# ANNALES 

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## SECTIO MATHEMATICA

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## ANNALES

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# THE ZARANKIEWICZ PROBLEM, CAGES, AND GEOMETRIES 

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Dedicated to the memories of András Gács and István Reiman.


#### Abstract

In the paper we consider some constructions of ( $k, 6$ )-graphs that are isomorphic to an induced subgraph of the incidence graph of a finite projective plane, and present some unifying concepts. Also, we obtain new bounds on and exact values of Zarankiewicz numbers, mainly when the parameters are close to those of a design.


## 1. Introduction

This paper is dedicated to the memory of András Gács and István Reiman. We wish to present results on two well-known extremal graph theoretic problems, $(k, g)$-graphs (related to cages) and the Zarankiewicz problem, that András worked on in the last period of his life. These topics in some cases have close relations to finite geometry, and design theory. The first, pioneering results in exploring these connections are due to István Reiman [37, 38] in case of the Zarankiewicz problem. Although we formulate some results in more general settings, we mainly focus on issues that are related to finite projective planes.

[^0]András had a major role in our work on $(k, g)$-graphs, and also took part in obtaining our first results on the Zarankiewicz problem. Those results have been improved later on, and we wish to publish them now.

In this section we give the preliminary definitions and notations, and introduce the two problems. In the paper we only consider finite structures, and all graphs are simple (without loops or multiple edges). The set of the neighbors of a vertex $v$ will be denoted by $N(v)$, and $|N(v)|$ will be referred to as the degree of $v$ or $\operatorname{deg}(v)$. A graph is $k$-regular if all of its vertices have degree $k$. The girth of a graph is the length of the shortest cycle in it. $K_{n, m}$ and $C_{n}$ denote the complete bipartite graph on $n+m$ vertices and the cycle of length $n$, respectively. Note that $K_{2,2}$ is isomorphic to $C_{4}$. The number of edges of a graph $G$ will be denoted by $e(G)$.

Definition 1.1. A $(k, g)$-graph is a $k$-regular graph of girth $g$. $\mathrm{A}(k, g)$-cage is a $(k, g)$-graph with as few vertices as possible. We denote the number of vertices of a $(k, g)$-cage by $c(k, g)$.

A bipartite graph $G$ with vertex classes $A$ and $B$, and edge-set $E$ will be denoted by $G=(A, B ; E)$; we may omit the edge-set and write simply $(A, B)$. We call $(|A|,|B|)$ the size of $G$; we may also say that $G$ is a bipartite graph on $(|A|,|B|)$ vertices.

Definition 1.2. A bipartite graph $G=(A, B ; E)$ is $K_{s, t}$-free if it does not contain $s$ nodes in $A$ and $t$ nodes in $B$ that span a subgraph isomorphic to $K_{s, t}$. The maximum number of edges a $K_{s, t}$-free bipartite graph of size $(m, n)$ may have is denoted by $Z_{s, t}(m, n)$, and is called a Zarankiewicz number.

Note that a $K_{s, t}$-free bipartite graph is not necessarily $K_{t, s}$-free if $s \neq t$.
We remark that Zarankiewicz's question in its original form was formulated via matrices in the following way: what is the minimum number of 1 's in an $m \times$ $\times n 0-1$ matrix that ensures the existence of an $s \times t$ submatrix all of whose entries are 1 s? This quantity clearly equals $Z_{s, t}(m, n)+1$, and it is also used as the definition of a Zarankiewicz number (e.g., in [23]).

Determining the exact values of $c(k, g)$ and $Z_{s, t}(m, n)$ is extremely hard in general. As a bipartite graph does not contain cycles of odd length, a $K_{2,2}=C_{4-}$ free bipartite graph automatically has girth at least 6 . In fact, the incidence graph of a finite projective plane of order $n$ is known to be an extremal $K_{2,2}$-free graph of size $\left(n^{2}+n+1, n^{2}+n+1\right)$, and it is an $(n+1,6)$-cage as well. Projective planes can be considered as designs or as generalized polygons as well, which
are incidence structures with special properties, and are also closely related to the Zarankiewicz problem and cage graphs, respectively.

An incidence structure $(\mathcal{P}, \mathcal{L}, \mathcal{I})$ is a triplet of the sets $\mathcal{P}, \mathcal{L}$, and $\mathcal{I} \subset \mathcal{P} \times$ $\times \mathcal{L}$. The elements of $\mathcal{P}$ and $\mathcal{L}$ are referred to as points and lines (or blocks; then we write $\mathcal{B}$ instead of $\mathcal{L}$ ), respectively, and $\mathcal{I}$ is called the incidence relation. The incidence (or Levi) graph of an incidence structure $(\mathcal{P}, \mathcal{L}, \mathcal{I})$ is the bipartite graph $(\mathcal{P}, \mathcal{L} ; \mathcal{I})$, that is, the two classes of vertices correspond to the point-set and the line-set of the structure, while edges are the flags (incident point-line pairs). As bipartite graphs and incidence structures are basically the same, we will mix the terminologies of the two notions without any further warning. In this manner, we may call the vertices of a graph a point or a line, or we may talk about a subgraph of an incidence structure. By the degree of a point or a line in an incidence structure we will mean the degree of the corresponding vertex in the incidence graph. The dual of the incidence structure $(\mathcal{P}, \mathcal{L}, \mathcal{I})$ is $\left(\mathcal{L}, \mathcal{P}, \mathcal{I}^{T}\right)$, where $(l, P) \in \mathcal{I}^{T} \Longleftrightarrow(P, l) \in \mathcal{I}$, that is, we only interchange the words point and line (block). We will usually omit the indication of the set $\mathcal{I}$ of incidences from the triplet, and we will use the notation $P \in l$ instead of $(P, l) \in \mathcal{I}$. Conventionally, a line $l \in \mathcal{L}$ (or block $B \in \mathcal{B}$ ) may be identified with the set of points it is incident with, and hence we may also write for example $|B|$ to indicate the size of a block $B$. Also, if the elements of $\mathcal{L}$ are considered as lines, then we say that the points $P_{1}, \ldots, P_{k}$ are collinear if there exists a line $l \in \mathcal{L}$ incident with each $P_{i}(1 \leq i \leq k)$.

Definition 1.3. Let $x, y \in \mathcal{P} \cup \mathcal{L}$ be two objects of some incidence structure $(\mathcal{P}, \mathcal{L}, \mathcal{I})$. Then the distance $d(x, y)$ of $x$ and $y$ is the distance of $x$ and $y$ in the incidence graph, that is, the length of the shortest path between $x$ and $y$. Should there be no such path, let $d(x, y)=\infty$.

Definition 1.4. Let $G=(V ; E)$ be a graph with vertex-set $V$. For two (finite) vertex-sets $X$ and $Y$ let $d(X, Y)=\min \{d(x, y): x \in X, y \in Y\}$. If $X$ or $Y$ has one element only, we write, for example, $d(x, Y)$ instead of $d(\{x\}, Y)$. A ball of center $v$ and radius $r$ is $B(v, r)=\{u \in V: d(v, u) \leq r\}$.

Definition 1.5 (Generalized polygon, GP). An incidence structure $(\mathcal{P}, \mathcal{L}, \mathcal{I})$ is a generalized $n$-gon of order $(s, t)$ if and only if the following hold:

GP1: every point is incident with $s+1$ lines;
GP2: every line is incident with $t+1$ points;
GP3: the diameter and the girth of the incidence graph is $n$ and $2 n$, respectively.

From GP3 it follows that if $d(x, y) \leq n-1$, then there is a unique path of length $\leq n-1$ connecting $x$ to $y$. Note that the axioms of generalized polygons are symmetric in points and lines, that is, the dual of a GP of order $(s, t)$ is a GP of order $(t, s)$. By definition, the incidence graph of a generalized $n$-gon of order $(q, q)$ is a $(q+1,2 n)$-graph; moreover, it is a cage. Generalized $n$-gons of order $(q, q)$ exist only if $n=3,4$ or 6 , and are called a generalized triangle or projective plane, a generalized quadrangle (GQ), and a generalized hexagon (GH) of order $q$, respectively. If $q$ is a power of a prime, such generalized polygons of order $q$ do exist, but none is known otherwise. We also mention that one can give alternative definitions of a GP. For example, a projective plane is commonly defined as an incidence structure satisfying the following three properties: (i) any two lines have a unique point in common; (ii) any two points have a unique line incident with both; (iii) there exist four points in general position (that is, no three of them are collinear). From these properties it follows that there exists a number $q$ such that our incidence structure is a generalized triangle of order $(q, q)$. In case of generalized quadrangles, GP3 is commonly rephrased as GQ3: for all $P \in \mathcal{P}$ and $l \in \mathcal{L}$ such that $P \notin l$, there exists a unique line $e \in \mathcal{L}$ such that $P \in e$ and $e$ intersects $l$.

Definition 1.6. Let $\emptyset \neq K \subset \mathbb{Z}^{+}$. An incidence structure $(\mathcal{P}, \mathcal{B})$ is called a $t-(v, K, \lambda)$ design, if $|\mathcal{P}|=v, \forall B \in \mathcal{B}:|B| \in K$, and every $t$ distinct points are contained in precisely $\lambda$ distinct blocks. If $K=\{k\}$, we write simply $t-(v, k, \lambda)$.

The total number $|\mathcal{B}|=b$ of blocks, and the number $r$ of blocks incident with an arbitrary fixed point in a $t-(v, k, \lambda)$ design are $b=\lambda\binom{v}{t} /\binom{k}{t}, r=$ $b k / v=\lambda\binom{v-1}{t-1} /\binom{k-1}{t-1}$, respectively. We always assume that $k<v$ and $\lambda \geq 1$.

The incidence graph of a $t-(v, k, \lambda)$ design is $K_{t, \lambda+1}$ free of size $(v, b)$ by definition, and they turn out to have the most possible number of edges among such graphs.

Definition 1.7. We call the parameters $(t, v, k, \lambda)$ admissible, if they are positive integers satisfying $2 \leq t, t \leq k<v$, furthermore, $b:=\lambda\binom{v}{t} /\binom{k}{t}$ and $r:=b k / v=\lambda\binom{v-1}{t-1} /\binom{k-1}{t-1}$ are also integers.

A projective plane of order $q$ can be considered as a generalized triangle of order $(q, q)$, or as a $2-\left(q^{2}+q+1, q+1,1\right)$ design. The main concept this paper considers is to look for small $(k, 6)$-graphs or $C_{4}$-free graphs with many edges as subgraphs of the incidence graph of a projective plane (or more generally, of a GP or a design), and we also propose the systematic study of this idea.

Section 2 is devoted to $(k, g)$-graphs $(g=6,8,12)$ as induced subgraphs of generalized polygons. Induced regular subgraphs of GPs are obtained by deleting vertices only from the incidence graph of the GP. In [19], $t$-good structures were introduced to examine this idea. We show that many former constructions that we are to list can be unified with this concept. We believe that $t$-good structures are useful to better understand the constructions obtained by several authors and different methods, and sometimes they even help to give new constructions.

One may look for non-induced regular subgraphs of a GP, that is, we are allowed to delete vertices and edges as well to obtain a regular graph from the incidence graph of the GP. Several recent papers use these kinds of ideas, see for example [3], [6]. This method might be examined through a natural generalization of $t$-good structures that is due to Araujo-Pardo and Balbuena [5]. In many cases the $(k, g)$-graphs obtained in this way are smaller than the induced ones. Also, one can extend the concept of $t$-good structures to obtain biregular graphs, which we will do only in order to give a better understanding of some 1-good structures in GQs. These ideas are rather unexplored yet, and will not be covered by this article. We wish only to detail the results in connection with $t$-good structures; for a general and recent survey on $(k, g)$ graphs, we refer to [15]. We do not consider constructions that use different ideas, like [16] or [1].

Section 3 is devoted to the Zarankiewicz problem, particularly the case of $K_{2,2}$-free graphs. Among others, we prove the following (more detailed formulation is given in Section 3).

Theorem 1.8. Assume that a projective plane of order $n$ exists, and let $n \geq 15$ in the first, and $n \geq 4$ in the fourth case. Then

$$
\begin{aligned}
& Z_{2,2}\left(n^{2}+n+1-c, n^{2}+n+1\right)=\left(n^{2}+n+1-c\right)(n+1) \\
&(0 \leq c \leq n / 2) \\
& Z_{2,2}\left(n^{2}+c, n^{2}+n\right)= n^{2}(n+1)+c n \\
&(0 \leq c \leq n+1) \\
& Z_{2,2}\left(n^{2}-n+c, n^{2}+n-1\right)=\left(n^{2}-n\right)(n+1)+c n \\
&(0 \leq c \leq 2 n) \\
& Z_{2,2}\left(n^{2}-2 n+1+c, n^{2}+n-2\right)=\left(n^{2}-2 n+1\right)(n+1)+c n \\
&(0 \leq c \leq 3(n-1))
\end{aligned}
$$

Other exact values of Zarankiewicz numbers are also obtained if the parameters are small, or they are close enough to those of a design.

## 2. $(k, g)$-graphs

For details and results on cages, we refer to the online available dynamic survey of Exoo and Jajcay [15]. Connections with the degree/diameter problem and Moore graphs can be found in [35].

A general lower bound on the number of vertices of a $(k, g)$-cage, known as the Moore bound, is a simple consequence of the fact that the vertices at distance $0,1, \ldots,\lfloor(g-1) / 2\rfloor$ from a vertex (if $g$ is odd), or an edge (if $g$ is even) must be distinct.

Proposition 2.1 (Moore bound).

$$
\begin{aligned}
c(k, g) & \geq M(k, g)= \\
& = \begin{cases}1+k+k(k-1)+\cdots+k(k-1)^{\frac{g-1}{2}-1} & \text { for } g \text { odd } \\
2\left(1+(k-1)+(k-1)^{2}+\cdots+(k-1)^{\frac{g}{2}-1}\right) & \text { for } g \text { even. }\end{cases}
\end{aligned}
$$

As $(k, 2 n+1)$-graphs with $M(k, 2 n+1)$ vertices coincide with Moore graphs of valency $k$ and diameter $n$, the term Moore graph is extended to any $(k, g)$ graph on $M(k, g)$ vertices. Such graphs may also be referred to as Moore cages. It is easy to see that $k+1$-regular Moore graphs with girth $2 n$ are precisely the incidence graphs of generalized $n$-gons of order $(k, k)$. Note that the cases $g=3$ and $g=4$ are trivial, the corresponding Moore cages are complete graphs and regular complete bipartite graphs, respectively.

### 2.1. Some constructions of $(k, g)$-graphs $(g=6,8,12)$

From now on we focus on constructions and results regarding generalized polygons, that is, the cases $g=6,8,12$.

Starting from a projective plane of order $q$, Brown ([11], 1967) constructed ( $k, 6$ )-graphs for arbitrary $4 \leq k \leq q$ by deleting some properly chosen points and lines from the plane, that is, by removing vertices from the incidence graph of the plane. This is equivalent to finding a $k$-regular induced subgraph of the incidence graph. The ( $k, 6$ )-graphs Brown obtained have $2 k q$ number of vertices,
hence from the distribution of primes it follows that $c(k, 6) \sim 2 k^{2}$. Although Brown himself only gave one specific construction, we refer to this construction method (deleting vertices from a projective plane of order $q$ to obtain a $(k, 6)$ graph, $k \leq q$ ) as Brown's method. It may be generalized to the idea of finding $\left(k^{\prime}, g\right)$-graphs as induced subgraphs of $(k, g)$-cages, $k^{\prime}<k$.

In 1997, Lazebnik, Ustimenko, and Woldar [33] proved the following.
Result 2.2. Let $k \geq 2$ and $g \geq 5$ be integers, and let $q$ denote the smallest odd prime power for which $k \leq q$. Then

$$
c(k, g) \leq 2 k q^{\frac{3}{4} g-a},
$$

where $a=4,11 / 4,7 / 2,13 / 4$ for $g \equiv 0,1,2,3(\bmod 4)$, respectively.
In particular, for $g=6,8,12$ this gives $c(k, 6) \leq 2 k q, c(k, 8) \leq 2 k q^{2}$, $c(k, 12) \leq 2 k q^{5}$, where $q$ is the smallest odd prime power not smaller than $k$. Combined with the Moore bound, this yields $c(k, 8) \sim 2 k^{3}$.

Using the addition and multiplication tables of $\operatorname{GF}(q)$, Abreu, Funk, Labbate and Napolitano ([2], 2006) constructed two infinite families of ( $k, 6$ ), $k \leq q$ graphs via their incidence matrices. The number of vertices of the graphs in the first and the second family are $2 k q$ and $2(k q+(q-1-k))$, respectively. The second construction yields a graph smaller than the previously known ones for $k=q$, resulting $c(q, 6) \leq 2\left(q^{2}-1\right)$ for any prime power $q$. Moreover, Abreu et al. settled a conjecture on the incidence matrices of $\operatorname{PG}(2, q), q$ square, in connection with the partition of the point-set and line-set of $\mathrm{PG}(2, q)$ into Baer subplanes. They verified the conjecture for $q=4,9$, and 16 , which allowed them to construct $(k, 6)$ graphs of $\operatorname{size} 2(k q-(q-k)(\sqrt{q}+1)-\sqrt{q}) \geq c(k, 6)$ for $q=4,9,16$ and $k \leq q$.

Deleting vertices from the incidence graph of a generalized quadrangle or hexagon, Araujo, González, Montellano-Ballesteros and Serra ([7], 2007) showed $c(k, 8) \leq 2 k q^{2}$ and also $c(k, 12) \leq 2 k q^{4}, k \leq q, q$ a prime power. Their construction uses only elementary combinatorial properties of generalized polygons. Their upper bound on $c(k, 8)$ is the same as that of Lazebnik et al.'s [33], but the bound on $c(k, 12)$ is better, and leads to $c(k, 12) \sim 2 k^{5}$.

Note that the above results yield $c(k, 2 n) \sim 2 k^{n-1}$ for $n=2,3,4,6$.

### 2.2. Brown's method reformulated: $t$-good structures, a unifying concept.

Regarding the cases $g=6,8$, and 12, Gács and Héger [19] (2008) present a point of view that unifies all the above constructions (except Lazebnik, Ustimenko, and Woldar's for $g=12$ ) using the concept of a $t$-good structure, and also started to study them systematically.

Definition 2.3. A t-good structure in a generalized polygon is a pair $\mathcal{T}=\left(\mathcal{P}_{0}, \mathcal{L}_{0}\right)$ consisting of a proper subset of points $\mathcal{P}_{0}$ and a proper subset of lines $\mathcal{L}_{0}$, with the property that there are exactly $t$ lines in $\mathcal{L}_{0}$ through any point not in $\mathcal{P}_{0}$, and exactly $t$ points in $\mathcal{P}_{0}$ on any line not in $\mathcal{L}_{0}$.

Removing the points and lines of a $t$-good structure $\mathcal{T}=\left(\mathcal{P}_{0}, \mathcal{L}_{0}\right)$ from the incidence graph of a generalized $n$-gon of order $q$ results in a $(q+1-t)$-regular graph of girth at least $2 n$, and hence provides an upper bound on $c(q+1-t, 2 n)$. It is easy to see that $\left|\mathcal{P}_{0}\right|=\left|\mathcal{L}_{0}\right|$ for every $t$-good structure $\mathcal{T}$, hence the size of $\mathcal{T}$ is defined as $\left|\mathcal{P}_{0}\right|$, and may be denoted by $|\mathcal{T}|$. Trivially, the larger $t$-good structure we find for a fixed $t$, the smaller $(q+1-t)$-regular graph we obtain. Note that this concept works in any GP.

Most known $t$-good structures follow the same, general pattern we give here.
The neighboring balls construction. Recall that $d(x, y)$ denotes the distance of $x$ and $y$. Let $\mathcal{L}^{*}=\left\{l_{1}, \ldots, l_{t}\right\}$ and $\mathcal{P}^{*}=\left\{P_{1}, \ldots, P_{t}\right\}$ be a collection of distinct lines and points such that $\forall 1 \leq i<j \leq t$ the following hold:
(i) $d\left(l_{i}, l_{j}\right)=2$ (the lines are pairwise intersecting);
(ii) the unique point at distance one from $l_{i}$ and $l_{j}$ (their intersection point) is an element of $\mathcal{P}^{*}$;
(i') $d\left(P_{i}, P_{j}\right)=2$ (the points are pairwise collinear);
(ii') the unique line at distance one from $P_{i}$ and $P_{j}$ (the line joining them) is an element of $\mathcal{L}^{*}$.

Proposition 2.4. Let $\left(\mathcal{P}^{*}, \mathcal{L}^{*}\right)$ satisfy the conditions above, and let $\mathcal{T}=\left(\mathcal{P}_{0}, \mathcal{L}_{0}\right)$ be the collection of points and lines that are at distance at most $n-2$ from some element of $\mathcal{P}^{*}$ or $\mathcal{L}^{*}$, that is, $\mathcal{P}_{0} \cup \mathcal{L}_{0}=\bigcup_{i=1}^{t}\left\{B\left(P_{i}, n-2\right)\right\} \cup \bigcup_{i=1}^{t}\left\{B\left(l_{i}, n-2\right)\right\}$. Then $\mathcal{T}$ is $t$-good.

Proof. Let $Q \notin \mathcal{P}_{0}$. Then for every $i(1 \leq i \leq t), d\left(Q, l_{i}\right)=n-1$ or $n$, and $d\left(Q, P_{i}\right)=n$ or $n-1$, depending on $n$ being even or odd, respectively. We may assume that $n$ is even (the odd case is analogous). Then for all $i(1 \leq i \leq t)$
there is a unique a line $e_{i}$ such that $d\left(Q, e_{i}\right)=1$ and $d\left(e_{i}, l_{i}\right)=n-2$, and these are precisely the lines of $\mathcal{L}_{0}$ that are incident with $Q$. Hence we must show that these are distinct. Suppose to the contrary that $e_{i}=e_{j}=e$ for some $i \neq j$. Let $P \in \mathcal{P}^{*}$ be the point incident with $l_{i}$ and $l_{j}$. Since $d(Q, P)=n, d(P, e)=n-1$. But then there are two distinct paths of length $n-1$ from $P$ to $e$, one through $l_{i}$ and another one through $l_{j}$, a contradiction. The same (dual) arguments hold for lines.

Note that if we allow $\mathcal{P}^{*}$ and $\mathcal{L}^{*}$ to have different sizes, $s$ and $t$ respectively, and define $\mathcal{T}$ in the same way, then the same arguments show that after deleting $\mathcal{T}$, every point not in $\mathcal{T}$ has degree $q+1-s$ or $q+1-t$, and line not in $\mathcal{T}$ has degree $q+1-t$ or $q+1-s$, depending on $n$ being odd or even, respectively. Hence in order to obtain biregular graphs, we could define $(s, t)$-good structures, as we will do in Subsection 2.2.2, but mainly restrict its use to construct 1 -good structures.

We will use the next definition usually in the context of a $t$-good structure.
Definition 2.5. Let $\mathcal{T}=\left(\mathcal{P}_{0}, \mathcal{L}_{0}\right)$ be a pair of a point-set and a line-set in a GP $(\mathcal{P}, \mathcal{L})$. Then a point $P$ is $\mathcal{T}$-complete, if $P \in \mathcal{P}_{0}$, and every line incident with $P$ is in $\mathcal{L}_{0}$. We define a $\mathcal{T}$-complete line dually.

### 2.2.1. $t$-good structures in projective planes

In the $n=3$ case, that is, if we start from an arbitrary projective plane, the conditions (i) and (i') of the general construction hold automatically, while conditions (ii) and (ii') claim that ( $\mathcal{P}^{*}, \mathcal{L}^{*}$ ) should be a (possibly degenerate) subplane. We call a set of points and lines a degenerate subplane, if the intersection point of its lines and the lines joining two of its points belong to it, but it does not have four points in general position. Note that in a projective plane $d(x, y) \leq n-2=1$ means that $x=y$ or $x$ is incident with $y$. Hence $\left(\mathcal{P}_{0}, \mathcal{L}_{0}\right)$ consists of points and lines that are incident with a subplane, that is, we put the points and the lines of $\mathcal{P}^{*}$ and $\mathcal{L}^{*}$ completely into $\mathcal{T}$ and delete them; thus this construction is called a completely deleted subplane by Gács, Héger and Weiner [20].

There are two types of degenerate subplanes:

- type $\pi_{1}$ : there is an incident point-line pair $(P, l)$ such that all points are incident with $l$ and all lines are incident with $P$;
- type $\pi_{2}$ : there is a non-incident point-line pair $(P, l)$ such that every point except $P$ is incident with $l$ and every line except $l$ is incident with $P$.

In a degenerate subplane of type $\pi_{1}$ and $\pi_{2}$ there are at most two or three points in general position, respectively. Brown's construction [11] and the first infinite family of Abreu et al. [2] can be obtained by completely deleting degenerate subplanes (CDDS) of type $\pi_{1}$ from a finite projective plane, while the second family of Abreu et al. can be constructed by CDDS of type $\pi_{2}$, see [19]. We remark that the constructions of Abreu et al. [2] correspond to $t$-good structures in PG(2,q), while Brown's construction works in an arbitrary finite projective plane. Also, note that a subplane has the same number of points and lines except if it is degenerate of type $\pi_{1}$; in that case, it may have a different number of points and lines, hence it can be used to obtain biregular graphs.

A different construction is also given in [19]. Let $\mathcal{T}$ consist of the points and the lines of $t$ pairwise disjoint Baer subplanes. Then, using a result of Svéd [40], it can be shown that $\mathcal{T}$ is $t$-good. It is well known that $\operatorname{PG}(2, q), q$ square, can be partitioned into (pairwise) disjoint Baer subplanes, hence we may take $t$ of them to obtain a $t$-good structure. Note that if we take the union of $t$ disjoint subplanes from the partition, it is easily seen to be $t$-good without the result of Svéd. However, the disjoint Baer subplanes construction works for arbitrary disjoint Baer subplanes. This construction is independent from the conjecture of Abreu et al. [2], and extends their result to arbitrary square prime powers.

Regarding the sizes, the $t$-good structure resulting from a degenerate subplane of type $\pi_{1}$ or $\pi_{2}$, or a non-degenerate subplane of order $t_{1}$, where $t=t_{1}^{2}+$ $+t_{1}+1$, is of size $t q+1, t q-t+3$ and $t q-\left(t_{1}-1\right) t$, respectively. The disjoint Baer subplanes construction gives a $t$-good structure of size $t(q+\sqrt{q}+1)$.

Gács et al. in [19] and [20] show that if $t$ is small enough, then the Baer subplane construction is optimal. Moreover, there are no other $t$-good structures in $\mathrm{PG}(2, q)$ than the ones listed above. The precise results are the following.

Result 2.6. Let $\mathcal{T}$ be a $t$-good structure in a projective plane of order $q$, $t \leq 2 \sqrt{q}$. Then $|\mathcal{T}| \leq t(q+\sqrt{q}+1)$. If the plane is $\mathrm{PG}(2, q)$ and $t<\sqrt[4]{q} / 2$, then in case of equality $\mathcal{T}$ is the union of t disjoint Baer subplanes.

Result 2.7. Let $p$ be a prime and let $\mathcal{T}$ be a $t$-good structure in $\mathrm{PG}(2, q), q=p^{h}$; furthermore,

- for $h=1$ and $h=2$, let $t<p^{1 / 2} / 2$;
- for $h \geq 3$, let $t<\min \left\{p+1, c_{p} q^{1 / 6}-1, q^{1 / 4} / 2\right\}$, where $c_{2}=c_{3}=1 / 8$ and $c_{p}=1$ for $p>3$.

Then $\mathcal{T}$ is either a completely deleted degenerate subplane, or the union of $t$ disjoint Baer subplanes.

### 2.2.2. $t$-good structures in GQs and GHs

In the cases $n=4,6$, that is, generalized quadrangles and hexagons, two or more pairwise collinear points must all be incident with a fixed line $l_{1}$. Hence to use the neighboring balls construction for $t \geq 2$, the points of $\mathcal{P}^{*}$ are all incident with $l_{1}$, and $l_{1} \in \mathcal{L}^{*}$. Dually, the lines of $\mathcal{L}^{*}$ must all be incident with a point $P_{1} \in \mathcal{P}^{*}$, and hence $P_{1} \in l_{1}$. This construction, due to Araujo et al. [7], is analogous to the CDDS of type $\pi_{1}$ in a projective plane. In other words, it might be regarded as an extension of Brown's original construction from projective planes to generalized polygons. This gives a $t$-good structure of size $t q^{n-2}+$ $+q^{n-3}+\ldots+q+1$.

If $t=1$, we may choose $\mathcal{P}^{*}=\left\{P_{1}\right\}$ and $\mathcal{L}^{*}=\left\{l_{1}\right\}$ arbitrarily, the conditions on $\mathcal{P}^{*}$ and $\mathcal{L}^{*}$ are trivially satisfied; hence $P_{1} \notin l_{1}$ is also admissible [19]. In projective planes, this corresponds to a degenerate subplane of type $\pi_{2}$. This construction gives a 1 -good structure of size $q^{n-2}+2 q^{n-3}+q^{n-4}+\ldots+1$, which is greater than the former one by $q^{n-3}$.

We may also define $(s, t)$-good structures, that is, a pair of a point-set and a line-set $\mathcal{T}=\left(\mathcal{P}_{0}, \mathcal{L}_{0}\right)$ such that every line outside $\mathcal{L}_{0}$ intersects $\mathcal{P}_{0}$ in $s$ points, and every point outside $\mathcal{P}_{0}$ is covered by $t$ lines of $\mathcal{L}_{0}$. By definition, $\mathcal{T}$ is $t$-good if and only if it is $(t, t)$-good. It is also straightforward to check that the union $\mathcal{T}$ of an $\left(s_{1}, t_{1}\right)$-good structure $\mathcal{T}_{1}=\left(\mathcal{P}_{1}, \mathcal{L}_{1}\right)$ and an $\left(s_{2}, t_{2}\right)$-good structure $\mathcal{T}_{2}=\left(\mathcal{P}_{2}, \mathcal{L}_{2}\right)$ is $\left(s_{1}+s_{2}, t_{1}+t_{2}\right)$-good if and only if in $\mathcal{T}=\left(\mathcal{P}_{1} \cup \mathcal{P}_{2}, \mathcal{L}_{1} \cup\right.$ $\cup \mathcal{L}_{2}$ ) every point in $\mathcal{P}_{1} \cap \mathcal{P}_{2}$ and every line in $\mathcal{L}_{1} \cap \mathcal{L}_{2}$ is $\mathcal{T}$-complete. Note that the points of a $(0, t)$-good, and the lines of an $(s, 0)$-good structure must be $\mathcal{T}$-complete, hence their union is $(s, t)$-good. With this (unexplored) concept it is comfortable to construct 1 -good structures as the union of a $(0,1)$ and a $(1,0)$-good structure.

From now on we consider a generalized quadrangle $(\mathcal{P}, \mathcal{L})$ of order $q$. For $U \subset \mathcal{P}, U^{\perp}$ denotes the set of points collinear with all points of $U$, and $U^{\perp \perp}$ the set of points collinear with all points of $U^{\perp}$. (Every point is considered to be collinear with itself.) One can similarly define $W^{\perp}$ and $W^{\perp \perp}$ for a set $W$ of lines.

It is easy to see that for a pair of points $\{u, v\},\left|\{u, v\}^{\perp}\right|=q+1$. A noncollinear point-pair $u, v$ is called regular if $\left|\{u, v\}^{\perp \perp}\right|=q+1$ holds. The definition of a regular line pair is analogous.

Let $\left\{u_{0}, u_{1}\right\}$ be a regular point pair, and put $\left\{u_{0}, u_{1}\right\}^{\perp} \cup\left\{u_{0}, u_{1}\right\}^{\perp \perp}$ into $\mathcal{T}=\left(\mathcal{P}_{0}, \mathcal{L}_{0}\right)$ completely. In other words, let $\mathcal{P}_{0}=\left\{u_{0}, u_{1}\right\}^{\perp} \cup\left\{u_{0}, u_{1}\right\}^{\perp \perp}$, and let $\mathcal{L}_{0}$ consist of the lines that intersect $\mathcal{P}_{0}$. It is not hard to check that $\left(\mathcal{P}_{0}, \mathcal{L}_{0}\right)$ is $(0,1)$-good. Similarly, a regular line pair results in a $(1,0)$-good structure. It is
also easy to see that the points and the lines at distance at most $n-2=2$ from a fixed point $P$ or a fixed line $l$ (that is, a ball of radius two) form a $(1,0)$ or a ( 0,1 )good structure, respectively. Regular point or line pairs do not always exist, but if they do, we can use them to construct a 1-good structure as follows. These constructions can be found in [19], though not using the concept of $(s, t)$-good structures.

Suppose that there exists a $(0,1)$-good structure $\mathcal{T}=\left(\mathcal{P}_{0}, \mathcal{L}_{0}\right)$ arising from a regular point pair. Uniting $\mathcal{T}$ with a ball of center $P \notin \mathcal{T}$, we obtain a 1-good structure will be of size $q^{2}+3 q+1$. If we find a regular line pair such that the lines in the resulting $(1,0)$-good structure are not incident with any point from $\mathcal{P}_{0}$, their union will be of size $q^{2}+4 q+3$. In the classical generalized quadrangle $Q(4, q)$, the first construction always works, while the second works if $q>2$ is even.

Beukemann and Metsch ([10], 2011) studied one-good structures in arbitrary generalized quadrangles of order $q$, and in particular, in the classical one $Q(4, q)$. They give several examples that work for arbitrary prime power $q$ that can be phrased in terms of $(0,1)$ and $(1,0)$-good structures as above. Besides the two such structures above, they use an ovoid or a spread to construct 1-good structures. An ovoid in a GQ is a set of $q^{2}+1$ points that intersect every line in one point. A spread is the dual of an ovoid, that is, a set of $q^{2}+1$ lines that cover all point once. If $\mathcal{O}$ is an ovoid, then $(\mathcal{O}, \emptyset)$ is $(1,0)$-good, while for a spread $\mathcal{S}$, $(\emptyset, \mathcal{S})$ is $(0,1)$-good, hence can be used to obtain 1-good structures. However, they find no larger construction than the two in [19] that works for general $q$. For $q=3$, they find a sporadic example of size $22=q^{2}+4 q+2$. Moreover, Beukemann and Metsch prove the following upper bound on the size of a 1-good structure in a GQ.

Theorem 2.8 ([10]). Let $Q$ be a generalized quadrangle of order $q, q>1$, and let $\mathcal{T}$ be a 1 -good structure in $Q$. Then
(1) $|\mathcal{T}| \leq 2 q^{2}+2 q-1$;
(2) If $Q$ is $Q(4, q)$ and $q$ is even, then $|\mathcal{T}| \leq 2 q^{2}+q+1$.

It seems that understanding $t$-good structures in GQs is much more difficult than in projective planes. In the latter case the characterization of 1-good structures is almost immediate (cf. [19]).

### 2.2.3. The construction by Lazebnik et al. as $t$-good structures

Consider the construction of Lazebnik et al. [33]. In the cases $g=6$ and 8 , the graphs they construct are of the same size as Brown's [11] and Araujo et al.'s [7], respectively. We show that just as the latter two, Lazebnik et al.'s construction can also be interpreted as a special case of Brown's method, that is, it is isomorphic to a graph obtained by deleting a $t$-good structure from a projective plane or a GQ.

