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ON THE K FUNCTOR, AND TOPOLOGICAL ALGEBRAS

By
FERMÍN DALMAGRO

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Abstract. Let us consider the extension functor F of ring of scalars, associated to a surjective map $f\colon A \to \overline{A}$ of commutative rings with 1. Let $1+\operatorname{Ker} f$ consist of invertible elements. Then F induces an isomorphism of groups $K(A) \to K(\overline{A})$. Moreover, we investigate topological algebras, where the invertible elements form an open set, and prove for them some analogues of theorems for Banach algebras.

1. Introduction

Let $A \ni 1$ be a commutative complex algebra. (In the whole paper algebras are complex, and have $1(\neq 0)$). Let $P(A)$ be the category of all unitary projective, finitely generated A-modules. Then $P(A)$ is a (small) additive category and the set $\mu(A)$ of all classes of isomorphisms of objects in $P(A)$ is an abelian monoid.

Let us denote by $K(A)$ the symmetrization of the monoid $\mu(A)$ ([4, p. 53]). For a compact Hausdorff topological space we denote by $C(X)$ the algebra of all continuous complex functions with domain X. Whenever no misunderstanding arises, we shall write $K(X)$ instead of $K\big(C(X)\big)$.

A seminorm p on a topological algebra A is said to be submultiplicative if, for any elements x, y in A, we have that

$$P(x \cdot y) \leq p(x) \cdot p(y).$$

A locally convex algebra A is locally multiplicatively convex if the topology of A is defined by a separating and (upwards) directed family of submultiplicative seminorms. We shall, for brevity, call such an algebra a locally m-convex algebra and such base of seminorms an m-base.

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Let A be a locally m-convex algebra. Let P be an m-base of seminorms of A. For each $p \in P$ we define the set N_p by

$$N_p = \{x \in A : p(x) = 0\}.$$

Clearly, N_p is a closed two sided ideal of A. Also, for each x in A and each p in P, we denote by x_p the class of x in the quotient algebra $A_p := A/N_p$.

For each seminorm $p \in P$ we have that the function on A_p defined by

$$\|x_p\| = p(x)$$

is a norm and

$$\|x_p \cdot y_p\| \leq \|x_p\| \cdot \|y_p\|.$$

This norm is continuous with respect to the quotient topology and, consequently, endowing A_p with this norm, the natural homomorphism

$$\pi_p : A \to A_p$$

is continuous and onto.

Let θ_A be the set of invertible elements of the locally convex algebra $A \ni 1$. If θ_A is open in A (i.e. $1 \in \operatorname{int} \theta_A$) then A is called a Q-algebra.

Let A' be the topological dual of the locally convex algebra $A \ni 1$. The spectrum of A consists of all elements of A', which are algebra-homomorphisms preserving 1. This spectrum with the weak topology induced by A will be denoted by $m(A)$. More precisely, this is the weak topology induced by the maps $m(A) \ni \ni f \mapsto f(a) \in \mathbb{C}$, where $a \in A$.

It is well known that every Banach algebra is a Q-algebra (see [9, p. 16]). It is also known that if the Banach algebra A is commutative then $m(A)$ is compact Hausdorff ([9, p. 44]).

2. On the K-functor

Let us begin with some observations which are to be used in what follows.

Let $A \ni 1$ be a commutative ring, J an ideal of A and M a unitary A-module. By $\overline{A}$ we mean the ring A/J. The tensor product by M of the exact sequence of A-modules

$$0 \to J \to A \to \overline{A} \to 0$$

gives us the exact sequence of A-modules ([6, Ch. VI., §2])

$$M \otimes_A J \to M \otimes_A A \to M \otimes_A \overline{A} \to 0.$$

By $JM = \operatorname{Im}(M \otimes_A J \to M)$ we denote the submodule of M spanned by the elements of the form $a \cdot x$ with $a \in J$ and $x \in M$. Since $M \otimes_A A$ and M are isomorphic, the sequence

$$0 \to JM \to M \to M \otimes_A \overline{A} \to 0$$

of A-modules is exact. Therefore, $M \otimes_A \overline{A}$ and M/JM are isomorphic both as $\overline{A}$-modules and as A-modules.

The functor $(\cdot) \otimes_A \overline{A}$, from the category of A-modules to the category of $\overline{A}$-modules, is called the extension functor of the ring of scalars associated to the map $A \to \overline{A}$ (cf. [4, II. 1. 12]).

Let g be an endomorphism of $A^n = \underbrace{A \oplus \ldots \oplus A}_{n}$ and let $\overline{g} = g \otimes_A \mathrm{id}_{\overline{A}}$: $A^n \otimes_A \overline{A} \to A^n \otimes_A \overline{A}$ be the induced endomorphism. If we take the tensor product of the exact sequence

$$0 \to \operatorname{Ker} g \to A^n \to \operatorname{Im} g \to 0$$

by $\overline{A}$ we get the exact sequence ([6], above cited)

$$\operatorname{Ker} g \otimes_A \overline{A} \to A^n \otimes_A \overline{A} \xrightarrow{\overline{g}} \operatorname{Im} g \otimes \overline{A} \to 0,$$

whence, $\operatorname{Im}(\operatorname{Ker} g \otimes_A \overline{A} \to A^n \otimes_A \overline{A}) = \operatorname{Ker} \overline{g}$ and $\operatorname{Im} \overline{g} = \operatorname{Im} g \otimes_A \overline{A}$.

Theorem 2.1. *Let A be a commutative ring with unit 1. Let J be an ideal of A such that $1 + J$ is contained in θ_A. Then the extension functor of the ring of scalars, associated to the quotient mapping from A into $\overline{A} := A/J$ induces an isomorphism of groups between $K(A)$ and $K(\overline{A})$.*

Proof. The proof consists of two steps: the functor mentioned in the theorem induces an injection, and also a surjection between $K(A)$ and $K(\overline{A})$.

We prove first that if M and N are objects in $P(A)$ such that $M \otimes_A \overline{A}$ and $N \otimes_A \overline{A}$ are $\overline{A}$-isomorphic then M and N are A-isomorphic.

Let p, q be projections of A^n such that $\operatorname{Im} p \cong M$ and $\operatorname{Im} q \cong N$. For given $\varphi\colon A^n \to A^n$, we write $\overline{\varphi}\colon \overline{A}^n \to \overline{A}^n$ for the image of φ by the $(\cdot) \otimes_A \overline{A}$ functor (observe that $A^n \otimes_A \overline{A} = \overline{A}^n$). If $M \otimes_A \overline{A}$ is $\overline{A}$-isomorphic to $N \otimes_A \overline{A}$, then there exist $\overline{A}$-isomorphisms

$$f\colon\ \operatorname{Im} \overline{p} \to \operatorname{Im} \overline{q}, \qquad g\colon\ \operatorname{Im} \overline{q} \to \operatorname{Im} \overline{p}$$

such that

$$f \cdot g = \mathrm{id}, \qquad g \cdot f = \mathrm{id}\,.$$

We now extend f and g to endomorphisms f', g' of $\overline{A}^n$, letting $f'(\operatorname{Ker}\overline{p}) = g'(\operatorname{Ker}\overline{q}) = \{0\}$, as shown on the following commutative diagram,

$$\begin{array}{ccccccccc}
0 & \to & \operatorname{Ker}\overline{p} & \to & \overline{A}^n & \to & \operatorname{Im}\overline{p} & \to & 0 \\
 & & \downarrow & & f'\downarrow & & \downarrow f & & \\
0 & \to & \operatorname{Ker}\overline{q} & \to & \overline{A}^n & \to & \operatorname{Im}\overline{q} & \to & 0 \\
 & & \downarrow & & g'\downarrow & & \downarrow g & & \\
0 & \to & \operatorname{Ker}\overline{p} & \to & \overline{A}^n & \to & \operatorname{Im}\overline{p} & \to & 0
\end{array}$$

(the left hand side vertical arrows are 0's, $f' = 0 \oplus f\colon \operatorname{Ker}\overline{p} \oplus \operatorname{Im}\overline{p} \to \operatorname{Ker}\overline{q} \oplus$ $\oplus \operatorname{Im}\overline{q}$, and similarly for g').

From this it follows that the following diagram is commutative:

$$\begin{tikzcd}
\overline{A}^n \arrow[r, "f'"] \arrow[d, "\overline{p}"] & \overline{A}^n \arrow[r, "g'"] \arrow[d, "\overline{q}"] & \overline{A}^n \arrow[d, "\overline{p}"] \\
\overline{A}^n \arrow[r, "f'"'] \arrow[ur, "f'"] & \overline{A}^n \arrow[r, "g'"'] \arrow[ur, "g'"] & \overline{A}^n
\end{tikzcd}$$

and that $g'f' = \overline{p}, f'g' = \overline{q}$.

Let us choose endomorphisms f'' and g'' of A^n such that $\overline{f''} = f'$, $\overline{g''} = g'$, and take $\hat{f} = qf''p$, $\hat{g} = pg''q$. Then the following diagram is commutative

$$\begin{tikzcd}
A^n \arrow[r, "\hat{f}"] \arrow[d, "p"] & A^n \arrow[r, "\hat{g}"] \arrow[d, "q"] & A^n \arrow[d, "p"] \\
A^n \arrow[r, "\hat{f}"'] \arrow[ur, "\hat{f}"'] & A^n \arrow[r, "\hat{g}"'] \arrow[ur, "\hat{g}"] & A^n
\end{tikzcd}$$

in particular

$$\hat{f}(\operatorname{Im} p) \subset \operatorname{Im} q, \qquad \hat{g}(\operatorname{Im} q) \subset \operatorname{Im} p.$$

Let $\tilde{f}\colon \operatorname{Im} p \to \operatorname{Im} q$ and $\tilde{g}\colon \operatorname{Im} q \to \operatorname{Im} p$ be defined to be the restrictions of $\hat{f}$ and $\hat{g}$ to $\operatorname{Im} p$ and $\operatorname{Im} q$ respectively. Then we have that

$$\bar{\tilde{f}} = f, \qquad \bar{\tilde{g}} = g$$

whence,

$$\tilde{f} \cdot \tilde{g}(x) - x \in J \cdot \operatorname{Im} q, \qquad \text{for all } x \in \operatorname{Im} q$$

and

$$\tilde{g} \cdot \tilde{f}(x) - x \in J \cdot \operatorname{Im} p, \qquad \text{for all } x \in \operatorname{Im} p.$$

It can now be easily deduced from Nakayama's Lemma ([1, Ch. III, Prop. (2.2)]), that $\tilde{g} \cdot \tilde{f}$ and $\tilde{f} \cdot \tilde{g}$ are also isomorphisms. This, in turn, implies that $\tilde{f}$ and $\tilde{g}$ are isomorphisms.

Now we prove that, if N is an object in $P(\overline{A})$, then there exists an object M in $P(A)$ such that $M \otimes_A \overline{A}$ is $\overline{A}$-isomorphic to N. In fact, if N is an object of $P(\overline{A})$, then N is a direct summand of some finitely generated free A-module, therefore, there exists a short and split exact sequence

$$0 \to \operatorname{Ker} p \to \overline{A}^n \xrightarrow{p} N \to 0$$

with p a projection of $\overline{A}^n$.

Let g be an endomorphism of A^n such that $\overline{g} = p$. From the fact that the sequence

$$0 \to \operatorname{Ker} g \to A^n \xrightarrow{g} \operatorname{Im} g \to 0$$

is exact, the diagram

$$\begin{tikzcd} JA^n \arrow[r, "j"] \arrow[d, "i_1"] & J \cdot \operatorname{Im} g \arrow[d, "i_2"] & \\ A^n \arrow[r, "g"'] & \operatorname{Im} g \arrow[r] & 0 \end{tikzcd}$$

is commutative, where the vertical arrows represent the corresponding inclusions and j is the restriction of g to $J \cdot A^n$, and the fact that $\overline{g - g^2} = p - p^2 = 0$, we deduce that $g - g^2$ maps A^n onto $J \cdot \operatorname{Im} g$.

Since j is an epimorphism, there exists an A-homomorphism, say $\varrho\colon A^n \to$ $\to J \cdot A^n$, such that

$$i_3 \cdot j \cdot \varrho = g - g^2,$$

or, equivalently,

$$i_3 \cdot g \cdot i_1 \cdot \varrho = g - g^2,$$

where $i_3 : J \cdot \operatorname{Im} g \to A^n$ is the inclusion.

Let $h = \mathrm{id}_{A^n} - i_1 \cdot \varrho$ then $gh = g^2$ and, for each $x \in A^n$ we have that

$$h(x) - x = i_1 \cdot \varrho(x)$$

is in JA^n. From this fact and Nakayama's Lemma we deduce that h is an isomorphism.

Since $\overline{(h-g)^2-(h-g)}=0$, then proceeding as before we deduce that there exists an isomorphism k such that $(h-g)k=(h-g)^2$, i.e., $(h-g)k^{-1}=$ $=(h-g)^2k^{-2}$. Now, writing $q=(h-g)k^{-1}$ we have that $q^2=q$, and $k(x)-x\in$ $\in JA^n$, that is, q is a projection of A^n such that

$$\begin{aligned}\overline{q}&=\overline{(h-g)}=\\&=1-p.\end{aligned}$$

Finally, by taking M as the image of the projection $1-q$ we deduce that M is an object of $P(A)$ and that

$$M\otimes_A\overline{A}\cong N.$$ ∎

A simple consequence is the following

Corollary 2.2. *Let A, B be commutative rings. Let $\mu\colon A\to B$ be a ring epimorphism. If* $1+\operatorname{Ker}\mu\subset\theta_A$, *then*

$$K(A)\cong K(B).$$ ∎

Corollary 2.3. *Let A be a complex commutative locally m-convex Q-algebra with identity* 1. *Then for each m-base P of seminorms of A there exists $p_0\in P$ such that*

$$K(A)\cong K(A_p),\qquad \text{for each } p\geq p_0.$$

Proof. Let P be an m-base of seminorms in A. Let Γ be the family of all finite subsets of P. For each $H\in\Gamma$ and each $\varepsilon>0$, let us define the set $U(H,\varepsilon)$ by

$$U(H,\varepsilon)=\{x\in A\colon p(x)<\varepsilon,\ \text{for all } p\in H\}.$$

This family is a basis of neighborhoods of zero in A. If A is a Q-algebra, then there exist H in Γ and $\varepsilon>0$ such that

$$1+U(H,\varepsilon)\subset\theta_A.$$

Let p_0 be an upper bound of H, such that

$$1+N_{p_0}\subset 1+U(\{p_0\},\varepsilon)\subset 1+U(H,\varepsilon)\subset\theta_A.$$

Then we deduce from Theorem 2.1 that $K(A)\cong K(A_{p_0})$.

Moreover, if $p\geq p_0$ in P, then $N_p\subset N_{p_0}$ and, consequently, $K(A_p)=K(A)$. ∎

3. The spectrum $m(A)$ of A

Let us recall that for a commutative complex algebra A with identity 1 and for each $x \in A$ the set $\Sigma_A(x)$ is defined by

$$\Sigma_A(x) = \{\lambda \in \mathbb{C} \colon x - \lambda 1 \notin \theta_A\}.$$

This set, with the topology induced by that of $\mathbb{C}$, i.e., the subspace topology, is called the spectrum of x. If A is locally m-convex, then $\Sigma_A(x) \neq 0$ ([7, p. 10]).

The following proposition may be known. We include it here for the sake of completeness.

PROPOSITION 3.1. *If $A \ni 1$ is a commutative complex locally m-convex Q-algebra, then the set $\Sigma_A(x)$ is compact for each $x \in A$.*

PROOF. Let $\{\lambda_n\}_{n=1}^{\infty}$ be a sequence in $\Sigma_A(x)$ which converges to the point λ in the closure of $\Sigma_A(x)$. Then the sequence $\{x - \lambda_n 1\}_{n=1}^{\infty}$ converges to $x - \lambda 1$ in $A \setminus \theta_A$ and, since $A \setminus \theta_A$ is closed, $x - \lambda 1 \notin \theta_A$, that is, $\lambda \in \Sigma_A(x)$.

The set $\Sigma_A(x)$ being closed, it remains to prove that it is bounded. There is a convex, balanced, and absorbent neighborhood W of zero in A such that $W - 1 \subset$ $\subset \theta_A$. Consequently, there is $\lambda_0 \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}$, such that for any $\lambda \in \mathbb{C}^*$ for which $|\lambda| < |\lambda_0|$ we have that $\lambda x \in W$. This in turn implies that $\lambda x + 1 =$ $= \lambda(x + \lambda^{-1} 1) \in \theta_A$, that is $-\lambda^{-1} \notin \Sigma_A(x)$ and $\Sigma_A(x) \subset B_{\mathbb{C}}(0, |\lambda_0|^{-1})$. ∎

Let $\sigma_A(x)$, $x \in A$ be the function defined by

$$\sigma_A(x) = \sup\{|\lambda| \colon \lambda \in \Sigma_A(x)\}.$$

Let $\mathfrak{s}(A)$ be the set of all $x \in A$ such that $\sigma_A(x) \leq 1$. Then, since

$$\{f(x) \colon f \in m(A)\} \subset \Sigma_A(x),$$

we have that

$$|f(x)| \leq \sigma_A(x)$$

for each $f \in m(A)$ and

$$\mathfrak{s}(A) \subset \bigcap_{f \in m(A)} \{x \in A \colon |f(x)| \leq 1\}.$$

PROPOSITION 3.2. *Let A be a commutative complex locally m-convex Q-algebra. Then $\mathfrak{s}(A)$ is a neighborhood of zero in A.*

PROOF. Let W be a convex, balanced and absorbent neighborhood of 0 such that $W + 1 \subset \theta_A$. If W is not contained in $\mathfrak{s}(A)$, then there is an $x \in W$ and a $\lambda \in \mathbb{C}$ such that $|\lambda| > 1$ and $x - \lambda 1 \notin \theta_A$.

From the fact that W is balanced and the obvious equality $x - \lambda 1 = -\lambda(-\lambda^{-1}x + 1)$ we deduce that $-\lambda^{-1}x+1$ is in $W+1$ and $x-\lambda 1 \in \theta_A$. This is a contradiction. ∎

THEOREM 3.3. *Let A be a commutative complex locally m-convex Q-algebra. Then the set $m(A)$ is compact Hausdorff and equicontinuous.*

PROOF. Since, according to Proposition 3.2, the set $\mathfrak{s}(A)$ is a neighborhood of zero and for each $x \in \mathfrak{s}(A), f \in m(A)$ we have $|f(x)| \leq 1$, then we deduce that $m(A)$ is equicontinuous.

The set $m(A)$ is closed in $A' = \prod_{x \in A} \mathbb{C}x$ by equicontinuity. It follows from Proposition 3.1 and Tychonoff's theorem that the space $m(A)$ can be identified with a compact subset of A'. Thus $m(A)$ is compact Hausdorff. ∎

The continuous homomorphism $\pi_p \colon A \to A_p$ induces a continuous map $\pi_p^* \colon m(A_p) \to m(A)$.

THEOREM 3.4. *Let A be a commutative complex locally m-convex Q-algebra. Let P be an m-base of seminorms in A. Then, there exists an element q_0 in P such that, for every $p \geq q_0$, the mapping π_p^* is a homeomorphism from $m(A_p)$ onto $m(A)$.*

PROOF. Since the space $m(A)$ is equicontinuous, it follows from the fact that P is an m-base in A that there are an $\varepsilon > 0$ and a finite subset H of P such that for all $x \in U(H, \varepsilon)$ (defined in the proof of Corollary 2.3), $|f(x)| \leq 1$ for each $f \in m(A)$.

Let q_0 be an upper hound of H. Then

$$U(\{q_0\}, \varepsilon) \subset U(H, \varepsilon)$$

and consequently, we have that for each $x \in N_{q_0}$

$$\mathbb{C} \cdot x \subset N_{q_0} \subset U(\{q_0\}, \varepsilon).$$

Hence, for each $f \in m(A)$,

$$|f(\lambda x)| = |\lambda||f(x)| < 1$$

for all $\lambda \in \mathbb{C}$, that is $f(x) = 0$ on N_{q_0}. This in turn implies that $\pi_{q_0}^*$ is onto, and since it is injective, $\pi_{q_0}^*$ is a homeomorphism.

Finally, if $p \geq q_0$ then p is an upper bound for H and proceeding as above, we deduce that π_p^* is a homeomorphism. ∎

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Fermín Dalmagro
Departamento de Matemáticas
Facultad de Ciencias
Universidad Central de Venezuela
Apartado 21201, Caracas
Venezuela

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NON-FINITELY BASED VARIETIES OF WEAKLY ASSOCIATIVE LATTICES*

By
ERVIN FRIED

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Abstract. A directed graph is called a Weakly Associative Lattice (WAL), if every pair of elements a, b has a least common upper bound ($a \vee b$) and a greatest common lower bound ($a \wedge b$). This generalization of lattices includes a generalization of distributive lattices. Let $\mathcal{U}$ be the variety of WAL generated by the directed graphs in which every pair of distinct elements have a unique upper and unique lower bound. The subdirectly irreducible elements of variety $\mathcal{U}$ consist of all graphs such that every pair of elements has a unique common upper bound and a unique common lower bound, as well. This property is called UBP, as well as the graphs which satisfy UBP.

Let $\mathbb{I}$ be any set of integers greater than 2. Let $\mathcal{U}_{\mathbb{I}}$ denote the subvariety of $\mathcal{U}$ generated by those UBPs which do not contain cycles of length $n \in \mathbb{I}$. These varieties have finite equational basis if and only if $\mathbb{I}$ is finite. An equational basis is presented for every $\mathbb{I}$.

As a "byproduct" it is shown that every finite UBP, except the two-element lattice contain a "triangle" or a fourcycle. The problem regarding triangle, only, is still open.

0. Preliminaries

Tournaments are complete directed graphs. In a tournament one can define two operations $\vee$ and $\wedge$, as follows: Both operations are idempotent, commutative and for $a \to b$ let $a \vee b = b$ and $a \wedge b = a$. Either of these operations uniquely determine the graph[1]. In [15] the authors proved that the variety generated by

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[1] More details about this connection and about the concepts used here will be given in the introduction.

tournaments with one binary operation is not finitely based. R. McKenzie asked whether the same holds when we consider both operations. Though this variety is nicer, the problem seems to be more difficult.

For tournaments with a single operation the first step was, to have a close look at the free algebra on three and four generators ([16]). The first one has 15 elements and the second one a few millions. The variety generated by the tournaments as type $2,2$ algebras is congruence distributive. However the problem with this variety is, that the three-generated free algebra has 162 elements and the four-generated has more than 10^{11}. Indeed, these algebras are weakly associative lattices. The free algebra is a subdirect product of four-generated subdirect irreducible tournaments, considered as many times as many the number of essentially different products of four-generated tournaments having neither least nor greatest elements. Some of these are triangles. By [9] their subdirect product is direct, moreover a direct factor of the free algebra.

We get as many triangles as many distinct onto-projection the generators a, b, c, d have to the triangle $1, 2, 3$. Two of them – say a and b – must have the same image – say 1. Then c is mapped either to 2 or to 3, which are distinct cases because of $3 \to 1 \to 2$. Hence, the number of triangles are $\binom{4}{3} \cdot 3 \cdot 2 = 36$. Since we have a direct product, the number of elements is $3^{36} \approx 1{,}5 \cdot 10^{36} > 10^{11}$.

In [3] tournaments with two operations was the natural starting point of weakly associative lattices – a non-associative generalization of lattices –, with the idea, that tournaments are the generalization of chains, i.e., they generate a variety, a generalization of the class of distributive lattices.

It is natural to ask whether, if not the whole variety, at least a subvariety of this variety is non-finitely based. There is an easy way, to check the existence of such a subvariety. If a variety has a continuum of subvarieties, then "most of them" must be non-finitely based, since the number of finitely based subvarieties is countable[2]. In [6] the following method was used for establishing a continuum of subvarieties. Suppose, the subdirectly irreducible members of a congruence distributive variety "behave nicely" so that we can find an infinite set of finite ones such that no variety, generated by any subset of these algebras contains the others. Then we can conclude the existence of a continuum of subvarieties. Certain aspects of this "nice behavior" hold for tournaments. Namely, finite homomorphic images of subalgebras of ultraproducts of finite tournaments are tournaments, again. Due to the well-known behavior of ultraproducts the existence of a continuum of subvarieties is equivalent to the following question:

[2] We speak about varieties having finitely many finitary operations.

Problem. *Does there exist an infinite set of finite tournaments such that none of them is isomorphic to (a homomorphic image of) a subalgebra*[3] *of some other one?*

However, it is possible, that the variety of tournaments have countable many subvarieties, only. It is not difficult to see that this is, basically, equivalent to the following

Problem. *To each natural k does there exist a natural n, such that every finite simple tournament of cardinality greater than n contains an isomorphic copy of every tournament of cardinality at most k? (A tournament is simple if it contains no proper convex subsets. A subset $\mathcal{S}$ of a tournament $\mathcal{T}$ is convex if $a, b \in \mathcal{S}$, $a \to x \to b$ implies $x \in \mathcal{S}$. $\mathcal{S}$ is proper if $\mathcal{S} \neq \mathcal{T}$ and $|\mathcal{S}| \neq 1$.)*

In [10], [4], [5], [11], [12], [13], and [14] it became clear that the "appropriate" generalization of distributive lattices is the variety generated by those weakly associative lattices which satisfy the unique bound property. Following an other question of R. McKenzie, it was obvious to try to find subvarieties of this variety which are not finitely based. Actually, in [6] the existence of a continuum of such subvarieties of this variety was proved. We used the method outlined above, thus each of these subvarieties was generated by finite algebras. Of course, none of these subvarieties was "pointed out" to be non-finitely based.

In this paper, we shall produce a continuum of subvarieties of this variety, such that their finite members are only the distributive lattices. Using the results of [2] we shall be able to "point out" some of them. We shall give, as well, an infinite system of equations which define one of these varieties.

1. Introduction

Let $\mathcal{G} = \langle G; \leq \rangle$ be any directed graph, i.e., $\leq$ is a reflexive and antisymmetric relation on G. For $a \leq b$ and $a \neq b$ we shall use the notation $a \to b$. For any $a \in G$ let $U(a) = \{x | a \leq x\}$ and $L(a) = \{y | y \leq a\}$ denote the set of upper and lower bounds, respectively. If, for $a, b \in G$, there exist a $c \in U(a) \cap U(b)$ such that for every $x \in U(a) \cap U(b)$ we have $c \in L(x)$, then c is called the least upper bound of a and b and it is denoted by $a \vee b$. The greatest lower bound $a \wedge b$ is defined dually. If every pair of elements in G has least upper bound and greatest

[3] It is easy to see, that any homomorphic image of a tournament is isomorphic to a subalgebra of the given tournament.

lower bound, then $\mathcal{G} = \langle G, \leq \rangle$ is called a *Weakly Associative Lattice* (WAL for short). Obviously, a WAL is a lattice if and only if the relation $\leq$ is transitive.

We can give an equational form of these operations. They are idempotent, commutative, satisfy the absorption laws, just as in the case of lattices. But here we only have a weaker form of associativity (this explains the name):

$$x \vee [(x \wedge y) \vee (x \wedge z)] \;=\; x \wedge [(x \vee y) \wedge (x \vee z)] \;=\; x.$$

These equations imply that the relation $x \leq y$: $x \vee y = y$ gives a WAL in the above sense. Moreover, the WAL $\langle G; \leq \rangle$ and the algebra $\langle G; \vee, \wedge \rangle$ uniquely determine each other. Therefore, we may refer to these algebras as WALs, as well. Obviously, tournaments are WALs. The variety of WALs will be denoted by $\mathcal{W}$, and the variety of lattices by $\mathcal{L}$.

As in $\mathcal{L}$, the term $m(x,y,z) = [(x \vee y) \wedge (x \vee z)] \wedge (y \vee z)$ is a majority term, hence, $\mathcal{W}$ is *congruence distributive* (CD for short).

Several attempts were made to find an "appropriate" generalization of the variety $\mathcal{D}$ of distributive lattices within $\mathcal{W}$. The idea, that they satisfy the *Congruence Extension Property* (CEP for short), used in [10] turned out to be the best. In [3], [4], [11] many properties were discussed, and it was proved that they are equivalent to CEP.

In [10] a variety $\mathcal{T}$ was described, which covers $\mathcal{D}$. $\mathcal{T}$ is generated by the "triangle" $\mathfrak{T}$. It has three elements, satisfying $a \to b \to c \to a$. It is subdirectly irreducible, just as the two-element lattice.

None of the subdirectly irreducible members of subvarieties of $\mathcal{W}$ satisfying CEP contain a three-element chain, i.e., elements satisfying $a \to b \to c$ and $a \to c$. This property turned out to be equivalent to the *Unique Bound Property* (UBP for short). UBP means, that for distinct a and b both $U(a) \cap U(b)$ and $L(a) \cap L(b)$ have a single element. The WALs satisfying UBP will be called UBP, themselves. $\mathbf{U}$ will denote the class of all UBP and $\mathcal{U}$ will denote the variety, generated by all UBP.

As UBP can be defined by a first order universal sentence, $\mathbf{U}$ is closed under ultra-products and subalgebras. Since every UBP is simple, $\mathbf{U}$ is closed under proper homomorphic images, as well. Thus, all *Subdirectly Irreducible* (SI for short) members of $\mathcal{U}$ are in $\mathbf{U}$.

It was shown in [5] that $\mathbf{U}$ consists of all SI members of $\mathcal{W}$ satisfying CEP. It was proved in [12], that $\mathcal{U}$ is the largest subvariety of $\mathcal{W}$ enjoying CEP. In the same paper it was proved, as well, that $\mathcal{U}$ is the largest variety having *Restricted Equationally Definable Principal Congruences* (REDPC for short). This means, that there exists a (finite) set $\Sigma(x,y,u,v) = \{p_i(x,y,u,v) = q_i(x,y,u,v)\}$ of equations such that for $a,b,c,d \in \mathfrak{A} \in \mathcal{U}$ the congruence $c \equiv d\big(\Theta(a,b)\big)$ holds if and only if $\Sigma(a,b,c,d)$ is satisfied.

One more very important feature of $\mathcal{U}$ was discovered in [13]. A ternary function $g(x,y,z)$ is called the *dual discriminator* on a set A if

$$g(a,b,c) = \begin{cases} a & \text{if} \quad a = b \\ c & \text{if} \quad a \neq b \end{cases} \qquad \text{for} \quad a,b,c \in A.$$

A variety $\mathcal{V}$ is called a *dual discriminator variety* if there exists a term $q(x,y,z)$ in the language of $\mathcal{V}$ which yields the dual discriminator on each SI member of $\mathcal{V}$. In this case $q(x,y,z)$ is called a dual discriminator term. Dual discriminator terms are always majority terms. Dual discriminator varieties have REDPC.

$\mathcal{D}$ is the maximal dual discriminator variety of $\mathcal{L}$, with the majority term $m(x,y,z)$ as the dual discriminator term. $\mathcal{T}$ is a dual discriminator variety, as well. Here both

$$t(x,y,z) = \{[(z \wedge x) \vee y] \wedge x\} \vee (z \wedge y)$$

and its "dual"

$$t^*(x,y,z) = \{[(z \vee y) \wedge x] \vee y\} \wedge (z \vee x)$$

are dual discriminator terms, yielding the dual discriminator on the two-element lattice, as well as on the triangle (but not on other WALs).

$\mathcal{U}$ is, also, a dual discriminator variety, in fact the largest in $\mathcal{W}$. Here the construction of the dual discriminator term is more complicated. Let $u(x,y,z) = t(t(x,y,z), t^*(x,y,z), z)$. Then

$$q(x,y,z) = t(u(x, x \vee y, z), u(y, x \vee y, z), z)$$

is a dual discriminator term, yielding the dual discriminator function on each UBP.

In [14] a graph-theoretic description of UBPs, with at least three elements were given. Every UBP determines a projective plane, but distinct UBPs may give the same plane. There are two kinds of "UBP-graphs". Singular UBPs consist of a set X and two more elements 0 and 1, such that $0 \to 1$ and for every $x \in X$ the relations $1 \to x \to 0$ hold. Regular UBPs have the following property: There exists a cardinal λ, depending on the UBP only, such that for each element a in the underlying set of the UBP, $|U(a)| = |L(a)| = \lambda$ is satisfied. $\mathfrak{T}$ is the only UBP which is both singular and regular.

A directed graph $\mathcal{G} = \langle G, \leq \rangle$ is called a *partial UBP* if, for each $a, b \in G$ both $U(a) \cap U(b)$ and $L(a) \cap L(b)$ have at most one element. In [7] the following procedure was given to produce a UBP from a partial UBP.

Let $\mathcal{G} = \langle G, \leq \rangle$ be any partial UBP and consider all pairs $a, b \in G$ having no common upper bounds. Then extend G by new elements $[a, b, \vee]$, and for each pair $c, d \in G$ having no common lower bounds extend G by new elements

$[c,d,\wedge]$. Then add the new relations $a \to [a,b,\vee]$, $b \to [a,b,\vee]$, $[c,d,\wedge] \to$ $\to c$, and $[c,d,\wedge] \to d$. This way we get a new partial UBP $\mathcal{G}'$, where each pair of elements in G has both (unique) upper bound and (unique) lower bound. Starting with a given partial UBP $\mathcal{G}_0$ we can construct $\mathcal{G}_{n+1} = (\mathcal{G}_n)'$ for each $n \in \{0,1,\dots\}$. G_n will denote the underlying set of $\mathcal{G}_n$. Their union $\mathcal{G}^*$ with underlying set G^* is, obviously, a UBP, called the UBP *freely constructed* from $\mathcal{G}$. (This is obviously not a free algebra.)

2. Patterns in U

We start by observing some patterns in elements of $\mathbf{U}$, which will be needed in the sequel. Consider the free extension of a graph $\mathcal{G}_0$ with underlying set G_0, as defined in the previous section. The elements in G_0 will be called *elements of rank* 0 and the elements of $G_n \setminus G_{n-1}$ *elements of rank n*. $r(a)$ will denote the rank of a.

PROPOSITION 2.1. *Suppose $a \in G_{n+1} \setminus G_n$, then either $U(a) \cap G_n$ or $L(a) \cap G_n$ is empty, and the other has exactly two elements.*

PROOF. The statement follows immediately from the construction of the freely constructed UBP. ∎

DEFINITION 2.2. Let $\mathcal{G}$ be any directed graph and $a,u,v,w \in \mathcal{G}$. We shall say that a is oppressed by the elements u,v,w, if either $u \to a$, $v \to a$ and $w \to a$ or $a \to u$, $a \to v$ and $a \to w$. We shall say that a is oppressed by u,v if $u \to a \to v$.

We shall say that a finite directed graph is oppressed if every element of the graph is oppressed (in the graph).

We shall say that a directed graph is oppressed if it is the disjoint union of finite oppressed subgraphs.

PROPOSITION 2.3. *Suppose the element $a \in \mathcal{G}^*$ is oppressed by a set S. Then, at least one of the elements in S has greater rank than a.*

If a complete subgraph $\mathcal{S}$ is oppressed, then its underlying set S is contained in G_0.

PROOF. The first statement follows immediately from Proposition 2.1.

If $\mathcal{S}$ is finite, then it must contain at least one element of maximal rank. By the first statement of our proposition this rank must be 0.

In the general case every component is contained in G_0, hence the last statement follows. ∎

DEFINITION 2.4. Let P be a finite set, $\rightarrow \subseteq P \times P$ and $\not\rightarrow \subseteq P \times P$ disjoint subsets of $P \times P$, such that none of them contains the diagonal. The relational set $\mathcal{P} = \langle P| \rightarrow, \not\rightarrow\rangle$ will be called a pattern and $\mathcal{P} = \langle P| \rightarrow\rangle$ will be called a positive pattern.

A set $\mathcal{P}_{\mathbb{I}} = \{\mathcal{P}_i | i \in \mathbb{I}\}$ of positive patterns will be called disjoint if no member of $\mathcal{P}_{\mathbb{I}}$ is a subgraph of some other member.

A graph $\langle G| \rightarrow\rangle$ contains pattern $\mathcal{P}$ if there exists an embedding $\varphi\colon P \rightarrow G$, such that for $a, b \in P$ the relation $a \rightarrow b$ or the relation $a \not\rightarrow b$ implies that $\varphi(a) \rightarrow \varphi(b)$ holds or $\varphi(a) \rightarrow \varphi(b)$ does not hold, respectively.

A graph $\langle G| \rightarrow\rangle$ contains positive pattern $\mathcal{P}$ if there exists an embedding $\varphi\colon P \rightarrow G$, such that for $a, b \in P$ the relation $a \rightarrow b$ implies that $\varphi(a) \rightarrow \varphi(b)$ holds.

