to a manifold of quasi-constant curvature.
A. Gray [16] introduced two classes of Riemannian manifolds determined by the covariant differentiation of Ricci tensor. The class $A$ consisting of all Riemannian manifolds whose Ricci tensor S is a Codazzi type tensor,

$$
\text { i.e., } \quad\left(\nabla_{X} S\right)(Y, Z)=\left(\nabla_{Y} S\right)(X, Z)
$$

The class $B$ consisting of all Riemannian manifolds whose Ricci tensor is cyclic parallel,

$$
\text { i.e., } \quad\left(\nabla_{X} S\right)(Y, Z)+\left(\nabla_{Y} S\right)(Z, X)+\left(\nabla_{Z} S\right)(X, Y)=0
$$

The present paper is organised as follows:
Section 2 contains the proof of a theorem which proves the existence of a $M(Q E)_{n}$. In Section 3, we prove that a conformally flat $M(Q E)_{n}$ is a $M(Q C)_{n}$. In the next section we study sectional curvatures at a point of a conformally flat $M(Q E)_{n}$. In Section 5, we prove that $M(Q E)_{n}$ reduces to $(Q E)_{n}$ under certain conditions and also prove that the associated scalar b is less than $\frac{1}{\sqrt{ } 2} d$, where d is the length of the Ricci tensor S . Section 6 is devoted to study the nature of the associated 1-forms of a $M(Q E)_{n}$. Section 7 deals with the study of a $M(Q E)_{n}$ satisfying cyclic parallel Ricci tensor under certain conditions. In the next Section we study totally umbilical hypersurfaces of a $M(Q E)_{n}$. Finally, we construct two non-trivial examples of a $M(Q E)_{n}$.

## 2. Existence theorem of a mixed quasi-Einstein manifold

In this section we enquire under what condition a mixed quasi-Einstein manifold exists.

Theorem 2.1. If the Ricci tensor $S$ of a Riemannian manifold satisfies the relation
(10) $S(X, W) S(Y, Z)-g(X, W) g(Y, Z)=\mu[g(X, Z) S(Y, W)+g(Y, W) S(X, Z]$,
where $\mu$ is a non-zero scalar, then the manifold is a mixed quasi-Einstein manifold.

Proof. Let $U$ be a vector field defined by $g(X, U)=A(X), \forall X \in T M$.
Putting $X=W=U$ in (10), we obtain

$$
\begin{equation*}
t S(Y, Z)-|U|^{2} g(Y, Z)=\mu[A(Z) A(L Y)+A(Y) A(L Z)] \tag{11}
\end{equation*}
$$

where $S(U, U)=t$ and $g(U, U)=|U|^{2}$ and $L$ is the symmetric endomorphism of the tangent space at each point corresponding to the Ricci tensor $S$, and $g(L X, Y)=S(X, Y), \forall X, Y \in \chi(M)$.

That is,

$$
\begin{equation*}
S(Y, Z)=a g(Y, Z)+b[A(Y) B(Z)+A(Z) B(Y)] \tag{12}
\end{equation*}
$$

where $a=\frac{|U|^{2}}{t}, b=\frac{\mu}{t}$ and $B(X)=A(L X)$.
This shows that the manifold is a mixed quasi-Einstein manifold.

## 3. Conformally flat $M(Q E)_{n}(n>3)$

An $M(Q E)_{n}(n>3)$ is not, in general a $M(Q C)_{n}$. In this section we consider a conformally flat $M(Q E)_{n}(n>3)$ and show that such a $M(Q E)_{n}$ is a $M(Q C)_{n}$.

It is known [28] that in a conformally flat Riemannian manifold $\left(M^{n}, g\right)$ $(n>3)$ the curvature tensor $\hat{R}$ of type $(0,4)$ has the following form:

$$
\begin{align*}
\dot{R}(X, Y, Z, W) & =\frac{1}{n-2}[g(Y, Z) S(X, W)-g(X, Z) S(Y, W)+ \\
& +g(X, W) S(Y, Z)-g(Y, W) S(X, Z)]- \\
& -\frac{r}{(n-1)(n-2)}[g(Y, Z) g(X, W)-g(X, Z) g(Y, W)] \tag{13}
\end{align*}
$$

Using (6 in (13), we obtain

$$
\begin{align*}
\dot{R}(X, Y, Z, W) & =\frac{2 a(n-1)-r}{(n-1)(n-2)}[g(Y, Z) g(X, W)-g(X, Z) g(Y, W)]+ \\
& +\frac{b}{n-2}[g(Y, Z)\{A(X) B(W)+B(X) A(W)\}+ \\
& +g(X, W)\{A(Y) B(Z)+A(Z) B(Y)\}- \\
& -g(X, Z)\{A(Y) B(W)+A(W) B(Y)\}- \\
& -g(Y, W)\{A(X) B(Z)+B(X) A(Z)\}] \tag{14}
\end{align*}
$$

Thus we can state the theorem:
Theorem 3.1. Every conformally flat $M(Q E)_{n}$ is a $M(Q C)_{n}$.
REmARK. For $n=3$, the conformal curvature tensor vanishes identically. Hence a three-dimensional $M(Q E)_{n}$ is a $M(Q C)_{n}$.

## 4. Sectional curvatures at a point of a conformally flat $M(Q E)_{n}$ <br> $$
(n>3)
$$

Let $U^{\perp}$ denote the $(n-1)$-dimensional distribution in a conformally flat $M(Q E)_{n}(n \geq 3)$ orthogonal to U . Then for all $X \in U^{\perp}, g(X, U)=0$ i.e., $A(X)=0$.

In this section we shall determine sectional curvature $K$ at the plane determined by the vectors $X, Y \in U^{\perp}$ or by $X, U$.

Putting $Z=Y$ and $W=X$ in (14) we get

$$
\begin{equation*}
\dot{R}(X, Y, Y, X)=\frac{2 a(n-1)-r}{(n-1)(n-2)}\left[g(X, X) g(Y, Y)-\{g(X, Y)\}^{2}\right] \tag{15}
\end{equation*}
$$

Then,

$$
\begin{equation*}
K(X, Y)=\frac{2 a(n-1)-r}{(n-1)(n-2)} \tag{16}
\end{equation*}
$$

From (6) on contraction we get $r=a n$, where $r$ is the scalar curvature of the manifold.

Using $r=a n$ in (16) we obtain $K(X, Y)=\frac{a}{n-1}$.
Again we have

$$
\begin{equation*}
K(X, U)=\frac{\dot{R}(X, U, U, X)}{g(X, X) g(U, U)-\{g(X, U)\}^{2}} \tag{17}
\end{equation*}
$$

and

$$
\begin{equation*}
\dot{R}(X, U, U, X)=\frac{2 a(n-1)-r}{(n-1)(n-2)} g(U, U) g(X, X) \tag{18}
\end{equation*}
$$

Using the relations (17) and (18) and $r=a n$, we obtain $K(X, U)=\frac{a}{n-1}$.
Thus we can state the following:
THEOREM 4.1. In a conformally flat $M(Q E)_{n}(n>3)$, the sectional curvature of the plane determined by two vectors $X, Y \in U^{\perp}$ is $\frac{a}{n-1}$, while the sectional curvature of the plane determined by two vectors $X, U$ is also $\frac{a}{n-1}$.

## 5. Some geometric properties of $M(Q E)_{n}$

At first we suppose that $U$ and $V$ are parallel vector fields in a $M(Q E)_{n}$.
Then $\nabla_{X} U=0$ and $\nabla_{X} V=0$, which implies that $R(X, Y) U=0$ and $R(X, Y) V=0$. Hence it follows that $S(X, U)=0$ and $S(X, V)=0$.

Thus from (6) we obtain

$$
\begin{align*}
& a A(X)+b B(X)=0  \tag{19}\\
& b A(X)+a B(X)=0 \tag{20}
\end{align*}
$$

From (19) we get

$$
B(X)=-\frac{a}{b} A(X)
$$

Hence from (6) we obtain

$$
\begin{equation*}
S(X, Y)=a g(X, Y)+k_{1} A(X) A(Y) \tag{21}
\end{equation*}
$$

where $k_{1}=-\frac{a}{b}$.
Again from (20) we get

$$
B(X)=-\frac{b}{a} A(X)
$$

Hence from (6) we obtain

$$
\begin{equation*}
S(X, Y)=a g(X, Y)+k_{2} A(X) A(Y) \tag{22}
\end{equation*}
$$

where $k_{2}=-\frac{b}{a}$.
Thus we can state the following:
Theorem 5.1. In a $M(Q E)_{n}$ if the vector fields $U$ and $V$ are parallel, then $M(Q E)_{n}$ reduces to a $(Q E)_{n}$.

Again, from (6) we have

$$
\begin{align*}
& S(U, U)=a,  \tag{23}\\
& S(V, V)=a,  \tag{24}\\
& S(U, V)=b . \tag{25}
\end{align*}
$$

If $X$ is a unit vector field, then $S(X, X)$ is the Ricci curvature in the direction of $X$. Hence, from (23) and (24), it can be stated that a is the Ricci curvature in the directions of both $U$ and $V$.

Let $L$ be the symmetric endomorphism of the tangent space at each point corresponding to the Ricci tensor $S$.

Then,

$$
\begin{equation*}
S(X, Y)=g(L X, Y), \forall X, Y \in T M \tag{26}
\end{equation*}
$$

Let $d^{2}$ denote the square of the length of the Ricci tensor $S$.
Then,

$$
\begin{equation*}
d^{2}=S\left(L e_{i}, e_{i}\right), \tag{27}
\end{equation*}
$$

where $\left\{e_{i}\right\},(i=1,2, \ldots, n)$ is an orthonormal basis of the tangent space at each point of $M(Q E)_{n}$.

From (6) we have

$$
\begin{equation*}
S\left(L e_{i}, e_{i}\right)=n a^{2}+2 b^{2}>0 \tag{28}
\end{equation*}
$$

Therefore

$$
d^{2}>2 b^{2}
$$

which means that

$$
b<\frac{1}{\sqrt{ } 2} d .
$$

Thus we can state the following:
Theorem 5.2. In a $M(Q E)_{n}$, the associated scalar a is the Ricci curvature in the directions of both the generators $U$ and $V$ and the associated scalar $b$ is less than $\frac{1}{\sqrt{2}} d$, where $d$ is the length of the Ricci tensor $S$.

## 6. Nature of the associated 1-forms of a $M(Q E)_{n}$

A Riemannian manifold is said to satisfy Codazzi type of Ricci tensor [15] if its Ricci tensor $S$ satisfies the following condition

$$
\begin{equation*}
\left(\nabla_{X} S\right)(Y, Z)=\left(\nabla_{Y} S\right)(X, Z) . \tag{29}
\end{equation*}
$$

Let us suppose that the manifold under consideration satisfies Codazzi type of Ricci tensor and the associated scalars are constant. Then from (6) we obtain

$$
\begin{align*}
\left(\nabla_{X} S\right)(Y, Z) & =b\left[\left(\nabla_{X} A\right)(Y) B(Z)+\left(\nabla_{X} B\right)(Z) A(Y)+\right. \\
& \left.+\left(\nabla_{X} B\right)(Y) A(Z)+\left(\nabla_{X} A\right)(Z) B(Y)\right] \tag{30}
\end{align*}
$$

Hence from (29) and (30) we have

$$
\begin{align*}
& \left(\nabla_{X} A\right)(Y) B(Z)+\left(\nabla_{X} B\right)(Z) A(Y)+\left(\nabla_{X} B\right)(Y) A(Z)+ \\
& +\left(\nabla_{X} A\right)(Z) B(Y)-\left(\nabla_{Y} A\right)(X) B(Z)-\left(\nabla_{Y} B\right)(Z) A(X)- \\
& -\left(\nabla_{Y} B\right)(X) A(Z)-\left(\nabla_{Y} A\right)(Z) B(X)=0 \tag{31}
\end{align*}
$$

Putting $Z=U$ in (31) and using $\left(\nabla_{X} A\right)(U)=0$ and $\left(\nabla_{X} B\right)(U)=0$, since $U$ is a unit vector, we get

$$
\begin{gathered}
\left(\nabla_{X} B\right)(Y)-\left(\nabla_{Y} B\right)(X)=0 \\
\text { i.e., } \quad d B(X, Y)=0
\end{gathered}
$$

Again, putting $Z=V$ and using $\left(\nabla_{X} A\right)(V)=0$ and $\left(\nabla_{X} B\right)(V)=0$, since $V$ is a unit vector, we get

$$
\begin{gathered}
\left(\nabla_{X} A\right)(Y)-\left(\nabla_{Y} A\right)(X)=0 \\
\text { i.e., } \quad d A(X, Y)=0
\end{gathered}
$$

Thus we can state the following:
Theorem 6.1. If a $M(Q E)_{n}$ satisfies Codazzi type of Ricci tensor and the associated scalars are constant, then the associated 1 -forms $A$ and $B$ are closed.

## 7. Generators $U$ and $V$ as Killing vector fields

In this section let us consider the generators $U$ and $V$ of the manifold under consideration are Killing vector fields. We also suppose the associated scalars of the manifold are constant. Then we have $\left(£_{U} g\right)(X, Y)=0$ and $\left(£_{V} g\right)(X, Y)=0$, where $£$ denotes the Lie derivative.

Thus we get

$$
\begin{equation*}
g\left(\nabla_{X} U, Y\right)+g\left(X, \nabla_{Y} U\right)=0 \tag{32}
\end{equation*}
$$

and

$$
\begin{equation*}
g\left(\nabla_{X} V, Y\right)+g\left(X, \nabla_{Y} V\right)=0 \tag{33}
\end{equation*}
$$

Again, since $g\left(\nabla_{X} U, Y\right)=\left(\nabla_{X} A\right)(Y)$ and $g\left(\nabla_{X} V, Y\right)=\left(\nabla_{X} B\right)(Y)$, we obtain from (32) and (33)

$$
\begin{equation*}
\left(\nabla_{X} A\right)(Y)+\left(\nabla_{Y} A\right)(X)=0 \tag{34}
\end{equation*}
$$

and

$$
\begin{equation*}
\left(\nabla_{X} B\right)(Y)+\left(\nabla_{Y} B\right)(X)=0, \tag{35}
\end{equation*}
$$

for all $X, Y$.
Similarly, we have

$$
\begin{align*}
& \left(\nabla_{X} A\right)(Z)+\left(\nabla_{Z} A\right)(X)=0,  \tag{36}\\
& \left(\nabla_{X} B\right)(Z)+\left(\nabla_{Z} B\right)(X)=0,  \tag{37}\\
& \left(\nabla_{Y} A\right)(Z)+\left(\nabla_{Z} A\right)(Y)=0,  \tag{38}\\
& \left(\nabla_{Y} B\right)(Z)+\left(\nabla_{Z} B\right)(Y)=0, \tag{39}
\end{align*}
$$

for all $X, Y, Z$.
Now from (6) and using the relations (34), (35), (36), (37), (38) and (39) we have

$$
\left(\nabla_{X} S\right)(Y, Z)+\left(\nabla_{Y} S\right)(Z, X)+\left(\nabla_{Z} S\right)(X, Y)=0 .
$$

Thus we can state the following:
Theorem 7.1. If the generators of a $M(Q E)_{n}$ are Killing vector fields and the associated scalars are constant, then the manifold satisfies cyclic parallel Ricci tensor.

## 8. Totally umbilical hypersurfaces of $M(Q E)_{n}$

Let $(\bar{V}, \bar{g})$ be an ( $\mathrm{n}+1$ )-dimensional Riemannian manifold covered by a system of coordinate neighbourhoods $\left\{U, y^{\alpha}\right\}$. Let $(V, g)$ be a hypersurface of $(\bar{V}, \bar{g})$ defined in a local coordinate system by means of a system of parametric equation $y^{\alpha}=y^{\alpha}\left(x^{i}\right)$, where Greek indices take values $1,2, \ldots, n$ and Latin indices take the values $1,2, \ldots,(\mathrm{n}+1)$. Let $N^{\alpha}$ be the components of a local unit normal to $(V, g)$. Then we have

$$
\begin{gather*}
g_{i j}=\bar{g}_{\alpha \beta} y_{i}^{\alpha} y_{j}^{\beta},  \tag{40}\\
\bar{g}_{\alpha \beta} N^{\alpha} y_{j}^{\beta}=0, \quad \bar{g}_{\alpha \beta} N^{\alpha} N^{\beta}=1,  \tag{41}\\
y_{i}^{\alpha} y_{j}^{\beta} g^{i j}=\bar{g}^{\alpha \beta}-N^{\alpha} N^{\beta}, \quad y_{i}^{\alpha}=\frac{\partial y^{\alpha}}{\partial x^{i}} . \tag{42}
\end{gather*}
$$

The hypersurface $(V, g)$ is called a totally umbilical hypersurface $([5,14])$ of $(\bar{V}, \bar{g})$ if its second fundamental form $\Omega_{i j}$ satisfies

$$
\begin{equation*}
\Omega_{i j}=H g_{i j}, \quad y_{i, j}^{\alpha}=g_{i j} H N^{\alpha} \tag{43}
\end{equation*}
$$

where the scalar function H is called the mean curvature of $(V, g)$ given by

$$
H=\frac{1}{n} \Sigma g^{i j} \Omega_{i j}
$$

If, in particular, $H=0, \quad$ i.e.,

$$
\begin{equation*}
\Omega_{i j}=0 \tag{44}
\end{equation*}
$$

then the totally umbilical hypersurface is called a totally geodesic hypersurface of $(\bar{V}, \bar{g})$.

The equation of Weingarten for $(V, g)$ can be written as $N_{, j}^{\alpha}=-\frac{H}{n} y_{j}^{\alpha}$.
The structure equations of Gauss and Codazzi $([5,14])$ for $(V, g)$ and $(\bar{V}, \bar{g})$ are respectively given by

$$
\begin{gather*}
R_{i j k l}=\bar{R}_{\alpha \gamma \delta} F_{i j k l}^{\alpha \beta \gamma \delta}+H^{2} G_{i j k l},  \tag{45}\\
\bar{R}_{\alpha \beta \gamma \delta} F_{i j k}^{\alpha \beta \gamma} N^{\delta}=H_{, i} g_{j k}-H_{, j} g_{i k} \tag{46}
\end{gather*}
$$

where $R_{i j k l}$ and $\bar{R}_{\alpha \beta \gamma \delta}$ are curvature tensors of $(V, g)$ and $(\bar{V}, \bar{g})$ respectively, and

$$
F_{i j k l}^{\alpha \beta \gamma \delta}=F_{i}^{\alpha} F_{j}^{\beta} F_{k}^{\gamma} F_{l}^{\delta}, \quad F_{i}^{\alpha}=y_{i}^{\alpha}, \quad G_{i j k l}=g_{i l} g_{j k}-g_{i k} g_{j l}
$$

Also we have ([5, 14])

$$
\begin{gather*}
\bar{S}_{\alpha \delta} F_{i}^{\alpha} F_{j}^{\delta}=S_{i j}-(n-1) H^{2} g_{i j}  \tag{47}\\
\bar{S}_{\alpha \delta} N^{\alpha} F_{i}^{\delta}=(n-1) H_{, i}  \tag{48}\\
\bar{r}=r-n(n-1) H^{2} \tag{49}
\end{gather*}
$$

where $S_{i j}$ and $\bar{S}_{\alpha \delta}$ are the Ricci tensors of $(V, g)$ and $(\bar{V}, \bar{g})$ respectively and $r$ and $\bar{r}$ are the scalar curvatures of $(V, g)$ and $(\bar{V}, \bar{g})$ respectively.

In terms of local coordinates the relation (6) can be written as

$$
\begin{equation*}
S_{i j}=a g_{i j}+b\left[A_{i} B_{j}+A_{j} B_{i}\right] \tag{50}
\end{equation*}
$$

Let $(\bar{V}, \bar{g})$ be a $M(Q E)_{n}$. Then we get

$$
\begin{equation*}
\bar{S}_{\alpha \beta}=a \bar{g}_{\alpha \beta}+b\left[A_{\alpha} B_{\beta}+A_{\beta} B_{\alpha}\right] \tag{51}
\end{equation*}
$$

Multiplying both sides of (51) by $F_{i j}^{\alpha \beta}$ and then using (47) and (50), we obtain $H=0$, which implies that the manifold is a totally geodesic hypersurface.

Conversely, we now consider that the manifold $(V, g)$ is totally geodesic hypersurface, i.e.,

$$
\begin{equation*}
H=0 \tag{52}
\end{equation*}
$$

In view of (52), (47) yeilds

$$
\begin{equation*}
\bar{S}_{\alpha \delta} F_{i}^{\alpha} F_{j}^{\delta}=S_{i j} . \tag{53}
\end{equation*}
$$

Using (53) in (51), we have the relation (50). Thus we can state the following:
Theorem 8.1. The totally umbilical hypersurface of a $M(Q E)_{n}$ is a $M(Q E)_{n}$ if and only if the manifold is a totally geodesic hypersurface.

## 9. Examples of a $M(Q E)_{n}$

Example 1. We consider a Riemannian manifold $\left(M^{4}, g\right)$ endowed with the metric $g$ given by

$$
\begin{equation*}
d s^{2}=g_{i j} d x^{i} d x^{j}=\left(d x^{1}\right)^{2}+\left(x^{1}\right)^{2}\left(d x^{2}\right)^{2}+\left(x^{2}\right)^{2}\left(d x^{3}\right)^{2}+\left(d x^{4}\right)^{2} \tag{54}
\end{equation*}
$$

$i, j=1,2,3,4$.
The only non-vanishing components of Christoffel symbols, the curvature tensor and the Ricci tensor are

$$
\begin{gathered}
\Gamma_{22}^{1}=-x^{1}, \quad \Gamma_{33}^{2}=-\frac{x^{2}}{\left(x^{1}\right)^{2}}, \quad \Gamma_{12}^{2}=\frac{1}{x^{1}}, \quad \Gamma_{23}^{3}=\frac{1}{x^{2}} \\
R_{1332}=-\frac{x^{2}}{x^{1}}, \quad S_{12}=-\frac{1}{x^{1} x^{2}} .
\end{gathered}
$$

It can be easily shown that the scalar curvature of the manifold is zero. Therefore $R^{4}$ with the considered metric is a Riemannian manifold $\left(M^{4}, g\right)$ of vanishing scalar curvature. We shall now show that this $M^{4}$ is a $M(Q E)_{4}$ i.e., it satisfies the defining relation (6).

We take the associated scalars as follows:

$$
a=\frac{1}{x^{1}\left(x^{2}\right)^{2}}, \quad b=-\frac{2}{\left(x^{1}\right)^{2} x^{2}}
$$

We choose the 1 -forms as follows:

$$
A_{i}(x)= \begin{cases}x^{1}, & \text { for } i=2 \\ 0, & \text { for } i=1,3,4\end{cases}
$$

and

$$
B_{i}(x)= \begin{cases}\frac{1}{2}, & \text { for } i=1 \\ \frac{3^{1 / 2} x^{2}}{2}, & \text { for } i=3 \\ 0, & \text { for } i=2,4\end{cases}
$$

at any point $x \in M$. In our $\left(M^{4}, g\right)$, (6) reduces with these associated scalars and 1 -forms to the following equation:

$$
\begin{equation*}
S_{12}=a g_{12}+b\left[A_{1} B_{2}+A_{2} B_{1}\right] \tag{55}
\end{equation*}
$$

It can be easily proved that the equation (55) is true.
We shall now show that the 1 -forms are unit and orthogonal.
Here,

$$
g^{i j} A_{i} A_{j}=1, \quad g^{i j} B_{i} B_{j}=1, \quad g^{i j} A_{i} B_{j}=0
$$

So, the manifold under consideration is a $M(Q E)_{4}$.
Example 2. We consider a Riemannian manifold $\left(\mathbb{R}^{4}, g\right)$ endowed with the metric $g$ given by

$$
\begin{equation*}
d s^{2}=g_{i j} d x^{i} d x^{j}=\left(d x^{1}\right)^{2}+\left(x^{1}\right)^{2}\left(d x^{2}\right)^{2}+\left(x^{1} \sin x^{2}\right)^{2}\left(d x^{3}\right)^{2}+\left(d x^{4}\right)^{2} \tag{56}
\end{equation*}
$$

where $x^{1} \neq 0$ and $0<x^{2}<\frac{\pi}{2}$. Then the non-vanishing components of the Christoffel symbols and the curvature tensor are

$$
\begin{gathered}
\Gamma_{22}^{1}=-x^{1}, \quad \Gamma_{33}^{1}=-x^{1}\left(\sin x^{2}\right)^{2}, \quad \Gamma_{12}^{2}=\Gamma_{13}^{3}=\frac{1}{x^{1}} \\
\Gamma_{23}^{3}=\cot x^{2}, \quad \Gamma_{33}^{2}=-\sin x^{2} \cos x^{2}, \quad R_{2332}=-\left(x^{1} \sin x^{2}\right)^{2}
\end{gathered}
$$

and the components which can be obtained from these by the symmetry properties. Using the above relations, we can find the non-vanishing components of Ricci tensor as follows:

$$
S_{22}=-1, \quad S_{33}=-\left(\sin x^{2}\right)^{2}
$$

Also it can be easily found that the scalar curvature of the manifold is nonconstant and is equal to $-\frac{2}{\left(x^{1}\right)^{2}} \neq 0$.

We take the associated scalars as follows:

$$
a=-\frac{1}{\left(x^{1}\right)^{2}}, \quad b=x^{1} x^{2}
$$

We choose the 1 -forms as follows:

$$
A_{i}(x)= \begin{cases}x^{1} \sin x^{2}, & \text { for } i=3 \\ 0, & \text { for } i=1,2,4\end{cases}
$$

and

$$
B_{i}(x)= \begin{cases}x^{1}, & \text { for } i=2 \\ 0, & \text { for } i=1,3,4\end{cases}
$$

at any point $x \in M$. In our $\left(M^{4}, g\right)$, (6) reduces with these associated scalars and 1-forms to the following equations:

$$
\begin{equation*}
S_{22}=a g_{22}+b\left[A_{2} B_{2}+A_{2} B_{2}\right] \tag{57}
\end{equation*}
$$

$$
\begin{equation*}
S_{33}=a g_{33}+b\left[A_{3} B_{3}+A_{3} B_{3}\right] . \tag{58}
\end{equation*}
$$

It can be easily proved that the equations (57) and (58) are true.
We shall now show that the 1 -forms are unit and orthogonal.
Here,

$$
g^{i j} A_{i} A_{j}=1, \quad g^{i j} B_{i} B_{j}=1, \quad g^{i j} A_{i} B_{j}=0 .
$$

So, the manifold under consideration is a $M(Q E)_{4}$.

## References

[1] A. L.Besse, Einstein manifolds, Ergeb. Math. Grenzgeb.,3. Folge, Bd. 10, Springer-Verlag, Berlin, Heidelberg, New York, 1987.
[2] M. C. Chaki and R. K. Maity, On quasi Einstein manifolds, Publ. Math. Debrecen, 57 (2000), 297-306.
[3] M. C. Chaki, On generalized quasi-Einstein manifolds, Publ. Math. Debrecen, 58 (2001), 683-691.
[4] M. C.Chaki, On super quasi-Einstein manifolds, Publ. Math. Debrecen, 64 (2004), 481-488.
[5] B. Y. Chen, Geometry of sub-manifolds, Marcel-Deker, New York, 1973.
[6] B. Y. Chen and K. Yano, Special conformally flat spaces and canal hypersurfaces, Tohoku Math. J., 25 (1973), 177-184.
[7] U. C. De and G. C. Ghosh, On quasi-Einstein and special quasi-Einstein manifolds, Proc. of the Int. Conf. of Mathematics and its applications, Kuwait University, April 5-7, 2004, 178-191.
[8] U. C. De and G. C. Ghosh, On quasi-Einstein manifolds, Period. Math. Hungar., 48 (2004), 223-231.
[9] U. C. De and B. K. De, On quasi-Einstein manifolds, Commun. Korean Math. Soc., 23 (2008), 413-420.
[10] G. C. Ghosh, U. C. De and T. Q. Binh, Certain curvature restrictions on a quasiEinstein manifold, Publ. Math. Debrecen, 69 (2006), 209-217.
[11] U. C. De and G. C. Ghosh, On generalized quasi-Einstein manifolds, Kyungpook Math. J., 44 (2004), 607-615.
[12] U. C. De and G. C.Ghosh, Some global properties of generalized quasi-Einstein manifolds, Ganita 56, 1 (2005), 65-70.
[13] P. Debnath and A. Konar, On super quasi-Einstein manifolds, Publications de L'institut Mathematique, Nouvelle serie, Tome 89(103) (2011), 95-104.
[14] L. P. Eisenhart, Riemannian Geometry, Princeton University Press, 1949.
[15] D. Ferus, A remark on Codazzi tensors in constant curvature spaces, Lecture note in Mathematics, 838, Global Differential Geometry and Global Analysis, Springer Verlag, New York, (1981), 257.
[16] A. Gray, Einstein-like manifolds which are not Einstein, Geom. Dedicate, 7 (1998), No. 3, 259-280.
[17] A. L. Mocanu, Les varietes a courbure quasi-constant de type Vranceanu, Lucr. Conf. Nat. de. Geom. Si Top., Tirgoviste, 1987.
[18] C. Özgür, On a class of generalized quasi-Einstein manifolds, Applied Sciences, Balkan Society of Geometers, Geometry Balkan Press, 8 (2006), 138-141.
[19] C. ÖzGür, $N(k)$-quasi-Einstein manifolds satisfying certain conditions, Chaos, Solitons and Fractals, 38 (2008), 1373-1377.
[20] C. Özgür, On some classes of super quasi-Einstein manifolds, Chaos, Solitons and Fractals, 40 (2009), 1156-1161.
[21] H. G. Nagaraja, On $N(k)$-mixed quasi-Einstein manifolds, Eur. J. Pure Appl. Math., 3 (2010), 16-25.
[22] A. A. Shaikh and S. K. Hui, On some classes of generalized quasi-Einstein manifolds, Commun. Korean Math. Soc., 24 (2009), 415-424.
[23] A. A. Shaikh, On pseudo quasi-Einstein manifolds, Period. Math. Hungar., 59 (2009), 119-146.
[24] A. Taleshian and A. A. Hosseinzadeh, Investigation of some conditions on $N(k)$-quasi-Einstein manifolds, Bull. Malays. Math. Sci. Soc., 34 (2011), 455-464.
[25] A. Taleshian and A. A. Hosseinzadeh, On $W_{2}$-Curvature Tensor $N(k)$-Quasi Einstein manifolds, The Journal of Mathematics and Computer Science, 1 (2010), 28-32.
[26] L. Tamassay and T. Q. Binh, On weak symmetries of Einstein and Sasakian manifolds, Tensor, N.S., 53 (1993), 140-148.
[27] Gh. Vranceanu, Lecons des Geometrie Differential, Vol.4, Ed.de I'Academie, Bucharest, 1968.
[28] K. Yano and M. Kon, Structures on manifolds, vol. 40, World Scientific Press, 1989.

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# A LIE ALGEBRA APPROACH TO DIFFERENCE SETS: HOMAGE TO YAHYA OULD HAMIDOUNE 

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#### Abstract

We demonstrate how the adjoint representation of the general linear Lie algebra over a finite dimensional vector space may be used in the study of difference sets. This approach extends quite naturally to a purely matrix algebraic proof of the Cauchy-Davenport theorem given previously in the language of tensor algebra by Dias da Silva and Hamidoune.


## 1. Introduction

Given an abelian group $G \neq 0$, let $p(G)$ denote the smallest possible order of a nontrivial subgroup in $G$. In case $G=\mathbb{F}^{+}$is the additive group of a field $\mathbb{F}$, we simply write $p(\mathbb{F})$. Thus, $p(\mathbb{F})$ equals the characteristic of the field $\mathbb{F}$ if it is positive, otherwise $p(\mathbb{F})=\infty$.

For subsets $\mathcal{A}, \mathcal{B} \subseteq G$, their sumset is defined as

$$
\mathcal{A}+\mathcal{B}:=\{a+b \mid a \in \mathcal{A}, b \in \mathcal{B}\} .
$$

In the special case when $\mathcal{B}=-\mathcal{A}:=\{-a \mid a \in \mathcal{A}\}$ we simply write $\mathcal{A}-\mathcal{A}$ instead of $\mathcal{A}+(-\mathcal{A})$. A classical result of Cauchy [2] and Davenport [3] can be phrased as follows.

Theorem 1. Let $\mathcal{A}, \mathcal{B}$ be nonempty subsets of an abelian group $G$. Then

$$
|\mathcal{A}+\mathcal{B}| \geq \min \{|\mathcal{A}|+|\mathcal{B}|-1, p(G)\} .
$$

[^2]This theorem is originally claimed for the case when $G$ is the group of integers modulo a prime, but the combinatorial proof carries over to the general case without any difficulty, see [7]. As a particular case we have

Theorem 2. Let $\mathcal{A}$ be a nonempty subset of a field $\mathbb{F}$. Then

$$
|\mathcal{A}-\mathcal{A}| \geq \min \{2|\mathcal{A}|-1, p(\mathbb{F})\} .
$$

The aim of this short note is twofold. One is to disseminate the knowledge that difference sets arise naturally as spectra of the adjoint representation of diagonal matrices. We make use of this fact to give a proof of Theorem 2 in Section 2. Initiated by the seminal paper of Olson [10], various algebraic tools have been introduced to obtain finite addition theorems, ranging from polynomial and multilinear algebra through group extensions to group algebras, representation theory and even algebraic topology. To the best of our knowledge, a Lie algebra approach to additive combinatorics has never been investigated before. Thus we hope this idea might eventually lead to some novel developments.

Our other intention is to bring the masterpiece [5] of Dias da Silva and Hamidoune closer to the heart of the combinatorics community. The material of this note was presented at the Additive Combinatorics conference held in Paris dedicated to the memory of Yahya Ould Hamidoune, followed by an expressed interest in a written exposition. Our proof of Theorem 2 does not really rely on the Lie algebra structure. We find that a slight modification allows one to give a proof of Theorem 1, at least when $G=\mathbb{F}^{+}$, in purely matrix algebraic terms. This is the content of Section 3. In retrospect, it is only a somewhat simplified version of the proof of the Cauchy-Davenport theorem found by Dias da Silva and Hamidoune [4]; see the first remark at the end of the present paper. While the polynomial method, probably because of its relative simplicity, became very successful after the appearance of Alon's Nullstellensatz [1], the multilinear and in particular the exterior algebra method remained less exploited. We feel that there is still a lot of potential in these methods and hope that this writing together with [9] will make them more accessible to the readers interested in additive combinatorics.

Some argue that the polynomial and multilinear approaches to set addition are fundamentally the same, even though this similarity only seems to materialize in certain coefficients which apparently encode some combinatorial information. It would be very interesting to obtain a deeper understanding of this coincidence.