First they construct an incidence structure $D(q)$ as follows. Points and lines of $D(q)$ are written inside a parenthesis () or brackets [], respectively. Consider the vectors $(P)$ and $[l]$ of infinite length over $\operatorname{GF}(q)$ :

$$
\begin{aligned}
(P) & =\left(p_{1}, p_{11}, p_{12}, p_{21}, p_{22}^{\prime}, p_{23}, \ldots, p_{i i}, p_{i i}^{\prime}, p_{i, i+1}, p_{i+1, i}, \ldots\right) \\
{[l] } & =\left[l_{1}, l_{11}, l_{12}, l_{21}, l_{22}^{\prime}, l_{23}, \ldots, l_{i i}, l_{i i}^{\prime}, l_{i, i+1}, l_{i+1, i}, \ldots\right]
\end{aligned}
$$

A point $(P)$ and a line $[l]$ are incident if and only if the following infinite list of equations hold simultaneously:

$$
\begin{aligned}
l_{11}-p_{11} & =l_{1} p_{1} \\
l_{12}-p_{12} & =l_{11} p_{1} \\
l_{21}-p_{21} & =l_{1} p_{11} \\
l_{i i}-p_{i i} & =l_{1} p_{i-1, i} \\
l_{i i}^{\prime}-p_{i i}^{\prime} & =l_{i-1, i} p_{1} \\
l_{i, i+1}-p_{i, i+1} & =l_{i, i} p_{1} \\
l_{i+1, i}-p_{i+1, i} & =l_{1} p_{i i}^{\prime}
\end{aligned}
$$

where the last four equations are defined for all $i \geq 2$. For an integer $n \geq 2$, let $D(n, q)$ be derived from $D(q)$ by projecting every vector onto its initial $n$ coordinates. Then the point-set $\mathcal{P}_{n}$ and the line-set $\mathcal{L}_{n}$ of $D(n, q)$ both have $q^{n}$ elements, and incidence is defined by the first $n-1$ equations above. Note that those involve only the first $n$ coordinates of $(P)$ and $[l]$, hence apply to the points and lines of $D(n, q)$ unambiguously. $D(n, q)$ as a bipartite graph can be proved to be $q$-regular and have girth at least $n+4$ (thus at least $n+5$ if $n$ is odd).

Let $R, S \subset \operatorname{GF}(q)$, where $|R|=r \geq 1$ and $|S|=s \geq 1$, and let

$$
\begin{aligned}
\mathcal{P}_{R} & =\left\{(P) \in \mathcal{P}_{n}: p_{1} \in R\right\} \\
\mathcal{L}_{S} & =\left\{[l] \in \mathcal{L}_{n}: l_{1} \in S\right\}
\end{aligned}
$$

The graph $D(n, q, R, S)$ is defined as the subgraph of $D(n, q)$ induced by $\mathcal{P}_{R} \cup \mathcal{L}_{S}$. It can be shown that every vertex in $\mathcal{P}_{R}$ or $\mathcal{L}_{S}$ in $D(n, q, R, S)$ has degree $s$ and $r$, respectively.

In the case $n=2, \mathcal{P}_{2}=\left\{\left(p_{1}, p_{11}\right) \in \operatorname{GF}(q)^{2}\right\}$ and $\mathcal{L}_{2}=\left\{\left[l_{1}, l_{11}\right] \in\right.$ $\left.\in \operatorname{GF}(q)^{2}\right\}$, and a point $(x, y) \in \mathcal{P}_{2}$ is incident with the line $[m, b] \in \mathcal{L}_{2}$ if and only if $b-y=m x$. Let

$$
\begin{aligned}
\varphi: D(2, q) & \rightarrow \mathrm{AG}(2, q) \\
(x, y) & \mapsto(x, y) \\
{[m, b] } & \mapsto\{(x, y): y=-m x+b\} .
\end{aligned}
$$

The mapping $\varphi$ is clearly injective and preserves incidence, hence it is an embedding of $D(2, q)$ into $\mathrm{AG}(2, q) \subset \mathrm{PG}(2, q)$. Note that vertical lines are not in the image, hence $\varphi(D(2, q))$ can be obtained by deleting the ideal line together with its points and the vertical lines from $\operatorname{PG}(2, q)$. If we consider the induced subgraph $D(2, q, R, S)$, geometrically it means that we take points only on the vertical lines $X=x: x \in R$ and lines with slopes $-m \in S$. In other words, we delete (besides the formerly deleted points and lines) all the points of the vertical lines $X=x: x \notin R$, and we delete all lines having slopes $-m \notin S$; that is, we delete the lines that intersect the ideal line in a direction (or point) ( $m$ ) with $-m \notin S$. Hence this construction corresponds to a $(q+1-r, q+1-s)$-good CDDS of type $\pi_{1}$.

To see why the construction for $n=3$ (that is, $g=8$ ) is isomorphic to an $(s, t)$-good structure in a GQ, we give an explicit description of $\mathrm{PG}(3, q)$ and the classical generalized quadrangle $W(q)$ first.

The projective space $\operatorname{PG}(3, q)$ can be represented as the system of non-zero dimensional subspaces of $\mathrm{GF}(q)^{4}$, that is, the points, the lines and the planes of $\operatorname{PG}(3, q)$ correspond to the one, two and three dimensional subspaces of $\mathrm{GF}(q)^{4}$, respectively. Hence, a point of $\mathrm{PG}(3, q)$ can be represented by a nonzero vector of $\mathrm{GF}(q)^{4}$ that is defined up to a non-zero scalar multiplier. We write this representative as $(x: y: z: w)$, where the colons express that the coordinates are homogeneous. A line $l$ of $\operatorname{PG}(3, q)$ corresponds to a plane of $G F(q)^{4}$, and hence can be defined as the span of two vectors, that is, $l=\{\alpha(x: y: z: w)+$ $\left.+\beta\left(x^{\prime}: y^{\prime}: z^{\prime}: w^{\prime}\right) \mid(\alpha, \beta) \in \operatorname{GF}(q)^{2} \backslash\{(0,0)\}\right\}$ for some distinct points $(x: y: z: w)$ and $\left(x^{\prime}: y^{\prime}: z^{\prime}: w^{\prime}\right)$ of $\mathrm{PG}(3, q)$.

The generalized quadrangle $W(q)$ is defined by a non-degenerate symplectic form over $\operatorname{PG}(3, q)$. Let $q$ be an odd prime power. Take a matrix $A \in \operatorname{GF}(q)^{4 \times 4}$ such that $A^{T}=-A$, and for $x, y \in \mathrm{GF}(q)^{4}$, let $x \sim y(x$ perpendicular to $y)$
if and only if $x A y=0$. Note that the relation $\sim$ is well-defined over $\operatorname{PG}(3, q)$, and for all $x \in \operatorname{GF}(q)^{4}: x \sim x$. The points of $W(q)$ are those of $\mathrm{PG}(3, q)$, and the lines of $W(q)$ are those of $\mathrm{PG}(3, q)$ that are totally isotropic, that is, any two points of which are perpendicular. Note that if $x \sim y$, then $(\alpha x+\beta y) \sim(\gamma x+\delta y)$ for all $\alpha, \beta, \gamma, \delta \in \mathrm{GF}(q)$, hence two points $x$ and $y$ are collinear in $W(q)$ if and only if $x \sim y$. Thus a point is incident with a line in $W(q)$ if and only if it is perpendicular to at least two of its points (and hence to all of them). It can be proved that $W(q)$ is a generalized quadrangle of order $(q, q)$.

Now the graph $D(3, q)$ has point-set $\mathcal{P}_{3}=\left\{(x, y, z) \in \operatorname{GF}(q)^{3}\right\}$ and line-set $\mathcal{L}_{3}\left\{[a, b, c] \in \operatorname{GF}(q)^{3}\right\}$, where $(x, y, z) \in[a, b, c]$ if and only if $b-y=a x$ and $c-z=b x$. Now let

$$
\begin{aligned}
\varphi: D(3, q) & \rightarrow \mathrm{PG}(3, q) \\
(x, y, z) & \mapsto(x: y: z: 1) \\
{[a, b, c] } & \mapsto \\
& \mapsto\left\{\alpha(1:-a:-b: 0)+\beta(0: b: c: 1) \mid(\alpha, \beta) \in \operatorname{GF}(q)^{2} \backslash\{(0,0)\}\right\}
\end{aligned}
$$

furthermore, let

$$
A=\left(\begin{array}{cccc}
0 & 1 & 0 & 0 \\
-1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 1 & -1 & 0
\end{array}\right)
$$

We claim that $\varphi$ is an embedding of $D(3, q)$ into $W(q)$ defined by the symplectic form coming from $A$. It is clear that $\varphi$ is injective. Moreover, $(x, y, z) \in$ $\in[a, b, c] \Longleftrightarrow b-y=a x$ and $c-z=b x \Longleftrightarrow(x: y: z: 1) A(1:-a:-b:$ $0)=0$ and $(x: y: z: 1) A(0: b: c: 1)=0 \Longleftrightarrow(x: y: z: 1)$ is on the line spanned by $(1:-a:-b: 0)$ and $(0: b: c: 1)$, hence $\varphi$ preserves incidence.

Note that the $q^{2}+q+1$ points collinear with $P_{1}=(0: 0: 1: 0)$ in $W(q)$ (that is, points of form $(x: y: z: 0)$, or in other words, the points of the plane at infinity) are not in the image of $\varphi$; moreover, lines intersecting the line $l_{1}=\{(0: \alpha: \beta: 0)\}$ are also excluded (no lines in the image contain a point with first and fourth coordinates both 0 ). This means that $\varphi(D(3, q)) \subset$ $\subset W(q)$ is obtained from $W(q)$ by deleting every point collinear with $P_{1}$ and every line intersecting $l_{1}$. As $P_{1} \in l_{1}$, this corresponds to a 1 -good neighboring balls construction.

Now the points $(x: y: z: 1)$, with $x \notin R$ fixed, are precisely the $q^{2}$ points collinear to $P_{x}=(0: 1: x: 0) \in l_{1}$ not on $l_{1}$. The lines $\{\alpha(1:-a:-b: 0)+$ $+\beta(0: b: c: 1)\}$, with $a \notin S$ fixed, are precisely the $q^{2}$ lines intersecting the
line $l_{a}=\{\gamma(1:-a: 0: 0)+\delta(0: 0: 1: 0)\}$ not in $P_{1}$. Hence $\varphi(D(3, q, R, S))$ can be obtained by deleting the balls around $\mathcal{P}^{*}=\left\{P_{x}: x \notin R\right\} \cup\left\{P_{1}\right\}$ and $\mathcal{L}^{*}=\left\{l_{a}: a \notin S\right\} \cup\left\{l_{1}\right\}$.

## 3. The Zarankiewicz problem

In the Introduction (see Definition 1.2) we stated Zarankiewicz's problem. Here we focus on results for $s=t=2$, that is, determining the maximum number of edges in $K_{2,2}$-free bipartite graphs. The history of the problem and early results are collected by Guy [23], so we only discuss some of the results. Kővári, T. Sós and Turán [32] proved $Z_{2,2}(m, n)<\left[n^{3 / 2}\right]+2 n$ and $\lim _{n \rightarrow \infty} Z_{2,2}(m, n) / n^{3 / 2}=1$. They also observed, using finite affine planes, that $Z_{2,2}\left(p^{2}, p^{2}+p\right)=p^{2}(p+1)$ for $p$ prime. The case $m=n$ was studied in detail by Reiman.

Theorem 3.1 (Reiman [37]). Let $G$ be a $K_{2,2}$-free bipartite graph of size ( $n, n$ ). Then the number of edges in $G$ satisfies the inequality

$$
e(G) \leq \frac{n}{2}(1+\sqrt{4 n-3})
$$

Equality holds if and only if $n=k^{2}+k+1$ for some $k$ and $G$ is the incidence graph of a projective plane of order $k$.

In the same paper Reiman proved $Z_{2,2}(m, n) \leq \frac{1}{2}\left(n+\sqrt{n^{2}+4 n m(m-1)}\right)$ and clarified the connection of $Z_{2,2}\left(p^{2}, p^{2}+p\right)=p^{2}(p+1)$ with affine planes. Later Reiman [38] went on to study Zarankiewicz's problem for $s=2$ and larger $t$, and proved $Z_{2, \lambda+1}(m, n) \leq \frac{1}{2}\left(n+\sqrt{n^{2}+4 \lambda n m(m-1)}\right)$ with equality if and only if there is a $2-(m, k, \lambda)$-design, and the bipartite graph is the incidence graph of the design. Here $n=m(m-1) \lambda /(k(k-1))$ is the number of blocks in this design. This upper bound was also proved by Hyltén-Cavallius [25]. The connection of Zarankiewicz's problem for general $s, t$ and block designs was noted in a particular case by Kárteszi [29, 30], and done in detail by Roman [39] (see Theorem 3.5). We give two more early results that provide exact values for $Z_{s, t}(m, n)$ if $n$ is much larger than $m$.

Theorem 3.2 (Čulík [14]). If $1 \leq s \leq m$ and $n \geq(t-1)\binom{m}{s}$, then

$$
Z_{s, t}(m, n)=(s-1) n+(t-1)\binom{m}{s}
$$

Theorem 3.3 (Guy [23]). If $\ell(n, s, t) \leq n \leq(t-1)\binom{m}{s}+1$, then

$$
Z_{s, t}(m, n)=\left\lfloor\frac{\left(s^{2}-1\right) n+(t-1)\binom{m}{s}}{s}\right\rfloor,
$$

where $\ell(n, s, t)$ is approximately $(t-1)\binom{m}{s} /(s+1)$.
Irving [27] gave a method which can be used to explicitly calculate an upper bound for $Z_{s, t}(m, n)$ in case of given parameters; his idea was also investigated in [21]. One may also relate $s$ and $t$ to $n$ and $m$ (e.g., $s=n / 2, t=m / 2$ ); for such studies see [9], [22] and their references. For general bounds, we refer to Füredi [17, 18], Kollár-Rónyai-Szabó [31], Alon-Rónyai-Szabó [4], Nikiforov [36], and the references therein.

### 3.1. Roman's inequality

Let $I \subset \mathbb{R}$ be an interval, $f: I \rightarrow \mathbb{R}$ a strictly increasing convex function, $n \in \mathbb{N}, x_{1}, \ldots, x_{n} \in I \cap \mathbb{Z}, A:=\sum_{i=1}^{n} x_{i}=n p+r$ for some $p \in \mathbb{Z}, 0 \leq r<p$. Then Jensen's inequality for integers claims

$$
\begin{aligned}
\sum_{i=1}^{n} f\left(x_{i}\right) & \geq r f(p+1)+(n-r) f(p)= \\
& =(A-n p) f(p+1)+(n(p+1)-A) f(p)= \\
& =A(f(p+1)-f(p))-n(p f(p+1)-(p+1) f(p))
\end{aligned}
$$

that is,

$$
A \leq\left(\sum_{i=1}^{n} f\left(x_{i}\right)+n(p f(p+1)-(p+1) f(p))\right) /(f(p+1)-f(p)) .
$$

Roman's ideas [39] can be used to prove this inequality for general $p \in \mathbb{Z}$.
Theorem 3.4 (Roman's inequality). Let $I \subset \mathbb{R}$ be an interval, $f: I \rightarrow \mathbb{R}$ a strictly increasing convex or a strictly decreasing concave function, $n \in \mathbb{N}$,
$x_{1}, \ldots, x_{n}, p, p+1 \in I \cap \mathbb{Z}$. Then

$$
\sum_{i=1}^{n} x_{i} \leq \frac{\sum_{i=1}^{n} f\left(x_{i}\right)}{f(p+1)-f(p)}+n \cdot \frac{p f(p+1)-(p+1) f(p)}{f(p+1)-f(p)}
$$

Equality holds if and only if $x_{i} \in\{p, p+1\}$ for every $1 \leq i \leq n$ or $\left\{x_{1}, \ldots, x_{n}, p, p+1\right\} \subset I^{\prime}$ for an interval $I^{\prime}$ on which $f$ is linear.

It can be shown that the best choice of $p$ is indeed $\lfloor A / n\rfloor$, hence Roman's inequality follows from Jensen's one. We note that Irving's method [27] for $s=$ $t=2$ is nothing else but Jensen's inequality for integers; however, for higher values of $s$ and $t$ it may give much better results. The advantage of Roman's bound is that we may choose the parameter $p$ freely to obtain an upper bound on $A=\sum x_{i}$ in a comfortable way, while in Jensen's inequality one has to use $\lfloor A / n\rfloor$, where we are about to estimate $A$. We will use the following bound that was explicitly proved in [39].

TheOrem 3.5 (Roman's bound [39]). Let $G=(A, B ; E)$ be a $K_{s, t}$-free bipartite graph of size $(m, n)$, and let $p \geq s-1$. Then the number of edges in $G$ satisfy

$$
e(G) \leq \frac{(t-1)}{\binom{p}{s-1}}\binom{m}{s}+n \cdot \frac{(p+1)(s-1)}{s}
$$

Equality holds if and only if every vertex in $B$ has degree $p$ or $p+1$ and every $s$-tuple in $A$ has exactly $t-1$ common neighbors in $B$.

Definition 3.6. For $s, t, m, n, p \in \mathbb{N}, p \geq s-1$, let

$$
R(s, t, m, n, p):=\frac{(t-1)}{\binom{p}{s-1}}\binom{m}{s}+n \cdot \frac{(p+1)(s-1)}{s}
$$

Remark 3.7. If $(t, v, k, \lambda)$ are admissible parameters in the sense of Definition 1.7 , then $R(t, \lambda+1, v, b, k)=b k=r v$ is integer.

The incidence graphs of $t-(v,\{k, k+1\}, \lambda)$ designs are $K_{t, \lambda+1}$-free, and these are precisely the graphs that satisfy the conditions of equality in Roman's bound. Bipartite graphs that are in some sense very close to $2-(v,\{k, k+1\}, 1)$ designs were also considered in [12].

Example 3.8. a) If we delete one point arbitrarily from a $t-(v, k, \lambda)$ design $\mathcal{D}$, we obtain a $t-(v-1,\{k-1, k\}, \lambda)$ design $\mathcal{D}^{\prime}$.
b) Take a $2-(v, k, 1)$ design $\mathcal{D}$ and delete a block from it with all, or all but one of its points. The obtained structure $\mathcal{D}^{\prime}$ will be a $2-(v-k+a,\{k-1, k\}, 1)$ design, $a \in\{0,1\}$.
c) Delete two intersecting lines from an affine plane of order $n$ (a $2-\left(n^{2}, n, 1\right)$ design). In this way we get a $2-\left(n^{2}-2 n+1,\{n-2, n-1\}, 1\right)$ design.

### 3.2. Results on the Zarankiewicz problem

To prove our first result, we need a theorem of Metsch.
Result 3.9 (Metsch [34]). Let $n \geq 15,(\mathcal{P}, \mathcal{L}, \mathcal{I})$ be an incidence structure with $|\mathcal{P}|=n^{2}+n+1,|\mathcal{L}| \geq n^{2}+2$ such that every line in $\mathcal{L}$ is incident with $n+1$ points of $\mathcal{P}$ and every two lines have at most one point in common. Then a projective plane $\Pi$ of order $n$ exists and $(\mathcal{P}, \mathcal{L}, \mathcal{I})$ can be embedded into $\mathcal{P}$.

Lemma 3.10. Let $n \geq 15, G=(\mathcal{P}, \mathcal{L} ; \mathcal{I})$ be an incidence graph with $|\mathcal{P}|=n^{2}+$ $+n+1,|\mathcal{L}| \geq n^{2}+2$ such that every line in $\mathcal{L}$ is incident with at least $n+1$ points of $\mathcal{P}$, and every two lines have at most one point in common. Then a projective plane $\Pi$ of order $n$ exists, and $(\mathcal{P}, \mathcal{L}, \mathcal{I})$ can be embedded into $\mathcal{P}$; specially, every line in $\mathcal{L}$ is incident with exactly $n+1$ points of $\mathcal{P}$.

Proof. By deleting edges from $G$, we can obtain a graph $G^{\prime}=\left(\mathcal{P}, \mathcal{L}, \mathcal{I}^{\prime}\right)$ in which the vertices of $\mathcal{L}$ have degree exactly $n+1$. Then, by Theorem 3.9, $G^{\prime}$ is a subgraph of a projective plane $\Pi$ of order $n$. Now suppose that there is a line $l$ in $\mathcal{L}$ that has degree at least $n+2$ in $G$. This means that there exists a point $P$ such that $l$ is incident with $P$ in $G$, but not in $\Pi$. Then each of the $n+1$ lines passing through $P$ in $\Pi$ intersects $l$ in a point different from $P$. As $|\mathcal{L}| \geq n^{2}+1$, at least one of these lines is a line of $G$ as well, but it intersects $l$ in at least two points in $G$, a contradiction. Hence every line has $n+1$ points in $G$.

Theorem 3.11. Let $n \geq 15$, and $c \leq n / 2$. Then

$$
Z_{2,2}\left(n^{2}+n+1-c, n^{2}+n+1\right) \leq\left(n^{2}+n+1-c\right)(n+1) .
$$

Equality holds if and only if a projective plane of order $n$ exists. Moreover, graphs giving equality are subgraphs of the incidence graph of a projective plane of order $n$.

Proof. If a projective plane of order $n$ exists, deleting $c$ of its lines yields a graph on $\left(n^{2}+n+1-c, n^{2}+n+1\right)$ vertices and $\left(n^{2}+n+1-c\right)(n+1)$ edges.

Suppose that $G=(A, B ; E)$ is a $K_{2,2}$-free graph on $\left(n^{2}+n+1-c, n^{2}+\right.$ $+n+1)$ vertices and $e(G) \geq|A|(n+1)$ edges. Let $m$ be the number of vertices in $A$ of degree at most $n$ (low-degree vertices). Assume that $m \geq n-c$. Delete $(n-c)$ low-degree vertices to obtain a graph $G^{\prime}$ on $\left(n^{2}+1, n^{2}+n+1\right)$ vertices with at least $\left(n^{2}+1\right)(n+1)+(n-c)$ edges. By Roman's bound with $p=n$, $Z_{2,2}\left(n^{2}+1, n^{2}+n+1\right) \leq\left(n^{2}+1\right)(n+1)+(n-1) / 2$, hence $n-c \leq(n-1) / 2$. This contradicts $c \leq n / 2$, thus $m<n-c$ must hold.

Now delete all the low-degree vertices from $A$ to obtain a graph $G^{\prime}$ on the vertex sets $\left(A^{\prime}, B\right)$ with $\left|A^{\prime}\right| \geq n^{2}+2,|B|=n^{2}+n+1$. Then every vertex in $A^{\prime}$ has degree at least $n+1$, hence we can apply Lemma 3.10 to derive that $G^{\prime}$ can be embedded into a projective plane $\Pi$ of order $n$, therefore every vertex in $A^{\prime}$ has degree $n+1$, which combined with $e(G) \geq|A|(n+1)$ yields that every vertex in $A$ has degree $n+1$ (in $G$ ), thus $G$ itself can be embedded into $\Pi$.

Remark 3.12. If we knew $Z_{2,2}\left(n^{2}+1, n^{2}+n+1\right) \leq\left(n^{2}+1\right)(n+1)+\delta$, then the above argument would hold for $c<n-\delta$. Removing $n$ points (or lines) from a projective plane of order $n$ we get $Z_{2,2}\left(n^{2}+1, n^{2}+n+1\right) \geq\left(n^{2}+1\right)(n+$ +1 ). Note that an affine plane plus an extra line containing a single point shows $Z_{2,2}\left(n^{2}, n^{2}+n+1\right) \geq n^{2}(n+1)+1$.

Question 3.13. Is it true that $Z_{2,2}\left(n^{2}+1, n^{2}+n+1\right) \leq\left(n^{2}+1\right)(n+1)$ (if $n$ is large enough)?

Remark 3.14. The upper bound on the number of edges in Theorem 3.11 is a direct consequence of Roman's bound if $c(c-1)<2 n$ without assuming $n \geq 15$.

The next result is based on a very simple observation, which was also pointed out by Guy [23], p138, point C. Let $\mathcal{F}$ be a subgraph-closed family of bipartite graphs, that is, if $G \in \mathcal{F}$ and $H$ is a subgraph of $G$, then $H \in \mathcal{F}$. For example, $K_{s, t}$-free graphs clearly form a subgraph-closed family. Let $\mathcal{F}(m, n)=$ $\{G=(A, B ; E) \in \mathcal{F}:|A|=m,|B|=n\}$, and let $\operatorname{ex}_{\mathcal{F}}(m, n)=\max \{e(G): G \in$ $\in \mathcal{F}(m, n)\}$, and let $\operatorname{Ex}_{\mathcal{F}}(m, n)=\left\{G \in \mathcal{F}(m, n): e(G)=\operatorname{ex}_{\mathcal{F}}(m, n)\right\}$. Graphs of $\operatorname{Ex}_{\mathcal{F}}(m, n)$ are called extremal.

Theorem 3.15. Let $\mathcal{F}$ be a subgraph-closed family of bipartite graphs, suppose that $\operatorname{ex}_{\mathcal{F}}(m, n) \leq e$, and let $c \in \mathbb{N}$. Then
(1) $\operatorname{ex}_{\mathcal{F}}(m+c, n) \leq e+c\lfloor e / m\rfloor$;
(2) $\operatorname{ex}_{\mathcal{F}}(m, n+c) \leq e+c\lfloor e / n\rfloor$.

Moreover, if equality holds in, say, (1) for some $c \geq 1$, then equality holds for all $c^{\prime} \in \mathbb{N}, 0 \leq c^{\prime}<c$ as well, and any $G \in \operatorname{Ex}_{\mathcal{F}}(m+c, n)$ has an induced subgraph that is in $\operatorname{Ex}_{\mathcal{F}}(m+c-1, n)$.

Proof. It is enough to prove (1), as (2) is completely analogous. We prove the assertion by induction on $c$. The statement is trivial if $c=0$. Let $d=\lfloor e / m\rfloor$. Suppose $\operatorname{ex}_{\mathcal{F}}(m+c, n) \geq e+c d$, and let $G=(A, B ; E) \in \operatorname{Ex}_{\mathcal{F}}(m+c, n)$. There is no vertex of degree strictly smaller than $d$ in $A$, otherwise removing such a vertex we would obtain a graph in $\mathcal{F}(m+c-1, n)$ with more than $e+(c-1) d$ edges, which is not possible by the inductive hypothesis. Consider an arbitrary subgraph of $G$ on $(m, n)$ vertices. By the definition of $d$, we find a vertex in $A$ of degree $d$. Removing this vertex we obtain a graph of $\mathcal{F}(m+c-1, n)$ with at least, hence (by the inductive hypothesis) exactly $e+(c-1) d$ edges. Thus $\operatorname{ex}_{\mathcal{F}}(m+c-1, n)=e+(c-1) d$, and $\operatorname{ex}_{\mathcal{F}}(m+c, n)=e(G)=e+c d$.

For example, the above theorem can be used if we start from a design or a $2-(v,\{k, k+1\}, 1)$ design obtained by deleting a block from a $2-\left(v^{\prime}, k+1,1\right)$ (Example $3.8 \mathbf{b}$ )).

Corollary 3.16. (i) Let $(t, v, k, \lambda)$ be admissible parameters (with $b=$ $\left.=\lambda\binom{v}{t} /\binom{k}{t}, r=\lambda\binom{v-1}{t-1} /\binom{k-1}{t-1}\right)$, and let $0 \leq c \in \mathbb{N}$. Then

$$
\begin{equation*}
Z_{t, \lambda+1}(v+1+c, b) \leq r v+\lambda \frac{\binom{v}{t-1}}{\binom{k}{t-1}}+c(r-1) . \tag{3.1}
\end{equation*}
$$

(ii) Let $(2, v, k, 1)$ be admissible parameters. Then

$$
\begin{equation*}
Z_{2,2}(v-k+c, b-1) \leq(v-k) r+c(r-1) . \tag{3.2}
\end{equation*}
$$

Moreover, if a $2-(v, k, 1)$ design exists, then equality holds in (3.2) for all $0 \leq c \leq k$.

Proof. (i) We apply Theorem 3.15 with $m=v+1, n=b$. By Roman's bound we see $r v=R(t, \lambda+1, v, b, k)=\lambda\binom{v}{t} /\binom{k}{t-1}+b(k+1)(t-1) / t$, furthermore

$$
\begin{aligned}
& Z_{2, \lambda+1}(v+1, b) \leq e:=R(t, \lambda+1, v+1, b, k)= \\
& \quad=\lambda \frac{\binom{v+1}{t}}{\binom{k}{t-1}}+\frac{b(k+1)(t-1)}{t}=r v+\lambda \frac{\binom{v}{t-1}}{\binom{k}{t-1}} .
\end{aligned}
$$

It is easy to see that $r<\lambda \frac{\left(\begin{array}{c}v \\ t-1) \\ (t-1) \\ k\end{array}\right)}{}$, thus $\lfloor e /(v+1)\rfloor=r-1$.
(ii) Here $r=(v-1) /(k-1)$. Simple computations show that

$$
Z_{2,2}(v-k, b-1) \leq R(2,2, v-k, b-1, k-1)=r(v-k),
$$

thus the case $c=0$ is verified. As

$$
\begin{aligned}
Z_{2,2}(v-k+1, b-1) \leq e & :=R(2,2, v-k+1, b-1, k-1)= \\
& =r(v-k)+(v-k) /(k-1)<r(v-k)+r,
\end{aligned}
$$

Theorem 3.15 with $m=v-k+1, n=b-1$ proves the assertion.
We remark that a $t-(v, k, 1)$ design is also called a Steiner system; in particular, $2-(v, 3,1)$ and $3-(v, 4,1)$ designs are also known as Steiner triple systems (STS) and Steiner quadruple systems (SQS), respectively (see e.g. [13]). For $k=3,4$ or 5 , a $2-(v, k, 1)$ design exists whenever $v \equiv 1$ or $3(\bmod 6)$, $v \equiv 1$ or $4(\bmod 12)$, or $v \equiv 1$ or $5(\bmod 20)$, respectively. These can be used to obtain some exact values of $Z_{2,2}(m, n)$.

In case of affine planes, embeddability theorems are available, thus we can formulate stronger results. Recall that an affine plane of order $n$ is always embeddable into a projective plane of order $n$. Totten [41] also has a result on the complement of two lines in a projective plane (that is, we delete one line and all its points from an affine plane).

Result 3.17 (Totten [41]). Let $\mathcal{S}=(\mathcal{P}, \mathcal{L})$ be a finite linear space (that is, an incidence structure where any two distinct points are contained in a unique line) with $|\mathcal{P}|=n^{2}-n,|\mathcal{L}|=n^{2}+n-1,2 \leq n \neq 4$, and every point having degree $n+1$. Then $\mathcal{S}$ can be embedded into a projective plane of order $n$.

Corollary 3.18. Let $\mathcal{S}=(\mathcal{P}, \mathcal{L})$ be a finite partial linear space (that is, an incidence structure where any two distinct points are contained in at most one line) with $|\mathcal{P}|=n^{2}-n,|\mathcal{L}|=n^{2}+n-1, n>4$, in which the number of flags is at least $\left(n^{2}-n\right)(n+1)$. Then $\mathcal{S}$ is a linear space, and it can be embedded into a projective plane of order $n$.

Proof. As $R\left(2,2, n^{2}-n, n^{2}+n-1, n-1\right)=\left(n^{2}-n\right)(n+1)$, each line in $\mathcal{L}$ has degree $n-1$ or $n$, and any two distinct points must be contained in a unique line. The average degree of a point is $n+1$. Now suppose that there is a point $P$ of degree at least $n+2$. Then the number of points on the lines incident with $P$ is at least $1+(n+2)(n-2)=n^{2}-3>|\mathcal{P}|=n^{2}-n($ by $n>4)$. Hence every point has degree $n+1$, so by Totten's Result 3.17, $\mathcal{S}$ is the complement of two lines in a projective plane of order $n$.

Corollary 3.19. Let $c \in \mathbb{N}$. Then

$$
\begin{align*}
& Z_{2,2}\left(n^{2}+c, n^{2}+n\right) \leq n^{2}(n+1)+c n,  \tag{3.3}\\
& Z_{2,2}\left(n^{2}-n+c, n^{2}+n-1\right) \leq\left(n^{2}-n\right)(n+1)+c n,  \tag{3.4}\\
& Z_{2,2}\left(n^{2}-2 n+1+c, n^{2}+n-2\right) \leq\left(n^{2}-2 n+1\right)(n+1)+c n, \text { if } n \geq 4 \text {. } \tag{3.5}
\end{align*}
$$

Equality can be reached in all three inequalities if a projective plane of order $n$ exists and $c \leq n+1, c \leq 2 n$, or $c \leq 3(n-1)$, respectively.

Moreover, if $c \leq n+1$, or $c \leq 2 n$ and $n>4$, then graphs reaching the bound in (3.3) or (3.4), respectively, can be embedded into a projective plane of order $n$.

Proof. The parameters of an affine plane, $\left(2, n^{2}, n, 1\right)$ (with $b=n^{2}+n$, $r=n+1$ ) are admissible. Hence (3.3) and (3.4) follow from Corollary 3.16. To apply Theorem 3.15 in (3.5), simply calculate that

$$
R\left(2,2, n^{2}-2 n+1, n^{2}+n-2, n-2\right)=\left(n^{2}-2 n+1\right)(n+1)=e,
$$

and that

$$
\begin{aligned}
R\left(2,2, n^{2}-2 n+2, n^{2}+n-2, n-2\right) & =e+n+1 /(n-2)< \\
& <\left(n^{2}+2 n+2\right)(n+1) \quad(n \geq 4)
\end{aligned}
$$

By taking a projective plane of order $n$, and deleting one, two, or three of its lines and all but $c$ of their points each of which is contained in only one of the deleted lines, we can reach equality in (3.3), (3.4), and (3.5), respectively.

In (3.3), Theorem 3.15 also provides an affine plane of order $n$ as an induced subgraph in graphs obtaining equality. Now the $c$ extra points of degree $n$ must be incident with pairwise non-intersecting lines to avoid $C_{4}$ 's in the graph; that is, they can be considered as the common points of $c$ distinct parallel classes. Adding the missing $n+1-c$ ideal points and the line at infinity, we obtain a projective plane of order $n$.

In (3.4), Theorem 3.15 provides us an extremal $C_{4}$-free subgraph $G=$ $=(A, B)$ on $\left(n^{2}-n, n^{2}+n-1\right)$ vertices and $\left(n^{2}-n\right)(n+1)$ edges in graphs reaching equality. By Corollary 3.18, $G$ can be embedded into a projective plane of order $n$. As before, it is easy to see that the embedding extends to the $c$ extra points as well.

Next we prove a straightforward recursive inequality. For a bipartite graph $G=(A, B ; E)$ and vertex-sets $X \subset A$ and $Y \subset B$, let $G[X, Y]$ denote the subgraph of $G$ induced by $X \cup Y$.

Proposition 3.20. Let $U_{s, t}(m, n, \alpha, \beta)=Z_{s-\alpha, t}(m-\alpha, \beta)+Z_{s, t}(m-\alpha, n-\beta)+$ $+(\alpha-1) n+\beta$. Then

$$
\begin{aligned}
& Z_{s, t}(m, n) \leq \\
& \leq \min _{\alpha} \max _{\beta} \min \left\{Z_{\alpha, \beta+1}(m, n), U_{s, t}(m, n, \alpha, \beta): 1 \leq \alpha<s, t-1 \leq \beta \leq n\right\}
\end{aligned}
$$

Proof. Let $G=(A, B ; E)$ be a maximal $K_{s, t}$ free bipartite graph on $m+n$ vertices. Let $1 \leq \alpha<s$, and let $\beta$ be the largest integer for which $K_{\alpha, \beta}$ is a subgraph of $G$ (the ordering of the classes does matter). Then $|E| \leq Z_{\alpha, \beta+1}(m, n)$ follows from $G$ being $K_{\alpha, \beta+1}$-free. Now let $S \subset A$ and $T \subset B$ induce a $K_{\alpha, \beta}$, and let $U=A \backslash S, V=B \backslash T$. Then $G[U, T]$ must be $K_{s-\alpha, t}$-free, $G[U, V]$ is $K_{s, t}$-free; moreover, since no $K_{\alpha, \beta+1}$ can be found in $G$, every vertex in $V$ may have at most $\alpha-1$ neighbors in $S$. Summing up the maximum number of edges in each part, we get $|E| \leq \alpha \beta+Z_{s-\alpha, t}(m-\alpha, \beta)+Z_{s, t}(m-\alpha, n-\beta)+$ $+(\alpha-1)(n-\beta)=U_{s, t}(m, n, \alpha, \beta)$. As $G$ is maximal, it must contain a $K_{\alpha, t-1}$ for all $\alpha<s$, hence we have $\beta \geq t-1$.