A graph $\mathcal{G}$ is free of the (positive) pattern $\mathcal{P}$ if it does not contain pattern $\mathcal{P}$.

PROPOSITION 2.5. *Let $\mathcal{P}_{\mathbb{I}} = \{\mathcal{P}_i | i \in \mathbb{I}\}$ be a set of patterns. Let $\mathcal{G}$ be the ultraproduct of the graphs $\{\mathcal{G}_\lambda | \lambda \in \Lambda\}$. If each $\mathcal{G}_\lambda$ is free of every $\mathcal{P}_i$, then so is $\mathcal{G}$.*

If a UBP is free of the positive pattern in $\mathcal{P}_{\mathbb{I}}$ so is the homomorphic image of every subalgebra.

Let $\{\mathfrak{A}_\lambda | \lambda \in \Lambda\}$ be a collection of UBP, free of the positive pattern in $\mathcal{P}_{\mathbb{I}}$. Then each subdirectly irreducible member in the variety generated by them is free of these patterns, as well.

PROOF. Any pattern can be formulated by a first order sentence. Hence, an ultraproduct contains a pattern if and only if almost all components have it, proving our first statement for a single pattern. This, obviously, yields the statement for any set of patterns.

If a UBP $\mathfrak{A} \leq \mathfrak{B}$ contains a positive pattern, $\mathfrak{B}$ comtaines it, as well[4]. As every UBP is simple, i.e., every nontrivial homomorphic image is an isomorphic copy of it, the second statement holds, too.

These, together with Jónsson's Lemma yield the last statement. ∎

Observe, that the number of patterns is countable, so we may assume, that $\mathbb{I}$ is a subset of positive integers.

DEFINITION 2.6. Let $\mathcal{P}_{\mathbb{I}}$ be any set of positive patterns. Let $\mathbf{U}_{\mathcal{P}_{\mathbb{I}}}$ denote the class of all UBP, which are free of $\mathcal{P}_{\mathbb{I}}$. $\mathcal{U}_{\mathcal{P}_{\mathbb{I}}}$ denote the variety generated by $\mathbf{U}_{\mathcal{P}_{\mathbb{I}}}$.

[4]We need, in general, that the pattern is positive, because $a \rightarrow b$ might be true in a graph, but not in a subgraph.

PROPOSITION 2.7. *Let $\mathcal{P}_{\mathbb{I}}$ be any given set of disjoint positive patterns. For any distinct $\mathbb{J}, \mathbb{K} \subseteq \mathbb{I}$ the varieties $\mathcal{U}_{\mathcal{P}_{\mathbb{J}}}$ and $\mathcal{U}_{\mathcal{P}_{\mathbb{K}}}$ are different. If $\mathbb{I}$ is infinite, the number of subvarieties of $\mathcal{U}_{\mathcal{P}_{\mathbb{I}}}$ is continuum.*

PROOF. The first statement follows immediately from the last statment of proposition 2.5. The second stetement follows from the fact, that $\mathbb{I}$ is countable and the number of subsets of a countable set is continuum. ∎

For $n > 1$ a finite graph $\mathfrak{P}_n = \langle \{x_0, x_1, \ldots, x_n\} | \to \rangle\}$ of distinct elements will be called a *path of length n*, provided $x_0 \to x_1 \to \ldots \to x_n$.

For $n > 1$ a finite graph $\mathfrak{C}_n = \langle \{x_0, x_1, \ldots, x_n\} | \to \rangle\}$ of distinct elements will be called a *cycle of length n*, provided $x_0 \to x_1 \to \ldots \to x_n \to x_0$. Since any cycle is oppressed, we get, immediately

COROLLARY 2.8. *Any cycle in G^* is contained in G_0.* ∎

Consider the cycle $\mathfrak{C}_n$ of length n as a partial UBP. According to Corollary 2.8 we have:

PROPOSITION 2.9. *$\mathfrak{U}_n = \mathfrak{C}_n^*$ contains a unique cycle, namely $\mathfrak{C}_n$.* ∎

Proposition 2.7 and Proposition 2.9 yield immediately:

COROLLARY 2.10. *Let $\mathbb{I}$ be any subset of the set $\mathbb{N}_2$ of integers greater than 2. Consider the set $\mathcal{C}_{\mathbb{I}}$ of patterns consisting of all cycles of length $n \in \mathbb{I}$. Let $\mathbf{U}_{\mathcal{C}_{\mathbb{I}}}$ be the class of all UBP which are free of all cycles in $\mathcal{C}_{\mathbb{I}}$ and $\mathcal{U}_{\mathcal{C}_{\mathbb{I}}}$ be the variety generated by $\mathbf{U}_{\mathcal{C}_{\mathbb{I}}}$. These varieties are all distinct for different subsets $\mathbb{I}_1$ and $\mathbb{I}_2$. The number of these varieties are continuum.* ∎

Let us remark the following: If $\mathbb{I} = \emptyset$, then no UBP is excluded, therefore the variety $\mathcal{U}_{\mathcal{C}_{\emptyset}} = \mathcal{U}$. In case $\mathbb{I} = \mathbb{N}_2$ all UBPs are excluded, which contain a cycle. If we have no relations in the starting graph, but it has two vertices, we get an infinite UBP. It was shown in [6] that this UBP generates a variety $\mathcal{U}_\infty \leq \mathcal{U}_{\mathbb{N}_2}$, covering $\mathcal{D}$.

THEOREM 2.11. *Almost all $\mathcal{U}_{\mathcal{C}_{\mathbb{I}}}$ are non-finitely based.*

PROOF. Since the number of finitely based subvarieties is countable our assertion holds by Corollary 2.10. ∎

Observe, that the same argument holds if instead of $\mathbb{N}_2$ we start with any infinite subset of $\mathbb{N}_2$.

3. Non-finitely based varieties

We start this section with some results of GÁBOR BRAUN in [2]. His results will be listed as B.1., B.2., and B.3.. Let $\mathcal{A} = \langle A| \to\rangle$ be any finite directed graph. Let $v = v(\mathcal{A}) = |A|$ denote the number of vertices of $\mathcal{A}$ and $e = e(\mathcal{A})$ denote the number of edges in $\mathcal{A}$ (i.e., the number of pairs $a, b \in A$ such that $a \to b$ hold). We shall call the number $\mu = \mu(\mathcal{A}) = 2v - e$ the *weight* of $\mathcal{A}$.

Consider a finite graph $\mathcal{G}$ and the graph $\mathcal{G}^*$ defined in section 1. A finite subgraph $\mathcal{B} \leq \mathcal{G}^*$ will be called a *generating graph* of $\mathcal{G}^*$ if $\mathcal{G}^*$ is the least UBP in $\mathcal{G}^*$, containing $\mathcal{B}$. A generating graph $\mathcal{A}$ *freely generates* $\mathcal{G}^*$ if the natural embedding $\mathcal{A} \to \mathcal{G}^*$ has a unique extension to a graph-isomorphism $\mathcal{A}^* \to \mathcal{G}^*$.

A generating graph $\mathcal{I} \leq \mathcal{G}^*$ of size n will be called *independent (of size n)* if $e(\mathcal{I}) = 0$ and $v(\mathcal{I}) = n$.

THEOREM B.1. *Let $\mathcal{A}$ and $\mathcal{B}$ two generating graphs of $\mathcal{G}^*$. If $\mathcal{A}$ freely generates $\mathcal{G}^*$, then $\mu(\mathcal{A}) \leq \mu(\mathcal{B})$ (and equality holds if and only if $\mathcal{B}$ freely generates $\mathcal{G}^*$, as well).*

COROLLARY B.2. *If, for a cycle $\mathfrak{C}_n$ of length n, an independent $\mathcal{I}$ of size k is a generating graph of $\mathfrak{C}_n^*$, then $2k \geq n$.*

THEOREM B.3. *Let $\mathfrak{P}_n$ be a path of even length $n > 3$. Then $\mathfrak{P}_n^*$ has an independent generating graph.*

Now, we are going to prove a considerable generalization of Theorem 2.11:

THEOREM 3.1. *For infinite $\mathbb{I}$, $\mathcal{U}_{\mathcal{C}_\mathbb{I}}$ is non-finitely based.*

PROOF. Consider any finite set $\Sigma = \Sigma(x_1, \ldots, x_k)$ of identities. Choose any $n \in \in \mathbb{I}$ such that $n \geq 2k$. The mapping of the k free generators into $\mathfrak{C}_n^*$ can not generate this algebra, by Corollary B.2, hence $\mathfrak{C}_n^*$ does not satisfy Σ. On the other hand $\mathfrak{C}_n^* \in \mathcal{U}_{\mathcal{C}_1}$. Indeed, by Proposition 2.9, $\mathfrak{C}_n^*$ does not contain $\mathfrak{C}_m^*$ for $n < m \in \mathbb{I}$. This means Σ does not define this variety. ∎

COROLLARY 3.2. *$\mathcal{U}_{\mathcal{C}_{\mathbb{N}_2}}$ is non-finitely based.* ∎

THEOREM 3.3. *$\mathcal{U}_\infty$ is non-finitely based.*

PROOF. Consider any finite set $\Sigma = \Sigma(x_1, \ldots, x_k)$ of identities, satisfied in $\mathcal{U}_\infty$. Choose any even $n \in \mathbb{N}_2$ such that $n > 2k$. Let $\mathfrak{A} \leq \mathfrak{C}_n^*$ be a proper subalgebra. Then, $\mathfrak{C}_n \not\leq \mathfrak{A}$. Hence, there is a path $\mathfrak{P}_n \leq \mathfrak{C}_n$ such that the subalgebra $\mathfrak{B}$ generated by $\mathfrak{P}_n$ contains $\mathfrak{A}$. By Theorem B.3., $\mathfrak{B}$ has an independent generating graph.

It is easy to see, that in this case $\mathfrak{B}^*$ generates the variety $\mathcal{U}_\infty$.

Thus, $\mathfrak{C}_n^*$ satisfies Σ, but it is not in the variety generated by $\mathcal{U}_\infty$ ∎

4. Patterns and terms

In this section we aim to give terms, to exclude given patterns. We shall use a "peculiar" phenomenon of the dual discriminator, introduced in [8].

Suppose, we are given three terms $f = f(x_1, \dots, x_n)$, $g = g(x_1, \dots, x_n)$ and $h = h(x_1, \dots, x_n)$ in a dual discriminator variety $\mathcal{V}$, and let $q(x, y, z)$ denote the dual discriminator term. Consider any subdirectly irreducible member $\mathfrak{S}$ of the subvariety $\mathcal{V}'$ of $\mathcal{V}$, defined by the identity $q(f, g, h) = h$. For any sequence $a_1, \dots, a_n$ of elements in $\mathfrak{S}$ let $u = f(a_1, \dots, a_n)$, $v = g(a_1, \dots, a_n)$ and $w = h(a_1, \dots, a_n)$.

If $u = v$, then $u = q(u, v, w) = w$ yields $u = w$. Conversely, if $\mathfrak{S}$ is any subdirectly irreducible member of $\mathcal{V}$, satisfying the condition: "whenever $u = v$, we have $u = w$", then the initial identity must hold, i.e., it belongs to the subvariety.

Now, consider the variety $\mathcal{U}$ with the dual discriminator term $q(x, y, z)$ and define the following terms in variables $x_0, x_1, \dots, x_n, \dots$:

Let $y_0 = x_0$, and for the integers $i \geq 0$, $y_{i+1} = x_{i+1} \vee y_i$. Consider the identities:

$$q(x_0, x_0 \vee y_i, y_i) = y_i \qquad \text{for} \qquad i > 1. \qquad \#_i$$

Proposition 4.1. *A UBP $\mathfrak{A}$ satisfies the identity $\#_i$ if and only if it is free of n-cycles, for $n \leq i + 1$.*

Proof. Suppose, the UBP $\mathfrak{A}$ satisfies $\#_i$ and has elements $a_0 \leq a_1 \leq \dots a_i \to \to a_0$. Substituting x_j by a_j, for $0 \leq j \leq i$, we get, according to $\#_i$, that

$$a_0 = q(a_0, a_0, a_i) = q(a_0, a_0 \vee a_i, a_i) = a_i,$$

a contradiction.

Conversely, for any $a_0, \dots, a_i$, let $b_0 = a_0$, and for $0 \leq j < i$, let $b_{j+1} = a_{j+1} \vee b_j$. Then, we have $a_0 \leq \dots \leq a_i$ and $q(a_0, a_0 \vee a_i, a_i) = a_i$ implies $a_0 \neq a_0 \vee a_i$, i.e., $\mathfrak{A}$ does not contain a cycle of length at most $i + 1$. ∎

Theorem 4.2. *The identities $\#_i$ define the (non-finitely based) subvariety of $\mathcal{U}$ generated by all UBP, containing no cycles.*

Proof. Using proposition 4.1, it is enough to show, that if a member of $\mathcal{U}$ does not satisfy these identities, it is not in this subvariety. This, in turn follows from the fact that in this case at least one subdirectly irreducible component does not satisfy at least one of these identities. ∎

REMARK. Observe, that if the identity $\#_{i+1}$ is satisfied so is $\#_i$. Hence, this system of identities is not independent.

If all the cycles are excluded, then, obviously, all finite UBPs are excluded, as well. The following, still unsolved question was raised some 25 years ago:

PROBLEM. *Does there exist a finite UBP, containing no triangle?*

This is obviously equivalent to the question, whether $\#_2$ excludes all finite UBPs. We are going to prove:

PROPOSITION 4.3. *$\#_3$ excludes all finite UBPs.*

In fact, we prove more:

PROPOSITION 4.4. *Let a be any vertex of a finite UBP $\mathfrak{A}$ (having more then two elements). Then there are elements $a \leq b \leq c \to a$ in $\mathfrak{A}$.*

For $a \to b$ in $\mathfrak{A}$ there are elements $b \leq c \leq d \leq e \leq a$ in $\mathfrak{A}$.

PROOF. For singular UBPs (see the introduction) the statement is obvious. For regular ones, by [14] there exists a natural number r, such that $|U(x)| = |L(x)| = r$, for every $x \in \mathfrak{A}$. Since $x \cup U(x)$ can be considered as the lines of the defined projective plane, we have $|\mathfrak{A}| = r^2 + r + 1$.

Let $U^2(x)$ be the set consisting of all $z \in \mathfrak{A}$, for which there exists an element $y \in U(x)$, such that $y \to z$. Since $\mathfrak{A}$ contains no three-element chain, $U^2(x)$ is disjoint to $U(x)$. The intransitivity of $\to$ implies $x \notin U^2(x)$.

Now, we are going to count the number of elements of $U^2(x)$. Let $U(x) = \{y_1, \dots, y_r\}$. The number of upper bounds of y_1 is r. Counting the upper bounds of y_2, we get exactly $r - 1$ new elements, because we have already counted $y_1 \vee y_2$. Consider, recursively, the new elements in $U(y_i)$. This set itself has r elements. However, $y_1 \vee y_i, y_2 \vee y_i$, etc. were, already, counted. Since they need not be distinct, we know, only, that the number of newly added elements is at least $r - i + 1$. Hence, $|U^2(x)| \geq \binom{r+1}{2}$. Therefore, $|U(x) \cup U^2(x)| \geq \frac{r^2+3r}{2}$. Similarly, for any x', we get $|L(x') \cup L^2(x')| \geq \frac{r^2+3r}{2}$. As $r^2 + 3r > r^2 + r$, we get that $[U(x) \cup U^2(x)] \cap [L(x') \cup L^2(x')]$ is never empty, and this gives the desired result. ∎

5. Identities for subvarieties excluding cycles

We have already, given an equational basis for the subvariety generated by UBPs containing no cycles. For the cases, when infinitely many cycles are excluded, we proved that these varieties are not finitely based, however, we gave no equational basis, yet.

The method used in the previous section is, obviously, not applicable. To find a basis, however, it is sufficient to find a finite set of identities, excluding a cycle of given length. Firstly, we shall start with any dual discriminator variety.

PROPOSITION 5.1. *Let $\mathcal{Q}$ be any dual discriminator variety with $q(x,y,z)$ the dual discriminator term. Then, Baker's Principal Intersection Property [1] holds, in fact:*

$$\Theta(a,b) \cap \Theta(c,d) = \Theta(q(a,b,c), q(a,b,d)).$$

Moreover, for any natural n there exist terms $c_n = c_n(u_1,\ldots,u_n,v_1,\ldots,v_n)$, and $d_n = d_n(u_1,\ldots,u_n,v_1,\ldots,v_n)$ such that

$$\bigcap_{1\le i\le n} (\Theta(u_i,v_i)) = \Theta(c_n(u_1,\ldots,u_n,v_1,\ldots,v_n), d_n(u_1,\ldots,u_n,v_1,\ldots,v_n)).$$

PROOF. It is enough, to prove the first identity for subdirectly irreducible members $\mathfrak{S}$ of $\mathcal{Q}$. If $a = b$, then we have the least congruence on both sides, since in this case $q(a,a,x) = a$, for all $x \in \mathfrak{S}$. Otherwise, $\Theta(a,b)$ is the greatest congruence ($\mathfrak{A}$ is simple) and $q(a,b,x) = x$, for all $x \in \mathfrak{S}$. Thus, we have $\Theta(c,d)$ on both sides.

The second statement will be proved by induction on n. For $n = 1$, the statement is trivial ($c_1 = u_1$ and $d_1 = v_1$). For $n = 2$ this is a restatement of Baker's Theorem ($c_2 = q(u_2,v_2,u_1)$ and $d_2 = q(u_2,v_2,v_1)$).

Suppose, the terms c_n and d_n are, already, given.
Define $c_{n+1} = q(c_n,d_n,u_{n+1})$ and $d_{n+1} = q(c_n,d_n,v_{n+1})$. Using the case $n = 2$, a routine calculation shows that $\Theta(c_{n+1},d_{n+1}) = \Theta(c_n,d_n) \cap \Theta(u_{n+1},v_{n+1})$ holds. ∎

Let us remark, that the given congruence equation holds after any substitutions $u_i \mapsto a_i$ and $v_i \mapsto b_i$, due to Maltsev's Lemma.

THEOREM 5.2. *Let $\mathcal{Q}$ be a dual discriminator variety with a dual discriminator $q(x,y,z)$. Then, there are terms*

$$\begin{aligned}
q^*_{n,t} &= q^*_{n,t}(x_0,x_1,\ldots,x_n,y_0,y_1,\ldots,y_n),\\
p^*_{n,t} &= p^*_{n,t}(x_0,x_1,\ldots,x_n,y_0,y_1,\ldots,y_n)\\
q_{n,t} &= q_{n,t}(x,y,z_1,\ldots,z_n),\\
p_{n,t} &= p_{n,t}(x,y,z_1,\ldots,z_n)
\end{aligned}$$

for $t \in \{1,2\}$, such that for every $a_0,a_1,\ldots,a_n,b_0,b_1,\ldots,b_n$ and for every $a,b,e_1,\ldots,e_n$ in any subdirectly irreducible $\mathfrak{S} \in \mathcal{Q}$ the equations

$$q^*_{n,t}(a_0,a_1,\ldots,a_n,b_0,b_1,\ldots,b_n) = p^*_{n,t}(a_0,a_1,\ldots,a_n,b_0,b_1,\ldots,b_n)$$

hold if and only if the equality $a_0 = b_0$ implies at least one of the equalities $a_i = b_i$ $(0 < i \le n)$, and the equations

$$q_{n,t}(a,b,e_1,\ldots,e_n) = p_{n,t}(a,b,e_1,\ldots,e_n)$$

hold if and only if the equality $a = b$ implies at least one of the equalities $e_i = e_j$ $(0 < i < j \le n)$.

PROOF. First, we prove, that it is enough to deal with the terms q^* and p^*. Consider q_n and let $k = \binom{n}{2}$ and choose q^*_k. For each $1 \le \ell \le k$ consider a pair $i < j$ $(1 \le i,j \le n)$. Choose $x_\ell = z_i$ and $y_\ell = z_j$. Since the pairs x_ℓ, y_ℓ cover all the pairs z_i, z_j, the statement follows.

In what follows, we shall need a result from [8].

If $q(x,y,z)$ is a dual discriminator term of the variety $\mathcal{Q}$, then for every $a,b,c,d \in \mathfrak{A} \in \mathcal{Q}$ the congruence $c \equiv d(\Theta(a,b))$ holds if and only if $c = q(c,d,q(a,b,c))$ and $d = q(c,d,q(a,b,d))$ are both satisfied.

By Maltsev's lemma, if the congruence holds then it holds in any subdirectly irreducible member $\mathfrak{S}$ of the variety. If $c = d$, then the equalities hold, obviously. Otherwise, $c \ne d$ implies $a \ne b$, and we get $q(a,b,c) = c$ and $q(a,b,d) = d$, yielding the desired equalities, as q is a majority term. Conversely, suppose the equalities hold. Factor by the congruence $\Theta(a,b)$. Then for the corresponding images $b^* = a^*, c^*, d^*$ we have $c^* = q(c^*,d^*,q(a^*,a^*,c^*)) = q(c^*,d^*,a^*)$ and $d^* = q(c^*,d^*,q(a^*,a^*,d^*)) = q(c^*,d^*,a^*)$, i.e., $c^* = d^*$ therefore the desired congruence holds.

Now, we turn back to the proof of the theorem.

From the result above it follows that $c_n \equiv d_n\big(\Theta(a_0,b_0)\big)$ holds for c_n, d_n from Proposition 5.1. if and only if the equalities

$$c_n = q(c_n,d_n,q(a_0,b_0,c_n)) \text{ and } d_n = q(c_n,d_n,q(a_0,b_0,d_n))$$

are satisfied. Choose

$$p^*_{n,1} = c_n,\ q^*_{n,1} = q(c_n, d_n, q(a_0, b_0, c_n)),$$
$$p^*_{n,2} = d_n,\ q^*_{n,2} = q(c_n, d_n, q(a_0, b_0, d_n)).$$

We have to prove, that from these equalities it follows that $a_0 = b_0$ implies $a_i = b_i$ for some $0 < i \leq n$ in any subdirectly irreducible algebra. If $a_i = b_i$ does not hold for any $i > 0$, then the congruence $\Theta(a_i, b_i)$ is never the least congruence. Since every subdirectly irreducible member of a dual discriminator variety is simple, these congruences must be equal to the greatest congruence. Therefore, by proposition 5.1. $\Theta(c_n, d_n)$ is the greatest congruence. As $c_n \equiv$ $\equiv d_n\big(\Theta(a_0, b_0)\big)$ holds, $\Theta(a_0, b_0)$ is the greatest congruence, i.e., $a_0 \neq b_0$, a contradiction. ∎

THEOREM 5.3. *Let $\mathcal{U}$ be the variety generated by all UBP and let $\mathfrak{F} \in \mathcal{U}$ be the algebra freely generated by the set $x_0, \ldots, x_n, \ldots$. For each $n > 2$ there exists a pair of identities $\gamma_n = \gamma_n(x_0, \ldots, x_n)$ and $\delta_n = \delta_n(x_0, \ldots, x_n)$, which is satisfied in a UBP $\mathfrak{S}$ if and only if it does not contain an n-cycle.*

For any sequence $\mathbb{I} \subseteq \mathbb{N} \setminus \{0, 1, 2\}$ there is a system of identities $\Gamma_{\mathbb{I}} = \{\gamma_n \mid n \in \mathbb{I}\} \cup \{\delta_n \mid n \in \mathbb{I}\}$ which is satisfied in a subvariety $\mathcal{U}_{\mathbb{I}}$ if and only if no subdirectly irreducible members of the variety contain an n-cycle for $n \in \mathbb{I}$. These systems are finite if and only if $\mathbb{I}$ is finite. The elements of $\Gamma_{\mathbb{I}}$ are independent, i.e., omitting any of the pairs (γ_n, δ_n), the variety will be changed.

PROOF. Consider the terms

$$t_0 = x_0, t_1 = t_0 \vee x_1, \ldots, t_{i+1} = t_i \vee x_{i+1}, \ldots, t_n = t_{n-1} \vee x_n, \quad s = t_0 \vee t_n,$$

and (for $t = 1, 2$)

$$p_{n,t} = p_{n,t}(s_0, t_0, t_1, \ldots, t_n, s), \quad q_{n,t} = q_{n,t}(s_0, t_0, t_1, \ldots, t_n, s).$$

Define the identities

$$\gamma_n = \gamma_n(x_0, \ldots, x_n) \colon p_{n,1} = q_{n,1} \quad \text{and} \quad \delta_n = \delta_n(x_0, \ldots, x_n) \colon p_{n,2} = q_{n,2}$$

Let $\mathfrak{S}$ be subdirectly irreducible and $\pi \colon \mathfrak{F} \to \mathfrak{S}$ any projection. Further, let $a_i = \pi(t_i)$ and $b = \pi(s)$. Then, we have $a_0 \leq a_1 \leq \ldots \leq a_n \leq b$ and $a_0 \leq b$. By theorem 5.2, $a_0 = b$ implies at least one equality of the form $a_i = a_j$ for $i \neq j$. Hence, no cycle of the form $a_0 \to a_1 \to \ldots \to a_n \to b = a_0$ exists. On the other hand, if such a cycle exists, then for the appropriate projection the equality is not satisfied, i.e., γ_n, δ_n is not a pair of identities. ∎

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Ervin Fried

Loránd Eötvös University
H-1117 Budapest
Pázmány P. s. 1/c
Hungary
fried.ervin@renyi.elte.hu

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CIRCULAR QUARTICS IN ISOTROPIC PLANE OBTAINED AS PEDAL CURVES OF CONICS

By
E. JURKIN

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Abstract. In this paper circular quartics constructed as pedal curves of conics are classified depending on their position with respect to the absolute figure. It is shown that only 2, 3 and 4-circular quartics can be obtained by using this method.

1. Introduction

An *isotropic plane* $\mathcal{I}_2$ is a real projective plane where the metric is induced up to the constant by a real line f and a real point F, incidental with it, [6], [7]. The ordered pair (f, F) is called the *absolute figure* of the isotropic plane.

In the affine model of the isotropic plane where the affine coordinates of the points are given by

$$x = \frac{x_1}{x_0}, \quad y = \frac{x_2}{x_0},$$

where (x_0, x_1, x_2) are the homogeneous coordinates such that the absolute line f is determined by the equation $x_0 = 0$ and the absolute point F by the coordinates $(0, 0, 1)$.

Projective transformations that map the absolute figure into itself form a 5-parametric group $\mathcal{G}_5$. They have the equations of the form

$$\overline{x} = a + dx, \quad \overline{y} = b + cx + ey.$$

$\mathcal{G}_5$ is called the *group of similarities* of the isotropic plane, [1], [6]. Its subgroup $\mathcal{G}_3$ consisting of the transformations of the form

$$\overline{x} = a + x, \quad \overline{y} = b + cx + y,$$

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is called the *group of motions* of the isotropic plane. It preserves the quantities such as the distance between two points or the angle between two lines. So, it has been selected for the fundamental group of transformations.
The ordered pair $(\mathcal{I}_2, \mathcal{G}_3)$ is called the *isotropic geometry*.

All straight lines through the absolute point F are called isotropic lines and all points incidental with f are called isotropic points.

There are seven types of regular conics classified depending on their position with respect to the absolute figure, [1], [6]. An *ellipse (imaginary or real)* is a conic that intersects the absolute line in a pair of conjugate imaginary points. If a conic intersects the absolute line in two different real points, it is called a *hyperbola* (of *1st* or *2nd type*, depending on whether the absolute point is outside or inside the conic). A hyperbola passing through the absolute point is called a *special hyperbola* and a conic touching the absolute line is called a *parabola*. If a conic touches the absolute line at the absolute point, it is said to be a *circle*.

A curve in the isotropic plane is *circular* if it passes through the absolute point F, [10]. Its *degree of circularity* is defined as the intersection multiplicity of the curve and the absolute line f at the absolute point. If it does not share any common point with the absolute line except the absolute point, it is *entirely circular*, [5].

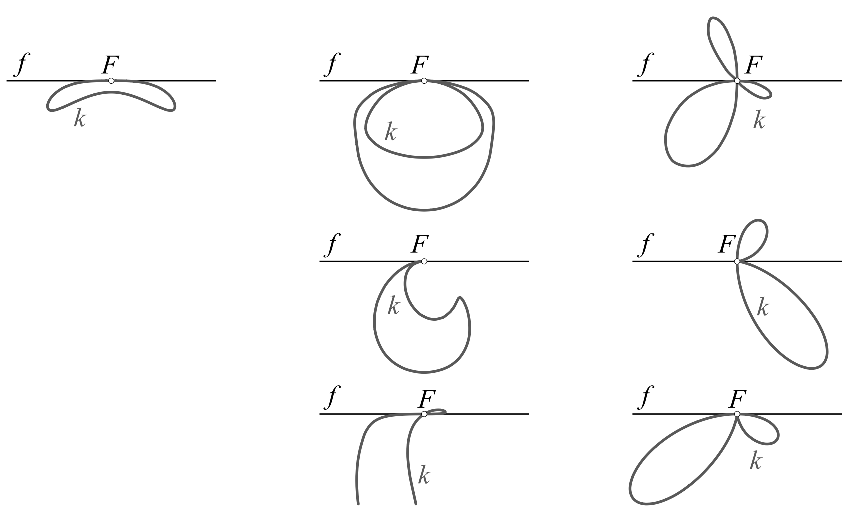

Figure 1

A circular curve of order four can be 1, 2, 3 and 4-circular. The absolute line can intersect it, touch it, osculate or hyperosculate it at the absolute point.

The absolute point can be simple, double or triple point of the curve. Due to that for each degree of circularity we distinguish several types of quartics. Just for the illustration types of entirely circular quartics are shown in Figure 1. The first column contains a quartic with a simple point at the absolute point F, the second column contains the quartics with a double point at F, and the quartics having a triple point at F are shown in the third column. The cases of isolated double points are also possible, but here they are not specially distinguished from the nodes.

In the Euclidean plane bicircular quartics (entirely circular quartics with double points in the absolute points of the plane) can be constructed as the pedal curves of an ellipse and a hyperbola, [9]. The aim of this paper is to construct circular quartics as pedal curves of conics and classify them with respect to the absolute figure. As it includes all four possible types of the pedal transformation, in that sense it is an extension of the paper [11] where only one of the types was studied.

2. Pedal transformation in isotropic plane

2.1. Definition and properties of pedal transformation

Definition 2.1. The pedal curve k_N of a given curve k with respect to a fixed conic q is the locus of the foot of the perpendicular to the tangent of the given curve k from the pole of the tangent with respect to the conic q, [10].

The conic q is called the *fundamental conic* of the pedal transformation. The polar line p of the absolute point F, called also the *pedal point*, with respect to the fundamental conic q intersects the conic in the points P_1 and P_2 (foci of the conic q). Their polar lines p_1 and p_2 are isotropic lines. The points $P = F$, P_1 and P_2 are called *fundamental points* and the lines p, p_1 and p_2 are called *fundamental lines* of the pedal transformation, [10].

The construction of the pedal curve k_N of the curve k should be done in the following way: Let t be a tangent of the curve k and let T be its pole with respect to the fundamental conic q. The isotropic line n through T meets t in the point T_N lying on the claimed curve, Figure 2.

It has been shown in the paper [10] that the pedal curve of a conic is a quartic.

There are two tangents l_1, l_2 of the conic k passing through the absolute point F. Their poles L_1, L_2 with respect to the fundamental conic lie on the line

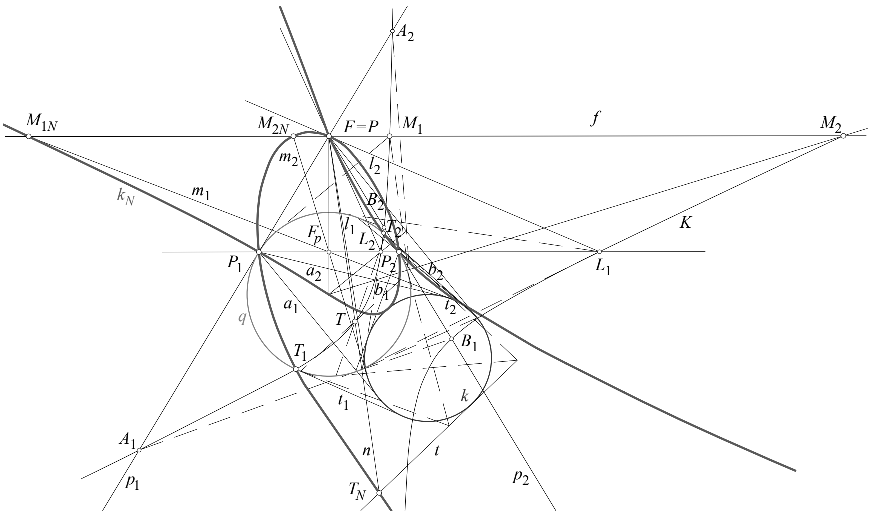

Figure 2

p. Isotropic lines through L_1, L_2 intersect the tangents l_1, l_2 in the point F. Therefore, the pedal point is double point of the quartic k_N and the normal lines FL_1 and FL_2 are tangents of k_N at the point F.

Two tangents a_1, a_2 of the conic k can be drawn from the fundamental point P_1. Their poles lie on the fundamental line p_1 being at the same time the normal of the tangents a_1, a_2. It follows that P_1 is a double point of the curve k_N. The same applies to P_2.

The construction of the tangents at the double points can be adopted from [3] where it has been done for the pedal transformation in the Euclidean plane. The equalities $(p, t_{a1}, p_1, a_1) = -1$, $(p, t_{a2}, p_1, a_2) = -1$ and $(p, t_{b1}, p_2, b_1) = -1$, $(p, t_{b2}, p_2, b_2) = -1$ hold, where t_{a1}, t_{a2} and t_{b1}, t_{b2} are the tangents of the curve k_N at the points P_1 and P_2, respectively, b_1 and b_2 are the tangents of k from P_2.

The quartic k_N has a node, a cusp or an isolated double point in the fundamental point depending on whether P is outside, on or inside the conic k. In other words, the type of the double point depends on the type of the tangents passing through it.

The fourth order curve with three double points is a curve of the sixth class. If some of the double points are cusps, the class is reduced and equals $6 - c$, where c is the number of cusps, [8], [9].

The poles T_1, T_2, T_3, T_4 of the common tangents of q and k are also their normal's pedals. So, they lie on the constructed quartic.

The curve q of order four and the curve k of order two have eight common points. Consequently, there are no other intersections beside those four points and twice counted fundamental points P_1, P_2 (points with intersection multiplicity 2).

If a fundamental line is a tangent of the conic k, the constructed quartic will split into the line and a cubic. In order to obtain an irreducible quartic, k must not touch any of the fundamental lines. Let K be the locus of the pole of the tangent of the conic k with respect to the conic q. The conic K should not contain any fundamental points.

There are two tangents m_1, m_2 of the conic k passing through the pole F_p of the absolute line f (the center of q). Their poles M_1, M_2 lie on the absolute line f being therefore the normal line of the tangents. Pedal points M_{1N}, M_{2N} together with the twice counted absolute point F are all common points of the quartic k_N and the line f. If the point F_p is located in the interior of the conic k, M_{1N}, M_{2N} form a pair of the conjugate imaginary points. If F_p lies on k, those two points coincide and form a contact point with the absolute line.

After connecting contact points of the tangents t of k with their poles $T \in K$ a curve of class four is obtained. Counting multiplicities, four tangent lines of the curve pass through every point of the plane. In particular, this holds for the pedal point P. One intersection of each of those four lines with the conic k is the contact point of the conic and the curve k_N.

A quartic can also be obtained as a pedal curve of a curve k of order higher than two. But in that case it is necessary for k to touch some of the fundamental lines. For example, curve k of order three touching two fundamental lines is transformed into a curve of order four. We will consider the quartics obtained as conic pedal curves.

2.2. Circular quartics obtained by pedal transformation

Since the absolute point F is the pedal point and due to this fact a double point of the pedal curve k_N of the conic k, only quartics the degree of circularity of which is at least 2 can be obtained by the derivation

It is necessary to determine the conditions that the fundamental elements and the conic k have to fulfill in order to obtain a circular quartic of a certain type.

Beside the pedal point $P = F$ two fundamental points P_1 and P_2 are double points of the curve k_N. If the pedal point lies on the fundamental conic, three fundamental points fall into one, being a singularity of higher order, [8], and four times counted intersection of the fundamental line $p = p_1 = p_2$ and the quartic k_N.

We distinguish four types of pedal transformation:

(1) The fundamental conic q is a 0-circular ellipse or hyperbola.
Three fundamental points are different and tree fundamental lines are different, too.

(2) The fundamental conic q is a 0-circular parabola.
Three fundamental points are different and three fundamental lines are different, too. The absolute line is one of the fundamental lines.