## 2. Difference Sets

Let $\mathcal{A}=\left\{a_{1}, \ldots, a_{n}\right\}$. (We assume that the $a_{i}$ are pairwise different.) Consider the general linear Lie algebra $\mathfrak{L}=\mathfrak{g l}(n, \mathbb{F})$, that is, the vector space $\mathbb{F}^{n \times n}$ of $n \times n$ matrices over $\mathbb{F}$ equipped with the binary operation $(X, Y) \rightarrow$ $\rightarrow[X Y]=X Y-Y X$. The elements of $\mathfrak{L}$ act on the vector space $V=\mathbb{F}^{n}$ the usual way. Then in the standard basis $e_{1}, \ldots, e_{n}$ the action of the diagonal matrix $A=\operatorname{diag}\left(a_{1}, \ldots, a_{n}\right)$ is described by

$$
A\left(\alpha_{1} e_{1}+\cdots+\alpha_{n} e_{n}\right)=\alpha_{1} a_{1} e_{1}+\cdots+\alpha_{n} a_{n} e_{n}
$$

Thus, the spectrum of $A$ is $\mathcal{A}$ and its minimal polynomial $m_{A} \in \mathbb{F}[x]$ is given by

$$
\begin{equation*}
m_{A}(x)=\left(x-a_{1}\right)\left(x-a_{2}\right) \ldots\left(x-a_{n}\right) . \tag{2.1}
\end{equation*}
$$

For any $X \in \mathfrak{L}$ the map $\operatorname{ad} X: \mathfrak{L} \rightarrow \mathfrak{L}$ defined by ad $X(Y)=[X Y]$ is an endomorphism of $\mathfrak{L}$; the adjoint representation of $\mathfrak{L}$ is obtained by the map $\mathfrak{L} \rightarrow$ $\rightarrow \operatorname{End}(\mathfrak{L})$ sending $X$ to ad $X$. The key observation (see [6, Exercise 1.6]) is that the spectrum of

$$
\varphi=\operatorname{ad} A \in \operatorname{End}(\mathfrak{L})
$$

is the difference set $\mathcal{A}-\mathcal{A}$. More precisely, consider the matrix $E_{i j} \in \mathfrak{L}$ whose $i j$ th entry is 1 and all other entries are 0 . Thanks to

$$
\varphi\left(E_{i j}\right)=A E_{i j}-E_{i j} A=\left(a_{i}-a_{j}\right) E_{i j},
$$

each matrix $E_{i j}(1 \leq i, j \leq n)$ is an eigenvector of $\varphi$. Since these matrices form a basis of $\mathfrak{L}, \mathcal{A}-\mathcal{A}$ is indeed the set of all eigenvalues of $\varphi$. Moreover $\varphi$ is a diagonal map, therefore

$$
\operatorname{deg} m_{\varphi}=|\operatorname{Spec}(\varphi)|=|\mathcal{A}-\mathcal{A}| .
$$

In summary, in order to prove Theorem 2 it is enough to control the degree of $m_{\varphi}$. The statement is obvious if $p(\mathbb{F})=2$, and in case $p(\mathbb{F})=2 k-1<2 n-1$ there exists $A^{\prime} \subset A$ with $\left|A^{\prime}\right|=k$. Since $\mathcal{A}^{\prime} \subset \mathcal{A}$ implies $\mathcal{A}^{\prime}-\mathcal{A}^{\prime} \subset \mathcal{A}-\mathcal{A}$, we may readily assume that $p(\mathbb{F}) \geq 2 n-1$. Accordingly, it will be enough to prove that the maps $\operatorname{id}_{\mathfrak{L}}, \varphi, \varphi^{2}, \ldots, \varphi^{2 n-2}$ are linearly independent in $\operatorname{End}(\mathfrak{L})$. This is done by exhibiting a matrix $X \in \mathfrak{L}$ such that

$$
X, \varphi(X), \varphi^{2}(X), \ldots, \varphi^{2 n-2}(X)
$$

are independent in $\mathfrak{L}$. To find such an $X$, first consider the vector

$$
v=e_{1}+\cdots+e_{n} \in V .
$$

Then $A^{i} v=a_{1}^{i} e_{1}+\cdots+a_{n}^{i} e_{n}$ and therefore the matrix formed by the column vectors $v, A v, A^{2} v, \ldots, A^{n-1} v \in \mathbb{F}^{n}$ is a Vandermonde matrix whose determinant

$$
\prod_{i<j}\left(a_{j}-a_{i}\right)
$$

is different from zero. It follows that $v, A v, A^{2} v, \ldots, A^{n-1} v$ form a basis of $V$. Consequently the matrices $\left(A^{i} v\right)\left(A^{j} v\right)^{\top}(0 \leq i, j \leq n-1)$ form a basis of $\mathfrak{L}$. We refer to the quantity $i+j$ as the height of the basis element $\left(A^{i} v\right)\left(A^{j} v\right)^{\top}$.

Now consider the matrix $X=v v^{\top}$ all whose elements are 1 . A simple induction reveals that

$$
\begin{align*}
\varphi^{k}(X) & =\sum_{i=0}^{k}(-1)^{i}\binom{k}{i} A^{k-i} X A^{i} \\
& =\sum_{i=0}^{k}(-1)^{i}\binom{k}{i}\left(A^{k-i} v\right)\left(A^{i} v\right)^{\top} \tag{2.2}
\end{align*}
$$

holds for every nonnegative integer $k$. Here $\left(A^{k-i} v\right)\left(A^{i} v\right)^{\top}$ is a basis element of height $k$ if and only if both $i$ and $k-i$ are smaller than $n$. Otherwise it is a linear combination of basis elements of height less than $k$, for (2.1) implies the relation

$$
\begin{equation*}
A^{n} v=\sum_{i=1}^{n}(-1)^{i-1} \sigma_{i}(\mathcal{A}) A^{n-i} v \tag{2.3}
\end{equation*}
$$

where $\sigma_{i}(\mathcal{A})$ stands for the $i$ th elementary symmetric polynomial of the elements of $\mathcal{A}$. Note that any relation

$$
c_{0} X+c_{1} \varphi(X)+c_{2} \varphi^{2}(X)+\cdots+c_{r} \varphi^{r}(X)=0
$$

with $r \leq 2 n-2, c_{i} \in \mathbb{F}, c_{r} \neq 0$ allows $\varphi^{2 n-2}(X)$ to be expressed as a linear combination of the matrices $\varphi^{i}(X), i=0,1, \ldots, 2 n-3$. Therefore to complete the proof of Theorem 2 it is enough to show that $\varphi^{2 n-2}(X)$ is not in the linear span of

$$
\left\{\varphi^{i}(X) \mid i=0,1, \ldots, 2 n-3\right\}
$$

Suppose that on the contrary

$$
\begin{equation*}
\varphi^{2 n-2}(X)=\sum_{k=0}^{2 n-3} \alpha_{k} \varphi^{k}(X) \tag{2.4}
\end{equation*}
$$

holds with some coefficients $\alpha_{k} \in \mathbb{F}$. In view of (2.2), the right hand side can be expressed as a linear combination of basis elements $\left(A^{i} v\right)\left(A^{j} v\right)^{\top}$ of height less
than $2 n-2$. On the other hand,

$$
\varphi^{2 n-2}(X)=(-1)^{n-1}\binom{2 n-2}{n-1}\left(A^{n-1} v\right)\left(A^{n-1} v\right)^{\top}+\ldots,
$$

where $\ldots$ represents a linear combination of basis elements of height less than $2 n-2$. Given that $p(\mathbb{F})>2 n-2$, the coefficient of $\left(A^{n-1} v\right)\left(A^{n-1} v\right)^{\top}$ is not zero, which contradicts (2.4).

## 3. The Cauchy-Davenport Theorem

We obtain a proof of Theorem 1 for the additive group $G$ of a field $\mathbb{F}$ with a slight modification of the above argument as follows. Keeping some notation from the previous section, let $\mathcal{B}=\left\{b_{1}, \ldots, b_{m}\right\}$ and $B=\operatorname{diag}\left(b_{1}, \ldots, b_{m}\right) \in$ $\in \mathbb{F}^{m \times m}$. Then the spectrum of $B$ is the set $\mathcal{B}$ and its minimal polynomial is

$$
\begin{equation*}
m_{B}(x)=\left(x-b_{1}\right)\left(x-b_{2}\right) \ldots\left(x-b_{m}\right) . \tag{3.1}
\end{equation*}
$$

Consider the vector space $\mathfrak{V}=\mathbb{F}^{n \times m}$ and the linear map $\varphi \in \operatorname{End}(\mathfrak{V})$ defined by $\varphi(X)=A X+X B$. Thanks to $\varphi\left(E_{i j}\right)=\left(a_{i}+b_{j}\right) E_{i j}$, this is a diagonal map whose spectrum is $\mathcal{A}+\mathcal{B}$. That is, $|\mathcal{A}+\mathcal{B}|=\operatorname{deg} m_{\varphi}$. Since we may assume that $p(G) \geq n+m-1$, it will be enough to find a matrix $X \in \mathfrak{V}$ such that

$$
X, \varphi(X), \varphi^{2}(X), \ldots, \varphi^{n+m-2}(X)
$$

are linearly independent in $\mathfrak{V}$. Denote by $f_{1}, \ldots, f_{m}$ the standard basis of $U=$ $\mathbb{F}^{m}$, and let $u=f_{1}+\cdots+f_{m}$. Then $u, B u, B^{2} u, \ldots, B^{m-1} u$ form a basis of $U$, therefore the matrices $\left(A^{i} v\right)\left(B^{j} u\right)^{\top}$ for $0 \leq i \leq n-1,0 \leq j \leq m-1$ form a basis of $\mathfrak{V}$. Let $X=v u^{\top}$, the $n \times m$ all 1 matrix. Then

$$
\begin{equation*}
\varphi^{k}(X)=\sum_{i=0}^{k}\binom{k}{i}\left(A^{k-i} v\right)\left(B^{i} u\right)^{\top} \tag{3.2}
\end{equation*}
$$

holds for every nonnegative integer $k$. Thanks to (2.3) and the relation

$$
B^{m} u=\sum_{i=1}^{m}(-1)^{i-1} \sigma_{i}(\mathcal{B}) B^{m-i} u
$$

implied by (3.1), each $\left(A^{k-i} v\right)\left(B^{i} u\right)^{\top}$ which is not a basis element can be expressed as a linear combination of basis elements of height less than $k$. Thus to check that $\varphi^{n+m-2}(X)$ is not in the linear span of

$$
\left\{\varphi^{i}(X) \mid i=0,1, \ldots, n+m-3\right\}
$$

one only has to observe that

$$
\varphi^{n+m-2}(X)=\binom{n+m-2}{m-1}\left(A^{n-1} v\right)\left(B^{m-1} u\right)^{\top}+\text { terms of lower height }
$$

where the coefficient $\binom{n+m-2}{m-1}$ is not zero due to the assumption $p(\mathbb{F})=p(G) \geq$ $\geq n+m-1$.

## 4. Concluding Remarks

I. TENSOR PRODUCTS: A DIFFERENT LANGUAGE. The vector space $\mathfrak{V}=\mathbb{F}^{n \times m}$ is isomorphic to the tensor product $V \otimes U=\mathbb{F}^{n} \otimes \mathbb{F}^{m}$ in a natural way, the matrices $E_{i j}=e_{i} f_{j}^{\top}$ corresponding to the elements $e_{i} \otimes f_{j}$. In general, matrices of the form $a b^{\top}$ represent elements $a \otimes b$, thus the matrix $X$ belongs to $v \otimes u$. Finally, the map $\varphi \in \operatorname{End}(\mathfrak{V})$ can be understood as the Kronecker sum $\varphi_{A} \otimes \mathrm{id}_{U}+\mathrm{id}_{V} \otimes \varphi_{B}$ of the linear maps $\varphi_{A} \in \operatorname{End}(V)$ and $\varphi_{B} \in \operatorname{End}(U)$, which denote left multiplication by $A$ and $B$ respectively. The independence of the vectors $v, A v, \ldots, A^{n-1} v$ means that the cyclic space of $\varphi_{A}$ generated by $v$ is the whole vector space $V$. Rewriting the previous section in this language we arrive at a somewhat simplified version of the proof given in [4]. In particular, the result is obtained by bounding the dimension of the cyclic space of the Kronecker sum generated by $v \otimes u$.
2. Restricted set addition. Consider the vector space $\mathfrak{W}$ of all skew symmetric $n \times n$ matrices over $\mathbb{F}$. Then $\mathfrak{W} \cong V \wedge V$. A standard basis for $\mathfrak{W}$ is the set of matrices $E_{i j}-E_{j i}(1 \leq i<j \leq n)$, which correspond to the elements $e_{i} \wedge e_{j}$. The map $\varphi \in \operatorname{End}(\mathfrak{W})$ defined by $\varphi(X)=A X+X A$ can be understood as the derivative of $\varphi_{A}$ on $V \wedge V$ given by $D \varphi_{A}=\varphi_{A} \wedge \mathrm{id}_{V}+\mathrm{id}_{V} \wedge \varphi_{A}$. The spectrum of this map is the restricted sumset $\mathcal{A} \dot{+} \mathcal{A}:=\left\{a+a^{\prime} \mid a, a^{\prime} \in \mathcal{A}, a \neq a^{\prime}\right\}$. Working with the matrix $X=\left(v v^{\top}\right) A-A\left(v v^{\top}\right)$ corresponding to $v \wedge A v$, the argument gives the $m=2$ case of the Dias da Silva-Hamidoune theorem as presented in [9] in the language of wedge products; namely that $|A \dot{+} \mathcal{A}| \geq \min \{p(\mathbb{F}), 2|A|-3\}$.
3. Towards structural results. Consider the proof of Theorem 2. Suppose that $p(\mathbb{F})>|\mathcal{A}-\mathcal{A}|=2 n-1$. This means that $\varphi^{2 n-1}(X)$ is a linear combination of the matrices $X, \varphi(X), \ldots, \varphi^{2 n-2}(X)$ with some coefficients in $\mathbb{F}$. In view of the relations (2.2) and (2.3) this implies a system of polynomial equations connecting the numbers $\sigma_{i}(\mathcal{A})$ and these coefficients. Eliminating the latter and assuming that $\mathbb{F}$ is algebraically closed it can be seen that the elements of $\mathcal{A}$ must form an arithmetic progression. This way we recover a special case of Vosper's
inverse theorem [11]. We do not elaborate on the heavy details here, see [8] for more substantial inverse theorems obtained using this idea.

## References

[1] N. Alon, Combinatorial Nullstellensatz, Combin. Prob. Comput. 8 (1999), 7-29
[2] A.L. Cauchy, Recherches sur les nombres, J. École Polytech. 9 (1813), 99-116
[3] H. Davenport, On the addition of residue classes, J. London Math. Soc. 10 (1935), 30-32
[4] J.A. Dias da Silva and Y.O. Hamidoune, A note on the minimal polynomial of the Kronecker sum of two linear operators, Linear Algebra Appl. 141 (1990), 283-287
[5] J.A. Dias da Silva and Y.O. Hamidoune, Cyclic spaces for Grassmann derivatives and additive theory, Bull. London Math. Soc. 26 (1994), 140-146
[6] J.E. Humphreys, Introduction to Lie Algebras and Representation Theory, revised edition, Graduate Texts in Mathematics, Vol. 9, Springer-Verlag, New YorkBerlin, 1978
[7] Gy. Károlyi, A compactness argument in the additive theory and the polynomial method, Discrete Math. 302 (2005), 124-144
[8] Gy. Károlyı, Structural results for set addition via derivations on tensor spaces and Grassmannians, submitted
[9] Gy. Károlyi and R. Paulin, On the exterior algebra method applied to restricted set addition, European J. Combin. 34 (2013), 1383-1389
[10] J.E. Olson, A combinatorial problem on finite abelian groups. I, J. Number Theory 1 (1969), 8-10
[11] A.G. Vosper, The critical pairs of subsets of a group of prime order, J. London Math. Soc. 31 (1956), 200-205, Addendum 280-282

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# ITERATE $(i, j)$ - $m$-STRUCTURES AND ITERATE (i,j)-m-CONTINUITY 

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#### Abstract

We introduce the notion of $(i, j) m I T$-open sets determined by operators $m_{X}^{i}$-Int and $m_{X}^{i}-\mathrm{Cl}(i=1,2)$ on a bi- $m$-space $\left(X, m_{X}^{1}, m_{X}^{2}\right)$. By using $(i, j) m I T$-open sets, we introduce and investigate a function $f:\left(X, m_{X}^{1}, m_{X}^{2}\right) \rightarrow\left(Y, \sigma_{1}, \sigma_{2}\right)$ called $(i, j) m I T$ continuous. As a special case of $(i, j) m I T$-continuous functions, we obtain $(i, j)$-mprecontinuous functions due to Carpintero et al. [7].


## 1. Introduction

The concepts of minimal structures (briefly $m$-structures) and minimal spaces (briefly $m$-spaces) are introduced by the present authors in [27] and [28]. In these papers, they introduced $M$-continuous functions and $m$-continuous functions and obtained their basic properties. Moreover, in [21] and [24], they extended the study of continuity between bitopological spaces to the study of $m$-continuity and $M$-continuity beteen minimal stuructures. Quite recently, in [14]-[18], Min and Kim introduced the notions of $m$-semi-open sets, $m$-preopen sets, $m$ - $\alpha$-open sets and $m$ - $\beta$-open sets which are generalizations of semi-open sets, preopen sets, $\alpha$-open sets and $\beta$-open sets, respectively. And also, they introduced the notions of $m$-semi-continuity, $m$-precontinuity, $m$ - $\alpha$-continuity and $m$ - $\beta$-continuity which are generalizations of the notions of semi-continuity, precontinuity, $\alpha$-continuity and $\beta$-continuity, respectively. In [6], [33] and [34], the notions of $m$-semi-open sets, $m$-preopen sets, $m$ - $\alpha$-open sets and $m$ - $\beta$-open sets are also introduced and studied. In [26], the present authors introduced the notions of iterate minimal structures and iterate $m$-continuity.

[^3]A set with two minimal structures is used in Theorems 4.1 and 4.2 of [31], Theorems 4.2 and 4.3 of [32], Theorems 7.4 and 7.5 of [22], Theorem 4.5 of [23], Theorems 4.2, 4.3 and 5.2 of [25]. The notion of bi- $m$-spaces is introduced in [20]. Quite recently, Carpintero et al. [7] introduced the notions of $(i, j)-m$ preopen sets and $(i, j)-m$-precontinuous functions. In the present paper, we introduce the notions of $(i, j) m I T$-open sets and $(i, j) m I T$-continuous functions which are generalizations of $(i, j)$ - $m$-preopen sets and $(i, j)$-m-precontinuous functions.

## 2. Preliminaries

Let $(X, \tau)$ be a topological space and $A$ a subset of $X$. The closure of $A$ and the interior of $A$ are denoted by $\mathrm{Cl}(A)$ and $\operatorname{Int}(A)$, respectively. We recall some generalized open sets in topological spaces.
Definition 2.1. Let $(X, \tau)$ be a topological space. A subset $A$ of $X$ is said to be
(1) $\alpha$-open [19] if $A \subset \operatorname{Int}(\mathrm{Cl}(\operatorname{Int}(A)))$,
(2) semi-open $[10]$ if $A \subset \mathrm{Cl}(\operatorname{Int}(A))$,
(3) preopen [12] if $A \subset \operatorname{Int}(\mathrm{Cl}(A))$,
(4) $\beta$-open [1] or semi-preopen [3] if $A \subset \mathrm{Cl}(\operatorname{Int}(\mathrm{Cl}(A)))$.

The family of all $\alpha$-open (resp. semi-open, preopen, $\beta$-open) sets in $(X, \tau)$ is denoted by $\alpha(X)$ (resp. $\mathrm{SO}(X), \mathrm{PO}(X), \beta(X)$ ).
Definition 2.2. Let $(X, \tau)$ be a topological space. A subset $A$ of $X$ is said to be $\alpha$-closed [13] (resp. semi-closed [8], preclosed [12], $\beta$-closed [1]) if the complement of $A$ is $\alpha$-open (resp. semi-open, preopen, $\beta$-open).
Definition 2.3. Let $(X, \tau)$ be a topological space and $A$ a subset of $X$. The intersection of all $\alpha$-closed (resp. semi-closed, preclosed, $\beta$-closed) sets of $X$ containing $A$ is called the $\alpha$-closure [13] (resp. semi-closure [8], preclosure [9], $\beta$-closure [2]) of $A$ and is denoted by $\alpha \mathrm{Cl}(A)$ (resp. $\mathrm{sCl}(A), \mathrm{pCl}(A),{ }_{\beta} \mathrm{Cl}(A)$ ).
Definition 2.4. Let $(X, \tau)$ be a topological space and $A$ a subset of $X$. The union of all $\alpha$-open (resp. semi-open, preopen, $\beta$-open) sets of $X$ contained in $A$ is called the $\alpha$-interior [13] (resp. semi-interior [8], preinterior [9], $\beta$-interior [2]) of $A$ and is denoted by $\alpha \operatorname{Int}(A)$ (resp. $\left.\operatorname{sInt}(A), \operatorname{pInt}(A),{ }_{\beta} \operatorname{Int}(A)\right)$.
Definition 2.5. A function $f:(X, \tau) \rightarrow(Y, \sigma)$ is said to be $\alpha$-continuous [13] (resp. semi-continuous [10], precontinuous [12], $\beta$-continuous [1]) at $x \in X$ if for each open set $V$ containing $f(x)$, there exists an $\alpha$-open (resp. semi-open, preopen, $\beta$-open) set $U$ of $X$ containing $x$ such that $f(U) \subset V$. The function $f$ is
said to be $\alpha$-continuous (resp. semi-continuous, precontinuous, $\beta$-continuous) if it has this property at each point $x \in X$.

## 3. Minimal structures and $m$-continuity

Definition 3.1. Let $X$ be a nonempty set and $\mathcal{P}(X)$ the power set of $X$. A subfamily $m_{X}$ of $\mathcal{P}(X)$ is called a minimal structure (briefly $m$-structure) on $X$ [27], [28] if $\emptyset \in m_{X}$ and $X \in m_{X}$.

By $\left(X, m_{X}\right)$, we denote a nonempty set $X$ with an $m$-structure $m_{X}$ on $X$ and call it an $m$-space. Each member of $m_{X}$ is said to be $m_{X}$-open (briefly $m$-open) and the complement of an $m_{X}$-open set is said to be $m_{X}$-closed (briefly $m$-closed).

Remark 3.1. Let $(X, \tau)$ be a topological space. The families $\tau, \alpha(X), \operatorname{SO}(X)$, $\mathrm{PO}(X)$ and $\beta(X)$ are all minimal structures on $X$.

Definition 3.2. Let $X$ be a nonempty set and $m_{X}$ an $m$-structure on $X$. For a subset $A$ of $X$, the $m_{X}$-closure of $A$ and the $m_{X}$-interior of $A$ are defined in [11] as follows:
(1) $\mathrm{mCl}(A)=\cap\left\{F: A \subset F, X \backslash F \in m_{X}\right\}$,
(2) $\operatorname{mInt}(A)=\cup\left\{U: U \subset A, U \in m_{X}\right\}$.

Remark 3.2. Let $(X, \tau)$ be a topological space and $A$ a subset of $X$. If $m_{X}=\tau$ (resp. $\mathrm{SO}(X), \mathrm{PO}(X), \alpha(X), \beta(X)$ ), then we have
(1) $\operatorname{mCl}(A)=\mathrm{Cl}(A)\left(\right.$ resp. $\left.\mathrm{sCl}(A), \mathrm{pCl}(A), \alpha \mathrm{Cl}(A),{ }_{\beta} \mathrm{Cl}(A)\right)$,
(2) $\operatorname{mInt}(A)=\operatorname{Int}(A)\left(\operatorname{resp} . \operatorname{sInt}(A), \operatorname{pInt}(A), \alpha \operatorname{Int}(A),{ }_{\beta} \operatorname{Int}(A)\right)$.

Lemma 3.1 (Maki et al. [11]). Let $X$ be a nonempty set and $m_{X}$ a minimal structure on $X$. For subsets $A$ and $B$ of $X$, the following properties hold:
(1) $\mathrm{mCl}(X \backslash A)=X \backslash \operatorname{mInt}(A)$ and $\operatorname{mInt}(X \backslash A)=X \backslash \operatorname{mCl}(A)$,
(2) If $(X \backslash A) \in m_{X}$, then $\mathrm{mCl}(A)=A$ and if $A \in m_{X}$, then $\operatorname{mInt}(A)=A$,
(3) $\mathrm{mCl}(\emptyset)=\emptyset, \mathrm{mCl}(X)=X, \operatorname{mInt}(\emptyset)=\emptyset$ and $\operatorname{mInt}(X)=X$,
(4) If $A \subset B$, then $\mathrm{mCl}(A) \subset \mathrm{mCl}(B)$ and $\operatorname{mInt}(A) \subset \operatorname{mInt}(B)$,
(5) $A \subset \mathrm{mCl}(A)$ and $\operatorname{mint}(A) \subset A$,
(6) $\mathrm{mCl}(\mathrm{mCl}(A))=\mathrm{mCl}(A)$ and $\operatorname{mInt}(\operatorname{mInt}(A))=\operatorname{mInt}(A)$.

Lemma 3.2. (Popa and Noiri [27]). Let $\left(X, m_{X}\right)$ be an $m$-space and $A$ a subset of $X$. Then $x \in \operatorname{mCl}(A)$ if and only if $U \cap A \neq \emptyset$ for each $U \in m_{X}$ containing $x$.
Definition 3.3. An $m$-structure $m_{X}$ on a nonempty set $X$ is said to have property $\mathcal{B}$ [11] if the union of any family of subsets belonging to $m_{X}$ belongs to $m_{X}$.

Remark 3.3. If $(X, \tau)$ is a topological space, then the $m$-structures $\mathrm{SO}(X)$, $\mathrm{PO}(X), \alpha(X)$ and $\beta(X)$ have property $\mathcal{B}$.
Lemma 3.3 (Popa and Noiri [30]). Let $X$ be a nonempty set and $m_{X}$ an $m$-structure on $X$ satisfying property $\mathcal{B}$. For a subset $A$ of $X$, the following properties hold:
(1) $A \in m_{X}$ if and only if $\operatorname{mint}(A)=A$,
(2) $A$ is $m_{X}$-closed if and only if $\mathrm{mCl}(A)=A$,
(3) $\operatorname{mInt}(A) \in m_{X}$ and $\mathrm{mCl}(A)$ is $m_{X}$-closed.

Definition 3.4. A function $f:\left(X, m_{X}\right) \rightarrow(Y, \sigma)$ is said to be $m$-continuous at $x \in X$ [28] if for each open set $V$ containing $f(x)$, there exists $U \in m_{X}$ containing $x$ such that $f(U) \subset V$. The function $f$ is $m$-continuous if it has this property at each $x \in X$.

Remark 3.4. Let $(X, \tau)$ be a topological space. If $f:\left(X, m_{X}\right) \rightarrow(Y, \sigma)$ is $m$ continuous and $m_{X}=\alpha(X)$ (resp. $\mathrm{SO}(X), \mathrm{PO}(X), \beta(X)$ ), then by Definition 3.4, we obtain Definition 2.5.

Theorem 3.1 (Popa and Noiri [28]). For a function $f:\left(X, m_{X}\right) \rightarrow(Y, \sigma)$, the following properties are equivalent:
(1) $f$ is $m$-continuous;
(2) $f^{-1}(V)=\operatorname{mInt}\left(f^{-1}(V)\right)$ for every open set $V$ of $Y$;
(3) $f^{-1}(F)=\mathrm{mCl}\left(f^{-1}(F)\right)$ for every closed set $F$ of $Y$;
(4) $\mathrm{mCl}\left(f^{-1}(B)\right) \subset f^{-1}(\mathrm{Cl}(B))$ for every subset $B$ of $Y$;
(5) $f(\mathrm{mCl}(A)) \subset \mathrm{Cl}(f(A))$ for every subset $A$ of $X$;
(6) $f^{-1}(\operatorname{Int}(B)) \subset \operatorname{mInt}\left(f^{-1}(B)\right)$ for every subset $B$ of $Y$.

Corollary 3.1 (Popa and Noiri [28]). For a function $f:\left(X, m_{X}\right) \rightarrow(Y, \sigma)$, where $m_{X}$ has property $\mathcal{B}$, the following properties are equivalent:
(1) $f$ is $m$-continuous;
(2) $f^{-1}(V)$ is $m$-open in $X$ for every open set $V$ of $Y$;
(3) $f^{-1}(F)$ is $m$-closed in $X$ for every closed set $F$ of $Y$.

For a function $f:\left(X, m_{X}\right) \rightarrow(Y, \sigma)$, we define $D_{m}(f)$ as follows:

$$
D_{m}(f)=\{x \in X: f \text { is not } m \text {-continuous at } x\} .
$$

Theorem 3.2 (Popa and Noiri [29]). For a function $f:\left(X, m_{X}\right) \rightarrow(Y, \sigma)$, the following properties hold:

$$
\begin{aligned}
D_{m}(f) & =\bigcup_{G \in \sigma}\left\{f^{-1}(G) \backslash \operatorname{mInt}\left(f^{-1}(G)\right)\right\} \\
& =\bigcup_{B \in \mathcal{P}(Y)}\left\{f^{-1}(\operatorname{Int}(B)) \backslash \operatorname{mInt}\left(f^{-1}(B)\right)\right\}
\end{aligned}
$$

$$
\begin{aligned}
& =\bigcup_{B \in \mathcal{P}(Y)}\left\{\operatorname{mCl}\left(f^{-1}(B)\right) \backslash f^{-1}(\mathrm{Cl}(B))\right\} \\
& =\bigcup_{A \in \mathcal{P}(X)}\left\{\operatorname{mCl}(A) \backslash f^{-1}(\mathrm{Cl}(f(A)))\right\} \\
& =\bigcup_{F \in \mathcal{F}}\left\{\operatorname{mCl}\left(f^{-1}(F)\right) \backslash f^{-1}(F)\right\}
\end{aligned}
$$

where $\mathcal{F}$ is the family of closed sets of $(Y, \sigma)$.

## 4. Iterate $(i, j)-m$-structures and iterate $(i, j)$-m-continuity

Definition 4.1. Let $X$ be a nonempty set and $m_{X}^{1}, m_{X}^{2}$ be minimal structures on $X$. The triple $\left(X, m_{X}^{1}, m_{X}^{2}\right)$ is called a bi-minimal space (briefly bi-m-space) [20] or a biminimal structure space [5].

Definition 4.2. Let $\left(X, m_{X}^{1}, m_{X}^{2}\right)$ be a bi- $m$-space. A subset $A$ of $X$ is said to be
(1) $(i, j)$-m- $\alpha$-open [4] if $A \subset m_{X}^{i} \operatorname{Int}\left(m_{X}^{j} \mathrm{Cl}\left(m_{X}^{i} \operatorname{Int}(A)\right)\right)$, where $i \neq j$, $i, j=1,2$,
(2) $(i, j)$-m-semiopen [4] if $\left.A \subset m_{X}^{i} \mathrm{Cl}_{( } m_{X}^{j} \operatorname{Int}(A)\right)$, where $i \neq j, i, j=1,2$,
(3) $(i, j)$-m-preopen [4], [7] if $A \subset m_{X}^{i} \operatorname{Int}\left(m_{X}^{j} \mathrm{Cl}(A)\right)$, where $i \neq j$, $i, j=1,2$,
(4) $(i, j)$-m- $\beta$-open [4] if $A \subset m_{X}^{i} \mathrm{Cl}\left(m_{X}^{j} \operatorname{Int}\left(m_{X}^{i} \mathrm{Cl}(A)\right)\right)$, where $i \neq j$, $i, j=1,2$.

The family of all $(i, j)-m$ - $\alpha$-open (resp. $(i, j)$ - $m$-semiopen, $(i, j)$-m-preopen, $(i, j)$ - $m$ - $\beta$-open) sets in a bi- $m$-space $\left(X, m_{X}^{1}, m_{X}^{2}\right)$ is denoted by $(i, j) m \alpha(X)$ (resp. $(i, j) m \mathrm{SO}(X),(i, j) m \mathrm{PO}(X),(i, j) m \beta(X))$.

Let $\left(X, m_{X}^{1}, m_{X}^{2}\right)$ be a bi- $m$-space. Then $(i, j) m \alpha(X),(i, j) m \operatorname{SO}(X)$, $(i, j) m \mathrm{PO}(X)$ and $(i, j) m \beta(X)$ are determined by iterating operators mInt and mCl . Hence, they are called $(i, j)$-m-iterate structures and are denoted by $(i, j) \mathrm{mIT}(X)$ (briefly $(i, j) \mathrm{mIT}$ ).
REMARK 4.1. (1) If $m_{X}^{1}=m_{X}^{2}=m_{X}$, we obtain the definition of iterate $m$ structures in [26].
(2) It follows from Lemma 3.1(3)(4) that $(i, j) m \alpha(X),(i, j) m \mathrm{SO}(X)$, $(i, j) m \mathrm{PO}(X)$ and $(i, j) m \beta(X)$ are minimal structures with property $\mathcal{B}$. If $(i, j) \mathrm{mIT}(X)=(i, j) m \mathrm{PO}(X)$, it is also shown in Theorem 3.7 of [7].

Since $(i, j) \mathrm{mIT}(X)$ is a minimal structure on $X$, we can define the $(i, j) m I T$ closure and $(i, j) m I T$-interior of a subset $A$ of $X$ as in Definition 3.2:
(1) $(i, j) m I T \mathrm{Cl}(A)=\cap\{F: A \subset F, X \backslash F \in(i, j) \operatorname{mIT}(X)\}$,
(2) $(i, j) m I T \operatorname{Int}(A)=\cup\{U: U \subset A, U \in(i, j) \operatorname{mIT}(X)\}$.

Remark 4.2. Let $\left(X, m_{X}^{1}, m_{X}^{2}\right)$ be a bi- $m$-space and $(i, j) \operatorname{mIT}(X)$ an $(i, j)$ - $m$ iterate structure on $X$. If $(i, j) \operatorname{mIT}(X)=(i, j) m \alpha(X)$ (resp. $(i, j) m \mathrm{SO}(X)$, $(i, j) m \mathrm{PO}(X),(i, j) m \beta(X))$, then we have
(1) $(i, j) m I T \mathrm{Cl}(A)=(i, j) m \alpha \mathrm{Cl}(A)$ (resp. $(i, j) m s \mathrm{Cl}(A),(i, j) m p \mathrm{Cl}(A)$, $(i, j) m \beta \mathrm{Cl}(A))$,
(2) $(i, j) m I T \operatorname{Int}(A)=(i, j) m \alpha \operatorname{Int}(A)($ resp. $(i, j) m s \operatorname{Int}(A),(i, j) m p \operatorname{Int}(A)$, $(i, j) m \beta \operatorname{Int}(A))$.
Remark 4.3. (1) If $(i, j) \operatorname{mIT}(X)=(i, j) m \mathrm{PO}(X)$, then by Lemmas 3.1 and 3.2, we obtain the results established in Theorem 3.12 (i)-(v) and Theorem 3.14 (i)-(v) of [7].
(2) By Lemma 3.1(1), we obtain Theorem 3.15 of [7].

Definition 4.3. A function $f:\left(X, m_{X}^{1}, m_{X}^{2}\right) \rightarrow\left(Y, \sigma_{1}, \sigma_{2}\right)$, where $\left(Y, \sigma_{1}, \sigma_{2}\right)$ is a bitopological space, is said to be ( $i, j$ )-m-precontinuous [7] at $x \in X$ if for each $\sigma_{i^{-}}$ open set $V$ containing $f(x)$, there exists an $(i, j)$-m-preopen set $U$ of $X$ containing $x$ such that $f(U) \subset V$. The function $f$ is said to be $(i, j)$-m-precontinuous if it has this property at each $x \in X$.
Definition 4.4. Let $\left(X, m_{X}^{1}, m_{X}^{2}\right)$ be a bi- $m$-space and $\left(Y, \sigma_{1}, \sigma_{2}\right)$ a bitopological space. A function $f:\left(X, m_{X}^{1}, m_{X}^{2}\right) \rightarrow\left(Y, \sigma_{1}, \sigma_{2}\right)$ is said to be $(i, j)$ mIT-continuous at $x \in X($ on $X)$ if $f:(X,(i, j) m I T(X)) \rightarrow\left(Y, \sigma_{i}\right)$ is $m$-continuous at $x \in X($ on $X)$.

Hence, $f:\left(X, m_{X}^{1}, m_{X}^{2}\right) \rightarrow\left(Y, \sigma_{1}, \sigma_{2}\right)$ is said to be $(i, j) m I T$-continuous at $x \in X$ if for each $\sigma_{i}$-open set $V$ of $Y$ containing $f(x)$, there exists $U \in$ $\in(i, j) \operatorname{mIT}(X)$ containing $x$ such that $f(U) \subset V$. The function $f$ is $(i, j) m I T$ continuous if it has this property at each $x \in X$.

Remark 4.4. (1) Since $(i, j) \mathrm{mIT}(X)$ has property $\mathcal{B}$, by Definition 4.4 we obtain Definition 4.3. Similarly, we can define $(i, j)$ - $m$-semi-continuity, $(i, j)-m-\alpha-$ continuity and $(i, j)-m-\beta$-continuity.
(2) If $m_{X}^{1}=m_{X}^{2}=m_{X}$, we obtain the definition of $m I T$-continuous functions in [26].

Since $(i, j) \operatorname{mIT}(X)$ has property $\mathcal{B}$, by Theorems 3.1 and 3.2 and Corollary 3.1 we have the following theorems and corollaries.

Theorem 4.1. For a function $f:\left(X, m_{X}^{1}, m_{X}^{2}\right) \rightarrow\left(Y, \sigma_{1}, \sigma_{2}\right)$, the following properties are equivalent:
(1) $f$ is $(i, j) m I T$-continuous;
(2) $f^{-1}(V)$ is $(i, j) m I T$-open for every $\sigma_{i}$-open set $V$ of $Y$;
(3) $f^{-1}(F)$ is $(i, j) m I T$-closed for every $\sigma_{i}$-closed set $F$ of $Y$;
(4) $(i, j) m I T \mathrm{Cl}\left(f^{-1}(B)\right) \subset f^{-1}\left(\sigma_{i}-\mathrm{Cl}(B)\right)$ for every subset $B$ of $Y$;
(5) $f((i, j) m I T \mathrm{Cl}(A)) \subset \sigma_{i}-\mathrm{Cl}(f(A))$ for every subset $A$ of $X$;
(6) $f^{-1}\left(\sigma_{i}-\operatorname{Int}(B)\right) \subset(i, j) m I T \operatorname{Int}\left(f^{-1}(B)\right)$ for every subset $B$ of $Y$.