Remark 3.21. In particular, the case $\alpha=1$ of this inequality investigates the vertex with largest degree. $Z_{s, t}(m, 0)$ is defined to be zero (which occurs above for $\beta=n$ ). Note that we may interchange the role of the classes, that is, write up the above inequality for $Z_{t, s}(n, m)$. We will call this the transpose of Proposition 3.20.

Remark 3.22. In case of $\alpha=s-1$, the function $U_{s, t}(m, n, s-1, \beta)$ is nonincreasing in $\beta(\beta \geq t-1)$, while $Z_{s-1, \beta+1}(m, n)$ is clearly non-decreasing. Thus the maximum of the minimum of these two values in $\beta$ can be found easily.

Proof.
$U_{s, t}(m, n, s-1, \beta)=Z_{1, t}(m-s+1, \beta)+Z_{s, t}(m-s+1, n-\beta)+(s-2) n+\beta=$

$$
(t-1)(m-s+1)+(s-2) n+\beta+Z_{s, t}(m-s+1, n-\beta) .
$$

By adding a vertex of degree $t-1$, we have $Z_{s, t}(m-s+1, n-\beta) \geq$ $Z_{s, t}(m-s+1, n-(\beta+1))+t-1$.

This recursion is useful in some cases. For example, Roman's bound with $p=4$ or 5 yields $Z_{3,3}(7,7) \leq 35$. We show $Z_{3,3}(7,7) \leq 33$. (Here, in fact, equality holds.) Let $\alpha=2$. For $\beta \leq 4$ we have $Z_{2, \beta+1}(7,7) \leq R(2,5,7,7,5)=33$, while $U_{3,3}(7,7,2,4)=Z_{1,3}(5,4)+Z_{3,3}(5,3)+7+4=33$. By Remark 3.22, we are done. Other examples that prove this recursion useful are the balanced $C_{4}$-free graphs.

Proposition 3.23. Let $2 \leq q \in \mathbb{N}, 3-q \leq c \leq 1+q$. Then

$$
Z_{2,2}\left(q^{2}+c, q^{2}+c\right) \leq\left(q^{2}+c\right)\left(q+\frac{1}{2}\right)+\left(\frac{c}{2}-1\right) q+\frac{c}{2}+\frac{(c-1)(c-2)}{2(q-1)} .
$$

Proof. Consider the bounds in Corollary 3.22 with $s=t=2$. If $\beta \leq q$, then $Z_{1, \beta+1}\left(q^{2}+c, q^{2}+c\right) \leq q\left(q^{2}+c\right)$, which is smaller than the bound stated provided that $c \geq 3-q$. Hence we may assume $\beta \geq q+1$. Then the second expression is $\left(q^{2}+c-1\right)+\beta+Z_{2,2}\left(q^{2}+c-1, q^{2}+c-\beta\right) \leq q^{2}+q+c+$ $+Z_{2,2}\left(q^{2}+c-1, q^{2}+c-q-1\right)$. Applying Roman's bound with $p=q-1$ to $Z_{2,2}\left(q^{2}+c-q-1, q^{2}+c-1\right)$, we get the desired result.

Remark 3.24. It is easy to calculate that for $3-q \leq c \leq 1+q$, Roman's upper bound on $Z_{2,2}\left(q^{2}+c, q^{2}+c\right)$ gives the best result if we set $p=q$. The bound in Proposition 3.23 is smaller than Roman's one by

$$
\frac{q-c}{2}+\frac{(2 q-c)(c-1)}{2 q(q-1)} .
$$

In the rest of this section we tackle Roman's bound and the recursive idea to establish some results that are tight if we are close to a design. Without a strong embedding theorem like Result 3.9, we obtain weaker results. The next proposition is a direct consequence of Roman's bound.

Proposition 3.25. Assume that the parameters $(t, v, k, \lambda)$ are admissible, and let $c_{0}$ be the largest integer such that $\lambda\left(\binom{v-c_{0}}{t}+c_{0}\binom{v-1}{t-1}-\binom{v}{t}\right)<\binom{k-1}{t-1}$. Then for every $0 \leq c \leq c_{0}$,

$$
Z_{t, \lambda+1}(v-c, b) \leq r(v-c)
$$

Equality can be reached if a $t-(v, k, \lambda)$-design exists. Moreover, if $c<c_{0}$, then in the graphs obtaining equality, the vertices in the class of size $v-c$ have degree $r$. In particular, the condition for $t=2$ is $c_{0}\left(c_{0}-1\right)<2(k-1) / \lambda$.

Proof. Removing $c$ points from the incidence graph of a $t-(v, k, \lambda)$ design we obtain a $K_{t, \lambda+1}$-free graph on $(v-c, b)$ nodes and $r(v-c)$ edges.

On the other hand, using $r v=b k$ and $b k / t=\lambda\binom{v}{t} /\binom{k-1}{t-1}$, Roman's bound with $p=k-1$ yields

$$
\begin{aligned}
Z_{t, \lambda+1}(v-c, b) & \leq\left\lfloor\frac{\lambda}{\binom{k-1}{t-1}}\binom{v-c}{t}+b \cdot \frac{k(t-1)}{t}\right\rfloor= \\
& =r(v-c)+\left\lfloor\frac{\lambda\left(\binom{v-c}{t}+c\binom{v-1}{t-1}-\binom{v}{t}\right)}{\binom{k-1}{t-1}}\right\rfloor .
\end{aligned}
$$

Suppose that $G=(A, B)$ is $K_{t, \lambda+1}$-free on $(v-c, b)$ vertices and $(v-c) r$ edges, $c<c_{0}$. Assume that there is a vertex $u \in A$ with degree smaller than $r$. Removing $u$ from $A$, we obtain a graph on $(v-c-1, b)$ vertices and more than $(v-c-1) r$ edges, which contradicts our upper bound.

The recursive inequality of Proposition 3.20 can be used to achieve another bound in a more special case.

Proposition 3.26. Let ( $2, v, k, 1$ ) be admissible parameters. Then

$$
Z_{2,2}(v+1, b) \leq b k+b-k(r-1) .
$$

Proof. Let $G=(A, B ; E)$ be an extremal $K_{2,2}$-free bipartite graph of size ( $v+$ $+1, b)$. Then there must be a vertex in $B$ with degree at least $k+1$. Thus by Remark 3.22, we may use the transpose of Proposition 3.20 with $\alpha=1, \beta=k+$ +1 to obtain

$$
e(G) \leq U_{2,2}(b, v+1,1, k+1)=(b-1)+k+1+Z_{2,2}(b-1, v-k) .
$$

Now $Z_{2,2}(b-1, v-k) \leq(v-k) r$, as deleting a block and its points from a $2-(v, k, 1)$ design would result in a structure seen in Example 3.8 (so $R(2,2, v-k, b-1, k-1)=(v-k) r)$. Hence $e(G) \leq k+b+(v-k) r=b k+$ $+b-k(r-1)$.

Corollary 3.27. Let $n \geq 2$. Then

$$
Z_{2,2}\left(n^{2}+n+2, n^{2}+n+1\right) \leq\left(n^{2}+n+1\right)(n+1)+1,
$$

and equality holds if and only if a projective plane of order $n$ exists. Moreover, any graph $G$ reaching equality can be obtained in the following way: take a projective plane $(\mathcal{P}, \mathcal{L})$ of order $n$, let $A=\mathcal{P} \cup\left\{u_{0}\right\}\left(u_{0} \notin \mathcal{L} \cup \mathcal{P}\right), B=\mathcal{L}$. Take any point $v \in \mathcal{L}$, and let $\left\{u_{1}, \ldots, u_{n+1}\right\}$ be its neighbors in $\mathcal{P}$. Let $H$ be any subset of the neighbors of $u_{1}$, for which $v \notin H$. Delete the edges $u_{1} v^{\prime}$ for all
$v^{\prime} \in H$, and add the edges $u_{0} v$ and $u_{0} v^{\prime}$ for all $v^{\prime} \in H$. In particular, there must be a vertex in $A$ with degree at most $n / 2+1$.

Proof. Proposition 3.26 applied to a projective plane of order $n$ (with parameters $v=b=n^{2}+n+1, t=2, \lambda=1, k=n+1$ ) yields

$$
Z_{2,2}\left(n^{2}+n+1, n^{2}+n+2\right) \leq\left(n^{2}+n+1\right)(n+1)+1
$$

Now let $G=(A, B)$ be a $C_{4}$-free graph on $\left(n^{2}+n+2, n^{2}+n+1\right)$ vertices and $\left(n^{2}+n+1\right)(n+1)+1$ edges. Then there must be a vertex $v \in B$ of degree at least $n+2$. Consider the proof of Proposition 3.26. As

$$
\begin{aligned}
U_{2,2}(b, v+1,1, k+2) & =n^{2}+n+n+3+Z_{2,2}\left(n^{2}+n, n^{2}\right) \leq \\
& \leq n^{2}+2 n+3+\left(n^{2}-1\right)(n+1)= \\
& =\left(n^{2}+n\right)(n+1)+2<\left(n^{2}+n+1\right)(n+1)+1
\end{aligned}
$$

$v$ must have degree $n+2$. To reach equality, the decomposition in the proof of Proposition 3.20 (with $\alpha=1, \beta=n+2$ ) assures that removing $v$ and its neighbors $N(v)=\left\{u_{0}, \ldots, u_{n+1}\right\}$ from $G$, we find an affine plane of order $n$, whose points and lines correspond to $A \backslash N(v)$ and $B \backslash\{v\}$, respectively; moreover, the degree of the vertices of $B \backslash\{v\}$ in $G$ is $n+1$. As these vertices have precisely $n$ neighbors in $A \backslash N(v)$, each one has to be adjacent to one of the $u_{i} \mathrm{~s}$. On the other hand, any $u_{i}(0 \leq i \leq n+1)$ may be adjacent only to the $n$ lines of one parallel class (besides $v$ ), hence $\operatorname{deg}\left(u_{i}\right) \leq n+1$. Let $\mathcal{L}_{i} \subset A \backslash\{v\}$ be the parallel classes of $\mathcal{L}(1 \leq i \leq n+1)$. We may assume that $N\left(u_{i}\right) \backslash\{v\} \subset \mathcal{L}_{i}$ for all $1 \leq i \leq n+1$. Let $H=N\left(u_{0}\right) \backslash\{v\}$; we may assume $H \subset \mathcal{L}_{1}$. Then $N\left(u_{i}\right)=\{v\} \cup \mathcal{L}_{i}$ for all $2 \leq i \leq n+1$, and $N\left(u_{1}\right)=\{v\} \cup \mathcal{L}_{1} \backslash H$. Then $\operatorname{deg}\left(u_{0}\right)+\operatorname{deg}\left(u_{1}\right)=n+2$.

Proposition 3.28. Let $c \geq 1$ and $n \geq 2$. Then

$$
Z_{2,2}\left(n^{2}+n+2+c, n^{2}+n+1\right) \leq\left(n^{2}+n+1\right)(n+1)+c n+1
$$

If $n \geq 3$, then

$$
Z_{2,2}\left(n^{2}+n+2+c, n^{2}+n+1\right) \leq\left(n^{2}+n+1\right)(n+1)+c n
$$

Proof. Let $\mathcal{F}$ be the family of $C_{4}$-free graphs. The first statement follows from Proposition 3.27 and Theorem 3.15 (with $m=n^{2}+n+2$ and $d=n$ ). Now suppose $n \geq 3$ and that equality holds for some $c \geq 1$, thus for $c=1$ as well. Then any $G \in \operatorname{Ex}_{\mathcal{F}}\left(n^{2}+n+3, n^{2}+n+1\right)$ induces a graph from
$\operatorname{Ex}_{\mathcal{F}}\left(n^{2}+n+2, n^{2}+n+1\right)$, which has a vertex with degree at most $n / 2+1$ by Proposition 3.27. Deleting this vertex from $G$ we would have

$$
\begin{aligned}
\operatorname{ex}_{\mathcal{F}}\left(n^{2}+n+2, n^{2}+n+1\right) & \geq\left(n^{2}+n+1\right)(n+1)+n+1-(n / 2+1) \\
& >\left(n^{2}+n+1\right)(n+1)+1,
\end{aligned}
$$

a contradiction.
There are ad hoc ideas that may help when determining Zarankiewicz numbers for small parameters, see Guy [23], p 138. The next proposition illustrates such a case.

Proposition 3.29. $Z_{2,2}(16,17) \leq 70$.
Proof. Suppose to the contrary that there exist a $C_{4}$-free bipartite graph $G=(A, B ; E)$, where $|A|=16,|B|=17,|E|=71$. As $Z_{2,2}(16,16)=$ $=Z_{2,2}(15,17)=67$ (see Table I), every vertex in $G$ has degree at least four. Corollary 3.22 yields that there can be no vertex of degree six. Hence the degree sequence of $A$ and $B$ are $\left\{4^{9}, 5^{7}\right\},\left\{4^{14}, 5^{3}\right\}$, where the superscripts denote the multiplicity of that degree. Let $v \in A, \operatorname{deg}(v)=5$, and let $N(v)=\left\{u_{1}, \ldots, u_{5}\right\}$. Then $\operatorname{deg}\left(u_{i}\right)=4$ for $1 \leq i \leq 5$, otherwise the pairwise disjoint sets $N\left(u_{i}\right) \backslash\{v\} \subset A \backslash\{v\}, 1 \leq i \leq 5$, would have more than 15 elements. Let $v_{i} \in A$ a vertex with degree $5,1 \leq i \leq 5$. Then $\left|N\left(v_{1}\right) \cup \ldots \cup N\left(v_{5}\right)\right| \geq 5+4+$ $+3+2+1=15$, but there are only 14 vertices of degree four in $B$.

### 3.3. Lower bounds for $s=t=2$

Now let us collect some constructions regarding the case $s=t=2$. As a general principle, if we have an extremal graph $G=(A, B)$, we can always delete the lowest degree vertex from $A$ (or $B$ ) to obtain a graph on $(|A|-1,|B|)$ (or $(|A|,|B|-1)$ ) vertices with many edges. This trivial method gives good results in many cases. Another simple idea is that if we find $k$ points in $A$ such that no two of them has a common neighbor, then we can add one vertex to $B$ and connect it with those vertices. Note that $k=1$ always works. Without the sake of completeness, we illustrate these methods in the upcoming propositions.

Proposition 3.30. $Z_{2,2}(14,25)=80$.
Proof. For basic facts about ovals we refer to [24]. Let $O$ be an oval in $\mathrm{PG}(2,5)$, and let $\mathcal{L}_{0}$ be the set of its six tangent lines. Let $\mathcal{P}_{0}$ be the set of $\binom{6}{2}=15$ outer points of $O$ together with two arbitrarily chosen points of $O$.

Delete $\mathcal{P}_{0}$ and $\mathcal{L}_{0}$ from $\mathrm{PG}(2,5)$. The resulting graph clearly has size $(14,25)$. Any inner point of $O$ is incident with zero tangent to $O$, whereas a point of $O$ is incident with precisely one tangent to $O$. Thus the number of edges is $4 \cdot 5+$ $+10 \cdot 6=80$. On the other hand, $R(2,2,14,25,3)<81$.

Proposition 3.31. Let $\mathcal{D}$ be a $2-(v, k, 1)$ design, and let $\ell^{\mathcal{D}}(i)$ be the least number of points that the union of $i$ blocks may cover in $\mathcal{D}$. Let $f^{\mathcal{D}}(c)$ be the maximal value of $i$ for which $\ell^{\mathcal{D}}(i) \leq c$. Then

$$
Z_{2,2}(v-c, b) \geq(v-c) r+f^{\mathcal{D}}(c) .
$$

Proof. By definition of $f^{\mathcal{D}}(c)$, we can delete $c$ points from $\mathcal{D}$ so that $f^{\mathcal{D}}(c)$ blocks become empty. We can connect these blocks with any one of the points without creating a $C_{4}$, so we can add altogether $f^{\mathcal{D}}(c)$ edges to the $(v-c) r$ edges that remain after the deletion.

| ${ }^{m}$ | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 7 | $\mathbf{2 1}$ |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |
| 8 | $\mathbf{2 2}$ | $\mathbf{2 4}$ |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |
|  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |
| 9 | $\mathbf{2 4}$ | $\mathbf{2 6}$ | $\mathbf{2 9}$ |  |  |  |  |  |  |  |  |  |  |  |  |  |  |
| 10 | $\mathbf{2 5}$ | $\mathbf{2 8}$ | $\mathbf{3 1}$ | $\mathbf{3 4}$ |  |  |  |  |  |  |  |  |  |  |  |  |  |
| 11 | $\mathbf{2 7}$ | $\mathbf{3 0}$ | $\mathbf{3 3}$ | $\mathbf{3 6}$ | $\mathbf{3 9}$ |  |  |  |  |  |  |  |  |  |  |  |  |
| 12 | $\mathbf{2 8}$ | $\mathbf{3 2}$ | $\mathbf{3 6}$ | $\mathbf{3 9}$ | $\mathbf{4 2}$ | $\mathbf{4 5}$ |  |  |  |  |  |  |  |  |  |  |  |
| 13 | $\mathbf{3 0}$ | $\mathbf{3 3}$ | $\mathbf{3 7}$ | $\mathbf{4 0}$ | $\mathbf{4 4}$ | $\mathbf{4 8}$ | $\mathbf{5 2}$ |  |  |  |  |  |  |  |  |  |  |
| 14 | $\mathbf{3 1}$ | $\mathbf{3 5}$ | $\mathbf{3 9}$ | $\mathbf{4 2}$ | $\mathbf{4 5}$ | $\mathbf{4 9}$ | $\mathbf{5 3}$ | $\mathbf{5 6}$ |  |  |  |  |  |  |  |  |  |
| 15 | $\mathbf{3 3}$ | $\mathbf{3 6}$ | $\mathbf{4 0}$ | $\mathbf{4 4}$ | $\mathbf{4 7}$ | $\mathbf{5 1}$ | $\mathbf{5 5}$ | $\mathbf{5 8}$ | $\mathbf{6 0}$ |  |  |  |  |  |  |  |  |
| 16 | $\mathbf{3 4}$ | $\mathbf{3 8}$ | $\mathbf{4 2}$ | $\mathbf{4 6}$ | $\mathbf{5 0}$ | $\mathbf{5 3}$ | $\mathbf{5 7}$ | $\mathbf{6 0}$ | $\mathbf{6 4}$ | $\mathbf{6 7}$ |  |  |  |  |  |  |  |
| 17 | $\mathbf{3 6}$ | $\mathbf{3 9}$ | $\mathbf{4 3}$ | $\mathbf{4 7}$ | $\mathbf{5 1}$ | $\mathbf{5 5}$ | $\mathbf{5 9}$ | $\mathbf{6 3}$ | $\mathbf{6 7}$ | $\mathbf{7 0}$ | $\mathbf{7 4}$ |  |  |  |  |  |  |
| 18 | $\mathbf{3 7}$ | $\mathbf{4 1}$ | $\mathbf{4 5}$ | $\mathbf{4 9}$ | $\mathbf{5 3}$ | $\mathbf{5 7}$ | $\mathbf{6 1}$ | $\mathbf{6 5}$ | $\mathbf{6 9}$ | $\mathbf{7 3}$ | $\mathbf{7 7}$ | $\mathbf{8 1}$ |  |  |  |  |  |
| 19 | $\mathbf{3 9}$ | $\mathbf{4 2}$ | $\mathbf{4 6}$ | $\mathbf{5 1}$ | $\mathbf{5 5}$ | $\mathbf{6 0}$ | $\mathbf{6 4}$ | $\mathbf{6 8}$ | $\mathbf{7 2}$ | $\mathbf{7 6}$ | $\mathbf{8 0}$ | $\mathbf{8 4}$ | $\mathbf{8 8}$ |  |  |  |  |
| 20 | $\mathbf{4 0}$ | $\mathbf{4 4}$ | $\mathbf{4 8}$ | $\mathbf{5 2}$ | $\mathbf{5 7}$ | $\mathbf{6 1}$ | $\mathbf{6 6}$ | $\mathbf{7 0}$ | $\mathbf{7 5}$ | $\mathbf{8 0}$ | $\mathbf{8 4}$ | $\mathbf{8 8}$ | $\mathbf{9 2}$ | $\mathbf{9 6}$ |  |  |  |
| 21 | $\mathbf{4 2}$ | $\mathbf{4 5}$ | $\mathbf{4 9}$ | $\mathbf{5 4}$ | $\mathbf{5 9}$ | $\mathbf{6 3}$ | $\mathbf{6 7}$ | $\mathbf{7 2}$ | $\mathbf{7 7}$ | $\mathbf{8 1}$ | 86 | $\mathbf{9 0}$ | $\mathbf{9 5}$ | $\mathbf{1 0 0}$ | $\mathbf{1 0 5}$ |  |  |
| 22 | $\mathbf{4 3}$ | $\mathbf{4 7}$ | $\mathbf{5 1}$ | $\mathbf{5 5}$ | $\mathbf{6 0}$ | $\mathbf{6 5}$ | $\mathbf{6 9}$ | $\mathbf{7 3}$ | $\mathbf{7 8}$ | 83 | 88 | 93 | 97 | $\mathbf{1 0 1}$ | $\mathbf{1 0 6}$ | 110 |  |
| 23 | $\mathbf{4 4}$ | $\mathbf{4 8}$ | $\mathbf{5 2}$ | $\mathbf{5 7}$ | $\mathbf{6 2}$ | $\mathbf{6 6}$ | $\mathbf{7 1}$ | $\mathbf{7 5}$ | $\mathbf{8 0}$ | 85 | 90 | 95 | 100 | 105 | 110 | 113 | 116 |
| 24 | $\mathbf{4 5}$ | $\mathbf{5 0}$ | $\mathbf{5 4}$ | $\mathbf{5 8}$ | $\mathbf{6 3}$ | $\mathbf{6 8}$ | $\mathbf{7 3}$ | $\mathbf{7 8}$ | 83 | 88 | 93 | 98 | 102 | 107 | 112 | 117 | 120 |
| 25 | $\mathbf{4 6}$ | $\mathbf{5 1}$ | $\mathbf{5 5}$ | $\mathbf{6 0}$ | $\mathbf{6 5}$ | $\mathbf{7 0}$ | $\mathbf{7 5}$ | $\mathbf{8 0}$ | $\mathbf{8 5}$ | $\mathbf{9 0}$ | 95 | 100 | 105 | 110 | 115 | 120 | 125 |
| 26 | $\mathbf{4 7}$ | $\mathbf{5 3}$ | $\mathbf{5 7}$ | $\mathbf{6 1}$ | $\mathbf{6 6}$ | $\mathbf{7 2}$ | $\mathbf{7 8}$ | $\mathbf{8 1}$ | $\mathbf{8 6}$ | $\mathbf{9 1}$ | $\mathbf{9 6}$ | $\mathbf{1 0 1}$ | $\mathbf{1 0 6}$ | $\mathbf{1 1 1}$ | $\mathbf{1 1 6}$ | 121 | 126 |
| 27 | $\mathbf{4 8}$ | $\mathbf{5 4}$ | $\mathbf{5 8}$ | $\mathbf{6 3}$ | $\mathbf{6 8}$ | $\mathbf{7 3}$ | $\mathbf{7 9}$ | $\mathbf{8 3}$ | $\mathbf{8 8}$ | $\mathbf{9 3}$ | $\mathbf{9 8}$ | $\mathbf{1 0 3}$ | $\mathbf{1 0 8}$ | $\mathbf{1 1 3}$ | $\mathbf{1 1 8}$ | $\mathbf{1 2 3}$ | $\mathbf{1 2 8}$ |
| 28 | $\mathbf{4 9}$ | $\mathbf{5 6}$ | $\mathbf{6 0}$ | $\mathbf{6 4}$ | $\mathbf{6 9}$ | $\mathbf{7 5}$ | $\mathbf{8 1}$ | $\mathbf{8 5}$ | 91 | $\mathbf{9 6}$ | $\mathbf{1 0 1}$ | $\mathbf{1 0 6}$ | $\mathbf{1 1 1}$ | $\mathbf{1 1 6}$ | $\mathbf{1 2 1}$ | $\mathbf{1 2 6}$ | $\mathbf{1 3 1}$ |
| 29 | $\mathbf{5 0}$ | $\mathbf{5 7}$ | $\mathbf{6 1}$ | $\mathbf{6 6}$ | $\mathbf{7 1}$ | $\mathbf{7 6}$ | $\mathbf{8 2}$ | 88 | 93 | 98 | 103 | 109 | $\mathbf{1 1 4}$ | $\mathbf{1 2 0}$ | $\mathbf{1 2 5}$ | $\mathbf{1 3 0}$ | $\mathbf{1 3 5}$ |
| 30 | $\mathbf{5 1}$ | $\mathbf{5 8}$ | $\mathbf{6 3}$ | $\mathbf{6 7}$ | $\mathbf{7 2}$ | $\mathbf{7 8}$ | $\mathbf{8 4}$ | 90 | 95 | 100 | 105 | 111 | 117 | 122 | 127 | $\mathbf{1 3 2}$ | $\mathbf{1 3 8}$ |
| 31 | $\mathbf{5 2}$ | $\mathbf{5 9}$ | $\mathbf{6 4}$ | $\mathbf{6 9}$ | $\mathbf{7 4}$ | $\mathbf{7 9}$ | $\mathbf{8 5}$ | 91 | 97 | 102 | 107 | 113 | 119 | 125 | 130 | 135 | 140 |

TABLE I. The table contains the best upper bounds on $Z_{2,2}(m, n)$ up to our knowledge. Bold numbers indicate equality. An exact value is in italic shape if it was not reported by Guy in [23].

In some cases we did rely on the exact values reported by Guy. Possibly undiscovered inaccuracies there may result in inaccurate values here as well.

Note that we can dualize the above proposition: if we delete vertices that represent blocks, we may add an edge to each of the points all of whose neighbors have been removed. Next we give the exact value of $\ell^{\mathcal{D}}(i)$ in some cases.

Remark 3.32. (1) For any $2-(v, k, 1)$ design $\mathcal{D}, \ell^{\mathcal{D}}(i)=i k-\binom{i}{2}$ for $1 \leq$ $i \leq 3$.
(2) Let $\mathcal{D}=\mathrm{PG}(2, q), i \leq q+1$. Then $\ell^{\mathcal{D}}(i)=i(q+1)-\binom{i}{2}$.

Let $\mathcal{D}=\operatorname{AG}(2, q), i \leq q$. Then $\ell^{\mathcal{D}}(i)=i q-\binom{i}{2}$.
Proof. In general, as any two blocks of a $2-(v, k, 1)$ design intersect in at most one point, $i \leq k+1$ blocks cover at least $k+(k-1)+\ldots+(k-i+1)=i k-\binom{i}{2}$ points. This can be reached if and only if there exist $i$ pairwise intersecting blocks in general position (no three of them have a common point). As $k \geq 2$, one can easily find three such blocks. In $\operatorname{PG}(2, q)$, a dual conic is well-known to be a set of $q+1$ lines in general position. One taken as the line at infinity, we obtain $q$ lines in general position in $\operatorname{AG}(2, q)$.

Proposition 3.33. Let $q$ be a square prime power, and let $v=q^{2}+q+1$, $w=q+\sqrt{q}+1$. Suppose that $1 \leq c \leq q-\sqrt{q}, 0 \leq d \leq c w, 0 \leq h \leq w-2$. Then
(1) $Z_{2,2}(v-c(w-1), v-d) \geq(v-c(w-1))(q+1)+c \sqrt{q}-d(q-\sqrt{q}+2-c)$;
(2) $Z_{2,2}(v-c(w-1)-h, v) \geq(v-c(w-1)-h)(q+1)+c \sqrt{q}$;
(3) $Z_{2,2}(v-c w, v-c w) \geq(v-c w)(q+1-c)$.

Proof. Let $\mathrm{PG}(2, q)=(\mathcal{P}, \mathcal{L})$, and let $\mathcal{B}_{1}=\left(\mathcal{P}_{1}, \mathcal{L}_{1}\right), \ldots, \mathcal{B}_{c}=\left(\mathcal{P}_{c}, \mathcal{L}_{c}\right)$ be $c$ pairwise disjoint Baer subplanes in it. Let $\mathcal{P}_{0}=\cup_{i=1}^{c} \mathcal{P}_{i}, \mathcal{L}_{0}=\cup_{i=1}^{c} \mathcal{L}_{i}$.
(1) Define $G=(A, B)$ in the following way. Let $A=\mathcal{P} \backslash \mathcal{P}_{0} \cup\left\{\mathcal{B}_{1}, \ldots, \mathcal{B}_{c}\right\}$ $(|A|=v-c w+c), B=\mathcal{L}$. The edges between $A \cap \mathcal{P}$ and $B$ are those defined by $\operatorname{PG}(2, q)$; furthermore, connect the vertex $\mathcal{B}_{i}$ to all the vertices of $\mathcal{L}_{i} \subset B$, $1 \leq i \leq c$. (That is, we contract the points of the Baer subplanes.) As any two lines of $\mathcal{L}_{i}$ had an intersection in $\mathcal{P}_{i}$, we do not create a $C_{4}$. Note that every $\mathcal{P}_{i}$ is a blocking set, so every line not in $\mathcal{L}_{0}$ looses precisely $c$ neighbors. Thus the $v-c w$ vertices of $A \cap \mathcal{P}$ have degree $q+1$, the $c$ new vertices have degree $w=q+\sqrt{q}+1$, thus there are $(v-c w+w)(q+1)+c \sqrt{q}$ edges in $G$. Let $\ell \in \mathcal{L}_{i} \subset \mathcal{L}_{0}$. Then $\left|\ell \cap \mathcal{P}_{j}\right|$ equals one for all $1 \leq j \leq c$ except for $j=i$, in which case it equals $\sqrt{q}+1$. Hence $\operatorname{deg}(\ell)=q+1-\sqrt{q}-(c-1)$ in $G$. There are $c(q+\sqrt{q}+1)$ lines in $\mathcal{L}_{0}$, so we may delete any $d$ of them to obtain a graph $G^{\prime}$ with the stated parameters.

| $m$ | $n$ | Lower b. | $Z_{2,2}$ |  | Upper b. | $m$ | $n$ | Lower b. | $Z_{2,2}$ |  | Upper b. |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 8 | 8 | $24^{\text {d }}$ | 24 | 24 | $\alpha=1, \beta=3$ | 13 | 13 | 52 | 52 | 52 | Re |
| 8 | 9 | $26^{\text {d }}$ | 26 | 26 | $\alpha=1, \beta=4$ | 13 | 14 | 53 p | 53 | 53 | $\alpha=1, \beta=5$ |
| 8 | 10 | 28 | 28 | 28 | g | 13 | 15 | 54 d | 55 | 55 | p |
|  |  |  |  |  |  | 13 | 16 | 57 | 57 | 58 | Re |
| 9 | 9 | $29{ }^{\text {d }}$ | 29 | 29 | $\alpha=1, \beta=4$ | 13 | 17 | 59 d | 59 | 59 | g |
| 9 | 10 | 31 d | 31 | 31 | $\alpha=1, \beta=4$ | 13 | 18 | 61 Aff | 61 | 61 | Aff |
| 9 | 11 | 33 | 33 | 33 | Aff | 13 | 19 | 64 Aff | 64 | 64 | Re |
|  |  |  |  |  |  | 13 | 20 | 66 B,d | 66 | 66 | Re |
| 10 | 10 | $34{ }^{\text {d }}$ | 34 | 34 | $\alpha=1, \beta=4$ | 13 | 21 | 67 B | 67 | 68 | Re |
| 10 | 11 | 36 d | 36 | 36 | Aff | 13 | 22 | 69 d | 69 | 70 | Re |
| 10 | 12 | 39 d | 39 | 39 | Re | 13 | 23 | 71 d | 71 | 72 | Re |
| 10 | 13 | $40{ }^{\text {d }}$ | 40 | 40 | $\alpha=1, \beta=4$ | 13 | 24 | 73 d | 73 | 73 | g |
| 10 | 14 | 42 d | 42 | 43 | Re | 13 | 25 | 75 d | 75 | 75 | g |
| 10 | 15 | 44 d | 44 | 44 | g |  |  |  |  |  |  |
| 10 | 16 | $46^{\text {d }}$ | 46 | 46 | Re | 14 | 14 | $56^{\text {B }}$ | 56 | 56 | $\alpha=1, \beta=4$ |
| 10 | 17 | 47 | 47 | 47 | g | 14 | 15 | 58 d | 58 | 58 | $\alpha=1, \beta=5$ |
|  |  |  |  |  |  | 14 | 16 | $60{ }^{\text {d }}$ | 60 | 61 | g |
| 11 | 11 | $39^{\text {d }}$ | 39 | 39 | Aff | 14 | 17 | 63 d | 63 | 63 | g |
| 11 | 12 | 42 d | 42 | 42 | Re | 14 | 18 | 65 Aff | 65 | 65 | Aff |
| 11 | 13 | 44 d | 44 | 44 |  | 14 | 19 | 68 Aff | 68 | 68 | $p=3$ |
| 11 | 14 | $45 \mathrm{p}, \mathrm{d}$ | 45 | 46 | Re | 14 | 20 | 70 d | 70 | 70 | $p=3$ |
| 11 | 15 | 47 d | 47 | 48 | Re | 14 | 21 | 72 B | 72 | 72 | $p=3$ |
| 11 | 16 | 50 d | 50 | 50 | Re | 14 | 22 | 73 d | 73 | 74 | $p=3$ |
| 11 | 17 | 51 d | 51 | 51 | g | 14 | 23 | 75 | 75 | 76 | $p=3$ |
| 11 | 18 | 53 Aff | 53 | 53 | Aff | 14 | 24 | 78 d | 78* | 78 | $p=3$ |
| 11 | 19 | $55^{\text {d }}$ | 55 | 55 | g | 14 | 25 | 80 d | 80* | 80 | $p=3$ |
|  |  |  |  |  |  | 14 | 26 | 81 d | 81 | 82 | $p=3$ |
| 12 | 12 | 45 d | 45 | 45 | Aff | 14 | 27 | 83 d | 83 | 84 | Re |
| 12 | 13 | 48 d | 48 | 48 | Re | 14 | 28 | $84^{\text {d }}$ | 85 | 86 | Re |
| 12 | 14 | $49 \mathrm{p}, \mathrm{d}$ | 49 | 49 | $\alpha=1, \beta=5$ |  |  |  |  |  |  |
| 12 | 15 | 51 d | 51 | 52 | Re | 15 | 15 | $60{ }^{\text {d }}$ | 60 | 62 | $\alpha=1, \beta=5$ |
| 12 | 16 | 53 d | 53 | 54 | Re | 15 | 16 | $64{ }^{\text {d }}$ | 64* | 64 | $\alpha=-1, \beta=4$ |
| 12 | 17 | 55 d | 55 | 55 | g | 15 | 17 | 67 d | 67* | 67 |  |
| 12 | 18 | 57 Aff | 57 | 57 | Aff | 15 | 18 | 69 Aff | 69 | 69 | Aff |
| 12 | 19 | 60 Aff | 60 | 60 |  | 15 | 19 | 72 Aff | 72 | 72 | Aff |
| 12 | 20 | $61{ }^{\text {d }}$ | 61 | 62 | Re | 15 | 20 | 75 d | 75 | 75 | Re |
| 12 | 21 | 63 d | 63 | 64 | Re | 15 | 21 | 77 B | 77 | 77 | Re |
| 12 | 22 | 64 d | 65 | 65 |  |  |  |  |  |  |  |
| 12 | 23 | $66^{\text {d }}$ | 66 | 67 |  | 16 | 20 | 80 | 80 | 80 |  |
| 12 | 24 | 68 d | 68 | 68 | g |  |  |  |  |  |  |

Table 2. The table contains the best lower and upper bounds on $Z_{2,2}(m, n)$ that can be obtained using the results presented in this paper. The parameters $n$ and $m$ range over the region where the general results 3.2 and 3.3 do not apply, but Guy published the exact values of $Z_{2,2}(m, n)$ in [23]. The marks are the following: ${ }^{\text {d }}$ : deletion principle (e.g., 3.31); ${ }^{\mathrm{B}}: 3.33 ;{ }^{\mathrm{P}}: 3.27$ and $3.28 ;^{\mathrm{Re}}:$ [37], [25] and [32]; ${ }^{p=k}$ : Roman's bound 3.5 (with $p=k$ ); ${ }^{\mathrm{g}}: 3.15$; Aff. 3.19 ; $^{\alpha=x, \beta=y}: 3.20$ (if $\alpha<0$, then the transposed version); ${ }^{*}$ : the value is inaccurate in [23]. If more than one bounds give the stated result, we refer to the historically first one.
(2) Every point of $A \cap \mathcal{P}$ has degree $q+1$ in $G$, so we may delete any $h$ of them. It is not worth deleting more than $w-2$ points since we can contract another Baer subplane.
(3) Consider the graph induced by $\mathcal{P} \backslash \mathcal{P}_{0}$ and $\mathcal{L} \backslash \mathcal{L}_{0}$. Here every vertex has degree $q+1-c$.