(3) The fundamental conic q is a 1-circular special hyperbola.
Three fundamental points coincide with the pedal point F and three fundamental lines coincide with the line p different from the absolute line.

(4) The fundamental conic q is 2-circular, i.e., a circle.
Three fundamental points coincide with the pedal point F and three fundamental lines coincide with the absolute line f.

Table 1 contains a complete list of circular quartics that could be generated as a conic pedal curve in the isotropic plane. For example, the 2-circular quartics that can be obtained by the pedal transformation of the type (1) are shown in the first column and the first row. Two main types are possible, either the quartic has a double point at F and intersects f at two different further points, or the quartic has a double point at F and touches f at a further point. The double point can be a node, an isolated double point or a cusp. In Table 1 only the cases of nodes and cusps are presented.

The following theorem is valid:

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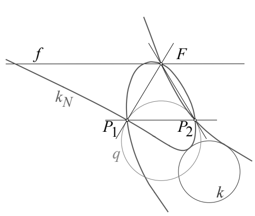
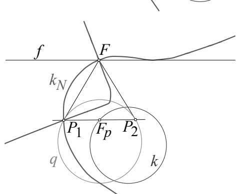
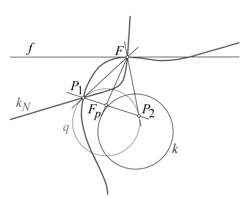
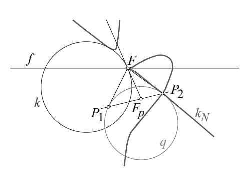
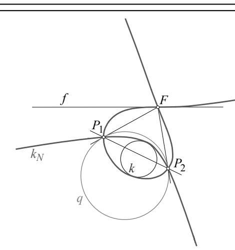
<tr><td></td><td>


</td><td>


</td><td>
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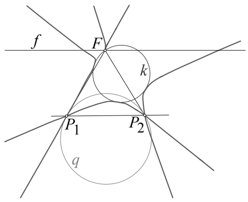
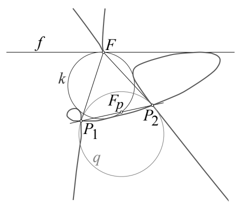
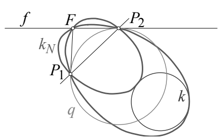
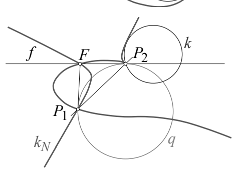
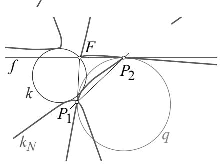
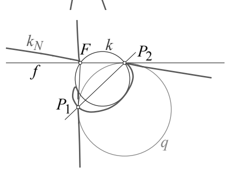
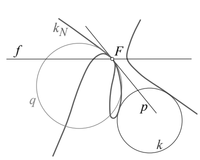
<tr><td>type 1</td><td>
</td><td></td><td></td></tr>
<tr><td>type 2</td><td>


</td><td></td><td></td></tr>
<tr><td></td><td></td><td></td><td></td></tr>
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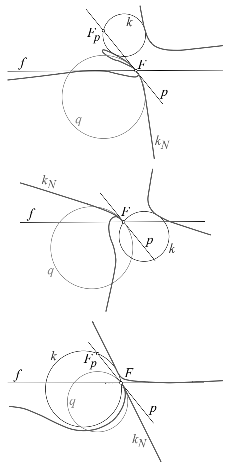
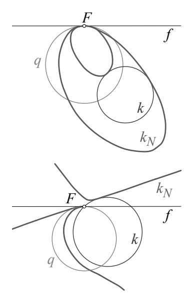
p.t.	2-circular	3-circular	4-circular
type 3			
type 4			

Table 1

THEOREM 2.1. *A pedal transformation in the isotropic plane maps a conic k, not touching any fundamental lines, into a circular quartic k_N the degree of circularity of which depends on the type of the pedal transformation and on the conic k as follows:*

- *If the fundamental conic q is a hyperbola or an ellipse, k_N is a 2-circular quartic in general case, 3-circular if FF_p is a tangent of the conic k or 4-circular if FF_p is the tangent of the conic at the point F_p, where F is the absolute point and F_p is the pole of the absolute line f with respect to q.*
- *If the fundamental conic q is a parabola or a special hyperbola, k_N is a 2-circular quartic.*

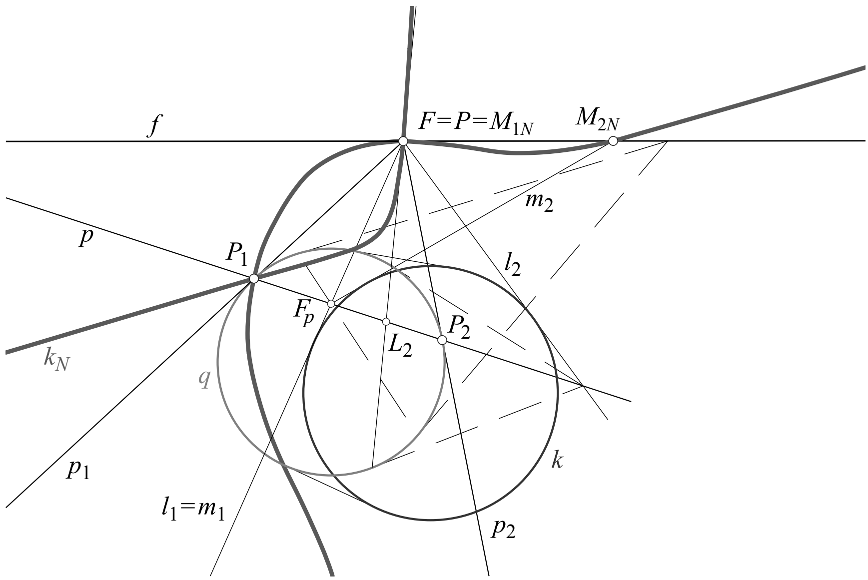


Figure 3

- *If the fundamental conic q is a circle, k_N is a 4-circular quartic.*

PROOF. We consider the above four types of pedal transformations separately.

Type (1)

A general case of this type of pedal transformation is shown in Figure 2. The pedal curve k_N of a conic k is at least 2-circular since it has a double point in the pedal point F. Two other intersections of quartic k_N and the absolute line f are the intersections M_{1N}, M_{2N} of f with tangents of k drawn from the pole F_p of the line f with respect to the fundamental conic q.

If the conic k touches the connecting line FF_p at the point different from F_p, then one of the points M_{1N}, M_{2N} falls into the absolute point. Consequently, the constructed quartic is 3-circular, Figure 3. The pole of the tangent $l_1 = FF_p$ lies on the line f which is therefore one of the quartic's tangents at the double point F. If the line FF_p touches the conic at the absolute point, both tangents at the point F coincide with the absolute line being the tangent at the cusp.

It is also possible to get entirely circular quartics by this type of pedal transformation. If the conic k touches the line FF_p at the point F_p, four common points of the absolute line and the quartic fall into the absolute point $F = M_{1N} = M_{2N}$. Thus, the line f osculates k_N at the double point F. The other tangent of the

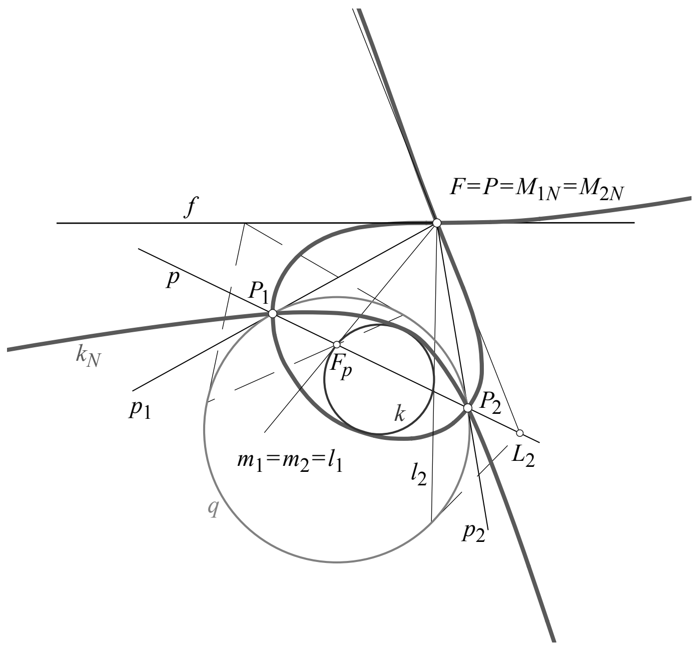


Figure 4

quartic at the point is the line joining F with the pole L_2 of the tangent l_2 of the conic k, Figure 4.

Type (2)

A quartic derived as a conic image by this type of pedal transformation is necessarily 2-circular. It intersects the absolute line at the fundamental points P and P_1. Each of them is a double point of the quartic.

Type (3)

If the fundamental conic q passes through the pedal point P, all three fundamental points coincide with P, and all three fundamental lines with the tangent p of q at P, Figure 5. A conic k is transformed into a 2-circular quartic k_N with a singularity in P. Both tangents at the point P coincide with the polar p of the point. If k is a special hyperbola, the pedal point P is a cusp. The other two common points of the absolute line f and the quartic k_N are the intersections M_{1N}, M_{2N} of the absolute line with tangents of the quartic passing through the pole F_p of the line.

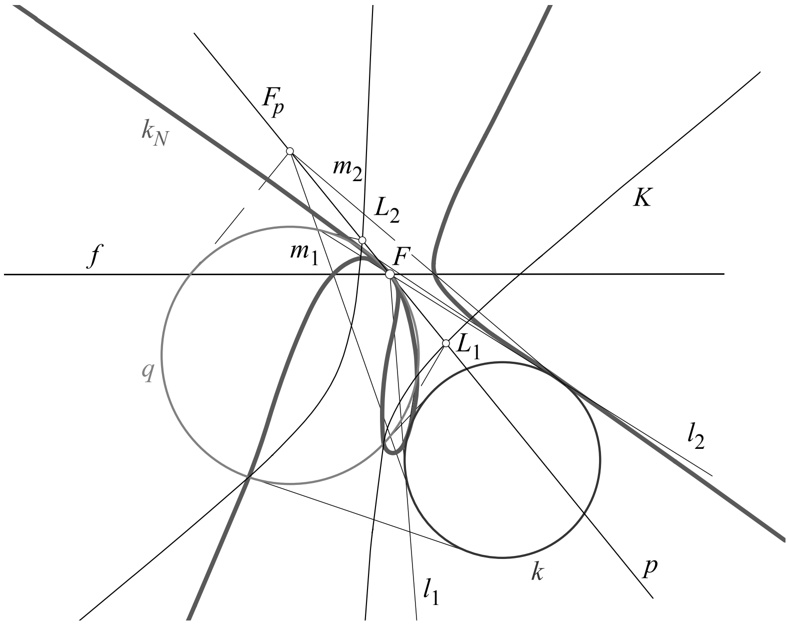


Figure 5

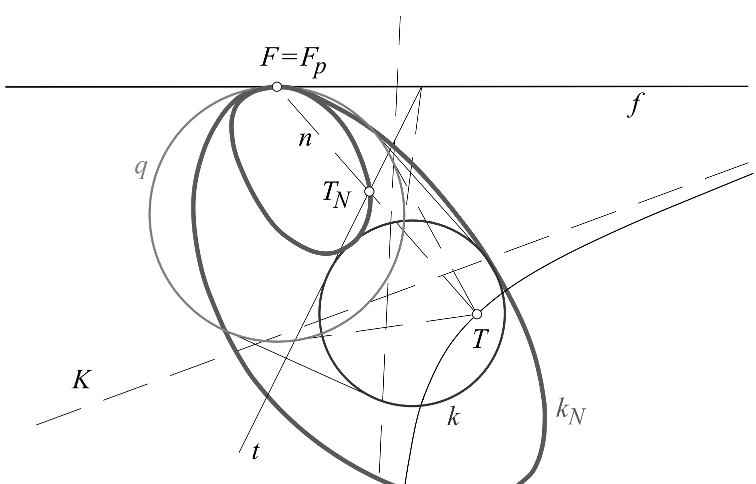


Figure 6

Type (4)
If the fundamental conic q is a circle, three fundamental points coincide with the pedal point F and three fundamental lines coincide with the absolute line f. Since, in this case, the pole F_p of the absolute line is the absolute point, all intersections of the absolute line with the quartic fall into F. The constructed quartic k_N is entirely circular. The poles of the tangents drawn from F lie on the line f so the absolute line is the two times counted tangent of the quartic at the absolute point, which is a node (Figure 6), an isolated double point or a cusp (Figure 7) depending on whether it is outside, inside or on the conic k.

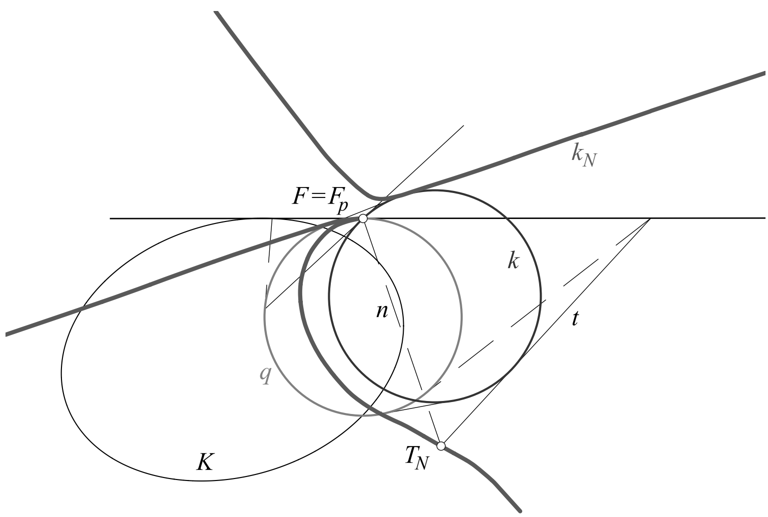


Figure 7

Type (1) on affine model of isotropic plane
The facts about the pedal transformation have been proved by using the synthetic method and have been illustrated on the projective model of an isotropic plane. The same conclusions can also be carried out by using analytical method on the affine (Euclidean) model of the plane. Since the approach is similar in all cases, we will consider here only the case of the pedal transformation of type (1).

Let us first determine the equation of the fundamental conic q which is in this case an ellipse or a hyperbola.
Every conic is given by the equation of the form

$$a_{00}x_0^2 + a_{11}x_1^2 + a_{22}x_2^2 + 2a_{01}x_0x_1 + 2a_{02}x_0x_2 + 2a_{12}x_1x_2 = 0,$$

or in the affine coordinates

$$a_{00} + a_{11}x^2 + a_{22}y^2 + 2a_{01}x + 2a_{02}y + 2a_{12}xy = 0.$$

Using the isotropic motion

$$\bar{x} = x - \frac{a_{01}a_{22} - a_{02}a_{12}}{a_{12}^2 - a_{11}a_{22}}, \quad \bar{y} = y - \frac{a_{02}a_{11} - a_{01}a_{12}}{a_{12}^2 - a_{11}a_{22}}$$

it is possible to get a simpler form of the equation

$$a_{11}x^2 + a_{22}y^2 + 2a_{12}xy + a = 0.$$

If we assume that the conic is an ellipse or a hyperbola intersecting the absolute line in the points $(0,1,\pm n)$, it follows that $a_{12} = 0$, $a_{11} = -n^2$ and the equation becomes

$$bx^2 + y^2 + a = 0.$$

The polar line p of the pedal point $F(0,0,1)$ has the equation $y = 0$, while the polar lines p_1, p_2 of two other fundamental points

$$P_1\left(-\sqrt{-\frac{a}{b}}, 0\right), \quad P_2\left(\sqrt{-\frac{a}{b}}, 0\right)$$

have the equations

$$x = -\sqrt{-\frac{a}{b}}, \quad x = \sqrt{-\frac{a}{b}}.$$

$F_p(1,0,0)$ is the pole of the absolute line $f\,[1,0,0]$, and the line F_pF joining it with the absolute point has the equation $x_1 = 0$, i.e. $x = 0$.

According to [4], the pole T of the given line $t\,[t_0, t_1, t_2]$, in the affine coordinates $t\,[u, v]$, with respect to the conic q is

$$A^{-1}t = \begin{bmatrix} \frac{1}{a} & 0 & 0 \\ 0 & \frac{1}{b} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} t_0 \\ t_1 \\ t_2 \end{bmatrix} = \begin{bmatrix} \frac{t_0}{a} \\ \frac{t_1}{b} \\ t_2 \end{bmatrix}.$$

The coordinates of the pedal $T_N(x,y)$ of the line perpendicular to t can be determined by solving the system of the linear equations

$$\begin{vmatrix} x_0 & x_1 & x_2 \\ \frac{t_0}{a} & \frac{t_1}{b} & t_2 \\ 0 & 0 & 1 \end{vmatrix} = 0$$

$$x_0t_0 + x_1t_1 + x_2t_2 = 0.$$

Thus,

$$x = \frac{au}{b}, \quad y = -\frac{b + au^2}{bv}. \tag{2.1}$$

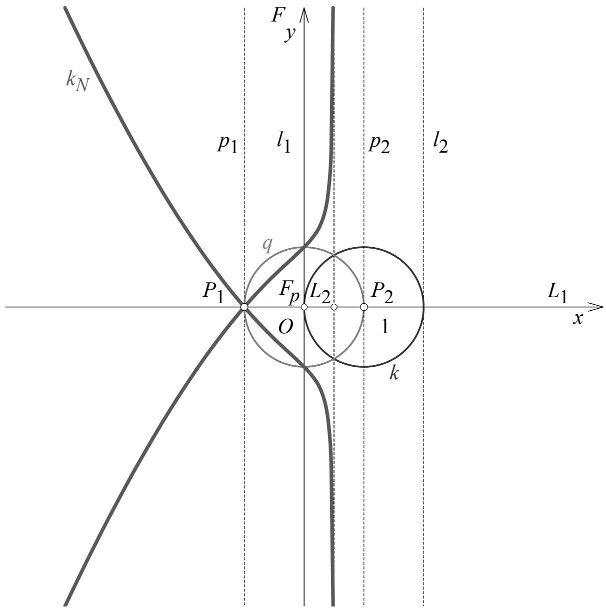


Figure 8

Let the conic k be given by point-coordinates

$$k \quad \dots \quad b_{00} + b_{11}x^2 + b_{22}y^2 + 2b_{01}x + 2b_{02}y + 2b_{12}xy = 0$$

and line-coordinates

$$(b_{00}b_{22} - b_{02}^2)u^2 + 2(b_{01}b_{02} - b_{00}b_{12})uv + (b_{00}b_{11} - b_{01}^2)v^2 +$$

$$+ 2(b_{02}b_{12} - b_{01}b_{22})u + 2(b_{01}b_{12} - b_{02}b_{11})v + b_{11}b_{22} - b_{12}^2 = 0.$$

Substituting u and v from (2.1) into the equation above, the equation of the quartic k_N is obtained

$$\begin{aligned}
&b^2 \left[(b_{00}b_{11} - b_{01}^2)x^4 + 2(b_{00}b_{12} - b_{01}b_{02})x^3y + (b_{00}b_{22} - b_{02}^2)x^2y^2\right] + \\
&\quad + 2ab\left[(b_{02}b_{11} - b_{01}b_{12})x^2y + (b_{02}b_{12} - b_{01}b_{22})xy^2\right] + \\
&\quad + 2ab(b_{00}b_{11} - b_{01}^2)x^2 + 2ab(b_{00}b_{12} - b_{01}b_{02})xy + \\
&\quad + a^2(b_{11}b_{22} - b_{12}^2)y^2 + \\
&\quad - 2a^2(b_{01}b_{12} - b_{02}b_{11})y + a^2(b_{00}b_{11} - b_{01}^2) = 0.
\end{aligned}$$

Let us now determine the intersection points of the absolute line $f \dots x_0 = 0$ with the quartic k_N. Their coordinates satisfy the equation

$$b^2 x_1^2 \left[(b_{00}b_{11} - b_{01}^2)x_1^2 + 2(b_{00}b_{12} - b_{01}b_{02})x_1x_2 + (b_{00}b_{22} - b_{02}^2)x_2^2 \right] = 0.$$

It is obvious that $x_1 = 0$ is double solution of the equation. Therefore, the absolute point $F(0,0,1)$ is a point with intersection multiplicity at least 2.

$x_1 = 0$ is at least a triple solution if and only if $b_{00}b_{22} - b_{02}^2 = 0$, thus if and only if the line $FF_p\,[0,1,0]$ is a tangent of the conic k.

$x_1 = 0$ is quadruple solution if and only if $b_{00}b_{22} - b_{02}^2 = 0$ and $b_{00}b_{12} - -b_{01}b_{02} = 0$. This holds when FF_p touches the conic k at the point F_p, Figure 8.

In the general case the other two intersection points are

$$M_{1N,2N}\Big(0, b_{00}b_{22} - b_{02}^2, b_{01}b_{02} - b_{00}b_{12} \pm \\ \pm \sqrt{(b_{01}b_{02} - b_{00}b_{12})^2 - (b_{00}b_{22} - b_{02}^2)(b_{00}b_{11} - b_{01}^2)}\Big).$$

It can be checked easily that those are the intersections of the absolute line with the tangents m_1, m_2 of the conic k drawn from the point F_p. The points M_{1N}, M_{2N} fall into one point if F_p lies on the conic k, in other words, if $b_{00} = 0$.

Therefore, k_N can be 2, 3 or 4-circular.

We compute the tangents of the quartic k_N at the point F: Any line through F different from the absolute one has the equation of the form $x_1 = mx_0$. The coordinates of its intersections with k_N satisfy the equation

$$x_0^2 \Big[(a + bm^2)^2(b_{00}b_{11} - b_{01}^2)x_0^2 + 2(a + bm^2)\Big(a(b_{02}b_{11} - b_{01}b_{12}) + \\ + bm(b_{00}b_{12} - b_{01}b_{02})\Big)x_0x_2 + \Big(a^2(b_{11}b_{22} - b_{12}^2) + \\ + 2abm(b_{02}b_{12} - b_{01}b_{22}) + b^2m^2(b_{00}b_{22} - b_{02}^2)\Big)x_2^2 \Big] = 0.$$

Since $x_0 = 0$ is a double solution of the equation for each m, $F(0,0,1)$ is a double point of the curve. In order to determine the tangent at that point, it is necessary to define the values of m for which $x_0 = 0$ is a triple solution. This is achieved when $a^2(b_{11}b_{22} - b_{12}^2) + 2abm(b_{02}b_{12} - b_{01}b_{22}) + b^2m^2(b_{00}b_{22} - b_{02}^2) = 0$.

In the general case the equations of two tangents are

$$x = \frac{a(b_{01}b_{22} - b_{02}b_{12}) \pm a\sqrt{(b_{02}b_{12} - b_{01}b_{22})^2 - (b_{00}b_{22} - b_{02}^2)(b_{11}b_{22} - b_{12}^2)}}{b(b_{00}b_{22} - b_{02}^2)}.$$

The equations above represent perpendiculars PL_1, PL_2 to the tangents l_1, l_2

$$x = \frac{b_{01}b_{22} - b_{02}b_{12} \pm \sqrt{(b_{02}b_{12} - b_{01}b_{22})^2 - (b_{00}b_{22} - b_{02}^2)(b_{11}b_{22} - b_{12}^2)}}{b_{12}^2 - b_{11}b_{22}}$$

of the conic k.

If $b_{00}b_{22} - b_{02}^2 = 0$, the conic k touches the line FF_p. This implies that one of the tangents of k_N at F coincides with the absolute line, while the other is

$$x = \frac{a(b_{11}b_{22} - b_{12}^2)}{2b(b_{01}b_{22} - b_{02}b_{12})}.$$

If $b_{00}b_{22} - b_{02}^2 = 0$ and $b_{01}b_{22} - b_{02}b_{12} = 0$, it necessary holds $b_{22} = 0$, as otherwise k would be a singular conic. The conic k now touches the line FF_p at the point F and the quartic k_N does not have any tangents at the absolute point except the absolute line.

Figure 8 displays the conic $k \ldots x^2 + y^2 - 2x = 0$ ($v^2 - 2u - 1 = 0$) and its pedal curve $k_N \ldots x^4 + 2xy^2 - 2x^2 - y^2 + 1 = 0$ in the case when the fundamental conic q is given by the equation $x^2 + y^2 - 1 = 0$. Beside the absolute point F the fundamental points are $P_{1,2}(\mp 1, 0)$. The fundamental lines are $y = 0$ and $x = \mp 1$. The conic k touches $FF_p \ldots x = 0$ at $F_p(0,0)$. Due to that k_N possesses a node in the absolute point at which one tangent coincides with the absolute line while the other has the equation $x = \frac{1}{2}$. The quartic is entirely circular. ∎

REMARK 2.1. Observing automorphic inversion, [2], and the pedal transformation in an isotropic plane, it can be noted that some of the circular quartics can be obtained by both transformations, while some can be obtained by one of them only. Moreover, the following statement is valid:

Let K be the reciprocal curve of a conic k with respect to a special hyperbola or a circle q. The pedal curve k_N of the conic k with respect to the conic q is identical to the inverse image of the conic K with respect to the fundamental conic q and the pole F, [10].

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E. Jurkin
University of Zagreb
Faculty of Mining
Geology and Petroleum Engineering
Pierottijeva 6, 10000 Zagreb
Croatia
ejurkin@rgn.hr

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ROUGH CLOSEDNESS, ROUGH CONTINUITY AND I_g-CLOSED SETS

By
ERDAL EKICI and SENA ÖZEN

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Abstract. In this paper, the notions of rough closedness and rough continuity related to I_g-closed sets in ideal topological spaces are introduced and studied. The relationships between rough closedness, I_g-closed sets and between rough continuity, I_g-closed sets are investigated. Various properties of rough closedness and rough continuity are discussed.

1. Introduction and preliminaries

In 1999, Dontchev et al. [1] discussed the notion of generalized closedness in ideal topological spaces and introduced I_g-closed sets in ideal topological spaces. In 2008, Navaneethakrishnan and Joseph [5] have studied properties of I_g-closed sets in ideal topological spaces. The main aim of this paper is to introduce and study the notions of rough closedness and rough continuity related to I_g-closed sets in ideal topological spaces. The relationships between rough closedness, I_g-closed sets and between rough continuity, I_g-closed sets and also various properties of rough closedness and rough continuity are investigated. It is shown in the present paper that the inverse image of any I_g-closed set under I_g-continuity and rough closedness is I_g-closed and the image of any I_g-closed set under rough continuity and $\star$-closedness is I_g-closed.

In this paper, we consider a topological space (X, τ) with no separation properties assumed by a space. For a subset U of a topological space (X, τ), $Cl(U)$ and $Int(U)$ will denote the closure and interior of U in (X, τ), respectively.

An ideal I on a topological space (X, τ) is a nonempty collection of subsets of X which satisfies

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(1) $U \in I$ and $V \subset U$ implies $V \in I$,

(2) $U \in I$ and $V \in I$ implies $U \cup V \in I$ [3].

For a topological space (X, τ) with an ideal I on X, a set operator $(.)^*: P(X) \to P(X)$ where $P(X)$ is the set of all subsets of X, called a local function [3] of U with respect to τ and I is defined as follows:

for $U \subset X$, $U^*(I, \tau) = \{x \in X\colon V \cap U \notin I \text{ for every } V \in \tau(x)\}$ where $\tau(x) = \{V \in \tau : x \in V\}$.

A Kuratowski closure operator $Cl^*(.)$ for a topology $\tau^*(I, \tau)$, called the $\star$-topology and finer than τ, is defined by $Cl^*(U) = U \cup U^*(I, \tau)$ [2]. We will simply write U^* for $U^*(I, \tau)$ and τ^* for $\tau^*(I, \tau)$. If I is an ideal on X, then (X, τ, I) is said to be an ideal topological space or simply an ideal space. A subset V of a topological space (X, τ) is called g-closed [4] if $Cl(V) \subset G$ whenever $V \subset G$ and G is open in (X, τ).

Definition 1. A subset U of an ideal topological space (X, τ, I) is called

(1) I_g-closed [1] if $U^* \subset V$ whenever $U \subset V$ and V is open in (X, τ, I).

(2) I_g-open [1] if $X \setminus U$ is I_g-closed.

Theorem 2. ([5]) *For an ideal topological space (X, τ, I) and $U \subset X$, the following are equivalent:*

(1) U is I_g-closed,

(2) $Cl^*(U) \subset V$ whenever $U \subset V$ and V is open in (X, τ, I).

Theorem 3. ([5]) *Let (X, τ, I) be an ideal topological space and $U \subset X$. Then U is I_g-open if and only if $K \subset \mathrm{Int}^*(U)$ whenever K is closed and $K \subset U$.*

2. Rough closedness and I_g-closed sets

Definition 4. A function $f\colon (X, \tau, I) \to (Y, \sigma, J)$ is called roughly closed if $f(K) \subset \mathrm{Int}^*(U)$ whenever K is a closed set in (X, τ, I) and U is an I_g-open set in (Y, σ, J) such that $f(K) \subset U$.

Remark 5. For a function $f\colon (X, \tau, I) \to (Y, \sigma, J)$, if f is closed, then it is roughly closed but the reverse implication is not true in general as shown in the following example:

Example 6. Let $X = \{a, b, c, d\}$, $\tau = \{X, \emptyset, \{a\}, \{b, c\}, \{a, b, c\}\}$ and $I = \{\emptyset, \{a\}, \{d\}, \{a, d\}\}$. Then the function $f\colon (X, \tau, I) \to (X, \tau, I)$ defined by $f(a) = a, f(b) = c, f(c) = b, f(d) = a$ is roughly closed but it is not closed.

DEFINITION 7. A function $f\colon (X,\tau,I) \to (Y,\sigma,J)$ is said to be I_g-continuous if for every $\star$-closed subset K of (Y,σ,J), $f^{-1}(K)$ is I_g-closed.

THEOREM 8. *Let $f\colon (X,\tau,I) \to (Y,\sigma,J)$ be a function. If f is I_g-continuous and roughly closed, then $f^{-1}(U)$ is an I_g-closed set in (X,τ,I) for each I_g-closed set U in (Y,σ,J).*

PROOF. Let U be an I_g-closed set in (Y,σ,J). Suppose that $f^{-1}(U) \subset V$ for an open set V in (X,τ,I). This implies that $X\setminus V \subset f^{-1}(Y\setminus U)$ and then $f(X\setminus V) \subset$ $\subset Y\setminus U$. Since f is roughly closed, then $f(X\setminus V) \subset \mathrm{Int}^*(Y\setminus U) = Y\setminus Cl^*(U)$. This implies $X\setminus V \subset X\setminus f^{-1}(Cl^*(U))$ and then $f^{-1}(Cl^*(U)) \subset V$. Since f is I_g-continuous, then $f^{-1}(Cl^*(U))$ is an I_g-closed set. Consequently,

$$Cl^*(f^{-1}(U)) \subset Cl^*(f^{-1}(Cl^*(U))) \subset V.$$

Thus, $Cl^*(f^{-1}(U)) \subset V$ and hence $f^{-1}(U)$ is an I_g-closed set in (X,τ,I). ∎

COROLLARY 9. *Let $f\colon (X,\tau,I) \to (Y,\sigma,J)$ be a function. If f is I_g-continuous and roughly closed, then $f^{-1}(U)$ is an I_g-open set in (X,τ,I) for each I_g-open set U in (Y,σ,J).*

COROLLARY 10. *Let $f\colon (X,\tau,I) \to (Y,\sigma,J)$ be a function. If f is I_g-continuous and closed, then*

(1) $f^{-1}(U)$ is an I_g-closed set in (X,τ,I) for each I_g-closed set U in (Y,σ,J).

(2) $f^{-1}(U)$ is an I_g-open set in (X,τ,I) for each I_g-open set U in (Y,σ,J).

PROOF. It follows from Theorem 8 and Remark 5. ∎

THEOREM 11. *Let $f\colon (X,\tau,I) \to (Y,\sigma,J)$ be a function. If $f(K)$ is a $\star$-open set in (Y,σ,J) for each closed set K in (X,τ,I), then f is a roughly closed function.*

PROOF. Suppose that $f(K)$ is a $\star$-open set in (Y,σ,J) for each closed set K in (X,τ,I). Let $f(V) \subset U$ for a closed set V in (X,τ,I) and an I_g-open set U in (Y,σ,J). This implies that $Int^*(f(V)) \subset \mathrm{Int}^*(U)$. Since $f(V)$ is a $\star$-open set in (Y,σ,J), then $f(V) \subset \mathrm{Int}^*(U)$. Hence, f is a roughly closed function. ∎

REMARK 12. The following example shows that the reverse of Theorem 11 is not true in general.

EXAMPLE 13. Let $X = \{a,b,c,d\}$, $\tau = \big\{X,\emptyset,\{a\},\{b,c\},\{a,b,c\}\big\}$ and $I = \big\{\emptyset,\{a\},\{d\},\{a,d\}\big\}$. Then the identity function $i\colon (X,\tau,I) \to (X,\tau,I)$ is roughly closed but $f(\{a,d\}) = \{a,d\}$ is not $\star$-open.

DEFINITION 14. ([1]) Let (X,τ,I) be an ideal topological space. Then (X,τ,I) is called a T_I-ideal space if each I_g-closed set in (X,τ,I) is $\star$-closed.

THEOREM 15. *Let* (Y, σ, J) *be an ideal topological space. The following properties are equivalent:*

(1) Each function $f\colon (X, \tau, I) \to (Y, \sigma, J)$ *is roughly closed for each ideal topological space* (X, τ, I)*,*

(2) (Y, σ, J) *is a* T_I*-ideal space.*

PROOF. (1) $\Rightarrow$ (2): Let $U \neq \emptyset$ be an I_g-closed set in (Y, σ, J). Suppose that (X, τ, I) where $X = Y$ and $I = J$ is an ideal topological space with the topology $\tau = \{X, \emptyset, U\}$. Let $f\colon (X, \tau, I) \to (Y, \sigma, J)$ be the identity function. Then $f\colon (X, \tau, I) \to (Y, \sigma, J)$ is roughly closed. Since $Y \setminus U$ is I_g-open in (Y, σ, J) and $X \setminus U$ is closed in (X, τ, I) and $f(X \setminus U) \subset Y \setminus U$, then we have $f(X \setminus U) \subset$ $\subset \operatorname{Int}^*(Y \setminus U)$. Since

$$f(X \setminus U) = Y \setminus U \subset \operatorname{Int}^*(Y \setminus U) = Y \setminus Cl^*(U),$$

then $Cl^*(U) \subset U$. Thus, U is a $\star$-closed set in (Y, σ, J). Hence, (Y, σ, J) is a T_I-ideal space.

(2) $\Rightarrow$ (1): Let V be an I_g-open set in (Y, σ, J) and $f(K) \subset V$ where K is a closed set in (X, τ, I). Since (Y, σ, J) is a T_I-ideal space, then V is $\star$-open. This implies that $f(K) \subset \operatorname{Int}^*(V)$. Thus, f is a roughly closed function. ∎

THEOREM 16. ([5]) *Each subset of an ideal topological space* (X, τ, I) *is* I_g*-closed if and only if each open set of* (X, τ, I) *is* $\star$*-closed.*

THEOREM 17. *Let* $f\colon (X, \tau, I) \to (Y, \sigma, J)$ *be a function where* $\sigma = \eta$ *and* η *is the family of all* $\star$*-closed sets of* (Y, σ, J)*. Then the following properties are equivalent:*

(1) f *is roughly closed,*

(2) $f(K)$ *is* $\star$*-open for each closed set* K *in* (X, τ, I)*.*

PROOF. (1) $\Rightarrow$ (2): Let $f\colon (X, \tau, I) \to (Y, \sigma, J)$ be a roughly closed function. Suppose that K is a closed set in (X, τ, I). This implies from Theorem 16 that $f(K)$ is an I_g-open set in (Y, σ, J). Since f is a roughly closed function, then $f(K) \subset \operatorname{Int}^*(f(K))$. Consequently, $f(K)$ is a $\star$-open set in (Y, σ, J).

(2) $\Rightarrow$ (1): It follows from Theorem 11. ∎

REMARK 18. The following example shows that the composition of two roughly closed functions is not roughly closed in general.