Remark 4.5. (1) If $(i, j) \mathrm{mIT}(X)=(i, j) m \mathrm{PO}(X)$, then by Theorem 4.1 we obtain Theorem 4.4 of [7].
(2) If $(i, j) \operatorname{mIT}(X)=(i, j) m \mathrm{SO}(X),(i, j) m \alpha(X)$ or $(i, j) m \beta(X)$, then we obtain the similar results. For example, if $(i, j) \mathrm{mIT}(X)=(i, j) m \mathrm{SO}(X)$, we obtain the following corollary.
Corollary 4.1. For a function $f:\left(X, m_{X}^{1}, m_{X}^{2}\right) \rightarrow\left(Y, \sigma_{1}, \sigma_{2}\right)$, the following properties are equivalent:
(1) $f$ is $(i, j)$-m-semi-continuous;
(2) $f^{-1}(V)$ is $(i, j)$-m-semi-open for every $\sigma_{i}$-open set $V$ of $Y$;
(3) $f^{-1}(F)$ is $(i, j)$-m-semi-closed for every $\sigma_{i}$-closed set $F$ of $Y$;
(4) $(i, j) m s \mathrm{Cl}\left(f^{-1}(B)\right) \subset f^{-1}\left(\sigma_{i}-\mathrm{Cl}(B)\right)$ for every subset $B$ of $Y$;
(5) $f((i, j) m s \mathrm{Cl}(A)) \subset \sigma_{i}-\mathrm{Cl}(f(A))$ for every subset $A$ of $X$;
(6) $f^{-1}\left(\sigma_{i}-\operatorname{Int}(B)\right) \subset(i, j) m s \operatorname{Int}\left(f^{-1}(B)\right)$ for every subset $B$ of $Y$.

For a function $f:\left(X, m_{X}^{1}, m_{X}^{2}\right) \rightarrow\left(Y, \sigma_{1}, \sigma_{2}\right)$, we define $D_{(i, j) m I T}(f)$ as follows:

$$
D_{(i, j) m I T}(f)=\{x \in X: f \text { is not }(i, j) m I T \text {-continuous at } x\} .
$$

By Theorem 3.2, we obtain Theorem 4.2.
Theorem 4.2. For a function $f:\left(X, m_{X}^{1}, m_{X}^{2}\right) \rightarrow\left(Y, \sigma_{1}, \sigma_{2}\right)$, the following properties hold:

$$
\begin{aligned}
D_{(i, j) m I T}(f) & =\bigcup_{G \in \sigma_{i}}\left\{f^{-1}(G) \backslash(i, j) m I T \operatorname{Int}\left(f^{-1}(G)\right)\right\} \\
& =\bigcup_{B \in \mathcal{P}(Y)}\left\{f^{-1}\left(\sigma_{i}-\operatorname{Int}(B)\right) \backslash(i, j) m I T \operatorname{Int}\left(f^{-1}(B)\right)\right\} \\
& =\bigcup_{B \in \mathcal{P}(Y)}\left\{(i, j) m I T \operatorname{Cl}\left(f^{-1}(B)\right) \backslash f^{-1}\left(\sigma_{i}-\mathrm{Cl}(B)\right)\right\} \\
& =\bigcup_{A \in \mathcal{P}(X)}\left\{(i, j) m I T \operatorname{Cl}(A) \backslash f^{-1}\left(\sigma_{i}-\mathrm{Cl}(f(A))\right)\right\} \\
& =\bigcup_{F \in \mathcal{F}}\left\{(i, j) m I T \mathrm{Cl}\left(f^{-1}(F)\right) \backslash f^{-1}(F)\right\},
\end{aligned}
$$

where $\mathcal{F}$ is the family of $\sigma_{i}$-closed sets of $\left(Y, \sigma_{i}\right)$.
If $(i, j) \mathrm{mIT}(X)=(i, j) m \mathrm{PO}(X)$, then we obtain the following corollary. For a function $f:\left(X, m_{X}^{1}, m_{X}^{2}\right) \rightarrow\left(Y, \sigma_{1}, \sigma_{2}\right)$, we define $D_{(i, j) m p}(f)$ as follows:

$$
D_{(i, j) m p}(f)=\{x \in X: f \text { is not }(i, j) \text {-m-precontinuous at } x\}
$$

Corollary 4.2. For a function $f:\left(X, m_{X}^{1}, m_{X}^{2}\right) \rightarrow\left(Y, \sigma_{1}, \sigma_{2}\right)$, the following properties hold:

$$
\begin{aligned}
D_{(i, j) m p}(f) & =\bigcup_{G \in \sigma_{i}}\left\{f^{-1}(G) \backslash(i, j) m p \operatorname{Int}\left(f^{-1}(G)\right)\right\} \\
& =\bigcup_{B \in \mathcal{P}(Y)}\left\{f^{-1}\left(\sigma_{i^{-}} \operatorname{Int}(B)\right) \backslash(i, j) m p \operatorname{Int}\left(f^{-1}(B)\right)\right\} \\
& =\bigcup_{B \in \mathcal{P}(Y)}\left\{(i, j) m p \operatorname{Cl}\left(f^{-1}(B)\right) \backslash f^{-1}\left(\sigma_{i^{-}} \mathrm{Cl}(B)\right)\right\} \\
& =\bigcup_{A \in \mathcal{P}(X)}\left\{(i, j) m p \mathrm{Cl}(A) \backslash f^{-1}\left(\sigma_{i}-\mathrm{Cl}(f(A))\right)\right\} \\
& =\bigcup_{F \in \mathcal{F}}\left\{(i, j) m p \mathrm{Cl}\left(f^{-1}(F)\right) \backslash f^{-1}(F)\right\}
\end{aligned}
$$

where $\mathcal{F}$ is the family of $\sigma_{i}$-closed sets of $\left(Y, \sigma_{i}\right)$.

## 5. Some properties of $(i, j) m I T$-continuous functions

Since the study of $(i, j) m I T$-continuity is reduced from the study of $m$ continuity, the properties of $(i, j) m I T$-continuous functions follow from the properties of $m$-continuous functions in [28].

Definition 5.1. An $m$-space $\left(X, m_{X}\right)$ is said to be $m$ - $T_{2}$ if for each distinct points $x, y \in X$, there exist $U, V \in m_{X}$ containing $x$ and $y$, respectively, such that $U \cap V=\emptyset$.

Definition 5.2. A bi- $m$-space $\left(X, m_{X}^{1}, m_{X}^{2}\right)$ is said to be $(i, j) m I T-T_{2}$ if an $m$ space $(X,(i, j) \operatorname{mIT}(X))$ is $m-T_{2}$ [28].

Hence, a bi- $m$-space $\left(X, m_{X}^{1}, m_{X}^{2}\right)$ is $(i, j) m I T-T_{2}$ if for each distinct points $x, y \in X$, there exist $U, V \in(i, j) \operatorname{mIT}(X)$ containing $x$ and $y$, respectively, such that $U \cap V=\emptyset$.

Remark 5.1. Let $\left(X, m_{X}^{1}, m_{X}^{2}\right)$ be a bi- $m$-space. If $(i, j) \operatorname{mIT}(X)=(i, j) m \mathrm{PO}(X)$, then by Definition 5.2 we obtain the definition of $(i, j)$ - $m$-pre- $T_{2}$-spaces in [7].

Lemma 5.1 (Popa and Noiri [28]). If $f:\left(X, m_{X}\right) \rightarrow(Y, \sigma)$ is an $m$-continuous injection and $(Y, \sigma)$ is a Hausdorff space, then $\left(X, m_{X}\right)$ is $m-T_{2}$.
Theorem 5.1. Iff $:\left(X, m_{X}^{1}, m_{X}^{2}\right) \rightarrow\left(Y, \sigma_{1}, \sigma_{2}\right)$ is an $(i, j) m I T$-continuous injection and $\left(Y, \sigma_{i}\right)$ is a Hausdorff space, then $\left(X, m_{X}^{1}, m_{X}^{2}\right)$ is $(i, j) m I T-T_{2}$.

Proof. The proof follows from Definition 5.2 and Lemma 5.1.
Remark 5.2. Let $\left(X, m_{X}^{1}, m_{X}^{2}\right)$ be a bi- $m$-space. If $(i, j) \operatorname{mIT}(X)=(i, j) m \mathrm{PO}(X)$, then by Theorem 5.1 we obtain Theorem 4.9 in [7].

Definition 5.3. An $m$-space ( $X, m_{X}$ ) is said to be $m$-compact [28] if every cover of $X$ by $m_{X}$-open sets of $X$ has a finite subcover.

A subset $K$ of an $m$-space ( $X, m_{X}$ ) is said to be $m$-compact [28] if every cover of $K$ by $m_{X}$-open sets of $X$ has a finite subcover.
Definition 5.4. A bi- $m$-space $\left(X, m_{X}^{1}, m_{X}^{2}\right)$ is said to be $(i, j) m I T$-compact if the $m$-space $(X,(i, j) \operatorname{mIT}(X))$ is $m$-compact.

A subset $K$ of a bi- $m$-space $\left(X, m_{X}^{1}, m_{X}^{2}\right)$ is said to be $(i, j) m I T$-compact if every cover of $K$ by $(i, j) m I T$-open sets of $X$ has a finite subcover.
Lemma 5.2 (Popa and Noiri [28]). Let $f:\left(X, m_{X}\right) \rightarrow(Y, \sigma)$ be an $m$-continuous function. If $K$ is an $m$-compact set of $X$, then $f(K)$ is compact.
Theorem 5.2. If $f:\left(X, m_{X}^{1}, m_{X}^{2}\right) \rightarrow\left(Y, \sigma_{1}, \sigma_{2}\right)$ is an $(i, j) m I T$-continuous function and $K$ is an $(i, j) m I T$-compact set of $X$, then $f(K)$ is $\sigma_{i}$-compact.

Proof. The proof follows from Definition 5.4 and Lemma 5.2.
If $(i, j) \mathrm{mIT}(X)=(i, j) m \mathrm{PO}(X)$, then by Theorem 5.2 we obtain the following corollary.

Corollary 5.1. If $f:\left(X, m_{X}^{1}, m_{X}^{2}\right) \rightarrow\left(Y, \sigma_{1}, \sigma_{2}\right)$ is an (i,j)-m-precontinuous function and $K$ is an $(i, j)$-m-precompact set of $X$, then $f(K)$ is $\sigma_{i}$-compact.
Definition 5.5. A function $f:\left(X, m_{X}\right) \rightarrow(Y, \sigma)$ is said to have a strongly $m$ closed graph (resp. m-closed graph) [28] if for each $(x, y) \in(X \times Y)-\mathrm{G}(f)$, there exist $U \in m_{X}$ containing $x$ and an open set $V$ of $Y$ containing $y$ such that $[U \times \mathrm{Cl}(V)] \cap \mathrm{G}(f)=\emptyset$ (resp. $[U \times V] \cap \mathrm{G}(f)=\emptyset)$.
Defintition 5.6. A function $f:\left(X, m_{X}^{1}, m_{X}^{2}\right) \rightarrow\left(Y, \sigma_{1}, \sigma_{2}\right)$ is said to have a strongly ( $i, j$ )mIT-closed graph (resp. (i,j)mIT-closed graph) if a function
$f:(X,(i, j) \operatorname{mIT}(X)) \rightarrow\left(Y, \sigma_{i}\right)$ has a strongly m-closed graph (resp. m-closed graph $)$.

Hence, a function $f:\left(X, m_{X}^{1}, m_{X}^{2}\right) \rightarrow\left(Y, \sigma_{1}, \sigma_{2}\right)$ has a strongly $(i, j) m I T$ closed graph (resp. ( $i, j$ )mIT-closed graph) if for each $(x, y) \in(X \times Y)-\mathrm{G}(f)$, there exist $U \in(i, j) \operatorname{mIT}(X)$ containing $x$ and a $\sigma_{i}$-open set $V$ of $Y$ containing $y$ such that $\left[U \times \sigma_{i}-\mathrm{Cl}(V)\right] \cap \mathrm{G}(f)=\emptyset$ (resp. $\left.[U \times V] \cap \mathrm{G}(f)=\emptyset\right)$.
Remark 5.3. Let $\left(X, m_{X}^{1}, m_{X}^{2}\right)$ be a bi- $m$-space. If $(i, j) \operatorname{mIT}(X)=(i, j) m \mathrm{PO}(X)$, then by Definition 5.6 we obtain Definition 4.5 of [7].
Lemma 5.3. (Popa and Noiri [28]) If $f:\left(X, m_{X}\right) \rightarrow(Y, \sigma)$ is an $m$-continuous function and $(Y, \sigma)$ is a Hausdorff space, then $f$ has a strongly $m$-closed graph.

Theorem 5.3. If $f:\left(X, m_{X}^{1}, m_{X}^{2}\right) \rightarrow\left(Y, \sigma_{1}, \sigma_{2}\right)$ is an (i,j)mIT-continuous function and $\left(Y, \sigma_{i}\right)$ is a Hausdorff space, then $G(f)$ is strongly $(i, j) m I T-c l o s e d$.

Proof. The proof follows from Definition 5.6 and Lemma 5.3.
If $(i, j) \mathrm{mIT}(X)=(i, j) m \mathrm{PO}(X)$, then by Theorem 5.3 we obtain the following corollary which is an improvement of Theorem 4.7 of [7].
Corollary 5.2. If $f:\left(X, m_{X}^{1}, m_{X}^{2}\right) \rightarrow\left(Y, \sigma_{1}, \sigma_{2}\right)$ is an (i,j)-m-precontinuous function and $\left(Y, \sigma_{i}\right)$ is a Hausdorff space, then $G(f)$ is strongly $(i, j)$-m-preclosed.

Lemma 5.4 (Popa and Noiri [28]). If $f:\left(X, m_{X}\right) \rightarrow(Y, \sigma)$ is a surjective function with a strongly m-closed graph, then $(Y, \sigma)$ is Hausdorff.
Theorem 5.4. If $f:\left(X, m_{X}^{1}, m_{X}^{2}\right) \rightarrow\left(Y, \sigma_{1}, \sigma_{2}\right)$ is a surjective function with a strongly $(i, j) m I T$-closed graph, then $\left(Y, \sigma_{i}\right)$ is Hausdorff.

Proof. The proof follows from Definition 5.6 and Lemma 5.4.
If $(i, j) \mathrm{mIT}(X)=(i, j) m \mathrm{PO}(X)$, then by Theorem 5.4 we obtain the following corollary.
Corollary 5.3. If $f:\left(X, m_{X}^{1}, m_{X}^{2}\right) \rightarrow\left(Y, \sigma_{1}, \sigma_{2}\right)$ is a surjective function with a strongly $(i, j)$-m-preclosed graph, then $\left(Y, \sigma_{i}\right)$ is Hausdorff.

Lemma 5.5 (Popa and Noiri [28]). Let ( $X, m_{X}$ ) be an $m$-space and $m_{X}$ have property $\mathcal{B}$. If $f:\left(X, m_{X}\right) \rightarrow(Y, \sigma)$ is an injective $m$-continuous function with an $m$-closed graph, then $X$ is $m-T_{2}$.
Theorem 5.5. If $f:\left(X, m_{X}^{1}, m_{X}^{2}\right) \rightarrow\left(Y, \sigma_{1}, \sigma_{2}\right)$ is an injective ( $\left.i, j\right) m I T$ continuous function with an $(i, j) m I T$-closed graph, then $X$ is $(i, j) m I T-T_{2}$.

Proof. Since $(i, j) \operatorname{mIT}(X)$ has property $\mathcal{B}$, the proof follows from Definition 5.6 and Lemma 5.5 .

Remark 5.4. Let $\left(X, m_{X}^{1}, m_{X}^{2}\right)$ be a bi- $m$-space. If $(i, j) \operatorname{mIT}(X)=(i, j) m \mathrm{PO}(X)$, then by Theorem 5.5 we obtain Theorem 4.10 of [7].
Definition 5.7. An $m$-space ( $X, m_{X}$ ) is said to be $m$-connected [28] if $X$ cannot be written as the union of two nonempty disjoint sets of $m_{X}$.
Definition 5.8. A bi- $m$-space $\left(X, m_{X}^{1}, m_{X}^{2}\right)$ is said to be $(i, j) m I T$-connected if an $m$-space $(X,(i, j) \operatorname{mIT}(X))$ is $m$-connected.

Hence, a bi- $m$-space $\left(X, m_{X}^{1}, m_{X}^{2}\right)$ is $(i, j) m I T$-connected if $X$ cannot be written as the union of two nonempty disjoint sets of $(i, j) \operatorname{mIT}(X)$.

Lemma 5.6. Let $f:\left(X, m_{X}\right) \rightarrow(Y, \sigma)$ be a function, where $m_{X}$ has property $\mathcal{B}$. If $f$ is an $m$-continuous surjection and $\left(X, m_{X}\right)$ is $m$-connected, then $(Y, \sigma)$ is connected.

Theorem 5.6. Iff: $\left(X, m_{X}^{1}, m_{X}^{2}\right) \rightarrow\left(Y, \sigma_{1}, \sigma_{2}\right)$ is an $(i, j) m I T$-continuous surjection and $\left(X, m_{X}^{1}, m_{X}^{2}\right)$ is $(i, j) m I T$-connected, then $\left(Y, \sigma_{i}\right)$ is connected.

Proof. The proof follows from Definition 5.8, Lemma 5.6 and the fact that $(i, j) \operatorname{mIT}(X)$ has property $\mathcal{B}$.

## References

[1] M. E. Abd El-Monsef, S. N. El-Deeb and R. A. Mahmoud, $\beta$-open sets and $\beta$-continuous mappings, Bull. Fac. Sci. Assiut Univ., 12 (1983), 77-90.
[2] M. E. Abd El-Monsef, R. A. Mahmoud and E. R. Lashin, $\beta$-closure and $\beta$-interior, J. Fac. Ed. Ain Shams Univ., 10 (1986), 235-245.
[3] D. Andrijević, Semi-preopen sets, Mat. Vesnik, 38 (1986), 24-32.
[4] C. Boonpok, Almost and weakly $M$-continuous functions in $m$-spaces, Far East J. Math. Sci., 43 (2010), 41-58.
[5] C. Boonpok, Biminimal structure spaces, Int. Math. Forum, 5 (2010), no. 15, 703-707.
[6] C. Carpintero, R. Rosas and M. Salas, $M$-minimal structures and separation axioms, Int. J. Pure Appl. Math., 34 (4) (2007), 439-448.
[7] C. Carpintero, N. Rajesh and R. Rosas, m-preopen sets in biminimal spaces, Demonstratio Math., 45 (2012), 953-961.
[8] S. G. Crossley and S. K. Hildebrand, Semi-closure, Texas J. Sci., 22 (1971), 99-112.
[9] S. N. El-Deeb, I. A. Hasanein, A. S. Mashhour and T. Noiri, On p-regular spaces, Bull. Math. Soc. Sci. Math. R. S. Roumanie, 27 (75) (1983), 311-315.
[10] N. Levine, Semi-open sets and semi-continuity in topological spaces, Amer. Math. Monthly, 70 (1963), 36-41.
[11] H. Maki, C. K. Rao and A. Nagoor Gani, On generalizaing semi-open and preopen sets, Pure Appl. Math. Sci., 49 (1999), 17-29.
[12] A. S. Mashhour, M. E. Abd El-Monsef and S. N. El-Deep, On precontinuous and weak precontinuous mappings, Proc. Math. Phys. Soc. Egypt, 53 (1982), 47-53.
[13] A. S. Mashhour, I. A. Hasanein and S. N. El-Deeb, $\alpha$-continuous and $\alpha$-open mappings, Acta Math. Hungar., 41 (1983), 213-218.
[14] W.K. Min, $m$-semiopen sets and $M$-semicontinuous functions on spaces with minimal structures, Honam Math. J., 31 (2009), 239-245.
[15] W. K. Min, $\alpha m$-open sets and $\alpha M$-continuous functions, Commun. Korean Math. Soc., 25 (2010), 251-256.
[16] W. K. Min, On minimal semicontinuous functions, Comm. Korean Math. Soc., 27(2) (2012), 341-345.
[17] W. K. Min and Y. K. Kim, m-preopen sets and $M$-precontinuity on spaces with minimal structures, Adv. Fuzzy Sets Systems, 4 (2009), 237-245.
[18] W. K. Min and Y. K. Kim, On minimal precontinuous functions, J. Chungcheong Math. Soc., 22 (2009), 667-673.
[19] O. Njåstad, On some classes of nearly open sets, Pacific J. Math., 15 (1965), 961-970.
[20] T. Noiri, The further unified theory for modifications of $g$-closed sets, Rend. Circ. Mat. Palermo, 57 (2008), 411-421.
[21] T. Noiri and V. Popa, A new viewpoint in the study of irresoluteness forms in bitopological spaces, J. Math. Anal. Approx. Theory, 1 (2006), 1-9.
[22] T. Noiri and V. Popa, A unified theory of weak continuity for multifunctions, Stud. Cerc. St. Ser. Mat. Univ. Bacău, 16 (2006), 167-200.
[23] T. Noiri and V. Popa, A generalization of nearly continuous multifunctions, Annal. Univ. Sci. Budapest., 50 (2007), 59-74.
[24] T. Noiri and V. Popa, A new viewpoint in the study of continuity forms in bitopological spaces, Kochi J. Math., 2 (2007), 95-106.
[25] T. Noiri and V. Popa, A unified theory of upper and lower almost nearly continuous multifunctions, Math. Balkanica, 23 (2009), 51-72.
[26] T. Noiri and V. Popa, On iterate minimal structures and iterate $m$-continuous functions, Annal. Univ. Sci. Budapest., 55 (2012), 953-961.
[27] V. Popa and T. Noiri, On M-continuous functions, Anal. Univ. "Dunarea de Jos" Galati, Ser. Mat. Fiz. Mec. Teor., Fasc. II, 18 (23) (2000), 31-41.
[28] V. Popa and T. Noiri, On the definitions of some generalized forms of continuity under minimal conditions, Mem. Fac. Sci. Kochi Univ. Ser. Math., 22 (2001), 9-18.
[29] V. Popa and T. Noiri, On the point of continuity and discontinuity, Bull. U. P. G. Ploieşti, Ser. Mat. Fiz. Inform., 53 (2001), 95-100.
[30] V. Popa and T. Noiri, A unified theory of weak continuity for functions, Rend. Circ. Mat. Palermo (2), 51 (2002), 439-464.
[31] V. Popa and T. Noiri, On almost $m$-continuous functions, Math. Notae, 40 (1999-2002), 75-94.
[32] V. Popa and T. Noiri, On weakly m-continuous functions, Mathematica (Cluj), 45 (68) (2003), 53-67.
[33] E. Rosas, N. Rajesh and C. Carpintero, Some new types of open and closed sets in minimal structures, I, II, Int. Math. Forum, 4 (2009), 2169-2184, 2185-2198.
[34] L. Vasquez, M. S. Brown and E. Rosas, Functions almost contra-supercontinuity in $m$-spaces, Bol. Soc. Paran. Mat., 29 (2) (2011), 15-36.

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# MIXED CONNECTEDNESS IN GTS VIA HEREDITARY CLASSES 

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(Received July 26, 2013)


#### Abstract

We introduce six forms of connected sets on a generalized topological space with a hereditary class and investigate their relations and also their unified properties.


## 1. Introduction

The notion of ideal topological spaces was studied by Kuratowski [9] and Vaidyanathswamy [14]. The notion was further investigated by Janković and Hamlett [7]. Recently, the notion of $*$-connected ideal topological spaces has been introduced and studied in $[6,13,10]$.

Császár [5] introduced the notion of a generalized topological space with hereditary class. This is a generalization of an ideal topological space. In this paper, we introduce six forms of connected sets on a generalized topological space with a hereditary class and investigate their relations and also their unified properties.

## 2. Preliminaries

Let $X$ be a nonempty set and $P(X)$ the power set of $X$. A subset $\mu$ of $P(X)$ is called a generalized topology (GT) $[1,2,3]$ if $\emptyset \in \mu$ and the arbitrary union of members of $\mu$ is in $\mu$. A generalized topology $\mu$ is called a quasi-topology [4] on $X$ if $U, V \in \mu$ implies $U \cap V \in \mu$. A nonempty subset $\mathcal{H}$ of $P(X)$ is called a hereditary class [5] of $X$ if $A \subset B, B \in \mathcal{H}$ implies $A \in \mathcal{H}$. For each subset $A$ of $X$, a subset $A^{*}(\mathcal{H})$ (briefly $A^{*}$ ) of $X$ is defined in [5] as follows: $A^{*}(\mathcal{H})=\{x \in$ $\in X: U \cap A \notin \mathcal{H}$ for every $U \in \mu$ containing $x\}$. If $c_{\mu^{*}}(A)=A \cup A^{*}$ for each
subset $A$ of $X$, then $\mu^{*}=\left\{A \subset X: c_{\mu^{*}}(X-A)=X-A\right\}$ is a generalized topology on $X$ finer than $\mu$ [5].

Let us recall some properties established in [5].
Lemma 2.1. [5] For a subset $A$ of $X$, the following properties hold:
(1) $A \subset B$ implies $A^{*} \subset B^{*}$,
(2) $A^{*}$ is a $\mu$-closed, that is, $X-A^{*} \in \mu$,
(3) $A^{*} \subset c_{\mu}(A)$, where $c_{\mu}(A)=\bigcap\{F \subset X: A \subset F, X-F \in \mu\}$.

Lemma 2.2. [5] The family $\mathcal{B}=\{M-H: M \in \mu, H \in \mathcal{H}\}$ is a base for $\mu^{*}$.
In the sequel, a generalized topological space $(X, \mu)$ with a hereditary class $\mathcal{H}$ is dented by $(X, \mu, \mathcal{H})$ and is called a GTSH. Let $(X, \mu, \mathcal{H})$ be a GTSH. The closure of a subset $A$ of $X$ in $\left(X, \mu^{*}\right)$ is denoted by $c_{\mu^{*}}(A)$.

## 3. Mixed separated sets

Definition 3.1. Let $(X, \mu, \mathcal{H})$ be a generalized topological space with a hereditary class $\mathcal{H}$. Nonempty disjoint subsets $A, B$ of $(X, \mu, \mathcal{H})$ are said to be
(1) $c_{\mu^{-}}-c_{\mu^{*}}$-separated if $c_{\mu}(A) \cap c_{\mu^{*}}(B)=\emptyset=c_{\mu^{*}}(A) \cap c_{\mu}(A)$,
(2) $c_{\mu}$-I-separated if $c_{\mu}(A) \cap B=\emptyset=A \cap c_{\mu}(B)$,
(3) $c_{\mu^{*}}-I$-separated if $c_{\mu^{*}}(A) \cap B=\emptyset=A \cap c_{\mu^{*}}(B)$,

(5) $c_{\mu^{*}-*-s e p a r a t e d ~ i f ~}^{\mu_{\mu^{*}}}(A) \cap B^{*}=\emptyset=A^{*} \cap c_{\mu^{*}}(B)$,
(6) $*$-I-separated if $A^{*} \cap B=\emptyset=A \cap B^{*}$.

Theorem 3.2. For a subset of $(X, \mu, \mathcal{H})$, the following implications hold:


Proof. Since $\mu \subset \mu^{*}, A \cup A^{*}=c_{\mu^{*}}(A) \subset c_{\mu}(A)$ for every subset $A$ of $X$.
For converse implications we shall discuss in the following example. It is also shown in (1) and (2) that " $c_{\mu}-I$-separated" and " $c_{\mu^{*-*}}$-separated" are independent.

## Examples.

(1) Let $X=\{a, b, c\}, \mu=\{\emptyset,\{a\},\{c\},\{a, c\}, X\}$ and $\mathcal{H}=\{\emptyset\}$. Then $\mu-$ closed sets are: $X,\{b, c\},\{a, b\}, \emptyset$ and $\{b\}$. Now $c_{\mu}(\{a\}) \cap(\{c\})=(\{a, b\}) \cap$ $\cap(\{c\})=\emptyset=(\{a\}) \cap(\{b, c\})=(\{a\}) \cap c_{\mu}(\{c\})$ and hence $\{a\}$ and $\{c\}$ are $c_{\mu}-I$-separated. Further $(\{a\})^{*}=\{a, c\},(\{b\})^{*}=\{b, c\}$ and $(\{a\})^{*} \cap$ $\cap(\{c\})=\emptyset=(\{a\}) \cap(\{c\})^{*}$. But $c_{\mu}(\{a\}) \cap c_{\mu^{*}}(\{c\})=(\{a, b\}) \cap(\{b, c\}) \neq$ $\neq \emptyset$ and $(\{a\})^{*} \cap c_{\mu^{*}}(\{c\})=\{a, c\} \cap\{b, c\}=\{c\} \neq \emptyset$. Therefore, $\{a\}$
 $*-I$ separated.
(2) Let $X=\{a, b, c\}, \mu=\{\emptyset,\{a, b\}, X\}, \mathcal{H}=\{\emptyset,\{a\},\{b\},\{a, b\}\}$. Now $(\{c\})^{*}=\{c\} ; c_{\mu}(\{a, b\})=X$. Therefore, $\{c\}$ and $\{a, b\}$ are neither $c_{\mu^{-*-}}$ separated nor $c_{\mu}-I$-separated. Again $(\{c\})^{*} \cap c_{\mu^{*}}(\{a, b\})=\{c\} \cap\{a, b\}=\emptyset=$ $c_{\mu^{*}}(\{c\}) \cap(\{a, b\})^{*}=\{c\} \cap \emptyset=\emptyset$ (since $\left.(\{a, b\})^{*}=\emptyset\right)$. Here $\{c\}$ and $\{a, b\}$ are $c_{\mu^{*}-* \text {-separated. }}$
(3) Let $X=\{a, b, c, d\}, \mu=\{\emptyset, X,\{a\},\{a, b\},\{a, c, d\}\}$ and $\mathcal{H}=$ $\{\emptyset,\{a\},\{b\},\{a, b\}\}$. Now $(\{a\})^{*}=\emptyset$, and $c_{\mu^{*}}(\{a\}) \cap(\{b, c, d\})=\{a\} \cap$ $\cap\{b, c, d\}=\emptyset=(\{a\}) \cap c_{\mu^{*}}(\{b, c, d\})$. Then $\{a\}$ and $\{b, c, d\}$ are $c_{\mu^{*}-I-}$ separated but not $c_{\mu}-I$-separated.
(4) Let $X=\{a, b\}, \mu=\{\emptyset, X,\{a\}\}, \mathcal{H}=\{\emptyset,\{a\},\{b\}\}$. Now $(\{a\})^{*}=$ $\emptyset,(\{b\})^{*}=\emptyset$. Therefore $\{a\}$ and $\{b\}$ are $c_{\mu}-*$-separated. Again $\{a\}$ and $\{b\}$ are not $c_{\mu}-I$-separated and hence not $c_{\mu^{-}}-c_{\mu^{*}}$-separated, because $c_{\mu}(\{a\})=X$.

Theorem 3.3. Let $A, B$ be nonempty disjoint subsets of $(X, \mu, \mathcal{H})$. Then $A$ and


Proof. Let $A$ and $B$ be $*-I$-separated. Then $A^{*} \cap B=\emptyset=A \cap B^{*}$. Hence $c_{\mu^{*}}(A) \cap$ $\cap B=\left(A^{*} \cup A\right) \cap B=\left(A^{*} \cap B\right) \cup(A \cap B)=\emptyset$. Similarly, $A \cap c_{\mu^{*}}(B)=\emptyset$. Therefore, $A$ and $B$ are $c_{\mu^{*}}-I$-separated. The converse is obvious by Theorem 3.2.

Theorem 3.4. Let $A$ and $B$ be $c_{\mu}-I$-separated (resp. $c_{\mu}-c_{\mu^{*}}$-separated, $c_{\mu}-*-$
 If $C \subset A$ and $D \subset B$, then $C$ and $D$ are also $c_{\mu}-I$-separated (resp. $c_{\mu}-c_{\mu^{*}}$


Proof. The proof is obvious from the fact that $c_{\mu}, c_{\mu^{*}}$ and ()* are enlarging operators [3].

Theorem 3.5. Let $(X, \mu, \mathcal{H})$ be a GTSH. If $A$ and $B$ are nonempty disjoint $\mu$ open sets, then $A$ and $B$ are $c_{\mu}-I$-separated and hence they are $*-I$-separated.

Proof. The proof is obvious from Theorem 3.2 and the following facts:
$c_{\mu}(A) \cap B=\emptyset$ since $c_{\mu}(A) \subset c_{\mu}(X-B)$ and similarly $c_{\mu}(B) \cap A=\emptyset$ since $c_{\mu}(B) \subset c_{\mu}(X-A)$.
Theorem 3.6. Let $(X, \mu, \mathcal{H})$ be a GTSH. If $A$ and $B$ are nonempty disjoint $\mu^{*}$ -


Proof. The proof is obvious from Theorem 3.2 and the following facts:
$c_{\mu^{*}}(A) \cap B=\emptyset$ since $c_{\mu^{*}}(A) \subset c_{\mu^{*}}(X-B)$ and similarly $c_{\mu^{*}}(B) \cap A=\emptyset$ since $c_{\mu^{*}}(B) \subset c_{\mu^{*}}(X-A)$.
Lemma 3.7. Let $(X, \mu, \mathcal{H})$ be a GTSH with a quasi-topology $\mu$. If $U \in \mu$ and $V \in \mu^{*}$, then $U \cap V \in \mu^{*}$.

Proof. Let $x \in U \cap V$. Then $x \in U$ and $x \in V \in \mu^{*}$. By Lemma 2.2, there exist $M \in \mu$ and $H \in \mathcal{H}$ such that $x \in M-H \subset V$. Then, we have $x \in U \cap$ $\cap(M-H)=U \cap M \cap(X-H)=(U \cap M) \cap(U \cap(X-H))=(U \cap M) \cap$ $\cap(U-H)=(U \cap M) \cap(U-(U \cap H))=(U \cap M)-(U \cap H)$, where $U \cap M \in \mu$ and $U \cap H \in \mathcal{H}$. Since $x \in(U \cap M)-(U \cap H) \subset U \cap V, U \cap V \in \mu^{*}$.

Theorem 3.8. Let $(X, \mu, \mathcal{H})$ be a GTSH with a qusi-topology $\mu$. If $A$ and $B$ are
 $A, B \in \mu^{*}$.

Proof. We prove this theorem for $c_{\mu^{*}-}-$-separated sets.
 $A \cup B \in \mu$ and $c_{\mu^{*}}(A)$ is $\mu^{*}$-closed in $X$, by Lemma $3.7 B$ is $\mu^{*}$-open. By the similar way, we obtain that $A$ is $\mu^{*}$-open.

For the next theorem, we define the following:
Definition 3.9. A subset $A$ of a $\operatorname{GTSH}(X, \mu, \mathcal{H})$ is called $\mu^{*}$-dense-in-itself if $A=A^{*}$.

Theorem 3.10. Let $(X, \mu, \mathcal{H})$ be a GTSH with a quasi-topology $\mu$. If $A$ and $B$
 itself and $A \cup B \in \mu$, then $A$ and $B$ are $\mu^{*}$-open.

Proof. We shall prove the theorem for the case of $c_{\mu^{*}-* \text {-separated sets. }}$
Since $A$ and $B$ are $c_{\mu^{*-*}}$-separated, then $A=(A \cup B) \cap\left(X-B^{*}\right)$. Since $A \cup B \in \mu$ and $X-B^{*}$ is $\mu$-open in $X$, then $A$ is $\mu$-open and hence $\mu^{*}$-open. Again from $B=(A \cup B) \cap\left(X-c_{\mu^{*}}(A)\right)$, by Lemma 3.7 B is also $\mu^{*}$-open.

## 4. Mixed connectedness

Definition 4.1. A subset $A$ of a $\operatorname{GTSH}(X, \mu, \mathcal{H})$ is said to be $P$ - $Q$-connected if $A$ is not the union of two $P$ - $Q$-separated sets in $(X, \mu, \mathcal{H})$, where $P$ and $Q$ denote the operations in Definition 3.1.

For example, in case $P=c_{\mu^{*}}$ and $Q=*,(X, \mu, \mathcal{H})$ is said to be $c_{\mu^{*-*}}$ connected if $X$ cannot be written as the disjoint union of two nonempty $c_{\mu^{*-*-}}$ separated sets.

Theorem 4.2. For a subset of $(X, \mu, \mathcal{H})$, the following implications hold:


Proof. This is an immediate consequence of Theorem 3.2 and Theorem 3.3.
From Example (2), we have every $c_{\mu^{-}-\text {-connected space need not be }}$
 connected space need not be a $c_{\mu^{*}-I \text {-connected space in general. Again from }}$ Example (4), every $c_{\mu^{-}} c_{\mu^{*}}$-connected space need not be a $c_{\mu^{-}-\text {-connected space. }}$ Further from Example (1), every $c_{\mu}-c_{\mu^{*}-c o n n e c t e d ~ s p a c e ~ n e e d ~ n o t ~ b e ~ a ~}^{c_{\mu}-I-}$ connected space. By Examples (1) and (2), $c_{\mu}-I$-connectedness and $c_{\mu^{*-*-}}$ connectedness are independent.