### 3.4. Some remarks and open problems

For small values of $m$ and $n$, we have computed the best results one can obtain on $C_{4}$-free graphs using these ideas. These values can be found in Tables 1 and 2.

Illés and Krarup [26] use the formulation of Zarankiewicz's problem in terms of integer programming. They introduce Problem (R), that is, to find $r(n)=\max \left\{\sum_{j=1}^{n} x_{j}: \sum_{j=1}^{n}\binom{x_{j}}{2} \leq\binom{ n}{2}\right.$, where $x_{j} \geq 0, x_{j} \in \mathbb{Z}$ for all $\left.1 \leq j \leq n\right\}$.

The cost of a solution $\mathbf{x}=\left(x_{1}, \ldots, x_{n}\right)$ is $\sum_{j}\binom{x_{j}}{2}$. They call a solution $\mathbf{x}$ realizable if there exists an $n \times n J_{2}=\left(\begin{array}{ll}1 & 1 \\ 1 & 1\end{array}\right)$-free $0-1$ matrix in which the $j$ th column contains $x_{j}$ ones. In Remark 6, page 129 they claim: "It is conjectured that a necessary condition for realizability is that the corresponding optimal solution to ( R ) is a least cost solution." Note that the transpose of an optimal $n \times n J_{2}$-free $0-1$ matrix is also an optimal matrix of that kind, hence the conjecture claims that the rows also correspond to a least cost optimal solution. As $\binom{x}{2}$ is convex, the cost of a solution is minimal if and only if $\left|x_{i}-x_{j}\right| \leq 1$ for all $1 \leq i<j \leq n$. In terms of $C_{4}$-free bipartite graphs of size $(n, n)$, this is equivalent with saying that if such a graph has the maximum possible number of edges, then the degrees inside both classes must differ by at most one. This conjecture is false. Let $n=8$. Then $Z_{2,2}(8,8)=24$. Let $G=(A, B)$ be the incidence graph of the Fano plane, and let $a \in A$ and $b \in B$ two non-adjacent vertices. Add two new vertices, $u$ and $v$ to $A$ and $B$, respectively, and let $\{u, v\},\{a, v\},\{u, b\}$ be edges. The resulting graph is $C_{4}$-free, has $21+3=24$ edges, and the degrees in both classes take the values 2,3 and 4 . However, deleting a line $l$ and a point $P$ not on $l$, together with all the points and lines incident with $l$ and $P$ from $\operatorname{PG}(2,3)$, we obtain a three-regular bipartite graph on $(8,8)$ vertices.

We say that a vertex class of a bipartite graph is nearly regular, if the degrees in that class differ by at most one. We end this section by posing some questions
that, to the best of our knowledge, are open. Let $2 \leq t \leq n \leq m$ be arbitrary integers.

Question 3.34. Does there exist an extremal $K_{t, t}$-free graph on $(n, n)$ vertices whose classes are both nearly regular?

Question 3.35. Does there exist an extremal $K_{t, t}$-free graph on $(n, m)$ vertices with at least one nearly regular class?

Corollary 3.27 shows that extremal $C_{4}$-free bipartite graphs on

$$
\left(n^{2}+n+1, n^{2}+n+2\right)
$$

vertices, $n$ a power of a prime, can not have two nearly regular classes.
Question 3.36 (See [21]). Is it true that

$$
Z_{t, t}(n, m) \leq Z_{t, t}(\lfloor(n+m) / 2\rfloor,\lceil(n+m) / 2\rceil) ?
$$

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# TOTALLY $(\mu, \lambda)$-CONTINUOUS AND SLIGHTLY $(\mu, \lambda)$-CONTINUOUS FUNCTIONS IN GENERALIZED TOPOLOGICAL SPACES 

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#### Abstract

In this paper, totally $(\mu, \lambda)$-continuity and slightly $(\mu, \lambda)$-continuity are introduced and studied. Furthermore, basic properties and preservation theorems of totally $(\mu, \lambda)$-continuous and slightly $(\mu, \lambda)$-continuous functions are investigated and the relationships between these functions and their relationships with some other functions are investigated.


## 1. Introduction and preliminaries

In [1]-[12], Á. Császár founded the theory of generalized topological spaces, and studied the elementary character of these classes. Especially he introduced the notions of continuous functions on generalized topological spaces, and investigated characterizations of generalized continuous functions $(=(\mu, \lambda)$ continuous functions in [3]). We recall some notions defined in [3]. Let $X$ be a non-empty set and $\exp X$ the power set of $X$. We call a class $\mu \subseteq \exp X$ a generalized topology [3] if $\phi \in \mu$ and the arbitrary union of elements of $\mu$ belongs to $\mu$. A set $X$ with a generalized topology $\mu$ on it is called a generalized topological space and is denoted by $(X, \mu)$.

For a generalized topological space $(X, \mu)$, the elements of $\mu$ are called $\mu$-open sets and the complements of $\mu$-open sets are called $\mu$-closed sets. For $A \subseteq X$, we denote by $c_{\mu}(A)$ the intersection of all $\mu$-closed sets containing $A$, i.e., the smallest $\mu$-closed set containing $A$; and by $i_{\mu}(A)$ the union of all $\mu$-open
sets contained in $A$, i.e., the largest $\mu$-open set contained in $A$ (see [3], [9]). According to [8], for $A \subseteq X$ and $x \in X$, we have $x \in c_{\mu}(A)$ if and only if $x \in M \in \mu$ implies $M \cap A \neq \phi$.

Definition 1.1 ([13]). A generalized topological space $(X, \mu)$ is said to be $\mu-$ - $T_{0}$ if for any pair of distinct points of $X$, there exists a $\mu$-open set containing one of the points but not the other.

Definition 1.2 ([13]). A generalized topological space $(X, \mu)$ is said to be $\mu-$ - $T_{1}$ if for each pair of distinct points $x$ and $y$ of $X$, there exist $\mu$-open sets $U$ and $V$ containing $x$ and $y$, respectively such that $y \notin U$ and $x \notin V$.

Definition 1.3 ([13]). A generalized topological space ( $X, \mu$ ) is said to be $\mu-$ $-T_{2}$ if for each pair of distinct points $x$ and $y$ in $X$, there exist disjoint $\mu$-open sets $U$ and $V$ in $X$ such that $x \in U$ and $y \in V$.

Definition 1.4 ([3]). Let $f:(X, \mu) \rightarrow(Y, \lambda)$ be a function on generalized topological spaces. Then the function $f$ is said to be $(\mu, \lambda)$-continuous if $G \in \lambda$ implies $f^{-1}(G) \in \mu$.

## 2. Totally $(\mu, \lambda)$-continuous functions

In this section, the notion of totally $(\mu, \lambda)$-continuous functions is introduced. If $A$ is both $\mu$-open and $\mu$-closed, then it is said to be $\mu$-clopen.

Definition 2.1. Let $f:(X, \mu) \rightarrow(Y, \lambda)$ be a function on generalized topological spaces. Then the function $f$ is said to be totally $(\mu, \lambda)$-continuous if $f^{-1}(V)$ is $\mu$ clopen for each $\lambda$-open set $V$ of $Y$.

Remark 2.2. Every totally ( $\mu, \lambda$ )-continuous function is $(\mu, \lambda)$-continuous but the converse need not be true as it can be seen from the following example.

Example 2.3. Let $X=\{a, b, c\}, \mu=\{\phi,\{a\},\{a, c\}\}$ and $\lambda=\{\phi,\{a, b\}\}$. Let $f:(X, \mu) \rightarrow(X, \lambda)$ be a function defined as follows: $f(a)=a, f(b)=c, f(c)=$ $=b$. The inverse image of the $\lambda$-open set $\{a, b\}$ is $\{a, c\}$ which is $\mu$-open but it is not $\mu$-clopen. Then $f$ is $(\mu, \lambda)$-continuous but it is not totally $(\mu, \lambda)$-continuous.

Definition 2.4. A generalized topological space $(X, \mu)$ is called $\mu$ - connected if it is not the union of two nonempty disjoint $\mu$-open sets.

Theorem 2.5. If a generalized topological space $(X, \mu)$ is $\mu$ - connected then every totally $(\mu, \lambda)$-continuous function from ( $X, \mu$ ) into any $\lambda-T_{0}$-space $(Y, \lambda)$ is a constant map.

Proof. Suppose that $f:(X, \mu) \rightarrow(Y, \lambda)$ is a totally $(\mu, \lambda)$-continuous function, where $(Y, \lambda)$ is a $\lambda-T_{0}$-space. Assume that $f$ is not constant and $x, y \in X$ such that $f(x) \neq f(y)$. Since $(Y, \lambda)$ is $\lambda-T_{0}$, and $f(x)$ and $f(y)$ are distinct points in $Y$, then there is an open set $V$ in $(Y, \lambda)$ containing only one of the points $f(x), f(y)$. We take the case $f(x) \in V$ and $f(y) \notin V$. The proof of the other case is similar. Since $f$ is a totally $(\mu, \lambda)$-continuous function, $f^{-1}(V)$ is a $\mu$-clopen subset of $X$ and $x \in f^{-1}(V)$, but $y \notin f^{-1}(V)$. Since $X=f^{-1}(V) \cup\left(X-f^{-1}(V)\right), X$ is a union of two nonempty disjoint $\mu$-open subsets of $X$. Thus $(X, \mu)$ is not $\mu$-connected, which is a contradiction.

Theorem 2.6. Let $f:(X, \mu) \rightarrow(Y, \lambda)$ be a $(\mu, \lambda)$-continuous injection. If $Y$ is $\lambda-T_{0}$ then $(X, \mu)$ is $\mu-T_{2}$.

Proof. Let $x, y \in X$ with $x \neq y$. Since $f$ is injection, $f(x) \neq f(y)$. Since $Y$ is $\lambda-T_{0}$, there exists a $\lambda$-open subset $V$ of $Y$ containing $f(x)$ but not $f(y)$, or containing $f(y)$ but not $f(x)$. Thus for the first case we have, $x \in f^{-1}(V)$ and $y \notin f^{-1}(V)$. Since $f$ is totally $(\mu, \lambda)$-continuous and $V$ is a $\lambda$-open subset of $Y$, $f^{-1}(V)$ and $X-f^{-1}(V)$ are disjoint $\mu$-clopen subsets of $X$ containing $x$ and $y$, respectively. The second case is proved in the same way. Thus $X$ is $\mu-T_{2}$.

Definition 2.7. A function $f:(X, \mu) \rightarrow(Y, \lambda)$ is called a strongly $(\mu, \lambda)$ continuous function if the inverse image of every subset of $Y$ is a $\mu$-clopen subset of $X$.

Remark 2.8. Every strongly ( $\mu, \lambda$ )-continuous function is totally $(\mu, \lambda)$ continuous, but the converse need not be true as the following example shows.

Example 2.9. Let $X=\{a, b, c\}$ and $\mu=\lambda=\{X, \phi,\{b\},\{a, c\}\}$. Let $f:(X, \mu) \rightarrow(X, \lambda)$ be the identity function, then $f$ is totally $(\mu, \lambda)$-continuous but it is not strongly $(\mu, \lambda)$-continuous.

## 3. Slightly $(\mu, \lambda)$-continuous functions

In this section, the notion of slightly $(\mu, \lambda)$-continuous functions is introduced and characterizations and some relationships of slightly $(\mu, \lambda)$-continuous functions and basic properties of slightly $(\mu, \lambda)$-continuous functions are investigated and obtained.

Definition 3.1. Let $f:(X, \mu) \rightarrow(Y, \lambda)$ be a function on generalized topological spaces. Then the function $f$ is said to be slightly $(\mu, \lambda)$-continuous at a point $x \in$ $\in X$ if for each $\lambda$-clopen subset $V$ in $Y$ containing $f(x)$, there exists a $\mu$-open subset $U$ in $X$ containing $x$ such that $f(U) \subseteq V$. The function $f$ is said to be slightly $(\mu, \lambda)$-continuous if it has this property at each point of $X$.

Remark 3.2. Every ( $\mu, \lambda$ )-continuous function is slightly ( $\mu, \lambda$ )-continuous but the converse need not be true as it can be seen from the following example.

Example 3.3. Let $R$ and $N$ be the real numbers and natural numbers, respectively. Take two generalized topologies on $R$ as $\mu=\{R, \phi, N\}$ and $\lambda=$ $=\{R, \phi, R-N\}$. Let $f:(R, \mu) \rightarrow(R, \lambda)$ be an identity function. Then, $f$ is slightly $(\mu, \lambda)$-continuous, but it is not $(\mu, \lambda)$-continuous.

Remark 3.4. Since every totally $(\mu, \lambda)$-continuous function is $(\mu, \lambda)$-continuous then every totally $(\mu, \lambda)$-continuous function is slightly $(\mu, \lambda)$-continuous but the converse need not be true. The function $f$ in Example 3.3 is slightly $(\mu, \lambda)$ continuous but it is not totally $(\mu, \lambda)$-continuous.

Remark 3.5. Since every strongly $(\mu, \lambda)$-continuous function is totally $(\mu, \lambda)$ continuous then every strongly $(\mu, \lambda)$-continuous function is slightly $(\mu, \lambda)$ continuous but the converse need not be true. The function $f$ in Example 2.9 is slightly $(\mu, \lambda)$-continuous but it is not strongly $(\mu, \lambda)$-continuous.

Theorem 3.6. Let $(X, \mu)$ and $(Y, \lambda)$ be two generalized topological spaces. The following statements are equivalent for a function $f:(X, \mu) \rightarrow(Y, \lambda)$ :
(1) $f$ is slightly $(\mu, \lambda)$-continuous;
(2) for every $\lambda$-clopen set $V \subseteq Y, f^{-1}(V)$ is $\mu$-open;
(3) for every $\lambda$-clopen set $V \subseteq Y, f^{-1}(V)$ is $\mu$-closed;
(4) for every $\lambda$-clopen set $V \subseteq Y, f^{-1}(V)$ is $\mu$-clopen.

Proof. (1) $\Rightarrow(2)$ : Let $V$ be a $\lambda$-clopen subset of $Y$ and let $x \in f^{-1}(V)$. Since $f$ is slightly $(\mu, \lambda)$-continuous, by (1) there exists a $\mu$-open set $U_{x}$ in $X$ containing $x$ such that $f\left(U_{x}\right) \subseteq V$; hence $U_{x} \subseteq f^{-1}(V)$. We obtain that $f^{-1}(V)=\cup\left\{U_{x}: x \in\right.$ $\left.\in f^{-1}(V)\right\}$. Thus, $f^{-1}(V)$ is $\mu$-open.
(2) $\Rightarrow$ (3): Let $V$ be a $\lambda$-clopen subset of $Y$. Then $Y-V$ is $\lambda$-clopen. By (2) $f^{-1}(Y-V)=X-f^{-1}(V)$ is $\mu$-open. Thus $f^{-1}(V)$ is $\mu$-closed.
$(3) \Rightarrow(4)$ : It can be shown easily.
(4) $\Rightarrow(1)$ : Let $x \in X$ and $V$ be a $\lambda$-clopen subset in $Y$ with $f(x) \in V$. Let $U=f^{-1}(V)$. By assumption $U$ is $\mu$-clopen and so $\mu$-open. Also $x \in U$ and $f(U) \subseteq V$.

Theorem 3.7. Let $(X, \mu)$ and $(Y, \lambda)$ be two generalized topological spaces where $\lambda=\exp Y$, then every slightly $(\mu, \lambda)$-continuous function $f:(X, \mu) \rightarrow$ $\rightarrow(Y, \lambda)$ is strongly $(\mu, \lambda)$-continuous.

Proof. Let $A$ be any subset of $Y$. Then $A$ is a $\lambda$-clopen subset of $Y$. Hence $f^{-1}(A)$ is $\mu$-clopen in $X$. Thus $f$ is strongly $(\mu, \lambda)$-continuous.

Theorem 3.8. Let $(X, \mu),(Y, \lambda)$ and $(Z, \sigma)$ be generalized topological spaces. If $f:(X, \mu) \rightarrow(Y, \lambda)$ is $(\mu, \lambda)$-continuous and $g:(Y, \lambda) \rightarrow(Z, \sigma)$ is slightly $(\lambda, \sigma)$-continuous, then $g \circ f$ is slightly $(\mu, \sigma)$-continuous.

Proof. Let $V$ be any $\sigma$-clopen set in $Z$. Since $g$ is slightly $(\lambda, \sigma)$-continuous, $g^{-1}(V)$ is $\lambda$-open. Since $f$ is $(\mu, \lambda)$-continuous, $f^{-1}\left(g^{-1}(V)\right)=(g \circ f)^{-1}(V)$ is $\mu$-open. Therefore, $g \circ f$ is slightly $(\mu, \sigma)$-continuous.

Definition 3.9. A function $f:(X, \mu) \rightarrow(Y, \lambda)$ is called a $(\mu, \lambda)$-open function if the image of each $\mu$-open set in $X$ is a $\lambda$-open set in $Y$.

Theorem 3.10. Let $(X, \mu),(Y, \lambda)$ and $(Z, \sigma)$ be generalized topological spaces. Let $f:(X, \mu) \rightarrow(Y, \lambda)$ be a $(\mu, \lambda)$-continuous $(\mu, \lambda)$-open surjection and $g:(Y, \lambda) \rightarrow(Z, \sigma)$ be a function. Then $g$ is slightly $(\lambda, \sigma)$-continuous if and only if $g \circ f$ is slightly $(\mu, \sigma)$-continuous.

Proof. $\Rightarrow$ Let $g$ be slightly $(\lambda, \sigma)$-continuous. Then by Theorem 3.8, $g \circ f$ is slightly $(\mu, \sigma)$-continuous.
$\Leftarrow \operatorname{Let} g \circ f$ be slightly $(\mu, \sigma)$-continuous and $V$ be a $\sigma$-clopen set in $Z$. Then $(g \circ f)^{-1}(V)$ is $\mu$-open. Since $f$ is a $(\mu, \lambda)$-open surjection, then $f\left((g \circ f)^{-1}(V)\right)=$ $=g^{-1}(V)$ is $\lambda$-open in $Y$. This shows that $g$ is slightly $(\lambda, \sigma)$-continuous.

Theorem 3.11. Iff: $(X, \mu) \rightarrow(Y, \lambda)$ is a slightly $(\mu, \lambda)$-continuous surjection and $(X, \mu)$ is $\mu$ - connected, then $(Y, \lambda)$ is $\lambda$-connected.

Proof. Suppose that $(Y, \lambda)$ is not $\lambda$-connected. Then there exist nonempty disjoint $\lambda$-open sets $U$ and $V$ such that $Y=U \cup V$. Therefore, $U$ and $V$ are $\lambda$ clopen sets in $Y$. Since $f$ is slightly $(\mu, \lambda)$-continuous, $f^{-1}(U)$ and $f^{-1}(V)$ are $\mu-$ -open in $X$. Moreover, $f^{-1}(U)$ and $f^{-1}(V)$ are disjoint and $X=f^{-1}(U) \cup f^{-1}(V)$. Since $f$ is surjective, $f^{-1}(U)$ and $f^{-1}(V)$ are nonempty. Therefore, $(X, \mu)$ is not $\mu$ - connected. This is a contradiction and hence $(Y, \lambda)$ is $\lambda$ - connected.

Corollary 3.1. The inverse image of a $\lambda$-disconnected space under a slightly $(\mu, \lambda)$-continuous surjection is $\mu$-disconnected.

Definition 3.12. A generalized topological space $(X, \mu)$ is said to be:
(1) $\mu$-locally indiscrete if every $\mu$-open set of $X$ is $\mu$-closed in $X$.
(2) $\mu$-0-dimensional if for each $\mu$-open set $V$ and each $x \in V$ there exists a $\mu$-clopen set $U$ such that $x \in U \subseteq V$.

Theorem 3.13. If $f:(X, \mu) \rightarrow(Y, \lambda)$ is a slightly $(\mu, \lambda)$-continuous function and $(Y, \lambda)$ is $\lambda$-locally indiscrete, then $f$ is $(\mu, \lambda)$-continuous.

Proof. Let $V$ be any $\lambda$-open set of $Y$. Since $Y$ is $\lambda$-locally indiscrete, $V$ is $\lambda$ clopen and hence $f^{-1}(V)$ are $\mu$-open in $X$. Therefore, $f$ is $(\mu, \lambda)$-continuous.

Theorem 3.14. If $f:(X, \mu) \rightarrow(Y, \lambda)$ is a slightly $(\mu, \lambda)$-continuous function and $(Y, \lambda)$ is $\lambda$ - 0 -dimensional, then $f$ is $(\mu, \lambda)$-continuous.

Proof. Let $x \in X$ and $V \subseteq Y$ be any $\lambda$-open set containing $f(x)$. Since $Y$ is $\lambda$-0-dimensional, there exists a $\lambda$-clopen set $U$ containing $f(x)$ such that $U \subseteq V$. But $f$ is slightly $(\mu, \lambda)$-continuous then there exists a $\mu$-open set $G$ containing $x$ such that $f(x) \in f(G) \subseteq U \subseteq V$. Hence $f$ is $(\mu, \lambda)$-continuous.

Theorem 3.15. Let $f:(X, \mu) \rightarrow(Y, \lambda)$ be a slightly $(\mu, \lambda)$-continuous injection and $(Y, \lambda)$ is $\lambda$ - 0 -dimensional. If $Y$ is $\lambda-T_{1}$ (resp. $\lambda-T_{2}$ ), then $X$ is $\mu-T_{1}$ (resp. $\mu-T_{2}$ ).

Proof. We prove only the second statement, the prove of the first being analogous. Let $Y$ be $\lambda-T_{2}$. Since $f$ is injective, for any pair of distinct points $x, y \in X$, $f(x) \neq f(y)$. Since $Y$ is $\lambda-T_{2}$, there exist $\lambda$-open sets $V_{1}, V_{2}$ in $Y$ such that $f(x) \in V_{1}, f(y) \in V_{2}$ and $V_{1} \cap V_{2}=\phi$. Since $Y$ is $\lambda$ - 0 -dimensional, there exist $\lambda$-clopen sets $U_{1}, U_{2}$ in $Y$ such that $f(x) \in U_{1} \subseteq V_{1}$ and $f(y) \in U_{2} \subseteq V_{2}$. Consequently $x \in f^{-1}\left(U_{1}\right) \subseteq f^{-1}\left(V_{1}\right), y \in f^{-1}\left(U_{2}\right) \subseteq f^{-1}\left(V_{2}\right)$ and $f^{-1}\left(U_{1}\right) \cap$ $\cap f^{-1}\left(U_{2}\right)=\phi$. Since $f$ is slightly $(\mu, \lambda)$-continuous, $f^{-1}\left(U_{1}\right)$ and $f^{-1}\left(U_{2}\right)$ are $\mu$-open sets and this implies that $X$ is $\mu-T_{2}$.

Definition 3.16. A generalized topological space $(X, \mu)$ is said to be:
(i) $\mu$-clopen $T_{1}$ if for each pair of distinct points $x$ and $y$ of $X$, there exist $\mu$-clopen sets $U$ and $V$ containing $x$ and $y$, respectively such that $y \notin U$ and $x \notin V$.
(ii) $\mu$-clopen $T_{2}$ if for each pair of distinct points $x$ and $y$ in $X$, there exist disjoint $\mu$-clopen sets $U$ and $V$ in $X$ such that $x \in U$ and $y \in V$.

THEOREM 3.17. Iff: $(X, \mu) \rightarrow(Y, \lambda)$ is a slightly $(\mu, \lambda)$-continuous injection and $(Y, \lambda)$ is $\lambda$-clopen $T_{1}$, then $(X, \mu)$ is $\mu-T_{1}$.

Proof. Suppose that $Y$ is $\lambda$-clopen $T_{1}$. For any distinct points $x$ and $y$ in $X$, there exist $\lambda$-clopen sets $V$ and $W$ such that $f(x) \in V, f(y) \notin V$ and $f(y) \in W, f(x) \notin W$. Since $f$ is slightly $(\mu, \lambda)$-continuous, $f^{-1}(V)$ and $f^{-1}(W)$ are $\mu$-open subsets of $X$ such that $x \in f^{-1}(V), y \notin f^{-1}(V)$ and $y \in f^{-1}(W), x \notin f^{-1}(W)$. This shows that $X$ is $\mu-T_{1}$.

Theorem 3.18. Let $f:(X, \mu) \rightarrow(Y, \lambda)$ is a slightly $(\mu, \lambda)$-continuous injection and $(Y, \lambda)$ is $\lambda$-clopen $T_{2}$, then $(X, \mu)$ is $\mu-T_{2}$.

Proof. For any pair of distinct points $x$ and $y$ in $X$, there exist disjoint $\lambda$-clopen sets $U$ and $V$ in $Y$ such that $f(x) \in U$ and $f(y) \in V$. Since $f$ is slightly $(\mu, \lambda)$ continuous, $f^{-1}(U)$ and $f^{-1}(V)$ are $\mu$-open subsets of $X$ containing $x$ and $y$, respectively. Also, $f^{-1}(U) \cap f^{-1}(V)=\phi$ because $U \cap V=\phi$. This shows that $X$ is $\mu-T_{2}$.

Definition 3.19. A generalized topological space $(X, \mu)$ is called $\mu$-clopen regular (respectively $\mu$-regular) if for each $\mu$-clopen (respectively $\mu$-closed) set $F$ and each point $x \notin F$, there exist disjoint $\mu$-open sets $U$ and $V$ such that $F \subseteq U$ and $x \in V$.

Definition 3.20. A generalized topological space ( $X, \mu$ ) is called $\mu$-clopen normal (respectively $\mu$-normal) if for every pair of disjoint $\mu$-clopen (respectively $\mu$-closed) subsets $A$ and $B$ of $X$, there exist disjoint $\mu$-open sets $U$ and $V$ such that $A \subseteq U$ and $B \subseteq V$.

Theorem 3.21. If $f:(X, \mu) \rightarrow(Y, \lambda)$ is a slightly $(\mu, \lambda)$-continuous injective ( $\mu, \lambda$ )-open function from a $\mu$-regular space $(X, \mu)$ onto a space $(Y, \lambda)$, then $(Y, \lambda)$ is $\lambda$-clopen regular.

Proof. Let $F$ be a $\lambda$-clopen set in $Y$ and $y \notin F$. Take $y=f(x)$. Since $f:(X, \mu) \rightarrow$ $\rightarrow(Y, \lambda)$ is a slightly $(\mu, \lambda)$-continuous, $f^{-1}(F)$ is a $\mu$-closed set. Take $G=$ $=f^{-1}(F)$. We have $x \notin G$. Since $X$ is $\mu$-regular, there exist disjoint $\mu$-open sets $U$ and $V$ such that $G \subseteq U$ and $x \in V$. We obtain that $F=f(G) \subseteq f(U)$ and $y=f(x) \in f(V)$ such that $f(U)$ and $f(V)$ are disjoint $\lambda$-open sets. This shows that $Y$ is $\lambda$-clopen regular.

Theorem 3.22. If $f:(X, \mu) \rightarrow(Y, \lambda)$ is a slightly $(\mu, \lambda)$-continuous injective ( $\mu, \lambda$ )-open function from a $\mu$-normal space $(X, \mu)$ onto a space $(Y, \lambda)$, then $(Y, \lambda)$ is $\lambda$-clopen normal.

Proof. Let $A$ and $B$ be disjoint $\lambda$-clopen subsets of $Y$. Since $f:(X, \mu) \rightarrow(Y, \lambda)$ is a slightly $(\mu, \lambda)$-continuous, $f^{-1}(A)$ and $f^{-1}(B)$ are $\mu$-closed sets. Take $U=$ $=f^{-1}(A)$ and $V=f^{-1}(B)$. We have $U \cap V=\phi$. Since $X$ is $\mu$-normal, there exist disjoint $\mu$-open sets $G$ and $H$ such that $U \subseteq G$ and $V \subseteq H$. We obtain that $A=f(U) \subseteq f(G)$ and $B=f(V) \subseteq f(H)$ such that $f(G)$ and $f(H)$ are disjoint $\lambda$-open sets. Thus, $Y$ is $\lambda$-clopen normal.

Definition 3.23. A generalized topological space ( $X, \mu$ ) is said to be $\mu$-mildly compact (resp. $\mu$-mildly Lindelöf) if every $\mu$-clopen cover of $X$ has a finite (resp. countable) subcover.

Definition 3.24. A generalized topological space ( $X, \mu$ ) is said to be $\mu$-compact (resp. $\mu$-Lindelöf) if every $\mu$-open cover of $X$ has a finite (resp. countable) subcover.

Theorem 3.25. Let $f:(X, \mu) \rightarrow(Y, \lambda)$ be a slightly $(\mu, \lambda)$-continuous surjection, then the following statements hold:
(1) if $(X, \mu)$ is $\mu$-compact, then $(Y, \lambda)$ is $\lambda$-mildly compact.
(2) if $(X, \mu)$ is $\mu$-Lindelöf, then $(Y, \lambda)$ is $\lambda$-mildly Lindelöf.

Proof. We prove (1), the proof of (2) being entirely analogous.
Let $\left\{V_{\alpha}: \alpha \in \Delta\right\}$ be a $\lambda$-clopen cover of $Y$. Since $f$ is slightly $(\mu, \lambda)$ continuous, $\left\{f^{-1}\left(V_{\alpha}\right): \alpha \in \Delta\right\}$ is a $\mu$-open cover of $X$. Since $X$ is $\mu$-compact, there exists a finite subset $\Delta_{0}$ of $\Delta$ such that $X=\cup\left\{f^{-1}\left(V_{\alpha}\right): \alpha \in \Delta_{0}\right\}$. Thus we have $Y=\cup\left\{V_{\alpha}: \alpha \in \Delta_{0}\right\}$ which means that $(Y, \lambda)$ is $\lambda$-mildly compact.

Definition 3.26. A generalized topological space ( $X, \mu$ ) is called $\mu$-closed compact (resp. $\mu$-closed Lindelöf) if every cover of $X$ by $\mu$-closed sets has a finite (resp. countable) subcover.

Theorem 3.27. Let $f:(X, \mu) \rightarrow(Y, \lambda)$ be a slightly $(\mu, \lambda)$-continuous surjection, then the following statements hold:
(1) if $(X, \mu)$ is $\mu$-closed compact, then $(Y, \lambda)$ is $\lambda$-mildly compact.
(2) if $(X, \mu)$ is $\mu$-closed Lindelöf, then $(Y, \lambda)$ is $\lambda$-mildly Lindelöf.

Proof. It can be obtained similarly as Theorem 3.26.
Definition 3.28. A subset $A$ of a generalized topological space $(X, \mu)$ is said to be $\mu \delta^{*}$-open if for each $x \in A$ there exists a $\mu$-clopen subset $G$ of $X$ such that $x \in G \subset A$. The complement of a $\mu \delta^{*}$-open set is called $\mu \delta^{*}$-closed.

If $A \subseteq X$, then $c_{\mu \delta^{*}}(A)$ denotes the intersection of all $\mu \delta^{*}$-closed sets containing $A$.

The following theorem gives a new set of conditions which characterize slightly $(\mu, \lambda)$-continuous functions.

Theorem 3.29. For a function $f:(X, \mu) \rightarrow(Y, \lambda)$ the following are equivalent:
(a) $f$ is slightly $(\mu, \lambda)$-continuous;
(b) $f^{-1}(V)$ is $\mu$-open for every $\lambda \delta^{*}$-open set $V$ in $Y$;
(c) $f^{-1}(C)$ is $\mu$-closed for every $\lambda \delta^{*}$-closed set $C$ in $Y$;
(d) $f\left(c_{\mu}(A)\right) \subseteq c_{\lambda \delta^{*}}(f(A))$ for every subset $A$ of $X$;
(e) $c_{\mu}\left(f^{-1}(B)\right) \subset f^{-1}\left(c_{\lambda \delta^{*}}(B)\right)$ for every subset $B$ of $Y$.

Proof. $(a) \Rightarrow(b)$ : Let $V$ be a $\lambda \delta^{*}$-open set in $Y$ and let $x \in f^{-1}(V)$. Then $f(x) \in$ $\in V$. The $\lambda \delta^{*}$-openness of $V$ gives a $\lambda$-clopen set $U$ in $Y$ such that $f(x) \in U \subset V$. This implies that $x \in f^{-1}(U) \subset f^{-1}(V)$. Since $f$ is slightly $(\mu, \lambda)$-continuous,
$f^{-1}(U)$ is a $\mu$-open set in $X$. Hence $f^{-1}(V)$ is a $\mu$-neighbourhood of each of its points. Consequently, $f^{-1}(V)$ is a $\mu$-open set in $X$.
(b) $\Rightarrow(c)$ : It is obvious from the fact that the complement of a $\lambda \delta^{*}$-closed set is $\lambda \delta^{*}$-open.
$(c) \Rightarrow(d)$ : Let $A$ be a subset of $X$. We have, $c_{\lambda \delta^{*}}(f(A))=\cap\{F: f(A) \subset F$ and $F$ is $\lambda \delta^{*}$-closed in $\left.Y\right\}$ is a $\lambda \delta^{*}$-closed set in $Y$. Thus $A \subset f^{-1}\left(c_{\lambda \delta^{*}}(f(A))=\right.$ $=\cap\left\{f^{-1}(F): f(A) \subset F\right.$ and $F$ is $\lambda \delta^{*}$-closed in $\left.Y\right\}$. But $\cap\left\{f^{-1}(F): f(A) \subset F\right.$ and $F$ is $\lambda \delta^{*}$-closed in $\left.Y\right\}$ is $\mu$-closed in $X$, so we obtain $c_{\mu}(A) \subset f^{-1}\left(c_{\lambda \delta^{*}}(f(A))\right.$. Hence, $f\left(c_{\mu}(A)\right) \subset c_{\lambda \delta^{*}}(f(A))$.
$(d) \Rightarrow(e)$ : Let $B$ be a subset of $Y$. We have $f\left(c_{\mu}\left(f^{-1}(B)\right)\right) \subset$ $\subset c_{\lambda \delta^{*}}\left(f\left(f^{-1}(B)\right)\right) \subset c_{\lambda \delta^{*}}(B)$ and hence, we obtain, $c_{\mu}\left(f^{-1}(B)\right) \subset$ $\subset f^{-1}\left(c_{\lambda \delta^{*}}(B)\right)$.
$(e) \Rightarrow(a)$ : Let $V$ be a $\lambda$-clopen set in $Y$. Then $V$ is $\lambda \delta^{*}$-closed in $Y$. Thus $c_{\mu}\left(f^{-1}(B)\right) \subset f^{-1}\left(c_{\lambda \delta^{*}}(B)\right)=f^{-1}(B)$. Therefore, $f^{-1}(B)$ is $\mu$-closed. Hence, $f$ is slightly $(\mu, \lambda)$-continuous.