EXAMPLE 19. Let $X = Y = \{a, b, c, d\}$, $\tau = \big\{X, \emptyset, \{a\}, \{a, b\}, \{c, d\}, \{a, c, d\}\big\}$, $I = J = \big\{\emptyset, \{a\}, \{d\}, \{a, d\}\big\}$ and $\sigma = \big\{Y, \emptyset, \{a\}, \{b, c\}, \{a, b, c\}\big\}$. Then for the functions $f\colon (X, \tau, I) \to (Y, \sigma, J)$ defined by $f(a) = d, f(b) = a,$

$f(c) = c, f(d) = d$ and $g\colon (Y,\sigma,J) \to (Y,\sigma,J)$ defined by $g(a) = a$, $g(b) = c$, $g(c) = b$, $g(d) = a$, $g \circ f$ is not roughly closed but f and g are roughly closed.

THEOREM 20. *Let $f\colon (X,\tau,I) \to (Y,\sigma,J)$ and $g\colon (Y,\sigma,J) \to (Z,\eta,\ell)$ be two functions. If f is closed and g is roughly closed, then $g \circ f\colon (X,\tau,I) \to (Z,\eta,\ell)$ is a roughly closed function.*

PROOF. Let K be a closed set in (X,τ,I) and U be an I_g-open set in (Z,η,ℓ) such that $(g \circ f)(K) \subset U$. Since f is a closed function, then $f(K)$ is a closed set in (Y,σ,J). Since g is a roughly closed function, then $g(f(K)) \subset \mathrm{Int}^*(U)$. Thus, $(g \circ f)(K) \subset \mathrm{Int}^*(U)$ and hence $g \circ f$ is roughly closed. ∎

DEFINITION 21. A function $f\colon (X,\tau,I) \to (Y,\sigma,J)$ is called

(1) $\star$-open if $f(K)$ is $\star$-open for every $\star$-open subset K of (X,τ,I).

(2) I_g-irresolute if $f^{-1}(K)$ is I_g-closed for every I_g-closed subset K of (Y,σ,J).

THEOREM 22. *Let $f\colon (X,\tau,I) \to (Y,\sigma,J)$ and $g\colon (Y,\sigma,J) \to (Z,\eta,\ell)$ be two functions. If f is roughly closed and $g\colon (Y,\sigma,J) \to (Z,\eta,\ell)$ is $\star$-open and I_g-irresolute, then $g \circ f\colon (X,\tau,I) \to (Z,\eta,\ell)$ is a roughly closed function.*

PROOF. Let K be a closed set in (X,τ,I) and U be an I_g-open set in (Z,η,ℓ) such that $(g \circ f)(K) \subset U$. This implies that $f(K) \subset g^{-1}(U)$. Since $g^{-1}(U)$ is an I_g-open set and f is roughly closed, then $f(K) \subset \mathrm{Int}^*(g^{-1}(U))$. We have $(g \circ f)(K) = g(f(K)) \subset g(\mathrm{Int}^*(g^{-1}(U)))$. Since $Int^*(g^{-1}(U)) \subset g^{-1}(U)$, then $g(Int^*(g^{-1}(U))) \subset g(g^{-1}(U)) \subset U$. Since g is a $\star$-open function, then $g(\mathrm{Int}^*(g^{-1}(U))) \subset \mathrm{Int}^*(U)$. Thus, $(g \circ f)(K) \subset \mathrm{Int}^*(U)$ and hence, $g \circ f\colon (X,\tau,I) \to (Z,\eta,\ell)$ is a roughly closed function. ∎

Recall that for an ideal topological space (X,τ,I) and $V \subset X$, (V,τ_V,I_V), where τ_V is the relative topology on V and $I_V = \{V \cap J\colon J \in I\}$ is an ideal topological space [2].

THEOREM 23. *Let $f\colon (X,\tau,I) \to (Y,\sigma,J)$ be a function. If f is roughly closed and V is a closed set in (X,τ,I), then its restriction $f|_V\colon (V,\tau_V,I_V) \to (Y,\sigma,J)$ is a roughly closed function.*

PROOF. Let U be a closed set in (V,τ_V,I_V) and G be an I_g-open set in (Y,σ,J) such that $f|_V(U) \subset G$. This implies that U is a closed set in (X,τ,I) and $f|_V(U) = f(U) \subset G$. Since f is roughly closed, then we have $f|_V(U) = f(U) \subset \subset \mathrm{Int}^*(G)$. Hence, $f|_V$ is a roughly closed function. ∎

REMARK 24. The following example shows that any restriction of a roughly closed function is not roughly closed in general.

EXAMPLE 25. Let $X = Y = \{a,b,c,d\}$, $\tau = \{X, \emptyset, \{a\}, \{a,b\}, \{c,d\}, \{a,c,d\}\}$, $I = J = \{\emptyset, \{a\}, \{d\}, \{a,d\}\}$ and $\sigma = \{Y, \emptyset, \{a\}, \{b,c\}, \{a,b,c\}\}$. Then for the function $f\colon (X,\tau,I) \to (Y,\sigma,J)$ defined by $f(a) = c, f(b) = d, f(c) = a, f(d) = d$ and $A = \{a,c,d\}$, f is roughly closed but $f|_A\colon (A,\tau_A,I_A) \to$ $\to (Y,\sigma,J)$ is not roughly closed.

3. Rough continuity and I_g-closed sets

DEFINITION 26. A function $f\colon (X,\tau,I) \to (Y,\sigma,J)$ is called roughly continuous if $V^* \subset f^{-1}(U)$ whenever U is an open set in (Y,σ,J) and V is an I_g-closed set in (X,τ,I) such that $V \subset f^{-1}(U)$.

THEOREM 27. *Let $f\colon (X,\tau,I) \to (Y,\sigma,J)$ be a function. The following are equivalent:*

(1) f is roughly continuous,

(2) $Cl^*(V) \subset f^{-1}(U)$ whenever U is an open set in (Y,σ,J) and V is an I_g-closed set in (X,τ,I) such that $V \subset f^{-1}(U)$.

PROOF. (1) $\Rightarrow$ (2): Suppose that f is roughly continuous. Let U be an open set in (Y,σ,J) and V be an I_g-closed set in (X,τ,I) such that $V \subset f^{-1}(U)$. By (1), we have $V^* \subset f^{-1}(U)$. This implies that $V^* \cup V = Cl^*(V) \subset f^{-1}(U)$.

(2) $\Rightarrow$ (1): Let U be an open set in (Y,σ,J) and V be an I_g-closed set in (X,τ,I) such that $V \subset f^{-1}(U)$. By (2), $Cl^*(V) \subset f^{-1}(U)$. Then, we have $Cl^*(V) = V^* \cup V \subset f^{-1}(U)$. Thus, $V^* \subset f^{-1}(U)$ and hence, f is roughly continuous. ∎

REMARK 28. Let $f\colon (X,\tau,I) \to (Y,\sigma,J)$ be a function. If f is continuous, then f is roughly continuous. The following example shows that the reverse of this implication is not true in general.

EXAMPLE 29. Let $X = \{a,b,c,d\}$, $\tau = \{X, \emptyset, \{a\}, \{b,c\}, \{a,b,c\}\}$ and $I =$ $= \{\emptyset, \{a\}, \{d\}, \{a,d\}\}$. Then the function $f\colon (X,\tau,I) \to (X,\tau,I)$ defined by $f(a) = d, f(b) = c, f(c) = b, f(d) = a$ is roughly continuous but it is not continuous.

THEOREM 30. *Let $f\colon (X,\tau,I) \to (Y,\sigma,J)$ be a bijective function. Then the following are equivalent:*

(1) f is roughly closed,

(2) f^{-1} is roughly continuous.

PROOF. Suppose that $f\colon (X,\tau,I) \to (Y,\sigma,J)$ is a roughly closed function. Let U be an open set in (X,τ,I) and V be an I_g-closed set in (Y,σ,J) such that $V \subset f(U)$. This implies that $X \setminus U$ is a closed set in (X,τ,I) and $Y \setminus V$ be an I_g-open set in (Y,σ,J) such that $Y \setminus f(U) = f(X \setminus U) \subset Y \setminus V$. Since f is roughly closed, then $f(X \setminus U) \subset \mathrm{Int}^*(Y \setminus V)$. We have $f(X \setminus U) = Y \setminus f(U) \subset Y \setminus Cl^*(V)$ and then $Cl^*(V) \subset f(U)$. Thus, f^{-1} is roughly continuous. The converse is similar. ∎

DEFINITION 31. A function $f\colon (X,\tau,I) \to (Y,\sigma,J)$ is called $\star$-closed if $f(K)$ is $\star$-closed for each $\star$-closed set K in (X,τ,I).

THEOREM 32. *Let $f\colon (X,\tau,I) \to (Y,\sigma,J)$ be a function. If f is roughly continuous and $\star$-closed, then $f(V)$ is an I_g-closed set in (Y,σ,J) for each I_g-closed set V in (X,τ,I).*

PROOF. Let V be an I_g-closed set in (X,τ,I) and U be an open set in (Y,σ,J) such that $f(V) \subset U$. We have $V \subset f^{-1}(U)$. Since f is roughly continuous, then $Cl^*(V) \subset f^{-1}(U)$. This implies $f(Cl^*(V)) \subset U$. Since f is $\star$-closed, then we have

$$Cl^*(f(V)) \subset Cl^*(f(Cl^*(V))) = f(Cl^*(V)) \subset U.$$

Consequently, $Cl^*(f(V)) \subset U$. Thus, $f(V)$ is an I_g-closed set in (Y,σ,J). ∎

COROLLARY 33. *Let $f\colon (X,\tau,I) \to (Y,\sigma,J)$ be a function. If f is continuous and $\star$-closed, then $f(V)$ is an I_g-closed set in (Y,σ,J) for each I_g-closed set V in (X,τ,I).*

PROOF. It follows from Theorem 32 and Remark 28. ∎

THEOREM 34. *Let $f\colon (X,\tau,I) \to (Y,\sigma,J)$ be a function. If $f^{-1}(U)$ is a $\star$-closed set in (X,τ,I) for each open set U in (Y,σ,J), then f is a roughly continuous function.*

PROOF. Let U be an open set in (Y,σ,J) and K be an I_g-closed set in (X,τ,I) such that $K \subset f^{-1}(U)$. Then we have $Cl^*(K) \subset Cl^*(f^{-1}(U)) = f^{-1}(U)$. Hence, f is a roughly continuous function. ∎

REMARK 35. The reverse of Theorem 34 is not true in general as shown in the following example.

EXAMPLE 36. Let $X = \{a,b,c,d\}$, $\tau = \big\{X, \emptyset, \{a\}, \{b,c\}, \{a,b,c\}\big\}$ and $I = \big\{\emptyset, \{a\}, \{d\}, \{a,d\}\big\}$. Then the identity function $i\colon (X,\tau,I) \to (X,\tau,I)$ is roughly continuous but $f^{-1}(\{a,b,c\}) = \{a,b,c\}$ is not $\star$-closed.

Theorem 37. *Let (X,τ,I) be an ideal topological space and $R \subset S \subset X$. If R is an I_g-closed set relative to S and S is g-closed relative to (X,τ^*), then R is I_g-closed relative to (X,τ,I).*

Proof. Let R be an I_g-closed set relative to S and S be g-closed relative to (X,τ^*). Since S is g-closed relative to (X,τ^*), then S is I_g-closed in X. So, by Theorem 2.8 of [1], R is I_g-closed relative to (X,τ,I). ∎

Theorem 38. *Let $f\colon (X,\tau,I) \to (Y,\sigma,J)$ be a function. If f is roughly continuous and R is g-closed relative to (X,τ^*), then the restriction function $f\,|_R\colon (R,\tau_R,I_R) \to (Y,\sigma,J)$ is roughly continuous.*

Proof. Let S be an I_g-closed set in (R,τ_R,I_R) and U be an open set in (Y,σ,J) such that $S \subset (f\,|_R)^{-1}(U)$. This implies that $S \subset f^{-1}(U)\cap R$. Since S is I_g-closed relative to (X,τ,I) by Theorem 37 and f is roughly continuous, then $Cl^*(S) \subset$ $\subset f^{-1}(U)$. We have $Cl^*(S)\cap R \subset f^{-1}(U)\cap R$. Thus, $Cl^*_R(S) \subset (f\,|_R)^{-1}(U)$ and hence, $f\,|_R\colon (R,\tau_R,I_R) \to (Y,\sigma,J)$ is roughly continuous. ∎

Theorem 39. *Let (X,τ,I) be an ideal topological space. The following properties are equivalent:*

(1) Each function $f\colon (X,\tau,I) \to (Y,\sigma,J)$ is roughly continuous for each ideal topological space (Y,σ,J),

(2) (X,τ,I) is a T_I-ideal space.

Proof. (1) ⇒ (2): Let U be a nonempty I_g-closed set in (X,τ,I). Suppose that $Y = X$, $I = J$ and $\sigma = \{Y,\emptyset,U\}$ and $f\colon X \to Y$ is the identity function. Since f is roughly continuous, U is an I_g-closed set in (X,τ,I) and an open set in (Y,σ,J) such that $U \subset f^{-1}(U)$, then $Cl^*(U) \subset f^{-1}(U) = U$. Consequently, U is a $\star$-closed set in (X,τ,I). Thus, (X,τ,I) is a T_I-ideal space.

(2) ⇒ (1): Let K be an I_g-closed set in (X,τ,I) and V be an open set in (Y,σ,J) such that $K \subset f^{-1}(V)$. Since (X,τ,I) is a T_I-ideal space, then K is a $\star$-closed set. Thus, $Cl^*(K) \subset f^{-1}(V)$ and hence f is roughly continuous. ∎

Theorem 40. *Let (X,τ,I) be an ideal topological space such that τ and the family of $\star$-closed sets of (X,τ,I) coincide. Then a function $f\colon (X,\tau,I) \to (Y,\sigma,J)$ is roughly continuous if and only if $f^{-1}(U)$ is $\star$-closed for each open set U in (Y,σ,J).*

Proof. (⇒): Let $f\colon (X,\tau,I) \to (Y,\sigma,J)$ be a roughly continuous function. Let U be an open set in (Y,σ,J). By Theorem 16, $f^{-1}(U)$ is an I_g-closed set in (X,τ,I). Since f is roughly continuous, $Cl^*(f^{-1}(U)) \subset f^{-1}(U)$. Hence, $f^{-1}(U)$ is a $\star$-closed set in (X,τ,I).

($\Leftarrow$): It follows from Theorem 34. ∎

REMARK 41. The following example shows that any composition of two roughly continuous functions is not roughly continuous in general.

EXAMPLE 42. Let $X = Y = \{a,b,c,d\}$, $\tau = \{X, \emptyset, \{a\}, \{b,c\}, \{a,b,c\}\}$, $I = J = \{\emptyset , \{a\}, \{d\}, \{a,d\}\}$ and $\sigma = \{X, \emptyset, \{a\}, \{a,b\}, \{c,d\}, \{a,c,d\}\}$. Then for the functions $f\colon (X,\tau,I) \to (X,\tau,I)$ defined by $f(a) = d, f(b) = b$, $f(c) = c, f(d) = a$ and $g\colon (X,\tau,I) \to (Y,\sigma,J)$ defined by $g(a) = a$, $g(b) = a$, $g(c) = b$, $g(d) = b$, $g \circ f$ is not roughly continuous but f and g are roughly continuous.

THEOREM 43. *If $f\colon (X,\tau,I) \to (Y,\sigma,J)$ is a roughly continuous function and $g\colon (Y,\sigma,J) \to (Z,\eta,\ell)$ is a continuous function, then $g\circ f\colon (X,\tau,I) \to (Z,\eta,\ell)$ is roughly continuous.*

PROOF. Let V be an I_g-closed set in (X,τ,I) and U be an open set in (Z,η,ℓ) such that $V \subset (g\circ f)^{-1}(U)$. Since g is continuous, then $g^{-1}(U)$ is an open set. Since f is roughly continuous, then $Cl^*(V) \subset f^{-1}(g^{-1}(U)) = (g\circ f)^{-1}(U)$. Thus, $Cl^*(V) \subset (g\circ f)^{-1}(U)$ and hence $g\circ f$ is roughly continuous. ∎

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Erdal Ekici
Department of Mathematics,
Canakkale Onsekiz Mart University,
Terzioglu Campus, 17020
Canakkale, Turkey
eekici@comu.edu.tr

Sena Özen
Department of Mathematics,
Canakkale Onsekiz Mart University,
Terzioglu Campus, 17020
Canakkale, Turkey

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ON COVERING PROPERTIES VIA GENERALIZED OPEN SETS

By
BISHWAMBHAR ROY and S. JAFARI

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Abstract. We characterize some properties of generalized compactness and generalized closedness of generalized topological spaces by using closure operator defined on such a space. It is also shown that many results done in this area in some previous papers can be considered as special cases of our results.

1. Introduction

In the past few years, several forms of open sets have been studied. Recently, Á. Császár found the theory of generalized topology in [8], studying the extremely elementary character of these classes. Especially, the author defined some basic operators on generalized topological spaces.

It is observed that a large number of papers is devoted to the study of compactness and closedness of a space, containing the class of open sets and possessing properties more or less similar to those of open sets. For example, [13] has introduced semi-compact and s-closed [22], strongly compact [23] and p-closed [12], δ_p-closed [26], δ-semi-compact [14], α-compact [21] and α-closed [16], β-compact [3] and β-closed [4] topological spaces. It is interesting to mention that Ganster [15] has shown that infinite β-compact spaces do not exist.

Owing to the fact that corresponding definitions have many features in common, it is quite natural to conjecture that they can be obtained and a considerable part of the properties of generalized open sets can be deduced from suitable more general definitions. The purpose of this paper is to show that this is possible and

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the generality obtained in this manner helps us to point out extremely elementary character of the proofs and to get many unknown results by special choice of the GT.

We recall some notions defined in [8]. Let X be a non-empty set, $\exp X$ denotes the power set of X. We call a class $g \subseteq \exp X$ a generalized topology [8], (briefly, GT) if $\emptyset \in g$ and union of elements of g belongs to g. A set with a GT is said to be a generalized topological space (briefly, GTS). A generalized topology g on X is called strong [9] if $X \in g$. We note that for any topological space (X, τ), the collection of all open sets denoted by τ (preopen sets [1] denoted by $PO(X)$, semi-open sets [20] denoted by $SO(X)$, δ-open sets [30] denoted by $\delta O(X)$, δ-preopen [27, 26] sets denoted by δ-$PO(X)$, δ-semiopen sets [25] denoted by δ-$SO(X)$, α-open sets [24] denoted by $\alpha O(X)$, β-open sets [2] denoted by $\beta O(X)$) form a GT.

For a GTS (X, g), the elements of g are called g-open sets and the complements of g-open sets are called g-closed sets. For $A \subseteq X$, we denote by $c_g(A)$ the intersection of all g-closed sets containing A, i.e., the smallest g-closed set containing A; and by $i_g(A)$ the union of all g-open sets contained in A, i.e., the largest g-open set contained in A (see [8, 10]). It is easy to observe that i_g and c_g are idempotent and monotonic, where $\gamma\colon \exp X \to \exp X$ is said to be idempotent iff for each $A \subseteq X$, $\gamma(\gamma(A)) = \gamma(A)$, and monotonic iff $\gamma(A) \subseteq \gamma(B)$ whenever $A \subseteq B \subseteq X$. It is also well known from [10, 11] that if g is a GT on X and $A \subseteq X$, $x \in X$, then $x \in c_g(A)$ iff $x \in M \in g \Rightarrow M \cap A \neq \emptyset$ and $c_g(X \setminus A) = X \setminus i_g(A)$.

In this paper we use the concepts of g-open sets to introduce g-compact and g-closed spaces. It is shown that many results in previous papers can be considered as special cases of our results.

2. g-compactness in generalized topological spaces

Definition 2.1. Let (X, g) be a GTS. A non-empty subset A of X is called generalized compact relative to X or in short, g-compact relative to X if every cover of A by g-open sets of X (such a cover will henceforth be called a g-open cover of A) has a finite subfamily which covers A. If, in addition, $A = X$ then X is called g-compact.

Remark 2.1. Let (X, g_1) and (X, g_2) be two GTS's such that $g_1 \subseteq g_2$ and $X \in \in g_1$. If (X, g_2) is g_2-compact then (X, g_1) is g_1-compact.

For any topological space (X, τ), since $\tau \subseteq \alpha O(X) \subseteq PO(X) \subseteq \beta O(X)$ and $\tau \subseteq \alpha O(X) \subseteq SO(X) \subseteq \beta O(X)$, it follows from the above theorem that

$$\begin{array}{ccc} \beta\text{-compact} & \Rightarrow \text{semicompact} \Rightarrow \alpha\text{-compact} \Rightarrow \text{compact} \\ \Downarrow & \\ \text{strongly compact or pre-compact} \Rightarrow & \alpha\text{-compact} \Rightarrow \text{compact.} \end{array}$$

DEFINITION 2.2. [29] Let (X, g) be a GTS and $x \in X$. A net $\{x_\alpha\}_{\alpha\in D}$ in X is said to g-converge to x in X if for every g-open set U containing x there exists $\alpha_0 \in D$ such that $x_\alpha \in U$, for each $\alpha \geq \alpha_0$.

DEFINITION 2.3. Let (X, g) be a GTS. Let $\{x_\alpha\}_{\alpha\in D}$ be a net in X. Then,

(i) a point $x \in X$ is called a g-cluster point of $\{x_\alpha\}_{\alpha\in D}$ iff $\{x_\alpha\}_{\alpha\in D}$ is frequently in every g-open set containing x. We denote the set of all g-cluster points of $\{x_\alpha\}_{\alpha\in D}$ by $g\text{-}cp\{x_\alpha\}_{\alpha\in D}$.

(ii) $\{x_\alpha\}_{\alpha\in D}$ is said to satisfy $P(I)$ iff there exists $y \in g\text{-}cp\{x_\alpha\}_{\alpha\in D}$ such that $\{\alpha \in D\colon x_\alpha = y\}$ is cofinal in D.

(iii) $\{x_\alpha\}_{\alpha\in D}$ is said to satisfy $P(II)$ iff there exists $y \in g\text{-}cp\{x_\alpha\}_{\alpha\in D}$ such that if $U, V \in g$ and $y \in U \cap V$, then $(U \cap V) \setminus \{y\} \neq \emptyset$.

THEOREM 2.1. *For a GTS (X, g), the following properties are equivalent:*

(i) (X, g) is g-compact;

(ii) Every family of g-closed subsets of X with finite intersection property has non-empty intersection;

(iii) $g\text{-}cp\{x_\alpha\}_{\alpha\in D} \neq \emptyset$, for each net $\{x_\alpha\}_{\alpha\in D}$ in X.

PROOF. The proof is straightforward. ∎

This theorem contains several results in literature. For example, in a topological space, (X, τ), if we take $g = \tau$ then we get the Theorems 5.1 and 5.2 of [18], if $g = PO(X)$, we get the Theorem 2.1. of [23], if $g = SO(X)$, we get the Theorem 3.3 of [13].

THEOREM 2.2. *Let (X, g) be a strong GTS in which no net satisfy $P(II)$. Then X is singleton.*

PROOF. Let $y \in X$. Then y is a g-cluster point of the constant net $\{y\}$. Take $U = V = X \in g$. Since no net in X has the property $P(II)$, we have $(U \cap V) \setminus \{y\} = \emptyset$. Hence $X = U \cap V = \{y\}$. ∎

This theorem contains several results in literature. For example, in a topological space (X, τ), if $g = PO(X)$, we get Theorem 2.8 of [23], if we take $g = SO(X)$, we get Proposition 3 of [28].

THEOREM 2.3. *Let (X, g) be a GTS and $\{x_\alpha\}_{\alpha\in D}$ be a net in X such that $g\text{-}cp\{x_\alpha\}_{\alpha\in D}$ is finite. Then $\{x_\alpha\}_{\alpha\in D}$ satisfies $P(I)$ or $P(II)$.*

Proof. We shall prove the Theorem by the method of induction. Let $\{x_\alpha\}_{\alpha\in D}$ be a net in X such that g-$cp\{x_\alpha\}_{\alpha\in D} = \{x\}$. If possible, let $\{x_\alpha\}_{\alpha\in D}$ does not satisfy $P(I)$ and $P(II)$. Then there exist $U, V \in g$ such that $x \in U\cap V$ and $U\cap V\setminus\{x\} = \emptyset$. Let $B = \{\alpha \in D\colon x_\alpha \in U\}$, then it is cofinal in D. Then $\{x_\beta\}_{\beta\in B}$ is a net in X. Since $\{x_\alpha\}_{\alpha\in D}$ does not satisfy $P(I)$, $\{\beta \in B\colon x_\beta = x\}$ is residual in B and for $x \in V \in g$, $\{x_\beta\}_{\beta\in B}$ is not frequently in V which implies that $x \notin g$-$cp\{x_\beta\}_{\beta\in B}$. Since g-$cp\{x_\beta\}_{\beta\in B} \subseteq g$-$cp\{x_\alpha\}_{\alpha\in D} = \{x\}$, g-$cp\{x_\beta\}_{\beta\in B} = \emptyset$-a contradiction.

Assume that the statement is true for all natural numbers less than k, where $k > 1$. Let $\{x_\alpha\}_{\alpha\in D}$ be a net in X such that g-$cp\{x_\alpha\}_{\alpha\in D}$ has exactly k elements. We need to show that $\{x_\alpha\}_{\alpha\in D}$ satisfy $P(I)$ or $P(II)$. For suppose not, let $x \in g$-$cp\{x_\alpha\}_{\alpha\in D}$. Then there exist $U, V \in g$ such that $x \in U \cap V$ and $U\cap V\setminus\{x\} = \emptyset$. Let $B = \{\alpha \in D\colon x_\alpha \in U\}$, then it is cofinal in D. Since $\{x_\alpha\}_{\alpha\in D}$ does not satisfy $P(I)$, then $\{\beta \in B\colon x_\beta = x\}$ is residual in B and $V \in g$ such that $x \in V$ and $\{x_\beta\}_{\beta\in B}$ is not frequently in V which implies that $x \notin g$-$cp\{x_\beta\}_{\beta\in B}$. Since g-$cp\{x_\beta\}_{\beta\in B} \subseteq g$-$cp\{x_\alpha\}_{\alpha\in D}$, $x \in g$-$cp\{x_\alpha\}_{\alpha\in D}$ and $x \notin g$-$cp\{x_\beta\}_{\beta\in B}$, then g-$cp\{x_\beta\}_{\beta\in B}$ has less than k elements, which implies that $\{x_\beta\}_{\beta\in B}$ satisfies $P(I)$ or $P(II)$-a contradiction. ∎

This theorem contains several results in literature. For example, in a topological space (X,τ), if $g = PO(X)$, we get Theorem 2.2 of [23], if we take $g = SO(X)$, we get Theorem 3.5 of [13].

Definition 2.4. A GTS (X, g) is said to be anti-g-compact if the only g-compact subsets of X are those which have to be g-compact because of their cardinality, i.e., the finite subsets.

Theorem 2.4. *A GTS (X, g) is anti-g-compact iff for each point $x \in X$ and each infinite subset A of X, there is a g-open set U containing x such that $A \setminus U$ is not g-compact relative to X.*

Proof. Let (X, g) be anti-g-compact. Since $A \cup \{x\}$ is not g-compact, there is a cover $\mathcal{U}$ of $A \cup \{x\}$ by g-open subsets of X which has no finite subcover. Then there is some $U \in \mathcal{U}$ such that $x \in U$. Then $\mathcal{U} \setminus \{U\}$ is a cover of $A \setminus U$ by g-open subsets of X which has no finite subcover. Thus $A \setminus U$ is not g-compact.

Conversely, let A be a g-compact subset of X. If A is finite, then there is nothing to show. Let A be infinite and $x \in A$. By hypothesis, there is a g-open set U containing x such that $A \setminus U$ is not g-compact. Hence there is a cover $\mathcal{U}$ of $A \setminus U$ by g-open subsets of X which has no finite subcover. Then $\mathcal{U} \cup \{U\}$ is a cover of A by g-open subsets of X which has no finite subcover. Hence A is not g-compact. ∎

If we consider a topological space (X,τ) with $g = SO(X)$, then we get Proposition 8 of [28].

THEOREM 2.5. *A GTS (X,g) which is not g-compact has an infinite subset which is anti-g-compact.*

PROOF. Let $\mathcal{U}$ be a cover of X by g-open subsets, which has no finite subcover. Let $x_1 \in X$. Then there exists $U_1 \in \mathcal{U}$ such that $x_1 \in U_1$. Then there is a point $x_2 \in X \setminus U_1$ and a set $U_2 \in \mathcal{U}$ such that $x_2 \in U_2$. There is a point $x_3 \in X \setminus (U_1 \cup U_2)$ and so on, by induction for each positive integer n, there is a point $x_n \in U_n \in \mathcal{U}$ and $x_{n+1} \in X \setminus \cup_{i=1}^{n} U_i$. Then $\{x_n : n \in N\}$ is infinite and anti-g-compact. ∎

This theorem contains several results in literature. For example, in a topological space (X,τ), if we take $g = SO(X)$ it gives that any space (X,τ) which is not semi-compact has an infinite subset which is anti-semi-compact, Proposition 9 of [28].

THEOREM 2.6. *Let (X,g_1) and (X,g_2) be two GTS's such that $g_1 \subseteq g_2$ and (X,g_1) is anti-g_1-compact. Then (X,g_2) is anti-g_2-compact.*

PROOF. Suppose A is an infinite g_2-compact subset of X. We claim that A is a g_1-compact subset of X. Let $\{V_\alpha : \alpha \in D\}$ be a cover of A by g_1-open subsets of X. Since $g_1 \subseteq g_2$, $\{V_\alpha : \alpha \in D\}$ is a family of g_2-open subsets of X. By g_2-compactness of A, there must exist a finite subset D_0 of D such that $A \subseteq$ $\subseteq \cup\{V_\alpha : \alpha \in D_0\}$. This shows that A is g_1-compact – a contradiction. ∎

DEFINITION 2.5. [8] Let (X,g_1) and (Y,g_2) be two GTS's. A mapping $f : (X,g_1) \to (Y,g_2)$ is said to be (g_1,g_2)-continuous iff $f^{-1}(G_2) \in g_1$ for each $G_2 \in g_2$.

THEOREM 2.7. *Let (X,g_1) and (Y,g_2) be two GTS's and $f : (X,g_1) \to (Y,g_2)$ be a (g_1,g_2)-continuous surjection. If (X,g_1) is g_1-compact, then (Y,g_2) is g_2-compact.*

PROOF. Straightforward and hence omitted. ∎

THEOREM 2.8. *Let $f : (X,g_1) \to (Y,g_2)$ be a (g_1,g_2)-continuous map. If $\{x_\alpha\}_{\alpha\in D}$ be a net in X which g_1-converges to x, then the net $\{f(x_\alpha)\}_{\alpha\in D}$ g_2-converges to $f(x)$.*

PROOF. Straightforward and hence omitted. ∎

That the converse is false follows from the next example.

EXAMPLE 2.1. Let $X = \{a,b,c\}$ and $g_1 = \{\emptyset, \{a\}, \{a,b\}, \{a,c\}, X\}$ and $g_2 =$ $= \{\emptyset, \{a\}, \{b\}, \{c\}, \{b,c\}, \{a,c\}, \{a,b\}, X\}$ be two GT's on X. Consider the function $f\colon (X, g_1) \to (X, g_2)$ defined by $f(x) = x$, for all $x \in X$. Then for every constant net $\{x_\alpha\}$ in X g_1-converging to x, $f(\{x_\alpha\})$ g_2-converges to $f(x)$, but clearly, f is not (g_1, g_2)-continuous.

3. gH-closedness in generalized topological spaces

DEFINITION 3.1. A non-void subset A of a generalized topological space (X, g) is said to be g-closed relative to X if for every cover $\{U_\alpha\colon \alpha \in \Lambda\}$ (Λ being some index set) of A by g-open sets of X, there exists a finite subset Λ_0 of Λ such that $A \subseteq \cup\{c_g U_\alpha\colon \alpha \in \Lambda_0\}$. If, in addition, $A = X$, then X is called a g-closed space.

DEFINITION 3.2. A filter base $\mathcal{F}$ on a GTS (X, g) is said to be g-convergent to a point $x \in X$ if for each $V \in g$ containing x, there exists $F \in \mathcal{F}$ such that $F \subseteq c_g V$ and a filter base $\mathcal{F}$ is said to g-accumulate at $x \in X$ if $c_g V \cap F \neq \emptyset$, for each $x \in V \in g$ and each $F \in \mathcal{F}$.

THEOREM 3.1. *For a GTS (X, g) the followings are equivalent:*

(i) (X, g) is g-closed;

(ii) Every maximal filter-base g-converges to some point of X;

(iii) Every filter-base accumulates at some point of X;

(iv) For every family $\{V_\alpha\colon \alpha \in D\}$ of g-closed subsets such that $\cap\{V_\alpha\colon \alpha \in D\} = \emptyset$, there exists a finite subset D_0 of D such that $\cap\{i_g V_\alpha\colon \alpha \in$ $\in D_0\} = \emptyset$.

PROOF. (i) $\Rightarrow$ (ii): Let $\mathcal{F}$ be a maximal filter-base on X. Suppose $\mathcal{F}$ does not g-converge to any point of X. Since $\mathcal{F}$ is maximal, $\mathcal{F}$ does not g-accumulate to any point of X. Then for each $x \in X$, there exist $F_x \in \mathcal{F}$ and $V_x \in g$ containing x such that $c_g V_x \cap F_x = \emptyset$. The family $\{V_x\colon x \in X\}$ is a cover of X by g-open sets of X. Then by (i), there exist finite number of points $x_1, x_2, \ldots, x_n$ in X such that $X = \cup\{c_g V_{x_i}\colon i = 1, 2, \ldots, n\}$. Since $\mathcal{F}$ is a filter-base on X, there exists $F_0 \in \mathcal{F}$ such that $F_0 \subseteq \cap\{F_{x_i}\colon i = 1, 2, \ldots, n\}$. Therefore we obtain $F_0 = \emptyset$-a contradiction.

(ii) $\Rightarrow$ (iii): Let F be any filter-base on X. Then there exists a maximal filter-base $\mathcal{F}_0$ such that $\mathcal{F} \subseteq \mathcal{F}_0$. By (ii), $\mathcal{F}_0$ g-converges to some point $x \in X$. Then for every $F \in \mathcal{F}$ and every $V \in g$ with $x \in V$, there exists $F_0 \in \mathcal{F}_0$ such that $F_0 \subseteq c_g V$, hence $\emptyset \neq F_0 \cap F \subseteq c_g V \cap F$. This shows that $\mathcal{F}$ accumulates at x.

(iii) $\Rightarrow$ (i): Suppose X is not g-closed. Then there exists a g-open cover $\{V_\alpha\colon \alpha \in D\}$ of X such that $\cup\{c_g V_\alpha\colon \alpha \in D_0\} \neq X$ for each finite subset D_0 of

D. Let $G(D)$ denotes the ideal of all finite subsets of D. Since $\cap\{X \setminus c_g V_\alpha \colon D_0 \in \in G(D)\} \neq \emptyset$, the family $\mathcal{F} = \{\cap(X \setminus c_g V_\alpha) \colon \alpha \in D_0\}$ is a filter- base on X. By (iii), $\mathcal{F}$ accumulates at some point $x \in X$. Since $\cup\{V_\alpha \colon \alpha \in D\}$ is a cover of X, $x \in V_{\alpha_0}$ for some $\alpha_0 \in D$. Therefore we obtain $V_{\alpha_0} \in g$ with $x \in V_{\alpha_0}$, $X \setminus c_g V_{\alpha_0} \in \mathcal{F}$ and $c_g V_{\alpha_0} \cap (X \setminus c_g V_{\alpha_0}) = \emptyset$ – a contradiction.

(iii) $\Rightarrow$ (iv): Let $\{V_\alpha \colon \alpha \in D\}$ be a family of g-closed subsets of X such that $\cap\{V_\alpha \colon \alpha \in D\} = \emptyset$. Let $G(D)$ denotes the ideal of all finite subsets of D. Assume $\cap\{i_g V_\alpha \colon \alpha \in D_0\} \neq \emptyset$ for each $D_0 \in G(D)$. Then the family $\mathcal{F} = = \cap_{D_0 \in G(D)}\{i_g V_\alpha \colon \alpha \in D_0\}$ is a filter-base on X. By (iii), $\mathcal{F}$ accumulates at some point $x \in X$. Since $\{X \setminus V_\alpha \colon \alpha \in D\}$ is a cover of X, $x \in X \setminus V_{\alpha_0}$ for some $\alpha_0 \in D$. Therefore we obtain $x \in X \setminus V_{\alpha_0} \in g$, $i_g V_{\alpha_0} \in \mathcal{F}$ and $c_g(X \setminus V_{\alpha_0}) \cap \cap i_g V_{\alpha_0} = \emptyset$-a contradiction.