Theorem 4.3. Let $(X, \mu, \mathcal{H})$ be a GTSH. If $A$ is a $c_{\mu}-I$-connected (resp. $c_{\mu}-c_{\mu^{*-}}$
 subset of $X$ and $H, G$ are $c_{\mu}$-I-separated (resp. $c_{\mu^{-}}-c_{\mu^{*}}$-separated, $c_{\mu^{-*}}$-separated,
 then either $A \subset H$ or $A \subset G$.

Proof. We shall prove this theorem only for $c_{\mu^{*-*-c o n n e c t e d n e s s . ~}}$
Let $A$ be a $c_{\mu^{*-*}}$-connected set. Let $A \subset H \cup G$. Since $A=(A \cap H) \cup(A \cap G)$, then $c_{\mu^{*}}(A \cap G) \cap(A \cap H)^{*} \subset c_{\mu^{*}}(G) \cap H^{*}=\emptyset$. By the similar way, we have $(A \cap G)^{*} \cap c_{\mu^{*}}(A \cap H)=\emptyset$. Suppose $A \cap H$ and $A \cap G$ are nonempty. Then $A$ is not $c_{\mu^{*-} *-c o n n e c t e d . ~ T h i s ~ i s ~ a ~ c o n t r a d i c t i o n . ~ T h u s, ~ e i t h e r ~} A \cap H=\emptyset$ or $A \cap G=\emptyset$. This implies that $A \subset H$ or $A \subset G$.

Theorem 4.4. If $A$ and $B$ are $c_{\mu}$-I-connected (resp. $c_{\mu}-c_{\mu^{*}}$-connected, $c_{\mu}-*-$ connected, $c_{\mu^{*}-*-c o n n e c t e d, ~}$-I-connected, $c_{\left.\mu^{*}-I-c o n n e c t e d\right) ~ s e t s ~ o f ~ a ~ G T S H ~}^{\text {-ch }}$ $(X, \mu, \mathcal{H})$ such that none of them is $c_{\mu}-I$-separated (resp. $c_{\mu}-c_{\mu}$-separated,
$c_{\mu^{-*} \text {-separated, }} c_{\mu^{*-*}}$-separated, $*-I$-separated, $c_{\mu^{*}-I \text {-separated), then } A \cup B \text { is }}$ $c_{\mu}-I$-connected (resp. $c_{\mu}-c_{\mu^{*}}$-connected, $c_{\mu}-*$-connected, $c_{\mu^{*-}-\text {-connected, } *-I-}$ connected, $c_{\mu^{*}}-I$-connected).

Proof. Let $A$ and $B$ be $c_{\mu}-I$-connected in $X$. Suppose $A \cup B$ is not $c_{\mu}-I$-connected. Then, there exist two nonempty disjoint $c_{\mu}-I$-separated sets $G$ and $H$ such that $A \cup B=G \cup H$. Since $A$ and $B$ are $c_{\mu}-I$-connected, by Theorem 4.3, either $A \subset$ $\subset G$ and $B \subset H$ or $B \subset G$ and $A \subset H$. Now if $A \subset G$ and $B \subset H$, then $A \cap H=B \cap G=\emptyset$. Therefore, $(A \cup B) \cap G=(A \cap G) \cup(B \cap G)=(A \cap$ $\cap G) \cup \emptyset=A \cap G=A$. Also, $(A \cup B) \cap H=(A \cap H) \cup(B \cap H)=B \cap H=B$. Now, $B \cap c_{\mu}(A)=((A \cup B) \cap H) \cap c_{\mu}((A \cup B) \cap G)=H \cap c_{\mu}(G)=\emptyset$ and $c_{\mu}(B) \cap A=c_{\mu}((A \cup B) \cap H) \cap((A \cup B) \cap G)=c_{\mu}(H) \cap G=\emptyset$. Thus, $A$ and $B$ are $c_{\mu}-I$-separated, which is a contradiction. Hence, $A \cup B$ is $c_{\mu}-I$-connected.

The proof of other connectedness are obvious from the fact that $c_{\mu},()^{*}$ and $c_{\mu^{*}}$ are enlarging operator.
THEOREM 4.5. If $\left\{M_{i}: i \in I\right\}$ is a nonempty family of $c_{\mu}$-I-connected (resp.
 connected) sets of a GTSH $(X, \mu, \mathcal{H})$ with $\cap_{i \in I} M_{i} \neq \emptyset$, then $\cup_{i \in I} M_{i}$ is $c_{\mu}-I$-connected (resp. $c_{\mu}-c_{\mu^{*}}$-connected, $c_{\mu}-*$-connected, $c_{\mu^{*-}}$-connected, $*-I-$ connected, $c_{\mu^{*}}-I$-connected).

Proof. We shall prove this theorem only for $c_{\mu}-c_{\mu^{*}}$-connectedness.
Suppose $\cup_{i \in I} M_{i}$ is not $c_{\mu^{-}} c_{\mu^{*}}$-connected. Then we have $\cup_{i \in I} M_{i}=H \cup G$, where $H$ and $G$ are $c_{\mu}-c_{\mu^{*}}$-separated sets in $X$. Since $\cap_{i \in I} M_{i} \neq \emptyset$, we have a point $x \in \cap_{i \in I} M_{i}$. Since $x \in \cup_{i \in I} M_{i}$, either $x \in H$ or $x \in G$. Suppose that $x \in H$. Since $x \in M_{i}$ for each $i \in I$, then $M_{i}$ and $H$ intersect for each $i \in I$. By Theorem 4.3, $M_{i} \subset H$ or $M_{i} \subset G$. Since $H$ and $G$ are disjoint, $M_{i} \subset H$ for all $i \in I$ and hence $\cup_{i \in I} M_{i} \subset H$. This implies that $G$ is empty. This is a contradiction. Suppose that $x \in G$. By the similar way, we have that $H$ is empty. This is a contradiction. Thus, $\cup_{i \in I} M_{i}$ is $c_{\mu^{-}} c_{\mu^{*}}$-connected.
Theorem 4.6. Let $(X, \mu, \mathcal{H})$ be a GTSH, $\left\{A_{\alpha}: \alpha \in \triangle\right\}$ be a family of $c_{\mu}-I$-connected (resp. $c_{\mu}-c_{\mu^{*}}$-connected, $c_{\mu^{-*}}$-connected, $c_{\mu^{*-}-\text {-connected, } *-I-}$ connected, $c_{\left.\mu^{*}-I-c o n n e c t e d\right) ~ s e t s ~ a n d ~} A$ be a $c_{\mu}-I$-connected (resp. $c_{\mu}-c_{\mu^{*}-}$ connected, $c_{\mu^{-}}$-connected, $c_{\mu^{*-*}}$-connected, $*-I$-connected, $c_{\left.\mu^{*}-I-c o n n e c t e d\right) ~}^{\text {- }}$ set. If $A \cap A_{\alpha} \neq \emptyset$ for every $\alpha \in \triangle$, then $A \cup\left(\cup_{\left.\alpha \in \triangle A_{\alpha}\right)}\right)$ is $c_{\mu}-I$-connected (resp. $c_{\mu}-c_{\mu^{*}}$-connected, $c_{\mu^{-*}}$-connected, $c_{\mu^{*-*}}$-connected, $*-I$ connected, $c_{\mu^{*}}$ I-connected).

Proof. We shall prove the theorem for only $*-I$-connected sets. Since $A \cap A_{\alpha} \neq$ $\neq \emptyset$ for each $\alpha \in \triangle$, by Theorem 4.3, $A \cup A_{\alpha}$ is $*-I$-connected for each $\alpha \in \triangle$. Moreover, $A \cup\left(\cup A_{\alpha}\right)=\cup\left(A \cup A_{\alpha}\right)$ and $\cap\left(A \cup A_{\alpha}\right) \supset A \neq \emptyset$. Thus by Theorem 4.5, $A \cup\left(\cup A_{\alpha}\right)$ is $*-I$-connected.

THEOREM 4.7. If $A$ is a $c_{\mu}-c_{\mu^{*}}$-connected subset of $(X, \mu, \mathcal{H})$ and $A \subset B \subset$ $\subset c_{\mu^{*}}(A)$, then $B$ is also a $c_{\mu^{-}} c_{\mu^{*}}$-connected subset of $X$.

Proof. Suppose $B$ is not a $c_{\mu}-c_{\mu^{*}}$-connected subset of $(X, \mu, \mathcal{H})$ then there exist $c_{\mu^{-}} c_{\mu^{*}}$-separated sets $H$ and $G$ such that $B=H \cup G$. This implies that $H$ and $G$ are nonempty and $c_{\mu}(G) \cap c_{\mu^{*}}(H)=\emptyset=G \cap H$. By Theorem 4.3, we have that either $A \subset H$ or $A \subset G$. Suppose that $A \subset H$. Then $c_{\mu^{*}}(A) \subset c_{\mu^{*}}(H)$ and $G \cap c_{\mu^{*}}(A) \subset c_{\mu}(G) \cap c_{\mu^{*}}(H)=\emptyset$. This implies that $G \subset B \subset c_{\mu^{*}}(A)$ and $G=c_{\mu^{*}}(A) \cap G=\emptyset$. Thus $G$ is an empty set. Since $G$ is nonempty, this is a contradiction. Hence, $B$ is $c_{\mu^{-}} c_{\mu^{*}}$-connected.

Corollary 4.8. If $A$ is a $c_{\mu^{*}-I-c o n n e c t e d ~ s u b s e t ~ o f ~}(X, \mu, \mathcal{H})$ and $A \subset B \subset$ $\subset c_{\mu^{*}}(A)$, then $B$ is also a $c_{\mu^{*}-I-c o n n e c t e d ~ s u b s e t ~ o f ~} X$.
Theorem 4.9. If $A$ is a $c_{\mu}$-I-connected subset of $(X, \mu, \mathcal{H})$ and $A \subset B \subset c_{\mu}(A)$, then $B$ is also a $c_{\mu}-I$-connected subset of $X$.

Proof. The proof is similar with Theorem 4.4.
Theorem 4.10. If $A$ is a $c_{\mu^{-}}$-connected (resp. $c_{\mu^{*-*}}$-connected, $*-I$-connected) subset of $(X, \mu, \mathcal{H})$ and $A \subset B \subset A^{*}$, then $B$ is also a $c_{\mu^{-*}}$-connected (resp. $c_{\mu^{*-*}}$-connected, $*-I$-connected) subset of $X$.

Proof. We shall prove this theorem only for $c_{\mu^{-}-* \text {-connectedness. }}$
Suppose $B$ is not a $c_{\mu^{-*}}$-connected subset of $(X, \mu, \mathcal{H})$ then there exist $c_{\mu^{-*-}}$ separated sets $H$ and $G$ such that $B=H \cup G$. This implies that $H$ and $G$ are nonempty and $c_{\mu}(G) \cap H^{*}=\emptyset=G^{*} \cap c_{\mu}(H)=G \cap H$. By Theorem 4.3, we have that either $A \subset H$ or $A \subset G$. Suppose that $A \subset H$. Then $A^{*} \subset H^{*}$. This implies that $G \subset B \subset A^{*}$ and $c_{\mu}(G)=c_{\mu}\left(A^{*}\right) \cap c_{\mu}(G)=A^{*} \cap c_{\mu}(G) \subset$ $\subset H^{*} \cap c_{\mu}(G)=\emptyset$. Thus $G$ is an empty set. Since $G$ is nonempty, this is a contradiction. Hence, $B$ is $c_{\mu^{-}}$-connected.
Corollary 4.11. Let $(X, \mu, \mathcal{H})$ be a GTSH. For a subset $A$ of $X$, the following properties hold:
(1) If $A$ is $c_{\mu}$-I-connected, then $c_{\mu}(A)$ is $c_{\mu}-I$-connected.
(2) If $A$ is $c_{\mu^{*}-I-c o n n e c t e d, ~ t h e n ~} c_{\mu^{*}}(A)$ is $c_{\mu^{*}-I-c o n n e c t e d . ~}^{\text {- }}$
(3) If $A$ is $c_{\mu}-c_{\mu^{*}}$-connected, then $c_{\mu^{*}}(A)$ is $c_{\mu}-c_{\mu^{*}}$-connected.
(4) If $A$ is $c_{\mu^{-}}$-connected and $A \cap A^{*} \neq \emptyset$, then $c_{\mu^{*}}(A)$ is $c_{\mu^{-*}}$-connected.
(5) If $A$ is $c_{\mu^{*-}}$-connected and $A \cap A^{*} \neq \emptyset$, then $c_{\mu^{*}}(A)$ is $c_{\mu^{*-*}}$-connected.
(6) If $A$ is $*-I$-connected and $A \cap A^{*} \neq \emptyset$, then $c_{\mu^{*}}(A)$ is $*-I$-connected.

## 5. Mixed components

Definition 5.1. Let $(X, \mu, \mathcal{H})$ be a GTSH and $x \in X$. Then union of all $c_{\mu}-I$-connected (resp. $c_{\mu}-c_{\mu^{*}}$-connected, $c_{\mu}-*$-connected, $c_{\mu^{*}-* \text {-connected, } *-I-}$ connected, $c_{\mu^{*}}-I$-connected) subsets of $X$ containing $x$ is called the $c_{\mu}-I$ component (resp. $c_{\mu^{-}} c_{\mu^{*}}$-component, $c_{\mu^{-*}}$-component, $c_{\mu^{*}-* \text {-component, } *-I \text { - }}$ component, $c_{\left.\mu^{*}-I-c o m p o n e n t\right) ~ o f ~} X$ containing $x$.
Theorem 5.2. Each $c_{\mu}-I$-component (resp. $c_{\mu^{-}} c_{\mu^{*}}$-component, $c_{\mu}-*$-component,
 a maximal $c_{\mu}-I$-connected (resp. $c_{\mu}-c_{\mu^{*}}$-connected, $c_{\mu^{-*}}$-connected, $c_{\mu^{*-*}}$ connected, $*-I$-connected, $c_{\left.\mu^{*}-I-c o n n e c t e d\right) ~ s e t ~ o f ~} X$.
Theorem 5.3. The set of all distinct $c_{\mu}-I$-components (resp. $c_{\mu^{-}} c_{\mu^{*}}$-components, $c_{\mu}-*$-components, $c_{\mu^{*}-* \text {-components, }} *-I$-components, $c_{\mu^{*}-I \text {-components) }}$ forms a partition of $X$.
Theorem 5.4. Each $c_{\mu^{-}} c_{\mu^{*}}$-component (resp. $c_{\mu^{-*}}$-component, $c_{\mu^{*-}}$-component, $*-I$-component, $c_{\left.\mu^{*}-I-c o m p o n e n t\right) ~ o f ~ a ~ G T S H ~ i s ~} \mu^{*}$-closed in $X$.

Proof. The proof follows from Corollary 4.11.

## Conclusions

If $\mu$ and $\mathcal{H}$ in $\operatorname{GTSH}(X, \mu, \mathcal{H})$ are replaced by the topology $\tau$ and the ideal $I$ in an ideal topological space $(X, \mu, I)$, then we obtain the following:
(1) $c_{\mu}-I$-connectedness coincides with connectedness in $(X, \tau)$,
(2) $c_{\mu^{-}}-$-connectedness coincides with $*$-Cl-connectedness in [10],
(3) $c_{\mu^{*}-* \text {-connectedness coincides with } *-C l^{*} \text {-connectedness in [10], }}^{\text {, }}$
(4) $*-I$-connectedness coincides with $*_{*}$-connectedness in [10],
(5) $c_{\mu^{*}}-I$-connectedness coincides with connectedness of $\left(X, \tau^{*}(I)\right)$.

## References

[1] Á. CsÁszÁr, Generalized topology, generalized continuity, Acta Math. Hungar, 96 (4) (2002), 351-357.
[2] Á. CsÁszÁr, Generalized open sets in generalized topologies, Acta Math. Hungar., 106 (2005), 53-66.
[3] Á. CsÁszÁr, Futher remarks on the formula for $\gamma$-interior, Acta Math. Hungar., 113 (2006), 325-332.
[4] Á. CsÁszÁr, Remarks on quasi topologies, continuity, Acta Math. Hungar., 119 (2007), 29-36.
[5] Á. CsÁszár, Modification of generalized topologies via hereditary classes, Acta Math. Hungar., 115 (1-2) (2008), 197-200.
[6] E. Ekici and T. Noiri, Connectedness in ideal topological spaces, Novi Sad J. Math., 38 (2) (2008), 65-70.
[7] D. Janković and T. R. Hamlett, New topologies from old via ideals, Amer. Math. Monthly, 97 (1990), 295-310.
[8] Y. K. Kim and W. K. Min, On operations induced by hereditary classes on generalized topological spaces, Acta Math. Hungar., 137 (1-2) (2012), 130-138.
[9] K. Kuratowski, Topology, Vol I, Academic Press, New York, 1966.
[10] S. Modak and T. Noiri, Connectedness of ideal topological spaces, (submitted).
[11] V. Renukadevi and K. Karuppayi, On modifications of generalized topologies via hereditary, J. Adv. Res. Pure Math., 2 (2) (2010), 14-20.
[12] V. Renukadevi and P. Vimaladevi, Note on generalized topological spaces with hereditary calsses, Bol. Soc. Paran. Mat. (3s), 32 (1) (2014), 89-97.
[13] N. Sathiyasundari and V. Renukadevi, Note on $*$-connected ideal spaces, Novi Sad J. Math., 42 (1) (2012), 15-20.
[14] R. Vaidyanathaswamy, Set Topology, Chelsea Publishing Company, 1960.

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# ON WEAKLY-Q-SYMMETRIC MANIFOLDS 

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(Received August 6, 2013)


#### Abstract

The object of the present paper is to study weakly- $Q$-symmetric manifolds $(W Q S)_{n}$. At first some geometric properties of $(W Q S)_{n}(n>2)$ have been studied. Next we consider the decomposability of $(W Q S)_{n}$. Finally, we give two examples of the $(W Q S)_{4}$.


## 1. Introduction

As is well known, symmetric spaces play an important role in differential geometry. The study of Riemannian symmetric spaces was initiated in the late twenties by Cartan [4], who, in particular, obtained a classification of those spaces. Let $\left(M^{n}, g\right),(n=\operatorname{dim} M)$ be a Riemannian manifold, i.e., a manifold $M$ with the Riemannian metric $g$, and let $\nabla$ be the Levi-Civita connection of $\left(M^{n}, g\right)$. A Riemannian manifold is called locally symmetric [4] if $\nabla R=0$, where $R$ is the Riemannian curvature tensor of $\left(M^{n}, g\right)$. This condition of local symmetry is equivalent to the fact that at every point $P \in M$, the local geodesic symmetry $F(P)$ is an isometry [18]. The class of Riemannian symmetric manifolds is very natural generalization of the class of manifolds of constant curvature. During the last six decades the notion of locally symmetric manifolds have been weakened by many authors in several ways to different extent such as conformally symmetric manifolds by Chaki and Gupta [5], recurrent manifolds introduced by Walker [25], conformally recurrent manifolds by Adati and Miyazawa [1], pseudo symmetric manifolds by Chaki [6], weakly symmetric manifolds by Tamássy and Binh [23] etc.

A non-flat Riemannian manifold $\left(M^{n}, g\right)(n>2)$ is called weakly symmetric [23] if the curvature tensor $R$ of type $(0,4)$ satisfies the condition

$$
\begin{align*}
& \left(\nabla_{X} R\right)(Y, Z, U, V)=A(X) R(Y, Z, U, V)+B(Y) R(X, Z, U, V)+ \\
& +C(Z) R(Y, X, U, V)+D(U) R(Y, Z, X, V)+E(V) R(Y, Z, U, X) \tag{1.1}
\end{align*}
$$

where $R(Y, Z, U, V)=g(\mathcal{R}(Y, Z) U, V), \mathcal{R}$ is the curvature tensor of type $(1,3)$ and $A, B, C, D$ and $E$ are 1-forms respectively which are non-zero simultaneously. Such a manifold is denoted by $(W S)_{n}$. It was proved in [8] that the 1-forms must be related as $B=C$ and $D=E$.

That is, the weakly symmetric manifold is characterized by the condition

$$
\begin{align*}
& \left(\nabla_{X} R\right)(Y, Z, U, V)=A(X) R(Y, Z, U, V)+B(Y) R(X, Z, U, V)+ \\
& +B(Z) R(Y, X, U, V)+D(U) R(Y, Z, X, V)+D(V) R(Y, Z, U, X) \tag{1.2}
\end{align*}
$$

The 1-forms $A, B$ and $D$ are called the associated 1-forms. If in (1.2) the 1-form $A$ is replaced by $2 A ; B$ and $D$ are replaced by $A$, then the manifold $\left(M^{n}, g\right)$ reduces to a pseudo symmetric manifold in the sense of Chaki [6].

Again if $A=B=D=0$, the manifold defined by (1.2) reduces to a symmetric manifold in the sense of Cartan. The existence of a $(W S)_{n}$ was proved by Prvanović [21] and a concrete example is given by De and Bandyopadhyay ([8], [9]).

Weakly symmetric manifolds have been studied by several authors ([2], [3], [7], [10], [11], [13], [14], [15], [16], [19], [20]) and many others.

Let $\rho_{1}, \rho_{2}$ and $\rho_{3}$ are the basic vectors corresponding to the 1 -forms $A, B$ and $D$ respectively, that is

$$
\begin{equation*}
g\left(X, \rho_{1}\right)=A(X), g\left(X, \rho_{2}\right)=B(X) \text { and } g\left(X, \rho_{3}\right)=D(X) \quad \text { for all } X \tag{1.3}
\end{equation*}
$$

A. Gray [12] introduced the notion of cyclic parallel Ricci tensor and Codazzi type of Ricci tensor. A Riemannian manifold is said to satisfy cyclic parallel Ricci tensor if its Ricci tensor $S$ of type ( 0,2 ) is non-zero and satisfy the condition

$$
\begin{equation*}
\left(\nabla_{X} S\right)(Y, Z)+\left(\nabla_{Y} S\right)(Z, X)+\left(\nabla_{Z} S\right)(X, Y)=0 \tag{1.4}
\end{equation*}
$$

Again a Riemannian manifold is said to satisfy Codazzi type of Ricci tensor if its Ricci tensor $S$ of type $(0,2)$ is non-zero and satisfy the following condition

$$
\begin{equation*}
\left(\nabla_{X} S\right)(Y, Z)=\left(\nabla_{Y} S\right)(X, Z) \tag{1.5}
\end{equation*}
$$

In a recent paper Mantica and Suh [17] introduced a new curvature tensor of type $(1,3)$ in an $n$-dimensional Riemannian manifold $\left(M^{n}, g\right)(n>2)$ denoted by $\mathcal{Q}$ and defined by

$$
\begin{equation*}
\mathcal{Q}(X, Y) W=\mathcal{R}(X, Y) W-\frac{\psi}{(n-1)}[g(Y, W) X-g(X, W) Y] \tag{1.6}
\end{equation*}
$$

where $\psi$ is an arbitrary scalar function. Such a tensor $\mathcal{Q}$ is known as $Q$-curvature tensor. The notion of $Q$ tensor is also suitable to reinterpret some differential structures on a Riemannian manifold.

Now (1.6) can be expressed as

$$
\begin{gather*}
Q(X, Y, W, U)= \\
=R(X, Y, W, U)-\frac{\psi}{(n-1)}[g(Y, W) g(X, U)-g(X, W) g(Y, U)] \tag{1.7}
\end{gather*}
$$

where $Q(X, Y, W, U)=g(\mathcal{Q}(X, Y) W, U)$. Since the $Q$-curvature tensor satisfies the properties of the Riemannian curvature tensor, therefore in a similar way as in [8] we can prove that weakly- $Q$-symmetric manifolds is characterized by the condition

$$
\begin{align*}
& \left(\nabla_{X} Q\right)(Y, W, U, V)=A(X) Q(Y, W, U, V)+B(Y) Q(X, W, U, V)+ \\
& +B(W) Q(Y, X, U, V)+D(U) Q(Y, W, X, V)+D(V) Q(Y, W, U, X) \tag{1.8}
\end{align*}
$$

where the 1 -forms $A, B$ and $D$ are non-zero simultaneously. Such a manifold is denoted by $(W Q S)_{n}$. If $\psi=0$, then the $(W Q S)_{n}$ reduces to a $(W S)_{n}$. Recently, Mantica and Molinari [14] have studied weakly-Z-symmetric manifolds. On the otherhand, Mantica and Suh ([15], [17]) have studied pseudo-Z-symmetric Riemannian manifolds with harmonic curvature tensors, pseudo- $Q$-symmetric Riemannian manifolds. Motivated by the above studies in the present paper we have studied a type of non-flat Riemannian manifold defined by (1.8).

We also have a very useful lemma as follows:
WaLKER's Lemma [25]. If $a_{i j}, b_{i}$ are numbers satisfying $a_{i j}=a_{j i}$, and $a_{i j} b_{k}+$ $+a_{j k} b_{i}+a_{k i} b_{j}=0$, for $i, j, k=1,2, \ldots, n$, then either all $a_{i j}=0$ or all $b_{i}$ are zero.

The paper is organized as follows:
After preliminaries, in Section 3, some curvature properties of $(W Q S)_{n}$ have been studied. Among others it is proved that if a $(W Q S)_{n}(n>2)$ is also a $(W S)_{n}$, then the sum of the associated 1 -forms is closed, provided $\psi \neq 0$. Moreover, if $\psi$ is a non-zero constant, then the sum of the associated 1-forms is zero. Next we prove that if in a $(W Q S)_{n}(n>2)$, the tensor $\tilde{Q}$ defined by (2.5) is cyclic parallel, then the manifold is either an Einstein manifold or the sum of the associated 1forms is zero. Section 4 is devoted to study decomposability of $(W Q S)_{n}(n>2)$. In this section we obtain the nature of the decomposable spaces. Also we prove that if $\left(M^{n}, g\right)$ is a Riemannian manifold such that $M=M_{1}^{p} \times M_{2}^{n-p},(2 \leq p \leq$ $\leq n-2)$ and $M$ is a $(W Q S)_{n}$ with non-vanishing $\psi$, then both decomposable spaces are $Q$-recurrent. Finally, we give two examples of the $(W Q S)_{4}$.

## 2. Preliminaries

Let $S$ and $r$ denote the Ricci tensor of type $(0,2)$ and the scalar curvature respectively and $L$ denotes the symmetric endomorphism of the tangent space at each point corresponding to the Ricci tensor $S$, that is,

$$
\begin{equation*}
g(L X, Y)=S(X, Y) \tag{2.1}
\end{equation*}
$$

In this section, some formulas are derived, which will be useful to the study of $(W Q S)_{n}$. Let $\left\{e_{i}\right\}$ be an orthonormal basis of the tangent space at each point of the manifold where $1 \leq i \leq n$.

From (1.6) we can easily verify that the tensor $\mathcal{Q}$ satisfies the following properties:

$$
\begin{align*}
\text { i) } & \mathcal{Q}(Y, W) U=-\mathcal{Q}(W, Y) U \\
\text { ii) } & \mathcal{Q}(Y, W) U+\mathcal{Q}(W, U) Y+\mathcal{Q}(U, Y) W=0 . \tag{2.2}
\end{align*}
$$

Also from (1.7) we have

$$
\begin{equation*}
\sum_{i=1}^{n} Q\left(X, Y, e_{i}, e_{i}\right)=0=\sum_{i=1}^{n} Q\left(e_{i}, e_{i}, W, U\right) \tag{2.3}
\end{equation*}
$$

and

$$
\begin{gather*}
\sum_{i=1}^{n} Q\left(e_{i}, Y, W, e_{i}\right)=\sum_{i=1}^{n} Q\left(Y, e_{i}, e_{i}, W\right)= \\
=S(Y, W)-\psi g(Y, W) \tag{2.4}
\end{gather*}
$$

where $r=\sum_{i=1}^{n} S\left(e_{i}, e_{i}\right)$ is the scalar curvature.
Let

$$
\begin{equation*}
\tilde{Q}(Y, W)=S(Y, W)-\psi g(Y, W) \tag{2.5}
\end{equation*}
$$

Therefore,

$$
\begin{equation*}
\tilde{Q}\left(e_{i}, e_{i}\right)=r-n \psi . \tag{2.6}
\end{equation*}
$$

From (1.7) and (2.2) it follows that
(i) $Q(X, Y, W, U)=-Q(Y, X, W, U)$,
(ii) $Q(X, Y, W, U)=-Q(X, Y, U, W)$,
(iii) $Q(X, Y, W, U)=Q(W, U, X, Y)$,
(iv) $\quad Q(X, Y, W, U)+Q(Y, W, X, U)+Q(W, X, Y, U)=0$,
where $Q(X, Y, W, U)=g(\mathcal{Q}(X, Y) W, U)$.

## 3. Some curvature properties of $(W Q S)_{n}(n>2)$

In general, the $Q$-curvature tensor $Q(Y, W, U, V)$ does not satisfy Bianchi's 2nd identity

$$
\begin{equation*}
\left(\nabla_{X} Q\right)(Y, W, U, V)+\left(\nabla_{Y} Q\right)(X, W, U, V)+\left(\nabla_{W} Q\right)(X, Y, U, V)=0 . \tag{3.1}
\end{equation*}
$$

We suppose that the condition (3.1) holds in the investigated weakly- $Q$ symmetric manifolds.

Now using (2.2), (2.7) and (3.1) we get from (1.8)

$$
\begin{equation*}
G(X) Q(Y, W, U, V)+G(Y) Q(W, X, U, V)+G(W) Q(X, Y, U, V)=0, \tag{3.2}
\end{equation*}
$$

where $G(X)=A(X)-2 B(X)$ and $\rho$ is a vector field defined by

$$
g(X, \rho)=G(X) .
$$

Contracting (3.2) over $Y$ and $V$ we get

$$
\begin{gather*}
G(X)[S(W, U)-\psi \boldsymbol{g}(W, U)]+G(\mathcal{Q}(W, X) U)- \\
-G(W)[S(X, U)-\psi \boldsymbol{g}(X, U)]=0 . \tag{3.4}
\end{gather*}
$$

Again contracting $W, U$ in (3.4) we get

$$
G(X)(r-n \psi)-2[G(L X)-\psi G(X)]=0 .
$$

or,

$$
\begin{equation*}
G(L X)=\frac{r-n \psi+2 \psi}{2} G(X) . \tag{3.5}
\end{equation*}
$$

or,

$$
\begin{equation*}
S(X, \rho)=\frac{r-n \psi+2 \psi}{2} g(X, \rho) . \tag{3.6}
\end{equation*}
$$

Thus we can state the following theorem.
Theorem 3.1. If the $Q$-curvature tensor of a $(W Q S)_{n}$ satisfies Bianchi's 2nd identity, then the Ricci tensor $S$ in $(W Q S)_{n}$ has eigen value $\frac{(r-n \psi+2 \psi)}{2}$ corresponding to the eigen vector $\rho$ defined by (3.3).

If in particular $\psi=\frac{r}{n}$, then we have the following corollary from (3.6).
Corollary 3.1. If the $Q$-curvature tensor of a $(W Q S)_{n}$ satisfies Bianchi's 2nd identity, then the Ricci tensor $S$ in $(W Q S)_{n}$ has eigen value $\frac{r}{n}$ corresponding to the eigen vector $\rho$ defined by (3.3) provided $\psi=\frac{r}{n}$.

Contracting (1.8) over $Y$ and $V$ we get

$$
\begin{align*}
& \left(\nabla_{X} \tilde{Q}\right)(W, U)=\{A(X) \tilde{Q}(W, U)+B(W) \tilde{Q}(X, U)+ \\
& +D(U) \tilde{Q}(W, X)\}+B(\mathcal{Q}(X, W) U)-D(\mathcal{Q}(U, X) W) \tag{3.7}
\end{align*}
$$

Again contracting (3.7) over $W$ and $U$ we get

$$
\begin{equation*}
A(X)(r-n \psi)+2\left\{\tilde{Q}\left(X, \rho_{2}\right)+\tilde{Q}\left(X, \rho_{3}\right)\right\}=0 \tag{3.8}
\end{equation*}
$$

Now using (2.5) in (3.8) we get

$$
\begin{gather*}
A(X)(r-n \psi)+2\left[\left\{S\left(X, \rho_{2}\right)+S\left(X, \rho_{3}\right)\right\}-\right. \\
\left.-\psi\left\{g\left(X, \rho_{2}\right)+g\left(X, \rho_{3}\right)\right\}\right]=0 \tag{3.9}
\end{gather*}
$$

Since, in a $(W Q S)_{n}, A(X) \neq 0$ so, if $\psi=\frac{r}{n}$ then from (3.9) we get

$$
\begin{equation*}
S(X, \bar{\rho})=\psi g(X, \bar{\rho}) \tag{3.10}
\end{equation*}
$$

where

$$
\begin{equation*}
g(X, \bar{\rho})=F(X)=B(X)+D(X), \quad \bar{\rho}=\rho_{2}+\rho_{3} \tag{3.11}
\end{equation*}
$$

Thus we can state the following theorem.
Theorem 3.2. In a $(W Q S)_{n}$, the Ricci tensor $S$ has eigen value $\psi$ corresponding to the eigen vector $\bar{\rho}$ defined by (3.11) provided $\psi=\frac{r}{n}$.

From (1.7) we get

$$
\begin{gather*}
\left(\nabla_{X} Q\right)(Y, W, U, V)=\left(\nabla_{X} R\right)(Y, W, U, V)- \\
-\frac{(X \psi)}{(n-1)}[g(W, U) g(Y, V)-g(Y, U) g(W, V)] . \tag{3.12}
\end{gather*}
$$

Using (3.12) in (1.8) we get

$$
\begin{gather*}
\left(\nabla_{X} R\right)(Y, W, U, V)-\frac{(X \psi)}{(n-1)}[g(W, U) g(Y, V)-g(Y, U) g(W, V)]= \\
=A(X) Q(Y, W, U, V)+B(Y) Q(X, W, U, V)+B(W) Q(Y, X, U, V)+  \tag{3.13}\\
+D(U) Q(Y, W, X, V)+D(V) Q(Y, W, U, X)
\end{gather*}
$$

Using (1.7) in (3.13) we get

$$
\begin{gather*}
\quad\left(\nabla_{X} R\right)(Y, W, U, V)-\frac{(X \psi)}{(n-1)}[g(W, U) g(Y, V)-g(Y, U) g(W, V)]= \\
=A(X) R(Y, W, U, V)+B(Y) R(X, W, U, V)+B(W) R(Y, X, U, V)+ \\
+D(U) R(Y, W, X, V)+D(V) R(Y, W, U, X)-\frac{\psi}{(n-1)}[\{g(W, U) g(Y, V)- \\
-g(Y, U) g(W, V)\} A(X)+\{g(W, U) g(X, V)-g(X, U) g(W, V)\} B(Y)+ \\
+\{g(X, U) g(Y, V)-g(Y, U) g(X, V)\} B(W)+\{g(W, X) g(Y, V)- \\
-  \tag{3.14}\\
-g(Y, X) g(W, V)\} D(U)+\{g(W, U) g(Y, X)-g(Y, U) g(W, X)\} D(V)]
\end{gather*}
$$

If a $(W Q S)_{n}$ is also $(W S)_{n}$, then using (1.2) in (3.14) we get

$$
\begin{align*}
- & \frac{(X \psi)}{(n-1)}[g(W, U) g(Y, V)-g(Y, U) g(W, V)]= \\
=- & \frac{\psi}{(n-1)}[\{g(W, U) g(Y, V)-g(Y, U) g(W, V)\} A(X)+ \\
& +\{g(W, U) g(X, V)-g(X, U) g(W, V)\} B(Y)+  \tag{3.15}\\
& +\{g(X, U) g(Y, V)-g(Y, U) g(X, V)\} B(W)+ \\
& +\{g(W, X) g(Y, V)-g(Y, X) g(W, V)\} D(U)+ \\
& +\{g(W, U) g(Y, X)-g(Y, U) g(W, X)\} D(V)]
\end{align*}
$$

Now contracting (3.15) over $Y$ and $V$ we get

$$
\begin{gather*}
-(X \psi) g(W, U)=-\frac{\psi}{(n-1)}[(n-1) A(X) g(W, U)+ \\
+B(X) g(W, U)-B(W) g(X, U)+(n-1) B(W) g(X, U)+  \tag{3.16}\\
+(n-1) D(U) g(W, X)+D(X) g(W, U)-D(U) g(W, X)]
\end{gather*}
$$

Again contracting (3.16) over $W$ and $U$ we get

$$
\begin{equation*}
-n(X \psi)=-\psi[n A(X)+2 B(X)+2 D(X)] \tag{3.17}
\end{equation*}
$$

Similarly, contracting (3.16) over $X$ and $W$ we get

$$
\begin{equation*}
-(U \psi)=-\psi[A(U)+B(U)+(n-1) D(U)] \tag{3.18}
\end{equation*}
$$

Replacing $U$ by $X$ in (3.18) we get

$$
\begin{equation*}
-(X \psi)=-\psi[A(X)+B(X)+(n-1) D(X)] \tag{3.19}
\end{equation*}
$$

Again contracting (3.16) over $X$ and $U$ we get

$$
\begin{equation*}
-(W \psi)=-\psi[A(W)+(n-1) B(W)+D(W)] \tag{3.20}
\end{equation*}
$$

Replacing $W$ by $X$ in (3.20) we get

$$
\begin{equation*}
-(X \psi)=-\psi[A(X)+(n-1) B(X)+D(X)] \tag{3.21}
\end{equation*}
$$

Now adding (3.17), (3.19) and (3.21) we get

$$
\begin{equation*}
(n+2)(X \psi)=(n+2) \psi[A(X)+B(X)+D(X)] \tag{3.22}
\end{equation*}
$$

Since $n>2$ so we have from (3.22)

$$
\begin{equation*}
(X \psi)=\psi[A(X)+B(X)+D(X)] \tag{3.23}
\end{equation*}
$$

which implies that the sum of the associated 1-forms is closed, provided $\psi \neq 0$. However if $\psi$ is a non-zero constant, then the sum of the associated 1-forms is zero.