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# $\wedge_{w}$-SETS AND $\vee_{w}$-SETS IN WEAK STRUCTURES 

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#### Abstract

In this paper we introduce the concepts of $\wedge_{w}$-sets and $\vee_{w}$-sets in a weak structure space due to Császár. It is shown that many results in previous papers can be considered as special cases of our results.


## 1. Introduction

The notion of $\wedge$-sets was introduced by Maki [5] in 1986. A subset $A$ of a topological space is called a $\wedge$-set if it is the intersection of all open sets containing $A$. Recently many authors have introduced and studied modifications of $\wedge$-sets. By using a minimal structure, Cammaroto and Noiri [1] introduced the notions of $\wedge_{m}$-sets and $\vee_{m}$-sets as unified forms of these modifications. Furthermore, recently Ekici and Roy [4] have introduced and investigated the notions of $\wedge_{\mu}$-sets and $\vee_{\mu}$-sets on a generalized topological space ( $X, \mu$ ) due to Császár [2]. Quite recently, Császár [3] has introduced the notion of weak structures and obtained several fundamental properties of weak structures.

In this paper, we introduce the notions of $\wedge_{w}$-sets and $\vee_{w}$-sets on a weak structure space $(X, w)$ and investigate the properties of sets and spaces related to $\wedge_{w}$-sets and $\vee_{w}$-sets.

## 2. Preliminaries

Let $X$ be a nonempty set and $\mathcal{P}(X)$ the power set of $X$. A subfamily $w$ of $\mathcal{P}(X)$ is called a weak structure (briefly, $W S$ ) [3] if $\phi \in w$. The pair $(X, w)$ is called a weak structure ( $W S$ ) space. Each member of a $W S w$ is said to be $w$ open [3] and the complement of a $w$-open set is said to be $w$-closed. Let $A$ be a subset of $X$. The union of all $w$-open sets contained in $A$ is called the $w$-interior of $A$ and is denoted by $i_{w}(A)$ [3]. The intersection of all $w$-closed sets containing $A$ is called the $w$-closure of $A$ and is denoted by $c_{w}(A)$.

For the $w$-interior and the $w$-closure, the following lemmas are useful in the sequel.

Lemma 2.1. [3] Let $w$ be a $W$ S on $X$ and $A, B$ subsets of $X$, then
(1) $i_{w}(A) \subseteq A \subseteq c_{w}(A)$.
(2) If $A \subseteq B$ implies that $i_{w}(A) \subseteq i_{w}(B)$ and $c_{w}(A) \subseteq c_{w}(B)$.
(3) $i_{w}\left(i_{w}(A)\right)=i_{w}(A)$ and $c_{w}\left(c_{w}(A)\right)=c_{w}(A)$.
(4) $i_{w}(X-A)=X-c_{w}(A)$ and $c_{w}(X-A)=X-i_{w}(A)$.

Lemma 2.2. [3] Let $w$ be a $W S$ on $X$, then
(1) $x \in i_{w}(A)$ if and only if there exists $W \in w$ such that $x \in W \subseteq A$.
(2) $x \in c_{w}(A)$ if and only if $W \cap A \neq \emptyset$ whenever $x \in W \in w$.
(3) If $A \in w$, then $A=i_{w}(A)$ and if $A$ is $w$-closed, then $A=c_{w}(A)$.

Remark 2.3. If $w$ is a $W S$ on $X$, then
(1) $i_{w}(\emptyset)=\emptyset$ and $c_{w}(X)=X$.
(2) $i_{w}(X)$ is the union of all $w$-open sets in $X$.
(3) $c_{w}(\emptyset)$ is the intersection of all $w$-closed sets in $X$.

We call a class $\mu \subseteq \mathcal{P}(X)$ a generalized topology [2] (briefly, GT) if $\phi \in \mu$ and the arbitrary union of elements of $\mu$ belongs to $\mu$. A set $X$ with a GT $\mu$ on it is called a generalized topological space (briefly, GTS) and is denoted by $(X, \mu)$.

## 3. $\wedge_{w}$-sets and $\vee_{w}$-sets

Definition 3.1. Let $w$ be a WS on a set $X$ and $A \subseteq X$. Then the subsets $\wedge_{w}(A)$ and $\vee_{w}(A)$ are defined as follows:

$$
\wedge_{w}(A)= \begin{cases}\cap\{G: A \subseteq G, G \in w\}, & \text { if there exists } G \in w \text { such that } A \subseteq G \\ X, & \text { otherwise }\end{cases}
$$

and

$$
\vee_{w}(A)= \begin{cases}\cup\{H: H \subseteq A, X-H \in w\}, & \text { if there exists } H \text { such that } \\ & X-H \in w \text { and } H \subseteq A \\ \phi, & \text { otherwise }\end{cases}
$$

Proposition 3.2. Let $A, B$ and $\left\{C_{\alpha}: \alpha \in \Delta\right\}$ be subsets of a $W$ space $(X, w)$. Then the following properties hold:
(1) $B \subseteq \wedge_{w}(B)$.
(2) If $A \subseteq B$, then $\wedge_{w}(A) \subseteq \wedge_{w}(B)$.
(3) $\wedge_{w}\left(\wedge_{w}(B)\right)=\wedge_{w}(B)$.
(4) $\cup_{\alpha \in \Delta}\left(\wedge_{w}\left(C_{\alpha}\right)\right) \subseteq \wedge_{w}\left(\cup_{\alpha \in \Delta} C_{\alpha}\right)$.
(5) $\wedge_{w}\left(\cap_{\alpha \in \Delta} C_{\alpha}\right) \subseteq \cap_{\alpha \in \Delta}\left(\wedge_{w}\left(C_{\alpha}\right)\right)$.
(6) If $A \in w$, then $A=\wedge_{w}(A)$.
(7) $\wedge_{w}(X-B)=X-\vee_{w}(B)$.
(8) $\vee_{w}(B) \subseteq B$.
(9) If $X-B \in w$, then $B=\vee_{w}(B)$.
(10) If $A \subseteq B$, then $\vee_{w}(A) \subseteq \vee_{w}(B)$.
(11) $\vee_{w}\left(\cup_{\alpha \in \Delta} C_{\alpha}\right) \supseteq \cup_{\alpha \in \Delta}\left(\vee_{w}\left(C_{\alpha}\right)\right)$.

Proof. (1), (6) and (8) are clear.
(2) If there does not exist any $U \in w$ such that $B \subseteq U$ then the proof is trivial. Suppose there exist $V \in w$ such that $B \subseteq V$ and that $x \notin \wedge_{w}(B)$. Then there exist a subset $U \in w$ such that $B \subseteq U$ with $x \notin U$. Since $A \subseteq B$, then $x \notin \wedge_{w}(A)$ and thus $\wedge_{w}(A) \subseteq \wedge_{w}(B)$.
(3) By (1), we have $\wedge_{w}\left(\wedge_{w}(B)\right) \supseteq \wedge_{w}(B)$. Suppose that $x \notin \wedge_{w}(B)$. Then there exists $U \in w$ such that $B \subseteq U$ and $x \notin U$. Since $B \subseteq \wedge_{w}(B) \subseteq U$, we have $x \notin \wedge_{w}\left(\wedge_{w}(B)\right)$ and hence $\wedge_{w}\left(\wedge_{w}(B)\right) \subseteq \wedge_{w}(B)$.
(4) The proof follows from (2).
(5) Suppose that $x \notin \cap_{\alpha \in \Delta}\left(\wedge_{w}\left(C_{\alpha}\right)\right)$. There exists $\alpha_{0} \in \Delta$ such that $x \notin \wedge_{w}\left(C_{\alpha_{0}}\right)$ and there exists a $w$-open set $U$ such that $x \notin U$ and $C_{\alpha_{0}} \subseteq U$. Since $\cap_{\alpha \in \Delta} C_{\alpha} \subseteq C_{\alpha_{0}}$ we have $x \notin \wedge_{w}\left(\cap_{\alpha \in \Delta} C_{\alpha}\right)$ and hence $\wedge_{w}\left(\cap_{\alpha \in \Delta} C_{\alpha}\right) \subseteq$ $\subseteq \cap_{\alpha \in \Delta}\left(\wedge_{w}\left(C_{\alpha}\right)\right)$.
(7) $X-\vee_{w}(B)=\cap\{X-F: X-B \subseteq X-F, X-F \in w\}=\wedge_{w}(X-B)$.
(9) If $X-B \in w$, then by (6) and (7) $X-B=\wedge_{w}(X-B)=X-\vee_{w}(B)$. Hence $B=\vee_{w}(B)$.
(10) This follows from (2) and (7).
(11) This follows from (10).

In (4), (5) and (11) of Proposition 3.2, the equality does not necessarily hold as shown in the next example.

Example 3.3. (1) Let $X=\{a, b, c\}$. Consider the WS $w=\{\phi,\{a\},\{b\}\}$ on $X$. Let $A=\{a, b\}$ and $B=\{a, c\}$. Then $\wedge_{w}(A)=X, \wedge_{w}(B)=X$ and $\wedge_{w}(A \cap B)=\{a\}$. Thus $\wedge_{w}(A \cap B) \neq \wedge_{w}(A) \cap \wedge_{w}(B)$.
(2) Let $X=\{a, b, c\}$. Consider the WS $w=\{\phi,\{a\},\{b\}\}$ on $X$. Let $A=$ $\{a\}$ and $B=\{b\}$. Then $\wedge_{w}(A)=\{a\}, \wedge_{w}(B)=\{b\}$ and $\wedge_{w}(A \cup B)=X$. Thus $\wedge_{w}(A \cup B) \neq \wedge_{w}(A) \cup \wedge_{w}(B)$.
(3) Let $X=\{a, b, c\}$. Consider the WS $w=\{\phi,\{a\},\{b, c\}\}$ on $X$. Let $A=\{b\}$ and $B=\{c\}$. Then $\vee_{w}(A)=\phi, \vee_{w}(B)=\phi$ and $\vee_{w}(A \cup B)=\{b, c\}$. Thus $\vee_{w}(A \cup B) \neq \vee_{w}(A) \cup \vee_{w}(B)$.

Definition 3.4. In a $W S$ space $(X, w)$ a subset $A$ is called a $\wedge_{w}$-set (resp. $\vee_{w}$-set) if $\wedge_{w}(A)=A\left(\right.$ resp. $\left.\vee_{w}(A)=A\right)$. By $\wedge_{w}\left(\right.$ resp. $\vee_{w}$ ), we denote the family of all $\wedge_{w}$-sets (resp. $\vee_{w}$-sets) of the WS space $(X, w)$.

Remark 3.5. It follows from Proposition 3.2 (6) and (9) that in a WS $w$ if $A \in w$, then $A$ is a $\wedge_{w}$-set and if $X-A \in w$ then $A$ is a $\vee_{w}$-set. Also it is easy to observe from Definition 3.1 that, $X$ is a $\wedge_{w}$-set and $\phi$ is a $\vee_{w}$-set.

Theorem 3.6. If $w$ is a $W S$ on $X$, then
(1) $\phi$ and $X$ are $\vee_{w}$-sets ( $\phi$ and $X$ are $\wedge_{w}$-sets).
(2) The union of $\vee_{w}$-sets is a $\vee_{w}$-set.
(3) The intersection of $\wedge_{w}$-sets is a $\wedge_{w}$-set.

Proof. (1) This follows from Remark 3.5.
(2) Let $\left\{C_{\alpha}: \alpha \in \Omega\right\}$ be a family of $\vee_{w}$-sets in a $W S$ on $X$. Then by Proposition 3.2 and Definition 3.4, $\cup_{\alpha \in \Omega} C_{\alpha}=\cup_{\alpha \in \Omega}\left[\vee_{w}\left(C_{\alpha}\right)\right] \subseteq \vee_{w}\left[\cup_{\alpha \in \Omega}\left(C_{\alpha}\right)\right] \subseteq$ $\subseteq \cup_{\alpha \in \Omega}\left(C_{\alpha}\right)$. Hence $\cup_{\alpha \in \Omega} C_{\alpha}=\vee_{w}\left[\cup_{\alpha \in \Omega}\left(C_{\alpha}\right)\right]$.
(3) Let $\left\{C_{\alpha}: \alpha \in \Omega\right\}$ be a family of $\wedge_{w}$-sets in a $W S$ on $X$. Then by Proposition 3.2 and Definition 3.4, $\cap_{\alpha \in \Omega} C_{\alpha}=\cap_{\alpha \in \Omega}\left[\wedge_{w}\left(C_{\alpha}\right)\right] \supseteq \wedge_{w}\left[\cap_{\alpha \in \Omega}\left(C_{\alpha}\right)\right] \supseteq$ $\supseteq \cap_{\alpha \in \Omega}\left(C_{\alpha}\right)$. Hence $\cap_{\alpha \in \Omega} C_{\alpha}=\wedge_{w}\left[\cap_{\alpha \in \Omega}\left(C_{\alpha}\right)\right]$.

Definition 3.7. A $W S$ space $(X, w)$ is said to be $w-T_{1}$ if for any pair of distinct points $x$ and $y$ of $X$, there exist a $w$-open set $U$ of $X$ containing $x$ but not $y$ and a $w$-open set $V$ of $X$ containing $y$ but not $x$.

Theorem 3.8. For a WS space $(X, w)$, the implications $(2) \Rightarrow(3) \Rightarrow(1)$ hold. If $w$ is $G T$, then the following properties are equivalent:
(1) $(X, w)$ is $w-T_{1}$;
(2) For each $x \in X$, the singleton $\{x\}$ is $w$-closed in $(X, w)$;
(3) For each $x \in X$, the singleton $\{x\}$ is a $\wedge_{w}$-set.

Proof. (1) $\Rightarrow$ (2): Let $y$ be any point of $X$ and $x \in X-\{y\}$. There exists $V_{x} \in w$ such that $x \in V_{x}$ and $y \notin V_{x}$. Hence we have $X-\{y\}=\cup_{x \in X-\{y\}} V_{x}$. Therefore, the singleton $\{y\}$ is $w$-closed in $(X, w)$.
$(2) \Rightarrow(3)$ : Let $x$ be any point of $X$ and $y \in X-\{x\}$. Then $x \in X-\{y\} \in w$ and $\wedge_{w}(\{x\}) \subseteq X-\{y\}$. Therefore, $y \notin \wedge_{w}(\{x\})$ and $\wedge_{w}(\{x\}) \subseteq\{x\}$. This shows that $\wedge_{w}(\{x\})=\{x\}$. Therefore, the singleton $\{x\}$ is a $\wedge_{w}$-set.
$(3) \Rightarrow(1)$ : Suppose that the singleton $\{x\}$ is a $\wedge_{w}$-set for each $x \in X$. Let $x$ and $y$ be any distinct points. Then $y \notin \wedge_{w}(\{x\})$ and there exists a $w$-open set $U_{x}$ such that $x \in U_{x}$ and $y \notin U_{x}$. Similarly, $x \notin \wedge_{w}(\{y\})$ and there exists a $w$-open set $U_{y}$ such that $y \in U_{y}$ and $x \notin U_{y}$. This shows that $(X, w)$ is $w-T_{1}$.

Theorem 3.9. For a WS space $(X, w)$, the implications $(2) \Leftrightarrow(3) \Rightarrow(1)$ hold. If $W$ is a GT, then the following properties are equivalent:
(1) $(X, w)$ is $w-T_{1}$.
(2) Every subset of $X$ is a $\wedge_{w}$-set.
(3) Every subset of $X$ is a $\vee_{w}$-set.

Proof. It is obvious that $(2) \Leftrightarrow(3)$.
$(1) \Rightarrow(3)$ : Let $A$ be any subset of $X$. Since $A=\cup\{\{x\}: x \in A\}$, by Theorem 3.8 $A$ is the union of $w$-closed sets, hence $A$ is a $\vee_{w}$-set (by Remark 3.5 and Theorem 3.6).
(2) $\Rightarrow(1)$ : Let $x \in X$. Then by (2), $\{x\}$ is a $\wedge_{w}$-set. Let $p, q$ be any two distint points of $X$. Then $q \notin \wedge_{w}(\{p\})=\{p\}$. So by definition of $\wedge_{w}$-sets, there exists a $w$-open set $U$ such that $p \in U$ but $q \notin U$. Similarly the other case can done. Thus $(X, w)$ is $w-T_{1}$.

## 4. Generalized $\wedge_{w}$-sets and generalized $\vee_{w}$-sets

Definition 4.1. In a $W S$ space $(X, w)$, a subset $B$ is called a generalized $\wedge_{w^{-}}$ set (briefly $g$. $\wedge_{w}$-set) if $\wedge_{w}(B) \subseteq F$ whenever $B \subseteq F$ and $F$ is $w$-closed. The complement of a $g . \wedge_{w}$-set is called a $g . \vee_{w}$-set.

Proposition 4.2. In a $W S$ space $(X, w)$, the following properties hold:
(1) Every $\wedge_{w}$-set is a $g . \wedge_{w}$-set;
(2) Every $\vee_{w}$-set is a $g . \vee_{w}$-set.

Proof. (1) This follows from Definitions 3.4 and 4.1.
(2) Let $B$ be a $\vee_{w}$-set subset of $X$. Then $B=\vee_{w}(B)$. By Proposition 3.2 (7), $\wedge_{w}(X-B)=X-\vee_{w}(B)=X-B$. Thus by (1) and Definition 4.1, $B$ is a g. $\vee_{w}$-set.

Proposition 4.3. Let $(X, w)$ be a WS space. For each $x \in X$, the following properties hold:
(1) $\{x\}$ is $w$-open or $X-\{x\}$ is a $g . \wedge_{w}$-set.
(2) $\{x\}$ is $w$-open or $\{x\}$ is a $g . \vee_{w}$-set.

Proof. (1) Suppose $\{x\}$ is not a $w$-open set. Then the only $w$-closed set $F$ containing $X-\{x\}$ is $X$. Thus $\wedge_{w}(X-\{x\}) \subseteq F=X$ and thus $X-\{x\}$ is a $g . \wedge_{w}$-set of $X$.
(2) This follows from (1) and Definition 4.1.

Proposition 4.4. If $A$ is a $g . \wedge_{w}$-set of a $W S$ space $(X, w)$ and $A \subseteq B \subseteq \wedge_{w}(A)$, then $B$ is a $g . \wedge_{w}$-set of $(X, w)$.

Proof. Since $A \subseteq B \subseteq \wedge_{w}(A)$, by Proposition 3.2 (2), (3) $\wedge_{w}(A)=\wedge_{w}(B)$. Let $F$ be any $w$-closed subset of $X$ such that $B \subseteq F$. Then, $\wedge_{w}(B)=\wedge_{w}(A) \subseteq F$, since $A$ is a $g . \wedge_{w}$-set.

Proposition 4.5. A subset $B$ of a $W S$ space $(X, w)$ is a $g . \vee_{w}$-set if and only if $U \subseteq \vee_{w}(B)$ whenever $U \subseteq B$ and $U \in w$.

Proof. Let $U$ be a $w$-open subset of $(X, w)$ such that $U \subseteq B$. Then since $X-U$ is $w$-closed and $X-B \subseteq X-U$, we have $\wedge_{w}(X-B) \subseteq X-U$ by Definition 4.1. Hence by Proposition 3.2 (7) $X-\vee_{w}(B) \subseteq X-U$. Thus $U \subseteq \vee_{w}(B)$.

Conversely, let $F$ be a $w$-closed subset of $X$ such that $X-B \subseteq F$. Since $X-F$ is $w$-open and $X-F \subseteq B$, by assumption we have $X-F \subseteq \vee_{w}(B)$. Then $\wedge_{w}(X-B)=X-\vee_{w}(B) \subseteq F$ by Proposition 3.2 (7). Thus $X-B$ is a $g . \wedge_{w}$-set and hence $B$ is a $g . \vee_{w}$-set.

Corollary 4.6. Let $B$ be a $g . \vee_{w}$-set in a $W S$ space $(X, w)$. Then for every $w$ closed set $F$ such that $\vee_{w}(B) \cup(X-B) \subseteq F, X=F$ holds.

Proof. The assumption $\vee_{w}(B) \cup(X-B) \subseteq F$ implies that $X-F \subseteq\left(X-\vee_{w}(B)\right) \cap$ $\cap B$. Since $B$ is a $g . \vee_{w}$-set, then by Proposition 4.5, we have $X-F \subseteq \vee_{w}(B)$. On the other hand, $X-F \subseteq \vee_{w}(B) \cap\left(X-\vee_{w}(B)\right)=\phi$. Therefore, we have $X=F$.

Corollary 4.7. Let $B$ be a $g . \vee_{w}$-set in a $W S$ space $(X, w)$. Then $\vee_{w}(B) \cup(X-B)$ is a $w$-closed set if and only if $B$ is a $\vee_{w}(B)$-set.

Proof. Suppose that $\vee_{w}(B)=B$, then $\vee_{w}(B) \cup(X-B)=X$ is $w$-closed.
Conversely, by Corollary 4.6, $X=(X-B) \cup \vee_{w}(B)$. Thus $\left(X-\vee_{w}(B)\right) \cap$ $\cap B=\phi$. Hence by Proposition $3.2(8), \vee_{w}(B)=B$.

Definition 4.8. Let $w$ be a weak structure $(W S)$ on $X$. Then $A \subseteq X$ is called a $w$ generalized closed set (or simply w $g$-closed set) if $c_{w}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is $w$-open. The complement of a $w g$-closed set is called a $w$-generalized open (or simply wg-open) set.

Theorem 4.9. Let $(X, w)$ be a $W S$ space such that $H \cap c_{w}(K)$ is $w$-closed for any $w$-closed set $H$ and any subset $K$ of $X$. Then a subset $A$ of $X$ is wg-closed if and only if $c_{w}(A)-A$ contains no nonempty $w$-closed sets.

Proof. Suppose that $A$ is $w g$-closed. Let $F$ be a $w$-closed subset of $c_{w}(A)-A$. Since $A \subseteq X-F$ and $A$ is $w g$-closed, $c_{w}(A) \subseteq X-F$ and so $F \subseteq X-c_{w}(A)$. Therefore, $F=\phi$. Conversely, suppose the condition holds and $A \subseteq M$ and $M \in w$. If $c_{w}(A) \nsubseteq M$, then $c_{w}(A) \cap(X-M)$ is a nonempty $w$-closed subset of
$c_{w}(A)-A$. This contradicts the hypothesis. Therefore, $c_{w}(A) \subseteq M$ which implies that $A$ is $w g$-closed.

Definition 4.10. A $W S$ space $(X, w)$ is said to be $w-T_{\frac{1}{2}}^{*}$ if every wg-closed subset of $X$ is $w$-closed.

Theorem 4.11. For a WS space $(X, w)$, if $X \in w$ the implications (1) $\Rightarrow$ (2) hold. If $w$ is GT, then the following statements are equivalent:
(1) $(X, w)$ is $w-T_{\frac{1}{2}}^{*}$,
(2) For each $x \in{ }^{2} X$ the singleton $\{x\}$ is $w$-closed or $w$-open.

Proof. (1) $\Rightarrow$ (2). Suppose that $(X, w)$ is $w-T_{\frac{1}{2}}^{*}$ and let $x \in X$. If $\{x\}$ is not $w$-closed, then $X-\{x\}$ is not $w$-open, and thus $X$ is the only possible $w$-open set containing $X-\{x\}$. Thus $X-\{x\}$ is wg-closed. By assumption, $X-\{x\}$ is $w$-closed, that is $\{x\}$ is $w$-open.
(2) $\Rightarrow(1)$. Suppose that every singleton of X is $w$-open or $w$-closed and let $A$ be a $w g$-closed subset of $X$. Let $x \in c_{w}(A)$. We discuss the following two cases:
(a) $\{x\}$ is $w$-open. Then $\{x\} \cap A \neq \phi$, that is $x \in A$.
(b) $\{x\}$ is $w$-closed. Since $A$ is wg-closed, it follows from Theorem 4.9 that $x \notin c_{w}(A)-A$ and so $x \in A$.

Thus in both cases, $x \in A$. Therefore, $c_{w}(A)=A$, that is, $A$ is $w$-closed. Hence, $(X, w)$ is $w-T_{\frac{1}{2}}^{*}$.

Theorem 4.12. For a WS space $(X, w)$, if $X \in w$ the implications $(1) \Rightarrow(2) \Leftrightarrow$ $\Leftrightarrow(3)$ hold. If $w$ is $G T$, then the following statements are equivalent:
(1) $(X, w)$ is $w-T_{\frac{1}{2}}^{*}$.
(2) Every $g . \wedge_{w}$-set is a $\wedge_{w}$-set.
(3) Every $g . \vee_{w}$-set is a $\vee_{w}$-set.

Proof. (1) $\Rightarrow$ (2). Suppose that $(X, w)$ is $w-T_{\frac{1}{2}}^{*}$. If $A$ is a $g . \wedge_{w}$-set which is not a $\wedge_{w}$-set, then since $A \subseteq \wedge_{w}(A)$, there exists $x \in \wedge_{w}(A)$ such that $x \notin A$. By Theorem 4.11, $\{x\}$ is $w$-open or $w$-closed. We discuss two cases:
(a) $\{x\}$ is $w$-open. Then $X-\{x\}$ is a $w$-closed set containing $A$ and $A$ is a $g . \wedge_{w}$-set. Hence $\wedge_{w}(A) \subseteq X-\{x\}$, that is, $x \notin \wedge_{w}(A)$. This is a contradiction.
(b) $\{x\}$ is $w$-closed. Then $X-\{x\}$ is a $w$-open set containing $A$, and $\wedge_{w}(A) \subseteq$ $\subseteq X-\{x\}$. This is contray that $x \in \wedge_{w}(A)$. This contradiction proves the implication (1) $\Rightarrow$ (2).
$(2) \Rightarrow(1)$. Suppose that every $g . \wedge_{w}$-set is a $\wedge_{w}$-set and let $x \in X$. We will prove that $\{x\}$ is $w$-open or $w$-closed. If $\{x\}$ is not $w$-open, then $X-\{x\}$ is not $w$-closed, and so the only $w$-closed set containing $X-\{x\}$ is $X$. Thus, $X-\{x\}$ is a $g . \wedge_{w}$-set. By assumption, $X-\{x\}$ is a $\wedge_{w}$-set. Therefore, $X-\{x\}$ is $w$-open, that is, $\{x\}$ is $w$-closed. Hence by Theorem 4.11, $(X, w)$ is $w-T_{\frac{1}{2}}^{*}$.
(2) $\Leftrightarrow(3)$. This is obvious.

## Conclusion

The investigation enables us to obtain a unified theory of notions related to different sets for example $\wedge$-sets, $\vee$-sets, semi- $\wedge$-sets, semi- $\vee$-sets, pre- $\wedge$-sets, pre- $\vee$-sets in topological spaces, $\wedge_{m}$-sets and $\vee_{m}$-sets in $m$-spaces and $\wedge_{\mu}$-sets and $\vee_{\mu}$-sets in GT spaces.

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# SUPRA $\beta$-OPEN SETS AND SUPRA $\beta$-CONTINUITY ON TOPOLOGICAL SPACES 

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#### Abstract

In this paper, a new class of sets and maps between topological spaces called supra $\beta$-open sets and supra $\beta$-continuous maps, respectively are introduced and studied. Furthermore, the concepts of supra $\beta$-open maps and supra $\beta$-closed maps in terms of supra $\beta$-open sets and supra $\beta$-closed sets, respectively, are introduced and several properties are investigated.


## 1. Introduction and preliminaries

The concept of supra topology is fundamental with respect to the investigation of general topological spaces. Extensive research was done by many mathematicians in supra topology [1, 2, 3, 4]. They generalized the concept of openness such as supra-open, supra $\alpha$-open, supra-preopen, supra $b$-openness and obtained many important results analogous to topological spaces. In 1983, Mashhour et al. [1] initiated the study of the so-called supra topological spaces and studied $S$-continuous maps and $S^{*}$-continuous maps. We will use the term supra-continuous maps instead of $S$-continuous maps.

In 2008, Devi et al. [2] introduced and studied a class of sets and maps between topological spaces called supra $\alpha$-open sets and supra $\alpha$-continuous maps, respectively. Recently, Sayed and Noiri [3] introduced and investigated the notions of supra b-continuity, supra b-openness and supra b-closedness in terms of supra b-open set and supra b-closed set, respectively and most recently Sayed [4] introduced and investigated the notion of supra-pre-continuity, suprapre openness and supra-pre closedness sets in terms of supra pre-open set and
supra pre-closed respectively. Now, we introduce the concept of supra $\beta$-open sets and study some basic properties of it. Also, we introduce the concepts of supra $\beta$-continuous maps, supra $\beta$-open maps and supra $\beta$-closed maps and investigate several properties for these class of maps. In particular, we study the relation between supra $\beta$-continuous maps and supra $\beta$-open maps.

Throughout this paper, $(X, \tau),(Y, \sigma)$ and $(Z, v)$ (or simply $X, Y$ and $Z$ ) denote topological spaces on which no separation axioms are assumed unless explicitly stated. All sets are assumed to be subset of topological spaces. The closure and the interior of a set $A$ are denoted by $\mathrm{Cl}(A)$ and $\operatorname{Int}(A)$, respectively. The complement of the subset $A$ of $X$ is denoted by $X \sim A$. A subcollection $\mu \subset 2^{X}$ is called a supra topology [1] on $X$ if $X \in \mu$ and $\mu$ is closed under arbitrary union. $(X, \mu)$ is called a supra topological space. The elements of $\mu$ are called supra open in $(X, \mu)$ and the complement of a supra open set is called a supra closed set. The supra closure of a set $A$, denoted by $\mathrm{Cl}^{\mu}(A)$, is the intersection of the supra closed sets including $A$. The supra interior of a set $A$, denoted by $\operatorname{Int}^{\mu}(A)$, is the union of the supra open sets included in $A$. The supra topology $\mu$ on $X$ is associated with the topology $\tau$ if $\tau \subset \mu$. A set $A$ is called supra $\alpha$-open [1] (resp. supra $b$-open [3], supra pre open [4]) if $A \subseteq \operatorname{Int}^{\mu}\left(\mathrm{Cl}^{\mu}\left(\operatorname{Int}^{\mu}(A)\right)\right.$ ) (resp. $A \subseteq \mathrm{Cl}^{\mu}\left(\operatorname{Int}^{\mu}(A)\right) \cup \operatorname{Int}^{\mu}\left(\mathrm{Cl}^{\mu}(A)\right), A \subseteq \operatorname{Int}^{\mu}\left(\mathrm{Cl}^{\mu}(A)\right)$.

## 2. Supra $\beta$-open sets

In this section, we introduce a new class of open sets called supra $\beta$-open sets and study some of their basic properties.

Definition 2.1. A set $A$ is supra $\beta$-open if $A \subseteq \mathrm{Cl}^{\mu}\left(\operatorname{Int}^{\mu}\left(\mathrm{Cl}^{\mu}(A)\right)\right)$.
The complement of supra $\beta$-open set is called supra $\beta$-closed. Thus $A$ is supra $\beta$-closed if and only if $\operatorname{Int}^{\mu}\left(\mathrm{Cl}^{\mu}\left(\operatorname{Int}^{\mu}(A)\right)\right) \subseteq A$.

Theorem 2.1. (i) Every supra $\alpha$-open set is supra $\beta$-open.
(ii) Every supra pre-open set is supra b-open and hence supra $\beta$-open.

Proof. Obvious.
The following example shows that supra $\beta$-open set need not be supra $b$ open.

Example 2.1. Let $(X, \mu)$ be a supra topological space, where $X=R$ and $\mu=$ usual topology. Then $[0,1) \cap Q$, where $Q$ is set of rationals, is supra $\beta$-open but not supra $b$-open.

The following diagram which is a continuation of diagram in [4] shows how supra $\beta$-open sets are related to some similar types of supra-open sets.
Supra-open $\rightarrow$ supra $\alpha$-open $\rightarrow$ supra pre-open $\rightarrow$ supra $b$-open $\rightarrow$ supra $\beta$-open
Theorem 2.2. (i) Arbitrary union of supra $\beta$-open sets is always supra $\beta$ open.
(ii) Finite intersection of supra $\beta$-open sets may fail to be supra $\beta$-open.
(iii) $X$ is a supra $\beta$-open set.

Proof. (i) Let $\left\{A_{\lambda}: \lambda \in \Lambda\right\}$ be the family of supra $\beta$-open sets in a topological space $X$. Then for any $\lambda \in \Lambda$, we have $A_{\lambda} \subseteq \mathrm{Cl}^{\mu}\left(\operatorname{Int}^{\mu}\left(\mathrm{Cl}^{\mu}\left(A_{\lambda}\right)\right)\right)$. Hence

$$
\begin{aligned}
\bigcup_{\lambda \in \Lambda} A_{\lambda} & \subseteq \bigcup_{\lambda \in \Lambda}\left(\mathrm{Cl}^{\mu}\left(\operatorname{Int}^{\mu}\left(\mathrm{Cl}^{\mu}\left(A_{\lambda}\right)\right)\right)\right) \subseteq \mathrm{Cl}^{\mu}\left(\bigcup_{\lambda \in \Lambda}\left(\operatorname{Int}^{\mu}\left(\mathrm{Cl}^{\mu}\left(A_{\lambda}\right)\right)\right)\right) \\
& \subseteq \mathrm{Cl}^{\mu}\left(\operatorname{Int}^{\mu}\left(\bigcup_{\lambda \in \Lambda} \mathrm{Cl}^{\mu}\left(A_{\lambda}\right)\right)\right) \subseteq \mathrm{Cl}^{\mu}\left(\operatorname{Int}^{\mu}\left(\mathrm{Cl}^{\mu}\left(\bigcup_{\lambda \in \Lambda} A_{\lambda}\right)\right)\right)
\end{aligned}
$$

Therefore $\bigcup_{\lambda \in \Lambda} A_{\lambda}$ is a supra $\beta$-open set.
(ii) Let $(X, \mu)$ be supra topological space, where $X=\{a, b, c\}$ and $\mu=$ $\{\phi, X,\{a\},\{a, b\},\{b, c\}\}$. Then both $\{a, c\}$ and $\{b, c\}$ are supra $\beta$-open, but their intersection $\{c\}$ is not supra $\beta$-open.
Also, $\{a, c\}$ is neither supra-open nor supra $b$-open.
(iii) Obvious.

Theorem 2.3. (i) Arbitrary intersection of supra $\beta$-closed sets is always supra $\beta$-closed.
(ii) Finite union of supra $\beta$-closed sets may fail to be supra $\beta$-closed.