(iv) $\Rightarrow$ (i): Let $\{V_\alpha \colon \alpha \in D\}$ be a cover of X by g-open subsets of X. Then $\{X \setminus V_\alpha \colon \alpha \in D\}$ is a family of g-closed subsets of X such that $\cap\{X \setminus V_\alpha \colon \alpha \in D\} = \emptyset$. By (iv), there exists a finite subset D_0 of D such that $\cap\{i_g(X \setminus V_\alpha) \colon \alpha \in \in D_0\} = \emptyset$, hence $X = \cup\{c_g V_\alpha \colon \alpha \in D_0\}$. This shows that X is g-closed. ∎

This theorem unifies many results of topology. For example, in a topological space (X, τ), if $g = \tau$, we get Theorem 2.1 of [31], if $g = \theta O(X)$, we get Theorem 5.10 of [19] if $g = PO(X)$, we get Theorem 2.8 of [12], if $g = \delta$-$PO(X)$ we get Theorem 3.3 of [27], if $g = \beta O(X)$, we get Theorem 3.5 of [4].

THEOREM 3.2. *For a GTS (X, g) the followings are equivalent:*

(i) A is g-closed relative to X;

(ii) Every maximal filter-base on X which meets A, g-converges to some point of A;

(iii) Every filter-base on X which meets A, g-accumulates at some point of A;

(iv) For every family $\{V_\alpha \colon \alpha \in D\}$ of g-closed subsets of (X, g), such that $\cap\{V_\alpha \colon \alpha \in D\} \cap A = \emptyset$, there exists a finite subset D_0 of D such that $\cap\{i_g V_\alpha \colon \alpha \in D_0\} \cap A = \emptyset$.

PROOF. The proof is similar to that of Theorem 3.3. ∎

THEOREM 3.3. *Let (X, g_1) and (X, g_2) be two GTS's such that $g_2 \subseteq g_1$. If (X, g_1) is g_1-closed, then (X, g_2) is g_2-closed.*

PROOF. Let $\{V_\alpha \colon \alpha \in D\}$ be a cover of X by g_2-open subsets of X. Since $g_2 \subseteq g_1$, $\{V_\alpha \colon \alpha \in D\}$ is a family of g_1-open cover of X. By hypothesis, there exists a finite subset D_0 of D such that $X = \cup\{c_{g_1} V_\alpha \colon \alpha \in D_0\} \subseteq \cup\{c_{g_2} V_\alpha \colon \alpha \in D_0\}$. ∎

From this theorem we observe that for any topological space (X, τ), every p-closed space is quasi H-closed [12] (by taking $g_2 = \tau$ and $g_1 = PO(X)$), every s-closed space is quasi H-closed [12] (by taking $g_1 = SO(X)$ and $g_2 = \tau$),

δ_p-closed space is quasi H-closed [27] (by taking $g_1 = \delta\text{-}PO(X)$ and $g_2 = \tau$), every β-closed space is quasi H-closed [4] (by taking $g_2 = \tau$ and $g_1 = \beta O(X)$), δ_p-closed space is p-closed [12] (by taking $g_1 = \delta\text{-}PO(X)$ and $g_2 = PO(X)$).

THEOREM 3.4. *Let the GTS (X, g) be a g-compact space. Then (X, g) is g-closed.*

PROOF. Let $\{V_\alpha : \alpha \in D\}$ be a g-open cover of X. Then by g-compactness of (X, g), there exists a finite subfamily D_0 of D such that $X = \cup\{V_\alpha : \alpha \in D_0\} \subseteq \subseteq \cup\{c_g V_\alpha : \alpha \in D_0\}$. This shows that (X, g) is g-closed. ∎

From this theorem we can conclude that in any topological space (X, τ), every compact space is quasi H-closed [5] (by taking $g = \tau$), every strongly compact space is p-closed [12] (by taking $g = PO(X)$), every semi-compact space is s-closed [6] (by taking $g = SO(X)$), every α-compact space is α-closed (by taking $g = \alpha O(X)$), every β-compact space is β-closed (by taking $g = = \beta O(X)$) [4].

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Bishwambhar Roy
Department of Mathematics
Women's Christian College
6, Greek Church Row
Kolkata-700026
India
bishwambhar_roy@yahoo.co.in

S. Jafari
College of Vestsjaelland South
Herrestraede 11
4200 Slagelse
Denmark
jafari@stofanet.dk

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ON ITERATE MINIMAL STRUCTURES AND ITERATE m-CONTINUOUS FUNCTIONS

By
TAKASHI NOIRI and VALERIU POPA

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Abstract. We introduce the notion of mIT-structures determined by operators mInt and mCl on an m-space (X, m_X). By using mIT-structures, we introduce and investigate a function $f\colon (X, \mathrm{mIT}) \to (Y, \sigma)$ called mIT-continuous. As special cases of mIT-continuity, we obtain m-semi-continuity [16] and m-precontinuity [18].

1. Introduction

Semi-open sets, preopen sets, α-open sets, β-open sets and b-open sets play an important role in the researches of generalizations of continuity in topological spaces. By using these sets, several authors introduced and studied various types of non-continuous functions. Certain of these non-continuous functions have properties similar to those of continuous functions and they hold, in many part, parallel to the theory of continuous functions.

In [20] and [21], the present authors introduced and studied the notions of minimal structures, m-spaces, m-continuity and M-continuity. Quite recently, in [14], [15], [16], [17], and [18], Min and Kim introduced the notions of m-semi-open sets, m-preopen sets, m-α-open sets, m-β-open sets which generalize the notions of m-open sets and m-semi-continuity, m-precontinuity, m-α-continuity, m-β-continuity generalizing the notions of m-continuity. In [6], [24] and [25], the notions of m-semi-open sets, m-preopen sets, m-α-open sets and m-β-open sets are also introduced and studied.

In the present paper, we introduce the notions of iterate m-structures and *mIT*-continuity and reduce the study of m-precontinuity and m-semi-continuity to the study of m-continuity.

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2. Preliminaries

Let (X, τ) be a topological space and A a subset of X. The closure of A and the interior of A are denoted by $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$, respectively. We recall some generalized open sets in topological spaces.

Definition 2.1. Let (X, τ) be a topological space. A subset A of X is said to be

(1) *α-open* [19] if $A \subset \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(A)))$,
(2) *semi-open* [10] if $A \subset \mathrm{Cl}(\mathrm{Int}(A))$,
(3) *preopen* [12] if $A \subset \mathrm{Int}(\mathrm{Cl}(A))$,
(4) *b-open* [4] if $A \subset \mathrm{Int}(\mathrm{Cl}(A)) \cup \mathrm{Cl}(\mathrm{Int}(A))$,
(5) *β-open* [1] or *semi-preopen* [3] if $A \subset \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(A)))$.

The family of all α-open (resp. semi-open, preopen, b-open, β-open) sets in (X, τ) is denoted by $\alpha(X)$ (resp. $\mathrm{SO}(X)$, $\mathrm{PO}(X)$, $\mathrm{BO}(X)$, $\beta(X)$).

Definition 2.2. Let (X, τ) be a topological space. A subset A of X is said to be *α-closed* [13] (resp. *semi-closed* [7], *preclosed* [12], *b-closed* [4], *β-closed* [1]) if the complement of A is α-open (resp. semi-open, preopen, b-open, β-open).

Definition 2.3. Let (X, τ) be a topological space and A a subset of X. The intersection of all α-closed (resp. semi-closed, preclosed, b-closed, β-closed) sets of X containing A is called the *α-closure* [13] (resp. *semi-closure* [7], *preclosure* [9], *b-closure* [4], *β-closure* [2]) of A and is denoted by $\alpha\,\mathrm{Cl}(A)$ (resp. $\mathrm{sCl}(A)$, $\mathrm{pCl}(A)$, $\mathrm{bCl}(A)$, ${}_{\beta}\,\mathrm{Cl}(A)$).

Definition 2.4. Let (X, τ) be a topological space and A a subset of X. The union of all α-open (resp. semi-open, preopen, b-open, β-open) sets of X contained in A is called the *α-interior* [13] (resp. *semi-interior* [7], *preinterior* [9], *b-interior* [4], *β-interior* [2]) of A and is denoted by $\alpha\,\mathrm{Int}(A)$ (resp. $\mathrm{sInt}(A)$, $\mathrm{pInt}(A)$, $\mathrm{bInt}(A)$, ${}_{\beta}\,\mathrm{Int}(A)$).

Definition 2.5. A function $f\colon (X, \tau) \to (Y, \sigma)$ is said to be *semi-continuous* [10] (resp. *precontinuous* [12], *α-continuous* [13], *b-continuous* [8], *β-continuous* [1]) at $x \in X$ if for each open set V containing $f(x)$, there exists a semi-open (resp. preopen, α-open, b-open, β-open) set U of X containing x such that $f(U) \subset V$. The function f is said to be *semi-continuous* (resp. *precontinuous*, *α-continuous*, *b-continuous*, *β-continuous*) if it has this property at each point $x \in X$.

3. Minimal structures and m-continuity

Definition 3.1. Let X be a nonempty set and $\mathcal{P}(X)$ the power set of X. A subfamily m_X of $\mathcal{P}(X)$ is called a *minimal structure* (briefly *m-structure*) on X [20], [21] if $\emptyset \in m_X$ and $X \in m_X$.

By (X, m_X), we denote a nonempty set X with an m-structure m_X on X and call it an m-space. Each member of m_X is said to be *m_X-open* (briefly m-open) and the complement of an m_X-open set is said to be *m_X-closed* (briefly m-closed).

Remark 3.1. Let (X, τ) be a topological space. The families τ, $\alpha(X)$, $\mathrm{SO}(X)$, $\mathrm{PO}(X)$, $\mathrm{BO}(X)$ and $\beta(X)$ are all minimal structures on X.

Definition 3.2. Let X be a nonempty set and m_X an m-structure on X. For a subset A of X, the *m_X-closure* of A and the *m_X-interior* of A are defined in [11] as follows:

(1) $\mathrm{mCl}(A) = \cap\{F\colon A \subset F, X \setminus F \in m_X\}$,
(2) $\mathrm{mInt}(A) = \cup\{U\colon U \subset A, U \in m_X\}$.

Remark 3.2. Let (X, τ) be a topological space and A a subset of X. If $m_X = \tau$ (resp. $\mathrm{SO}(X)$, $\mathrm{PO}(X)$, $\alpha(X)$, $\mathrm{BO}(X)$, $\beta(X)$), then we have

(1) $\mathrm{mCl}(A) = \mathrm{Cl}(A)$ (resp. $\mathrm{sCl}(A)$, $\mathrm{pCl}(A)$, $\alpha\,\mathrm{Cl}(A)$, $\mathrm{bCl}(A)$, ${}_\beta\,\mathrm{Cl}(A)$),
(2) $\mathrm{mInt}(A) = \mathrm{Int}(A)$ (resp. $\mathrm{sInt}(A)$, $\mathrm{pInt}(A)$, $\alpha\,\mathrm{Int}(A)$, $\mathrm{bInt}(A)$, ${}_\beta\,\mathrm{Int}(A)$).

Lemma 3.1 (Maki et al. [11]). *Let X be a nonempty set and m_X a minimal structure on X. For subsets A and B of X, the following properties hold:*

(1) $\mathrm{mCl}(X \setminus A) = X \setminus \mathrm{mInt}(A)$ *and* $\mathrm{mInt}(X \setminus A) = X \setminus \mathrm{mCl}(A)$,
(2) If $(X \setminus A) \in m_X$, *then* $\mathrm{mCl}(A) = A$ *and if* $A \in m_X$, *then* $\mathrm{mInt}(A) = A$,
(3) $\mathrm{mCl}(\emptyset) = \emptyset$, $\mathrm{mCl}(X) = X$, $\mathrm{mInt}(\emptyset) = \emptyset$ *and* $\mathrm{mInt}(X) = X$,
(4) If $A \subset B$, *then* $\mathrm{mCl}(A) \subset \mathrm{mCl}(B)$ *and* $\mathrm{mInt}(A) \subset \mathrm{mInt}(B)$,
(5) $A \subset \mathrm{mCl}(A)$ *and* $\mathrm{mInt}(A) \subset A$,
(6) $\mathrm{mCl}(\mathrm{mCl}(A)) = \mathrm{mCl}(A)$ *and* $\mathrm{mInt}(\mathrm{mInt}(A)) = \mathrm{mInt}(A)$.

Lemma 3.2 (Popa and Noiri [20]). *Let (X, m_X) be an m-space and A a subset of X. Then $x \in \mathrm{mCl}(A)$ if and only if $U \cap A \neq \emptyset$ for each $U \in m_X$ containing x.*

Definition 3.3. A minimal structure m_X on a nonempty set X is said to have *property $\mathcal{B}$* [11] if the union of any family of subsets belonging to m_X belongs to m_X.

REMARK 3.3. If (X, τ) is a topological space, then $\mathrm{SO}(X)$, $\mathrm{PO}(X)$, $\alpha(X)$, $\mathrm{BO}(X)$ and $\beta(X)$ have property $\mathcal{B}$.

LEMMA 3.3 (Popa and Noiri [23]). *Let X be a nonempty set and m_X an m-structure on X satisfying property $\mathcal{B}$. For a subset A of X, the following properties hold:*

(1) $A \in m_X$ if and only if $\mathrm{mInt}(A) = A$,
(2) A is m_X-closed if and only if $\mathrm{mCl}(A) = A$,
(3) $\mathrm{mInt}(A) \in m_X$ and $\mathrm{mCl}(A)$ is m_X-closed.

DEFINITION 3.4. A function $f\colon (X, m_X) \to (Y, \sigma)$ is said to be *m-continuous* at $x \in X$ [21] if for each open set $V \in \sigma$ containing $f(x)$, there exists $U \in m_X$ containing x such that $f(U) \subset V$. The function f is said to be m-continuous if it has this property at each $x \in X$.

REMARK 3.4. Let (X, τ) be a topological space. If $f\colon (X, m_X) \to (Y, \sigma)$ is m-continuous and $m_X = \mathrm{SO}(X)$ (resp. $\mathrm{PO}(X)$, $\alpha(X)$, $\mathrm{BO}(X)$ and $\beta(X)$), then we obtain Definition 2.5.

THEOREM 3.1 (Popa and Noiri [21]). *For a function $f\colon (X, m_X) \to (Y, \sigma)$, the following properties are equivalent:*

(1) f is m-continuous;
(2) $f^{-1}(V) = \mathrm{mInt}(f^{-1}(V))$ for every open set V of Y;
(3) $f^{-1}(F) = \mathrm{mCl}(f^{-1}(F))$ for every closed set F of Y;
(4) $\mathrm{mCl}(f^{-1}(B)) \subset f^{-1}(\mathrm{Cl}(B))$ for every subset B of Y;
(5) $f(\mathrm{mCl}(A)) \subset \mathrm{Cl}(f(A))$ for every subset A of X;
(6) $f^{-1}(\mathrm{Int}(B)) \subset \mathrm{mInt}(f^{-1}(B))$ for every subset B of Y.

COROLLARY 3.1 (Popa and Noiri [21]). *For a function $f\colon (X, m_X) \to$ $\to (Y, \sigma)$, where m_X has property $\mathcal{B}$, the following properties are equivalent:*

(1) f is m-continuous;
(2) $f^{-1}(V)$ is m_X-open in X for every open set V of Y;
(3) $f^{-1}(F)$ is m_X-closed in X for every closed set F of Y.

For a function $f\colon (X, m_X) \to (Y, \sigma)$, we define $D_m(f)$ as follows:

$$D_m(f) = \{x \in X\colon f \text{ is not } m\text{-continuous at } x\}.$$

THEOREM 3.2 (Popa and Noiri [22]). *For a function $f\colon (X, m_X) \to (Y, \sigma)$, the following properties hold:*

$$D_m(f) = \bigcup_{G \in \sigma} \{f^{-1}(G)\text{-}\,\mathrm{mInt}(f^{-1}(G))\} =$$

$$= \bigcup_{B \in \mathcal{P}(Y)} \{f^{-1}(\mathrm{Int}(B))\text{-}\, \mathrm{mInt}(f^{-1}(B))\} =$$
$$= \bigcup_{B \in \mathcal{P}(Y)} \{\mathrm{mCl}(f^{-1}(B)) - f^{-1}(\mathrm{Cl}(B))\} =$$
$$= \bigcup_{A \in \mathcal{P}(X)} \{\mathrm{mCl}(A) - f^{-1}(\mathrm{Cl}(f(A)))\} =$$
$$= \bigcup_{F \in \mathcal{F}} \{\mathrm{mCl}(f^{-1}(F)) - f^{-1}(F)\},$$

where $\mathcal{F}$ is the family of closed sets of (Y, σ).

4. Iterate m-structures and iterate m-continuity

DEFINITION 4.1. Let (X, m_X) be an m-space. A subset A of X is said to be

(1) *m-α-open* [15] if $A \subset \mathrm{mInt}(\mathrm{mCl}(\mathrm{mInt}(A)))$,
(2) *m-semi-open* [14] if $A \subset \mathrm{mCl}(\mathrm{mInt}(A))$,
(3) *m-preopen* [17] if $A \subset \mathrm{mInt}(\mathrm{mCl}(A))$,
(4) *m-β-open* [5], [25] if $A \subset \mathrm{mCl}(\mathrm{mInt}(\mathrm{mCl}(A)))$,
(5) *m-b-open* if $A \subset \mathrm{mInt}(\mathrm{mCl}(A)) \cup \mathrm{mCl}(\mathrm{mInt}(A))$.

The family of all m-α-open (resp. m-semi-open, m-preopen, m-β-open, m-b-open) sets in (X, m_X) is denoted by $m\alpha(X)$ (resp. $\mathrm{mSO}(X)$, $\mathrm{mPO}(X)$, $m\beta(X)$, $\mathrm{mBO}(X)$).

REMARK 4.1. Similar definitions of m-semi-open sets, m-preopen sets, m-α-open sets, m-β-open sets are provided in [6], [24] and [25].

Let (X, m_X) be an m-space. Then $m\alpha(X)$, $\mathrm{mSO}(X)$, $\mathrm{mPO}(X)$, $m\beta(X)$ and $\mathrm{mBO}(X)$ are determined by iterating operators mInt and mCl. Hence, they are called *m-iterate structures* and are denoted by $\mathrm{mIT}(X)$ (briefly mIT).

REMARK 4.2.

(1) It easily follows from Lemma 3.1(3)(4) that $m\alpha(X)$, $\mathrm{mSO}(X)$, $\mathrm{mPO}(X)$, $m\beta(X)$ and $\mathrm{mBO}(X)$ are minimal structures with property $\mathcal{B}$. They are also shown in Theorem 3.5 of [14], Theorem 3.4 of [17] and Theorem 3.4 of [15].
(2) Let (X, m_X) be an m-space and $\mathrm{mIT}(X)$ an iterate structure on X. If $\mathrm{mIT}(X) = \mathrm{mSO}(X)$ (resp. $\mathrm{mPO}(X)$, $m\alpha(X)$, $m\beta(X)$, $\mathrm{mBO}(X)$), then we obtain the following definitions provided in [14] (resp. [17], [15],

[18]): $\mathrm{mIT}\,\mathrm{Cl}(A) = \mathrm{msCl}(A)$ (resp. $\mathrm{mpCl}(A)$, $m\alpha\,\mathrm{Cl}(A)$, $m\beta\,\mathrm{Cl}(A)$, $\mathrm{mbCl}(A)$); $\mathrm{mIT}\,\mathrm{Int}(A) = \mathrm{msInt}(A)$ (resp. $\mathrm{mpInt}(A)$, $m\alpha\,\mathrm{Int}(A)$, $m\beta\,\mathrm{Int}(A)$, $\mathrm{mbInt}(A)$).

REMARK 4.3.

(1) By Lemmas 3.1 and 3.3, we obtain Theorems 3.7 and 3.8 of [17] and Theroems 3.8 and 3.9 of [15].

(2) By Lemma 3.2, we obtain Lemma 3.9 of [17] and Theorem 3.10 of [15].

DEFINITION 4.2. A function $f\colon (X, m_X) \to (Y, \sigma)$ is said to be *m-semi-continuous* [16] (resp. *m-precontinuous* [18]) at $x \in X$ if for each open set V containing $f(x)$, there exists m-semi-open set (resp. m-preopen) set U of X containing x such that $f(U) \subset V$. The function f is said to be *m-semi-continuous* (resp. *m-precontinuous*) if it has this property at each $x \in X$.

REMARK 4.4. By Definition 4.2 and Remark 4.2, it follows that a function $f\colon (X, m_X) \to (Y, \sigma)$ is m-semi-continuous (resp. m-precontinuous) if the function $f\colon (X, \mathrm{mSO}(X)) \to (Y, \sigma)$ (resp. $f\colon (X, \mathrm{mPO}(X)) \to (Y, \sigma)$) is m-continuous.

DEFINITION 4.3. A function $f\colon (X, m_X) \to (Y, \sigma)$ is said to be mIT-*continuous* at $x \in X$ (on X) if $f\colon (X, \mathrm{mIT}(X)) \to (Y, \sigma)$ is m-continuous at $x \in X$ (on X).

REMARK 4.5. Let (X, m_X) be a minimal space. If $\mathrm{mIT}(X) = \mathrm{mSO}(X)$ (resp. $\mathrm{mPO}(X)$, $m\alpha(X)$, $m\beta(X)$, $\mathrm{mBO}(X)$) and $f\colon (X, m_X) \to (Y, \sigma)$ is mIT-continuous, then f is m-semi-continuous [16] (resp. m-precontinuous [18]), m-α-continuous, m-β-continuous, m-b-continuous).

Since $\mathrm{mIT}(X)$ has property $\mathcal{B}$, by Theorems 3.1 and 3.2 and Corollary 3.1 we have the following theorems.

THEOREM 4.1. *For a function $f\colon (X, m_X) \to (Y, \sigma)$, the following properties are equivalent:*

(1) f is mIT-continuous;

(2) $f^{-1}(V)$ is mIT-open for every open set V of Y;

(3) $f^{-1}(F)$ is mIT-closed for every closed set F of Y;

(4) $\mathrm{mITCl}(f^{-1}(B)) \subset f^{-1}(\mathrm{Cl}(B))$ for every subset B of Y;

(5) $f(\mathrm{mITCl}(A)) \subset \mathrm{Cl}(f(A))$ for every subset A of X;

(6) $f^{-1}(\mathrm{Int}(B)) \subset \mathrm{mITInt}(f^{-1}(B))$ for every subset B of Y.

REMARK 4.6. If $\mathrm{mIT}(X) = \mathrm{mSO}(X)$ (resp. $\mathrm{mPO}(X)$), then by Theorem 4.1 we obtain Theorem 3.3 of [16] (resp. Theorem 3.3 of [18]).

For a function $f\colon (X, m_X) \to (Y, \sigma)$, we define $D_{\mathrm{mIT}}(f)$ as follows:

$$D_{\mathrm{mIT}}(f) = \{x \in X\colon f \text{ is not } \mathrm{mIT}\text{-continuous at } x\}.$$

THEOREM 4.2. *For a function $f\colon (X, m_X) \to (Y, \sigma)$, the following properties hold:*

$$\begin{aligned} D_{\mathrm{mIT}}(f) &= \bigcup_{G\in\sigma}\{f^{-1}(G)\text{-}\,\mathrm{mITInt}(f^{-1}(G))\} = \\ &= \bigcup_{B\in\mathcal{P}(Y)}\{f^{-1}(\mathrm{Int}(B))\text{-}\,\mathrm{mITInt}(f^{-1}(B))\} = \\ &= \bigcup_{B\in\mathcal{P}(Y)}\{\mathrm{mITCl}(f^{-1}(B)) - f^{-1}(\mathrm{Cl}(B))\} = \\ &= \bigcup_{A\in\mathcal{P}(X)}\{\mathrm{mITCl}(A) - f^{-1}(\mathrm{Cl}(f(A)))\} = \\ &= \bigcup_{F\in\mathcal{F}}\{\mathrm{mITCl}(f^{-1}(F)) - f^{-1}(F)\}, \end{aligned}$$

where $\mathcal{F}$ is the family of closed sets of (Y, σ).

5. Some properties of mIT-continuous functions

Since the study of mIT-continuity is reduced from the study of m-continuity, the properties of mIT-continuous functions follow from the properties of m-continuous functions in [21].

DEFINITION 5.1. An m-space (X, m_X) is said to be m-T_2 [21] if for each distinct points $x, y \in X$, there exist $U, V \in m_X$ containing x and y, respectively, such that $U \cap V = \emptyset$.

DEFINITION 5.2. An m-space (X, m_X) is said to be mIT-T_2 if the m-space $(X, \mathrm{mIT}(X))$ is m-T_2.

Hence, an m-space (X, m_X) is mIT-T_2 if for each distinct points $x, y \in X$, there exist $U, V \in \mathrm{mIT}(X)$ containing x and y, respectively, such that $U \cap V = \emptyset$.

REMARK 5.1. Let (X, m_X) be an m-space. If $\mathrm{mIT}(X) = \mathrm{mSO}(X)$ (resp. $\mathrm{mPO}(X)$), then by Definition 5.2 we obtain the definition of m-semi-T_2 spaces in [16] (resp. m-pre-T_2-spaces in [18]).

LEMMA 5.1 (Popa and Noiri [21]). *If $f\colon (X, m_X) \to (Y, \sigma)$ is an m-continuous injection and (Y, σ) is a Hausdorff space, then (X, m_X) is m-T_2.*

THEOREM 5.1. *If* $f\colon (X, m_X) \to (Y, \sigma)$ *is an* mIT*-continuous injection and* (Y, σ) *is a Hausdorff space, then* X *is* mIT-T_2.

PROOF. The proof follows from Definition 5.2 and Lemma 5.1. ∎

REMARK 5.2. Let (X, m_X) be an m-space. If $\mathrm{mIT}(X) = \mathrm{mSO}(X)$ (resp. $\mathrm{mPO}(X)$), then by Theorem 5.1 we obtain Theorem 3.12 in [16] (resp. Theorem 3.10 in [18]).

DEFINITION 5.3. An m-space (X, m_X) is said to be *m-compact* [21] if every cover of X by m_X-open sets of X has a finite subcover.

A subset K of an m-space (X, m_X) is said to be *m-compact* [21] if every cover of K by m_X-open sets of X has a finite subcover.

DEFINITION 5.4. An m-space (X, m_X) is said to be mIT*-compact* if the m-space $(X, \mathrm{mIT}(X))$ is m-compact.

A subset K of an m-space (X, m_X) is said to be mIT*-compact* if every cover of K by mIT-open sets of X has a finite subcover.

REMARK 5.3. Let (X, m_X) be an m-space. If $\mathrm{mIT}(X) = \mathrm{mSO}(X)$ (resp. $\mathrm{mPO}(X)$), then by Definition 5.4 we obtain the definition of m-semicompact spaces in [16] (resp. m-precompact spaces in [18]).

LEMMA 5.2 (Popa and Noiri [21]). *Let* $f\colon (X, m_X) \to (Y, \sigma)$ *be an* m*-continuous function. If* K *is an* m*-compact set of* X*, then* $f(K)$ *is compact.*

THEOREM 5.2. *If* $f\colon (X, m_X) \to (Y, \sigma)$ *is an* mIT*-continuous function and* K *is an* mIT*-compact set of* X*, then* $f(K)$ *is compact.*

PROOF. The proof follows from Definition 5.4 and Lemma 5.2. ∎

REMARK 5.4. Let (X, m_X) be an m-space. If $\mathrm{mIT}(X) = \mathrm{mSO}(X)$ (resp. $\mathrm{mPO}(X)$), then by Theorem 5.2 we obtain Theorem 3.16 of [16] (resp. Theorem 3.13 in [18]).

DEFINITION 5.5. A function $f\colon (X, m_X) \to (Y, \sigma)$ is said to have a *strongly m-closed graph* (resp. *m-closed graph*) [21] if for each $(x, y) \in (X \times Y) - \mathrm{G}(f)$, there exist $U \in m_X$ containing x and an open set V of Y containing y such that $[U \times \mathrm{Cl}(V)] \cap \mathrm{G}(f) = \emptyset$ (resp. $[U \times V] \cap \mathrm{G}(f) = \emptyset$).

DEFINITION 5.6. A function $f\colon (X, m_X) \to (Y, \sigma)$ is said to have a *strongly* mIT*-closed graph* (resp. mIT*-closed graph*) if a function $f\colon (X, \mathrm{mIT}(X)) \to (Y, \sigma)$ has a *strongly m-closed graph* (resp. *m-closed graph*).

Hence, a function $f\colon (X, m_X) \to (Y, \sigma)$ has a strongly mIT-closed graph (resp. mIT-closed graph) if for each $(x, y) \in (X \times Y) - \mathrm{G}(f)$, there exist $U \in \mathrm{mIT}(X)$ containing x and an open set V of Y containing y such that $[U \times \mathrm{Cl}(V)] \cap \mathrm{G}(f) = \emptyset$ (resp. $[U \times V] \cap \mathrm{G}(f) = \emptyset$).

REMARK 5.5. Let (X, m_X) be an m-space. If $\mathrm{mIT}(X) = \mathrm{mSO}(X)$ (resp. $\mathrm{mPO}(X)$), then by Definition 5.6 we obtain Definition 3.6 of [16] (resp. Definition 3.6 of [18]).

LEMMA 5.3 (Popa and Noiri [21]). *If $f\colon (X, m_X) \to (Y, \sigma)$ is an m-continuous function and (Y, σ) is a Hausdorff space, then f has a strongly m-closed graph.*

THEOREM 5.3. *If $f\colon (X, m_X) \to (Y, \sigma)$ is an mIT-continuous function and (Y, σ) is a Hausdorff space, then f has a strongly mIT-closed graph.*

PROOF. The proof follows from Definition 5.6 and Lemma 5.3. ∎

REMARK 5.6. Let (X, m_X) be an m-space. If $\mathrm{mIT}(X) = \mathrm{mSO}(X)$ (resp. $\mathrm{mPO}(X)$), then by Theorem 5.3 we obtain Theorem 3.8 of [16] (resp. Theorem 3.8 in [18]).

LEMMA 5.4 (Popa and Noiri [21]). *If $f\colon (X, m_X) \to (Y, \sigma)$ is a surjective function with a strongly m-closed graph, then (Y, σ) is Hausdorff.*

THEOREM 5.4. *If $f\colon (X, m_X) \to (Y, \sigma)$ is a surjective function with a strongly mIT-closed graph, then (Y, σ) is Hausdorff.*

PROOF. The proof follows from Definition 5.6 and Lemma 5.4. ∎

REMARK 5.7. Let (X, m_X) be an m-space. If $\mathrm{mIT}(X) = \mathrm{mSO}(X)$, then by Theorem 5.4 we obtain Theorem 3.10 of [16].

LEMMA 5.5 (Popa and Noiri [21]). *Let (X, m_X) be an m-space and m_X have property $\mathcal{B}$. If $f\colon (X, m_X) \to (Y, \sigma)$ is an injective m-continuous function with an m-closed graph, then X is m-T_2.*

THEOREM 5.5. *If $f\colon (X, m_X) \to (Y, \sigma)$ is an injective mIT-continuous function with an mIT-closed graph, then X is mIT-T_2.*

PROOF. The proof follows from Definition 5.6 and Lemma 5.5. ∎

REMARK 5.8. Let (X, m_X) be an m-space. If $\mathrm{mIT}(X) = \mathrm{mSO}(X)$ (resp. $\mathrm{mPO}(X)$), then by Theorem 5.5 we obtain Theorem 3.13 of [16] (resp. Theorem 3.11 in [18]).

DEFINITION 5.7. An m-space (X, m_X) is said to be *m-connected* [21] if X cannot be written as the union of two nonempty sets of m_X.

DEFINITION 5.8. An m-space (X, m_X) is said to be mIT-*connected* if an m-space $(X, \mathrm{mIT}(X))$ is m-connected.

Hence, an m-space $(X, \mathrm{mIT}(X))$ is m-connected if X cannot be written as the union of two nonempty sets of $\mathrm{mIT}(X)$.

LEMMA 5.6. *Let $f\colon (X, m_X) \to (Y, \sigma)$ be a function, where m_X has property $\mathcal{B}$. If f is an m-continuous surjection and (X, m_X) is m-connected, then (Y, σ) is connected.*

THEOREM 5.6. *Let $f\colon (X, m_X) \to (Y, \sigma)$ be a function. If f is an mIT-continuous surjection and (X, m_X) is mIT-connected, then (Y, σ) is connected.*

PROOF. The proof follows from Definition 5.8, Lemma 5.6 and the fact that $\mathrm{mIT}(X)$ has property $\mathcal{B}$. ∎

6. mIT-open sets for $m_X = \mathrm{SO}(X)$, $\mathrm{PO}(X)$

Let (X, τ) be a topological space. Among generalized open sets in Definition 2.1, the following implications are well-known:

DIAGRAM I

$$\begin{array}{ccccccc} \text{open} & \Rightarrow & \alpha\text{-open} & \Rightarrow & \text{semi-open} & & \\ & & \Downarrow & & \Downarrow & & \\ & & \text{preopen} & \Rightarrow & b\text{-open} & \Rightarrow & \beta\text{-open} \end{array}$$

Let (X, m_X) be an m-space. Similarly, among mIT-open sets in Definition 4.1, the following implications hold:

DIAGRAM II

$$\begin{array}{ccccccc} m\text{-open} & \Rightarrow & m\text{-}\alpha\text{-open} & \Rightarrow & m\text{-semi-open} & & \\ & & \Downarrow & & \Downarrow & & \\ & & m\text{-preopen} & \Rightarrow & m\text{-}b\text{-open} & \Rightarrow & m\text{-}\beta\text{-open} \end{array}$$

In this section, we investigate the properties of inverse implications in Diagram II for $m_X = \mathrm{SO}(X)$, $\mathrm{PO}(X)$.

LEMMA 6.1 (Andrijević [3], [4]). *For a subset A of a topological space (X, τ), the following properties hold:*

(1) $\mathrm{sInt}(A) = A \cap \mathrm{Cl}(\mathrm{Int}(A))$, $\mathrm{pInt}(A) = A \cap \mathrm{Int}(\mathrm{Cl}(A))$ *(Theorem 1.5 of [3]),*

(2) $\mathrm{sInt}(\mathrm{sCl}(A)) = \mathrm{sCl}(A) \cap \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(A)))$ *(Theorem 3.1 of [3]),*
(3) $\mathrm{sCl}(\mathrm{sInt}(\mathrm{sCl}(A))) = \mathrm{sInt}(\mathrm{sCl}(A))$ *(Corollary 3.2 of [3]),*
(4) $\mathrm{pCl}(\mathrm{pInt}(\mathrm{pCl}(A))) = \mathrm{pCl}(\mathrm{pInt}(A))$ *(Corollary 3.8 of [3]),*
(5) A is b-open if and only if $A \subset \mathrm{pCl}(\mathrm{pInt}(A))$ (Proposition 2.1 of [4]).

THEOREM 6.1. *Let (X, τ) be a topological space and $m_X = \mathrm{SO}(X)$. Then the following properties hold:*
(1) m-open $\Leftrightarrow$ m-α-open $\Leftrightarrow$ m-semi-open,
(2) m-preopen $\Leftrightarrow$ m-b-open $\Leftrightarrow$ m-β-open,
(3) m-preopen $\Leftrightarrow$ β-open.

PROOF.
(1) Let A be an m-semi-open set. Then, by Lemma 6.1(1), we have $A \subset \subset \mathrm{mCl}(\mathrm{mInt}(A)) = \mathrm{sCl}(\mathrm{sInt}(A)) \subset \mathrm{Cl}(\mathrm{sInt}(A)) = \mathrm{Cl}(A \cap \cap \mathrm{Cl}(\mathrm{Int}(A))) \subset \mathrm{Cl}(\mathrm{Int}(A))$. Therefore, A is semi-open, that is, m-open.
(2) Let A be an m-β-open set. Then, by Lemma 6.1(3), we have $A \subset \subset \mathrm{mCl}(\mathrm{mInt}(\mathrm{mCl}(A))) = \mathrm{sCl}(\mathrm{sInt}(\mathrm{sCl}(A))) = \mathrm{sInt}(\mathrm{sCl}(A))) = = \mathrm{mInt}(\mathrm{mCl}(A)))$. Therefore, A is m-preopen.
(3) Let A be an m-preopen set. Then, by Lemma 6.1(2), we have $A \subset \subset \mathrm{mInt}(\mathrm{mCl}(A)) = \mathrm{sInt}(\mathrm{sCl}(A)) = \mathrm{sCl}(A) \cap \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(A))) \subset \subset \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(A)))$. Therefore, A is β-open.