Hence we have the following theorem.
THEOREM 3.3. If a $(W Q S)_{n}(n>2)$ is also a $(W S)_{n}$, then the sum of the associated 1 -forms is closed, provided $\psi \neq 0$. Moreover, if $\psi$ is a non-zero constant, then the sum of the associated 1 -forms is zero.

Now writing (3.7) cyclically and adding we obtain

$$
\begin{align*}
& {\left[\left(\nabla_{X} \tilde{Q}\right)(W, U)+\left(\nabla_{W} \tilde{Q}\right)(U, X)+\left(\nabla_{U} \tilde{Q}\right)(X, W)\right]=} \\
& =\{A(X)+B(X)+D(X)\} \tilde{Q}(W, U)+ \\
& +\{A(W)+B(W)+D(W)\} \tilde{Q}(X, U)+  \tag{3.24}\\
& +\{A(U)+B(U)+D(U)\} \tilde{Q}(W, X)+ \\
& +\{B(\mathcal{Q}(X, W) U)+B(\mathcal{Q}(W, U) X)+B(\mathcal{Q}(U, X) W)\}- \\
& -\{D(\mathcal{Q}(X, W) U)+D(\mathcal{Q}(W, U) X)+D(\mathcal{Q}(U, X) W)\}]
\end{align*}
$$

Using (2.2) and (2.7) in (3.24) we get

$$
\begin{align*}
& \left(\nabla_{X} \tilde{Q}\right)(W, U)+\left(\nabla_{W} \tilde{Q}\right)(U, X)+\left(\nabla_{U} \tilde{Q}\right)(X, W)= \\
& =E(X) \tilde{Q}(W, U)+E(W) \tilde{Q}(U, X)+E(U) \tilde{Q}(X, W) \tag{3.25}
\end{align*}
$$

where

$$
\begin{equation*}
E(X)=A(X)+B(X)+D(X) \tag{3.26}
\end{equation*}
$$

Now if the $(W Q S)_{n}$ has $\tilde{Q}$-cyclic parallel tensor, then we have

$$
\begin{equation*}
\left(\nabla_{X} \tilde{Q}\right)(W, U)+\left(\nabla_{W} \tilde{Q}\right)(U, X)+\left(\nabla_{U} \tilde{Q}\right)(X, W)=0 \tag{3.27}
\end{equation*}
$$

Also we have from (2.5)

$$
\begin{equation*}
\tilde{Q}(X, Y)=\tilde{Q}(Y, X) \tag{3.28}
\end{equation*}
$$

Using (3.27) in (3.25) we get

$$
\begin{equation*}
E(X) \tilde{Q}(W, U)+E(W) \tilde{Q}(U, X)+E(U) \tilde{Q}(X, W)=0 \tag{3.29}
\end{equation*}
$$

Then by Walker's lemma we can see that either, $E(X)=0$ or, $\tilde{Q}(W, U)=0$ for all $X, W, U$.

Thus we have either,

$$
\begin{equation*}
A(X)+B(X)+D(X)=0 \tag{3.30}
\end{equation*}
$$

or,

$$
\begin{equation*}
S(X, W)=\psi g(X, W) \tag{3.31}
\end{equation*}
$$

Thus we can state the following:

THEOREM 3.4. If in a $(W Q S)_{n}(n>2)$, the tensor $\tilde{Q}$ defined by (2.5) is cyclic parallel, then the manifold is either an Einstein manifold or the sum of the associated 1 -forms is zero.

## 4. Decomposable $(W Q S)_{n}$

A Riemannian manifold $\left(M^{n}, g\right)$ is said to be decomposable or a product manifold ([22], [24]) if it can be expressed as $M_{1}^{p} \times M_{2}^{n-p}$ for $2 \leq p \leq(n-2)$, that is, in some coordinate neighbourhood of the Riemannian manifold $\left(M^{n}, g\right)$, the metric can be expressed as

$$
\begin{equation*}
d s^{2}=g_{i j} d x^{i} d x^{j}=\bar{g}_{a b} d x^{a} d x^{b}+g_{\alpha \beta}^{*} d x^{\alpha} d x^{\beta} \tag{4.1}
\end{equation*}
$$

where $\bar{g}_{a b}$ are functions of $x^{1}, x^{2}, \ldots, x^{p}$ denoted by $\bar{x}$ and $g_{\alpha \beta}^{*}$ are functions of $x^{p+1}, x^{p+2}, \ldots, x^{n}$ denoted by $x^{*} ; a, b, c, \ldots$ run from 1 to $p$ and $\alpha, \beta, \gamma, \ldots$ run from $p+1$ to $n$.

The two parts of (4.1) are the metrics of $M_{1}^{p}(p \geq 2)$ and $M_{2}^{n-p}(n-p \geq 2)$ which are called the components of the decomposable manifold $M^{n}=M_{1}^{p} \times$ $\times M_{2}^{n-p}(2 \leq p \leq n-2)$.

Let $\left(M^{n}, g\right)$ be a Riemannian manifold such that $M_{1}^{p}(p \geq 2)$ and $M_{2}^{n-p}$ $(n-p \geq 2)$ are two components of this manifold. Here throughout this section each object denoted by a 'bar' is assumed to be from $M_{1}$ and each object denoted by 'star' is assumed to be from $M_{2}$.

Let $\bar{X}, \bar{Y}, \bar{Z}, \bar{U}, \bar{V} \in \chi\left(M_{1}\right)$ and $X^{*}, Y^{*}, Z^{*}, U^{*}, V^{*} \in \chi\left(M_{2}\right)$. Then in a decomposable Riemannian manifold $M^{n}=M_{1}^{p} \times M_{2}^{n-p}(2 \leq p \leq n-2)$, the following relations hold [26]:

$$
\begin{aligned}
& R\left(X^{*}, \bar{Y}, \bar{Z}, \bar{U}\right)=0=R\left(\bar{X}, Y^{*}, \bar{Z}, U^{*}\right)=R\left(\bar{X}, Y^{*}, Z^{*}, U^{*}\right) \\
& \left(\nabla_{X^{*}} R\right)(\bar{Y}, \bar{Z}, \bar{U}, \bar{V})=0=\left(\nabla_{\bar{X}} R\right)\left(\bar{Y}, Z^{*}, \bar{U}, V^{*}\right)=\left(\nabla_{X^{*}} R\right)\left(\bar{Y}, Z^{*}, \bar{U}, V^{*}\right), \\
& R(\bar{X}, \bar{Y}, \bar{Z}, \bar{U})=\bar{R}(\bar{X}, \bar{Y}, \bar{Z}, \bar{U}) ; R\left(X^{*}, Y^{*}, Z^{*}, U^{*}\right)=R^{*}\left(X^{*}, Y^{*}, Z^{*}, U^{*}\right), \\
& S(\bar{X}, \bar{Y})=\bar{S}(\bar{X}, \bar{Y}) ; S\left(X^{*}, Y^{*}\right)=S^{*}\left(X^{*}, Y^{*}\right), \\
& \left(\nabla_{\bar{X}} S\right)(\bar{Y}, \bar{Z})=\left(\bar{\nabla}_{\bar{X}} S\right)(\bar{Y}, \bar{Z}) ;\left(\nabla_{X^{*}} S\right)\left(Y^{*}, Z^{*}\right)=\left(\nabla_{X^{*}}^{*} S\right)\left(Y^{*}, Z^{*}\right), \\
& \text { and } r=\bar{r}+r^{*},
\end{aligned}
$$

where $r, \bar{r}$ and $r^{*}$ are scalar curvatures of $M, M_{1}$ and $M_{2}$ respectively.
Let us consider a Riemannian manifold $\left(M^{n}, g\right)$, which is a decomposable $(W Q S)_{n}$.

Then $M^{n}=M_{1}^{p} \times M_{2}^{n-p}(2 \leq p \leq n-2)$.

Now from (1.7), we get

$$
\begin{gather*}
Q\left(Y^{*}, \bar{Z}, \bar{U}, \bar{V}\right)=0=Q\left(\bar{Y}, Z^{*}, U^{*}, V^{*}\right)= \\
=Q\left(\bar{Y}, Z^{*}, \bar{U}, \bar{V}\right)=Q\left(\bar{Y}, \bar{Z}, U^{*}, \bar{V}\right),  \tag{4.2}\\
Q\left(Y^{*}, \bar{Z}, \bar{U}, V^{*}\right)=-\frac{\psi}{(n-1)}\left[g(\bar{Z}, \bar{U}) g\left(Y^{*}, V^{*}\right)\right],  \tag{4.3}\\
Q\left(Y^{*}, Z^{*}, \bar{U}, \bar{V}\right)=0=Q\left(\bar{Y}, \bar{Z}, U^{*}, V^{*}\right), \tag{4.4}
\end{gather*}
$$

and

$$
\begin{equation*}
Q\left(Y^{*}, \bar{Z}, U^{*}, \bar{V}\right)=\frac{\psi}{(n-1)}\left[g\left(Y^{*}, U^{*}\right) g(\bar{Z}, \bar{V})\right] \tag{4.5}
\end{equation*}
$$

Again from (1.8), we get

$$
\begin{align*}
& \left(\nabla_{\bar{X}} Q\right)(\bar{Y}, \bar{Z}, \bar{U}, \bar{V})=A(\bar{X}) Q(\bar{Y}, \bar{Z}, \bar{U}, \bar{V})+B(\bar{Y}) Q(\bar{X}, \bar{Z}, \bar{U}, \bar{V})+ \\
& +B(\bar{Z}) Q(\bar{Y}, \bar{X}, \bar{U}, \bar{V})+D(\bar{U}) Q(\bar{Y}, \bar{Z}, \bar{X}, \bar{V})+D(\bar{V}) Q(\bar{Y}, \bar{Z}, \bar{U}, \bar{X}) \tag{4.6}
\end{align*}
$$

Replacing $\bar{X}$ by $X^{*}$ in (4.6) we get

$$
\begin{equation*}
A\left(X^{*}\right) Q(\bar{Y}, \bar{Z}, \bar{U}, \bar{V})=0 \tag{4.7}
\end{equation*}
$$

Similarly we have

$$
\begin{align*}
& B\left(Y^{*}\right) Q(\bar{X}, \bar{Z}, \bar{U}, \bar{V})=0  \tag{4.8}\\
& D\left(U^{*}\right) Q(\bar{Y}, \bar{Z}, \bar{X}, \bar{V})=0 \tag{4.9}
\end{align*}
$$

Now putting $\bar{X}=X^{*}, \bar{Y}=Y^{*}$ in (4.6) we get

$$
\begin{equation*}
\psi[D(\bar{U}) g(\bar{Z}, \bar{V})-D(\bar{V}) g(\bar{Z}, \bar{U})]=0 \tag{4.10}
\end{equation*}
$$

Similarly putting $\bar{X}=X^{*}, \bar{U}=U^{*}$ in (4.6) we obtain

$$
\begin{equation*}
\psi[B(\bar{Y}) g(\bar{Z}, \bar{V})-B(\bar{Z}) g(\bar{Y}, \bar{V})]=0 \tag{4.11}
\end{equation*}
$$

Also putting $\bar{Y}=Y^{*}, \bar{Z}=Z^{*}$ and $\bar{U}=U^{*}$ in (4.6) we have

$$
\begin{equation*}
\psi\left[B\left(Z^{*}\right) g\left(Y^{*}, U^{*}\right)-B\left(Y^{*}\right) g\left(Z^{*}, U^{*}\right)\right]=0 \tag{4.12}
\end{equation*}
$$

In the similar way, from (4.6) we have the following

$$
\begin{equation*}
\psi\left[D\left(U^{*}\right) g\left(Z^{*}, V^{*}\right)-D\left(V^{*}\right) g\left(Z^{*}, U^{*}\right)\right]=0 \tag{4.13}
\end{equation*}
$$

Also from (1.8), we obtain

$$
\begin{align*}
& \left(\nabla_{X^{*}} Q\right)\left(Y^{*}, Z^{*}, U^{*}, V^{*}\right)=A\left(X^{*}\right) Q\left(Y^{*}, Z^{*}, U^{*}, V^{*}\right)+ \\
& +B\left(Y^{*}\right) Q\left(X^{*}, Z^{*}, U^{*}, V^{*}\right)+B\left(Z^{*}\right) Q\left(Y^{*}, X^{*}, U^{*}, V^{*}\right)+  \tag{4.14}\\
& +D\left(U^{*}\right) Q\left(Y^{*}, Z^{*}, X^{*}, V^{*}\right)+D\left(V^{*}\right) Q\left(Y^{*}, Z^{*}, U^{*}, X^{*}\right)
\end{align*}
$$

From (4.14), it follows that

$$
\begin{align*}
& A(\bar{X}) Q\left(Y^{*}, Z^{*}, U^{*}, V^{*}\right)=0  \tag{4.15}\\
& B(\bar{Y}) Q\left(X^{*}, Z^{*}, U^{*}, V^{*}\right)=0  \tag{4.16}\\
& D(\bar{U}) Q\left(Y^{*}, Z^{*}, X^{*}, V^{*}\right)=0 \tag{4.17}
\end{align*}
$$

From (4.7)to (4.9) we have two cases, namely,
I) $A=B=D=0$ on $M_{2}$,
II) $M_{1}$ is $Q$-flat.

Firstly, we consider the case (I). Then from (4.14), it follows that

$$
\left(\nabla_{X^{*}} Q\right)\left(Y^{*}, Z^{*}, U^{*}, V^{*}\right)=0
$$

that is,

$$
\begin{gather*}
\left(\nabla_{X^{*}} R\right)\left(Y^{*}, Z^{*}, U^{*}, V^{*}\right)- \\
-\frac{\left(X^{*} \psi\right)}{(n-1)}\left[g\left(Z^{*}, U^{*}\right) g\left(Y^{*}, V^{*}\right)-g\left(Y^{*}, U^{*}\right) g\left(Z^{*}, V^{*}\right)\right]=0 \tag{4.18}
\end{gather*}
$$

Setting $Z^{*}=U^{*}=e_{\alpha}^{*}$ in (4.18) and taking summation over $\alpha, p+1 \leq \alpha \leq n$, we obtain

$$
\begin{equation*}
\left(\nabla_{X^{*}} S\right)\left(Y^{*}, V^{*}\right)-\frac{\left(X^{*} \psi\right)}{n-1}\left[(n-p-1) g\left(Y^{*}, V^{*}\right)\right]=0 \tag{4.19}
\end{equation*}
$$

since $r=\bar{r}+r^{*}$ and if we take $\left(X^{*} \psi\right)=0$ in $M_{2}$, then from (4.19) we have

$$
\left(\nabla_{X^{*}} S\right)\left(Y^{*}, V^{*}\right)=0
$$

This implies that $M_{2}$ is Ricci symmetric manifold if $\psi$ is constant in $M_{2}$.
Secondly, we discuss the case (II). Since $M_{1}$ is $Q$-flat, therefore it is a manifold of constant curvature provided that $\psi=$ constant in $M_{2}$. Hence we can state the following:

THEOREM 4.1. Let $\left(M^{n}, g\right)$ be a Riemannian manifold such that $M=M_{1}^{p} \times M_{2}^{n-p}$, $(2 \leq p \leq n-2)$. If $M$ is a $(W Q S)_{n}$ then the following holds:
(I) In the case of $A=B=D=0$ on $M_{2}$, then the manifold $M_{2}$ is Ricci symmetric, provided that $\psi=$ constant in $M_{2}$.
(II) When $M_{1}$ is $Q$-flat and $\psi=$ constant in $M_{1}$, then $M_{1}$ is a manifold of constant curvature.

Similarly, from (4.15) to (4.17) we get
Theorem 4.2. Let $\left(M^{n}, g\right)$ be a Riemannian manifold such that $M=M_{1}^{p} \times M_{2}^{n-p}$, $(2 \leq p \leq n-2)$. If $M$ is a $(W Q S)_{n}$ then the following holds:
(I) In the case of $A=B=D=0$ on $M_{1}$, then the manifold $M_{1}$ is Ricci symmetric, provided that $\psi=$ constant in $M_{1}$.
(II) When $M_{2}$ is $Q$-flat and $\psi=$ constant in $M_{2}$, then $M_{2}$ is a manifold of constant curvature.

Next we consider the contraction with respect to $\bar{Z}$ and $\bar{V}$ in (4.10) and obtain

$$
(p-1) \psi D(\bar{U})=0
$$

Since $p \geq 2$. we have

$$
\begin{equation*}
\psi D(\bar{U})=0 \tag{4.20}
\end{equation*}
$$

If $\psi$ is non-vanishing scalar, then (4.20) yields

$$
\begin{equation*}
D(\bar{U})=0 \quad \text { for all } \quad \bar{U} \in \chi\left(M_{1}\right) \tag{4.21}
\end{equation*}
$$

Similarly, from (4.11) we have

$$
\begin{equation*}
B(\bar{Y})=0 \quad \text { for all } \quad \bar{Y} \in \chi\left(M_{1}\right) \tag{4.22}
\end{equation*}
$$

provided $\psi$ is non-vanishing scalar.
Thus if $\psi \neq 0$, then from (4.21) and (4.22) we have $B=0$ and $D=0$ on $M_{1}$ and hence from (4.6) we get

$$
\begin{equation*}
\left(\nabla_{\bar{X}} Q\right)(\bar{Y}, \bar{Z}, \bar{U}, \bar{V})=A(\bar{X}) Q(\bar{Y}, \bar{Z}, \bar{U}, \bar{V}) \tag{4.23}
\end{equation*}
$$

Again if we consider the contraction with respect to $Y^{*}$ and $U^{*}$ in (4.12), then we obtain

$$
\psi(n-p-1) B\left(Z^{*}\right)=0
$$

Since $(n-p) \geq 2$. we have

$$
\begin{equation*}
\psi B\left(Z^{*}\right)=0 \tag{4.24}
\end{equation*}
$$

If $\psi$ is non-zero, then from (4.24) we have

$$
\begin{equation*}
B\left(Z^{*}\right)=0 \quad \text { for all } \quad Z^{*} \in \chi\left(M_{2}\right) \tag{4.25}
\end{equation*}
$$

Similarly, if $\psi \neq 0$ from (4.13) we get

$$
\begin{equation*}
D\left(U^{*}\right)=0 \quad \text { for all } \quad U^{*} \in \chi\left(M_{2}\right) \tag{4.26}
\end{equation*}
$$

and hence from (4.14) we get

$$
\begin{equation*}
\left(\nabla_{X^{*}} Q\right)\left(Y^{*}, Z^{*}, U^{*}, V^{*}\right)=A\left(X^{*}\right) Q\left(Y^{*}, Z^{*}, U^{*}, V^{*}\right) \tag{4.27}
\end{equation*}
$$

Hence from (4.23) and (4.27) we can state the following:
Theorem 4.3. Let $\left(M^{n}, g\right)$ be a Riemannian manifold such that $M=M_{1}^{p} \times M_{2}^{n-p}$, $(2 \leq p \leq n-2)$. If $M$ is a $(W Q S)_{n}$ with non-vanishing $\psi$, then both the decompositions are $Q$-recurrent.

## 5. Example of a $(W Q S)_{4}$

In this section we give an example of $(W Q S)_{n}$, with the non-zero scalar curvature.

Example 5.1. [10] Let $\left(\mathbb{R}^{4}, g\right)$ be a 4-dimensional Riemannian manifold endowed with the Riemannian metric $g$ given by

$$
\begin{equation*}
d s^{2}=g_{i j} d x^{i} d x^{j}=(1+2 q)\left[\left(d x^{1}\right)^{2}+\left(d x^{2}\right)^{2}+\left(d x^{3}\right)^{2}+\left(d x^{4}\right)^{2}\right] \tag{5.1}
\end{equation*}
$$

where $(i, j=1,2,3,4), q=\frac{e^{x^{1}}}{k^{2}}$ and $k$ is a non-zero constant. Here the only nonvanishing components of the Christoffel symbols and the curvature tensors are respectively:

$$
\begin{array}{cl}
\Gamma_{11}^{1}=\Gamma_{12}^{2}=\Gamma_{13}^{3}=\Gamma_{14}^{4}=\frac{q}{1+2 q}, & \Gamma_{22}^{1}=\Gamma_{33}^{1}=\Gamma_{44}^{1}=-\frac{q}{1+2 q} \\
R_{1221}=R_{1331}=R_{1441}=\frac{q}{1+2 q}, & R_{2332}=R_{2442}=R_{3443}=\frac{q^{2}}{1+2 q}
\end{array}
$$

and the components obtained by the symmetry properties. The non-vanishing components of the Ricci tensors are:

$$
R_{11}=\frac{3 q}{(1+2 q)^{2}}, \quad R_{22}=R_{33}=R_{44}=\frac{q}{1+2 q},
$$

It can be easily shown that the scalar curvature $r$ of this $\left(\mathbb{R}^{4}, g\right)$ is $\frac{6 q(1+q)}{(1+2 q)^{3}}$, which is non-vanishing and non-constant. Let us choose the arbitrary scalar function, $\psi=\frac{3}{(1+2 q)^{3}}$. Therefore the non-vanishing components of the $Q$-curvature tensor and their covariant derivatives are respectively:

$$
\begin{aligned}
& Q_{1221}=Q_{1331}=Q_{1441}=\frac{q-1}{1+2 q} ; \\
& Q_{2332}=Q_{2442}=Q_{3443}=\frac{q^{2}-1}{1+2 q} ; \\
& Q_{1221,1}=Q_{1331,1}=Q_{1441,1}=\frac{3 q}{(1+2 q)^{2}} ; \\
& \\
& Q_{2332,1}=Q_{2442,1}=Q_{3443,1}=\frac{2 q\left(1+q+q^{2}\right)}{(1+2 q)^{2}} .
\end{aligned}
$$

Let us choose the associated 1-forms as follows:

$$
\begin{align*}
& A_{i}(x)= \begin{cases}\frac{2 q\left(1+q+q^{2}\right)}{\left(q^{2}-1\right)(1+2 q)} & \text { for } i=1 \\
0 & \text { otherwise },\end{cases}  \tag{5.2}\\
& B_{i}(x)= \begin{cases}\frac{q}{1+q} & \text { for } i=1 \\
0 & \text { otherwise }\end{cases}  \tag{5.3}\\
& D_{i}(x)= \begin{cases}\frac{-2 q}{1+q} & \text { for } i=1 \\
0 & \text { otherwise }\end{cases} \tag{5.4}
\end{align*}
$$

at any point $x \in \mathbb{R}^{4}$. Now the equation (1.8) reduces to the equations

$$
\begin{align*}
& Q_{1221,1}=A_{1} Q_{1221}+B_{1} Q_{1221}+B_{2} Q_{1121}+D_{2} Q_{1211}+D_{1} Q_{1221}  \tag{5.5}\\
& Q_{1331,1}=A_{1} Q_{1331}+B_{1} Q_{1331}+B_{3} Q_{1131}+D_{3} Q_{1311}+D_{1} Q_{1331}  \tag{5.6}\\
& Q_{2332,1}=A_{1} Q_{2332}+B_{2} Q_{1332}+B_{3} Q_{2132}+D_{3} Q_{2312}+D_{2} Q_{2331}  \tag{5.7}\\
& Q_{1441,1}=A_{1} Q_{1441}+B_{1} Q_{1441}+B_{4} Q_{1141}+D_{4} Q_{1411}+D_{1} Q_{1441}  \tag{5.8}\\
& Q_{2442,1}=A_{1} Q_{2442}+B_{2} Q_{1442}+B_{4} Q_{2142}+D_{4} Q_{2412}+D_{2} Q_{2441}  \tag{5.9}\\
& Q_{3443,1}=A_{1} Q_{3443}+B_{3} Q_{1443}+B_{4} Q_{3143}+D_{4} Q_{3413}+D_{3} Q_{3441} \tag{5.10}
\end{align*}
$$

since, for the other cases (1.8) holds trivially.
By (5.2), (5.3) and (5.4) we get the following relation for the right hand side(R.H.S.) and the left hand side(L.H.S.) of (5.5)

$$
\begin{aligned}
\text { R.H.S. of }(5.5) & =A_{1} Q_{1221}+B_{1} Q_{1221}+B_{2} Q_{1121}+D_{2} Q_{1211}+D_{1} Q_{1221} \\
& =\left[A_{1}+B_{1}+D_{1}\right] Q_{1221} \\
& =\left\{\frac{3 q(q+1)}{\left(q^{2}-1\right)(1+2 q)}\right\} \frac{(q-1)}{(1+2 q)} \\
& =\frac{3 q}{(1+2 q)^{2}} \\
& =Q_{1221,1} \\
& =\text { L.H.S. of }(5.5)
\end{aligned}
$$

By similar argument it can be shown that (5.6), (5.7), (5.8), (5.9) and (5.10) are true. So, $\mathbb{R}^{4}$ is a $(W Q S)_{n}$ whose scalar curvature is non-zero and non-constant and the manifold $\left(\mathbb{R}^{4}, g\right)$ is neither $Q$-flat nor $Q$-symmetric.

Example 5.2. We define a Riemannian metric on the 4-dimensional real number space $\mathbb{R}^{4}$ by the formula

$$
\begin{equation*}
d s^{2}=g_{i j} d x^{i} d x^{j}=\left(x^{1}\right)^{2}\left[\left(d x^{1}\right)^{2}+\left(d x^{2}\right)^{2}+\left(d x^{3}\right)^{2}+\left(d x^{4}\right)^{2}\right] \tag{5.11}
\end{equation*}
$$

where $i, j=1,2, \ldots, 4$; and $x^{1}>0$.
Then the non-vanishing components of the Christoffel symbols, the curvature tensor and the Ricci tensor are respectively:

$$
\begin{gathered}
\Gamma_{11}^{1}=-\Gamma_{22}^{1}=-\Gamma_{33}^{1}=-\Gamma_{44}^{1}=\Gamma_{12}^{2}=\Gamma_{13}^{3}=\Gamma_{14}^{4}=\frac{1}{x^{1}} \\
R_{1221}=R_{1331}=R_{1441}=-1, \quad R_{2332}=R_{2442}=R_{3443}=1 \\
R_{11}=-\frac{3}{\left(x^{1}\right)^{2}}, \quad R_{22}=R_{33}=R_{44}=\frac{1}{\left(x^{1}\right)^{2}}
\end{gathered}
$$

and the components which can be obtained from these by the symmetric properties. It can be easily shown that the scalar curvature $r$ of this $\left(\mathbb{R}^{4}, g\right)$ is zero. Let us choose the arbitrary scalar function, $\psi=\frac{1}{\left(x^{1}\right)^{3}}$. Therefore the non-vanishing components of the $Q$-curvature tensor and their covariant derivatives are respectively:

$$
\begin{aligned}
Q_{1221} & =Q_{1331}=Q_{1441}=-1-\frac{x^{1}}{3} \\
Q_{2332} & =Q_{2442}=Q_{3443}=1-\frac{x^{1}}{3} \\
Q_{1221,1} & =Q_{1331,1}=Q_{1441,1}=1+\frac{4}{x^{1}} \\
Q_{2332,1} & =Q_{2442,1}=Q_{3443,1}=1-\frac{4}{x^{1}} \\
Q_{1223,3} & =Q_{1224,4}=Q_{1332,2}=Q_{1334,4}=Q_{1442,2}=Q_{1443,3}=-\frac{2}{x^{1}}
\end{aligned}
$$

Let us choose the associated 1-forms as follows:

$$
\begin{align*}
& A_{i}(x)= \begin{cases}\frac{3\left(x^{1}-4\right)}{x^{1}\left(3-x^{1}\right)} & \text { for } i=1 \\
0 & \text { otherwise }\end{cases}  \tag{5.12}\\
& B_{i}(x)= \begin{cases}-\frac{6}{x^{1}\left(3-x^{1}\right)} & \text { for } i=1 \\
0 & \text { otherwise }\end{cases}  \tag{5.13}\\
& D_{i}(x)= \begin{cases}\frac{6\left(2 x^{1}+3\right)}{x^{1}\left\{9-\left(x^{1}\right)^{2}\right\}} & \text { for } i=1 \\
0 & \text { otherwise }\end{cases} \tag{5.14}
\end{align*}
$$

at any point $x \in \mathbb{R}^{4}$. Now the equation (1.8) reduces to the equations

$$
\begin{align*}
& Q_{1221,1}=A_{1} Q_{1221}+B_{1} Q_{1221}+B_{2} Q_{1121}+D_{2} Q_{1211}+D_{1} Q_{1221},  \tag{5.15}\\
& Q_{1331,1}=A_{1} Q_{1331}+B_{1} Q_{1331}+B_{3} Q_{1131}+D_{3} Q_{1311}+D_{1} Q_{1331},  \tag{5.16}\\
& Q_{1441,1}=A_{1} Q_{1441}+B_{1} Q_{1441}+B_{4} Q_{1141}+D_{4} Q_{1411}+D_{1} Q_{1441},  \tag{5.17}\\
& Q_{2332,1}=A_{1} Q_{2332}+B_{2} Q_{1332}+B_{3} Q_{2132}+D_{3} Q_{2312}+D_{2} Q_{2331}  \tag{5.18}\\
& Q_{2442,1}=A_{1} Q_{2442}+B_{2} Q_{1442}+B_{4} Q_{2142}+D_{4} Q_{2412}+D_{2} Q_{2441},  \tag{5.19}\\
& Q_{3443,1}=A_{1} Q_{3443}+B_{3} Q_{1443}+B_{4} Q_{3143}+D_{4} Q_{3413}+D_{3} Q_{3441},  \tag{5.20}\\
& Q_{1223,3}=A_{3} Q_{1223}+B_{1} Q_{3223}+B_{2} Q_{1323}+D_{2} Q_{1233}+D_{3} Q_{1223},  \tag{5.21}\\
& Q_{1224,4}=A_{4} Q_{1224}+B_{1} Q_{4224}+B_{2} Q_{1424}+D_{2} Q_{1244}+D_{4} Q_{1224},  \tag{5.22}\\
& Q_{1332,2}=A_{2} Q_{1332}+B_{1} Q_{2332}+B_{3} Q_{1232}+D_{3} Q_{1322}+D_{2} Q_{1332},  \tag{5.23}\\
& Q_{1334,4}=A_{4} Q_{1334}+B_{1} Q_{4334}+B_{3} Q_{1434}+D_{3} Q_{1344}+D_{4} Q_{1334},  \tag{5.24}\\
& Q_{1442,2}=A_{2} Q_{1442}+B_{1} Q_{2442}+B_{4} Q_{1242}+D_{4} Q_{1422}+D_{2} Q_{1442},  \tag{5.25}\\
& Q_{1443,3}=A_{3} Q_{1443}+B_{1} Q_{3443}+B_{4} Q_{1343}+D_{4} Q_{1433}+D_{3} Q_{1443}, \tag{5.26}
\end{align*}
$$

since, for the other cases (1.8) holds trivially.
By (5.12), (5.13) and (5.14) we get the following relation for the right hand side(R.H.S.) and the left hand side(L.H.S.) of (5.15)

$$
\begin{aligned}
\text { R.H.S. of }(5.15) & =A_{1} Q_{1221}+B_{1} Q_{1221}+B_{2} Q_{1121}+D_{2} Q_{1211}+D_{1} Q_{1221} \\
& =\left[A_{1}+B_{1}+D_{1}\right] Q_{1221} \\
& =\left\{-\frac{3\left(x^{1}+4\right)}{x^{1}\left(3+x^{1}\right)}\right\}\left[-1-\frac{x^{1}}{3}\right] \\
& =\left\{-\frac{3\left(x^{1}+4\right)}{x^{1}\left(3+x^{1}\right)}\right\}\left[-\frac{\left(3+x^{1}\right)}{3}\right] \\
& =\frac{\left(x^{1}+4\right)}{x^{1}} \\
& =1+\frac{4}{x^{1}} \\
& =Q_{1221,1} \\
& =\text { L.H.S. of }(5.15)
\end{aligned}
$$

By similar argument it can be shown that the relations from (5.16) to (5.26) are true. So, $\mathbb{R}^{4}$ is a $(W Q S)_{n}$ whose scalar curvature is zero and the manifold $\left(\mathbb{R}^{4}, g\right)$ is neither $Q$-flat nor $Q$-symmetric.

## References

[1] Adati, T. and Miyazawa, T., On a Riemannian space with recurrent conformal curvature, Tensor (N.S.), 18 (1967), 348-354.
[2] Bejan, C. L. and Crasmareanu, M., Weakly-symmetry of the Sasakian lifts on the tangent bundles, Publ. Math. Debrecen, 83 (2013), 63-69.
[3] Binh, T. Q., On weakly symmetric Riemannian spaces, Publ. Math. Debrecen 42 (1993), 103-107.
[4] Cartan, E., Sur une classes remarquable d'espaces de Riemannian, Bull. Soc. Math. France, 54 (1926), 214-264.
[5] Chaki, M. C. and Gupta,B., On conformally symmetric spaces, Indian J. Math., 5 (1963), 113-295.
[6] Chaкı, M. C., On pseudo symmetric manifolds, Ann. St. Univ. "Al I Cuza" Iasi, 33 (1987), 53-58.
[7] De, U. C., On weakly symmetric structures on Riemannian manifolds, Facta Univ. Ser. Mech. Automat. Control Robot., 3 (14) (2003) 805-819.
[8] De, U. C. and Bandyopadhyay, S., On weakly symmetric spaces, Publ. Math. Debrecen, 54 (1999), 377-381.
[9] De, U. C. and Bandyopadhyay, S., On weakly symmetric spaces, Acta Math. Hungarica, 83 (2000), 205-212.
[10] De, U. C. and Pal, P., On some classes of almost pseudo Ricci symmetric manifolds, Publ. Math. Debrecen, 83 (2013), 207-225.
[11] De, U. C. and Sengupta, J., On a weakly symmetric Riemannian manifold admitting a special type of semi-symmetric metric connection, Novi Sad. J. Math., 29 (1999), 89-95.
[12] Gray, A., Einstein-like manifolds which are not Einstein, Geom. Dedicata, 7 (1978), 259-280.
[13] Hui, S. K., On weakly $W_{3}$-symmetric manifolds, Acta Univ. Palacki. Olomuc. Fac. rer. nat., Mathematica, 50 (2011), 53-71.
[14] Mantica, C. A. and Molinari, L. G., Weakly Z-Symmetric manifolds, Acta Math. Hungarica, 135 (2012), 80-96.
[15] Mantica, C. A. and Y. J. Suh, Pseudo-Z-symmetric Riemannian manifolds with harmonic curvature tensors, Int. J. Geom. Meth. Mod. Phys., 9 (2012), 1250004.
[16] Mantica, C. A. and Y. J. Suh, Recurrent $Z$ forms on Riemannian and Kaehler manifolds, Int. J. Geom. Meth. Mod. Phys., 9 (2012), 1250059.
[17] Mantica, C. A. and Y. J. Suh, Pseudo- $Q$-symmetric Riemannian manifolds, Int. J. Geom. Meth. Mod. Phys., 10 (2013), 1350013.
[18] O'Neill, B., Semi-Riemannian Geometry with Applications to Relativity, Academic Press, New York-London, 1983.
[19] Ozen, F. and Altay, S., Weakly and pseudo-symmetric Riemannian spaces, Indian J. Pure appl. Math., 33 (2002), 1477-1488.
[20] Ozen, F. and Altay, S., On weakly and pseudo-concircular symmetric structures on a Riemannian manifold, Acta Univ. Palack. Olomuc., Fac. Rerum. natur. Math., 47 (2008) 129-138.
[21] Prvanović, M., On weakly symmetric Riemannian manifolds, Publ. Math. Debrecen, 46 (1995), 19-25.
[22] Schouten, J. A., Ricci-Calculus, An introduction to Tensor Analysis and its Geometrical Applications, Springer-Verlag, Berlin-Göttingen-Heidelberg, 1954.
[23] Tamássy, L. and Binh, T. Q., On weakly symmetric and weakly projectively symmetric Riemannian manifolds, Colloq. Math. Soc. Janos Bolyai, 56 (1989), 663-670.
[24] Tamássy, L. and Binh, T. Q., On weak symmetries of Einstein and Sasakian manifolds, Tensor, (N.S.), 53 (1993), 140-148.
[25] Walker, A. G., On Ruse's space of recurrent curvature, Proc. of London Math. Soc., 52 (1950), 36-54.
[26] Yano, K. and Kon, M., Structures on manifolds, World scientific, Singapore, 1984, 418-421.