Proof. (i) This follows immediately from Theorem 2.2.
(ii) In Example of Theorem 2.2 (ii), $\{a\}$ and $\{b\}$ are supra $\beta$-closed, but their union $\{a, b\}$ is not supra $\beta$-closed.

Definition 2.2. The supra $\beta$-closure of a set $A$, denoted by $\mathrm{Cl}_{\beta}^{\mu}(A)$, is the intersection of the supra $\beta$-closed sets including $A$. The supra $\beta$-interior of a set $A$, denoted by $\operatorname{Int}_{\beta}^{\mu}(A)$, is the union of the supra $\beta$-open sets included in $A$.

Remark 2.1. It is clear that $\operatorname{Int}_{\beta}^{\mu}(A)$ is a supra $\beta$-open set and $\mathrm{Cl}_{\beta}^{\mu}(A)$ is supra $\beta$-closed.

Theorem 2.4. (i) $A \subseteq \mathrm{Cl}_{\beta}^{\mu}(A)$ and $A=\mathrm{Cl}_{\beta}^{\mu}(A)$ iff $A$ is a supra $\beta$-closed set.
(ii) $\operatorname{Int}_{\beta}^{\mu}(A) \subseteq A$ and $\operatorname{Int}_{\beta}^{\mu}(A)=A$ iff $A$ is a supra $\beta$-open set.
(iii) $X \sim \operatorname{Int}_{\beta}^{\mu}(A)=\mathrm{Cl}_{\beta}^{\mu}(X \sim A)$.
(iv) $X \sim \operatorname{Cl}_{\beta}^{\mu}(X \sim A)=\operatorname{Int}_{\beta}^{\mu}(A)$.
(v) If $A \subseteq B$, then $\mathrm{Cl}_{\beta}^{\mu}(A) \subseteq \mathrm{Cl}_{\beta}^{\mu}(B)$ and $\operatorname{Int}_{\beta}^{\mu}(A) \subseteq \operatorname{Int}_{\beta}^{\mu}(B)$.

Proof. Obvious.
Theorem 2.5. (a) $\operatorname{Int}_{\beta}^{\mu}(A) \cup \operatorname{Int}_{\beta}^{\mu}(B) \subseteq \operatorname{Int}_{\beta}^{\mu}(A \cup B)$
(b) $\mathrm{Cl}_{\beta}^{\mu}(A \cap B) \subseteq \mathrm{Cl}_{\beta}^{\mu}(A) \cap \mathrm{Cl}_{\beta}^{\mu}(B)$.

Proof. Obvious.
The inclusions in (a) and (b) in Theorem 2.5 can not be replaced by equalities as it can be seen from the following example.

Let $(X, \mu)$ be a supra topological space where $X=\{a, b, c\}$ and $\mu=$ $\{\phi, X,\{a\},\{a, b\},\{b, c\}\}$. Where, if $A=\{b\}$ and $B=\{c\}$, then

$$
\operatorname{Int}_{\beta}^{\mu}(A)=\operatorname{Int}_{\beta}^{\mu}(B)=\phi \quad \text { and } \quad \operatorname{Int}_{\beta}^{\mu}(A \cup B)=\{a, b\} .
$$

Also, if $C=\{a, b\}$ and $D=\{a, b\}$, then $\mathrm{Cl}_{\beta}^{\mu}(C)=\mathrm{Cl}_{\beta}^{\mu}(D)=X$ and $\mathrm{Cl}_{\beta}^{\mu}(C \cap$ $\cap D)=\{a\}$.

Proposition 2.1. (i) The intersection of supra open and supra $\beta$-open set is supra $\beta$-open.
(ii) The intersection of supra $\alpha$-open and the supra $\beta$-open set is supra $\beta$-open.

## 3. Supra $\beta$-continuous maps

In this section, we introduce a new type of continuous maps called a supra $\beta$-continuous maps and obtain some of their properties and characterizations.

Definition 3.1. Let $(X, \tau)$ and $(Y, \sigma)$ be two topological spaces and $\mu$ be an associated supra topology with $\tau$. A map $f:(X, \tau) \rightarrow(Y, \sigma)$ is called
(i) supra continuous (resp., supra $\alpha$-continuous [2], supra pre-continuous [4], supra $b$-continuous [3]) if the inverse image of each open set in $Y$ is supra open (resp., supra $\alpha$-open, supra pre-open, supra $b$-open) in $X$.
(ii) supra $\beta$-continuous if the inverse image of each open set in $Y$ is supra $\beta$ open in $X$.

Theorem 3.1. Every continuous map is supra $\beta$-continuous.
Proof. Let $f: X \rightarrow Y$ be a continuous map and $A$ is open set in $Y$. Then $f^{-1}(A)$ is an open set in $X$. Since $\mu$ associated with $\tau$, then $\tau \subseteq \mu$. Therefore $f^{-1}(A)$ is a supra open set in $X$ and thus supra $\beta$-open set in $X$. Hence $f$ is supra $\beta$-continuous map.

The converse of the above theorem is not true as it is shown in the following example.

Example 3.1. Let $X=\{a, b, c\}$ and $\tau=\{X, \phi,\{a, b\}\}$ be a topology on $X$. The supra topology $\mu$ is defined as follows: $\mu=\{X, \phi,\{a\},\{a, b\}\}$. Let $f:(X, \tau) \rightarrow$ $\rightarrow(X, \tau)$ be a map defined as follows: $f(a)=b, f(b)=c, f(c)=a$. Since the inverse image of the open set $\{a, b\}$ is $\{a, c\}$ which is not an open set but it is a supra $\beta$-open set. Then $f$ is supra $\beta$-continuous map but not continuous map.

The following example shows that supra $\beta$-continuous map need not be supra $b$-continuous map.

Example 3.2. Let $\tau$ be the usual topology on $R$ and $\sigma=\{\phi, R,[0,1) \cap Q\}$. Then the identity function $f:(R, \tau) \rightarrow(R, \sigma)$ is supra $\beta$-continuous but not supra $b$-continuous.

It is shown in Example 4.1 in [1] that supra $\alpha$-continuous map need not be supra continuous. Also Example 3.2 and 3.3 in [4] shows that supra $\beta$-continuous map need not be supra $\alpha$-continuous and supra $b$-continuous map need not be supra $\beta$-continuous.

From the above facts we have the following diagram in which the converses of the implications need not be true (cont. is the abbreviation of continuity).
Supra-cont $\rightarrow$ supra $\alpha$-cont. $\rightarrow$ supra pre-cont. $\rightarrow$ supra $b$-cont. $\rightarrow$ supra $\beta$-cont.
Theorem 3.2. Let $(X, \tau)$ and $(Y, \sigma)$ be two topological spaces and $\mu$ be an associated supra topology with $\tau$. Letf be a map from $X$ into $Y$. Then the following are equivalent:
(i) $f$ is a supra $\beta$-continuous map;
(ii) The inverse image of a closed set in $Y$ is a supra $\beta$-closed set in $X$;
(iii) $C l_{\beta}^{\mu}\left(f^{-1}(A)\right) \subseteq f^{-1}(\mathrm{Cl}(A))$.
(iv) $f\left(\mathrm{Cl}_{\beta}^{\mu}(A)\right) \subseteq \mathrm{Cl}(f(A))$ for every set $A$ in $X$;
(v) $f^{-1}(\operatorname{Int}(B)) \subseteq \operatorname{Int}_{\beta}^{\mu}\left(f^{-1}(B)\right)$ for every set $B$ in $Y$.

Proof. (1) $\Rightarrow$ (2): Let $A$ be a closed set in $Y$, then $Y \sim A$ is an open set in $Y$. Then $f^{-1}(Y \sim A)=X \sim f^{-1}(A)$ is a supra $\beta$-open set in $X$. It follows that $f^{-1}(A)$ is a supra $\beta$-closed subset of $X$.
$(2) \Rightarrow(3)$ : Let $A$ be any subset of $Y$. Since $\mathrm{Cl}(A)$ is closed in $Y$, then $f^{-1}(\mathrm{Cl}(A))$ is a supra $\beta$-closed set in $X$. Therefore

$$
\mathrm{Cl}_{\beta}^{\mu}\left(f^{-1}(A)\right) \subseteq C l_{\beta}^{\mu}\left(f^{-1}(\mathrm{Cl}(A))\right)=f^{-1}(\mathrm{Cl}(A))
$$

$(3) \Rightarrow(4)$ : Let $A$ be any subset of $X$. By (3), we have

$$
f^{-1}(\mathrm{Cl}(f(A))) \supseteq \mathrm{Cl}_{\beta}^{\mu}\left(f^{-1}(f(A))\right) \supseteq \mathrm{Cl}_{\beta}^{\mu}(A) .
$$

Therefore $f\left(\operatorname{Cl}_{\beta}^{\mu}(A)\right) \subseteq \mathrm{Cl}(f(A))$.
(4) $\Rightarrow$ (5): Let $B$ be any subset of $Y$. By (4), $f\left(\mathrm{Cl}_{\beta}^{\mu}\left(X \sim f^{-1}(B)\right)\right) \subseteq$ $\subseteq \mathrm{Cl}\left(f\left(X \sim f^{-1}(B)\right)\right)$ and $f\left(X \sim \operatorname{Int}_{\beta}^{\mu}\left(f^{-1}(B)\right)\right) \subseteq \operatorname{Cl}(Y \sim B)=Y \sim \operatorname{Int}(B)$.

Therefore, we have $X \sim \operatorname{Int}_{\beta}^{\mu}\left(f^{-1}(B)\right) \subseteq f^{-1}(Y \sim \operatorname{Int}(B))$ and hence $f^{-1}(\operatorname{Int}(B)) \subseteq \operatorname{Int}_{\beta}^{\mu}\left(f^{-1}(B)\right)$.
$(5) \Rightarrow(1)$ : Let $B$ be an open set in $Y$ and $f^{-1}(\operatorname{Int}(B)) \subseteq \operatorname{Int}_{\beta}^{\mu}\left(f^{-1}(B)\right)$. Then $f^{-1}(B) \subseteq \operatorname{Int}_{\beta}^{\mu}\left(f^{-1}(B)\right)$. But,

$$
\operatorname{Int}_{\beta}^{\mu}\left(f^{-1}(B)\right) \subseteq f^{-1}(B)
$$

Hence $\operatorname{Int}_{\beta}^{\mu}\left(f^{-1}(B)\right)=f^{-1}(B)$.
Therefore $f^{-1}(B)$ is supra $\beta$-open set in $Y$.

THEOREM 3.3. Iff: $X \rightarrow Y$ is supra $\beta$-continuous and $g: Y \rightarrow Z$ is continuous, then $g \circ f: X \rightarrow Z$ is supra $\beta$ - continuous.

Proof. Obvious.
Theorem 3.4. Let $(X, \tau)$ and $(Y, \sigma)$ be two topological spaces and $\mu$ and $u$ be the associated supra topologies with $\tau$ and $\sigma$, respectively, Then $f: X \rightarrow Y$ is supra $\beta$-continuous if one of the following hold:
(1) $f^{-1}\left(\operatorname{Int}_{\beta}^{\mu}(B)\right) \subseteq \operatorname{Int}\left(f^{-1}(B)\right)$ for every set $B$ in $Y$.
(2) $C l\left(f^{-1}(B)\right) \subseteq f^{-1}\left(\mathrm{Cl}_{\beta}^{\mu}(B)\right)$ for every set $B$ in $Y$.
(3) $f(\mathrm{Cl}(A)) \subseteq \mathrm{Cl}_{\beta}^{\mu}(f(A))$ for every set $A$ in $X$.

Proof. Let $B$ be any open set of $Y$. If condition (1) is satisfied, then $f^{-1}\left(\operatorname{Int}_{\beta}^{\mu}(B)\right) \subseteq \operatorname{Int}\left(f^{-1}(B)\right)$. We get $f^{-1}(B) \subseteq \operatorname{Int}\left(f^{-1}(B)\right)$.

Therefore $f^{-1}(B)$ is an open set. Every open set is supra $\beta$-open. Hence $f$ is supra $\beta$-continuous.

If condition (2) is satisfied, then we can easily prove that $f$ is supra $\beta$ continuous.

Let condition (3) be satisfied and $B$ be any open set in $Y$. Then $f^{-1}(B)$ is a set in $X$ and $f\left(\mathrm{Cl}\left(f^{-1}(B)\right)\right) \subseteq \mathrm{Cl}_{\beta}^{\mu}\left(f\left(f^{-1}(B)\right)\right)$. This implies that $f\left(\mathrm{Cl}\left(f^{-1}(B)\right)\right) \subseteq$ $\subseteq \mathrm{Cl}_{\beta}^{\mu}(B)$. This is nothing but condition (2). Hence $f$ is supra $\beta$-continuous.

## 4. Supra $\beta$-open maps and supra $\beta$-closed maps

Definition 4.1. A map $f: X \rightarrow Y$ is called supra $\beta$-open (resp. supra $\beta$-closed) if the image of each open (resp. closed) set in $X$, is supra $\beta$-open (resp. supra $\beta$-closed) in $Y$.

Theorem 4.1. A map $f: X \rightarrow Y$ is supra $\beta$-open if and only if $f(\operatorname{Int}(A)) \subseteq$ $\subseteq \operatorname{Int}_{\beta}^{\mu}(f(A))$ for each set $A$ in $X$.

Proof. Suppose that $f$ is a supra $\beta$-open map. Since $\operatorname{Int}(A) \subseteq A$. Then $f(\operatorname{Int}(A)) \subseteq f(A)$. By hypothesis $f(\operatorname{Int}(A))$ is a supra $\beta$-open set and $\operatorname{Int}_{\beta}^{\mu}(f(A))$ is the largest supra $\beta$-open set contained in $f(A)$, then $f(\operatorname{Int}(A)) \subseteq \operatorname{Int}_{\beta}^{\mu}(f(A))$.

Conversely, suppose $A$ is an open set in $X$. Then $f(\operatorname{Int}(A)) \subseteq \operatorname{Int}_{\beta}^{\mu}(f(A))$. Since $\operatorname{Int}(A)=A$, then $f(A) \subseteq \operatorname{Int}_{\beta}^{\mu}(f(A))$.

Therefore $f(A)$ is a supra $\beta$-open set in $Y$ and $f$ is supra $\beta$-open.
Theorem 4.2. A map $f: X \rightarrow Y$ is supra $\beta$-closed if and only if $\mathrm{Cl}_{\beta}^{\mu}(f(A)) \subseteq$ $\subseteq f(\mathrm{Cl}(A))$ for each set $A$ in $X$.

Proof. Suppose $f$ is a supra $\beta$-closed map. Since for each set $A$ in $X, \operatorname{Cl}(A)$ is closed set in $X$, then $f(\mathrm{Cl}(A))$ is a supra $\beta$-closed set in $Y$. Also, since $f(A) \subseteq$ $\subseteq f(\mathrm{Cl}(A))$, then $\mathrm{Cl}_{\beta}^{\mu}(f(A)) \subseteq f(C l(A))$.

Conversely, let $A$ be a closed set in $X$. Since $\mathrm{Cl}_{\beta}^{\mu}(f(A))$ is the smallest supra $\beta$-closed set containing $f(A)$, then $f(A) \subseteq \mathrm{Cl}_{\beta}^{\mu}(f(A)) \subseteq f(\mathrm{Cl}(A))=f(A)$.

Thus $f(A)=\mathrm{Cl}_{\beta}^{\mu}(f(A))$.
Hence $f(A)$ is a supra $\beta$-closed set in $Y$. Therefore $f$ is a supra $\beta$-closed map.
Theorem 4.3. Let $f: X \rightarrow Y$ and $g: Y \rightarrow Z$ be two maps.
(i) Ifgof is supra $\beta$-open and $f$ is continuous surjective, then $g$ is supra $\beta$-open.
(ii) If $g \circ f$ is open and $g$ is $\beta$-continuous injective, then $f$ is supra $\beta$-open.

Proof. (i) Let $A$ be an open set in $Y$. Then $f^{-1}(A)$ is an open set in $X$. Since $g \circ f$ is a supra $\beta$-open map, then

$$
(g \circ f)\left(f^{-1}(A)\right)=g\left(f\left(f^{-1}(A)\right)\right)=g(A)
$$

(because $f$ is surjective) is a supra $\beta$-open set in $Z$. Therefore $g$ is a supra $\beta$-open map.
(ii) Let $A$ be an open set in $X$. Then, $g(f(A))$ is open set in $Z$. Therefore, $g^{-1}(g(f(A)))=f(A)$ (because $g$ is injective) is a supra $\beta$-open set in $Y$. Hence $f$ is a supra $\beta$-open map.

Theorem 4.4. Let $f: X \rightarrow Y$ be a map. Then the following are equivalent:
(i) $f$ is a supra $\beta$-open map;
(ii) $f$ is a supra $\beta$-closed map;
(iii) $f^{-1}$ is a supra $\beta$-continuous map.

Proof. (i) $\Rightarrow$ (ii). Suppose $B$ is a closed set in $X$. Then $X \sim B$ is an open set in $X$. By (1), $f(X \sim B)$ is a supra $\beta$-open set in $Y$. Since $f$ is bijective, then $f(X \sim B)=Y \sim f(B)$. Hence $f(B)$ is a supra $\beta$-closed set in $Y$. Therefore $f$ is a supra $\beta$-closed map.
(ii) $\Rightarrow$ (iii). Let $f$ be a supra $\beta$-closed map and $B$ a closed set in $X$. Since $f$ is bijective, then $\left(f^{-1}\right)^{-1}(B)=f(B)$ which is a supra $\beta$-closed set in $Y$. By Theorem 3.2, $f$ is a supra $\beta$-continuous map.
(iii) $\Rightarrow$ (i). Let $A$ be an open set in $X$. Since $f^{-1}$ is a supra $\beta$ - continuous map, then $\left(f^{-1}\right)^{-1}(A)=f(A)$ is a supra $\beta$-open set in $Y$. Hence $f$ is supra $\beta$-open.

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# ABSOLUTE CONVERGENCE OF WALSH-FOURIER SERIES 

By<br>R. G. VYAS<br>(Received December 30, 2011)


#### Abstract

For the classes of functions of $\Lambda B V(p(n) \uparrow \infty, \varphi)$ and $B V \cap \operatorname{Lip}(\alpha, p)$, we obtain sufficiency conditions for the convergent of series $\sum_{n=1}^{\infty} n^{\alpha}|\hat{f}(n)|^{\beta},(\alpha \geq 0$, $0<\beta \leq 2$ ), where $\hat{f}(n)$ are Walsh-Fourier coefficients of $f$.


## 1. Introduction

In 1949, N. J. Fine [1] estimated the order of magnitude of Walsh-Fourier coefficients of a function satisfies Lipschitz condition of order $\alpha, 0<\alpha \leq 1$. In 2001, U. Goginava [2] obtained sufficiency condition for the uniform convergence of Walsh-Fourier series of functions of the generalized Wiener class $B V(p(n) \uparrow \infty)$. Peter Simon ([4], [5]) has studied summability of Walsh-Fourier series. Recently, Móricz [3] obtained sufficiency condition for the absolute convergence of Walsh-Fourier series. Here, we obtain sufficiency conditions for the generalized $\beta$-absolute convergence of Walsh-Fourier series of classes functions of $\Lambda B V(p(n) \uparrow \infty, \varphi)$ and $B V \cap \operatorname{Lip}(\alpha, p)$.

Let $f$ be a function defined on $(-\infty, \infty)$ with period $1 . \mathbf{P}$ is said to be a partition with period 1 if

$$
\mathbf{P}: \ldots<x_{-1}<x_{0}<x_{1}<\ldots<x_{m}<\ldots
$$

satisfies $x_{k+m}=x_{k}+1$ for $k=0, \pm 1, \pm 2, \ldots$, where $m$ is a positive integer.
DEFINITION 1.1. Let $\varphi(n)$ be a real sequence such that $\varphi(1) \geq 2$ and $\lim _{n \rightarrow \infty} \varphi(n)=\infty$. For a sequence $\Lambda=\left\{\lambda_{m}\right\}(m=1,2, \ldots)$ of non-decreasing
positive real numbers $\lambda_{m}$ such that $\sum_{m=1}^{\infty} \frac{1}{\lambda_{m}}$ diverges and $1 \leq p(n) \uparrow p$ as $n \rightarrow$ $\rightarrow \infty$, where $1 \leq p \leq \infty$, we say that $f \in \Lambda B V(p(n) \uparrow p, \varphi)$ (that is, $f$ is a function of $p(n)-\Lambda$-bounded variation over $[0,1])$ if

$$
V_{\Lambda}(f, p(n), \varphi)=\sup _{n \geq 1} \sup _{\mathbf{P}}\left\{\left(V_{\Lambda}(\mathbf{P}, f, p(n))\right): \rho\{\mathbf{P}\} \geq \frac{1}{\varphi(n)}\right\}<\infty
$$

where

$$
\begin{gathered}
V_{\Lambda}(\mathbf{P}, f, p(n))=\left(\sum_{k=1}^{m} \frac{\left|f\left(x_{k}\right)-f\left(x_{k-1}\right)\right|^{p(n)}}{\lambda_{k}}\right)^{1 / p(n)}, \\
\rho\{\mathbf{P}\}=\inf _{k}\left|x_{k}-x_{k-1}\right|
\end{gathered}
$$

Note that, if $\varphi(n)=2^{n}, \forall n$, and $p=\infty$ then one gets the class $\Lambda B V(p(n) \uparrow$ $\infty)$; if $\lambda_{m} \equiv 1, \forall m$, then one gets the class $B V(p(n) \uparrow p, \varphi)$; if $p(n)=p, \forall n$, then one gets the class $\Lambda B V^{(p)}$.

For $p=\infty$, we shall denote this class $\Lambda B V(p(n) \uparrow \infty, \varphi)$ by simply $\Lambda B V(p(n), \varphi)$.

Let $\left\{\phi_{n}\right\}\left(n \in \mathbb{N}_{0}=\{0,1,2, \ldots\}\right)$ denote the complete orthonormal Walsh system defined on the interval $[0,1]$ in the Paley enumeration.

The Walsh system [1] can be realized as the full set of characters of the dyadic group $G=Z_{2}^{\infty}$, in which $Z_{2}=\{0,1\}$ is the group under addition modulo 2 . We denote the operation of $G$ by $\dot{+} .(G, \dot{+})$ is identify with $([0,1], \dot{+})$ under the usual convention for the binary expansion of elements of $[0,1][1]$.

Any $x \in[0,1)$ can be written as

$$
x=\sum_{k=0}^{\infty} x_{k} 2^{-(k+1)}, \quad \text { each } x_{k}=0 \text { or } 1
$$

For any $x \in[0,1) \backslash Q$ there is only one expression of this form, where $Q$ is the class of dyadic rational in $[0,1)$. When $x \in Q$ there are two expression of this form, one which terminates in 0 's and one which terminates in 1 's. For any $x, y \in[0,1)$ their dyadic sum of is defined as

$$
x \dot{+} y=\sum_{k=0}^{\infty}\left|x_{k}-y_{k}\right| 2^{-(k+1)} .
$$

Observe that, for each $n \in \mathbb{N}, \phi_{n}(x+y)=\phi_{n}(x) \phi_{n}(y), x, y \in[0,1), x \dot{+} y \notin Q$.

The dyadic $p$-integral modulus of continuity $\omega^{(p)}(\gamma, f)$ of a function $f \in$ $\in L^{p}([0,1])(1 \leq p<\infty)$ is defined as

$$
\omega^{(p)}(\gamma, f)=\sup _{0 \leq h<\gamma}\left\{\left(\int_{0}^{1}|f(x+h)-f(x)|^{p} d x\right)^{1 / p}\right\} .
$$

For $p=\infty$, we omit writing p , the dyadic modulus of continuity of a function $f$ is defined as

$$
\omega(\gamma, f)=\sup \{|f(x+h)-f(x)|: x \in[0,1], 0 \leq h<\gamma\} .
$$

Note that for each $n \in \mathbb{N}, \gamma<\frac{1}{2^{n}}$ implies $\omega\left(\gamma, \phi_{n}\right)=0$. Thus the inequality

$$
\omega(2 \gamma, f) \leq 2 \omega(\gamma, f), \quad \gamma>0,
$$

does not hold for a function $f$.
For $\alpha>0, \operatorname{Lip}(\alpha)$ denotes the class of functions which satisfy the condition $\omega(\delta, f) \leq C \delta^{\alpha}, 0<\delta \leq 1$, where $C$ is a constant which depends on $f$. Similarly, for $\alpha>0$ and $1 \leq p<\infty, \operatorname{Lip}(\alpha, p)$ denotes the class of functions $f \in L^{p}([0,1])$ which satisfy the condition $\omega^{(p)}(\delta, f) \leq C \delta^{\alpha}, 0<\delta \leq 1$. Obviously; $\operatorname{Lip}(\alpha) \subset$ $\subset \operatorname{Lip}(\alpha, p)$.

For a 1-periodic function $f \in L^{1}[0,1]$, its Walsh-Fourier series is defined by

$$
\begin{equation*}
f(x) \sim \sum_{n \in \mathbb{N}_{0}} \hat{f}(n) \phi_{n}(x), \tag{1.1}
\end{equation*}
$$

where $\hat{f}(n)=\int_{0}^{1} f(x) \phi_{n}(x) d x, \forall n \in \mathbb{N}_{0}$, are the Walsh-Fourier coefficients of $f$.

Series (1.1) is said to be generalized $\beta$-absolute convergent if

$$
\begin{equation*}
\sum_{n \in \mathbb{N}_{0}} n^{\delta}|\hat{f}(n)|^{\beta}<\infty, \quad(\delta \geq 0,0<\beta \leq 2) \tag{1.2}
\end{equation*}
$$

For $\delta=0$ one gets the $\beta$-absolute convergence of Walsh-Fourier series; and for $\delta=0$ and $\beta=1$ one gets the absolute convergence of Walsh-Fourier series.

## 2. Statements of the results

We prove the following theorems.
Theorem 2.1. If a 1-periodic $f \in \Lambda B V(p(n), \varphi)$ over $[0,1]$ and

$$
\sum_{n=1}^{\infty}\left(\frac{\omega\left(\frac{1}{n}, f\right)}{n^{1-2 \delta / \beta}\left(\sum_{j=1}^{n} \frac{1}{\lambda_{j}}\right)^{1 / p(\tau(n))}}\right)^{\beta / 2}<\infty
$$

then (1.2) holds, where

$$
\begin{equation*}
\tau(m)=\min \{k: k \in \mathbb{N}, \varphi(k) \geq m\}, m \geq 1 \tag{2.1}
\end{equation*}
$$

Theorem 2.2. If a 1 -periodic $f \in B V \cap \operatorname{Lip}(\alpha, p)$ over $[0,1]$, for $\alpha>0$ and $p>2$, then its Walsh-Fourier series is $\beta$-absolutely convergent for $\beta>\frac{2(p-1)}{2 p+\alpha p-3}$.

Corollary 2.3. If a 1-periodic $f \in B V \cap \operatorname{Lip}(\alpha, p)$ over $[0,1]$ for $\alpha>0, p>2$ and $\alpha p>1$, then its Walsh-Fourier series converges absolutely.

We need the following lemma to prove the results.
Lemma 2.4. ([6, Lemma 3.1]) The class $\Lambda B V(p(n) \uparrow p, \varphi,[0,1])(1 \leq p \leq \infty)$ $\subseteq B[0,1]$, where $B[0,1]$ is the class of bounded functions over $[0,1]$.

## 3. Proof of the results

Proof of Theorem 2.1. In view of Lemma 2.4, $f \in \Lambda B V(p(n), \varphi)$ over $[0,1]$ implies $f$ is bounded and hence $f \in L^{2}[0,1]$.

Fix $k \in \mathbb{N}$ and $h=\frac{1}{2^{k+1}}$. Put

$$
g(x)=f(x+h)-f(x), \quad \text { for all } x .
$$

Then $g \in L^{2}[0,1]$. For any $n \in D_{k}$, where $D_{k}:=\left\{2^{k}, 2^{k}+1, \ldots, 2^{k+1}-1\right\}$, we have $\hat{g}(n)=\hat{f}(n) \phi_{n}(h)-\hat{f}(n)$.

Since $\phi_{n}(h)=-1$ for any $n \in D_{k}$, we get $\hat{g}(n)=-2 \hat{f}(n)$.

Parseval's equality implies

$$
\begin{equation*}
2\left(\sum_{n \in D_{k}}|\hat{f}(n)|^{2}\right)^{1 / 2}=\left(\sum_{n \in D_{k}}|\hat{g}(n)|^{2}\right)^{1 / 2} \leq\left(\int_{0}^{1}|g(x)|^{2} d x\right)^{1 / 2} \tag{3.1}
\end{equation*}
$$

For $j=1,2, \ldots, 2^{k}-1$, put $f_{j}(x)=f(x+2 j h)-f(x+(2 j-1) h)$. Since $f$ is 1-periodic, we get

$$
\int_{0}^{1}|g(x)|^{2} d x=\int_{0}^{1}\left|f_{j}(x)\right|^{2} d x, \quad j=1,2, \ldots, 2^{k}-1
$$

This together with (3.1) implies
(3.2) $\left(\sum_{n \in D_{k}}|\hat{f}(n)|^{2}\right)=O\left(\int_{0}^{1}\left|f_{j}(x)\right|^{2} d x\right), \quad$ for all $j=1,2, \ldots, 2^{k}-1$.

Multiplying both the sides of the above equation by $\frac{1}{\lambda_{j}}$ and then summing over $j=1$ to $2^{k}-1$, we get

$$
\begin{gathered}
S_{k} \equiv \sum_{n \in D_{k}}|\hat{f}(n)|^{2}=O\left(\frac{1}{\sum_{j=1}^{2^{k}-1} \frac{1}{\lambda_{j}}}\right)\left(\int_{0}^{1}\left(\sum_{j=1}^{2^{k}-1} \frac{\left|f_{j}(x)\right|^{2}}{\lambda_{j}}\right) d x\right)= \\
=O\left(\frac{\omega\left(\frac{1}{2^{k}}, f\right)}{\sum_{j=1}^{2^{k}-1} \frac{1}{\lambda_{j}}}\right)\left(\int_{0}^{1}\left(\sum_{j=1}^{2^{k}-1} \frac{\left|f_{j}(x)\right|}{\lambda_{j}}\right) d x\right) .
\end{gathered}
$$

Let $q(\tau(n))$ be the index conjugate of $p(\tau(n))$ for each $n \in \mathbb{N}$, then by applying Hölder's inequality on the right side of the above inequality, we have

$$
S_{k}=O\left(\frac{\omega\left(\frac{1}{2^{k}}, f\right)}{\sum_{j=1}^{2^{k}-1} \frac{1}{\lambda_{j}}}\right)\left(\int_{0}^{1}\left(\sum_{j=1}^{2^{k}-1} \frac{\left|f_{j}(x)\right|^{p\left(\tau\left(2^{k}\right)\right)}}{\lambda_{j}}\right)^{1 / p\left(\tau\left(2^{k}\right)\right)}\left(\sum_{j=1}^{2^{k}-1} \frac{1}{\lambda_{j}}\right)^{1 / q\left(\tau\left(2^{k}\right)\right)} d x\right)
$$

For any $x \in \mathbb{R}$, all these points $x+2 j h, x+(2 j-1) h$, for $j=1,2, \ldots, 2^{k}-1$ lie in the interval of length 1 . Thus, $f \in \Lambda B V(p(n), \varphi)$ over [ 0,1 ] implies $\left(\sum_{j=1}^{2^{k}-1} \frac{\left|f_{j}(x)\right| p^{\left(\tau\left(2^{k}\right)\right)}}{\lambda_{j}}\right)^{1 / p\left(\tau\left(2^{k}\right)\right)}=O(1)$. This together with $\sum_{j=1}^{2^{k}} \frac{1}{\lambda_{j}} \approx \sum_{j=1}^{2^{k}-1} \frac{1}{\lambda_{j}}$ and the above inequality implies

$$
S_{k}=O\left(\frac{\omega\left(\frac{1}{2^{k}}, f\right)}{\left(\sum_{j=1}^{2^{k}} \frac{1}{\lambda_{j}}\right)^{1 / p\left(\tau\left(2^{k}\right)\right)}}\right) .
$$

Suppose that $\sum^{*}$ indicates the summation over $2^{k} \leq n<2^{k+1}$. From Jensen's inequality, for the concave function $f(t)=t^{\beta / 2},(0<\beta \leq 2)$, we have

$$
\begin{gathered}
\sum^{*} n^{\delta}|\hat{f}(n)|^{\beta}=\sum^{*}\left(n^{2 \delta / \beta}\left(|\hat{f}(n)|^{2}\right)\right)^{\beta / 2} \leq \\
\leq 2^{k}\left(\sum^{*} 2^{-k} n^{2 \delta / \beta}|\hat{f}(n)|^{2}\right)^{\beta / 2}= \\
=O\left(2^{k(1+\delta-\beta / 2)}\right)\left(\sum^{*}|\hat{f}(n)|^{2}\right)^{\beta / 2}= \\
=O\left(2^{k(1+\delta-\beta / 2)}\right)\left(\frac{\omega\left(\frac{1}{2^{k}}, f\right)}{\left(\sum_{j=1}^{2^{k}} \frac{1}{\lambda_{j}}\right)^{1 / p\left(\tau\left(2^{k}\right)\right)}}\right)^{\beta / 2} .
\end{gathered}
$$

This proves the theorem.
Proof of Theorem 2.2. $f$ is of bounded variation over $[0,1]$ implies $f$ is bounded. Proceeding as in the proof of Theorem 2.1, we get (3.2). The Hölder's inequality gives

$$
\begin{gathered}
\left\|f_{j}\right\|_{2}^{2}=\left\|\left|f_{j}\right|^{2}\right\|_{1}=\left\|\left|f_{j}\right|^{\mid /(p-1)} \cdot\left|f_{j}\right|^{1 / q}\right\|_{1} \leq\left\|\left|f_{j}\right|^{p}\right\|_{1}^{1 /(p-1)}\left\|f_{j}\right\|_{1}^{1 / q}= \\
=\left\|f_{j}\right\|_{p}^{p /(p-1)}\left\|f_{j}\right\|_{1}^{1 / q},
\end{gathered}
$$

where $q$ is the index conjugate of $p-1$. This together with (3.2) implies

$$
\left(S_{k}\right)^{q}:=\left(\sum_{n \in D_{k}}|\hat{f}(n)|^{2}\right)^{q}=O(1)\left\|f_{j}\right\|_{p}^{p q /(p-1)}\left\|f_{j}\right\|_{1} .
$$

Summing both the sides of the above equation over $j=1$ to $2^{k}-1$, we have

$$
\begin{equation*}
\left(S_{k}\right)^{q}:=O\left(2^{-k}\right) \sum_{j=1}^{2^{k}-1}\left\|f_{j}\right\|_{p}^{p q /(p-1)}\left\|f_{j}\right\|_{1} \tag{3.3}
\end{equation*}
$$

Observe that $\left\|f_{j}\right\|_{p}=O\left(\omega^{(p)}\left(\frac{1}{2^{k}}, f\right)=O\left(\left(\frac{1}{2^{k}}\right)^{\alpha}\right)\right.$, for any $j$, as $f \in \operatorname{Lip}(\alpha, p)$ and

$$
\sum_{j}\left\|f_{j}\right\|_{1}=\left\|\sum_{j}\left|f_{j}\right|\right\|_{1}=O(1)
$$

as $f \in B V([0,1])$. Hence (3.3) implies

$$
S_{k}=O\left(2^{-k\left(1+\frac{\alpha p-1}{p-1}\right)}\right) .
$$

From Jensen's inequality, we have

$$
\begin{gathered}
\sum_{n \in D_{k}}|\hat{f}(n)|^{\beta}=\sum_{n \in D_{k}}\left(|\hat{f}(n)|^{2}\right)^{\beta / 2} \leq \\
\leq 2^{k}\left(2^{-k} \sum_{n \in D_{k}}|\hat{f}(n)|^{2}\right)^{\beta / 2}= \\
=2^{k\left(1-\frac{\beta}{2}\right)}\left(S_{k}\right)^{\beta / 2}=2^{k\left(1-\frac{\beta(2 p+\alpha p-3)}{2(p-1)}\right)} .
\end{gathered}
$$

Hence, the result follows.