Conversely, let A be a β-open set. Then, $A \subset \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(A)))$ and hence

$$A \subset \mathrm{sCl}(A) \cap \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(A))) = \mathrm{sInt}(\mathrm{sCl}(A)) = \mathrm{mInt}(\mathrm{mCl}(A)).$$

Therefore, A is m-preopen. ∎

COROLLARY 6.1. *Let (X, τ) be a topological space and $m_X = \mathrm{SO}(X)$. Then, a function $f\colon (X, \tau) \to (Y, \sigma)$ is semi-continuous if and only if $f\colon (X, m_X) \to (Y, \sigma)$ is m-semi-continuous.*

PROOF. This is an immediate consequence of Theorem 6.1. ∎

THEOREM 6.2. *Let (X, τ) be a topological space and $m_X = \mathrm{PO}(X)$. Then the following properties hold:*
(1) m-open $\Leftrightarrow$ m-α-open $\Leftrightarrow$ m-preopen,
(2) m-semi-open $\Leftrightarrow$ m-b-open $\Leftrightarrow$ m-β-open,
(3) m-semi-open $\Leftrightarrow$ b-open.

PROOF.

(1) Let A be an m-preopen set. Then, by Lemma 6.1(1), we have $A \subset \subset \mathrm{mInt}(\mathrm{mCl}(A)) = \mathrm{pInt}(\mathrm{pCl}(A)) \subset \mathrm{pInt}(\mathrm{Cl}(A)) = \mathrm{Cl}(A) \cap \cap \mathrm{Int}(\mathrm{Cl}(\mathrm{Cl}(A))) = \mathrm{Int}(\mathrm{Cl}(A))$. Therefore, A is preopen, that is, m-open.

(2) Let A be an m-β-open set. Then, by Lemma 6.1(4), we have $A \subset \subset \mathrm{mCl}(\mathrm{mInt}(\mathrm{mCl}(A))) = \mathrm{pCl}(\mathrm{pInt}(\mathrm{pCl}(A))) = \mathrm{pCl}(\mathrm{pInt}(A)) = = \mathrm{mCl}(\mathrm{mInt}(A)))$. Therefore, A is m-semi-open.

(3) By Lemma 6.1(5), the following properties are equivalent:
   (a) A is m-semi-open,
   (b) $A \subset \mathrm{mCl}(\mathrm{mInt}(A)) = \mathrm{pCl}(\mathrm{pInt}(A))$,
   (c) A is b-open.

∎

COROLLARY 6.2. *Let (X, τ) be a topological space and $m_X = \mathrm{PO}(X)$. Then, a function $f\colon (X, \tau) \to (Y, \sigma)$ is precontinuous if and only if $f\colon (X, m_X) \to \to (Y, \sigma)$ is m-precontinuous.*

PROOF. This is an immediate consequence of Theorem 6.2. ∎

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Takashi Noiri
2949-1 Shiokita-cho, Hinagu
Yatsushiro-shi, Kumamoto-ken
869-5142
Japan
t.noiri@nifty.com

Valeriu Popa
Department of Mathematics
Univ. Vasile Alecsandri of Bacău
600 115 Bacău
Rumania
vpopa@ub.ro

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FROM QUASI-ENTROPY

By
DÉNES PETZ*

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Abstract. The subject is the overview of the use of quasi-entropy in finite dimensional spaces. Matrix monotone functions and relative modular operators are used. The origin is the relative entropy and the f-divergence, monotone metrics, covariance and the χ^2-divergence are the most important particular cases.

Quasi-entropy was introduced by Petz in 1985 as the quantum generalization of Csiszár's f-divergence in the setting of matrices or von Neumann algebras. The important special case was the relative entropy of Umegaki and Araki. In this paper the applications are overviewed in the finite dimensional setting. Quasi-entropy has some similarity to the monotone metrics, in both cases the modular operator is included, but there is an essential difference: In the quasi-entropy two density matrices are included and for the monotone metric on foot-point density matrices. In this paper two density matrices are introduced in the monotone metric style.

1. The concept of quasi-entropy

Let $\mathcal{M}$ denote the algebra of $n \times n$ matrices with complex entries. For positive definite matrices $\rho_1, \rho_2 \in \mathcal{M}$, for $A \in \mathcal{M}$ and a function $f\colon \mathbb{R}^+ \to \mathbb{R}$, the

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quasi-entropy is defined as

$$S_f^A(\rho_1\|\rho_2) \quad := \langle A\rho_2^{1/2}, f(\Delta(\rho_1/\rho_2))(A\rho_2^{1/2})\rangle \tag{1}$$
$$= \mathrm{Tr}\,\rho_2^{1/2}(A^*f(\Delta(\rho_1/\rho_2))A\rho_2^{1/2}),$$

where $\langle B, C\rangle := \mathrm{Tr}\,B^*C$ is the so-called *Hilbert-Schmidt inner product* and $\Delta(\rho_1/\rho_2)\colon \mathcal{M} \to \mathcal{M}$ is a linear mapping acting on matrices:

$$\Delta(\rho_1/\rho_2)B = \rho_1 B\rho_2^{-1}.$$

This concept was introduced by Petz in 1985, see [19, 20], or Chapter 7 in [18]. (The relative modular operator $\Delta(\rho_1/\rho_2)$ was born in the context of von Neumann algebras and the paper of Araki [1] had a big influence even in the matrix case.) The quasi-entropy is the quantum generalization of the f-divergence of Csiszár used in classical information theory (and statistics) [2, 16]. Therefore the quantum f-divergence could be another terminology as in [10].

The definition of quasi-entropy can be formulated with mean. For a function f the corresponding mean is defined as $m_f(x,y) = f(x/y)y$ for positive numbers, or for commuting positive definite matrices. The linear mappings

$$\mathbb{L}_{\rho_1}X = \rho_1 X \quad \text{and} \quad R_{\rho_2}X = X\rho_2$$

are positive and commuting. The mean m_f makes sense and

$$S_f^A(\rho_1\|\rho_2) = \langle A, m_f(\mathbb{L}_{\rho_1}, R_{\rho_2})A\rangle. \tag{2}$$

Let $\alpha\colon \mathcal{M}_0 \to \mathcal{M}$ be a mapping between two matrix algebras. The dual $\alpha^*\colon \mathcal{M} \to \mathcal{M}_0$ with respect to the Hilbert-Schmidt inner product is positive if and only if α is positive. Moreover, α is unital if and only if α^* is trace preserving. $\alpha\colon \mathcal{M}_0 \to \mathcal{M}$ is called a *Schwarz mapping* if

$$\alpha(B^*B) \geq \alpha(B^*)\alpha(B) \tag{3}$$

for every $B \in \mathcal{M}_0$.

The quasi-entropies are monotone and jointly convex [18, 20].

THEOREM 1. *Assume that $f\colon \mathbb{R}^+ \to \mathbb{R}$ is an operator monotone function with $f(0) \geq 0$ and $\alpha\colon \mathcal{M}_0 \to \mathcal{M}$ is a unital Schwarz mapping. Then*

$$S_f^A(\alpha^*(\rho_1)\|\alpha^*(\rho_2)) \geq S_f^{\alpha(A)}(\rho_1\|\rho_2) \tag{4}$$

holds for $A \in \mathcal{M}_0$ and for invertible density matrices ρ_1 and ρ_2 from the matrix algebra $\mathcal{M}$.

PROOF. The proof is based on inequalities for operator monotone and operator concave functions. First note that

$$S_{f+c}^A(\alpha^*(\rho_1)\|\alpha^*(\rho_2)) = S_f^A(\alpha^*(\rho_1)\|\alpha^*(\rho_2)) + c\ \mathrm{Tr}\,\rho_1\alpha(A^*A))$$

and

$$S_{f+c}^{\alpha(A)}(\rho_1\|\rho_2) = S_f^{\alpha(A)}(\rho_1\|\rho_2) + c\ \mathrm{Tr}\,\rho_1(\alpha(A)^*\alpha(A))$$

for a positive constant c. Due to the Schwarz inequality (3), we may assume that $f(0) = 0$.

Let $\Delta := \Delta(\rho_1/\rho_2)$ and $\Delta_0 := \Delta(\alpha^*(\rho_1)/\alpha^*(\rho_2))$. The operator

$$VX\alpha^*(\rho_2)^{1/2} = \alpha(X)\rho_2^{1/2} \qquad (X \in \mathcal{M}_0) \tag{5}$$

is a contraction:

$$\begin{aligned}\|\alpha(X)\rho_2^{1/2}\|^2 &= \mathrm{Tr}\,\rho_2(\alpha(X)^*\alpha(X))\\ &\leq \mathrm{Tr}\,\rho_2(\alpha(X^*X) = \mathrm{Tr}\,\alpha^*(\rho_2)X^*X = \|X\alpha^*(\rho_2)^{1/2}\|^2\end{aligned}$$

since the Schwarz inequality is applicable to α. A similar simple computation gives that

$$V^*\Delta V \leq \Delta_0\,. \tag{6}$$

Since f is operator monotone, we have $f(\Delta_0) \geq f(V^*\Delta V)$. Recall that f is operator concave, therefore $f(V^*\Delta V) \geq V^*f(\Delta)V$ and we conclude

$$f(\Delta_0) \geq V^*f(\Delta)V\,. \tag{7}$$

Application to the vector $A\alpha^*(\rho_2)^{1/2}$ gives the statement. ∎

It is remarkable that for a multiplicative α we do not need the condition $f(0) \geq 0$. Moreover, $V^*\Delta V = \Delta_0$ and we do not need the matrix monotonicity of the function f. In this case the only condition is the matrix concavity, analogously to Theorem 1. If we apply the monotonicity (4) to the embedding $\alpha(X) = X \oplus X$ of $\mathcal{M}$ into $\mathcal{M} \oplus \mathcal{M}$ and to the densities $\rho_1 = \lambda E_1 \oplus (1-\lambda)F_1$, $\rho_2 = \lambda E_2 \oplus (1-\lambda)F_2$, then we obtain the joint concavity of the quasi-entropy:

THEOREM 2. *If $f\colon \mathbb{R}^+ \to \mathbb{R}$ is an operator convex, then $S_f^A(\rho_1\|\rho_2)$ is jointly convex in the variables ρ_1 and ρ_2.*

If we consider the quasi-entropy in the terminology of means, then we can have another proof. The joint convexity of the mean is the inequality

$$f(L_{(A_1+A_2)/2}R^{-1}_{(B_1+B_2)/2})R_{(B_1+B_2)/2} \leq \tfrac{1}{2}f(L_{A_1}R^{-1}_{B_1})R_{B_1} + \tfrac{1}{2}f(L_{A_2}R^{-1}_{B_2})R_{B_2}$$

which can be simplified as

$$\begin{aligned}&f(L_{A_1+A_2}R^{-1}_{B_1+B_2})\\ &\quad\leq R^{-1/2}_{B_1+B_2}R^{1/2}_{B_1}f(L_{A_1}R^{-1}_{B_1})R^{1/2}_{B_1}R^{-1/2}_{B_1+B_2} + R^{-1/2}_{B_1+B_2}R^{1/2}_{B_2}f(L_{A_2}R^{-1}_{B_2})R^{1/2}_{B_2}R^{-1/2}_{B_1+B_2}\\ &\quad\leq Cf(L_{A_1}R^{-1}_{B_1})C^* + Df(L_{A_2}R^{-1}_{B_2})D^*.\end{aligned}$$

Here $CC^* + DD^* = I$ and

$$C(L_{A_1}R_{B_1}^{-1})C^* + D(L_{A_2}R_{B_2}^{-1})D^* = L_{A_1+A_2}R_{B_1+B_2}^{-1}.$$

So the joint convexity of the quasi-entropy has the form

$$f(CXC^* + DYD^*) \le Cf(X)C^* + Df(Y)D^*$$

which is true for an operator convex function f [5, 24].

If f is operator monotone function, then it is operator concave and we have joint concavity in the previous theorem. The book [24] contains information about operator monotone functions. The standard useful properties are integral representations. The Löwner theorem is

$$f(x) = f(0) + \beta x + \int_0^\infty \frac{\lambda x}{\lambda + x}\, d\mu(\lambda)\,.$$

An operator monotone function $f\colon \mathbb{R}^+ \to \mathbb{R}^+$ will be called *standard* if $xf(x^{-1}) = f(x)$ and $f(1) = 1$. A standard function f admits a canonical representation

$$f(t) = \frac{1+t}{2} \exp \int_0^1 (1-t)^2 \frac{\lambda^2 - 1}{(\lambda + t)(1 + \lambda t)(\lambda + 1)^2} h(\lambda)\, d\lambda, \tag{8}$$

where $h\colon [0,1] \to [0,1]$ is a measurable function [6].

The concept of quasi-entropy includes many important special cases.

2. f-divergences

If ρ_2 and ρ_1 are different and $A = I$, then we have a kind of relative entropy. For $f(x) = x \log x$ we have Umegaki's relative entropy $S(\rho_1 \| \rho_2) = \mathrm{Tr}\, \rho_1(\log \rho_1 - \log \rho_2)$. (If we want a matrix monotone function, then we can take $f(x) = \log x$ and then we get $S(\rho_2 \| \rho_1)$.) Umegaki's relative entropy is the most important example, therefore the function f will be chosen to be matrix convex. This makes the probabilistic and non-commutative situation compatible as one can see in the next argument.

Let ρ_1 and ρ_2 be density matrices in $\mathcal{M}$. If in certain basis they have diagonal $p = (p_1.p_2, \dots, p_n)$ and $q = (q_1, q_2, \dots, q_n)$, then the monotonicity theorem gives the inequality

$$D_f(p\|q) \le S_f(\rho_1 \| \rho_2) \tag{9}$$

for a matrix convex function f. If ρ_1 and ρ_2 commute, them we can take the common eigenbasis and in (9) the equality appears. It is not trivial that otherwise the inequality is strict.

If ρ_1 and ρ_2 are different, then there is a choice for p and q such that they are different as well. Then

$$0 < D_f(p\|q) \leq S_f(\rho_1\|\rho_2).$$

Conversely, if $S_f(\rho_1\|\rho_2) = 0$, then $p = q$ for every basis and this implies $\rho_1 = \rho_2$. For the relative entropy, a deeper result is known. The *Pinsker-Csiszár inequality* says that

$$\|p - q\|_1^2 \leq 2D(p\|q). \tag{10}$$

This extends to the quantum case as

$$\|\rho_1 - \rho_2\|_1^2 \leq 2S(\rho_1\|\rho_2), \tag{11}$$

see [8], or [24, Chap. 3].

EXAMPLE 1. The f-divergence with $f(x) = x \log x$ is the relative entropy. It is rather popular the modification of the logarithm as

$$\log_\beta x = \frac{x^\beta - 1}{\beta} \qquad (\beta \in (0,1))$$

and the limit $\beta \to 0$ is the log. If we take $f_\beta(x) = x \log_\beta x$, then

$$S_\beta(\rho_1\|\rho_2) = \frac{\mathrm{Tr}\,\rho_1^{1+\beta}\rho_2^{-\beta} - 1}{\beta}.$$

Since f_β is operator convex, this is a good generalized entropy. It appeared in the paper [28], see also [18, Chap. 3], there γ is written instead of β and

$$S(\rho_1\|\rho_2) \leq S_\beta(\rho_1\|\rho_2) \qquad (\beta \in (0,1))$$

is proven.

The *relative entropies of degree* α

$$S_\alpha(\rho_2\|\rho_1) := \frac{1}{\alpha(1-\alpha)}\,\mathrm{Tr}(I - \rho_1^\alpha \rho_2^{-\alpha})\rho_2.$$

are essentially the same. ∎

The f-divergence is contained in details in the recent papers [26, 10].

3. WYD information

In the paper [12] the functions

$$g_p(x) = \begin{cases} \frac{1}{p(1-p)}(x - x^p) & \text{if } p \neq 1, \\ x \log x & \text{if } p = 1 \end{cases}$$

are used, this is a small reparametrization of Example 1. (Note that g_p is well-defined for $x > 0$ and $p \neq 0$.) The considered case is $p \in [1/2, 2]$, then g_p is operator concave.

For strictly positive A and B, Jenčová and Ruskai define

$$J_p(K, A, B) = \operatorname{Tr} \sqrt{B} K^* \, g_p\big(L_A R_B^{-1}\big)(K\sqrt{B})$$

which is the particular case of the quasi-entropy $S_f^K(A\|B)$ with $f = g_p$.

The joint concavity of $J_p(K, A, B)$ is stated in Theorem 2 in [12] and this is a particular case of Theorem 2 above. For $K = K^*$, we have

$$J_p(K, A, A) = -\frac{1}{2p(1-p)} \operatorname{Tr}[K, A^p][K, A^{1-p}]$$

which is the Wigner-Yanase-Dyson information (up to a constant) and extends it to the range $(0, 2]$.

4. Monotone metrics

Let $\mathcal{M}_n$ be the set of positive definite density matrices in $\mathbf{M}_n$. This is a manifold and the set of tangent vectors is $\{A = A^* \in \mathbf{M}_n \colon \operatorname{Tr} A = 0\}$. A Riemannian geometry is a set of real inner products $\gamma_D(A, B)$ on the tangent vectors [17]. By monotone metrics we mean inner product for all matrix spaces such that

$$\gamma_{\beta(D)}(\beta(A), \beta(A)) \leq \gamma_D(A, A) \tag{12}$$

for every completely positive trace preserving mapping $\beta \colon \mathbf{M}_n \to \mathbf{M}_m$.

Define $\mathbb{J}_D^f \colon \mathbf{M}_n \to \mathbf{M}_n$ as

$$\mathbb{J}_D^f = f(\mathbb{L}_D \mathbb{R}_D^{-1})\mathbb{R}_D = \mathbb{L}_D \, m_f \, \mathbb{R}_D\,, \tag{13}$$

where $f \colon \mathbb{R}^+ \to \mathbb{R}^+$ and m_f is the mean induced by the function f.

It was obtained in the paper [22] that monotone metrics with the property

$$\gamma_D(A, A) = \operatorname{Tr} D^{-1} A^2 \quad \text{if} \quad AD = DA \tag{14}$$

has the form

$$\gamma_D(A,B) = \operatorname{Tr} A(\mathbb{J}_D^f)^{-1}(B) \tag{15}$$

where f is a standard matrix monotone function. These monotone metrics are abstract Fisher informations, the condition (14) tells that in the commutative case the classical Fisher information is required. The popular case in physics corresponds to $f(x) = (1+x)/2$, this gives the SSA Fisher information.

Since

$$\operatorname{Tr} A(\mathbb{J}_D^f)^{-1}(B) = \langle (AD^{-1})D^{1/2}, \frac{1}{f}(\Delta(D/D))(AD^{-1})D^{1/2}\rangle,$$

we have

$$\gamma_D(A,A) = S_{1/f}^{AD^{-1}}(D\|D).$$

So the monotone metric is a particular case of the quasi-entropy, but there is another relation. The next example has been well-known.

EXAMPLE 2. The Boguliubov-Kubo-Mori Fisher information is induced by the function

$$f(x) = \frac{x-1}{\log x} = \int_0^1 x^t\, dt.$$

Then

$$\mathbb{J}_D^f A = \int_0^1 (\mathbb{L}_D \mathbb{R}_D^{-1})^t \mathbb{R}_D A\, dt = \int_0^1 D^t A D^{1-t}\, dt$$

and computing the inverse we have

$$\gamma_D^{BKM}(A,A) = \int_0^\infty \operatorname{Tr}(D+tI)^{-1}A(D+tI)^{-1}A\, dt.$$

A characterization is in the paper [4] and the relation with the relative entropy is

$$\gamma_D^{BKM}(A,B) = \frac{\partial^2}{\partial t \partial s} S(D+tA\|D+sB).$$

∎

Ruskai and Lesniewski discovered that all monotone Fisher informations are obtained from an f-divergence by derivation [14]:

$$\gamma_D^f(A,B) = \frac{\partial^2}{\partial t \partial s} S_F(D+tA\|D+sB)$$

The relation of the function F to the function f in this formula is

$$\frac{1}{f(t)} = \frac{F(t) + tF(t^{-1})}{(t-1)^2}. \tag{16}$$

If D runs on all positive definite matrices, conditions $\gamma_D(A,A) \in \mathbb{R}$ for self-adjoint A and (14) are not required, but the monotonicity (12) is assumed, then we have the generalized monotone metric characterized by Kumagai [13]. They have the form

$$K_\rho(A,B) = b(\operatorname{Tr}\rho)\operatorname{Tr}A^*\operatorname{Tr}B + c\langle A, (\mathbb{J}^f_\rho)^{-1}(B)\rangle,$$

where $f\colon \mathbb{R}^+ \to \mathbb{R}^+$ is matrix monotone, $f(1)=1$, $b\colon \mathbb{R}^+ \to \mathbb{R}^+$ and $c>0$.

Let $\beta\colon \mathbf{M}_n \otimes \mathbf{M}_2 \to \mathbf{M}_m$ be defined as

$$\begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} \mapsto B_{11} + B_{22}.$$

This is completely positive and trace-preserving, it is a so-called partial trace. For

$$D = \begin{bmatrix} \lambda D_1 & 0 \\ 0 & (1-\lambda)D_2 \end{bmatrix}, \qquad A = \begin{bmatrix} \lambda B & 0 \\ 0 & (1-\lambda)B \end{bmatrix}$$

the inequality (12) gives

$$\gamma_{\lambda D_1+(1-\lambda)D_2}(B,B) \le \gamma_{\lambda D_1}(\lambda B, \lambda B) + \gamma_{(1-\lambda)D_2}((1-\lambda)B, (1-\lambda)B).$$

Since $\gamma_{tD}(tA,tB) = t\gamma_D(A,B)$, we obtained the convexity.

THEOREM 3. *For a standard matrix monotone function f and for a self-adjoint matrix A the monotone metric $\gamma^f_D(A,A)$ is a convex function of D.*

This convexity relation can be reformulated from formula (15). We have the convexity of the operator $(\mathbb{J}^f_D)^{-1}$ in the positive definite D.

5. Generalized covariance

If $\rho_2 = \rho_1 = \rho$ and $A,B \in \mathcal{M}$ are arbitrary, then one can approach to the *generalized covariance* [23].

(17) $$\operatorname{qCov}^f_\rho(A,B) := \langle A\rho^{1/2}, f(\Delta(\rho/\rho))(B\rho^{1/2})\rangle - (\operatorname{Tr}\rho A^*)(\operatorname{Tr}\rho B).$$

is a generalized covariance. The first term is $\langle A, \mathbb{J}^f_\rho B\rangle$ and the covariance has some similarity to the monotone metrics.

If ρ, A and B commute, then this becomes $f(1)\operatorname{Tr}\rho A^*B - (\operatorname{Tr}\rho A^*)(\operatorname{Tr}\rho B)$. This shows that the normalization $f(1)=1$ is natural. The generalized covariance $\operatorname{qCov}^f_\rho(A,B)$ is a sesquilinear form and it is determined by $\operatorname{qCov}^f_\rho(A,A)$ when $\{A \in \mathcal{M}\colon \operatorname{Tr}\rho A = 0\}$. Formally, this is a quasi-entropy and Theorem 1 applies if f is matrix monotone. If we require the symmetry condition

$\mathrm{qCov}^f_\rho(A,A) = \mathrm{qCov}^f_\rho(A^*,A^*)$, then f should have the symmetry $xf(x^{-1}) = f(x)$.

Assume that $\mathrm{Tr}\,\rho A = \mathrm{Tr}\,\rho B = 0$ and $\rho = \mathrm{Diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$. Then

(18) $$\mathrm{qCov}^f_\rho(A,B) = \sum_{ij} \lambda_i f(\lambda_j/\lambda_i) A^*_{ij} B_{ij}.$$

The usual *symmetrized covariance* corresponds to the function $f(t) = (t+1)/2$:

$$\mathrm{Cov}_\rho(A,B) := \frac{1}{2}\,\mathrm{Tr}(\rho(A^*B + BA^*)) - (\mathrm{Tr}\,\rho A^*)(\mathrm{Tr}\,\rho B).$$

The interpretation of the covariances is not at all clear. In the next section they will be called *quadratic cost functions*. It turns out that there is a one-to-one correspondence between quadratic cost functions and Fisher informations.

THEOREM 4. *For a standard matrix monotone function f the covariance $\mathrm{qCov}^f_\rho(A,A)$ is a concave function of ρ for a self-adjoint A.*

PROOF. The argument similar to the proof of Theorem 3. Instead of the inequality $\beta^*(\mathbb{J}^f_{\beta(D)})^{-1}\beta \le (\mathbb{J}^f_D)^{-1}$ we use the inequality $\beta\mathbb{J}^f_D\beta^* \le \mathbb{J}^f_{\beta(D)}$ (see Theorem 1.2 in [27] or [23]). This gives the concavity of $\langle A, \mathbb{J}^f_\rho a\rangle$. The convexity of $(\mathrm{Tr}\,\rho A)^2$ is obvious. ∎

6. χ^2-divergence

The χ^2-divergence

$$\chi^2(p,q) = \sum_i \frac{(p_i - q_i)^2}{q_i} = \sum_i \left(\frac{p_i}{q_i} - 1\right)^2 q_i$$

was first introduced by Karl Pearson in 1900. Since

$$\left(\sum_i |p_i - q_i|\right)^2 = \left(\sum_i \left|\frac{p_i}{q_i} - 1\right| q_i\right)^2 \le \sum_i \left(\frac{p_i}{q_i} - 1\right)^2 q_i,$$

we have

(19) $$\|p - q\|_1^2 \le \chi^2(p,q).$$

We also remark that the χ^2-divergence is an f-divergence of Csiszár with $f(x) = (x-1)^2$ which is a (matrix) convex function. In the quantum case definition (1) gives

$$S_f(\rho,\sigma) = \mathrm{Tr}\,\rho^2\sigma^{-1} - 1.$$

Another quantum generalization was introduced very recently in [29]:

$$\chi^2_\alpha(\rho,\sigma) = \mathrm{Tr}\,\big((\rho-\sigma)\sigma^{-\alpha}(\rho-\sigma)\sigma^{\alpha-1}\big) = \mathrm{Tr}\,\rho\sigma^{-\alpha}\rho\sigma^{\alpha-1} - 1$$

where $\alpha \in [0,1]$. If ρ and σ commute, then this formula is independent of α. In the general case the above $S_f(\rho,\sigma)$ comes for $\alpha = 0$.

More generally, they defined

$$\chi^2_k(\rho,\sigma) := \Big\langle \rho-\sigma, \Omega^k_\sigma(\rho-\sigma)\Big\rangle,$$

where $\Omega^k_\sigma = R_\sigma^{-1}k(\Delta(\sigma/\sigma))$ and $1/k$ is a standard matrix monotone function. In the present notation $\Omega^k_\sigma = (\mathbb{J}^{1/k}_\sigma)^{-1}$ and for density matrices we have

$$\chi^2_k(\rho,\sigma) = \langle \rho, \Omega^k_\sigma\rho\rangle - 1 = \langle \rho, (\mathbb{J}^{1/k}_\sigma)^{-1}\rho\rangle - 1 = \gamma^{1/k}_\sigma(\rho,\rho) - 1.$$

Up to the additive constant this is a monotone metric. The monotonicity of the χ^2-divergence follows from (12) and monotonicity is stated as Theorem 4 in the paper [29], where the important function k is

$$k_\alpha(x) = \frac{1}{2}\left(x^{-\alpha} + x^{\alpha-1}\right) \quad \text{and} \quad \chi^2_{k_\alpha} = \chi^2_\alpha.$$

$1/k_\alpha$ is a standard matrix monotone function for $\alpha \in [0,1]$ and $k_\alpha(x)$ is convex in the variable α. The latter implies that χ^2_α is convex in α. The χ^2-divergence χ^2_α is minimal if $\alpha = 1/2$. (It is interesting that this appeared in [27] as Example 4.)

When $1/k(x) = (1+x)/2$ is the largest standard matrix monotone function, then the corresponding χ^2-divergence is the smallest and in the paper [29] the notation $\chi^2_{Bures}(\rho,\sigma)$ is used. Actually,

$$\chi^2_{Bures}(\rho,\sigma) = 2\int_0^\infty \mathrm{Tr}\,\rho\exp(-t\omega)\rho\exp(-t\omega)\,dt - 1,$$

see Example 1 in [27].

The monotonicity and the classical inequality (19) imply

$$\|\rho-\sigma\|_1^2 \le \chi^2(\rho,\sigma)$$

(when the conditional expectation onto the commutative algebra generated by $\rho - \sigma$ is used).

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Dénes Petz

Alfréd Rényi Institute of Mathematics
H-1364 Budapest, POB 127, Hungary
petz@math.bme.hu

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NONLINEAR TIME SERIES MODELS AND THEIR EXTREMES, WITH HYDROLOGICAL APPLICATIONS
Abstract of Ph.D. Thesis

By
PÉTER ELEK
SUPERVISOR: ANDRÁS ZEMPLÉNI
(Defended February 4, 2010)

1. Introduction

By combining the methods of time series analysis and extreme value theory I investigate probabilistic properties, extremal behaviour and parameter estimation of certain nonlinear time series models in this dissertation. My theoretical contributions were initially motivated by an applied hydrological project that – somewhat unusually in the hydrological literature – aimed to build models to capture both the times series dynamics and the extremal behaviour of water discharge data sets of Danube and Tisza. The coupling of the two theories may lead to a more precise description of time series extremes than their separate application would do. Purely time series models concentrate on the "typical" behaviour of the series and thus give too little weight to large observations, while purely extreme value models tend to neglect the information in the dynamics of the process, or make too general assumptions on it. The two methodologies are combined quite often in mathematical finance, actuarial studies, telecommunications but less usually in hydrological models.

The thesis summary follows the structure of the dissertation. I review the necessary preliminaries in section 2 and present my empirical findings on the properties of water discharge series in section 3. I give my theoretical and applied results on conditionally heteroscedastic models in section 4 and on Markov-switching models in section 5. Finally, section 6 reveals the relationship between the two model families and outlines directions of future research.

The results of section 4 are joint work with László Márkus, while those of section 5 are joint work with my supervisor András Zempléni.

2. Preliminaries

Although the dissertation uses preliminaries from both time series analysis and extreme value theory (EVT), I only review the basic concepts and scope of the latter field in the thesis summary. Time series analysis is a much more traditional research area hence its classical results can be found in numerous monographs.

Basically, EVT deals with two types of research problems.[1] First, the tail of a distribution (i.e. the rate of decay of the $\bar{F}(u) = 1 - F(u)$ survival function as $u \to \infty$) can be examined, and second, the clustering of high observations in a stationary series (i.e. to what extent they form groups) can also be analysed.

As far as the tail of a distribution is concerned, the theorem of Balkema-de Haan-Pickands is a basic result in EVT (Balkema and de Haan, 1974; Pickands, 1975). For any random variable X satisfying some general conditions there exists a measurable function $a(u)$ such that the distribution of normalised threshold exceedances tends to the generalised Pareto distribution (GPD) with shape parameter ξ as u tends to the upper end point of the support of the distribution:

$$((X - u)/a(u)) \mid (X > u) \to {}_d GPD_\xi .$$

The shape parameter of the GPD obtained in the limit greatly determines the extremal properties of the original distribution as well. If $\xi > 0$ the survival function is regularly varying with parameter $1/\xi$ (it follows that $E\left(X^+\right)^m = \infty$ for $m > 1/\xi$ values),[2] while for $\xi < 0$ the support of the distribution is bounded from above and the survival function (after a simple transformation) is regularly varying around the upper end point. The $\xi = 0$ case – when the obtained GPD is the exponential distribution – can be characterised with more difficulty. Although the distributions in this group share the common feature that all of their moments are finite one can find heavy and and also light tailed distributions among them.[3] The exponential and the normal laws are examples for the light-tailed case, while the Weibull distribution with exponent smaller than one for the heavy-tailed case.[4]

[1] The book of Embrechts et al. (1997) is a basic monograph of EVT.

[2] The notations $x^+ = \max(x, 0)$ and $x^- = \max(-x, 0)$ are used.

[3] A distribution is light-tailed if there exists an $s > 0$ such that the $L_X(s) = E\exp(sX)$ moment generating function is finite. If such an s does not exist the distribution is heavy-tailed.

[4] According to the terminology of the dissertation the survival function of the Weibull distribution is $\bar{F}(u) = \exp\left(-\lambda u^d\right)$ where d is the exponent parameter. A distribution has Weibull-like tail if there exist $K_1 > 0$, K_2, $\lambda > 0$ and $d > 0$ constants such that $\bar{F}(u) \sim K_1 u^{K_2} \exp\left(-\lambda u^d\right)$ as $u \to \infty$.

During the commonly used threshold-based estimation method high quantiles of a distribution are estimated from the parameters of a GPD fitted to exceedances above a high threshold. This is an appropriate procedure in the absence of other information but leads to highly variable or (if a relatively low reference threshold was chosen) biased quantile estimates. If additional information is available on the distribution (e.g. if we not only know that it belongs to the domain of attraction of the GPD with $\xi = 0$ but know its decay more accurately) then we can obtain more precise quantile estimates. Besides theoretical interest, this gives the motivation to study the rate of decay of the stationary distribution of certain theoretical time series models – a research area that has reached deep results e.g. in mathematical finance.

Turning to the other direction of EVT, clustering of large observations in a time series means more precisely the following. Let us examine a C_n extremal functional of a stationary process X_t :

$$C_n(u) = \sum_{t=1}^{n-m+1} g\left(X_t - u, \dots, X_{t+m-1} - u\right),$$

where g is a $\mathbb{R}^m \to \mathbb{R}_+$ function satisfying $g(\mathbf{x}) = 0$ for all $\mathbf{x} \notin \mathbb{R}_+^m$. Let us choose the $\{u_n\}$ sequence such that $\lim_{n\to\infty} n\bar{F}\left(u_n\right) \to \tau > 0$. Then under general conditions the distribution of $C_n(u_n)$ tends to the distribution of $C_1^* + C_2^* + \dots + C_L^*$, where L is a Poisson-distributed random variable and $C_1^*, C_2^*, \dots$ are i.i.d. variables, independent of L as well (Smith et al., 1997). Heuristically this means that large observations in a stationary time series occur in clusters, and these clusters are asymptotically independent of each other. Thus the distribution of C_i^* contains essential information about the time-dependence of extreme observations of the process. For instance, if $m = 1$ and $g(x) = \chi_{\{x>0\}}$ then C_i^* corresponds to the size of an extremal cluster in the limit (e.g. to the duration of a large flood in the hydrological context) and if $m = 1$ and $g(x) = x^+$ then the aggregate excess during an extreme event (e.g. the flood volume in hydrology) is obtained. In this case C^* will be denoted by W^*.

The simplest measure of extremal dependence is the extremal index (θ), which is obtained – under some assumptions – from the relationship $E\left(L\right) = \theta\tau$ or as the reciprocal of the expectation of the limiting cluster size: $\theta = E\left(C_i^*\right)^{-1}$ if $g(x) = \chi_{\{x>0\}}$. Apart from pathological cases $0 < \theta \leq 1$ and a smaller extremal index implies a stronger extremal dependence.

Since the extremal index and the other characteristics of extremal dependence are asymptotic concepts they can be estimated from finite samples (similarly to high quantiles) only with large uncertainty. Things are made even more complicated because – contrary to the GPD in the estimation of high quantiles

– the limiting cluster size and limiting aggregate excess distributions generally cannot be described with parametric families. The problem is often tackled by restricting the set of examined models, deriving theoretical extremal dependence properties for the special models and then developing methods for estimating cluster characteristics based on these results. If model specification is correct these procedures are more accurate than the methods applicable for more general series as well.

A good example for this research direction is the analysis of extremal dependence of Markov chains. Under general conditions Smith et al. (1997) have shown that a Markov chain with exponential tail behaves asymptotically (above high thresholds) as a random walk. Based on this result, an estimation and simulation method has been developed for the analysis of the extremal clusters of Markov chains.

3. Empirical features of water discharge data

In the dissertation I use daily water discharge data of three monitoring stations at river Danube (Komárom, Nagymaros, Budapest) and three at river Tisza (Tivadar, Vásárosnamény, Záhony). My first results concern the empirical properties of the series. First I fit ARMA models with seasonal components to the data and then generate synthetic water discharge series with the fitted models. Following the hydrological literature (e.g. Montanari et al., 1997), the ARMA innovations are not obtained from a parametric distribution but by the bootstrap method, i.e. by resampling the fitted innovations. My results show that the simulated processes substantially underestimate the high quantiles and do not reproduce the probability density of the observed data.

Since various heuristic estimators (correlogramm-based procedure, R/S statistic etc.) point to the presence of long memory I also estimate a fractional ARIMA model on the data. However, the fit of the high quantiles and the probability density improve only marginally. Hence, contrary to other rivers (e.g. Montanari et al., 1997), linear models do not describe accurately the behaviour of daily river flows of Danube and Tisza, which makes nonlinear modelling necessary.