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# SUFFICIENT CONDITIONS FOR SUPRA $\beta$-CONTINUITY 

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(Received September 27, 2013)


#### Abstract

We give two sufficient conditions for functions to be supra $\beta$ continuous. A notion of a new generalized derivative is introduced. The methods presented here can be used also for other kinds of generalized continuity.


## 1. Introduction

In this paper we give two sufficient conditions for functions to be supra $\beta$ continuous. One of the ways how we can see, that an object (e.g. a function) has some nice property is to compare it with another object with the same property. We do this kind of a comparison more often than we think. For example a differentiable real function is continuous, because the identity function id from $\mathbb{R}$ to $\mathbb{R}$ is continuous. Indeed - when differentiating, we are "comparing" small differences of the type $f(x+h)-f(x)$ and $(x+h)-(x)=i d(x+h)-i d(x)$ by calculating their quotient. And - in a way - every differentiable function $f$ will "inherit" the continuity of the identity function. Two sufficient conditions, presented in this paper, are based on this idea of comparison.

The classical notion of relative derivative replaces the identity function $i d: \mathbb{R} \rightarrow \mathbb{R}$ by a function $g: \mathbb{R} \rightarrow \mathbb{R}$ (e.g. in [1] or [8]). In this paper we are going to define a new notion of a generalized relative derivative.

[^4]
## 2. Generalized Derivative

In this paper we will use these notions: a net of points, a limit of a net (see e. g. [2] or [4]). When we say "a field", we mean the spaces $\mathbb{R}$ or $\mathbb{C}$. Now we define the notion of generalized derivative.

Definition 2.1. Let $(X, T)$ be a topological space, let $A \subset X$. Let $a$ be a limit point of $A$. Let $Y$ be a linear topological space defined over a field $F$. Let $f: X \rightarrow$ $\rightarrow Y$ and $g: X \rightarrow F$ be functions. We say that $f$ has a $g$-derivative (or generalized derivative with respect to $g$ ) at $a$ on $A$ if there exists an element $l$ of $Y$ such that

$$
l=\lim _{x \in A, x \rightarrow a}(f(x)-f(a))(g(x)-g(a))^{-1} .
$$

We denote such a limit by the symbol ${ }_{g / A} f^{\prime}(a)$. If $A=X$ we write $g f^{\prime}(a)$ instead of ${ }_{g / X} f^{\prime}(a)$.

In this article the set $A$ will be always the whole space $X$, because we are interested in global continuity of functions. $F$ will be always $\mathbb{R}$. Moreover, since we want to compare the behavior of two functions, we will use a special kind of topology on $X$.
Remark 2.2. We will use expressions $\frac{f(x)-f(a)}{g(x)-g(a)}$ instead of expressions of the type $(f(x)-f(a))(g(x)-g(a))^{-1}$.

This is our definition when using nets: $g / A f^{\prime}(a)$ exists if there exists a vector $l \in Y$ such that for every net $\left\{x_{\gamma}\right\}_{\gamma \in \Gamma}$ of points of $A-\{a\}$ converging to $a$ the net $\left\{\frac{f\left(x_{\gamma}\right)-f(a)}{g\left(x_{\gamma}\right)-g(a)}\right\}_{\gamma \in \Gamma}$ converges to $l$. We note this fact by writing $\lim _{\gamma \in \Gamma} \frac{f\left(x_{\gamma}\right)-f(a)}{g\left(x_{\gamma}\right)-g(a)}=l$.

It is easy to see that this new kind of derivative is a linear operator. When $X=A=Y=F=\mathbb{R}$ and $g(x) \equiv x$ we obtain the classical definition of the derivative. In general a function $f$ can have a $g$-derivative also when $f$ and $g$ are not continuous.

## 3. Sufficient Conditions for supra $\beta$-continuity

Be $X$ a set. A subcollection $\mu \subset 2^{X}$ is called a supra topology [9] on $X$ if $X \in \mu$ and $\mu$ is closed under arbitrary union. ( $X, \mu$ ) is called a supra topological space. The elements of $\mu$ are called supra open in $(X, \mu)$. The complement of a supra open set is called a supra closed set. The supra interior of a set $A \subset X$,
denoted by $\operatorname{Int}^{\mu}(A)$, is the union of the supra open sets included in $A$. The supra closure of a set $A \subset X$, denoted by $\mathrm{Cl}^{\mu}(A)$ is the intersection of the supra closed sets including $A$. A set $A$ is called supra $\alpha$-open [9] if $A \subseteq \operatorname{Int}^{\mu}\left(\mathrm{Cl}^{\mu}\left(\operatorname{Int}^{\mu}(A)\right)\right)$. In [6] Jafari and Tahiliani defined the notion of a supra $\beta$-open set. A set $A$ is supra $\beta$-open if $A \subseteq \mathrm{Cl}^{\mu}\left(\operatorname{Int}^{\mu}\left(\mathrm{Cl}^{\mu}(A)\right)\right)$. If $(X, T)$ is a topological space, the supra topology $\mu$ on $X$ is associated with the topology $T$ if $T \subset \mu$. Let $(X, T)$ and $(Y, S)$ be two topological spaces and $\mu$ be an associated supra topology with $T$. A map $f:(X, T) \rightarrow(Y, S)$ is called supra $\beta$-continuous [6] (supra $\alpha$-continuous [3]) if the inverse image of each open set in $Y$ is supra $\beta$-open in $X$ (is supra $\alpha$-open in $X$ ).

In our theorems we are going to deal with the supra $\beta$-continuity.
In our first theorem we are going to use a concrete type of the generalized relative derivative. Let us define it.

Definition 3.1. Let $(X, T)$ be a topological space. Let $f: X \rightarrow \mathbb{R}$ and $g: X \rightarrow$ $\rightarrow \mathbb{R}$ be functions. Denote $T_{f}=\left\{f^{-1}(V) ; V\right.$ open in $\left.\mathbb{R}\right\}$. We say that $g$ has an $r f$-derivative at a point $x$ from $X$ if there exists a real number $l$ such that

$$
l=\lim _{t \rightarrow x}(g(t)-g(x))(f(t)-f(x))^{-1}
$$

where the limit is taken with respect to the topology $T_{f}$. We denote this by

$$
r f g^{\prime}(x)=l
$$

REmARK 3.2. In other words: we say that $g$ has an $r f$-derivative at a point $x$ if there exists a real number $l$ such that for every net $\left\{x_{\gamma}\right\}_{\gamma \in \Gamma}$ of points of $X-\{x\}$ converging to $x$ in the topology $T_{f}$ the net $\left\{\frac{f\left(x_{\gamma}\right)-f(a)}{g\left(x_{\gamma}\right)-g(a)}\right\}_{\gamma \in \Gamma}$ converges to $l$ in $\mathbb{R}$.

Theorem 3.3. Let $(X, T)$ be a topological space and $\mu$ be an associated supra topology with $T$. Let $f: X \rightarrow \mathbb{R}$ and $g: X \rightarrow \mathbb{R}$ be functions. Let for each $x$ from $X$ there exist a finite ${ }_{r f} g^{\prime}(x)$. Let $f$ be supra $\beta$-continuous on $X$. Then $g$ is supra $\beta$-continuous on $X$ too.

Proof. Denote $T_{f}=\left\{f^{-1}(V) ; V\right.$ open in $\left.\mathbb{R}\right\}$. Since $f$ is supra $\beta$-continuous, we have $T_{f} \subset S B O$ where $S B O$ is the set of all supra $\beta$-open subsets of $X$. Note that the set $T_{f}$ is a topology on $X$. Therefore to prove the supra $\beta$-continuity of $g$ it suffices to show that $g:\left(X, T_{f}\right) \rightarrow \mathbb{R}$ is continuous.

Take an arbitrary point $x$ in $X$. To prove the continuity of $g$ at $x$ (with respect to the topology $T_{f}$ ) it suffices to prove
$(*)$ for every $\varepsilon$ positive there exists an open neighborhood $O \in T_{f}$ of $x$ such that for all $t$ from $O|g(t)-g(x)|<\varepsilon$.

We will prove $(*)$ by contradiction. Suppose $(*)$ is not true. Then for every neighborhood $V$ of $x$ there exists a point $x_{V}$ such that $\left|g\left(x_{V}\right)-g(x)\right| \geq \varepsilon$. This is equivalent with the fact that there exists a net $\left\{x_{\gamma}\right\}_{\gamma \in \Gamma}$ converging to $x$ such, that for all $\gamma$ from $\Gamma$ the inequality $\left|g\left(x_{\gamma}\right)-g(x)\right| \geq \varepsilon$ holds.

But since ${ }_{r f} g^{\prime}(x)$ exists, we have

$$
\begin{aligned}
& \lim _{\gamma \in \Gamma}\left(g\left(x_{\gamma}\right)-g(x)\right)= \\
& =\lim _{\gamma \in \Gamma} \frac{\left(g\left(x_{\gamma}\right)-g(x)\right)}{f\left(x_{\gamma}\right)-f(x)} \cdot \\
& =\left(f\left(x_{\gamma}\right)-f(x)\right)= \\
&
\end{aligned} \quad={ }_{{ }_{r f} g^{\prime}(x) \cdot \lim _{\gamma \in \Gamma}\left(f\left(x_{\gamma}\right)-f(x)\right)=0 .} \quad \begin{aligned}
\end{aligned}
$$

Of course, $f:\left(X, T_{f}\right) \rightarrow \mathbb{R}$ is continuous at $x$, so we have

$$
\lim _{\gamma \in \Gamma}\left(f\left(x_{\gamma}\right)=f(x)\right)
$$

The equalities above imply that $\lim _{\gamma \in \Gamma}\left(g\left(x_{\gamma}\right)-g(x)\right)=0$. We have obtained a contradiction.

We have just proved that $(*)$ is true. Since $x$ was na arbitrary point and $\varepsilon$ was an arbitrary positive number, we have just proved that the function $g:\left(X, T_{f}\right) \rightarrow$ $\rightarrow \mathbb{R}$ is continuous. So, as we have mentioned above, $g:(X, T) \rightarrow \mathbb{R}$ is supra $\beta$-continuous. This ends the proof.
Theorem 3.4. Let $(X, T)$ be a topological space and $\mu$ be an associated supra topology with $T$. Let $(Y, d)$ and $(Z, \varrho)$ be metric spaces. Let $f: X \rightarrow Y$ and $g: X \rightarrow Z$ be functions. Let $f$ be supra $\beta$-continuous on $X$. Let there exist a positive real number $K$ such that the following is true:

$$
\begin{equation*}
\forall x, y \in X \quad \varrho(g(x), g(y)) \leq K \cdot d(f(x), f(y)) \tag{**}
\end{equation*}
$$

Then $g$ is supra $\beta$-continuous on $X$ too.
Proof. The set $T_{f}=\left\{f^{-1}(V) ; V\right.$ open in $\left.\mathbb{R}\right\}$. is a subset of all supra $\beta$-open subsets of $X$. Since the set $T_{f}$ is a topology on $X$, to prove the supra $\beta$-continuity of $g$ it suffices to show that $g:\left(X, T_{f}\right) \rightarrow \mathbb{R}$ is continuous.

Take an arbitrary point $x$ in $X$. Take an arbirtrary net $\left\{x_{\gamma}\right\}_{\gamma \in \Gamma}$ converging to $x$. Since $f:\left(X, T_{f}\right) \rightarrow \mathbb{R}$ is continuous, the net $\left\{f\left(x_{\gamma}\right)\right\}_{\gamma \in \Gamma}$ convereges to $f(x)$. This means that

$$
\lim _{\gamma \in \Gamma} K \cdot d\left(f\left(x_{\gamma}\right), f(x)\right)=0
$$

Because of $(* *)$ we obtain that

$$
\lim _{\gamma \in \Gamma} K \cdot \varrho\left(g\left(x_{\gamma}\right), g(x)\right)=0
$$

too. But this is equivalent to the assertion

$$
\left.\lim _{\gamma \in \Gamma} g\left(x_{\gamma}\right)=g(x)\right)
$$

which was to be proved.
Remark 3.5. The methods presented here can be used also for other kinds of generalized continuity. For example for supra continuity or supra $\alpha$-continuity, (for definitions, see [6]), or even quasicontinuity (for a definition see [7]).

## References

[1] V. Aversa and D. Preis, Lusin's theorem for derivatives with respect to a continuous function, Proc. Amer. Math. Soc., 127 (1999), 3229-3235.
[2] K. Deimling, Nonlinear Functional Analysis, Springer-Verlag, Berlin, Heidelberg 1985.
[3] R. Devi, S. Sampathkumar and M. Caldas, On supra $\alpha$-open sets and $s \alpha$ continuous maps, General Mathematics, 16 (2008), 77-84.
[4] R. Engelking, General Topology, PWN Warszaw 1977.
[5] M. Grossman and R. Katz, Non-Newtonian Calculus, 7'th printing, Mathco, Rockport, Massachusetts 1979.
[6] S. Jafari and S. Tahiliani, Supra $\beta$-open sets and supra $\beta$-continuity on topological spaces, Annales Univ. Sci. Budapest, 56 (2013), 71-77.
[7] S. Kempisty, Sur les fonctions quasicontinues, Fund. Math., 19 (1932), 184-197.
[8] G L. Light, An Introductory Note on Relative Derivative and Proportionality, Int. J. Contemp. Math. Sci., Vol 1, No 7 (2006), 327-332.
[9] A. S. Mashour, A. A. Allam, F. S. Mahmoud and F. H. Khedr, On supra topological spaces, Indian J. Pure and Appl. Math., 14 (1983), 502-510.

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# ON UPPER AND LOWER ALMOST $M$-ITERATE CONTINUOUS MULTIFUNCTIONS 

By<br>TAKASHI NOIRI AND VALERIU POPA<br>(Received April 4, 2014)


#### Abstract

We introduce the notion of $m I T$-structures determined by operators mInt and mCl on an $m$-space $\left(X, m_{X}\right)$. By using $m I T$-structures, we introduce and investigate a multifunction $F:\left(X, m_{X}\right) \rightarrow(Y, \sigma)$ called upper/lower almost $m I T$-continuous. As special cases of upper/lower almost $m I T$-continuity, we obtain upper/lower almost $\gamma-M$ continuity [36] and upper/lower almost $\delta$ - $M$-precontinuity [37].


## 1. Introduction

Semi-open sets, preopen sets, $\alpha$-open sets, $\beta$-open sets, $\gamma$-open sets and $\delta$ open sets play an important role in the researches of generalizations of continuity in topological spaces. By using these sets, several authors introduced and studied various types of weak forms of continuity for functions and multifunctions. In 1968, Singal and Singal [33] introduced the notion of almost continuous functions. In 1982, Popa [24] introduced the concepts of upper/lower almost continuous multifunctions. In [9], [21], [25], [29], [31] and other papers, other forms of almost continuous multifunctions are introduced and investigated.

In [26] and [27], the present authors introduced and studied the notions of minimal structures, $m$-spaces, $m$-continuity, $M$-continuity and other notions. In [28], the notion of almost $m$-continuous functions is introduced and studied. Recently, in [23], a unified theory of almost continuity for multifunctions is obtained.

Quite recently, in [15], [16], [17], [18], and [19], Min and Kim introduced the notions of $m$-semi-open sets, $m$-preopen sets, $m$ - $\alpha$-open sets and $m$ - $\beta$-open sets which generalize the notions of semi-open sets, preopen sets, $\alpha$-open sets

[^5]and $\beta$-open sets, respectively. In [6], [7], [9] and [34], the notions of $m$-semiopen sets, $m$-preopen sets, $m$ - $\alpha$-open sets and $m$ - $\beta$-open sets are also introduced and studied. Quite recently, in [36] and [37], upper/lower almost $\gamma-M$ continuous multifunctions and upper/lower almost $\delta-M$-precontinuous multifunctions, respectively, are initiated and studied.

In the present paper, we introduce the notions of iterate $m$-structures and iterate upper/lower almost $m$-continuous multifunctions which generalize the notions of upper/lower almost $\gamma-M$ continuous multifunctions and upper/lower almost $\delta$ - $M$-precontinuous multifunctions. We obtain several characterizations of such multifunctions by generalizing the results established in [36] and [37].

## 2. Preliminaries

Let $(X, \tau)$ be a topological space and $A$ a subset of $X$. The closure of $A$ and the interior of $A$ are denoted by $\mathrm{Cl}(A)$ and $\operatorname{Int}(A)$, respectively. A subset $A$ of $X$ is said to be regular open (resp. regular closed) if $A=\operatorname{Int}(\mathrm{Cl}(A))$ (resp. $A=\mathrm{Cl}(\operatorname{Int}(A))$. We denote by $\mathrm{RO}(X)$ (resp. $\mathrm{RC}(X)$ ) the family of all regular open (resp. regular closed) sets of $X$.

A point $x \in X$ is called a $\delta$-cluster point of a subset $A$ if $\operatorname{Int}(\operatorname{Cl}((U)) \cap A \neq \emptyset$ for every open set $U$ containing $x$. The set of all $\delta$-cluster points of $A$ is called the $\delta$-closure of $A$ and is denoted by $\mathrm{Cl}_{\delta}(A)$. If $A=\mathrm{Cl}_{\delta}(A)$, then $A$ is said to be $\delta$-closed [35]. The complement of a $\delta$-closed set is said to be $\delta$-open. The union of all $\delta$-open sets contained in $A$ is called the $\delta$-interior of $A$ and is denoted by $\operatorname{Int}_{\delta}(A)$.

We recall some generalized open sets in topological spaces.
Definition 2.1. Let $(X, \tau)$ be a topological space. A subset $A$ of $X$ is said to be
(1) $\alpha$-open $[20]$ if $A \subset \operatorname{Int}(\operatorname{Cl}(\operatorname{Int}(A)))$,
(2) semi-open $[11]$ if $A \subset \mathrm{Cl}(\operatorname{Int}(A))$,
(3) preopen $[13]$ if $A \subset \operatorname{Int}(\mathrm{Cl}(A))$,
(4) b-open [5] or $\gamma$-open [3] if $A \subset \operatorname{Int}(\mathrm{Cl}(A)) \cup \mathrm{Cl}(\operatorname{Int}(A))$,
(5) $\beta$-open [1] or semi-preopen [4] if $A \subset \operatorname{Cl}(\operatorname{Int}(\operatorname{Cl}(A)))$.

The family of all $\alpha$-open (resp. semi-open, preopen, $\gamma$-open, $\beta$-open) sets in $(X, \tau)$ is denoted by $\alpha(X)$ (resp. $\mathrm{SO}(X), \mathrm{PO}(X), \gamma(X), \beta(X)$ ).
Definition 2.2. Let $(X, \tau)$ be a topological space. A subset $A$ of $X$ is said to be $\alpha$-closed [14] (resp. semi-closed [8], preclosed [13], $\gamma$-closed [3], $\beta$-closed [1]) if the complement of $A$ is $\alpha$-open (resp. semi-open, preopen, $\gamma$-open, $\beta$-open).

Definition 2.3. Let $(X, \tau)$ be a topological space and $A$ a subset of $X$. The intersection of all $\alpha$-closed (resp. semi-closed, preclosed, $\gamma$-closed, $\beta$-closed) sets of $X$ containing $A$ is called the $\alpha$-closure [14] (resp. semi-closure [8], preclosure [10], $\gamma$-closure [3], $\beta$-closure [2]) of $A$ and is denoted by $\alpha \mathrm{Cl}(A)($ resp. $\mathrm{sCl}(A)$, $\left.{ }_{p C l}(A), \gamma \mathrm{Cl}(A),{ }_{\beta} \mathrm{Cl}(A)\right)$.
Definition 2.4. Let $(X, \tau)$ be a topological space and $A$ a subset of $X$. The union of all $\alpha$-open (resp. semi-open, preopen, $\gamma$-open, $\beta$-open) sets of $X$ contained in $A$ is called the $\alpha$-interior [14] (resp. semi-interior [8], preinterior [10], $\gamma$-interior [3], $\beta$-interior [2]) of $A$ and is denoted by $\alpha \operatorname{Int}(A)(\operatorname{resp} \cdot \operatorname{sInt}(A)$, $\left.\operatorname{pInt}(A), \gamma \operatorname{Int}(A),{ }_{\beta} \operatorname{Int}(A)\right)$.

Throughout the present paper, $(X, \tau)$ and $(Y, \sigma)$ (or simply $X$ and $Y$ ) always denote topological spaces and $F:(X, \tau) \rightarrow(Y, \sigma)$ presents a multivalued function. For a multifunction $F: X \rightarrow Y$, we shall denote the upper and lower inverse of a set $B$ of $Y$ by $F^{+}(B)$ and $F^{-}(B)$, respectively, that is,

$$
F^{+}(B)=\{x \in X: F(x) \subset B\} \quad \text { and } \quad F^{-}(B)=\{x \in X: F(x) \cap B \neq \emptyset\}
$$

DEFINITION 2.5. A multifunction $f:(X, \tau) \rightarrow(Y, \sigma)$ is said to be
(1) upper almost continuous [24] (resp. upper almost quasicontinuous [25], upper almost precontinuous [31], upper almost $\alpha$-continuous [29], upper almost $\beta$-continuous [21], upper almost $\gamma$-continuous [9]) at $x \in X$ if for each open set $V$ of $Y$ containing $F(x)$, there exists an open (resp. semi-open, preopen, $\alpha$-open, $\beta$-open, $\gamma$-open) set $U$ of $X$ containing $x$ such that $F(U) \subset$ $\subset \operatorname{Int}(\mathrm{Cl}(V))$,
(2) lower almost continuous [24] (resp. lower almost quasicontinuous [25], lower almost precontinuous [31], lower almost $\alpha$-continuous [29], lower almost $\beta$-continuous [21], lower almost $\gamma$-continuous [9]) at $x \in X$ if for each open set $V$ of $Y$ such that $F(x) \cap V \neq \emptyset$, there exists an open (resp. semi-open, preopen, $\alpha$-open, $\beta$-open, $\gamma$-open) set $U$ of $X$ containing $x$ such that $F(u) \cap \operatorname{Int}(\mathrm{Cl}(V)) \neq \emptyset$ for each $u \in U$,
(3) upper/lower almost continuous [24] (resp. upper/lower almost quasicontinuous [25], upper/lower almost precontinuous [31], upper/lower almost $\alpha$ continuous [29], upper/lower almost $\beta$-continuous [21], upper/lower almost $\gamma$-continuous [9]) if it has the property at each point $x \in X$.

## 3. $m$-structures and upper/lower almost $m$-continuity

Definition 3.1. Let $X$ be a nonempty set and $\mathcal{P}(X)$ the power set of $X$. A subfamily $m_{X}$ of $\mathcal{P}(X)$ is called a minimal structure (briefly $m$-structure) on $X$ [26], [27] if $\emptyset \in m_{X}$ and $X \in m_{X}$.

By $\left(X, m_{X}\right)$, we denote a nonempty set $X$ with an $m$-structure $m_{X}$ on $X$ and call it an $m$-space. Each member of $m_{X}$ is said to be $m_{X}$-open (briefly $m$-open) and the complement of an $m_{X}$-open set is said to be $m_{X}$-closed (briefly $m$-closed).

Remark 3.1. Let $(X, \tau)$ be a topological space. The families $\tau, \mathrm{SO}(X), \mathrm{PO}(X)$, $\alpha(X), \beta(X)$ and $\gamma(X)$ are all minimal structures on $X$.

Definition 3.2. Let $X$ be a nonempty set and $m_{X}$ an $m$-structure on $X$. For a subset $A$ of $X$, the $m_{X}$-closure of $A$ and the $m_{X}$-interior of $A$ are defined in [12] as follows:
(1) $\operatorname{mCl}(A)=\cap\left\{F: A \subset F, X \backslash F \in m_{X}\right\}$,
(2) $\operatorname{mInt}(A)=\cup\left\{U: U \subset A, U \in m_{X}\right\}$.

Remark 3.2. Let $(X, \tau)$ be a topological space and $A$ a subset of $X$. If $m_{X}=\tau$ (resp. $\mathrm{SO}(X), \mathrm{PO}(X), \alpha(X), \beta(X), \gamma(X)$ ), then we have
(1) $\operatorname{mCl}(A)=\mathrm{Cl}(A)\left(\right.$ resp. $\left.\mathrm{sCl}(A), \operatorname{pCl}(A), \alpha \mathrm{Cl}(A),{ }_{\beta} \mathrm{Cl}(A), \gamma \mathrm{Cl}(A)\right)$,
(2) $\operatorname{mInt}(A)=\operatorname{Int}(A)\left(\right.$ resp. $\left.\operatorname{sInt}(A), \operatorname{pInt}(A), \alpha \operatorname{Int}(A),{ }_{\beta} \operatorname{Int}(A), \gamma \operatorname{Int}(A)\right)$.

Lemma 3.1 (Maki et al. [12]). Let $X$ be a nonempty set and $m_{X}$ a minimal structure on $X$. For subsets $A$ and $B$ of $X$, the following properties hold:
(1) $\mathrm{mCl}(X \backslash A)=X \backslash \operatorname{mInt}(A)$ and $\operatorname{mInt}(X \backslash A)=X \backslash \operatorname{mCl}(A)$,
(2) If $(X \backslash A) \in m_{X}$, then $\operatorname{mCl}(A)=A$ and if $A \in m_{X}$, then $\operatorname{mInt}(A)=A$,
(3) $\mathrm{mCl}(\emptyset)=\emptyset, \mathrm{mCl}(X)=X, \operatorname{mInt}(\emptyset)=\emptyset$ and $\operatorname{mInt}(X)=X$,
(4) If $A \subset B$, then $\operatorname{mCl}(A) \subset \operatorname{mCl}(B)$ and $\operatorname{mInt}(A) \subset \operatorname{mInt}(B)$,
(5) $A \subset \operatorname{mCl}(A)$ and $\operatorname{mint}(A) \subset A$,
(6) $\mathrm{mCl}(\mathrm{mCl}(A))=\mathrm{mCl}(A)$ and $\operatorname{mInt}(\operatorname{mInt}(A))=\operatorname{mInt}(A)$.

Lemma 3.2 (Popa and Noiri [26]). Let $\left(X, m_{X}\right)$ be an $m$-space and $A$ a subset of $X$. Then $x \in \operatorname{mCl}(A)$ if and only if $U \cap A \neq \emptyset$ for each $U \in m_{X}$ containing $x$.
Definition 3.3. A minimal structure $m_{X}$ on a nonempty set $X$ is said to have property $\mathcal{B}[12]$ if the union of any family of subsets belonging to $m_{X}$ belongs to $m_{X}$.

Remark 3.3. If $(X, \tau)$ is a topological space, then $\operatorname{SO}(X), \mathrm{PO}(X), \alpha(X), \gamma(X)$ and $\beta(X)$ have property $\mathcal{B}$.

Lemma 3.3 (Popa and Noiri [30]). Let $X$ be a nonempty set and $m_{X}$ an $m$-structure on $X$ satisfying property $\mathcal{B}$. For a subset $A$ of $X$, the following properties hold:
(1) $A \in m_{X}$ if and only if $\operatorname{mInt}(A)=A$,
(2) $A$ is $m_{X}$-closed if and only if $\mathrm{mCl}(A)=A$,
(3) $\operatorname{mInt}(A) \in m_{X}$ and $\mathrm{mCl}(A)$ is $m_{X}$-closed.

Definition 3.4. Let $\left(X, m_{X}\right)$ be an $m$-space and $(Y, \sigma)$ a topological space. A multifunction $F:\left(X, m_{X}\right) \rightarrow(Y, \sigma)$ is said to be
(1) upper almost m-continuous at $x \in X$ [23] if for each open set $V$ of $Y$ containing $F(x)$, there exists $U \in m_{X}$ containing $x$ such that $F(U) \subset \operatorname{Int}(\mathrm{Cl}(V))$,
(2) lower almost m-continuous at $x \in X$ [23] if for each open set $V$ of $Y$ such that $F(x) \cap V \neq \emptyset$, there exists $U \in m_{X}$ containing $x$ such that $F(u) \cap$ $\cap \operatorname{Int}(\mathrm{Cl}(V)) \neq \emptyset$ for every $u \in U$,
(3) upper/lower almost m-continuous if it has this property at each point $x \in X$.

THEOREM 3.1. For a multifunction $F:\left(X, m_{X}\right) \rightarrow(Y, \sigma)$, the following properties are equivalent:
(1) $F$ is upper almost m-continuous at $x \in X$;
(2) for every open set $V$ of $Y$ with $x \in F^{+}(V), x \in \operatorname{mInt}\left(F^{+}(\operatorname{Int}(\mathrm{Cl}(V)))\right)$;
(3) for every closed set $K$ of $Y$ with $x \in \operatorname{mCl}\left(F^{-}(\mathrm{Cl}(\operatorname{Int}(K)))\right), x \in F^{-}(K)$;
(4) for every subset $B$ of $Y$ with $x \in \operatorname{mCl}\left(F^{-}(\mathrm{Cl}(\operatorname{Int}(\mathrm{Cl}(B))))\right.$,

$$
x \in F^{-}(\mathrm{Cl}(B)) ;
$$

(5) for every subset $B$ of $Y$ with $x \in F^{+}(\operatorname{Int}(B))$,

$$
x \in \operatorname{mInt}\left(F^{+}(\operatorname{Int}(\mathrm{Cl}(\operatorname{Int}(B))))\right)
$$

(6) for every regular open set $V$ of $Y$ with $x \in F^{+}(V), x \in \operatorname{mInt}\left(F^{+}(V)\right)$;
(7) for every regular closed set $K$ of $Y$ with $\left.x \in \operatorname{mCl}\left(F^{-}(K)\right), x \in F^{-}(K)\right)$.

Proof. (1) $\Rightarrow$ (2): Let $V$ be any open set of $Y$ such that $x \in F^{+}(V)$, then $F(x) \subset V$. By (1), there exists $U \in m_{X}$ containing $x$ such that $F(U) \subset$ $\subset \operatorname{Int}(\mathrm{Cl}(V))$. Hence $x \in U \subset F^{+}(\operatorname{Int}(\mathrm{Cl}(V)))$. Since $U \in m_{X}$, we obtain $x \in \operatorname{mInt}\left(F^{+}(\operatorname{Int}(\operatorname{Cl}(V)))\right)$.
(2) $\Rightarrow$ (3): Let $K$ be any closed set of $Y$. Suppose that $x \notin F^{-}(K)$. Then $x \in X \backslash F^{-}(K)=F^{+}(Y \backslash K)$ and $Y \backslash K$ is open in $Y$. By (2) and Lemma 3.1, we have $x \in \operatorname{mInt}\left(F^{+}(\operatorname{Int}(\mathrm{Cl}(Y \backslash K)))\right)=\operatorname{mInt}\left(X \backslash F^{-}(\operatorname{Cl}(\operatorname{Int}(K)))\right)=X \backslash$ $\mathrm{mCl}\left(F^{-}(\mathrm{Cl}(\operatorname{Int}(K)))\right)$. Hence $x \notin \mathrm{mCl}\left(F^{-}(\mathrm{Cl}(\operatorname{Int}(K)))\right)$.
$(3) \Rightarrow$ (4): Let $B$ be any subset of $Y$. Then $\mathrm{Cl}(B)$ is a closed set of $Y$. By (3), we obtain that $x \in \mathrm{mCl}\left(F^{-}(\mathrm{Cl}(\operatorname{Int}(\mathrm{Cl}(B))))\right.$ implies $F^{-}(\mathrm{Cl}(B))$.
(4) $\Rightarrow$ (5): Let $B$ be any subset of $Y$ and $x \in F^{+}(\operatorname{Int}(B))$. Then we have $x \in X \backslash F^{-}(\mathrm{Cl}(Y \backslash B)) \subset X \backslash \mathrm{mCl}\left(F^{-}(\mathrm{Cl}(\operatorname{Int}(\mathrm{Cl}(Y \backslash B))))\right)=X \backslash \mathrm{mCl}\left(F^{-}(Y \backslash\right.$ $\operatorname{Int}(\operatorname{Cl}(\operatorname{Int}(B)))))=\operatorname{mint}\left(F^{+}(\operatorname{Int}(\mathrm{Cl}(\operatorname{Int}(B))))\right)$.
$(5) \Rightarrow(6)$ : Let $V$ be any regular open set of $Y$. By (5), we obtain that $x \in$ $\in F^{+}(V)$ implies $x \in \operatorname{mInt}\left(F^{+}(V)\right)$.
(6) $\Rightarrow$ (7): Let $K$ be a regular closed set of $Y$. Suppose that $x \notin F^{-}(K)$. Then, by (6) and Lemma 3.1, we obtain $x \in X \backslash F^{-}(K)=F^{+}(Y \backslash K)$. Then $x \in \operatorname{mInt}\left(F^{+}(Y \backslash K)\right)=X \backslash \operatorname{mCl}\left(F^{-}(K)\right)$. Hence, $x \notin \mathrm{mCl}\left(F^{-} K\right)$ ).
(7) $\Rightarrow$ (6): Let $V$ be any regular open set of $Y$ and $x \in F^{+}(V)$. Suppose that $x \notin \operatorname{mInt}\left(F^{+}(V)\right)$. Then $x \in X \backslash \operatorname{mInt}\left(F^{+}(V)\right)=\mathrm{mCl}\left(X \backslash F^{+}(V)\right)=$ $\mathrm{mCl}\left(F^{-}(Y \backslash V)\right)$, where $Y \backslash V$ is regular closed. $\mathrm{By}(7), x \in F^{-}(Y \backslash V)=X \backslash F^{+}(V)$. Hence $x \notin F^{+}(V)$.
(6) $\Rightarrow$ (1): Let $V$ be any open set of $Y$ containing $F(x)$. Then $x \in$ $\in F^{+}(V) \subset F^{+}(\operatorname{Int}(\mathrm{Cl}(V)))$. Since $\operatorname{Int}(\mathrm{Cl}(V))$ is regular open, by (6) $x \in$ $\in \operatorname{mInt}\left(F^{+}(\operatorname{Int}(\operatorname{Cl}(V)))\right)$. Hence there exists $U \in m_{X}$ containing $x$ such that $x \in U \subset F^{+}(\operatorname{Int}(\mathrm{Cl}(V)))$. Therefore, $F(U) \subset \operatorname{Int}(\mathrm{Cl}(V))$ and $F$ is upper almost $m$-continuous at $x$.

Theorem 3.2. For a multifunction $F:\left(X, m_{X}\right) \rightarrow(Y, \sigma)$, the following properties are equivalent:
(1) $F$ is lower almost $m$-continuous at $x \in X$;
(2) for every open set $V$ of $Y$ with $x \in F^{-}(V), x \in \operatorname{mInt}\left(F^{-}(\operatorname{Int}(\mathrm{Cl}(V)))\right)$;
(3) for every closed set $K$ of $Y$ with $x \in \operatorname{mCl}\left(F^{+}(\operatorname{Cl}(\operatorname{Int}(K)))\right), x \in F^{+}(K)$;
(4) for every subset $B$ of $Y$ with $x \in \operatorname{mCl}\left(F^{+}(\operatorname{Cl}(\operatorname{Int}(\mathrm{Cl}(B))))\right), x \in$ $\in F^{+}(\mathrm{Cl}(B))$;
(5) for every subset $B$ of $Y$ with $x \in F^{-}(\operatorname{Int}(B))$,

$$
x \in \operatorname{mInt}\left(F^{-}(\operatorname{Int}(\operatorname{Cl}(\operatorname{Int}(B))))\right) ;
$$

(6) for every regular open sets $V$ of $Y$ with $x \in F^{-}(V), x \in \operatorname{mInt}\left(F^{-}(V)\right)$;
(7) for every regular closed set $K$ of $Y$ with $\left.x \in \operatorname{mCl}\left(F^{+}(K)\right), x \in F^{+}(K)\right)$.