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# A NOTE ON SEPARATION PROPERTIES OF $\theta$-MODIFICATIONS OF TOPOLOGIES 

By<br>GUANG-FA HAN, GUI-RONG LI, AND PI-YU LI

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#### Abstract

In this note, we construct a Urysohn topology such that its $\theta$ modification $\theta(\tau)$ is not $T_{2}$, which answers a question in [1] (Problem 2.10 in [1]).


## 1. Introduction

By a space, we mean a topological space. For a subset $A$ in a space $(X, \tau)$, we denote by $\operatorname{int}(A)$ and $\operatorname{cl}(A)$ for the interior and the closure of $A$, respectively. A point $x$ of a space $X$ is called a $\theta$-cluster point [1] (also called $\theta$-adherent point in [5]) of a subset $A \subseteq X$ iff $\operatorname{cl}(U) \cap A \neq \emptyset$ whenever $U$ is an open neighbourhood of $x$. Let $\gamma_{\theta}(A)$ denote the set of all $\theta$-cluster points of $A$; $A$ is is called $\theta$-closed iff $A=\gamma_{\theta} A$. A subset $U$ is said to be $\theta$-open if its complement is $\theta$-closed. Clearly, a subset $U$ of $X$ is $\theta$-open iff for each point $x \in U$ there exists an open set $V$ containing $x$ such that $V \subseteq \operatorname{cl}(V) \subseteq U$.

The collection of all $\theta$-open sets forms a topology $\theta(\tau)$ on $X$. This topology is coarser than $\tau$ and called the $\theta$-modification [1] of the topology $\tau$. A topology $\tau$ is said to be Urysohn [1] iff $x, y \in X$ imply the existence of open sets $V$ and $W$ such that $x \in V, y \in W$ and $\operatorname{cl}(V) \cap \operatorname{cl}(W)=\emptyset$.

In [1], Á. Császár examined the relation of separation properties of $\tau$ and its modification $\theta(\tau)$. It has been proved that if $\theta(\tau)$ is $T_{2}$ then $\tau$ is Uryshon (see Theorem 2.6 in [1]). It is an open question whether the converse is true:

Problem 1.1. [1, Problem 2.10] Look for a Uryshon topology such that $\theta(\tau)$ is not $T_{2}$.

Throughout this paper, the set of all positive natural numbers is denoted by $\mathbb{N}^{+}$; the set of all real numbers is denoted by $R$; let $R^{2}$ be the set $R \times R$.

## 2. A Uryshon topology whose $\theta$-modification is not $T_{2}$

Example 2.1. There exists a Uryshon topology such that $\theta(\tau)$ is not $T_{2}$.
Proof. Let $X=R^{2}, p=\langle 0,-1\rangle \in X$.
We consider on $X$ a topology $\tau$ defined by neighborhood systems.
Namely,
When $y \neq 0$ and $y \neq-1$, the one point set $\{\langle x, y\rangle\}$ is open.
When $y=0$, there are two cases:
Case 1. The neighbourhood filter of the point $z=\langle 0,0\rangle$ is generated by the sets $U_{n}(z)$, where $n \in \mathbb{N}^{+}$and $U_{n}(z)=\left\{\langle x, y\rangle: x^{2}+y^{2}<\frac{1}{n}, y>0\right\} \cup\{\langle 0,0\rangle\} ;$

Case 2. For each point $z=\langle x, 0\rangle, x \neq 0$, we denote by $A(z)$ the set of all points $\langle x-y, y\rangle \in X$, where $-1<y \leq 0$. The neighbourhood filter of $z$ is generated by the sets $U_{n, F}(z)$, where $n \in \mathbb{N}^{+}, F$ is a finite subset of $X$ such that $z \notin F$ and $U_{n, F}(z)=\left(\left\{\langle x, y\rangle: x^{2}+y^{2}<\frac{1}{n}, y>0\right\} \cup A(z)\right) \backslash F$.

When $y=-1$, there are two cases:
Case 1. We denote by $L(x)$ the set of all points $\langle x, y\rangle \in X$, where $-1 \leq y<$ $<0$. For each point $z=\langle x,-1\rangle, x \neq 0$, the neighbourhood filter of $z$ is generated by the sets $U_{n, F}(z)$, where $n \in \mathbb{N}^{+}, F$ is a finite subset of $X$ such that $z \notin F$ and $U_{n, F}(z)=\left(\left\{\langle x, y\rangle: x^{2}+y^{2}<\frac{1}{n}, y<-1\right\} \cup L(x)\right) \backslash F$.

Case 2. The neighbourhood filter of the point $p=\langle 0,-1\rangle$ is generated by the sets $\{p\} \cup T_{n}$, where $n \in \mathbb{N}^{+}$and $T_{n}=\left(0, \frac{1}{n}\right) \times(-\infty,-1)$.

One can check that the topology $\tau$ on $X$ defined in this way is a Urysohn topology.

Next, we show that the point $\langle 0,0\rangle$ and the point $p$ can not be separated by disjoint $\theta$-open sets.

Suppose $U_{1}$ is an arbitrary $\theta$-open set which contains $\langle 0,0\rangle ; U_{2}$ is an arbitrary $\theta$-open set which contains $p$.

By the definition of the neighbourhood filter of $\langle 0,0\rangle$, we know that there that must be an open neighbourhood $V$ of $\langle 0,0\rangle$ satisfies $c l(V) \subseteq U_{1}$, where $V=\left\{\langle x, y\rangle: x^{2}+y^{2}<\frac{1}{n_{1}}, y>0\right\} \cup\{\langle 0,0\rangle\}$ for some $n_{1} \in \mathbb{N}^{+}$. Clearly, when $0<x<\frac{1}{n_{1}}$, we have $\langle x, 0\rangle \in \operatorname{cl}(V) \subseteq U_{1}$.

By the definition of the neighbourhood filter of $p$, we know that there must be an open neighbourhood $W$ of $p$ satisfies $c l(W) \subseteq U_{2}$, where $W=\left(0, \frac{1}{n_{2}}\right) \times$ $\times(-\infty,-1)$ for some $n_{2} \in \mathbb{N}^{+}$. Clearly, when $0<x<\frac{1}{n_{2}}$, we have $\langle x,-1\rangle \in$ $\in \operatorname{cl}(W) \subseteq U_{2}$.

Let $n=\max \left\{n_{1}, n_{2}\right\}$.
Then for each $0<x<\frac{1}{n}$, we have $\langle x, 0\rangle \in U_{1}$ and $\langle x,-1\rangle \in U_{2}$.
Let $z_{x}=\langle x, 0\rangle$. Then for each $0<x<\frac{1}{n}, a \in\left(x, \frac{1}{n}\right)$, we have $A\left(z_{x}\right) \cap$ $\cap L(a) \neq \emptyset$.

Let $\left\{q_{x_{i}}: q_{x_{i}}=\left\langle x_{i},-1\right\rangle, x_{i} \in\left(\frac{1}{2 n}, \frac{1}{n}\right), i \in \mathbb{N}^{+}\right\}$be an arbitrary countable infinite set. Clearly, for each $i \in \mathbb{N}^{+}, q_{x_{i}} \in U_{2}$ and there exists an open neighourhood $M_{q_{x_{i}}}$ of $q_{x_{i}}$ such that $c l\left(M_{q_{x_{i}}}\right) \subseteq U_{2}$. By the definition of the neighbourhood filter of $q_{x_{i}}$, we know that we can and only can remove a finite subset $F_{i}$ from $L\left(x_{i}\right)$ such that $\left(L\left(x_{i}\right) \backslash F_{i}\right) \subseteq M_{q_{x_{i}}}$. Clearly, $\cup_{i \in \mathbb{N}^{+}} F_{i}$ is a countable set.

For each point $r \in \cup_{i \in \mathbb{N}^{+}} F_{i}$, we denote by $z_{r}$ the intersection of the $x$-axis and the line passing through $r$ with slope -1 . Then $C=\left\{z_{r}\right\}_{r \in \cup_{i \in \mathbb{N}^{+}} F_{i}}$ is a countable set. So, $C \cap\left(0, \frac{1}{2 n}\right)$ is a countable set. It follows that there exists a point $z_{0} \in\left(0, \frac{1}{2 n}\right) \backslash\left(C \cap\left(0, \frac{1}{2 n}\right)\right)$.

Notice that for any $z_{1}=\left\langle x_{1}, 0\right\rangle, z_{2}=\left\langle x_{2}, 0\right\rangle$, if $A\left(z_{1}\right) \cap L(a)=A\left(z_{2}\right) \cap$ $\cap L(a) \neq \emptyset$, we have $z_{1}=z_{2}$. Thus, we can conclude that $A\left(z_{0}\right) \cap\left(L\left(x_{i}\right) \backslash F_{i}\right) \neq \emptyset$ for each $i \in \mathbb{N}^{+}$. Along with $\left(L\left(x_{i}\right) \backslash F_{i}\right) \subseteq M_{q_{x_{i}}} \subseteq U_{2}$, we can conclude that $U_{n, F}\left(z_{0}\right) \cap U_{2} \neq \emptyset$, for any $n$ and $F$, where $n \in \mathbb{N}^{+}, F$ is a finite subset of $X$ such that $z \notin F$. So, $U_{1} \cap U_{2} \neq \emptyset$.

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# BETWEEN $\delta$-CLOSED SETS AND $\delta-g$-CLOSED SETS 

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#### Abstract

Quite recently, by using $b$-open sets [2], Nagaveni and Narmadha [11] have introduced and investigated the notion of $r b$-closed sets in a topological space. These subsets place between $\delta$-closed sets and $\delta$ - $g$-closed sets due to Dontchev and Ganster [3]. In this paper, we introduce the notion of $m \delta g$-closed sets and obtain the unified theory for certain collections of subsets between $\delta$-closed sets and $\delta$ - $g$-closed sets.


## 1. Introduction

In 1970, Levine [7] introduced the notion of generalized closed ( $g$-closed) sets in topological spaces. Since then, many variations of $g$-closed sets are introduced and investigated. Dontchev and Ganster [3] introduced the notions of $\delta$ - $g$-closed sets and $T_{3 / 4}$-spaces. They showed that the digital line $(Z, \kappa)$ [5] is a $T_{3 / 4}$-space but it is not $T_{1}$. Quite recently, Nagaveni and Narmadha [11] have introduced the notion of $r b$-closed sets by using $b$-open sets and studied their basic properties and characterizations.

In this paper, we introduce the notion of $m \delta g$-closed sets in order to establish the unified theory for certain collections of subsets between $\delta$-closed sets and $\delta$ -$g$-closed sets. And we obtain the basic properties and characterizations of $m \delta g$ closed sets. In the last section, we define several new subsets which lie between $\delta$-closed sets and $\delta$ - $g$-closed sets.

## 2. Preliminaries

Let $(X, \tau)$ be a topological space and $A$ a subset of $X$. The closure of $A$ and the interior of $A$ are denoted by $\mathrm{Cl}(A)$ and $\operatorname{Int}(A)$, respectively. A subset $A$ is said to be regular closed (resp. regular open) if $\mathrm{Cl}(\operatorname{Int}(A))=A$ (resp. $\operatorname{Int}(\mathrm{Cl}(A))=A)$. A point $x \in X$ is called a $\delta$-cluster point of $A$ if $\operatorname{Int}(\mathrm{Cl}(V)) \cap$ $\cap A \neq \emptyset$ for every open set $V$ containing $x$. The set of all $\delta$-cluster points of $A$ is called the $\delta$-closure of $A$ and is denoted by $\mathrm{Cl}_{\delta}(A)$ [16]. The complement of a $\delta$-closed set is said to be $\delta$-open. The $\delta$-interior of $A$ is defined by the union of all regular open sets contained in $A$ and is denoted by $\operatorname{Int}_{\delta}(A)$.

Definition 2.1. A subset $A$ of a topological space $(X, \tau)$ is said to be semi-open [6] (resp. preopen [9], $\alpha$-open [10], $\beta$-open [1], $b$-open [2]) if $A \subset \mathrm{Cl}(\operatorname{Int}(A))$ (resp. $A \subset \operatorname{Int}(\mathrm{Cl}(A)), A \subset \operatorname{Int}(\mathrm{Cl}(\operatorname{Int}(A))), A \subset \mathrm{Cl}(\operatorname{Int}(\mathrm{Cl}(A)))), A \subset$ $\subset \operatorname{Int}(\mathrm{Cl}(A)) \cup \mathrm{Cl}(\operatorname{Int}(A)))$.

The family of all semi-open (resp. preopen, $\alpha$-open, $\beta$-open, $b$-open) sets in $X$ is denoted by $\mathrm{SO}(X)($ resp. $\mathrm{PO}(X), \alpha(X), \beta(X), \mathrm{BO}(X))$.

Remark 2.1. For the above generalizations of open sets, the following relations are well-known:

## 3. $m$-Structures

Definition 3.1. A subfamily $m_{X}$ of the power set $\mathcal{P}(X)$ of a nonempty set $X$ is called a minimal structure (briefly $m$-structure) [13] on $X$ if $\emptyset \in m_{X}$ and $X \in m_{X}$.

By $\left(X, m_{X}\right)$, we denote a nonempty set $X$ with a minimal structure $m_{X}$ on $X$ and call it an $m$-space. Each member of $m_{X}$ is said to be $m_{X}$-open (or briefly $m$ open) and the complement of an $m_{X}$-open set is said to be $m_{X}$-closed (or briefly $m$-closed).

Remark 3.1. Let $(X, \tau)$ be a topological space. Then the families $\mathrm{SO}(X)$, $\mathrm{PO}(X), \alpha(X), \mathrm{BO}(X)$ and $\beta(X)$ are all $m$-structures on $X$.

Definition 3.2. Let $\left(X, m_{X}\right)$ be an $m$-space. For a subset $A$ of $X$, the $m_{X}$-closure of $A$ and the $m_{X}$-interior of $A$ are defined in [8] as follows:
(1) $m_{X} \mathrm{Cl}(A)=\cap\left\{F: A \subset F, X-F \in m_{X}\right\}$,
(2) $m_{X} \operatorname{Int}(A)=\cup\left\{U: U \subset A, U \in m_{X}\right\}$.

Remark 3.2. Let $(X, \tau)$ be a topological space and $A$ a subset of $X$. If $m_{X}=\tau$ (resp. $\mathrm{SO}(X), \mathrm{PO}(X), \mathrm{BO}(X), \alpha(X), \beta(X)$ ), then we have
(1) $m_{X} \mathrm{Cl}(A)=\mathrm{Cl}(A)($ resp. $\mathrm{sCl}(A), \mathrm{pCl}(A), \mathrm{bCl}(A), \alpha \mathrm{Cl}(A), \beta \mathrm{Cl}(A))$,
(2) $m_{X} \operatorname{Int}(A)=\operatorname{Int}(A)($ resp. $\operatorname{sInt}(A), \operatorname{pInt}(A), b \operatorname{Int}(A), \alpha \operatorname{Int}(A)$ and $\beta \operatorname{Int}(A))$.

Lemma 3.1 (Popa and Noiri [13]). Let $\left(X, m_{X}\right)$ be an $m$-space and $A$ a subset of $X$. Then $x \in m_{X} \operatorname{Cl}(A)$ if and only if $U \cap A \neq \emptyset$ for every $U \in m_{X}$ containing $x$.

Definition 3.3. A minimal structure $m_{X}$ on a nonempty set $X$ is said to have property $\mathcal{B}$ [8] if the union of any family of subsets belonging to $m_{X}$ belongs to $m_{X}$.

Remark 3.3. Let $(X, \tau)$ be a topological space. Then the families $\mathrm{SO}(X)$, $\mathrm{PO}(X), \mathrm{BO}(X), \alpha(X)$ and $\beta(X)$ are all $m$-structures with property $\mathcal{B}$.

Lemma 3.2 (Popa and Noiri [14]). Let $X$ be a nonempty set and $m_{X}$ a minimal structure on $X$ satisfying property $\mathcal{B}$. For a subset $A$ of $X$, the following properties hold:
(1) $A \in m_{X}$ if and only if $m_{X} \operatorname{Int}(A)=A$,
(2) $A$ is $m_{X}$-closed if and only if $m_{X} \mathrm{Cl}(A)=A$,
(3) $m_{X} \operatorname{Int}(A) \in m_{X}$ and $m_{X} \mathrm{Cl}(A)$ is $m_{X}$-closed.

## 4. $m \delta g$-Closed sets

Definition 4.1. Let $(X, \tau)$ be a topological space and $m_{X}$ an $m$-structure on $X$. A subset $A$ is said to be $m \delta g$-closed if $\mathrm{Cl}_{\delta}(A) \subset U$ whenever $A \subset U$ and $U \in m_{X}$.

Remark 4.1. Let $(X, \tau)$ be a topological space and $A$ a subset of $X$.
(1) If $m_{X}=\tau($ resp. $\mathrm{BO}(X))$ and $A$ is $m \delta g$-closed, then $A$ is $\delta$ - $g$-closed [3] (resp. $r b$-closed [11]).
(2) In Definition 4.1, by setting $m_{X}=\mathrm{SO}(X)$ (resp. $\mathrm{PO}(X), \alpha(X), \beta(X)$ ), we can define a subset called $s \delta g$-closed (resp. $p \delta g$-closed, $\alpha \delta g$-closed, $\beta \delta g$-closed).
(3) By DIAGRAM I, we obtain the following diagram:

## DIAGRAM II

$\delta$ - $g$-closed $\Leftarrow \alpha \delta g$-closed $\Leftarrow p \delta g$-closed

$$
\stackrel{\Uparrow}{s \delta g \text {-closed }} \stackrel{\Uparrow}{\Uparrow} \stackrel{r b}{ } \text {-closed } \Leftarrow \beta \delta g \text {-closed } \Leftarrow \delta \text {-closed }
$$

In this section, let $(X, \tau)$ be a topological space and $m_{X}$ an $m_{X}$-structure on $X$. We obtain several basic properties of $m \delta g$-closed sets.

Proposition 4.1. Let $\tau \subset m_{X}$. Then the following implications hold:

$$
\delta \text {-closed } \Rightarrow m \delta g \text {-closed } \Rightarrow \delta \text {-g-closed }
$$

Proof. It is obvious that every $\delta$-closed set is $m \delta g$-closed. Suppose that $A$ is an $m \delta g$-closed set. Let $A \subset U$ and $U \in \tau$. Since $\tau \subset m_{X}, \mathrm{Cl}_{\delta}(A) \subset U$ and hence $A$ is $\delta$ - $g$-closed.

Proposition 4.2. If $A$ and $B$ are $m \delta g$-closed, then $A \cup B$ is $m \delta g$-closed.
Proof. Let $A \cup B \subset U$ and $U \in m_{X}$. Then $A \subset U$ and $B \subset U$. Since $A$ and $B$ are $m \delta g$-closed, we have $\mathrm{Cl}_{\delta}(A \cup B)=\mathrm{Cl}_{\delta}(A) \cup \mathrm{Cl}_{\delta}(B) \subset U$. Therefore, $A \cup B$ is $m \delta g$-closed.

Proposition 4.3. Let $\tau \subset m_{X}$ and $m_{X}$ have property $\mathcal{B}$. If $A$ is $m \delta g$-closed and $F$ is $\delta$-closed, then $A \cap F$ is $m \delta g$-closed.

Proof. Let $A \cap F \subset U$ and $U \in m_{X}$. Then $A \subset U \cup(X-F)$. Since $\tau \subset m_{X}$ and $m_{X}$ has property $\mathcal{B}, U \cup(X-F) \in m_{X}$ and hence $\mathrm{Cl}_{\delta}(A) \subset U \cup(X-F)$. Therefore, $\mathrm{Cl}_{\delta}(A \cap F) \subset \mathrm{Cl}_{\delta}(A) \cap \mathrm{Cl}_{\delta}(F)=\mathrm{Cl}_{\delta}(A) \cap F \subset[U \cup(X-F)] \cap F=U \cap F \subset U$. This shows that $A \cap F$ is $m \delta g$-closed.

Remark 4.2. It is shown in Theorem 3.11 of [3] that finite intersection of $\delta$ - $g$ closed sets may fail to be a $\delta$ - $g$-closed set. Therefore, in general, the intersection of two $m \delta g$-closed sets is not always $m \delta g$-closed.

Proposition 4.4. If $A$ is $m \delta$-closed and $m$-open, then $A$ is $\delta$-closed.
Proof. This is obvious.
Proposition 4.5. If $A$ is $m \delta g$-closed and $A \subset B \subset \mathrm{Cl}_{\delta}(A)$, then $B$ is $m \delta g$-closed.
Proof. Let $B \subset U$ and $U \in m_{X}$. Then $A \subset U$ and $A$ is $m \delta g$-closed. Hence $\mathrm{Cl}_{\delta}(B)=\mathrm{Cl}_{\delta}(A) \subset U$ and $B$ is $m \delta g$-closed.

Definition 4.2. Let $(X, \tau)$ be a topological space and $m_{X}$ an $m$-structure on $X$. A subset $A$ is said to be $m \delta g$-open if $X-A$ is $m \delta g$-closed.

Proposition 4.6. $A$ subset $A$ of $X$ is $m \delta g$-open if and only if $F \subset \operatorname{Int}_{\delta}(A)$ whenever $F \subset A$ and $F$ is $m$-closed.

Proof. Necessity. Suppose that $A$ is $m \delta g$-open. Let $F \subset A$ and $F$ be $m$-closed. Then $X-A \subset X-F \in m_{X}$ and $X-A$ is $m \delta g$-closed. Therefore, we have $X-\operatorname{Int}_{\delta}(A)=\mathrm{Cl}_{\delta}(X-A) \subset X-F$ and hence $F \subset \operatorname{Int}_{\delta}(A)$.

Sufficiency. Let $X-A \subset G$ and $G \in m_{X}$. Then $X-G \subset A$ and $X-G$ is $m$-closed. By hypothesis, we have $X-G \subset \operatorname{Int}_{\delta}(A)$ and hence $\mathrm{Cl}_{\delta}(X-A)=$ $=X-\operatorname{Int}_{\delta}(A) \subset G$. Therefore, $X-A$ is $m \delta g$-closed and $A$ is $m \delta g$-open.

Corollary 4.1. Let $\tau \subset m_{X}$. Then the following properties hold:
(1) Every $\delta$-open set is $m \delta g$-open and every $m \delta g$-open set is $\delta$ - $g$-open,
(2) If $A$ and $B$ are $m \delta g$-open, then $A \cap B$ is $m \delta g$-open,
(3) If $A$ is $m \delta g$-open and $F$ is $\delta$-open, then $A \cup F$ is $m \delta g$-open.
(4) If $A$ is $m \delta g$-open and $m$-closed, then $A$ is $\delta$-open,
(5) If $A$ is $m \delta g$-open and $\operatorname{Int}_{\delta}(A) \subset B \subset A$, then $B$ is $m \delta g$-open.

Proof. This follows immediately from Propositions 4.1-4.5.

## 5. Characterizations of $m \delta g$-closed sets

In this section, let $(X, \tau)$ be a topological space and $m_{X}$ an $m$-structure on $X$. We obtain some characterizations of $m \delta g$-closed sets.

Theorem 5.1. A subset $A$ of $X$ is $m \delta g$-closed if and only if $\mathrm{Cl}_{\delta}(A) \cap F=\emptyset$ whenever $A \cap F=\emptyset$ and $F$ is $m$-closed.

Proof. Necessity. Suppose that $A$ is $m \delta g$-closed. Let $A \cap F=\emptyset$ and $F$ be $m$ closed. Then $A \subset X-F \in m_{X}$ and $\mathrm{Cl}_{\delta}(A) \subset X-F$. Therefore, we have $\mathrm{Cl}_{\delta}(A) \cap F=\emptyset$.

Sufficiency. Let $A \subset U$ and $U \in m_{X}$. Then $A \cap(X-U)=\emptyset$ and $X-U$ is $m$-closed. By hypothesis, $\mathrm{Cl}_{\delta}(A) \cap(X-U)=\emptyset$ and hence $\mathrm{Cl}_{\delta}(A) \subset U$. Therefore, $A$ is $m \delta g$-closed.

Theorem 5.2. Let $\tau \subset m_{X}$ and $m_{X}$ have property $\mathcal{B}$. A subset $A$ of $X$ is $m \delta g-$ closed if and only if $\mathrm{Cl}_{\delta}(A)-A$ does not contain any nonempty $m$-closed set.

Proof. Necessity. Suppose that $A$ is $m \delta g$-closed. Let $F \subset \mathrm{Cl}_{\delta}(A)-A$ and $F$ be $m$-closed. Then $A \subset X-F \in m_{X}$ and hence $\mathrm{Cl}_{\delta}(A) \subset X-F$. Therefore, we have $F \subset X-\mathrm{Cl}_{\delta}(A)$. On the other hand, $F \subset \mathrm{Cl}_{\delta}(A)$ and $F \subset \mathrm{Cl}_{\delta}(A) \cap$ $\cap\left(X-\mathrm{Cl}_{\delta}(A)\right)=\emptyset$.

Sufficiency. Suppose that $A$ is not $m \delta g$-closed. Then $\emptyset \neq \mathrm{Cl}_{\delta}(A)-U$ for some $U \in m_{X}$ containing $A$. Since $\tau \subset m_{X}$ and $m_{X}$ has property $\mathcal{B}, \mathrm{Cl}_{\delta}(A)-U$ is $m$-closed. Moreover, we have $\mathrm{Cl}_{\delta}(A)-U \subset \mathrm{Cl}_{\delta}(A)-A$.

Theorem 5.3. Let $\tau \subset m_{X}$ and $m_{X}$ have property $\mathcal{B}$. A subset $A$ of $X$ is $m \delta g$ closed if and only if $\mathrm{Cl}_{\delta}(A)-A$ is $m \delta g$-open.

Proof. Necessity. Suppose that $A$ is $m \delta g$-closed. Let $F \subset \mathrm{Cl}_{\delta}(A)-A$ and $F$ be $m$-closed. By Theorem 5.2, we have $F=\emptyset$ and $F \subset \operatorname{Int}_{\delta}\left(\mathrm{Cl}_{\delta}(A)-A\right)$. It follows from Proposition 4.6 that $\mathrm{Cl}_{\delta}(A)-A$ is $m \delta g$-open.

Sufficiency. Let $A \subset U$ and $U \in m_{X}$. Then $\mathrm{Cl}_{\delta}(A) \cap(X-U) \subset \mathrm{Cl}_{\delta}(A)-A$ and $\mathrm{Cl}_{\delta}(A)-A$ is $m \delta g$-open. Since $\tau \subset m_{X}$ and $m_{X}$ has property $\mathcal{B}, \mathrm{Cl}_{\delta}(A) \cap$ $\cap(X-U)$ is $m$-closed and by Proposition 4.6 we have $\mathrm{Cl}_{\delta}(A) \cap(X-U) \subset$ $\subset \operatorname{Int}_{\delta}\left(\mathrm{Cl}_{\delta}(A)-A\right)$. Now, $\operatorname{Int}_{\delta}\left(\mathrm{Cl}_{\delta}(A)-A\right) \subset \mathrm{Cl}_{\delta}(A) \cap \operatorname{Int}_{\delta}(X-A)=\mathrm{Cl}_{\delta}(A) \cap$ $\cap\left(X-\mathrm{Cl}_{\delta}(A)\right)=\emptyset$. Therefore, we have $\mathrm{Cl}_{\delta}(A) \cap(X-U)=\emptyset$ and hence $\mathrm{Cl}_{\delta}(A) \subset U$. This shows that $A$ is $m \delta g$-closed.

Theorem 5.4. Let $m_{X}$ have property $\mathcal{B}$. A subset $A$ of $X$ is $m \delta g$-closed if and only if $m_{X} \mathrm{Cl}(\{x\}) \cap A \neq \emptyset$ for each $x \in \mathrm{Cl}_{\delta}(A)$.

Proof. Necessity. Suppose that $A$ is $m \delta g$-closed and $m_{X} \mathrm{Cl}(\{x\}) \cap A=\emptyset$ for some $x \in \mathrm{Cl}_{\delta}(A)$. By Lemma 3.2, $m_{X} \mathrm{Cl}(\{x\})$ is $m$-closed and $A \subset$ $\subset X-m_{X} \mathrm{Cl}(\{x\}) \in m_{X}$. Since $A$ is $m \delta g$-closed, $\mathrm{Cl}_{\delta}(A) \subset X-m_{X} \mathrm{Cl}(\{x\}) \subset$ $\subset X-\{x\}$. This contradicts that $x \in \mathrm{Cl}_{\delta}(A)$.

Sufficiency. Suppose that $A$ is not $m \delta g$-closed. Then $\emptyset \neq \mathrm{Cl}_{\delta}(A)-U$ for some $U \in m_{X}$ containing $A$. There exists $x \in \mathrm{Cl}_{\delta}(A)-U$. Since $x \notin U$, by Lemma $3.1 m_{X} \operatorname{Cl}(\{x\}) \cap U=\emptyset$ and hence $m_{X} \operatorname{Cl}(\{x\}) \cap A \subset m_{X} \operatorname{Cl}(\{x\}) \cap$ $\cap U=\emptyset$. This shows that $m_{X} \operatorname{Cl}(\{x\}) \cap A=\emptyset$ for some $x \in \mathrm{Cl}_{\delta}(A)$.

Corollary 5.1. Let $\tau \subset m_{X}$ and $m_{X}$ have property $\mathcal{B}$. For a subset $A$ of $X$, the following properties are equivalent:
(1) $A$ is $m \delta g$-open;
(2) $A-\operatorname{Int}_{\delta}(A)$ does not contain any nonempty $m$-closed set;
(3) $A-\operatorname{Int}_{\delta}(A)$ is $m \delta$-open;
(4) $m_{X}-\mathrm{Cl}(\{x\}) \cap(X-A) \neq \emptyset$ for each $x \in X-\operatorname{Int}_{\delta}(A)$.

Proof. This follows from Proposition 4.6 and Theorems 5.2, 5.3 and 5.4.

## 6. New forms of $m \delta g$-closed sets

First, we recall the $\theta$-closure of a subset in a topological space. Let $(X, \tau)$ be a topological space and $A$ a subset of $X$. A point $x \in X$ is called a $\theta$-cluster point of $A$ if $\mathrm{Cl}(V) \cap A \neq \emptyset$ for every open set $V$ containing $x$. The set of all $\theta$-cluster points of $A$ is called the $\theta$-closure of $A$ and is denoted by $\mathrm{Cl}_{\theta}(A)$ [16].

Definition 6.1. A subset of a topological space $(X, \tau)$ is said to be
(1) $\delta$-preopen [15] (resp. $\theta$-preopen [12]) if $A \subset \operatorname{Int}\left(\mathrm{Cl}_{\delta}(A)\right)$ (resp. $A \subset$ $\left.\subset \operatorname{Int}\left(\mathrm{Cl}_{\theta}(A)\right)\right)$,
(2) $\delta$ - $\beta$-open $[4]$ (resp. $\theta$ - $\beta$-open [12]) if $A \subset \mathrm{Cl}\left(\operatorname{Int}\left(\operatorname{Cl}_{\delta}(A)\right)\right)($ resp. $A \subset$ $\left.\subset \operatorname{Cl}\left(\operatorname{Int}\left(\mathrm{Cl}_{\theta}(A)\right)\right)\right)$.

By $\delta \mathrm{PO}(X)$ (resp. $\delta \beta(X), \theta \mathrm{PO}(X), \theta \beta(X)$ ), we denote the collection of all $\delta$-preopen (resp. $\delta$ - $\beta$-open, $\theta$-preopen, $\theta$ - $\beta$-open) sets of a topological space $(X, \tau)$. These four collections are $m$-structures with property $\mathcal{B}$. In [12], the following diagram is known:

## DIAGRAM III



For subsets of a topological space $(X, \tau)$, we can define many new variations of $\delta g$-closed sets. For example, in case $m_{X}=\delta \mathrm{PO}(X), \delta \beta(X), \theta \mathrm{PO}(X), \theta \beta(X)$, we can define new types of $\delta g$-closed sets as follows:

Definition 6.2. A subset of a topological space $(X, \tau)$ is said to be $\delta p \delta g$-closed (resp. $\theta p \delta g$-closed, $\delta \beta \delta$ g-closed, $\theta \beta \delta$ g-closed) if $\mathrm{Cl}_{\delta}(A) \subset U$ whenever $A \subset U$ and $U$ is $\delta$-preopen (resp. $\theta$-preopen, $\delta$ - $\beta$-open, $\theta$ - $\beta$-open) in $(X, \tau)$.

By DIAGRAM III and Definitions 6.2, we have the following diagram:

## DIAGRAM IV

$\delta$-g-closed $\Leftarrow \alpha \delta g$-closed $\Leftarrow p \delta$-closed $\Leftarrow \delta p \delta g$-closed $\Leftarrow \theta p \delta g$-closed


Theorem 6.1. Let $(X, \tau)$ be a regular space. For a subset of $X$, the following properties hold:
(1) $p \delta g$-closedness, $\delta p \delta g$-closedness and $\theta p \delta g$-closedness are equivalent,
(2) $\beta \delta g$-closedness, $\delta \beta \delta g$-closedness and $\theta \beta \delta g$-closedness are equivalent.

Proof. In a regular space, $\mathrm{Cl}(A)=\mathrm{Cl}_{\delta}(A)=\mathrm{Cl}_{\theta}(A)$ for every subset $A$ of $X$ and hence the following properties hold:
(1) preopenness, $\delta$-preopenness and $\theta$-preopenness are equal,
(2) $\beta$-openness, $\delta-\beta$-openness and $\theta$ - $\beta$-openness are equal.

Therefore, the proofs are obvious.
Conclusions. We can apply the results estabished in Sections 4 and 5 for the following collections:
(1) The subsets defined in Remark 4.1(2) and Definition 6.2,
(2) Variations of $m \delta g$-closed sets defined by $m$-structures.

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# FREE WEAK LATTICES 

By<br>ERVIN FRIED<br>(Received November 26, 2012<br>Revised February 17, 2013)

## 1. Introduction

There are many generalizations of the varieties of lattices, as weakly associative lettices (see e.g. [1], [2] and weak lattices [3]). The graphs in [2] have the property, that every pair of distinct elements have unique common upper and lower bounds (UBP). Any algebra satisfying UBP is subdirectly irreducible. (For comparable elements this means that the graph does not contains three-element chains. If a weak lattice satisfies UBP, then it is a weakly associative lattice. However, there are other weak lattices containing no three-element chain; for example the free weak lattices have this property.) UBP were strongly connected to projective planes [4]. The simplest one, having more than two elements is the triangle: $a \rightarrow b \rightarrow c \rightarrow a$. These were real generalizations of the two-element lattice. It is still an open question, whether finite ones always contain a triangle. It was given an infinite one in [5] such that subdirectly irreducible members of the variety generated by this algebra having more then two elements contain no triangle. A generalization of these graphs are the Dual discriminator algebras, see e.g. in [6].

In [2] free weakly associative lattices were constructed. Here we shall construct free weak lattices.