I also show in the chapter that – similarly to other rivers with medium or large catchments – the data sets for Danube and Tisza tend to belong to the domain of attraction of the GPD with $\xi = 0$ (the $\xi = 0$ hypothesis cannot be rejected for most monitoring stations and thresholds). The extremal index estimates reveal that observations above high thresholds are strongly clustering over time. These empirical results steer the modeler in the choice among the various possible nonlinear model families.

4. Conditionally heteroscedastic models

First I examine ARMA-β-TARCH models in the dissertation from a theoretical point of view:

$$X_t = c + \sum_{i=1}^{p} a_i(X_{t-i} - c) + \sum_{i=1}^{q} b_i\varepsilon_{t-i} \tag{1}$$

$$\varepsilon_t = \sigma(X_{t-1})Z_t, \tag{2}$$

$$\sigma^2(x) = \alpha_0 + \alpha_{1+}\left((x-m)^+\right)^{2\beta} + \alpha_{1-}\left((x-m)^-\right)^{2\beta}, \tag{3}$$

where I make the following assumptions:

ASSUMPTION 4.1. *Z_t is an independent identically distributed random sequence with zero mean and unit variance. The distribution of Z_t is absolutely continuous with respect to the Lebesgue-measure, and its support is the whole real line.*

ASSUMPTION 4.2. *For the characteristic polynomials*

$$\Phi(z) = 1 - \sum_{i=1}^{p} a_i z^i \neq 0 \quad \textit{and} \quad \Psi(z) = 1 + \sum_{i=1}^{q} b_i z^i \neq 0 \quad \textit{if} \quad |z| \leq 1,$$

and $\Phi(z)$ and $\Psi(z)$ have no common zeros.

ASSUMPTION 4.3. $0 < \beta < 1$.

ASSUMPTION 4.4. $\alpha_0 > 0$*,* $\alpha_{1+} \geq 0$ *and* $\alpha_{1-} \geq 0$.

Thus, in contrast to the usual (T)ARCH-type processes, the variance of the ε_t innovation depends on X_{t-1} and not on ε_{t-1}, and the function describing this relationship tends to infinity in a slower than quadratic rate (since $\beta < 1$).

If $\beta = 1$ the model given by (1)–(3) does not have a stationary solution for every $\alpha_{1+} \geq 0$ and $\alpha_{1-} \geq 0$, and the domain of stationarity depends not only on the variance parameters but on the coefficients of the ARMA equation, too. If the stationary solution exists its survival function is polynomially decaying even for light-tailed (e.g. normally distributed) Z_t noises, hence the distribution belongs to the domain of attraction of a GPD with $\xi > 0$ (see e.g. Embrechts et al. (1997) in a special case).

Using the drift condition for the stability of Markov chains (Meyn and Tweedie, 1993) I prove that the $0 < \beta < 1$ case is substantially different from the quadratic case:[5]

THEOREM 1. *If Assumptions 4.1–4.4 hold the X_t process defined by* (1)–(3) *is geometrically ergodic and has a unique stationary distribution. If, moreover, $E(|Z_t|^r) < \infty$ for an $r \geq 2$ real, then $E\left(|X_t|^r\right) < \infty$ under the stationary distribution.*

It follows that if all moments of the generating noise are finite then the stationary distribution of X_t can only belong to the domain of attraction of the GPD with $\xi = 0$. In the case without ARMA-terms, if the generating noise has Weibull-like tail with exponent parameter γ then I prove a more accurate statement: the tail of the stationary distribution can be approximated by Weibull-like distributions with parameter $\gamma(1-\beta)$.

ASSUMPTION 4.5. *Z_t is an i.i.d. sequence and there exist $u_0 > 0, \gamma > 0, K_1 > 0$ and K_2 such that its probability density satisfies*

$$f_{Z_t}(u) = K_1 |u|^{K_2} \exp\left(-\kappa |u|^{\gamma}\right)$$

for every $|u| > u_0$.

THEOREM 2. *Assume $a_i = 0$ and $b_i = 0$ $(i = 1, \ldots, \max(p,q))$, Assumption 4.5, $\alpha_0 > 0$, $\alpha_{1+} > 0$, $\alpha_{1-} > 0$ and $0 < \beta < 1$. Then, using the notation $\alpha_1^{\max} = \max(\alpha_{1+}, \alpha_{1-})$,*

$$\exp\left(-\frac{\left(\alpha_1^{\max}\right)^{-\gamma/2} \kappa\gamma\beta^{-\frac{\beta}{1-\beta}}}{2} u^{\gamma(1-\beta)} + O\left(u^{\gamma(1-\beta)/2}\right)\right) \leq \bar{F}_{X_t}(u)$$
$$\leq \exp\left(-\frac{\left(\alpha_1^{\max} + \alpha_0\right)^{-\gamma/2} \kappa\gamma\beta^{-\frac{\beta}{1-\beta}}}{2} u^{\gamma(1-\beta)} + O\left(u^{\gamma(1-\beta)/2}\right)\right).$$

PROPOSITION 3. *If the assumptions of the previous theorem hold but $\alpha_{1-} > 0$ is replaced to $\alpha_{1-} = 0$, then for every $\delta > 0$ there exists a $K > 0$ such that $\exp\left(-Ku^{(1+\delta)\gamma(1-\beta)}\right) \leq \bar{F}_{X_t}(u)$.*

Finishing the probabilistic analysis, I illustrate the conjecture that – unlike in the $\beta = 1$ case – the process has a unit extremal index thus asymptotically the exceedances do not form clusters. (This does not rule out, however, clustering above large but finite thresholds.)

[5]Unless stated otherwise, all probability statements in the sequel hold under the stationary distribution.

Turning to parameter estimation, let $\boldsymbol{\theta} = \left(a_1, \ldots, a_p, b_1, \ldots, b_q\right)$ and $\boldsymbol{\alpha} = (\alpha_0, \alpha_{1+}, \alpha_{1-})$. Denote the true ARMA parameter vector by $\boldsymbol{\theta}^0$ and the true parameters of the variance equation by $\boldsymbol{\alpha}^0$.

For given β and m I estimate the ARMA parameters by least squares and the parameters of the variance equation by QML (quasi maximum likelihood, i.e. by maximising the likelihood function obtained under the assumption of normally distributed noise sequence). In the latter part of the estimation the $\hat{\varepsilon}_t$ innovations calculated from the ARMA fit are used. I prove the consistency and asymptotic normality of the procedure under the following assumptions.

ASSUMPTION 4.6. *There exists a $\delta > 0$ such that $\boldsymbol{\theta}^0 \in \boldsymbol{\Theta}_\delta$ where*

$$\boldsymbol{\Theta}_\delta = \{\boldsymbol{\theta} \in \mathbf{R}^{p+q} : \text{the roots of } \Phi_{\boldsymbol{\theta}}(x) \text{ and } \Psi_{\boldsymbol{\theta}}(x) \text{ have moduli } \geq 1 + \delta\}.$$

Moreover, $\boldsymbol{\alpha}^0 \in \operatorname{int}(\mathbf{K})$, where $\mathbf{K}$ is a compact subset of $\mathbf{R}_{++} \times \mathbf{R}_+ \times \mathbf{R}_+$.

ASSUMPTION 4.7. *$E(|Z_t|^{4+2\eta}) < \infty$ holds for some $\eta > 0$.*

THEOREM 4. *Under Assumptions 4.1–4.3 and 4.6 the QML estimator is consistent, i.e. $\hat{\boldsymbol{\alpha}}_n \to \boldsymbol{\alpha}^0$ a.s. If in addition Assumption 4.7 holds, the resulting estimator is asymptotically normally distributed, i.e.*

$$\sqrt{n}(\hat{\boldsymbol{\alpha}}_n - \boldsymbol{\alpha}^0) \to_d N\left(0, \mathbf{H}^{-1}\left(\boldsymbol{\alpha}^0\right) \mathbf{V}\left(\boldsymbol{\alpha}^0\right) \mathbf{H}^{-1}\left(\boldsymbol{\alpha}^0\right)\right)$$

where

$$\mathbf{V}(\boldsymbol{\alpha}) = E_\pi \left(\frac{\partial l(\varepsilon_t, X_{t-1}, \boldsymbol{\alpha})}{\partial \boldsymbol{\alpha}} \frac{\partial l(\varepsilon_t, X_{t-1}, \boldsymbol{\alpha})}{\partial \boldsymbol{\alpha}}^T \right)$$
$$\mathbf{H}(\boldsymbol{\alpha}) = E_\pi \left(-\frac{\partial^2 l(\varepsilon_t, X_{t-1}, \boldsymbol{\alpha})}{\partial^2 \boldsymbol{\alpha}} \right).$$

The $\mathbf{H}(\boldsymbol{\alpha}^0)$ and $\mathbf{V}(\boldsymbol{\alpha}^0)$ matrices can be consistently estimated by the empirical counterparts of $\mathbf{H}(\hat{\boldsymbol{\alpha}}_n)$ and $\mathbf{V}(\hat{\boldsymbol{\alpha}}_n)$, with expectations replaced by sample averages.

Finally I fit the ARMA-β-TARCH model (with $\beta = 1/2$) to the water discharge data and obtain highly significant heteroscedasticity for all monitoring stations. Synthetic river flow series are generated from the fitted model in such a way that the noise sequences are obtained by resampling the $\{\hat{Z}_t\}$ calculated noises. The simulated time series approximate the probability densitis and high quantiles of observed water discharges much better than the simulations from the linear models do.

5. Markov-switching autoregressive models

In this chapter the Markov-switching (MS-) AR(1) model family is examined:

$$X_t = a_1 X_{t-1} + \varepsilon_{1,t} \text{ ha } \quad I_t = 1, \tag{4}$$

$$X_t = a_0 X_{t-1} + \varepsilon_{0,t} \text{ ha } \quad I_t = 0. \tag{5}$$

Here I_t is a two-state discrete time Markov chain with $p_i = P(I_t = 1 - i | I_{t-1} = i)$ transition probabilities $(i = 0, 1)$, and $\{\varepsilon_{1,t}\}$, $\{\varepsilon_{0,t}\}$ are i.i.d. sequences, independent of each other and from $\{I_t\}$ but not necessarily identically distributed with each other. Let us also assume that $|a_1| \geq |a_0|$.

The EVT literature had earlier dealt mainly with the $|a_0| < 1 < a_1$ case. Under this assumption the stationary distribution – if it exists – is heavier tailed than the generating $\varepsilon_{1,t}$ noise, moreover, the survival function is polynomially decaying under general assumptions (Saporta, 2005). The extremal index of the model is smaller than one. If, in contrast, $|a_0| \leq |a_1| < 1$ the stationary distribution can take many forms but it is certainly light-tailed with a unit extremal index in case of light-tailed $\varepsilon_{i,t}$ $(i = 0, 1)$ noises.

I analyse the case not examined previously from an extremal point of view where the first regime is a random walk and the second is a stationary autoregression (Assumption 5.1). I also assume that $\varepsilon_{1,t}$ and $\varepsilon_{0,t}$ are light-tailed.

Assumption 5.1. $a_1 = 1$ *and* $0 \leq a_0 < 1$.

Assumption 5.2. *The distribution of* $\varepsilon_{1,t}$ *is absolutely continuous with respect to the Lebesgue-measure and* $E|\varepsilon_{1,t}| < \infty$. *Moreover, there exists a* $\kappa > 0$ *such that* $(1 - p_1) L_{\varepsilon_{1,t}}(\kappa) = 1$ *and* $L'_{\varepsilon_{1,t}}(\kappa) < \infty$.

Assumption 5.3. *The distribution of* $\varepsilon_{0,t}$ *is absolutely continuous with respect to the Lebesgue-measure and its support is the whole real line. There exists an* $s_0 > \kappa$ *such that* $L_{|\varepsilon_{0,t}|}(s_0) < \infty$.

If Assumption 5.1 holds the model always has a stationary solution. Assumption 5.2 plays a crucial role in examining the extremal behaviour. Let

$$S_0 = 0, \qquad S_n = S_{n-1} + \varepsilon_n \quad (n = 1, 2, \dots)$$

be a random walk where the distribution of $\{\varepsilon_n\}$ is the same as the distribution of $\{\varepsilon_{1,t}\}$. Furthermore, let T be a Geom (p_1)-distributed random variable, independent of $\{\varepsilon_n\}$. I prove a Cramer-Lundberg-type approximation for the maximum of the random walk stopped at time $T - 1$ and then show the following:

PROPOSITION 5. *Under Assumption 5.2 there exists a* $K > 0$ *such that*

$$P(S_T > u) \sim K \exp(-\kappa u).$$

This statement, combined with the drift condition for Markov chains, eventually leads to the following theorem:

THEOREM 6. *If Assumptions 5.1--5.3 hold there exists a* $K > 0$ *such that*

$$P(X_t > u) \sim K \exp(-\kappa u).$$

Let τ_u denote the entrance time to (u, ∞) and define the $B_u = S_{\tau_u} - u$ overshoot on the $(\tau_u < \infty)$ event. Assumption 5.2 ensures that $B_u | (\tau_u < \infty) \to$ $\to_d B_\infty$ as $u \to \infty$. Define the $\{S_n^*\}$ random walk as

$$S_0^* = B_\infty, \qquad S_n^* = S_{n-1}^* + \varepsilon_n \quad (n = 1, 2, \dots),$$

where the B_∞ variable is chosen independently of the $\{\varepsilon_n\}$ sequence. With these notations I prove the following proposition on the extremal clustering behaviour of the MS-AR(1) model:

PROPOSITION 7. *Let* $g(x) = 0$ *for* $x < 0$ *and* $g(x) = o(\exp(\kappa x))$ *as* $x \to \infty$. *Then as* $n \to \infty$ $C_n(u_n)$ *converges to a Poisson sum of independent random variables, distributed as* C^* *where*

$$C^* = \sum_{k=0}^{T-1} g(S_k^*).$$

The θ *extremal index is given by*

$$\theta = \int_{-\infty}^{0} \kappa \exp(\kappa x) Q(x) dx$$

where $Q(x)$ *is the solution of the Wiener-Hopf-equation:*

$$Q(x) = p_1 + (1 - p_1) \int_0^\infty Q(y) f_{\varepsilon_{1,t}}(x - y) dy.$$

Hence, as far as extremes are concerned, this parameter choice lies between the two previously mentioned cases ($a_1 > 1$ and $|a_1| < 1$): the stationary distribution has an exponential tail but the extremal index is smaller than one. The exponent κ can be explicitly calculated in some special cases (e.g. for normally or Gamma-distributed noise) and the extremal index in others (e.g. for nonnegative or Laplace-distributed noise).

If, for instance, $\varepsilon_{1,t} \geq 0$ a.s. the limiting cluster size distribution is geometric with p_1 parameter and the extremal index is p_1. In a more special, hydrologically important case I prove a theorem about the limiting aggregate excess distribution using Laplace's method for sums:

Theorem 8. *If Assumptions 5.1 and 5.3 hold and* $\varepsilon_{1,t} \sim$ *Gamma* (α, λ) *then there exist* $K_i > 0$ $(i = 1, 2)$ *constants such that*

$$K_1 \exp\left(-2^{3/2}\left(\lambda_0^{-1} - \alpha\lambda_0\right)(\lambda y)^{1/2}\right) \leq \bar{F}_{W^*}(y) \leq$$
$$\leq K_2 \exp\left(-2\left(\lambda_0^{-1} - \alpha\lambda_0\right)(\lambda y)^{1/2}\right),$$

where λ_0 *is the unique real number satisfying*

$$\lambda_0^{-2} - 2\alpha \log \lambda_0 + \log(1 - p_1) - \alpha(1 + \log \alpha) = 0.$$

The examined MS-AR(1) model is interesting not only on its own right but in the analysis of extremal dependence of more general processes as well. Let X_t satisfy the following conditions:

Assumption 5.4. *Let* I_t *be a discrete time Markov chain as above. Let* X_t *be a stationary process whose conditional distribution, provided that* I_t *is known, only depends on the value of* X_{t-1} *(i.e.* X_t *is conditionally Markov in each regime). Formally, for* $A_t \subset \mathbf{R}$ *Borel-sets and* $j_t \in \{0, 1\}$,

$$P\left(X_t \in A_t | I_t = j_t, X_{t-i} \in A_{t-i}, I_{t-i} = j_{t-i}, i = 1, 2, \dots\right) =$$
$$= P\left(X_t \in A_t | X_{t-1} \in A_{t-1}, I_t = j_t\right).$$

Moreover, for each t, conditionally on $(I_1, I_2, \dots, I_t)$, *the set of random variables* $(X_1, X_2, \dots, X_t)$ *is independent of* $(I_{t+1}, I_{t+2}, \dots)$.

Assumption 5.5. *The stationary distribution of* X_t *is absolutely continuous with repsect to the Lebesgue-measure and there exist* $0 < a \leq 1$, $K_0 > 0$, $K_1 > 0$ *constants such that* $\bar{F}_1(u) \sim K_1 e^{-\kappa u}$ *and* $\bar{F}_0(u) \sim K_0 e^{-\kappa u/a}$.

Let $a_1 = 1$ and $a_0 = a$, and use the following notations for $j = 0, 1$:

$$F_j^u(z) = P\left(X_t < a_j u + z | X_{t-1} = u, I_t = j\right).$$

If Assumptions 5.4 and 5.5 are satisfied and the $(X_{t-1}, X_t) \mid (I_t = j)$ distributions $(j = 0, 1)$ belong to the domain of attraction of a bivariate extreme value law then (under further regularity conditions) $F_j^u(z)$ can be shown to have a limit for all z as $u \to \infty$. Instead of stating the regularity conditions precisely I formulate this as an assumption.

Assumption 5.6. *The joint distributions* $(X_{t-1}, X_t) \mid (I_t = j)$ $(j = 0, 1)$ *are absolutely continuous with respect to the Lebesgue-measure. There exist (possibly improper) distribution functions* $F_j^*(z)$ *such that* $F_j^u(z) \to F_j^*(z)$ *as* $u \to$ $\to \infty$ *uniformly on all compact intervals* $(j = 0, 1)$. *Moreover, if* $F_j^*(-\infty) =$

$= \lim_{z\to-\infty} F_j^*(z) > 0$ *for a* j*, then*

$$\lim_{M\to\infty} \limsup_{u\to\infty} \sup_{y\geq M} P\left(X_t > a_i u | X_{t-1} = u - y, I_t = i\right) = 0$$

is satisfied for $i = 0, 1$.

Let us define Y_t as an MS-AR(1) process with $a_1 = 1$, $a_0 = a$ and F_j^*-distributed $\varepsilon_{j,t}$ noises. I prove that the behaviours of X_t and Y_t do not differ substantially from each other above high thresholds:

PROPOSITION 9. *If Assumptions 5.4–5.6 hold then for all* $p, j_t \in \{0,1\}$ *and* y_t $(t = 1, \ldots, p)$

$$\lim_{u\to\infty} \left| P\left(X_t < \left(\prod_{i=1}^{t} a_{j_i}\right) u + y_t \,(t = 1, \ldots, p)\,| X_0 = u, I_t = j_t \,(t = 1, \ldots, p)\right) - \right.$$
$$\left. - P\left(Y_t < \left(\prod_{i=1}^{t} a_{j_i}\right) u + y_t \,(t = 1, \ldots, p)\,| Y_0 = u, I_t = j_t \,(t = 1, \ldots, p)\right) \right| = 0.$$

Hence (under regularity conditions similar to those routinely applied in the statistical practice) the high-level clustering of Markov-switching, conditionally Markov processes with exponential tail can be approximated by the similar behaviour of MS-AR models. Therefore, if the aim is to estimate the extremal dependence structure of a time series then it may be more effective to fit an MS-AR model only to high-level exceedances rather than to the whole series. This way, only the conditional Markov and not the conditional AR structure is assumed for the process. In the dissertation I develop an approximate maximum likelihood procedure for the threshold-based estimation of the MS-AR model, examine the properties of the estimator by simulation and apply it to the water discharge data at Tivadar. The flood maxima and flood volumes simulated from the estimated model fit well to the corresponding characteristics of the observed series, implying that the extremes of the river flow data can be adequately described by a simple conditionally Markov model.

6. Conclusions

Among the models presented in the dissertation the ARMA-β-TARCH model generalises the linear family in a statistically motivated way, while the

MS-AR process gives more "structural" insight into the behaviour of hydrological time series. Due to this reason, and also because of its better fitting theoretical extremal properties, the latter model is more appropriate for hydrological purposes. It is not surprising, however, that a purely statistical fitting procedure may easily lead to an ARCH-type process: the weak ARMA representation of MS-AR models can be shown to be conditionally heteroscedastic, moreover, the variance depends on the lagged value of the process approximately linearly in a wide range. This relationship connects the two model families.

Journal articles of the author

The dissertation is based on the following four peer-reviewed journal articles and on one yet unpublished manuscript:

- P. Elek, L. Márkus, A long-range dependent model with nonlinear innovations for simulating river flows, *Natural hazards and earth systems sciences,* **4** (2004), 277–283.
- P. Elek, L. Márkus, A light-tailed conditionally heteroscedastic model with applications to river flows, *Journal of Time Series Analysis,* **29** (2008), 14–36.
- P. Elek, A. Zempléni, Tail behaviour and extremes of two-state Markov-switching autoregressive processes, *Computers and Mathematics with Applications,* **55** (2008), 2839–2855.
- P. Elek, A. Zempléni, Modelling extremes of time-dependent data by Markov-switching structures, *Journal of Statistical Planning and Inference,* **139** (2009), 1953–1967.
- P. Elek, L. Márkus, Tail behaviour of β-TARCH processes, Manuscript, submitted.

The author has also published two conference proceedings and various conference abstracts in the topic of the dissertation, and coauthors further three journal articles in applied mathematics or statistics, which are more or less connected to the topic.

- M. Arató, D. Bozsó, P. Elek, A. Zempléni, Forecasting and simulating mortality tables, *Mathematical and Computer Modelling,* **49** (2008), 805–813.

- A. Bíró, P. Elek, J. Vincze, Model-based sensitivity analysis of the Hungarian economy to shocks and uncertainties, *Acta Oeconomica,* **58** (2008), 367–401.
- K. Vasas, P., Elek, L. Márkus, A two-state regime switching autoregressive model with application to river flow analysis, *Journal of Statistical Planning and Inference,* **137** (2007), 3113–3126.

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DISCRETE METHODS IN GEOMETRIC MEASURE THEORY
Abstract of Ph.D. Thesis

By
VIKTOR HARANGI
SUPERVISOR: TAMÁS KELETI

(Defended November 9, 2011)

1. Introduction

The thesis addresses problems from the field of geometric measure theory. It turns out that discrete methods can be used efficiently to solve these problems.

In Chapter 2 we investigate the following question proposed by Tamás Keleti. How large (in terms of Hausdorff dimension) can a compact set $A \subset \mathbb{R}^n$ be if it does not contain some given angle α, that is, it does not contain distinct points P, Q, $R \in A$ with $\angle PQR = \alpha$? Or equivalently, how large dimension guarantees that our set must contain α?

We also study an approximate version of this problem, where we only want our set to contain angles close to α rather than contain the exact angle α. This version turns out to be completely different from the original one, which is best illustrated by the case $\alpha = \pi/2$. If the dimension of our set is greater than 1, then it must contain angles arbitrarily close to $\pi/2$. However, if we want to make sure that it contains the exact angle $\pi/2$, then we need to assume that its dimension is greater than $n/2$.

Another interesting phenomenon is that different angles show different behaviour. In the approximate version the angles $\pi/3$, $\pi/2$ and $2\pi/3$ play special roles, while in the original version $\pi/2$ seems to behave differently than other angles.

The investigation of the above problems led us to the study of the so-called acute sets. A finite set $\mathcal{H}$ in $\mathbb{R}^n$ is called an acute set if any angle determined by three points of $\mathcal{H}$ is acute. Chapter 3 of the thesis studies the maximal cardinality $\alpha(n)$ of an n-dimensional acute set. The exact value of $\alpha(n)$ is known only for

$n \leq 3$. For each $n \geq 4$ we improve on the best known lower bound for $\alpha(n)$. We present different approaches. On one hand, we give a probabilistic proof that $\alpha(n) > c \cdot 1.2^n$. (This improves a random construction given by Erdős and Füredi.) On the other hand, we give an *almost* exponential constructive example which outdoes the random construction in low dimension ($n \leq 250$). Both approaches use the small dimensional examples that we found partly by hand ($n = 4, 5$), partly by computer ($6 \leq n \leq 10$).

Finally, in Chapter 4 we show that the Koch curve is tube-null, that is, it can be covered by strips of arbitrarily small total width.

Chapter 2 is based on [1] and [2]. The latter is a joint paper with Keleti, Kiss, Maga, Máthé, Mattila and Strenner. Chapter 3 and 4 are based on [3] and [4], respectively. (For the sake of completeness some constructions due to Máthé are also included in the thesis.)

2. How large dimension guarantees a given angle?

An easy consequence of Lebesgue's density theorem claims that for any Lebesgue measurable set $A \subset \mathbb{R}^n$ with positive Lebesgue measure it holds that a similar copy of any finite configuration of points can be found in A.

What can be said about infinite configurations? Erdős asked whether there is a sequence $x_n \to 0$ such that a similar copy of this sequence can be found in every measurable set $A \subset \mathbb{R}$ with $\lambda(A) > 0$. This question is usually referred to as Erdős similarity problem and still unsolved.

And what about finite configurations in null sets? The following problem was also posed by Erdős. How large (in terms of Hausdorff dimension) can a set $A \subset \mathbb{R}^2$ be if there is no equilateral triangle with all three vertices in A? Falconer answered this question by showing that there exists a compact set A on the plane with Hausdorff dimension 2 such that A does not contain three points that form an equilateral triangle. In fact, it was shown in [9, 12, 13] that for any three points in $\mathbb{R}$ or in $\mathbb{R}^2$ there exists a compact set (in $\mathbb{R}$ or in $\mathbb{R}^2$) of full Hausdorff dimension, which does not contain a similar copy of the three points. It is open whether the analogous result holds in higher dimension.

It would be interesting to find *patterns*, which can be found in every full dimensional set. In this chapter we investigate such a pattern. We say that a set $A \subset \mathbb{R}^n$ contains the angle α if there exist distinct points $P, Q, R \in A$ such that $\angle PQR = \alpha$. Keleti posed the following question: how large can a set $A \subset \mathbb{R}^n$ be if it does not contain α? If there is no restriction on A, then for any given $\alpha \in$ $\in [0, \pi]$ one can use transfinite recursion to construct a full dimensional set not containing α, see Theorem 2.14. The problem is more interesting, though, if we restrict ourselves to, for example, compact sets. What is the smallest s for which $\dim(A) > s$ implies that A must contain α provided that $A \subset \mathbb{R}^n$ is compact? (Or equivalently, what is the maximal Hausdorff dimension s of a compact set $A \subset \mathbb{R}^n$ with the property that A does not contain the angle α?) This minimal (maximal) value of s will be denoted by $C(n, \alpha)$. It is not hard to show that $C(n, \alpha) \leq n - 1$ for arbitrary α, in other words, if the Hausdorff dimension of a compact set $A \subset \mathbb{R}^n$ is greater than $n-1$, then A contains every angle $\alpha \in [0, \pi]$.

As far as lower bounds are concerned, the line segment shows that $C(n, \alpha) \geq 1$ for any $\alpha \in (0, \pi)$. Our first goal is to improve on this obvious lower bound by constructing a compact set of Hausdorff dimension greater than 1 which does not contain some angle $\alpha \in (0, \pi)$.

THEOREM 2.1. *There is a* $\delta_0 > 0$ *such that for any* $0 < \delta \leq \delta_0$ *there exists a self-similar set in* $\mathbb{R}^n$ *of dimension at least*

$$c_\delta n = c\delta^2 \log^{-1}(1/\delta) \cdot n$$

such that the angle determined by any three points of the set is in the δ*-neighbourhood of the set* $\{0, \pi/3, \pi/2, 2\pi/3, \pi\}$.

The above theorem readily implies that $C(n, \alpha) \geq c(\alpha)n$ given that $\alpha \in$ $\in (0, \pi)$ and $\alpha \neq \pi/3, \pi/2, 2\pi/3$. The construction uses the following result due to Erdős and Füredi [8]. For any $\delta > 0$ there exist at least $\left(1 + c\delta^2\right)^n$ points in $\mathbb{R}^n$ such that the distance of any two is between 1 and $1 + \delta$. (This result is also related to the problems studied in Chapter 3.)

What about the exceptional angles $\pi/3$, $\pi/2$, $2\pi/3$? Our next goal is to prove that there exist self-similar sets in $\mathbb{R}^n$ with large dimension that contain neither $\pi/3$, nor $2\pi/3$. We start with constructing a discrete set of points. We need to find as many points P_i as possible such that any angle determined by them is in a small neighbourhood of $\pi/3$ but avoids an even smaller neighbourhood of $\pi/3$. We were inspired by the following r-colouring of the complete graph on 2^r vertices. Let $C_1, \ldots, C_r$ denote the colours and let us associate to each vertex a 0-1 sequence of length r. Consider the edge between the vertices corresponding to the sequences $i_1, \ldots, i_r$ and $j_1, \ldots, j_r$. We colour this edge with C_k where k denotes the first index where the sequences differ, that is, $i_1 = j_1, \ldots, i_{k-1} = j_{k-1}$, $i_k \neq j_k$. Let us denote this coloured graph by $\mathcal{G}_r$. This colouring has the property that there is no monochromatic triangle in the graph. Moreover, every triangle has two sides with the same colour and a third side with a different colour of higher index. (This is a folklore graph colouring showing that the multicolour Ramsey number $R_r(3)$ is greater than 2^r.)

The idea is to realize $\mathcal{G}_r$ geometrically in the following manner: the vertices of the graph will be represented by points of a Euclidean space and edges with the same colour will correspond to equal distances. The next lemma claims that $\mathcal{G}_r$ can be represented in the above sense.

LEMMA 2.2. *Let $l_1 \geq l_2 \geq \ldots \geq l_r > 0$ be a decreasing sequence of positive reals. By $\mathcal{I}_r$ we denote the set of 0-1 sequences of length r. Then 2^r points $P_{i_1,\ldots,i_r}$, $(i_1,\ldots,i_r) \in \mathcal{I}_r$ can be given in some Euclidean space in such a way that for two distinct 0-1 sequences $(i_1,\ldots,i_r) \neq (j_1,\ldots,j_r)$ the distance of $P_{i_1,\ldots,i_r}$ and $P_{j_1,\ldots,j_r}$ is equal to l_k where k denotes the first index where the sequences differ, that is, $i_1 = j_1, \ldots, i_{k-1} = j_{k-1}, i_k \neq j_k$.*

The proof of the next theorem uses the above lemma as well as the well-known Johnson-Lindenstrauss lemma [11].

THEOREM 2.3. *There exist absolute constants c, $C > 0$ such that for any $0 < \delta < \varepsilon < 1$ with $\varepsilon/\delta > C$ there exists a self-similar set of dimension*

$$s \geq \frac{c\varepsilon/\delta}{\log(1/\delta)}$$

in a Euclidean space of dimension

$$n \leq \frac{C\varepsilon}{\delta^3}$$

such that any angle determined by three points of the set is inside the ε-neighbourhood of $\{0, \pi/3, \pi/2, 2\pi/3, \pi\}$ but outside the δ-neighbourhood of $\{\pi/3, 2\pi/3\}$.

By fixing a small ε and setting $\delta = c/\sqrt[3]{n}$ in the above theorem, we obtain the following corollaries. As we will see, the second one is surprisingly sharp.

COROLLARY 2.4. *A self-similar set $K \subset \mathbb{R}^n$ can be given such that the dimension of K is at least*

$$s \geq \frac{c\sqrt[3]{n}}{\log n},$$

and K does not contain the angle $\pi/3$ and $2\pi/3$ (moreover, K does not contain any angle in the $c/\sqrt[3]{n}$-neighbourhood of $\pi/3$ and $2\pi/3$).

COROLLARY 2.5. *For any $0 < \delta < 1$ there exists a self-similar set K of dimension at least $\frac{c}{\delta}/\log(\frac{1}{\delta})$ in some Euclidean space such that K does not contain any angle in $(\pi/3 - \delta, \pi/3 + \delta) \cup (2\pi/3 - \delta, 2\pi/3 + \delta)$.*

The rest of this chapter is joint work with Keleti, Kiss, Maga, Máthé, Mattila and Strenner. According to the following results large dimensional sets always contain angles close to $\pi/3$, $\pi/2$ and $2\pi/3$.

THEOREM 2.6. *Any set A in $\mathbb{R}^n$ $(n \geq 2)$ with Hausdorff dimension greater than 1 contains angles arbitrarily close to the right angle.*

THEOREM 2.7. *There exists an absolute constant C such that whenever $\dim(A) > \frac{C}{\delta}\log(\frac{1}{\delta})$ for some set $A \subset \mathbb{R}^n$ and $\delta > 0$ the following holds: A contains three points that form a δ-almost regular triangle, that is, the ratio of the longest and shortest side is at most $1 + \delta$.*

As an immediate consequence, we can find angles close to $\pi/3$.

COROLLARY 2.8. *Suppose that $\dim(A) > \frac{C}{\delta}\log(\frac{1}{\delta})$ for some set $A \subset \mathbb{R}^n$ and $\delta > 0$. Then A contains angles from the interval $(\pi/3 - \delta, \pi/3]$ and also from $[\pi/3, \pi/3 + \delta)$.*

REMARK 2.9. The above theorem and even the corollary is essentially sharp, see Corollary 2.5.

We mention that the above results remain valid even under somewhat weaker conditions (when Hausdorff dimension is replaced with upper Minkowski dimension).

To sum up the results we introduce the following function $\tilde{C}$ depending on an angle $\alpha \in [0, \pi]$ and a small positive δ.

$$\begin{aligned}\tilde{C}(\alpha,\delta) \stackrel{\text{def}}{=} \sup\{\dim(A) : A \subset \mathbb{R}^n \text{ for } \textbf{some } n; A \text{ is analytic};\\ A \text{ does not contain any angle from } (\alpha-\delta,\alpha+\delta)\}.\end{aligned}$$

It is shown in the thesis that $\tilde{C}$ satisfies the symmetry property

$$\tilde{C}(\alpha,\delta) = \tilde{C}(\pi - \alpha, \delta).$$

The above constructions and results give essentially all the values of $\tilde{C}(\alpha,\delta)$, see Table 1.

Table 1. Smallest dimensions that guarantee angle in $(\alpha - \delta, \alpha + \delta)$

α	$\tilde{C}(\alpha,\delta)$	
$0, \pi$	$= 0$	
$\pi/2$	$= 1$	
$\pi/3, 2\pi/3$	$\approx 1/\delta$	apart from a multiplicative error $C \cdot \log(1/\delta)$
other angles	$= \infty$	provided that δ is sufficiently small

Let us now return to estimates on $C(n, \alpha)$. First we give a precise definition.

Definition 2.10. If $n \geq 2$ is an integer and $\alpha \in [0, \pi]$, then let

$$C(n, \alpha) = \sup\{s : \exists A \subset \mathbb{R}^n \text{ compact such that } \dim(A) = s \text{ and } A \text{ does not contain the angle } \alpha\}.$$

We mention that we get the same definition if we consider analytic sets instead of compact sets.

The next theorem says that if we have an analytic set in $\mathbb{R}^n$ of Hausdorff dimension greater than $n-1$, then it must contain every angle $\alpha \in [0, \pi]$.

Theorem 2.11. *If $n \geq 2$ and $\alpha \in [0, \pi]$, then $C(n, \alpha) \leq n - 1$.*

We can prove a better upper bound for $C(n, \pi/2)$.

Theorem 2.12. *If n is even, then $C(n, \pi/2) \leq n/2$. If n is odd, then $C(n, \pi/2) \leq (n+1)/2$.*

For the sake of completeness we mention a construction due to András Máthé. Both constructions that we have seen so far (the one for general angles and the one for $\pi/3$, $2\pi/3$) have the property that they avoid not only the given angle α but also a small neighbourhood of α. The following construction does not have this property: even though the constructed set contains angles arbitrarily close to $\pi/2$, it succeeds to avoid $\pi/2$. It is based on number theoretic methods.

Theorem 2.13 (Máthé, [14]). *There exists a compact set $K \subset \mathbb{R}^n$ such that $\dim(K) = n/2$ and K does not contain the angle $\pi/2$.*

It follows from Theorem 2.12 that this result is sharp given that n is even.

We gathered the best known bounds for $C(n, \alpha)$ in Table 2.