Proof. The proof is similar to that of Theorem 3.1.
The following theorems are proved in [23].
Theorem 3.3. For a multifunction $F:\left(X, m_{X}\right) \rightarrow(Y, \sigma)$, the following properties are equivalent:
(1) $F$ is upper almost $m$-continuous;
(2) $F^{+}(V) \subset \operatorname{mInt}\left(F^{+}(\operatorname{Int}(\mathrm{Cl}(V)))\right)$ for every open set $V$ of $Y$;
(3) $\mathrm{mCl}\left(F^{-}(\mathrm{Cl}(\operatorname{Int}(K)))\right) \subset F^{-}(K)$ for every closed set $K$ of $Y$;
(4) $\mathrm{mCl}\left(F^{-}(\mathrm{Cl}(\operatorname{Int}(\mathrm{Cl}(B))))\right) \subset F^{-}(\mathrm{Cl}(B))$ for every subset $B$ of $Y$;
(5) $F^{+}(\operatorname{Int}(B)) \subset \operatorname{mInt}\left(F^{+}(\operatorname{Int}(\mathrm{Cl}(\operatorname{Int}(B))))\right)$ for every subset $B$ of $Y$;
(6) $F^{+}(V)=m \operatorname{Int}\left(F^{+}(V)\right)$ for every regular open set $V$ of $Y$;
(7) $F^{-}(K)=\mathrm{mCl}\left(F^{-}(K)\right)$ for every regular closed set $K$ of $Y$.

Theorem 3.4. For a multifunction $F:\left(X, m_{X}\right) \rightarrow(Y, \sigma)$, the following properties are equivalent:
(1) $F$ is lower almost $m$-continuous;
(2) $F^{-}(V) \subset \operatorname{mInt}\left(F^{-}(\operatorname{Int}(\mathrm{Cl}(V)))\right)$ for every open set $V$ of $Y$;
(3) $\mathrm{mCl}\left(F^{+}(\mathrm{Cl}(\operatorname{Int}(K)))\right) \subset F^{+}(K)$ for every closed set $K$ of $Y$;
(4) $\mathrm{mCl}\left(F^{+}(\mathrm{Cl}(\operatorname{Int}(\mathrm{Cl}(B))))\right) \subset F^{+}(\mathrm{Cl}(B))$ for every subset $B$ of $Y$;
(5) $F^{-}(\operatorname{Int}(B)) \subset \operatorname{mInt}\left(F^{-}(\operatorname{Int}(\mathrm{Cl}(\operatorname{Int}(B))))\right)$ for every subset $B$ of $Y$;
(6) $F^{-}(V)=m \operatorname{Int}\left(F^{-}(V)\right)$ for every regular open set $V$ of $Y$;
(7) $F^{+}(K)=\mathrm{mCl}\left(F^{+}(K)\right)$ for every regular closed set $K$ of $Y$.

Corollary 3.1. For a multifunction $F:\left(X, m_{X}\right) \rightarrow(Y, \sigma)$, where $m_{X}$ has property $\mathcal{B}$, the following properties are equivalent:
(1) $F$ is upper almost $m$-continuous;
(2) $F^{+}(V)$ is $m$-open for every regular open set $V$ of $Y$;
(3) $F^{-}(K)$ is $m$-closed for every regular closed set $K$ of $Y$.

Proof. The proof follows from Theorem 3.3 and Lemma 3.3.
Corollary 3.2. For a multifunction $F:\left(X, m_{X}\right) \rightarrow(Y, \sigma)$, where $m_{X}$ has property $\mathcal{B}$, the following properties are equivalent:
(1) $F$ is lower almost $m$-continuous;
(2) $F^{-}(V)$ is $m$-open for every regular open set $V$ of $Y$;
(3) $F^{+}(K)$ is $m$-closed for every regular closed set $K$ of $Y$.

Proof. The proof follows from Theorem 3.4 and Lemma 3.3.
For a multifunction $F:\left(X, m_{X}\right) \rightarrow(Y, \sigma)$, we define $D_{a m}^{+}(F)$ and $D_{a m}^{-}(F)$ as follows:

$$
\begin{aligned}
& D_{a m}^{+}(F)=\{x \in X: F \text { is not upper almost } m \text {-continuous at } x\}, \\
& D_{a m}^{-}(F)=\{x \in X: F \text { is not lower almost } m \text {-continuous at } x\} .
\end{aligned}
$$

Theorem 3.5. For a multifunction $F:\left(X, m_{X}\right) \rightarrow(Y, \sigma)$, the following equalities hold:

$$
\begin{aligned}
D_{a m}^{+}(F) & =\bigcup_{G \in \sigma}\left\{F^{+}(G) \backslash \operatorname{mInt}\left(F^{+}(\operatorname{Int}(\mathrm{Cl}(G)))\right)\right\}= \\
& =\bigcup_{K \in \mathcal{F}}\left\{\mathrm{mCl}\left(F^{-}(\mathrm{Cl}(\operatorname{Int}(K)))\right) \backslash F^{-}(K)\right\}
\end{aligned}
$$

$$
\begin{aligned}
& =\bigcup_{B \in \mathcal{P}(Y)}\left\{\operatorname{mCl}\left(F^{-}(\mathrm{Cl}(\operatorname{Int}(\mathrm{Cl}(B))))\right) \backslash F^{-}(\mathrm{Cl}(B))\right\}= \\
& =\bigcup_{B \in \mathcal{P}(Y)}\left\{F ^ { + } ( \operatorname { I n t } ( B ) ) \backslash \operatorname { m I n t } \left(F^{+}(\operatorname{Int}(\mathrm{Cl}(\operatorname{Int}(B))\}=\right.\right. \\
& =\bigcup_{V \in R O(Y)}\left\{F^{+}(V) \backslash \operatorname{mInt}\left(F^{+}(V)\right)\right\}= \\
& =\bigcup_{K \in R C(Y)}\left\{\mathrm{mCl}\left(F^{-}(K)\right) \backslash F^{-}(K)\right\}
\end{aligned}
$$

where $\mathcal{F}$ is the family of closed sets of $(Y, \sigma)$.
Proof. We shall show only the first equality since the proofs of other are similar. Let $x \in D_{a m}^{+}(F)$. By Theorem 3.1, there exists an open set $V$ of $Y$ such that $x \in F^{+}(V)$ and $x \notin \operatorname{mInt}\left(F^{+}(\operatorname{Int}(\mathrm{Cl}(V)))\right)$. Hence we have $x \in F^{+}(V) \backslash$ $\operatorname{mInt}\left(F^{+}(\operatorname{Int}(\mathrm{Cl}(V)))\right) \subset \bigcup_{G \in \sigma}\left\{F^{+}(G) \backslash \operatorname{mInt}\left(F^{+}(\operatorname{Int}(\mathrm{Cl}(G)))\right)\right\}$.

Conversely, let $x \in \bigcup_{G \in \sigma}\left\{F^{+}(G) \backslash \operatorname{mInt}\left(F^{+}(\operatorname{Int}(\mathrm{Cl}(G)))\right)\right\}$. Then there exists $V \in \sigma$ such that $x \in F^{+}(V) \backslash \operatorname{mInt}\left(F^{+}(\operatorname{Int}(\mathrm{Cl}(V)))\right)$. By Theorem 3.1, we obtain $x \in D_{a m}^{+}(F)$.

THEOREM 3.6. For a multifunction $F:\left(X, m_{X}\right) \rightarrow(Y, \sigma)$, the following equalities hold:

$$
\begin{aligned}
D_{a m}^{-}(F) & =\bigcup_{G \in \sigma}\left\{F^{-}(G) \backslash \operatorname{mInt}\left(F^{-}(\operatorname{Int}(\operatorname{Cl}(G)))\right)\right\}= \\
& =\bigcup_{K \in \mathcal{F}}\left\{\operatorname{mCl}\left(F^{+}(\mathrm{Cl}(\operatorname{Int}(K)))\right) \backslash F^{+}(K)\right\} \\
& =\bigcup_{B \in \mathcal{P}(Y)}\left\{\operatorname{mCl}\left(F^{+}(\mathrm{Cl}(\operatorname{Int}(\mathrm{Cl}(B))))\right) \backslash F^{+}(\mathrm{Cl}(B))\right\}= \\
& =\bigcup_{B \in \mathcal{P}(Y)}\left\{F^{-}(\operatorname{Int}(B)) \backslash \operatorname{mInt}\left(F^{-}(\operatorname{Int}(\operatorname{Cl}(\operatorname{Int}(B))))\right)\right\}= \\
& =\bigcup_{V \in R O(Y)}\left\{F^{-}(V) \backslash \operatorname{mInt}\left(F^{-}(V)\right)\right\}= \\
& =\bigcup_{K \in R C(Y)}\left\{\operatorname{mCl}\left(F^{+}(K)\right) \backslash F^{+}(K)\right\}
\end{aligned}
$$

where $\mathcal{F}$ is the family of closed sets of $(Y, \sigma)$.
Proof. The proof is similar to that of Theorem 3.5.

## 4. Iterate $m$-structures and iterate almost $m$-continuous multifunctions

Definition 4.1. Let $\left(X, m_{X}\right)$ be an $m$-space. A subset $A$ of $X$ is said to be
(1) $m$ - $\alpha$-open $[16]$ if $A \subset \operatorname{mInt}(\operatorname{mCl}(\operatorname{mInt}(A)))$,
(2) m-semi-open $[15]$ if $A \subset \operatorname{mCl}(\operatorname{mInt}(A))$,
(3) m-preopen [6], [18] if $A \subset \operatorname{mInt}(\operatorname{mCl}(A))$,
(4) $m$ - $\beta$-open $[34]$ if $A \subset \mathrm{mCl}(\operatorname{mInt}(\mathrm{mCl}(A)))$,
(5) $m$ - $\gamma$-open $[36]$ if $A \subset \operatorname{mInt}(\operatorname{mCl}(A)) \cup \mathrm{mCl}(\operatorname{mInt}(A))$,
(6) $m$-regular open (resp. m-regular closed) [37] if $A=\operatorname{mInt}(\mathrm{mCl}(A))$ (resp. $A=\mathrm{mCl}(\operatorname{mInt}(A)))$.

Let $A$ be a subset of an $m$-space $\left(X, m_{X}\right)$. The union of all $m$-regular open sets of $X$ contained in $A$ is called the $m$ - $\delta$-interior of $A$ and is denoted by $m \delta \operatorname{Int}(A)$. A subset $A$ is said to be $m-\delta$-open if $A=m \delta \operatorname{Int}(A)$. The complement of an $m$ -$\delta$-open set is said to be $m-\delta$-closed. The intersection of all $m-\delta$-closed sets of $X$ containing $A$ is called the $m$ - $\delta$-closure of $A$ and is denoted by $m \delta \mathrm{Cl}(A)$. A subset $A$ of $X$ is said to be $m$ - $\delta$-preopen [37] if $A \subset \operatorname{mInt}(m \delta \mathrm{Cl}(A))$. The complement of an $m-\delta$-preopen set is said to be $m$ - $\delta$-preclosed.

The family of all $m$ - $\alpha$-open (resp. $m$-semi-open, $m$-preopen, $m$ - $\beta$-open, $m$ -$\gamma$-open, $m$ - $\delta$-preopen) sets in $\left(X, m_{X}\right)$ is denoted by $m \alpha(X)$ (resp. $m \mathrm{SO}(X)$, $m \mathrm{PO}(X), m \beta(X), m \gamma(X), m \delta \mathrm{PO}(X))$.

Remark 4.1. Similar definitions of $m$-semi-open sets, $m$-preopen sets, $m$ - $\alpha$-open sets, $m$ - $\beta$-open sets are provided in [7], [32] and [34].

Let $\left(X, m_{X}\right)$ be an $m$-space. Then $m \alpha(X), m \mathrm{SO}(X), m \mathrm{PO}(X), m \beta(X)$, $m \gamma(X)$ and $m \delta \mathrm{PO}(X)$ are all minimal structures on $X$ and are determined by iterating operators $\mathrm{mInt}, \mathrm{mCl}$ and $m \delta \mathrm{Cl}$. Hence, they are called $m$-iterate structures and are denoted by $\mathrm{mIT}(X)$ (briefly mIT ).

Remark 4.2. (1) It easily follows from Lemma 3.1(1), (3), (4) that $m \alpha(X)$, $m \mathrm{SO}(X), m \mathrm{PO}(X), m \beta(X), m \gamma(X), m \delta \mathrm{PO}(X)$ are all minimal structures on $X$ with property $\mathcal{B}$. They are also shown in Theorem 3.5 of [15], Theorem 3.4 of [18], Theorem 3.4 of [16], Proposition 3.5 of [36].
(2) Let $\left(X, m_{X}\right)$ be an $m$-space and $\operatorname{mIT}(X)$ an iterate structure on $X$. If $\operatorname{mIT}(X)=m \mathrm{SO}(X)($ resp. $m \mathrm{PO}(X), m \alpha(X), m \beta(X)), m \gamma(X), m \delta \mathrm{PO}(X)$ ), then we obtain the following definitions provided in [15], [19], [36], [37]:
$\operatorname{mITCl}(A)=\operatorname{msCl}(A)($ resp. $\operatorname{mpCl}(A), m \alpha \mathrm{Cl}(A), m \beta \mathrm{Cl}(A), m \gamma \mathrm{Cl}(A)$, $m \delta \mathrm{pCl}(A))$,
$\operatorname{mITInt}(A)=\operatorname{msInt}(A)($ resp. $\operatorname{mpInt}(A), m \alpha \operatorname{Int}(A), m \beta \operatorname{Int}(A)$, $m \gamma \operatorname{Int}(A), m \delta \mathrm{pInt}(A))$.

Remark 4.3. (1) By Lemmas 3.1 and 3.3, we obtain Theorems 3.7 and 3.8 of [18], Theroems 3.8 and 3.9 of [16], Remark 3.10 of [36].
(2) By Lemma 3.2, we obtain Lemma 3.9 of [18] and Theorem 3.10 of [16].

Definition 4.2. Let $\left(X, m_{X}\right)$ be an $m$-space and $(Y, \sigma)$ a topological space. A multifunction $F:\left(X, m_{X}\right) \rightarrow(Y, \sigma)$ is said to be
(1) upper almost $\gamma$-M-continuous [36] (resp. upper almost $\delta$-M-precontinuous [37]) at $x \in X$ if for each open set $V$ of $Y$ containing $F(x)$, there exists $U \in$ $\in m \gamma(X)($ resp. $m \delta \mathrm{PO}(X))$ containing $x$ such that $F(U) \subset \operatorname{Int}(\mathrm{Cl}(V))$,
(2) lower almost $\gamma$-M-continuous [36] (resp. lower almost $\delta$ - $M$-precontinuous [37]) at $x \in X$ if for each open set $V$ of $Y$ such that $F(x) \cap V \neq \emptyset$, there exists $U \in m \gamma(X)($ resp. $m \delta \mathrm{PO}(X))$ containing $x$ such that $F(u) \cap \operatorname{Int}(\mathrm{Cl}(V)) \neq \emptyset$ for every $u \in U$,
(3) upper/lower almost $\gamma$-M-continuous [36] (resp. upper/lower almost $\delta$-Mprecontinuous [37]) if it has this property at each $x \in X$.

Remark 4.4. By Definition 4.2 and Remark 4.2, it follows that a multifunction $F:\left(X, m_{X}\right) \rightarrow(Y, \sigma)$ is upper/lower almost $\gamma$ - $M$-continuous (resp. upper/lower almost $\delta$-M-precontinuous) at $x$ (on $X$ ) if and only if a multifunction $F:(X, m \gamma(X)) \rightarrow(Y, \sigma)$ (resp. $F:(X, m \delta \mathrm{PO}(X)) \rightarrow(Y, \sigma))$ is upper/lower almost $m$-continuous at $x$ (on $X$ ).

DEFINITION 4.3. A multifunction $F:\left(X, m_{X}\right) \rightarrow(Y, \sigma)$ is said to be upper/lower almost mIT-continuous at $x \in X$ (on $X$ ) if $F:(X, \operatorname{mIT}(X)) \rightarrow(Y, \sigma)$ is upper/lower almost $m$-continuous at $x \in X$ (on $X$ ).
Remark 4.5. Let $\left(X, m_{X}\right)$ be a minimal space. If $\operatorname{mIT}(X)=m \mathrm{SO}(X)$ (resp. $m \mathrm{PO}(X), m \alpha(X), m \beta(X), m \gamma(X), m \delta \mathrm{PO}(X))$ and $F:\left(X, m_{X}\right) \rightarrow(Y, \sigma)$ is upper/lower almost $m I T$-continuous, then $F$ is upper/lower almost $m$-semicontinuous (resp. upper/lower almost $m$-precontinuous, upper/lower almost $m$ -$\alpha$-continuous, upper/lower almost $m$ - $\beta$-continuous, upper/lower almost $\gamma-M$ continuous [36], upper/lower almost $\delta$ - $M$-precontinuous [37]).

Remark 4.6. By Definition 4.3, it follows that the study of upper/lower almost $m I T$-continuity is reduced to the study of upper/lower almost $m$-continuity.

Since $\operatorname{mIT}(X)$ has property $\mathcal{B}$, Theorem 4.1 (resp. Theorem 4.2) follows from Theorem 3.3 and Corollary 3.1 (resp. Theorem 3.4 and Corollary 3.2). Theorems 4.3 and 4.4 follow from Theorems 3.1 and 3.2, respectively.

Theorem 4.1. For a multifunction $F:\left(X, m_{X}\right) \rightarrow(Y, \sigma)$, the following properties are equivalent:
(1) $F$ is upper almost mIT-continuous;
(2) $F^{+}(V) \subset \operatorname{mITInt}\left(F^{+}(\operatorname{Int}(\mathrm{Cl}(V)))\right)$ for every open set $V$ of $Y$;
(3) $\operatorname{mITCl}\left(F^{-}(\operatorname{Cl}(\operatorname{Int}(K)))\right) \subset F^{-}(K)$ for every closed set $K$ of $Y$;
(4) $\operatorname{mITCl}\left(F^{-}(\mathrm{Cl}(\operatorname{Int}(\mathrm{Cl}(B))))\right) \subset F^{-}(\mathrm{Cl}(B))$ for every subset $B$ of $Y$;
(5) $F^{+}(\operatorname{Int}(B)) \subset \operatorname{mITInt}\left(F^{+}(\operatorname{Int}(\mathrm{Cl}(\operatorname{Int}(B))))\right)$ for every subset $B$ of $Y$;
(6) $F^{+}(V)$ is $m I T$-open for every regular open set $V$ of $Y$;
(7) $F^{-}(K)$ is $m I T$-closed for every regular closed set $K$ of $Y$.

Theorem 4.2. For a multifunction $F:\left(X, m_{X}\right) \rightarrow(Y, \sigma)$, the following properties are equivalent:
(1) $F$ is lower almost mIT-continuous;
(2) $F^{-}(V) \subset \operatorname{mITInt}\left(F^{-}(\operatorname{Int}(\mathrm{Cl}(V)))\right)$ for every open set $V$ of $Y$;
(3) $\operatorname{mITCl}\left(F^{+}(\mathrm{Cl}(\operatorname{Int}(K)))\right) \subset F^{+}(K)$ for every closed set $K$ of $Y$;
(4) $\mathrm{mITCl}\left(F^{+}(\mathrm{Cl}(\operatorname{Int}(\mathrm{Cl}(B))))\right) \subset F^{+}(\mathrm{Cl}(B))$ for every subset $B$ of $Y$;
(5) $F^{-}(\operatorname{Int}(B)) \subset \operatorname{mITInt}\left(F^{-}(\operatorname{Int}(\mathrm{Cl}(\operatorname{Int}(B))))\right)$ for every subset $B$ of $Y$;
(6) $F^{-}(V)$ is $m I T$-open for every regular open set $V$ of $Y$;
(7) $F^{+}(K)$ is mIT-closed for every regular closed set $K$ of $Y$.

Theorem 4.3. For a multifunction $F:\left(X, m_{X}\right) \rightarrow(Y, \sigma)$, the following properties are equivalent:
(1) $F$ is upper almost mIT-continuous at $x \in X$;
(2) for every open set $V$ of $Y$ with $x \in F^{+}(V), x \in \operatorname{mITInt}\left(F^{+}(\operatorname{Int}(\mathrm{Cl}(V)))\right)$;
(3) for every closed set $K$ of $Y$ with $x \in \operatorname{mITCl}\left(F^{-}(\operatorname{Cl}(\operatorname{Int}(K)))\right), x \in F^{-}(K)$;
(4) for every subset $B$ of $Y$ with $x \in \operatorname{mTCl}\left(F^{-}(\operatorname{Cl}(\operatorname{Int}(\mathrm{Cl}(B))))\right), x \in$ $\in F^{-}(\mathrm{Cl}(B))$;
(5) for every subset $B$ of $Y$ with $x \in F^{+}(\operatorname{Int}(B))$,

$$
x \in \operatorname{mITInt}\left(F^{+}(\operatorname{Int}(\operatorname{Cl}(\operatorname{Int}(B))))\right) ;
$$

(6) for every regular open set $V$ of $Y$ with $x \in F^{+}(V), x \in \operatorname{mITInt}\left(F^{+}(V)\right)$;
(7) for every regular closed set $K$ of $Y$ with $\left.x \in \operatorname{mTCl}\left(F^{-}(K)\right), x \in F^{-}(K)\right)$.

Theorem 4.4. For a multifunction $F:\left(X, m_{X}\right) \rightarrow(Y, \sigma)$, the following properties are equivalent:
(1) $F$ is lower almost mIT-continuous at $x \in X$;
(2) for every open set $V$ of $Y$ with $x \in F^{-}(V), x \in \operatorname{mITInt}\left(F^{-}(\operatorname{Int}(\mathrm{Cl}(V)))\right)$;
(3) for every closed set $K$ of $Y$ with $x \in \operatorname{mITCl}\left(F^{+}(\operatorname{Cl}(\operatorname{Int}(K)))\right), x \in F^{+}(K)$;
(4) for every subset $B$ of $Y$ with $x \in \operatorname{miTCl}\left(F^{+}(\mathrm{Cl}(\operatorname{Int}(\mathrm{Cl}(B))))\right)$, $x \in F^{+}(\mathrm{Cl}(B)) ;$
(5) for every subset $B$ of $Y$ with $x \in F^{-}(\operatorname{Int}(B))$,

$$
x \in \operatorname{mITInt}\left(F^{-}(\operatorname{Int}(\operatorname{Cl}(\operatorname{Int}(B))))\right) ;
$$

(6) for every regular open set $V$ of $Y$ with $x \in F^{-}(V), x \in \operatorname{mITInt}\left(F^{-}(V)\right)$;
(7) for every regular closed set $K$ of $Y$ with $\left.x \in \operatorname{mITCl}\left(F^{+}(K)\right), x \in F^{+}(K)\right)$.

Remark 4.7. (1) If $m I T(X)=m \gamma(X)$ (resp. $m \delta \mathrm{PO}(X)$ ), then by Theorems 4.1 and 4.3 we obtain the results established in Theorem 4.13 (1), (2), (3), (5), (6), (7), (8) of [36] (resp. Theorem 4.2 (1), (2), (3), (5), (6), (7), (8) of [37]),
(2) If $\operatorname{mIT}(X)=m \gamma(X)$ (resp. $m \delta \mathrm{PO}(X)$ ), then by Theorems 4.2 and 4.4 we obtain the results established in Theorem 4.14 (1), (2), (3), (5), (6), (7), (8) of [36] (resp. Theorem 4.4 (1), (2), (3), (5), (6), (7), (8) of [37]).

For a multifunction $F:\left(X, m_{X}\right) \rightarrow(Y, \sigma)$, we define $D_{a m I T}^{+}(F)$ and $D_{a m I T}^{-}(F)$ as follows:

$$
\begin{aligned}
& D_{a m I T}^{+}(F)=\{x \in X: F \text { is not upper almost } m I T \text {-continuous at } x\} \\
& D_{a m I T}^{-}(F)=\{x \in X: F \text { is not lower almost } m I T \text {-continuous at } x\}
\end{aligned}
$$

Theorem 4.5. For a multifunction $F:\left(X, m_{X}\right) \rightarrow(Y, \sigma)$, the following equalities hold:

$$
\begin{aligned}
D_{a m I T}^{+}(F) & =\bigcup_{G \in \sigma}\left\{F^{+}(G) \backslash \operatorname{mITInt}\left(F^{+}(\operatorname{Int}(\operatorname{Cl}(G)))\right)\right\}= \\
& =\bigcup_{K \in \mathcal{F}}\left\{\operatorname{mITCl}\left(F^{-}(\mathrm{Cl}(\operatorname{Int}(K)))\right) \backslash F^{-}(K)\right\} \\
& =\bigcup_{B \in \mathcal{P}(Y)}\left\{\operatorname{mITCl}\left(F^{-}(\operatorname{Cl}(\operatorname{Int}(\operatorname{Cl}(B))))\right) \backslash F^{-}(\mathrm{Cl}(B))\right\}= \\
& =\bigcup_{B \in \mathcal{P}(Y)}\left\{F ^ { + } ( \operatorname { I n t } ( B ) ) \backslash \operatorname { m I T I n t } \left(F^{+}(\operatorname{Int}(\operatorname{Cl}(\operatorname{Int}(B))\}=\right.\right. \\
& =\bigcup_{V \in R O(Y)}\left\{F^{+}(V) \backslash \operatorname{mITInt}\left(F^{+}(V)\right)\right\}= \\
& =\bigcup_{K \in R C(Y)}\left\{\operatorname{mITCl}\left(F^{-}(K)\right) \backslash F^{-}(K)\right\},
\end{aligned}
$$

where $\mathcal{F}$ is the family of closed sets of $(Y, \sigma)$.
Proof. The proof follows from Theorem 3.5.

Theorem 4.6. For a multifunction $F:\left(X, m_{X}\right) \rightarrow(Y, \sigma)$, the following equalities hold:

$$
\begin{aligned}
D_{a m I T}^{-}(F) & =\bigcup_{G \in \sigma}\left\{F^{-}(G) \backslash \operatorname{mITInt}\left(F^{-}(\operatorname{Int}(\operatorname{Cl}(G)))\right)\right\} \\
& =\bigcup_{K \in \mathcal{F}}\left\{\operatorname{mITCl}\left(F^{+}(\mathrm{Cl}(\operatorname{Int}(K)))\right) \backslash F^{+}(K)\right\} \\
& =\bigcup_{B \in \mathcal{P}(Y)}\left\{\operatorname{mITCl}\left(F^{+}(\mathrm{Cl}(\operatorname{Int}(\operatorname{Cl}(B))))\right) \backslash F^{+}(\mathrm{Cl}(B))\right\} \\
& =\bigcup_{B \in \mathcal{P}(Y)}\left\{F ^ { - } ( \operatorname { I n t } ( B ) ) \backslash \operatorname { m I T I n t } \left(F^{-}(\operatorname{Int}(\mathrm{Cl}(\operatorname{Int}(B))\}\right.\right. \\
& =\bigcup_{V \in R O(Y)}\left\{F^{-}(V) \backslash \operatorname{mITInt}\left(F^{-}(V)\right)\right\} \\
& =\bigcup_{K \in R C(Y)}\left\{\operatorname{mITCl}\left(F^{+}(K)\right) \backslash F^{+}(K)\right\}
\end{aligned}
$$

where $\mathcal{F}$ is the family of closed sets of $(Y, \sigma)$.
Proof. The proof follows from Theorem 3.6.
Remark 4.8. Some other characterizations of upper/lower almost mITcontinuous multifunctions are obtained by Theorems 3.7-3.10 and Theorems $4.1-4.5$ of [23].

## References

[1] M. E. Abd El-Monsef, S. N. El-Deeb and R. A. Mahmoud, $\beta$-open sets and $\beta$-continuous mappings, Bull. Fac. Sci. Assiut Univ., 12 (1983), 77-90.
[2] M. E. Abd El-Monsef, R. A. Mahmoud and E. R. Lashin, $\beta$-closure and $\beta$-interior, J. Fac. Ed. Ain Shams Univ., 10 (1986), 235-245.
[3] M. E. Abd El-Monsef and A. A. Nasef, On multifunctions, Chaos, Solitons and Fractals, 12 (2001), 2387-2394.
[4] D. Andrijević, Semi-preopen sets, Mat. Vesnik, 38 (1986), 24-32.
[5] D. Andrijević, On b-open sets, Mat. Vesnik, 48 (1996), 59-64.
[6] C. Boonpok, Almost and weakly $M$-continuous functions in $m$-spaces, Far East J. Math. Sci., 43 (2010), 41-58.
[7] C. Carpintero, E. Rosas and M. Salas, Minimal structures and separation properties, Int. J. Pure Appl. Math., 34 (4) (2007), 473-488.
[8] S. G. Crossley and S. K. Hildebrand, Semi-closure, Texas J. Sci., 22 (1971), 99-112.
[9] E. Ekici and J. H. Park, A weak forms of some types of continuous multifunctions, Filomat, 20 (2006), 13-22.
[10] S. N. El-Deeb, I. A. Hasanein, A. S. Mashhour and T. Noiri, On p-regular spaces, Bull. Math. Soc. Sci. Math. R. S. Roumanie, 27(75) (1983), 311-315.
[11] N. Levine, Semi-open sets and semi-continuity in topological spaces, Amer. Math. Monthly, 70 (1963), 36-41.
[12] H. Maki, C. K. Rao and A. Nagoor Gani, On generalizing semi-open and preopen sets, Pure Appl. Math. Sci., 49 (1999), 17-29.
[13] A. S. Mashhour, M. E. Abd El-Monsef and S. N. El-Deep, On precontinuous and weak precontinuous mappings, Proc. Math. Phys. Soc. Egypt, 53 (1982), 47-53.
[14] A. S. Mashhour, I. A. Hasanein and S. N. El-Deeb, $\alpha$-continuous and $\alpha$-open mappings, Acta Math. Hungar., 41 (1983), 213-218.
[15] W. K. Min, $m$-semiopen sets and $M$-semicontinuous functions on spaces with minimal structures, Honam Math. J., 31 (2009), 239-245.
[16] W. K. Min, $\alpha m$-open sets and $\alpha M$-continuous functions, Commun. Korean Math. Soc., 25 (2) (2010), 251-256.
[17] W. K. Min, On minimal semicontinuous functions, Commun. Korean Math. Soc., 27 (2012), 341-345.
[18] W. K. Min and Y. K. Kim, m-preopen sets and $M$-precontinuity on spaces with minimal structures, Adv. Fuzzy Sets Systems, 4 (2009), 237-245.
[19] W. K. Min and Y. K. Kim, On minimal precontinuous functions, J. Chungcheong Math. Soc., 24 (2009), 667-673.
[20] O. Njåstad, On some classes of nearly open sets, Pacific J. Math., 15 (1965), 961-970.
[21] T. Noiri and V. Popa, On upper and lower almost $\beta$-continuous multifunctions, Acta Math. Hungar., 82 (1999), 57-73.
[22] T. Noiri and V. Popa, A unified theory of weak continuity for multifunctions, Stud. Cerc. St. Ser. Math. Univ. Bacău, 16 (2006), 167-200.
[23] T. Noiri and V. Popa, A unified theory of almost continuity for multifunctions, Sci. Stud. Res. Ser. Math. Inform., Vasile Alecsandri Univ. Bacău, 20 (2010), 185-214.
[24] V. Popa, Almost continuous multifunctions, Mat. Vesnik, 6(9)(34) (1982), 75-84.
[25] V. Popa and T. Noiri, On upper and lower almost quasicontinuous multifunctions, Bull. Inst. Math. Acad. Sinica, 21 (1993), 337-349.
[26] V. Popa and T. Noiri, On $M$-continuous functions, Anal. Univ. "Dunarea de Jos" Galaţi, Ser. Mat. Fiz. Mec. Teor. (2), 18 (23) (2000), 31-41.
[27] V. Popa and T. Noiri, On the definitions of some generalized forms of continuity under minimal conditions, Mem. Fac. Sci. Kochi Univ. Ser. Math., 22 (2001), 9-18.
[28] V. Popa and T. Noiri, On almost $m$-continuous functions, Math. Notae, 40 (1999/2002), 75-94.
[29] V. Popa and T. Noiri, On upper and lower weakly $\alpha$-continuous multifunctions, Novi Sad J. Math., 32 (2002), 7-24.
[30] V. Popa and T. Noiri, A unified theory of weak continuity for functions, Rend Circ. Mat. Palermo (2), 51 (2002), 439-464.
[31] V. Popa, T. Noiri and M. Ganster, On upper and lower almost precontinuous multifunctions, Far East J. Math. Sci. Special Vol., (1997), Part I, 49-68.
[32] E. Rosas, N. Rajesh and C. Carpintero, Some new types of open and closed sets in minimal structures, I, II, Int. Math. Forum, 4 (2009), 2169-2184, 2185-2198.
[33] M. K. Singal and A. R. Singal, Almost continuous mappings, Yokohama Math. J., 16 (1968), 63-73.
[34] L. Vasquez, M. S. Brown and E. Rosas, Functions almost contra-supercontinuity in $m$-spaces, Bol. Soc. Paran. Mat. (3s.), 29 (2011), 15-36.
[35] N. V. Veličкo, H-closed topological spaces, Amer. Math. Transl. (2), 78 (1968), 103-118.
[36] C. Viriyapong, C. Boonpok, N. Viriyapong and P. Pue-on, On upper and lower almost $\gamma-M$ continuous multifunctions, Intern. J. Math. Analysis, 6 (2012), 909-922.
[37] N. Viriyapong, C. Boonpok, C. Viriyapong and P. Pue-on, Almost $\delta$ - $M$ precontinuous multifunctions, Intern. J. Math. Analysis, 6 (2012), 897-907.

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# ON GENERALIZED SIDON SETS WHICH ARE ASYMPTOTIC BASES 

By<br>SÁNDOR Z. KISS<br>(Received May 15, 2014)


#### Abstract

Let $h$ and $k$ be positive integers. We say a set $\mathcal{A}$ of positive integers is an asymptotic basis of order $k$ if every large enough positive integer can be represented as the sum of $k$ terms from $\mathcal{A}$. A set of positive integers $\mathcal{A}$ is called $B_{h}[g]$ set if all positive integers can be represented as the sum of $h$ terms from $\mathcal{A}$ at most $g$ times. In this paper we prove the existence of $B_{h}[g]$ sets which are asymptotic bases of order $k$, if $3 \leq h<k$ by using probabilistic methods.


## 1. Introduction

Let $\mathbb{N}$ denote the set of positive integers. Let $h$ and $k$ be positive integers satisfying $3 \leq h<k$. Let $\mathcal{A} \subset \mathbb{N}$ be an infinite set of positive integers and let $R_{h}(\mathcal{A}, n)$ denote the number of solutions of the equation

$$
\begin{equation*}
a_{1}+a_{2}+\cdots+a_{h}=n, \quad a_{1} \in \mathcal{A}, \ldots, a_{h} \in \mathcal{A}, \quad a_{1} \leq a_{2} \leq \ldots \leq a_{h} \tag{1}
\end{equation*}
$$

where $n \in \mathbb{N}$. A set of positive integers $\mathcal{A}$ is called $B_{h}[g]$ set if for every $n \in \mathbb{N}$, the number of representations of $n$ as the sum of $h$ terms in the form (1) is at most $g$, that is $R_{h}(\mathcal{A}, n) \leq g$. We say a set $\mathcal{A} \subset \mathbb{N}$ is an asymptotic basis of order $k$, if $R_{k}(\mathcal{A}, n)>0$ for all large enough positive integer $n$, i.e., if there exists a positive integer $n_{0}$ such that $R_{k}(\mathcal{A}, n)>0$ for $n>n_{0}$. In [5] and [6] P. Erdős, A. Sárközy and V. T. Sós asked if there exists a Sidon set (or $B_{2}[1]$ set) which is an asymptotic basis of order 3. The problem also appears in [13] (with a typo in it: order 2 is written instead of order 3). It is easy to see that a Sidon set cannot be an asymptotic basis of order 2 (see in [8]). Recently J. M. Deshouillers and A. Plagne in [2] constructed a Sidon set which is an asymptotic basis of order at

[^6]most 7. In [11] I proved the existence of Sidon sets which are asymptotic bases of order 5 by using probabilistic methods. In [12] we improve on this result by proving the existence of a Sidon set which is an asymptotic basis of order 4. It could be also proved that there exists a positive integer $g$ and a $B_{2}[g]$ set which is an asymptotic basis of order 3. As Erdős claimed in [3] without proof: "We proved by the probabilistic method that there is a basis of order three for which the number of solutions of $a_{i}+a_{j}=n$ is $\leq c$ but we do not know that smallest value of $c$ ". In [1] Cilleruelo proved the existence of a $B_{2}[2]$ set which is an asymptotic basis of order 3. In this paper I will prove a similar but more general theorem for $B_{h}[g]$ sets:

THEOREM 1. For every positive integer $h$ and $k$ satisfying $3 \leq h<k$ there exists a positive integer $g$ and a $B_{h}[g]$ set which is an asymptotic basis of order $k$.

Before we prove the above theorem, we give a short survey of the probabilistic method we are working with.