## 2. Preliminaries

Weak lattices are the largest class of the generalizations of the lattices, which can be translated to the language of directed graphs. A weak lattice is an algebra $\mathfrak{A}=\{A \mid \wedge, \vee\}$, where the to binary operations $\wedge$ (called meet) and $\vee$ (called join) satisfy all the possible absorption laws:

$$
a \vee(a \wedge b)=a \vee(b \wedge a)=(a \wedge b) \vee a=(b \wedge a) \vee a=a ;
$$

and dually

$$
a \vee(a \wedge b)=(a \wedge b) \vee a=a \wedge(a \vee b)=(a \vee b) \wedge a
$$

Lemma 2.1. The following are equivalent in any weak lattice:

$$
a \vee b=b, \quad b \vee a=b, \quad a \wedge b=a, \quad a \wedge b=a .
$$

This condition will be denoted by $a \rightarrow b$ and $A^{\rightarrow}=\langle A \mid \rightarrow\rangle$ will be called the underlying graph of $\mathfrak{A}$.

Proof. Suppose, $a \vee b=b$. Then, using the absorption laws we have $a \wedge b=$ $=a \wedge(a \vee b)=a$ and $b \wedge a=(a \vee b) \wedge a=a$. Duality finishes the proof.

Observe, that both operations are idempotent. Indeed, we have $a=$ $=a \wedge(a \vee a)$, yielding $a \vee a=a \vee[a \wedge(a \vee a)]$. Now, for $b=(a \vee a)$ we get, $a \vee a=a \vee[a \wedge b]=a$. Similarly, $a \wedge a=a$.

Constructing the free weak lattices we shall follow the way used in [2]. To this end we shall investigate partial weak lattices. In some sense it is easier as in the case of weakly associateve lattices. However, we must work more carefully.

## 3. Partial weak lattices

Definition 3.1. A set $A$ with two partial binary operations $\vee$ and $\wedge$ will be called a partial weak lattice, if for some $a, b \in A$ there exist $a \vee b \in A$ or $a \wedge b \in$ $\in A$, satisfying the following conditions:

Whenever $a \vee b$ exists so do $a \wedge(a \vee b)$ and $(a \vee b) \wedge a$ and both equal to $a$. Similar conditions hold for $a \wedge b$.

Remark 3.1. We shall suppose that for each $a \in A$ both $a \vee a$ and $a \wedge a$ exist. However, this is not necessary when constructing the free weak lattice because it follows from the construction.

Observe, that the Lemma holds for partial weak lattices, as well, and we can define the underlying directed graph, too. Our definition implies, that we may imagine having a loop at every vertex, i.e. $a \rightarrow a$ holds for each $a \in A$.

Definition 3.2. Let $A$ be a partial weak lattice, and $a, b \in A$. These elements are comparable if either $a \rightarrow b$ or $b \rightarrow a$ hold. Otherwise these elements are incomparable.

Let $A$ be a partial weak lattice with the partial operations $\vee$ and $\wedge$.
Consider every incomparable pairs $a, b \in A$. Manufacture the triplets $(a, b, \vee)$ and $(a, b, \wedge)$. Extend $A$ by these elements and add the following new operations:
(1) $a \vee b=(a, b, \vee)$,
(2) $a \vee(a, b, \vee)=b \vee(a, b, \vee)=(a, b, \vee)$,
(3) $a \wedge(a, b, \vee)=(a, b, \vee) \wedge a=a$,
and their dual.
Observe, that - according to the definition $-(a, b, \vee)$ and $(b, a, \vee)$ are different, just as $(a, b, \wedge)$ and $(b, a, \wedge)$.

Extension Lemma. The new set $A^{\prime}=B$ with all the operations is a partial weak lattice, having $A$ as a sub--partial weak lattice. Every element $c \in B \backslash A$ has a unique representation, either as a meet or as a join of two elements in $A$. For elements $c, d \in B \backslash A$ neither $c \vee d$ nor $c \wedge d$ are defined.

Proof. Every $c \in B \backslash A$ is either of the form $(a, b, \vee)$ or $(a, b, \wedge)$, but not of the other. We have $c=a \vee b$ in the first case and $c=a \wedge b$ in the second. The element $c$ uniquely define the elements $a, b$ and their order. Let $c, d \in B$. If they are in $A$ and they are comparable, then the operations are defined for them, and the result is in $A$. If they are in $A$ and they are not comparable, then the operations are manufactured in the extension, then the results are in $B \backslash A$. If one of them is in $A$ and the other is in $B \backslash A$, then the operations are defined according to the extension. Finally, if none of them are in $A$, then none of the operations are defined for them.

## 4. Free weak lattices

Definition 4.1. Let $X$ be a subset of $A$, the underlying set of the weak lattice $\mathfrak{A}$. We say, that $X$ freely generates $\mathfrak{A}$, if every mapping $f: X \rightarrow B$ of the underlying set $B$ of any weak lattice $\mathfrak{B}$ uniquely extends to a homomorphism $\varphi: \mathfrak{A} \rightarrow \mathfrak{B}$.

Definition 4.2. Homomorhism of partial weak lattices and their extension. Let $A$ be a partial weak lattice and $\mathfrak{B}$ a weak lattice. We say, that $\varphi: A \rightarrow \mathfrak{B}$ is a homomorphism if it sends $A$ into the underlying set $B$ of $\mathfrak{B}$ and it preserves all the existing operations.

Номомоrhism Extension Lemma. Let $A$ be a weak lattice and $B$ its extension in the sense of the extension lemma. Then, for any weak lattice $\mathfrak{W}$, every homomorphism $\varphi: A \rightarrow \mathfrak{W}$ has a unique extension to a homomorphism $\psi: B \rightarrow \mathfrak{W}$.

Proof. Suppose we are given a homomorphism $\varphi: A \rightarrow \mathfrak{W}$. Then, any extension $\psi$ equals to $\varphi$ when restricted to $A$, i.e., for $a \in A$ we have $\psi(a)=\varphi(a)$. Let $c=(a, b, \vee) \in B \backslash A$. By the definition of the extension of homomorphisms, we must have

$$
\psi[(a, b, \vee)]=\varphi(a) \vee \varphi(b)=\psi(a) \vee \psi(b),
$$

and the meet is preserved, similarly. Hence $\psi: B \rightarrow \mathfrak{W}$ is a homomorphism, indeed.

Definition 4.3. Let $A$ be a partial weak lattice contained in the weak lattice $\mathfrak{A}$, generated by $A$. We say, that $A$ freely generates $\mathfrak{A}$ if for any weak lattice $\mathfrak{W}$ every homomorphism $\varphi: A \rightarrow \mathfrak{W}$ uniquely extends to a homomorphism $\psi: \mathfrak{A} \rightarrow \mathfrak{W}$.

Freeness Lemma. Let $A$ be a partial weak lattice contained in the weak lattice $\mathfrak{A}$, generated by $A$, and let $B=A^{\prime}$ be the partial weak lattice constructed in the extension lemma. Then we have a homomorphic embedding $B$ into $\mathfrak{A}$, such that the image of $B$ freely generates $\mathfrak{A}$.

Proof. Since the identical embedding sends $A$ into $\mathfrak{A}$ by the homomorphic extension lemma we have a homomorphic embedding $B$ into $\mathfrak{A}$. Therefore, we may identify $B$ with its image. Having a homomorphism $\varphi: B \rightarrow \mathfrak{A}$ the restriction uniquely extends to a homomorphism, i.e., $\varphi$ is unique.

Theorem 4.1. Let $A=A_{0}$ be a partial weak lattice. Define $A_{n+1}=A_{n}^{\prime}$ for the natural numbers $n$, and let $\mathfrak{A}=A^{*}$ their union. Then, $A$ freely generates $\mathfrak{A}$.

Proof. It follows, immediately from the freeness lemma.
Definition 4.4. For any positive integer $n$, weak lattice freely generated by the $n$-element set $X_{n}=\left\{x_{1}, \ldots, x_{n}\right\}$ is called the free $n$-generated weak lattice.

Observe, that the free weak lattice generated by one elements has no other elements.

Corollary 4.1. The two-generated free weak lattice contains a free weak lattice generated by a countable infinite set.

Proof. Let $\mathfrak{F}_{2}$ freely generated by $\{x, y\}$. The free extension contains their two joins $a=x \vee y, b=y \vee x$ and their two meets $c=x \wedge y, d=y \wedge x$. These four elements are incomparable. Due to the extension lemma the subalgebra freely generated by $\{a, b\}$ and the subalgebra freely generated by $\{c, d\}$ are disjoint. Continuing the procedure, when starting with $\left\{x_{0}, y_{0}\right\}$, we get $\left\{x_{1}, y_{1} u_{1}, v_{1}\right\}$ and after the $n^{\text {th }}$ step:

$$
\left\{x_{1}, y_{1} u_{1}, v_{1}, \ldots, x_{n}, y_{n} u_{n}, v_{n}\right\} .
$$

These elements are pairwise incomparable. Therefore, the weak lattice generated by the elements $u_{1}, u_{2}, \ldots, u_{n}, \ldots$ is freely generated by them.

## 5. An example

Observe, that the weak lattice uniquely determine the underlying graph, however the same underlying graph may belong to different weak lattices. Consider, e.g. the underlying graph of the five-element lattice consisting of $\{0=a \wedge b, a, b, u=a \vee b<1\}$. We can change the operations, defining $a<u$, $b<u, a \vee b=b \vee a=1$. Or we can keep all of these relations, except $b \vee a=1$ and declair $b<a$. This latter algebra is generated by $a, b$, so it is a homomorphic image of the two-generated free weak lattice. I do not know how it looks like.

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# GRAPH POLYNOMIALS AND GRAPH TRANSFORMATIONS IN ALGEBRAIC GRAPH THEORY Abstract of Ph.D. Thesis 

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## 1. Introduction

The algebraic graph theory has a long history due to its intimate relationship with chemistry and (statistical) physics. In these fields one often describes a system, state or a molecule by an appropriate parameter. Then it gives rise to the purely mathematical problem to determine what the extremal values of this parameter are. In the dissertation we give some general methods to attack these kinds of extremal problems. In the first, bigger half of the thesis we study two graph transformations, the so-called Kelmans transformation, and the generalized tree shift introduced by the author. The Kelmans transformation can be applied to all graphs, while one can apply the generalized tree shift only to trees. The importance of these transformations lies in the fact that surprisingly many natural graph parameters increase (or decrease) along these transformations. This way we gain a considerable amount of information on the extremal values of the studied parameter.

In the second half of the dissertation we study a purely extremal graph theoretic problem, the so-called density Turán problem which, however, turns out to be strongly related to algebraic graph theory in several ways. As a by-product of the efforts we did to solve the problem we give a solution to a longstanding open problem concerning trees having only integer eigenvalues.

## 2. Notations and basic definitions

Before we start to state our results, we introduce the most important notations.

We will follow the usual notation: $G$ is a simple graph, $V(G)$ is the set of its vertices, $E(G)$ is the set of its edges. In general, $|V(G)|=n$ and $|E(G)|=$ $=e(G)=m$. We will use the notation $N(x)$ for the set of the neighbors of the vertex $x,\left|N\left(v_{i}\right)\right|=\operatorname{deg}\left(v_{i}\right)=d_{i}$ denote the degree of the vertex $v_{i}$. We will also use the notation $N[v]$ for the closed neighborhood $N(v) \cup\{v\}$. The complement of the graph $G$ will be denoted by $\bar{G}$, while $\tau(G)$ stands for the number of spanning trees of the graph $G$.
$K_{n}$ will denote the complete graph on $n$ vertices, while $K_{n, m}$ stands for the complete bipartite graph with color classes of size $n$ and $m$. Let $P_{n}$ and $S_{n}$ denote the path and the star on $n$ vertices, respectively.

Let $M_{1}$ and $M_{2}$ be two graphs with distinguished vertices $u_{1}$ of $M_{1}$ and $u_{2}$ of $M_{2}$. Let $M_{1}: M_{2}$ be the graph obtained from $M_{1}$ and $M_{2}$ by identifying the vertices of $u_{1}$ and $u_{2}$. Thus $\left|V\left(M_{1}: M_{2}\right)\right|=\left|V\left(M_{1}\right)\right|+\left|V\left(M_{2}\right)\right|-1$ and $E\left(M_{1}: M_{2}\right)=E\left(M_{1}\right) \cup E\left(M_{2}\right)$. Note that this operation depends on the vertices $u_{1}, u_{2}$, but in general we do not indicate it in the notation.

The matrix $A(G)$ will denote the adjacency matrix of the graph $G$, i.e., $A(G)_{i j}$ is the number of edges going between the vertices $v_{i}$ and $v_{j}$. Since $A(G)$ is symmetric, its eigenvalues are real and we will denote them by $\mu_{1} \geq \mu_{2} \geq \ldots \geq \mu_{n}$. We will also use the notation $\mu(G)$ for the largest eigenvalue and we will call it the spectral radius of the graph $G$. The characteristic polynomial of the adjacency matrix will be denoted by

$$
\phi(G, x)=\operatorname{det}(x I-A(G))=\prod_{i=1}^{n}\left(x-\mu_{i}\right)
$$

We will simply call $\phi(G, x)$ the adjacency polynomial.
The Laplacian matrix of $G$ is $L(G)=D(G)-A(G)$ where $D(G)$ is the diagonal matrix for which $D(G)_{i i}=d_{i}$, the degree of the vertex $v_{i}$. We will denote the eigenvalues of the Laplacian matrix by $\lambda_{1} \geq \lambda_{2} \geq \ldots \lambda_{n-1} \geq \lambda_{n}=0$. The characteristic polynomial of $L(G)$ will be denoted by

$$
L(G, x)=\operatorname{det}(x I-L(G))=\prod_{i=1}^{n}\left(x-\lambda_{i}\right) .
$$

We will simply call it the Laplacian polynomial.
Let $m_{r}(G)$ denote the number of independent edge set of size $r$ (i.e., the $r$-matchings) in the graph $G$. We define the matching polynomial of $G$ as

$$
M(G, x)=\sum_{r=0}(-1)^{r} m_{r}(G) x^{n-2 r} .
$$

The roots of this polynomial are real, and we will denote the largest root by $t(G)$.
Let $i_{k}(G)$ denote the number of independent sets of size $k$. The independence polynomial of the graph $G$ is defined as

$$
I(G, x)=\sum_{K=0}^{n}(-1)^{k} i_{k}(G) x^{k}
$$

Let $\beta(G)$ denote the smallest real root of $I(G, x)$. It exists and it satisfies the inequality $0<\beta(G) \leq 1$.

Let $\operatorname{ch}(G, \lambda)$ be the chromatic polynomial of $G$, i.e. , for a positive integer $\lambda$ the value $\operatorname{ch}(G, \lambda)$ is the number of proper colorings of the graph $G$ with $\lambda$ colors. It is indeed a polynomial in $\lambda$ and it can be written in the form

$$
\operatorname{ch}(G, x)=\sum_{k=1}^{n}(-1)^{n-k} c_{k}(G) x^{k},
$$

where $c_{k}(G) \geq 0$.

## 3. The Kelmans transformation

We define the Kelmans transformation as follows.
Definition 3.1. Let $u, v$ be two vertices of the graph $G$. We obtain the Kelmans transformation of $G$ as follows: we erase all edges between $v$ and $N(v) \backslash(N(u) \cup$ $\cup\{u\})$ and add all edges between $u$ and $N(v) \backslash(N(u) \cup\{u\})$. The obtained graph has the same number of edges as $G$; in general, we will denote it by $G^{\prime}$ without referring to the vertices $u$ and $v$.

Kelmans studied the following problem when he introduced his transformation. Let $R_{q}^{k}(G)$ be the probability that if we remove the edges of the graph $G$ with probability $q$, independently of each other, then the obtained random
graph has at most $k$ components. Kelmans showed that the Kelmans transformation decreases this probability for every $q$, in other words, $R_{q}^{k}\left(G^{\prime}\right) \leq R_{q}^{k}(G)$. Satyanarayana, Schoppmann and Suffel [11] rediscovered this result and they proved that the Kelmans transformation decreases the number of spanning trees: $\tau\left(G^{\prime}\right) \leq \tau(G)$. We proved the following results.


Figure 1. The Kelmans transformation.

Theorem 3.2. [1] Let $G^{\prime}$ be obtained from $G$ by a Kelmans transformation. Let $\mu(G)$ and $\mu\left(G^{\prime}\right)$ denote the largest eigenvalue of the adjacency matrix of $G$ and $G^{\prime}$, respectively. Then $\mu\left(G^{\prime}\right) \geq \mu(G)$.

This result enabled us to attain a breakthrough in an old problem of Eva Nosal. In this problem one has to bound the expression $\mu(G)+\mu(\bar{G})$ in terms of the number of vertices. We managed to prove the following theorem which was a significant improvement of the previous results.

Theorem 3.3. [1] Let $G$ be a graph on $n$ vertices. Then

$$
\mu(G)+\mu(\bar{G}) \leq \frac{1+\sqrt{3}}{2} n .
$$

We also managed to prove the following theorems concerning the connection of the Kelmans transformation and graph polynomials.

Theorem 3.4. [5] Let $M(G, x)$ be the matching polynomial of the graph $G$ :

$$
M(G, x)=\sum_{r=0}(-1)^{r} m_{r}(G) x^{n-2 r} .
$$

Let $t(G)$ denote the largest root of the matching polynomial. Let $G^{\prime}$ be obtained from $G$ by a Kelmans transformation.

Then $m_{k}\left(G^{\prime}\right) \leq m_{k}(G)$ holds for every $1 \leq k \leq n / 2$ and $t\left(G^{\prime}\right) \geq t(G)$.

TheOrem 3.5. [5] Let $I(G, x)$ be the independence polynomial of the graph $G$ :

$$
I(G, x)=\sum_{k=0}(-1)^{k} i_{k}(G) x^{k}
$$

Let $\beta(G)$ denote the smallest root of the independence polynomial. Let $G^{\prime}$ be obtained from $G$ by a Kelmans transformation.

Then $i_{k}\left(G^{\prime}\right) \geq i_{k}(G)$ holds for every $1 \leq k \leq n$ and $\beta\left(G^{\prime}\right) \leq \beta(G)$.
THEOREM 3.6. [5] Let $L(G, x)=\sum_{k=1}^{n}(-1)^{n-k} a_{k}(G) x^{k}$ be the Laplacian polynomial of the graph $G$. Let $G^{\prime}$ be obtained from $G$ by a Kelmans transformation.

Then $a_{k}\left(G^{\prime}\right) \leq a_{k}(G)$ for $1 \leq k \leq n$.
THEOREM 3.7. [5] Letch $(G, x)=\sum_{k=1}^{n}(-1)^{n-k} c_{k}(G) x^{k}$ be the chromatic polynomial of the graph $G$. Let $G^{\prime}$ be obtained from $G$ by a Kelmans transformation.

Then $c_{k}\left(G^{\prime}\right) \leq c_{k}(G)$ for $1 \leq k \leq n$.

## 4. Generalized tree shift

We define the generalized tree shift as follows.

Definition 4.1. [2] Let $T$ be a tree and let $x$ and $y$ be vertices such that all the interior points of the path $x P y$ (if they exist) have degree 2 in $T$. The generalized tree shift (GTS) of $T$ is the tree $T^{\prime}$ obtained from $T$ as follows: let $z$ be the neighbor of $y$ lying on the path $x P y$, let us erase all the edges between $y$ and $N_{T}(y) \backslash\{z\}$ and add the edges between $x$ and $N_{T}(y) \backslash\{z\}$. We will denote the obtained tree by $T^{\prime}$ without referring to the role of $x$ and $y$. We call the generalized tree shift proper if $T$ and $T^{\prime}$ are not isomorphic.

Notations: In what follows we assume that the path $x P y$ has exactly $k$ vertices. The set $A \subset V(T)$ consists of the vertices which can be reached with a path from $k$ only through 1 , and similarly the set $B \subset V(T)$ consists of those vertices which can be reached with a path from 1 only through $k$. Let $H_{1}$ be the tree induced by the vertices of $A \cup\{1\}$ in $T$, similarly let $H_{2}$ denote the tree induced by the vertices of $B \cup\{k\}$ in $T$. Note that $H_{1}$ and $H_{2}$ are both subtrees of $T$ as well.

This transformation determines a partially ordered set on the set of trees on $n$ vertices.


Figure 2. The generalized tree shift.

Definition 4.2. [2] Let us say that $T^{\prime}>T$ if $T^{\prime}$ can be obtained from $T$ by some proper generalized tree shift. The relation $>$ induces a poset on the trees on $n$ vertices. Indeed, the number of leaves of $T^{\prime}$ is greater than the number of leaves of $T$, more precisely the two numbers differ by one. Hence the relation $>$ is extendable. We call this poset the induced poset of the generalized tree shift.


Figure 3. The poset of trees on 6 vertices.
The following observation is very simple.

Theorem 4.3. [2] The minimal element of the the induced poset of the generalized tree shift is the path on $n$ vertices, its maximal element is the star on $n$ vertices.

Remark 4.4. So whenever we prove that the generalized tree shift increases a certain parameter we immediately obtain that the maximum of this parameter is attained at the star and the minimum of this parameter is attained at the path among the trees on $n$ vertices.

In the sequel we collect some of the most important properties of the generalized tree shift.

Theorem 4.5. [2, 4] Let $T$ be a tree and let $T^{\prime}$ be obtained from $T$ by a generalized tree shift.

Then $m_{k}\left(T^{\prime}\right) \leq m_{k}(T)$ for $1 \leq k \leq n / 2$. Furthermore, $\mu\left(T^{\prime}\right) \geq \mu(T)$ and $\mu\left(\overline{T^{\prime}}\right) \geq \mu(\bar{T})$.

Remark 4.6. In the case of trees the adjacency polynomial and the matching polynomial coincide, in other words, $\phi(T, x)=M(T, x)$ and so $t(T)=\mu(T)$.

Theorem 4.7. [4] Let $L(G, x)=\sum_{k=1}^{n}(-1)^{n-k} a_{k}(G) x^{k}$ be the Laplacian polynomial of the graph $G$. Let $\lambda_{1}(G) \geq \lambda_{2}(G) \geq \ldots \lambda_{n-1}(G) \geq \lambda_{n}(G)=0$ be the roots of $L(G, x)$, in other words, the eigenvalues of the Laplacian matrix. Let $T$ be a tree and let $T^{\prime}$ be obtained from $T$ by a generalized tree shift.

Then $a_{k}\left(T^{\prime}\right) \leq a_{k}(T)$ for $1 \leq k \leq n, \lambda_{1}\left(T^{\prime}\right) \geq \lambda_{1}(T)$ and $\lambda_{n-1}\left(T^{\prime}\right) \geq$ $\geq \lambda_{n-1}(T)$.

Theorem 4.8. [4] Let $I(G, x)$ be the independence polynomial of the graph $G$ :

$$
I(G, x)=\sum_{k=0}(-1)^{k} i_{k}(G) x^{k} .
$$

Let $\beta(G)$ denote the smallest root of the independence polynomial. Let $T$ be a tree and let $T^{\prime}$ be obtained from $T$ by a generalized tree shift.

Then $i_{k}\left(T^{\prime}\right) \geq i_{k}(T)$ for $1 \leq k \leq n$ and $\beta\left(T^{\prime}\right) \leq \beta(T)$.
Remark 4.9. There is a common phenomenon in the background of the above mentioned theorems, namely, all of the above mentioned graph polynomials satisfy a certain recursive formula. As a consequence one can factorize the expres$\operatorname{sion} f\left(T^{\prime}, x\right)-f(T, x)$ in a special form. One can prove all the above mentioned
results by this factorisation together with some "monotonicity" property of the studied parameter.

In what follows, the notation $g(H \mid u, x)$ means that the graph polynomial $g$ may depend on the graph $H$ and a specified vertex $u$ of it.

Lemma 4.10. [4] Assume that the graph polynomials $f$ and $g$ satisfy the following recursive formula.

$$
\begin{aligned}
& f\left(M_{1}: M_{2}, x\right)=c_{1} f\left(M_{1}, x\right) f\left(M_{2}, x\right)+c_{2} f\left(M_{1}, x\right) g\left(M_{2} \mid u_{2}, x\right)+ \\
& \quad+c_{2} g\left(M_{1} \mid u_{1}, x\right) f\left(M_{2}, x\right)+c_{3} g\left(M_{1} \mid u_{1}, x\right) g\left(M_{2} \mid u_{2}, x\right),
\end{aligned}
$$

where $c_{1}, c_{2}, c_{3}$ are rational functions of $x$. Furthermore, assume that $c_{2} f\left(K_{2}\right)+$ $+c_{3} g\left(K_{2} \mid 1\right) \neq 0$. Then

$$
\begin{gathered}
f(T)-f\left(T^{\prime}\right)=c_{4}\left(c_{2} f\left(P_{k}\right)+c_{3} g\left(P_{k} \mid 1\right)\right)\left(c_{2} f\left(H_{1}\right)+\right. \\
\left.+c_{3} g\left(H_{1} \mid 1\right)\right)\left(c_{2} f\left(H_{2}\right)+c_{3} g\left(H_{2} \mid k\right)\right),
\end{gathered}
$$

where

$$
c_{4}=\frac{g\left(P_{3} \mid 1\right)-g\left(P_{3} \mid 2\right)}{\left(c_{2} f\left(K_{2}\right)+c_{3} g\left(K_{2} \mid 1\right)\right)^{2}} .
$$

The original application of the generalized tree shift was the following theorem. (Unlike the other theorems, this statement has a purely combinatorial proof.)

Theorem 4.11. [2] Let $W_{k}(G)$ denote the number of closed walks of length $k$. Let $T$ be a tree and let $T^{\prime}$ be obtained from $T$ by a generalized tree shift.

Then $W_{k}\left(T^{\prime}\right) \geq W_{k}(T)$ for every $k \geq 1$.
Remark 4.12. Let $\mu_{1} \geq \mu_{2} \geq \ldots \geq \mu_{n}$ be the eigenvalues of the adjacency matrix of the graph $G$. Then

$$
W_{k}(G)=\sum_{j=1}^{n} \mu_{j}^{k} .
$$

Thus the following theorem is an easy consequence of the previous statement.
Corollary 4.13. [2] Let $E E(G)=\sum_{j=1}^{n} e^{\mu_{j}}$ denote the Estrada index of the graph $G$. Let $T$ be a tree and let $T^{\prime}$ be obtained from $T$ by a generalized tree shift.

Then $E E\left(T^{\prime}\right) \geq E E(T)$.

Remark 4.14. The previous corollary implies that the Estrada index attains its minimal value at the path among the connected graphs on $n$ vertices and it is maximal for the star among the trees on $n$ vertices. This was conjectured by J. A. de la Peña, I. Gutman and J. Rada [7]. This conjecture prompted V. Nikiforov to state the conjecture that the minimal value of $W_{k}(G)$ is attained at the path on $n$ vertices among the trees on $n$ vertices. The generalized tree shift was developed to attack this conjecture.

## 5. Density Turán problem

The following problem was studied in Zoltán L. Nagy's master thesis [9]. This part is based on a joint work with him.

Given a simple, connected graph $H$, define the blown-up graph $G[H]$ of $H$ as follows. Replace all vertices $v_{i} \in V(H)$ by a cluster $A_{i}$ and we draw some edges between the clusters $A_{i}$ and $A_{j}$ (not necessarily all) if $v_{i}$ and $v_{j}$ were adjacent in $H$. Question: what kind of edge densities we have to require between the clusters so that $G[H]$ surely contains a graph isomorphic to $H$ such that the vertex corresponding to $v \in V(H)$ is in the cluster corresponding to $v$. In this case we say that $H$ is a transversal of $G[H]$.


Figure 4. A blown-up graph of the diamond containing the diamond as a transversal

In the sequel we need some technical definitions.

Definition 5.1. [9, 10] A weighted blown-up graph is a blown-up graph where a non-negative weight $w(u)$ is assigned to each vertex $u$ such that the total weight of each cluster is 1 . The density between two clusters is

$$
d_{i j}=\sum_{\substack{(u, v) \in E \\ u \in A_{i}, v \in A_{j}}} w(u) w(v) .
$$

Definition 5.2. [9, 10] We define the critical edge density $d_{\text {crit }}(H)$ of the graph $H$ as follows. The number $d_{c r i t}(H)$ is the smallest number $d$ for which it is true that whenever the edge density between any two clusters of $G[H]$ is larger than $d$ then $H$ is surely a transversal of $G[H]$.

Definition 5.3. [6] Let $x_{e}$ 's be variables assigned to each edge of a graph. The multivariate matching polynomial $F$ is defined as follows:

$$
F\left(\underline{x_{e}}, t\right)=\sum_{M \in \mathcal{M}}\left(\prod_{e \in M} x_{e}\right)(-t)^{|M|},
$$

where the summation goes over the matchings of the graph including the empty matching.

Now we are ready to state our results. First we study the case when the graph $H$ is a tree.

Theorem 5.4. [6] Let $T$ be a tree.
(a) Assume that the edge density between the clusters $A_{i}, A_{j}$ of the blown-up graph $G[H]$ is $\gamma_{i j}=1-r_{i j}$. Assume that $F_{T}\left(\underline{r_{e}}, t\right)>0$ for all $t \in[0,1]$. Then $G[T]$ surely contains $T$ as a transversal.
(b) If for the numbers $\gamma_{i j}=1-r_{i j}$, the polynomial $F_{T}\left(r_{e}, t\right)$ has a root in the interval $[0,1]$ then there exists weighted blown-up graph $G[T]$ of the tree $T$ such that the edge density between the clusters $A_{i}, A_{j}$ is $\gamma_{i j}$ for each $1 \leq i, j \leq n$, still $G[T]$ does not contain $T$ as a transversal.

Corollary 5.5. [10] Let $T$ be a tree and $\mu(T)$ be the largest eigenvalue of the adjacency matrix of $T$. Then

$$
d_{c r i t}(T)=1-\frac{1}{\mu(T)^{2}} .
$$

If $H$ is an arbitrary graph then the following statements remain true from the above theorems.

Theorem 5.6. [6] Let $H$ be a simple graph. Assume that the edge density between the clusters $A_{i}, A_{j}$ of the blown-up graph $G[H]$ is $\gamma_{i j}=1-r_{i j}$. Assume that $F_{H}\left(\underline{r_{e}}, t\right)>0$ for all $t \in[0,1]$. Then $G[H]$ surely contains $H$ as a transversal.

Theorem 5.7. [6] Let $H$ be a simple graph and let $t(H)$ denote the largest root of the matching polynomial of $H$. Then

$$
d_{c r i t}(H) \leq 1-\frac{1}{t(H)^{2}}
$$

Corollary 5.8. [6] Let $H$ be a simple graph of largest degree $\Delta>1$. Then

$$
1-\frac{1}{\Delta} \leq d_{c r i t}(H)<1-\frac{1}{4(\Delta-1)}
$$

As a lower bound we managed to prove the following theorem. Before we give the statement we need some definitions.

Definition 5.9. A proper labeling of the vertices of the graph $H$ is a bijective function $f$ from $\{1,2, \ldots, n\}$ to the set of vertices such that the vertex set $\{f(1), \ldots, f(k)\}$ induces a connected subgraph of $H$ for all $1 \leq k \leq n$. The set of the proper labelings will be denoted by $\mathcal{S}(H)$.

Let $f \in \mathcal{S}(H)$. The monotone-path tree $T_{f}(H)$ of $H$ is defined as follows. The vertices of this graph are the paths of the form $f\left(i_{1}\right) f\left(i_{2}\right) \ldots f\left(i_{k}\right)$ where $1=i_{1}<$ $<i_{2}<\cdots<i_{k}$ and two such paths are connected if one is the extension of the other with exactly one new vertex.


Figure 5. A monotone-path tree of the wheel on 5 vertices.

Theorem 5.10. [6]

$$
d_{\text {crit }}(H) \geq \max _{f \in S(H)}\left\{1-\frac{1}{\mu\left(T_{f}(H)\right)^{2}}\right\} .
$$

REMARK 5.11. In the case of the complete bipartite graph, for arbitrary proper labeling the largest eigenvalue of the monotone-path tree is $\sqrt{m+n-1}$. So the following conjecture is very natural.

Conjecture 5.12. [6]

$$
d_{c r i t}\left(K_{n, m}\right)=1-\frac{1}{m+n-1}
$$

## 6. Integral trees

We call a tree an integral tree if all the eigenvalues of the tree are integers. Integral trees are extremely rare, among the trees on at most 50 vertices only 28 are integral. Among the 2262366343746 trees on 35 vertices there is only one tree which is integral. In spite of this fact, several infinite class of integral trees were known and all of them had diameter at most 10 . It was an open problem for more than 30 years whether there exist integral trees of arbitrarily large diameters. We managed to answer this question affirmatively.

Theorem 6.1. [3] For every finite set $S$ of positive integers there exists a tree whose positive eigenvalues are exactly the elements of $S$. If the set $S$ is different from the set $\{1\}$ then the constructed tree will have diameter $2|S|$.

In the previous section we have seen that the monotone-path tree of the complete bipartite graph $K_{n, m}$ has spectral radius $\sqrt{n+m-1}$. In fact, the following stronger statement is also true.

THEOREM 6.2. Let $f$ be a proper labeling of the complete bipartite graph $K_{n, m}$. Then all the eigenvalues of the monotone-path tree $T_{f}\left(K_{n, m}\right)$ have the form $\pm \sqrt{q}$ where $q$ is a non-negative integer.

Hence, in order to prove Theorem 6.1, all we have to prove is that one can put perfect squares under the square roots. This can be done indeed.

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# BOOK REVIEW 

By<br>KRISTÓF BÉRCZI

András Frank, Connections in combinatorial optimization, Oxford University Press 2011, Oxford Lecture Series in Mathematics and its Applications, 38, 664 pages.

The title of the book represents a major aspect of Frank's research interest: revealing and taking advantage of unexpected links between apparently unrelated results and methods within Combinatorial Optimization (CO), as well as between CO and other fields. The aim of this approach is to find good characterizations, min-max theorems, and polynomial time algorithms for CO problems that are applicable in practice. The title also refers to the extensive theory of graph and hypergraph connectivity, including results on paths, cuts, trees, flows, bipartite matchings, as well as on the numerous variations of higher order connections of graphs, digraphs, and hypergraphs. The book also shows the intimate relationship of submodular functions and network optimization. These results are not only interesting from a theoretical viewpoint, but also wildly used in real life applications.

The book consists of three parts. Part I gives a comprehensive overview of the basics of combinatorial optimization, such as results on paths, bipartite matchings, and network optimization. It also provides an introduction to polyhedral combinatorics and matroid theory. The list of well-known classical results is enriched with interesting applications that may be new even for more experienced readers.

Part II covers more recent topics, like structures of cuts, orientations of graphs and hypergraphs, packings of and coverings by trees, forests, arborescences, and branchings. The usage of fundamental methods is emphasized here, such as the splitting-off technique, the uncrossing procedure, or the push-relabel algorithm.

In Part III, polyhedral and submodular optimization methods are discussed. Many problems of Parts I and II are re-investigated on a higher, abstract level. This approach exemplifies the concept that looking at problems in their abstract form may give rise to simpler and shorter proofs.

The book is not only readable for students or researchers as a source of advanced material, but it also provides efficient tools for engineers and practitioners.

A few words about the author. Working for an industrial research institute, learning from L. Lovász, and paying a 4-month visit to J. Edmonds in 1980 were altogether decisive in forming the profile of Frank's research interest. His algorithms for weighted matroid intersection, for optimal chain and antichain families, for submodular flows, for network augmentations, for routing problems, his discrete separation theorem and weight-splitting theorem for matroid intersection constitute the standard starting point of further investigations. He founded the EGRES combinatorial optimization group at the Eötvös University which serves as a forum for young researchers to work together on CO problems.

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