## 2. Probabilistic tools

To prove Theorem 1 we use the probabilistic method due to Erdős and Rényi. There is an excellent summary of this method in the book of Halberstam and Roth [9]. In this paper we denote the probability of an event by $P$, and the expectation of a random variable $\zeta$ by $E(\zeta)$. Let $\Omega$ denote the set of the strictly increasing sequences of positive integers.

Lemma 2. Let

$$
\alpha_{1}, \alpha_{2}, \alpha_{3} \ldots
$$

be real numbers satisfying

$$
0 \leq \alpha_{n} \leq 1 \quad(n=1,2, \ldots)
$$

Then there exists a probability space $(\Omega, X, P)$ with the following two properties:
(i) For every natural number $n$, the event $\mathcal{E}^{(n)}=\{\mathcal{A}: \mathcal{A} \in \Omega, n \in \mathcal{A}\}$ is measurable, and $P\left(\mathcal{E}^{(n)}\right)=\alpha_{n}$.
(ii) The events $\mathcal{E}^{(1)}, \mathcal{E}^{(2)}, \ldots$ are independent.

See Theorem 13. in [9], p. 142. We denote the characteristic function of the event $\mathcal{E}^{(n)}$ by $\varrho(\mathcal{A}, n)$ :

$$
\varrho(\mathcal{A}, n)=\left\{\begin{array}{l}
1, \text { if } n \in \mathcal{A} \\
0, \text { if } n \notin \mathcal{A}
\end{array}\right.
$$

Furthermore, for some $\mathcal{A}=\left\{a_{1}, a_{2}, \ldots\right\} \in \Omega$ we denote the number of solutions of $a_{i_{1}}+a_{i_{2}}+\ldots+a_{i_{h}}=n$ with $a_{i_{1}} \in \mathcal{A}, a_{i_{2}} \in \mathcal{A}, \ldots, a_{i_{h}} \in \mathcal{A}, 1 \leq a_{i_{1}}<$ $<a_{i_{2}} \ldots<a_{i_{h}}<n$ by $r_{h}(n)$. Then

$$
\begin{equation*}
r_{h}(\mathcal{A}, n)=r_{h}(n)=\sum_{\substack{\left(a_{1}, a_{2}, \ldots, a_{h}\right) \in \mathbb{N}^{h} \\ 1 \leq a_{1}<\ldots<a_{h}<n \\ a_{1}+a_{2}+\ldots+a_{h}=n}} \varrho\left(\mathcal{A}, a_{1}\right) \varrho\left(\mathcal{A}, a_{2}\right) \ldots \varrho\left(\mathcal{A}, a_{h}\right) \tag{2}
\end{equation*}
$$

Let $r_{h}^{*}(n)$ denote the number of those representations of $n$ in the form (1) in which there are at least two equal terms. Thus we have

$$
\begin{equation*}
R_{h}(\mathcal{A}, n)=R_{h}(n)=r_{h}(n)+r_{h}^{*}(n) \tag{3}
\end{equation*}
$$

It is easy to see from (2) that $r_{h}(n)$ is the sum of random variables. However for $h>2$ these variables are not independent because the same $\varrho\left(\mathcal{A}, a_{i}\right)$ may appear in many terms. To overcome this problem we need the following inequality due to S. Janson [7], [10], [14] which plays an important role in our proof.

Consider a set $\left\{t_{i}\right\}_{i \in Q}$ of independent random indicator variables and for an index set $\Gamma$ a family $\{Q(\gamma)\}_{\gamma \in \Gamma}$ of subsets of the index set $Q$, and define $I_{\gamma}=\prod_{i \in Q(\gamma)} t_{i}$ and $N=\sum_{\gamma \in \Gamma} I_{\gamma}$. (In other words $N$ counts the number of the given sets $\{Q(\gamma)\}$ that are contained in the random set $\left\{i \in Q: t_{i}=1\right\}$.) Let us write $\gamma \sim \delta$ if $Q(\gamma) \cap Q(\delta) \neq \emptyset$ but $\gamma \neq \delta$, and define

$$
\begin{gathered}
p_{\gamma}=E\left(I_{\gamma}\right) \\
\lambda=E(N)=\sum_{\gamma} p_{\gamma} \\
\Delta=\frac{1}{\lambda} \sum_{\gamma \sim \delta} E\left(I_{\gamma} I_{\delta}\right)
\end{gathered}
$$

Lemma 3. (Janson) With notations as above, if $0 \leq \varepsilon \leq 1$, then

$$
P(N \leq(1-\varepsilon) \lambda) \leq \exp \left(-\frac{1}{2(1+\Delta)} \varepsilon^{2} \lambda\right)
$$

In the proof of Theorem 1 we use the following lemma:
Lemma 4. (Borel--Cantelli) Let $X_{1}, X_{2}, \ldots$ be a sequence of events in a probability space. If

$$
\sum_{j=1}^{+\infty} P\left(X_{j}\right)<\infty
$$

then with probability 1 , at most a finite number of the events $X_{j}$ can occur.
See [9], p. 135. We also need the following lemma due to Erdős and Tetali:

Lemma 5. Let $Y_{1}, Y_{2}, \ldots$ be a sequence of events in a probability space. If $\sum_{i} P\left(Y_{i}\right) \leq \mu$, and $\kappa$ is a positive integer then

$$
\sum_{\substack{\left(Y_{1}, \ldots, Y_{\kappa}\right) \\ \text { independent }}} P\left(Y_{1} \cap \ldots \cap Y_{\kappa}\right) \leq \mu^{\kappa} / \kappa!
$$

See [7] for the proof. (We say the events $W_{1}, \ldots, W_{n}$ are independent if for all subsets $I \subseteq\{1, \ldots, n\}, P\left(\cap_{i \in I} W_{i}\right)=\prod_{i \in I} P\left(W_{i}\right)$.) Finally we need the following combinatorial result due to Erdős and Rado, see [4]. Let $r$ be a positive integer, $r \geq 3$. A collection of sets $L_{1}, L_{2}, \ldots, L_{r}$ is said to form a Delta - system if the sets have pairwise the same intersection.

Lemma 6. If $H$ is a collection of sets of size at most $m$ and $|H|>(r-1)^{m} m$ ! then $H$ contains $r$ sets forming a Delta - system.

## 3. Proof of Theorem 1

Let $k$ be fixed. Let $\alpha$ be any real number satisfying $\frac{1}{k}<\alpha<\frac{3}{3 k-1}$. Define the sequence $\alpha_{n}$ in Lemma 1 by

$$
\alpha_{n}=\frac{1}{n^{1-\alpha}},
$$

so that $P(\{\mathcal{A}: \mathcal{A} \in \Omega, n \in \mathcal{A}\})=\frac{1}{n^{1-\alpha}}$. The proof of Theorem 1 consists of two parts. In the first part we prove similarly as in [7] that with probability $1, \mathcal{A}$ is an asymptotic basis of order $k$, i.e., with probability $1, R_{k}(n)>0$ if $n$ is large enough. In the second part we show that with probability $1, \mathcal{A}$ is a $B_{h}[g]$ set.

Let $T_{1}=\left\{a_{1}, \ldots, a_{k}\right\}, T_{2}=\left\{b_{1}, \ldots, b_{k}\right\}$, two different representations of $n$, that is $T_{1} \neq T_{2}, T_{1}, T_{2} \subset \mathcal{A}$ and

$$
a_{1}+\ldots+a_{k}=b_{1}+\ldots+b_{k}=n .
$$

We say $T_{1}$ and $T_{2}$ are disjoint if they share no element in common. To prove that $\mathcal{A}$ is an asymptotic basis of order $k$ we apply Lemma 2 . We use the theorem with $Q=\mathbb{N}$, and $t_{i}$ is $\varrho(\mathcal{A}, i)$. For a fixed $n$ the sets $\{Q(\gamma)\}_{\gamma \in \Gamma}$ denote all the representations of $n$ as the sum of $k$ distinct positive integers, i.e.,

$$
\{Q(\gamma)\}_{\gamma \in \Gamma}=\left\{\left(a_{1}, \ldots, a_{k}\right): a_{1}+\ldots+a_{k}=n, 1 \leq a_{1}<\ldots<a_{k}<n\right\} .
$$

Thus $I_{\gamma}=\prod_{a_{i} \in Q(\gamma)} \varrho\left(a_{i}, \mathcal{A}\right)$. In other words $I_{\gamma}$ is the indicator variable that $Q(\gamma)$ i.e., a representation of $n$ as the sum of $k$ terms is in $\mathcal{A}$. Then it is clear that

$$
\begin{gathered}
N=\sum_{\gamma \in \Gamma} I_{\gamma}=\sum_{\gamma \in \Gamma} \prod_{a_{i} \in Q(\gamma)} \varrho\left(a_{i}, \mathcal{A}\right)= \\
=\sum_{\substack{\left(a_{1}, a_{2}, \ldots, a_{k}\right) \in \mathbb{N}^{k} \\
1 \leq a_{1}<\ldots<a_{k}<n \\
a_{1}+\ldots+a_{k}=n}} \varrho\left(\mathcal{A}, a_{1}\right) \varrho\left(\mathcal{A}, a_{2}\right) \ldots \varrho\left(\mathcal{A}, a_{k}\right)=r_{k}(n) .
\end{gathered}
$$

If $Q(\gamma), Q(\delta)$ are two different representations of $n$ as the sum of $k$ terms and $\gamma \neq \delta$, then $\gamma \sim \delta$ implies that they have at least 1 but at most $k-2$ common terms. It is clear that $E\left(I_{\gamma} I_{\delta}\right)=P(\{Q(\gamma) \in \mathcal{A}\} \cap\{Q(\delta) \in \mathcal{A}\})$. To apply Lemma 2 we have to estimate $E\left(r_{k}(n)\right)$ and calculate $\Delta$. First we give lower estimation for $E\left(r_{k}(n)\right)$. Let $a$ be a small positive constant. By $a_{k}<n$, we have

$$
\begin{gather*}
E\left(r_{k}(n)\right)=\sum_{\substack{a_{1}+\ldots+a_{k}=n \\
1 \leq a_{1}<\ldots<a_{k}<n}} P\left(a_{1} \in \mathcal{A}\right) \ldots P\left(a_{k} \in \mathcal{A}\right)=  \tag{4}\\
=\sum_{\substack{a_{1}+\ldots+a_{k}=n \\
1 \leq a_{1}<\ldots<a_{k}<n}} \frac{1}{\left(a_{1} \ldots a_{k}\right)^{1-\alpha}} \geq \sum_{\substack{a_{1}+\ldots+a_{k}=n \\
n^{a} \leq a_{1}<\ldots<a_{k}<n}} \frac{1}{\left(a_{1} \ldots a_{k}\right)^{1-\alpha}}> \\
>\frac{1}{n^{1-\alpha}} \sum_{n^{a}<a_{1}<\frac{n}{k(k-1)}} \frac{1}{a_{1}^{1-\alpha}} \sum_{\frac{n}{k(k-1)}<a_{2}<\frac{2 n}{k(k-1)}}^{a_{2}^{1-\alpha} \ldots \sum_{\frac{(k-2) n}{k(k-1)}<a_{k-1}<\frac{(k-1) n}{k(k-1)}}^{a_{k-1}^{1-\alpha}}=} \\
\left.=\frac{1}{n^{1-\alpha}}\left[\int_{n^{a}}^{\frac{n}{k(k-1)}} \frac{1}{a_{1}^{1-\alpha}}+O(1)\right]\left[\int_{\frac{n}{k(k-1)}}^{\int_{2}^{k(k-1)}} \frac{1}{a_{2}^{1-\alpha}}+O(1)\right] \ldots \int_{\frac{(k-2) n}{k(k-1)}}^{a_{k-1}^{k(k-1)}} \frac{1}{a_{k-\alpha}^{1-\alpha}}+O(1)\right]= \\
=\frac{1}{n^{1-\alpha}}\left[\frac{n^{\alpha}}{[k(k-1)]^{\alpha} \alpha}-\frac{n^{a \alpha}}{\alpha}+O(1)\right]\left[\frac{n^{\alpha}\left(2^{\alpha}-1\right)}{[k(k-1)]^{\alpha} \alpha}+O(1)\right] \ldots \times \\
\times\left[\frac{n^{\alpha}\left((k-2)^{\alpha}-(k-3)^{\alpha}\right)}{[(k-1)(k-2)]^{\alpha} \alpha}+O(1)\right]\left[\frac{n^{\alpha}\left((k-1)^{\alpha}-(k-2)^{\alpha}\right)}{[k(k-1)]^{\alpha} \alpha}+O(1)\right]= \\
\\
=\frac{1}{n^{1-\alpha} n^{(k-1) \alpha}(1+o(1)) c_{1}\left(1-n^{\alpha(a-1)}\right)>c_{2} n^{k \alpha-1},}
\end{gather*}
$$

if $n$ is large enough, and $c_{1}, c_{2}$ are constants depending on $\alpha$.
For $1 \leq l \leq k-1$ we denote by $r_{l}(n)$ the number of representations of $n$ as the sum of $l$ distinct numbers from $\mathcal{A}$. Let $E\left(r_{l}(n)\right)=\lambda_{l}(n)$. In the next step we
give upper estimation for $E\left(r_{l}(n)\right)$. By $n / l<a_{l}$, we have
(5)

$$
\begin{gathered}
\lambda_{l}(n)=E\left(r_{l}(n)\right)=\sum_{\substack{a_{1}+a_{2}+\ldots+a_{l}=n \\
1 \leq a_{1}<a_{2}<\ldots<a_{l}<n}} P\left(a_{1} \in \mathcal{A}\right) P\left(a_{2} \in \mathcal{A}\right) \ldots P\left(a_{l} \in \mathcal{A}\right)= \\
=\sum_{\substack{a_{1}+a_{2}+\ldots+a_{l}=n \\
1 \leq a_{1}<a_{2}<\ldots<a_{l}<n}} \frac{1}{\left(a_{1} \ldots a_{l}\right)^{1-\alpha}} \leq \\
\leq n^{-1+\alpha+o(1)} \sum_{\substack{a_{1}+a_{2}+\ldots+a_{l}=n \\
1 \leq a_{1}<a_{2}<\ldots<a_{l}<n}} \frac{1}{\left(a_{1} \ldots a_{l-1}\right)^{1-\alpha}} \leq \\
\leq n^{-1+\alpha+o(1)} \sum_{\substack{1 \leq a_{i} \leq n \\
i=1 \ldots l-1}} \frac{1}{\left(a_{1} \ldots a_{l-1}\right)^{1-\alpha}} \leq \\
\leq n^{-1+\alpha+o(1)}\left(\sum_{1 \leq a_{1} \leq n} \frac{1}{\left.a_{1}^{1-\alpha}\right)^{l-1}}=\right. \\
=n^{-1+\alpha+o(1)}\left(n^{\alpha+o(1))^{l-1}}=n^{-1+l \alpha+o(1)} .\right.
\end{gathered}
$$

Let $Q(i)$ and $Q(j)$ be two different representations of $n$ as the sum of $k$ terms. Let $F_{i}$ denote the event that $Q(i) \subset \mathcal{A}$. The following lemma shows that the above events have low correlation in the following sense:

Lemma 7.

$$
\sum_{i \sim j} P\left(F_{i} \cap F_{j}\right)=o(1) .
$$

Proof. The proof of this lemma is similar to Lemma 11 in [7] and Lemma 5 in [11]. For the sake of completeness I present the proof. Note that $i \sim j$ implies that $Q(i)$ and $Q(j)$ share at least 1 number and at most $k-2$ numbers.

$$
\sum_{i \sim j} P\left(F_{i} \cap F_{j}\right)=\sum_{l=1}^{k-2} \sum_{|Q(i) \cap Q(j)|=l} P\left(F_{i} \cap F_{j}\right)
$$

Consider $Q(i), Q(j)$ such that $|Q(i) \cap Q(j)|=l$. Say,

$$
Q(i)=\left(z_{1}, \ldots, z_{l}, x_{1}, x_{2}, \ldots, x_{k-l}\right)
$$

and

$$
Q(j)=\left(z_{1}, \ldots, z_{l}, y_{1}, y_{2}, \ldots, y_{k-l}\right)
$$

Let $\sum_{i} z_{i}=m$. Then $\sum_{i} x_{i}=\sum_{i} y_{i}=n-m$. Write $P\left(x_{i} \in \mathcal{A}\right)=P\left(x_{i}\right)$. So

$$
\begin{gathered}
\sum_{|Q(i) \cap Q(j)|=l} P\left(F_{i} \cap F_{j}\right)= \\
=\sum_{m} \sum_{\substack{z_{1}+\ldots+z_{l}=m \\
x_{1}+\ldots+x_{k-l}=n-m \\
y_{1}+\ldots+y_{k-l}=n-m}}\left(P\left(z_{1}\right) \ldots P\left(z_{l}\right)\right)\left(P\left(x_{1}\right) \ldots P\left(x_{k-l}\right)\right)\left(P\left(y_{1}\right) \ldots P\left(y_{k-l}\right)\right)= \\
=\sum_{m}\left(\sum_{z_{1}+\ldots+z_{l}=m} P\left(z_{1}\right) \ldots P\left(z_{l}\right)\right)\left(\sum_{x_{1}+\ldots+x_{k-l}=n-m} P\left(x_{1}\right) \ldots P\left(x_{k-l}\right)\right)^{2}= \\
=\sum_{m} \lambda_{l}(m)\left(\lambda_{k-l}(n-m)\right)^{2} .
\end{gathered}
$$

We already made the estimates in (5) that $\lambda_{l}(n)<n^{-1+l \alpha+o(1)}$, for $1 \leq l \leq k-1$. Fix $\varepsilon<1 /(6 k-2)$. Then there exists an $m_{0}$ such that

$$
\lambda_{l}(m)<m^{-1+l \alpha+\varepsilon}
$$

for $m>m_{0}$. Since $m_{0}$ is a constant, $\lambda_{l}(m)<C$, where $C$ is a constant, for $m \leq m_{0}$. We split the above summation in four parts:

$$
\begin{gathered}
\sum_{m} \lambda_{l}(m)\left(\lambda_{k-l}(n-m)\right)^{2}= \\
=\sum_{m \leq m_{0}} \lambda_{l}(m)\left(\lambda_{k-l}(n-m)\right)^{2}+\sum_{m_{0}<m \leq n / 2} \lambda_{l}(m)\left(\lambda_{k-l}(n-m)\right)^{2}+ \\
+\sum_{n / 2<m \leq n-m_{0}} \lambda_{l}(m)\left(\lambda_{k-l}(n-m)\right)^{2}+\sum_{n-m_{0}<m} \lambda_{l}(m)\left(\lambda_{k-l}(n-m)\right)^{2}= \\
=\Delta_{1}+\Delta_{2}+\Delta_{3}+\Delta_{4}
\end{gathered}
$$

First we estimate $\Delta_{1}$ :

$$
\begin{gathered}
\Delta_{1}=\sum_{m \leq m_{0}} \lambda_{l}(m)\left(\lambda_{k-l}(n-m)\right)^{2}< \\
<\left(n^{-1+(k-l) \alpha+o(1)}\right)^{2} \sum_{m \leq m_{0}} C=n^{-2-2 l \alpha+2 k \alpha+o(1)}=o(1)
\end{gathered}
$$

In the next step we estimate $\Delta_{2}$ :

$$
\begin{gathered}
\Delta_{2}=\sum_{m_{0}<m \leq n / 2} \lambda_{l}(m)\left(\lambda_{k-l}(n-m)\right)^{2}< \\
<\left(n^{-1+(k-l) \alpha+o(1)}\right)^{2} \sum_{m_{0}<m \leq n / 2} m^{-1+l \alpha+\varepsilon}= \\
=n^{-2-2 l \alpha+2 k \alpha+o(1)} \sum_{m_{0}<m \leq n / 2} m^{-1+l \alpha+\varepsilon}
\end{gathered}
$$

Now we estimate by integral over the full range:

$$
\begin{gathered}
\Delta_{2}<n^{-2-2 l \alpha+2 k \alpha+o(1)}\left(\int_{0}^{n} m^{-1+l \alpha+\varepsilon} d m+O(1)\right)= \\
=n^{-2-l \alpha+2 k \alpha+o(1)+\varepsilon}=o(1)
\end{gathered}
$$

In the next step we estimate $\Delta_{3}$ :

$$
\begin{gathered}
\Delta_{3}=\sum_{n / 2<m \leq n-m_{0}} \lambda_{l}(m)\left(\lambda_{k-l}(n-m)\right)^{2}< \\
<\left(n^{-1+l \alpha+o(1)}\right) \sum_{n / 2<m \leq n-m_{0}}\left((n-m)^{-1+(k-l) \alpha+\varepsilon}\right)^{2}= \\
=\left(n^{-1+l \alpha+o(1)}\right) \sum_{n / 2<m \leq n-m_{0}}(n-m)^{-2-2 l \alpha+2 k \alpha+2 \varepsilon} .
\end{gathered}
$$

Once again estimating by integral over the full range:

$$
\begin{aligned}
\Delta_{3}<n^{-1+l \alpha+o(1)} & \left(\int_{0}^{n}(n-m)^{-2-2 l \alpha+2 k \alpha+2 \varepsilon} d m+O(1)\right)= \\
= & n^{-2-l \alpha+2 k \alpha+2 \varepsilon+o(1)}=o(1)
\end{aligned}
$$

In the last step we estimate $\Delta_{4}$ :

$$
\begin{gathered}
\Delta_{4}=\sum_{n-m_{0}<m} \lambda_{l}(m)\left(\lambda_{k-l}(n-m)\right)^{2}< \\
<\left(n^{-1+l \alpha+o(1)}\right) \sum_{n-m_{0}<m} C^{2}=n^{-1+l \alpha+o(1)}=o(1) .
\end{gathered}
$$

Thus we have

$$
\sum_{i \sim j} P\left(F_{i} \cap F_{j}\right)=\Delta_{1}+\Delta_{2}+\Delta_{3}+\Delta_{4}=o(1)
$$

The proof of Lemma 6 is completed.

Then it follows from Lemma 2 that for $0 \leq c_{3} \leq 1$ constant, we have

$$
P\left(r_{k}(n) \leq c_{3} \lambda\right) \leq e^{-1 / 2(1+\Delta)\left(1-c_{3}\right)^{2} \lambda} .
$$

It follows from Lemma 6 that $\Delta=o(1)$. Thus in view of (4) it follows from Lemma 2 that

$$
P\left(r_{k}(n) \leq c_{3} E\left(r_{k}(n)\right)\right) \leq e^{-1 / 2(1+o(1))\left(1-c_{3}\right)^{2} c_{2} n^{k \alpha-1}}<e^{-c_{4} \log n}
$$

where $c_{4}$ is a constant. Note that $c_{3}$ can be chosen arbitrarily small, thus if $c_{4}$ is large enough we have

$$
P\left(r_{k}(n) \leq c_{3} E\left(r_{k}(n)\right)\right) \leq n^{-2+o(1)} .
$$

Thus by (4) and the Borel-Cantelli lemma we get that with probability 1 , there exists an $n_{0}=n_{0}(\mathcal{A})$ such that

$$
\begin{equation*}
r_{k}(n)>c_{3} n^{k \alpha-1} \quad \text { for } \quad n>n_{0} . \tag{6}
\end{equation*}
$$

It is clear from (3) that with probability $1, R_{k}(n)>c_{3} n^{k \alpha-1}$ for $n>n_{0}$, thus with probability $1, \mathcal{A}$ is an asymptotic basis of order $k$.

In the next section we will prove that with probability $1, \mathcal{A}$ is a $B_{h}[g]$ set for some $g$ positive integer. For $3 \leq h \leq k-1$, let $f_{h}(n)$ denote the size of a maximal collection of pairwise disjoint representations of $n$ as the sum of $h$ terms from $\mathcal{A}$. We show similarly as in [7] that with probability $1, f_{h}(n)$ is bounded by a constant. Let

$$
\begin{gathered}
\mathcal{B}=\left\{\left(a_{1}, \ldots, a_{h}\right): a_{1}+\ldots+a_{h}=n, a_{1} \in \mathcal{A}, \ldots, a_{h} \in \mathcal{A},\right. \\
\left.1 \leq a_{1}<\ldots<a_{h}<n\right\},
\end{gathered}
$$

and let $H(\mathcal{B})=\{\mathcal{T} \subset \mathcal{B}$ : all the $S \in \mathcal{T}$ are pairwise disjoint $\}$.
It is clear that pairwise disjointness of the sets implies the independence of the associated events, i.e., if $S_{1}$ and $S_{2}$ are pairwise disjoint representations as the sum of $h$ terms, the events $S_{1} \subset \mathcal{A}, S_{2} \subset \mathcal{A}$ are independent. Thus by (5) and Lemma 4 we have

$$
\begin{aligned}
& P\left(f_{h}(n)>2 k\right) \leq P\left(\bigcup_{\substack{\mathcal{T} \subset H(\mathcal{B}) \\
|\mathcal{T}|=2 k+1}} \bigcap_{\substack{S \in \mathcal{T}}} S\right) \leq \sum_{\substack{\mathcal{T} \subset H(\mathcal{B}) \\
|\mathcal{T}|=2 k+1}} P\left(\bigcap_{S \in \mathcal{T}} S\right)= \\
(7) \quad & \sum_{\substack{\left(S_{1}, \ldots, S_{2 k+1}\right) \\
\text { Pairwise disjoint }}} P\left(S_{1} \cap \ldots \cap S_{2 k+1}\right) \leq \frac{1}{(2 k+1)!}\left(E\left(f_{h}(n)\right)\right)^{2 k+1} \leq \\
\leq & \frac{1}{(2 k+1)!}\left(E\left(r_{h}(n)\right)\right)^{2 k+1} \leq \frac{1}{(2 k+1)!} c_{5}^{2 k+1} n^{(2 k+1)(-1+h \alpha+o(1))},
\end{aligned}
$$

where $c_{5}$ is a constant. By $h \leq(k-1)$ it follows that

$$
P\left(f_{h}(n)>2 k\right)<n^{-2+o(1)} .
$$

Thus by the Borel-Cantelli lemma with probability 1, the above assertion implies that almost always for $3 \leq h \leq(k-1)$ there exists $n_{h}$ such that if $n>n_{h}$, then $f_{h}(n) \leq 2 k$. But for any finite $n_{h}$, there are at most a finite number of representations as a sum of $h$ numbers. Therefore, almost always for $3 \leq h \leq(k-1)$ there exists a $C_{h}$ such that for every $n, f_{h}(n)<C_{h}$. Set $c_{\max }=\max _{h}\left\{C_{h}\right\}$. Now we show similarly as in [7] that almost always there exists a constant $c=c(\mathcal{A})$ such that for every $n$,

$$
\begin{equation*}
r_{h}(n)<c . \tag{8}
\end{equation*}
$$

The proof of (8) is purely combinatorial. We show that (whenever every $C_{h}$ exists), for every $n$

$$
\begin{equation*}
r_{h}(n) \leq\left(c_{\max }\right)^{h} h! \tag{9}
\end{equation*}
$$

We prove by contradiction. Suppose (9) is false for some $n=n^{\prime}$, i.e.,

$$
\begin{equation*}
r_{h}\left(n^{\prime}\right)>\left(c_{\max }\right)^{h} h!. \tag{10}
\end{equation*}
$$

We will apply Lemma 5 . Let $H$ be the set of representations of $n^{\prime}$ as the sum of $h$ distinct numbers from $\mathcal{A}$. Clearly $|H|=r_{h}\left(n^{\prime}\right)$, thus by (10) and applying Lemma 5 we get that $H$ contains $c_{\max }+1$ representations of $n^{\prime}$ as the sum of $h$ distinct numbers which form a Delta - system $\left\{S_{1}^{h}, \ldots, S_{c_{\max }+1}^{h}\right\}$. If the common intersection of these sets is empty then this $c_{\max }+1$ set form a family of disjoint $h$ representations of $n^{\prime}$, which contradicts the definition of $c_{\max }$. Otherwise let the common intersection of the system be $\left\{x_{1}, \ldots, x_{q}\right\}$, where $0 \leq q \leq h-2$. If $\sum_{i} x_{i}=m$, then removing the common intersection each set will yield $f_{h-q}\left(n^{\prime}-\right.$ $m) \geq c_{\max }+1$. This is impossible in view of $f_{h}(n)<C_{h}$ and the definition of $c_{\text {max }}$. This proves (9), and in fact, also shows that $r_{h}(n)$ is bounded by a constant, namely $r_{h}(n)<c_{\max }^{h} h$ !.

In the last section we will give an upper estimation for $r_{h}^{*}(n)$. This is almost the same as in the previous paragraph. For the sake of completeness I will present the proof and leave the details to the reader. If we collect the equal terms, we have

$$
\begin{equation*}
u_{1} a_{1}+u_{2} a_{2}+\ldots+u_{t} a_{t}=n \tag{11}
\end{equation*}
$$

where the $u_{i}$ 's are natural numbers, and

$$
\begin{equation*}
u_{1}+u_{2}+\ldots+u_{t}=h \tag{12}
\end{equation*}
$$

Thus $r_{h}^{*}(n)$ denotes the number of representations of $n$ in the form (11), where the $a_{i}$ 's are different. Similarly to the estimate of $r_{h}(n)$, we show that with probability $1, r_{h}^{*}(n)$ is also bounded by a constant. Let $2 \leq t \leq h-1$ be fixed. For a fixed
$u_{1}, \ldots, u_{t}$ denote $s_{t}(n)$ the number of representations of $n$ in the form (11). We show that with probability $1, s_{t}(n)$ is bounded by a constant. (Note that in the previous section we proved the case when all $u_{i}$ 's equal to one, and $t=h$ ). First we will give an upper estimation for $E\left(s_{t}(n)\right)$, with a calculation similar to (5). Using the definition, and $n / k<a_{t}$, we have

$$
\begin{gather*}
E\left(s_{t}(n)\right)=\sum_{\substack{u_{1} a_{1}+u_{2} a_{2}+\ldots+u_{t} a_{t}=n \\
1 \leq a_{1}<a_{2}<\ldots<a_{t}<n}} P\left(a_{1} \in \mathcal{A}\right) P\left(a_{2} \in \mathcal{A}\right) \ldots P\left(a_{t} \in \mathcal{A}\right)= \\
=\sum_{\substack{u_{1} a_{1}+u_{2} a_{2}+\ldots+u_{t} a_{t}=n \\
1 \leq a_{1}<a_{2}<\ldots<a_{t}<n}} \frac{1}{\left(a_{1} \ldots a_{t}\right)^{1-\alpha}} \leq  \tag{13}\\
\leq n^{-1+\alpha+o(1)} \sum_{\substack{u_{1} a_{1}+u_{2} a_{2}+\ldots+u_{t} a_{t}=n \\
1 \leq a_{1}<a_{2}<\ldots<a_{t}<n \\
\left(a_{1} \ldots a_{t-1}\right)^{1+\alpha}}}^{<n^{-1+h \alpha+o(1)} .}
\end{gather*}
$$

Let $s_{t}^{*}(n)$ denote the size of a maximal collection of pairwise disjoint representations in the form (11). The same argument as in (7) shows that almost always there exists a $d_{h}$ constant such that for every large enough $n, s_{t}^{*}(n) \leq d_{h}$. In view of (13), and applying Lemma 4 we have

$$
P\left(s_{t}^{*}(n)>d_{h}\right)<n^{-2+o(1)}
$$

thus by the Borel-Cantelli lemma we get that with probability $1, s_{t}^{*}(n) \leq d_{h}$ if $n$ is large enough. We say that a $m$ - tuple $\left(a_{1}, \ldots, a_{m}\right)(m \leq t)$ is an $m-$ representation of $n$ in the form (11) if there is a permutation $\pi$ of the numbers $\{1,2, \ldots, m\}$ such that $\sum_{i=1}^{m} u_{\pi(i)} a_{i}=n$. In the last step we apply Lemma 5 to prove that $s_{t}(n)$ is bounded by a constant. Let $D=\left(\max _{h}\left\{d_{h}\right\}\right)^{t} t!$. Let $H$ in Lemma 5 be the collection of representations of $n$ in the form (11). Clearly $|H|=s_{t}(n)$. If $s_{t}(n)>D$, and $n$ is sufficiently large then by Lemma 5, $H$ contains a Delta - system with $\max _{h}\left\{d_{h}\right\}+1$ sets. If the intersection of these sets is empty, then they form a family of disjoint $t$-representations in the form (11). Otherwise let the common intersection of the sets be $\left\{y_{1}, \ldots, y_{w}\right\}$, where $1 \leq w \leq t-$ 1. By the pigeon hole principle, there exists a permutation $\pi$ of the numbers $\{1,2, \ldots, t\}$ such that we can find $\max _{h}\left\{d_{h}\right\}+1(h-w)$ - representations of $n^{\prime \prime}=n-\sum_{i=1}^{w} u_{\pi(i)} y_{i}$. These $\max _{h}\left\{d_{h}\right\}+1$ sets are disjoint, thus in both cases we obtain a contradiction. Since there are only finitely many partitions of $h$ in the form (11), we get that with probability $1, r_{h}^{*}(n)$ is bounded by a constant. From (3) we get that with probability $1, R_{h}(n)$ is also bounded by a constant.

## References

[1] J. Cilleruelo, Sidon basis, preprint.
[2] J. M. Deshouillers, A. Plagne, A Sidon basis, Acta Mathematica Hungarica, 123 (2009), 233-238.
[3] P. Erdős, The probability method: Successes and limitations, Journal of Statistical Planning and Inference, 72 (1998), 207-213.
[4] P. Erdős, R. Rado, Intersection theorems for system of sets, Journal of London Mathematical Society, 35 (1960), 85-90.
[5] P. Erdős, A. Sárközy, V. T. Sós, On additive properties of general sequences, Discrete Mathematics, 136 (1994), 75-99.
[6] P. Erdős, A. Sárкözy, V. T. Sós, On sum sets of Sidon sets I., Journal of Number Theory, 47 (1994), 329-347.
[7] P. Erdős, P. Tetali, Representations of integers as the sum of $k$ terms, Random Structures and Algorithms, 1 (1990), 245-261.
[8] G. Grekos, L. Haddad, C. Helou, J. Pihko, Representation functions, Sidon sets and bases, Acta Arithmetica, 130 (2007), 149-156.
[9] H. Halberstam, K. F. Roth, Sequences, Springer-Verlag, New York, 1983.
[10] S. Janson, Poisson approximation for large deviations, Random Structures and Algorithms, 1 (1990), 221-229.
[11] S. Z. Kiss, On Sidon sets which are asymptotic bases, Acta Mathematica Hungarica, 128 (2010), 46-58.
[12] S. Z. Kiss, E. Rozgonyı, Cs. SÁndor, On Sidon sets which are asymptotic bases of order 4, Functiones et Approximatio Commentarii Mathematici, 51 (2014), 393-413.
[13] A. SÁrközy, Unsolved problems in number theory, Periodica Mathematica Hungarica, 42 (2001), 17-35.
[14] T. Tao, V. H. Vu, Additive Combinatorics, Cambridge University Press, 2006.
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ISSN 0524-9007
Address:
MATHEMATICAL INSTITUTE EÖTVÖS LORÁND UNIVERSITY ELTE TTK KARI KÖNYVTÁR MATEMATIKAI TUDOMÁNYÁGI SZAKGYÚJTEMÉNY PÁZMÁNY PÉTER SÉTÁNY 1/c 1117 BUDAPEST, HUNGARY Műszaki szerkesztő:

Fried Katalin PhD
A kiadásért felelős: az Eötvös Loránd Tudományegyetem rektora
A kézirat a nyomdába érkezett: 2014. december.
Készült a $\mathrm{XeT}_{\mathrm{E}} \mathrm{X}$ szedőprogram felhasználásával
az MSZ 5601-59 és 5602-55 szabványok szerint


[^0]:    AMS Subject Classification (2000): 51M10, 51M25; 26B15, 57M25.

    * Work performed under the auspices of the G.N.S.A.G.A. of I.N.d.A.M. (Italy) and the University of Bologna, funds for selected research topics, INTAS (grant 03-51-3663), Fondecyt (grants 7050189,1060378 ) and by the Russian Foundation for the Basic Researches (grant 06-01-00153).

[^1]:    AMS Subject Classification (2000): 54B99, 54C25

    * Work supported by the Carathéodory Programme of the University of Patras.

[^2]:    AMS Subject Classification (2000): 11B75, 15A18, 15A69, 15A75, 17B10
    Work supported by the Australian Research Council and by Hungarian Scientific Research Grant OTKA K100291.

[^3]:    AMS Subject Classification (2000): 54C08, 54E55

[^4]:    AMS Subject Classification (2000): Primary 54C08; Secondary 00A05, 26A06

[^5]:    AMS Subject Classification (2000): 54C08, 54C60.

[^6]:    AMS Subject Classification (2000): primary: 11B13; secondary: 11B75.
    The author was supported by the OTKA Grant No. NK105645.