Table 2. Best known bounds for $C(n, \alpha)$

α	lower bound	upper bound
$0, \pi$	$n-1$	$n-1$
$\alpha \in (0, \pi)$; $\alpha \neq \pi/2$	cn	$n-1$
$\pi/2$	$n/2$	$\lceil n/2 \rceil$

Finally, the next theorem shows that if there was no restriction on A in Definition 2.10, then $C(n, \alpha)$ would be n for any α.

Theorem 2.14. *Let $n \geq 2$. For any $\alpha \in [0, \pi]$ there exists $H \subset \mathbb{R}^n$ such that H does not contain the angle α, and H has positive Lebesgue outer measure. In particular, $\dim(H) = n$.*

3. Acute sets in Euclidean spaces

Around 1950 Erdős conjectured that given more than 2^d points in $\mathbb{R}^d$ there must be three of them determining an obtuse angle. The vertices of the d-dimensional cube show that 2^d points exist such that the angle determined by any three of them is at most $\pi/2$.

In 1962 Danzer and Grünbaum proved this conjecture [7]. They posed the following question in the same paper: what is the maximal number of points in $\mathbb{R}^d$ such that all angles determined are acute (in other words, this time we want to exclude right angles as well as obtuse angles). A set of such points will be called an *acute set* or *acute d-set* in the sequel.

The exclusion of right angles seemed to decrease the maximal number of points dramatically: they could only give $2d-1$ points, and they conjectured that this is the best possible. However, this was only proved for $d = 2, 3$ [10].

Then in 1983 Erdős and Füredi disproved the conjecture of Danzer and Grünbaum. They used the probabilistic method to show the existence of an acute d-set of cardinality exponential in d. Their idea was to choose random points from the vertex set of the d-dimensional unit cube, that is $\{0,1\}^d$. Actually they even proved the following result: for any fixed $\delta > 0$ there exist exponentially many points in $\mathbb{R}^d$ with the property that the angle determined by any three points is less than $\pi/3+\delta$. We used this result in the previous chapter to construct large dimensional sets such that each angle contained by the sets is close to one of the angles 0, $\pi/3$, $\pi/2$, $2\pi/3$, π.

We denote the maximal size of acute sets in $\mathbb{R}^d$ and in $\{0,1\}^d$ by $\alpha(d)$ and $\kappa(d)$, respectively; clearly $\alpha(d) \geq \kappa(d)$. Our goal in this chapter is to give good bounds for $\alpha(d)$ and $\kappa(d)$. The random construction of Erdős and Füredi implied the following lower bound for $\kappa(d)$ (thus for $\alpha(d)$ as well)

$$\kappa(d) > \frac{1}{2}\left(\frac{2}{\sqrt{3}}\right)^d > 0.5 \cdot 1.154^d. \tag{1}$$

The best known lower bound both for $\alpha(d)$ and for $\kappa(d)$ (for large values of d) is due to Ackerman and Ben-Zwi from 2009 [5]. They improved (1) with a factor $\sqrt{d}$:

$$\alpha(d) \geq \kappa(d) > c\sqrt{d}\left(\frac{2}{\sqrt{3}}\right)^d. \tag{2}$$

We modify the random construction of Erdős and Füredi to obtain the following theorem.

THEOREM 3.1.

$$\alpha(d) > c\left(\sqrt[10]{\frac{144}{23}}\right)^d > c \cdot 1.2^d,$$

that is, there exist at least $c \cdot 1.2^d$ *points in* $\mathbb{R}^d$ *such that any angle determined by three of these points is acute. (If* d *is divisible by* 5*, then* c *can be chosen to be* $1/2$*, for general* d *we need to use a somewhat smaller* c*.)*

We present another approach where we recursively construct acute sets. These constructions outdo Theorem 3.1 up to dimension 250. We show that this constructive lower bound is *almost* exponential in the following sense. Given any positive integer r, for infinitely many values of d we have an acute d-set of cardinality at least

$$\exp(d/\underbrace{\log\log\cdots\log}_{r}(d)).$$

Both the probabilistic and the constructive approach use small dimensional acute sets as building blocks. So it is crucial for us to construct small dimensional acute sets of large cardinality. In the thesis we present an acute set of 8 points in $\mathbb{R}^4$ and an acute set of 12 points in $\mathbb{R}^5$ (disproving the conjecture of Danzer and Grünbaum for $d \geq 4$ already). We used computer to find acute sets in dimension $6 \leq d \leq 10$.

As far as $\kappa(d)$ is concerned, in large dimension (2) is still the best known lower bound. Bevan used computer to determine the exact values of $\kappa(d)$ for $d \leq 9$ [6]. He also gave a recursive construction improving upon the random constructions in low dimension. Our constructive approach yields a lower bound not only for $\alpha(d)$ but also for $\kappa(d)$, which further improves the bounds of Bevan in low dimension.

The following notion plays an important role in both approaches.

DEFINITION 3.2. A triple A, B, C of three points in $\mathbb{R}^d$ will be called *bad* if for each integer $1 \leq i \leq d$ the i-th coordinate of B equals the i-th coordinate of A or C.

We denote by $\kappa_n(d)$ the maximal size of a set $S \subset \{0, 1, \ldots, n-1\}^d$ that contains no *bad triples*. It is easy to see that $\kappa_2(d) = \kappa(d)$ but our main motivation to investigate $\kappa_n(d)$ is that we can use sets without bad triples to construct acute sets recursively. We give an upper bound and two different lower bounds for $\kappa_n(d)$.

THEOREM 3.3. *For even* d

$$\kappa_n(d) \leq 2n^{d/2},$$

and for odd d

$$\kappa_n(d) \le n^{(d+1)/2} + n^{(d-1)/2}.$$

THEOREM 3.4.

$$\kappa_n(d) > \frac{1}{2}\left(\frac{n^2}{2n-1}\right)^{\frac{d}{2}} > \frac{1}{2}\left(\frac{n}{2}\right)^{\frac{d}{2}} = \left(\frac{1}{2}\right)^{\frac{d+2}{2}} n^{\frac{d}{2}}.$$

THEOREM 3.5. *If $d \ge 2$ is an integer and $n \ge d$ is a prime power, then*

$$\kappa_n(d) \ge n^{\lceil \frac{d}{2} \rceil}.$$

Setting $n = 2$ and using that $\kappa_2(d) = \kappa(d)$ the next corollary readily follows from Theorem 3.3.

COROLLARY 3.6. *For even d*

$$\kappa(d) \le 2^{(d+2)/2} = 2\left(\sqrt{2}\right)^d,$$

and for odd d

$$\kappa(d) \le 2^{(d+1)/2} + 2^{(d-1)/2} = \frac{3}{\sqrt{2}}\left(\sqrt{2}\right)^d.$$

This corollary improves the upper bound $\sqrt{2}(\sqrt{3})^d$ given by Erdős and Füredi in [8].

4. The Koch curve is tube-null

Theorem 4.1 answers the following question posed by, among others, Marianna Csörnyei: is the Koch snowflake curve tube-null?

THEOREM 4.1. *The Koch curve K is tube-null, that is, it can be covered by strips of arbitrarily small total width.*

Moreover, there exists a decomposition $K = K_0 \cup K_1 \cup K_2$ and projections π_0, π_1, π_2 such that the Hausdorff dimension of $\pi_i(K_i)$ is less than 1 for $i = 0, 1, 2$.

The proof contains geometric, combinatorial, algebraic and probabilistic arguments.

The thesis is based on the following papers

[H1] V. Harangi, Large dimensional sets not containing a given angle, *Cent. Eur. J. Math.,* to appear.
[H2] V. Harangi, T. Keleti, G. Kiss, P. Maga, A. Máthé, P. Mattila, B. Strenner, How large dimension guarantees a given angle?, arXiv:1101.1426.
[H3] V. Harangi, Acute sets in Euclidean spaces, submitted.
[H4] V. Harangi, The Koch snowflake curve is tube-null, *Proc. Amer. Math. Soc.,* **139** (2011), 1375–1381.

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[4] V. Harangi, The Koch snowflake curve is tube-null, *Proc. Amer. Math. Soc.,* **139** (2011), 1375–1381.
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ADDITIVE REPRESENTATION FUNCTIONS
Abstract of Ph.D. Thesis

By
SÁNDOR Z. KISS
SUPERVISOR: ANDRÁS SÁRKÖZI
(Defended June 18, 2010)

Abstract. Let $\mathcal{A}$ be an infinite sequence of positive integers and let $k \geq 2$ be a fixed integer. Let $R_1(\mathcal{A},n,k)$ the number of solutions of $a_1 + a_2 + \cdots + a_k = n$, $a_1 \in \in \mathcal{A}, a_2 \in \mathcal{A}, \ldots a_k \in \mathcal{A}$. In a series of papers P. Erdős, A. Sárközy and V. T. Sós studied the regularity properties and the monotonicity of the representation function $R_1(\mathcal{A},n,2)$. In some of my papers I extended and generalized their results to any $k > 2$. The aim of this paper is to survey these results. In the last part of the paper I study the connection between Sidon sets and asymptotic bases.

1. Introduction

Let $\mathbb{N}$ denote the set of positive integers, and let $k \geq 2$ be a fixed integer. Let $\mathcal{A}$ be an infinite sequence of positive integers. For any positive integer n let $R_1(\mathcal{A},n,k)$, $R_2(\mathcal{A},n,k)$, $R_3(\mathcal{A},n,k)$ denote the number of solutions of the equations

$$a_1 + a_2 + \cdots + a_k = n, \quad a_1 \in \mathcal{A}, \ldots, a_k \in \mathcal{A},$$

$$a_1 + a_2 + \cdots + a_k = n, \quad a_1 \in \mathcal{A}, \ldots, a_k \in \mathcal{A}, \quad a_1 < a_2 < \ldots < a_k,$$

and

$$a_1 + a_2 + \cdots + a_k = n, \quad a_1 \in \mathcal{A}, \ldots, a_k \in \mathcal{A}, \quad a_1 \leq a_2 \leq \ldots \leq a_k,$$

respectively. If $F(n) = O(G(n))$ then we write $F(n) << G(n)$. We put

$$A(n) = \sum_{\substack{a \in \mathcal{A} \\ a \leq n}} 1.$$

The research of the additive representation functions began in the 1950's. Starting from a problem of Sidon, P. Erdős proved that there exists a sequence $\mathcal{A} \subset \mathbb{N}$

so that there are two constans c_1 and c_2 for which for every n

$$c_1 \log n < R_1(\mathcal{A},n,2) < c_2 \log n.$$

On the other hand an old conjecture of Erdős states that for no sequence $\mathcal{A}$ can we have

$$\frac{R_1(\mathcal{A},n,2)}{\log n} \to c \quad (0 < c < +\infty).$$

There are some related questions in [3], [12] and [22]. These problems led P. Erdős, A. Sárközy and V. T. Sós to study the regularity property and the monotonicity of the function $R_1(\mathcal{A},n,2)$ see in [6], [7], [8], [9]. In this paper I survey my results about the regularity properties and the monotonicity of the representation function $R_1(\mathcal{A},n,k)$ for $k > 2$ integer. I extended and generalized some result of P. Erdős, A. Sárközy and V. T. Sós by using the generator function method and the probabilistic method.

2. The methods

In my papers I used the generating function method. We start out from the generating function of the sequence $\mathcal{A}$:

$$f(z) = \sum_{a\in\mathcal{A}} z^{a}.$$

It is easy to see that

$$f^k(z) = \sum_{n=1}^{\infty} R_1(\mathcal{A},n,k) z^{n}.$$

We used the generating function method to prove the results about the monotonicity. We also used the Hölder – inequality, the Cauchy – inequality and the Parseval – formula. In the next step I tell a few words about the probabilistic method [26]. An important problem in additive number theory is to prove that a sequene with certain properties exists. One of the essential ways to obtain an affirmative answer for such a problem is to use the probabilistic method due to Erdős and Rényi [3]. There is an excellent summary of this method in the Halberstam – Roth book [12]. To show that a sequence with a property $\mathcal{P}$ exists, it sufficies to show that a properly defined random sequence satisfies $\mathcal{P}$ with positive probability. Usually the property $\mathcal{P}$ requires that for all sufficiently large $n \in \mathbb{N}$, some relation $\mathcal{P}(n)$ holds. The general strategy to handle this situation is the following. For each n one first shows that $\mathcal{P}(n)$ fails with a small probability, say p_n. If p_n is sufficiently small so that $\sum_{n=1}^{+\infty} p_n$ converges, then by the Borel – Cantelli lemma, $\mathcal{P}(n)$ holds for all sufficiently large n with probability 1.

Note that in my proofs the additive representation function is the sum of random variables. However for $k > 2$ these variables are not independent. To overcome this problems in my thesis I apply the theorems of J. H. Kim and V. H. Vu, [15], [25], [26], [27], the Janson inequality [14] and the method of Erdős and Tetali [2], [10].

3. The Results

Starting from a problem of Sidon, P. Erdős proved that there exists a sequence $\mathcal{A} \subset \mathbb{N}$ so that there are two constans c_1 and c_2 for which for every n

$$c_1 \log n < R_1(\mathcal{A},n,2) < c_2 \log n.$$

In one of their paper [5] Erdős and Sárközy extended the above problem by estimating $|R_1(\mathcal{A},n,2) - F(n)|$ for nice arithmetic functions $F(n)$. In [13] G. Horváth extended their theorem to any $k > 2$ integer by estimating $|R_1(\mathcal{A},n,k) - F(n)|$. Later Erdős and Sárközy proved in [7] that if $F(n)$ is a nice arithmetic function, then there exists a sequence $\mathcal{A}$ such that

$$|R_1(\mathcal{A},n,2) - F(n)| << \Big(F(n)\log n\Big)^{1/2}.$$

In [19] I generalized their theorem by using the probabilistic method:

THEOREM 1. *If $k > 2$ is a positive integer, c_8 is a constant large enough in terms k, $F(n)$ is an arithmetic function satisfying*

$$F(n) > c_8 \log n \quad \text{for} \quad n > n_{0,}$$

and there exists a real function $g(x)$, defined for $0 < x < +\infty$, and real numbers $x_{0,}n_1$ and $c_{7,}c_9$ costants such that

(i) $0 < g(x) \le \frac{(\log x)^{\frac{1}{k}}}{x^{1-\frac{k+1}{k^2}}} < 1$ *for* $x \ge x_0$,

(ii) $\left|F(n) - k!\sum_{\substack{x_1+x_2+\ldots+x_k=n \\ 1\le x_1<x_2<\ldots<x_k<n}} g(x_1)g(x_2)\ldots g(x_k)\right| < c_7(F(n)\log n)^{1/2}$ *for* $n > n_1$,

then there exists a sequence $\mathcal{A}$ such that

$$|R_1(\mathcal{A},n,k) - F(n)| < c_9(F(n)\log n)^{1/2} \quad \text{for} \quad n > n_{2.}$$

Let $l \ge 1$ be a fixed integer. If $s_{0,}s_{1,}s_2 \ldots$ is a given sequence of real numbers, then let $\Delta_l s_n$ denote the l-th difference of the sequence $s_{0,}s_{1,}s_2 \ldots$ defined by $\Delta_1 s_n = s_{n+1} - s_n$ and $\Delta_l s_n = \Delta_1(\Delta_{l-1} s_n)$. It is well-known and it is easy to see by induction that

$$\Delta_l s_n = \sum_{i=0}^{l} (-1)^{l-i} \binom{l}{i} s_{n+i}.$$

Let $B(\mathcal{A},N)$ denote the number of blocks formed by consecutive integers in $\mathcal{A}$ up to N, i.e.,

$$B(\mathcal{A},N) = \sum_{\substack{a \le N \\ a-1 \notin \mathcal{A}, a \in \mathcal{A}}} 1.$$

P. Erdős, A. Sárközy and V. T. Sós studied the following problem: what condition is needed to ensure

$$\limsup_{n \to +\infty} |R_1(\mathcal{A},n+1,2) - R_1(\mathcal{A},n,2)| = +\infty?$$

They proved in [8] that if $\lim_{N\to\infty} \frac{B(\mathcal{A},N)}{\sqrt{N}} = \infty$, then the above holds. They also proved that their result is nearly best possible.

In [16] I extended their Theorem to any $k > 2$:

THEOREM 2. *If $k \ge 2$ is an integer and $\lim_{N\to\infty} \frac{B(\mathcal{A},N)}{\sqrt[k]{N}} = \infty$, and $l \le k$, then $|\Delta_l R_1(\mathcal{A},n,k)|$ cannot be bounded.*

I also proved in [20] that the above result is nearly best possible:

THEOREM 3. *For all $\varepsilon > 0$, there exists an infinite sequence $\mathcal{A}$ such that*

(i) $B(\mathcal{A},N) \gg N^{1/k-\varepsilon}$,
(ii) $R_1(\mathcal{A},n,k)$ is bounded so that also $\Delta_l R_1(\mathcal{A},n,k)$ is bounded if $l \le k$.

In the case $l > k$ I have only a partial result [17]:

THEOREM 4. *If $l \ge 2$ an integer and $\lim_{N\to\infty} \frac{B(\mathcal{A},N)}{\sqrt{N}} = \infty$, then $|\Delta_l(R_1(\mathcal{A},n,2))|$ cannot be bounded.*

For $i = 1,2,3$ we say $R_i(\mathcal{A},n,k)$ is monotonous increasing in n from a certain point on, if there exists an integer n_0 with

$$R_i(\mathcal{A},n+1,k) \ge R_i(\mathcal{A},n,k) \quad for \quad n \ge n_0.$$

In a series of papers P. Erdős, A. Sárközy and V. T. Sós studied the monotonicity properties of the three representation functions $R_1(\mathcal{A},n,2)$, $R_2(\mathcal{A},n,2)$, $R_3(\mathcal{A},n,2)$. A. Sárközy proposed the study of the monotonicity of the functions $R_i(\mathcal{A},n,k)$ for $k > 2$ [24, Problem 5]. He conjectured [23, p. 337] that for any $k \ge 2$ integer, if $R_i(\mathcal{A},n,k)$ $(i = 1,2,3)$ is monotonous increasing in n from a certain point on, then $A(n) = O(n^{2/k-\varepsilon})$ cannot hold. I proved in [18] and Min Tang proved independently the following slightly stronger result on $R_1(\mathcal{A},n,k)$:

THEOREM 5. *If $k \in \mathbb{N}$, $k \geq 2$, $\mathcal{A} \subset \mathbb{N}$ and $R_1(\mathcal{A},n,k)$ is monotonous increasing in n from a certain point on, then*

$$A(n) = o\left(\frac{n^{2/k}}{(\log n)^{2/k}}\right)$$

cannot hold.

We say a set $\mathcal{A}$ of positive integers is an asymptotic basis of order h if every large enough positive integer can be represented as the sum of h terms from $\mathcal{A}$. In other words $\mathcal{A}$ is an asymptotic bases of order h if there exists an n_0 positive integer such that $R_3(\mathcal{A},n,2) > 0$ for $n > n_0$. A set of positive integers $\mathcal{A}$ is called Sidon set if all the sums $a+b$ with $a \in \mathcal{A}$, $b \in \mathcal{A}$, $a \leq b$ are distinct. In other words $\mathcal{A}$ is a Sidon set if $R_3(\mathcal{A},n,2) \leq 1$. In [4] and [5] P. Erdős, A. Sárközy and V. T. Sós asked if there exists a Sidon set which is an asymptotic basis of order 3. The problem also appears in [24] (with a typo in it: order 2 is written instead of order 3). In [11] G. Grekos, L. Haddad, C. Helou and J. Pihko proved that a Sidon set cannot be an asymptotic basis of order 2. Recently J. M. Deshouillers and A. Plagne in [1] constructed a Sidon set which is an asymptotic basis of order at most 7. In [21] I improve on this result by proving:

THEOREM 6. *There exists an asymptotic basis of order 5 which is a Sidon set.*

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OPERATOR PRECONDITIONING IN HILBERT SPACE
Abstract of Ph.D. Thesis

By
TAMÁS KURICS
SUPERVISOR: JÁNOS KARÁTSON
(Defended December 21, 2010)

1. Introduction

The numerical solution of linear elliptic partial differential equations consists of two main steps: discretization and iteration, where generally some conjugate gradient method is used for solving the finite element discretization of the problem. However, when for elliptic problems the discretization parameter tends to zero, the required number of iterations for a prescribed tolerance tends to infinity. The remedy is suitable preconditioning. This can rely on the functional analytic background of the corresponding elliptic operators, which means that the preconditioning process takes place on the operator level. That is, we look for a suitable preconditioning operator for the operator equation, which is close to the original one in some sense, and use its discretization as a preconditioning matrix. Here we use the generalized conjugate gradient, least square methods (GCG-LS(s)) and the conjugate gradient normal (CGN) algorithm. These algorithms, the coincidence of the full GCG-LS and the truncated GCG-LS(0) methods and the related convergence theorems are discussed in [1, 2, 3, 4].

2. Preliminaries and short summary of the results

A general theory has been developed for such preconditioning using the notion of equivalent operators, which has been introduced and investigated in the aspect of linear convergence in [6]. As a further step, mesh independent superlinear convergence has been proved for the GCG-LS method for elliptic equations with homogeneous Dirichlet and mixed boundary conditions under FEM discretization, with severe restrictions on the coefficients (see [3, 7]). In [4] the notion of compact-equivalence has been introduced, summarized in [5], and superlinear mesh independence has been proved for the CGN algorithm without any restrictions (with the exception of the usual smoothness and coercivity conditions).

Based on these papers, we have compared the relation between the known theoretical convergence estimate and the numerical results and we have shown that the convergence rate remains valid even in cases not covered by the theory (cf. [14, 15]). Then we have extended the scope of the theoretical results to cases that have not been considered before: first we have dealt with symmetric preconditioning for elliptic systems using the compact normal operator framework and the GCG-LS algorithms (see [9, 12]). Then we have considered equations with nonhomogeneous mixed boundary conditions using operator pairs and the notion of compact-equivalence with the CGN method (cf. [13]). In contrast with finite element discretizations which fits in naturally with the Hilbert space background, there is no such abstract background for finite difference discretization, only a case-by-case study is possible. We have investigated a special model problem (see [11]) and we have derived a convergence estimate analogous to the finite element case. In [10] we have shown that the use of nonsymmetric preconditioners is more advantageous for singularly perturbed problems than symmetric preconditioning. Finally we have applied these results to nonlinear elliptic and time-dependent problems (cf. [8, 13]).

3. Summary of the applied methods

3.1. Compact-equivalence and the convergence of the CGN algorithm

Let H be a real Hilbert space and consider the operator equation

$$Lu = g \tag{1}$$

with a linear unbounded operator L in H, where $g \in H$ is given. We would like to consider its preconditioned form in weak sense in an energy space of a suitable symmetric operator S. The set of S-bounded and S-coercive operators is denoted by $BC_S(H)$ (see [4]).

Definition 1. (cf. [4]) For a given operator $L \in BC_S(H)$, we call $u \in H_S$ the weak solution of equation (1) if

$$\langle L_S u, v\rangle_S = \langle g, v\rangle \qquad \forall\, v \in H_S, \tag{2}$$

where $L_S \in B(H_S)$ represents the unique extension of the bounded bilinear form $(u,v) \mapsto \langle Lu, v\rangle$ from $D(L)$ to H_S.

DEFINITION 2. (cf. [4]) The operators $L, K \in BC_S(H)$ are compact-equivalent in H_S if $L_S = \mu K_S + Q_S$ for some constant $\mu > 0$ and compact operator $Q_S \in B(H_S)$.

As an important special case, we can consider compact-equivalence with $\mu = 1$ for the operators S and $L \in BC_S(H)$. Then

$$L_S = I + Q_S \tag{3}$$

holds with some compact operator Q_S.

Let us consider the operator equation (1) where $L \in BC_S(H)$, $g \in H$ and $u \in H_S$ is the weak solution defined in (2). To solve it numerically, let $V_h = \operatorname{span}\{\varphi_1, \dots, \varphi_n\} \subset H_S$ be a finite dimensional subspace of dimension n and $\mathbf{L}_h = \{\langle L_S\varphi_i, \varphi_j\rangle_S\}_{i,j=1}^n$, $\mathbf{g}_h = \{\langle g, \varphi_j\rangle\}_{j=1}^n$. Then the discrete solution $u_h \in V_h$ is $u_h = \sum_{i=1}^n c_i\varphi_i$, where $\mathbf{c} = (c_1, \dots, c_n) \in \mathbb{R}^n$ is the solution of the linear system $\mathbf{L}_h\mathbf{c} = \mathbf{g}_h$, which is the discretized form of (2). Now assume that L and S are compact-equivalent with $\mu = 1$, i.e. relation (3) holds. If S is used as a preconditioner, then the discretized form of the operator decomposition (3) becomes $\mathbf{L}_h = \mathbf{S}_h + \mathbf{Q}_h$, and the corresponding preconditioned form of $\mathbf{L}_h\mathbf{c} = \mathbf{g}_h$ has the form

$$\left(\mathbf{I}_h + \mathbf{S}_h^{-1}\mathbf{Q}_h\right)\mathbf{c} = \mathbf{S}_h^{-1}\mathbf{g}_h, \tag{4}$$

where $\mathbf{S}_h = \{\langle\varphi_i, \varphi_j\rangle_S\}_{i,j=1}^n$, $\mathbf{Q}_h = \{\langle Q_S\varphi_i, \varphi_j\rangle_S\}_{i,j=1}^n$.

THEOREM 3. *(cf. [4, Thm. 4.1]) Assume that $L \in BC_S(H)$, L and S are compact-equivalent with $\mu = 1$, i.e.* (3) *holds. Then the CGN algorithm for system* (4) *yields*

$$\left(\frac{\|r_k\|_{\mathbf{S}_h}}{\|r_0\|_{\mathbf{S}_h}}\right)^{1/k} \le \frac{2}{m^2}\left(\frac{1}{k}\sum_{i=1}^{k}\left(|\lambda_i(Q_S^* + Q_S)| + \lambda_i(Q_S^*Q_S)\right)\right) \xrightarrow{k\to\infty} 0, \tag{5}$$

where r_k is the residual vector, $m > 0$ comes from the S-coercivity of L and the right-hand side is independent of the subspace V_h.

3.2. The compact normal operator approach and the convergence of the GCG-LS algorithm

Let H be a complex Hilbert space and consider the operator equation (1) with an unbounded linear operator $L : D \subset H \to H$ defined on a dense domain D, and with some $g \in H$. Equation (1) is assumed to satisfy the following

ASSUMPTIONS 4.

(i) *The operator* L *is decomposed in* $L = S + Q$ *on its domain* D *where* S *is a self-adjoint operator in* H;

(ii) S *is a strongly positive operator, i.e.* $\exists\, p > 0$ *such that* $\langle Su, u\rangle \geq \geq p\,\|u\|^2 \quad \forall\, u \in D$;

(iii) *there exists* $\varrho > 0$ *such that* $\operatorname{Re}\langle Lu, u\rangle \geq \varrho\,\langle Su, u\rangle \quad \forall\, u \in D$;

(iv) *the operator* Q *can be extended to the energy space* H_S, *and then* $S^{-1}Q$ *is assumed to be a compact normal operator on* H_S.

Now we replace equation (1) by its preconditioned form

$$S^{-1}Lu = f \equiv S^{-1}g \iff \left(I + S^{-1}Q\right)u = f \equiv S^{-1}g. \tag{6}$$

THEOREM 5. *(cf. [3, Thm. 3]) Let Assumptions 4 hold. Then the GCG-LS algorithm applied for equation* (6) *in* H_S *yields for all* $k \in \mathbb{N}$

$$\left(\frac{\|e_k\|_L}{\|e_0\|_L}\right)^{1/k} \leq \frac{2}{\varrho}\left(\frac{1}{k}\sum_{i=1}^{k}\left|\lambda_i\left(S^{-1}Q\right)\right|\right) \xrightarrow{k\to\infty} 0, \tag{7}$$

where $e_k = u_k - u^*$ *is the error vector,* $\lambda_k\left(S^{-1}Q\right)$ $(k \in \mathbb{N})$ *are the ordered eigenvalues of the compact normal operator* $S^{-1}Q$.

Equation (1) can be solved numerically by using Galerkin discretization. Let us consider the finite dimensional subspace $V_h = \operatorname{span}\{\varphi_1, \ldots, \varphi_n\} \subset \subset H_S$ of dimension n and $\mathbf{S}_h = \left\{\left\langle\varphi_i, \varphi_j\right\rangle_S\right\}_{i,j=1}^{n}$, $\mathbf{Q}_h = \left\{\left\langle Q\varphi_i, \varphi_j\right\rangle\right\}_{i,j=1}^{n}$, $\mathbf{g}_h = \left\{\left\langle g, \varphi_j\right\rangle\right\}_{j=1}^{n}$. Then the discrete solution $u_h \in V_h$ is $u_h = \sum_{i=1}^{n} c_i\varphi_i$, where $\mathbf{c} = (c_1, \ldots, c_n) \in \mathbb{R}^n$ is the solution of the linear system $\mathbf{L}_h\mathbf{c} = \mathbf{g}_h$, where $\mathbf{L}_h = \mathbf{S}_h + \mathbf{Q}_h$. If the operator S is used as a preconditioner, then the discretized form of the preconditioned operator equation (6) becomes

$$\left(\mathbf{I}_h + \mathbf{S}_h^{-1}\mathbf{Q}_h\right)\mathbf{c} = \mathbf{S}_h^{-1}\mathbf{g}_h. \tag{8}$$

THEOREM 6. *(cf. [3, Cor. 4]) Suppose that* H *is a complex separable Hilbert space, Assumptions 4 are satisfied and the matrix* $\mathbf{S}_h^{-1}\mathbf{Q}_h$ *is* $\mathbf{S}_h$*-normal. Then the GCG-LS algorithm for system* (8) *yields*

$$\left(\frac{\|e_k\|_{\mathbf{L}_h}}{\|e_0\|_{\mathbf{L}_h}}\right)^{1/k} \leq \frac{2}{\varrho}\left(\frac{1}{k}\sum_{i=1}^{k}\left|\lambda_i(S^{-1}Q)\right|\right) \xrightarrow{k\to\infty} 0, \tag{9}$$

where the right-hand side is independent of the subspace V_h.

4. Main results

4.1. Equations with homogeneous boundary conditions

Using the theoretical background of Subsection 3.2 for elliptic convection-diffusion problems with homogeneous Dirichlet and mixed boundary conditions, first we have investigated the relation between the known theoretical convergence estimate and the numerical results. We have confirmed the mesh independent superlinear convergence property of the GCG-LS(0) when symmetric part preconditioning has been applied to the FEM discretization of the boundary value problem. We have shown that the convergence rate remains valid even in cases not covered by the theory, i.e. when another symmetric operator is used as a preconditioner, not only the symmetric part of the operator. The numerical computations have also yielded better results than the theoretical estimate (9).

4.2. Equations with nonhomogeneous mixed boundary conditions

Let us consider the elliptic boundary value problem

$$\left.\begin{array}{ll} -\operatorname{div}(A\,\nabla u) + \mathbf{b}\cdot\nabla u + cu & = g \\ u\big|_{\Gamma_D} = 0, \qquad \frac{\partial u}{\partial \nu_A} + \alpha u\big|_{\Gamma_N} & = \gamma \end{array}\right\} \tag{10}$$

satisfying the following assumptions:

ASSUMPTIONS 7.

(i) $\Omega \subset \mathbb{R}^d$ *is a bounded piecewise* C^1 *domain;* Γ_D, Γ_N *are disjoint open measurable subparts of* $\partial\Omega$ *such that* $\partial\Omega = \overline{\Gamma}_D \cup \overline{\Gamma}_N$*;*

(ii) $A \in L^\infty(\overline{\Omega}, \mathbb{R}^{d\times d})$ *and for all* $x \in \overline{\Omega}$ *the matrix* $A(x)$ *is symmetric; further,* $\mathbf{b} \in W^{1,\infty}(\Omega)^d$, $c \in L^\infty(\Omega)$, $\alpha \in L^\infty(\Gamma_N)$*;*

(iii) we have the coercivity properties

$$\exists\, p > 0 \text{ such that } A(x)\xi\cdot\xi \geq p\,|\xi|^2 \qquad \forall\, x \in \overline{\Omega},\ \xi \in \mathbb{R}^d, \tag{11}$$

$$\hat{c} := c - \frac{1}{2}\,\mathrm{div}\,\mathbf{b} \geq 0 \text{ in } \Omega, \qquad \hat{\alpha} := \alpha + \frac{1}{2}(\mathbf{b}\cdot\nu) \geq 0 \text{ on } \Gamma_N; \tag{12}$$

(iv) either $\Gamma_D \neq \emptyset$*, or* $\hat{c}$ *or* $\hat{\alpha}$ *has a positive lower bound.*

The definition of the operator L which corresponds to equation (10) has to be understood as a pair of operators: one acts on Ω and the other one acts on the Neumann boundary. Formally we have

$$L \equiv \begin{pmatrix} M \\ P \end{pmatrix}, \quad L\begin{pmatrix} u \\ \eta \end{pmatrix} = \begin{pmatrix} Mu \\ P\eta \end{pmatrix} = \begin{pmatrix} -\operatorname{div}(A\,\nabla u) + \mathbf{b}\cdot\nabla u + cu \\ \frac{\partial \eta}{\partial \nu_A} + \alpha\eta\big|_{\Gamma_N} \end{pmatrix}. \tag{13}$$

Let us define a symmetric elliptic operator on the same domain in an analogous way:

$$S \equiv \begin{pmatrix} N \\ R \end{pmatrix}, \quad S \begin{pmatrix} u \\ \eta \end{pmatrix} = \begin{pmatrix} Nu \\ R\eta \end{pmatrix} = \begin{pmatrix} -\operatorname{div}(G\,\nabla u) + \sigma u \\ \frac{\partial \eta}{\partial \nu_G} + \beta \eta\big|_{\Gamma_N} \end{pmatrix} \tag{14}$$

satisfying similar assumptions as of L:

ASSUMPTIONS 8.

(i) substituting G for A, Ω, Γ_D, Γ_N and G satisfy Assumptions 7;
(ii) $\sigma \in L^\infty(\Omega)$, $\sigma \geq 0$, $\beta \in L^\infty(\Gamma_N)$, $\beta \geq 0$; further, if $\Gamma_D \neq \emptyset$, then σ or β has a positive lower bound.

Let us consider the differential equation (10) with given functions $g \in \in L^2(\Omega), \gamma \in L^2(\Gamma_N)$. We are interested in solving the analogous operator equation

$$L \begin{pmatrix} u \\ u\big|_{\Gamma_N} \end{pmatrix} = \begin{pmatrix} g \\ \gamma \end{pmatrix}, \tag{15}$$

which is the appropriately modified version of the operator equation (1). Here the Hilbert space H is defined as the product space $H = L^2(\Omega) \times L^2(\Gamma_N)$.

THEOREM 9. *If Assumptions 7-8 hold, then the operator L is S-bounded and S-coercive in H, i.e. $L \in BC_S\left(L^2(\Omega) \times L^2(\Gamma_N)\right)$.*

Consider again the differential equation (10) with the corresponding operator L in (13) and preconditioner S in (14) and assume that $A = G$, then it follows from [4] that L and S are compact-equivalent with $\mu = 1$, thus (3) holds. On the discrete level – with the notations of Subsection 3.1 – the finite element solution $u_h \in V_h$ is obtained by solving the linear system $\mathbf{L}_h \mathbf{c} = \mathbf{d}_h$, where

$$(\mathbf{L}_h)_{ij} = \int_\Omega \left(A\,\nabla\varphi_i \cdot \nabla\varphi_j + \left(\mathbf{b}\cdot\nabla\varphi_j\right)\varphi_i + c\varphi_i\varphi_j\right) + \int_{\Gamma_N} \alpha\varphi_i\varphi_j,$$

$$(\mathbf{d}_h)_j = \int_\Omega g\varphi_j + \int_{\Gamma_N} \gamma\varphi_j.$$

Let us take the symmetric operator S and introduce its stiffness matrix in H_S as

$$(\mathbf{S}_h)_{ij} = \left\langle \varphi_i, \varphi_j \right\rangle_S = \int_\Omega \left(G\,\nabla\varphi_i \cdot \nabla\varphi_j + \sigma\varphi_i\varphi_j\right) + \int_{\Gamma_N} \beta\varphi_i\varphi_j,$$

and consider the preconditioned equation

$$\left(\mathbf{I}_h + \mathbf{S}_h^{-1}\mathbf{Q}_h\right)\mathbf{c} = \mathbf{S}_h^{-1}\mathbf{d}_h, \tag{16}$$

where $\mathbf{L}_h$ and $\mathbf{S}_h$ come from the elliptic operators L and S and $\mathbf{Q}_h = \mathbf{L}_h - \mathbf{S}_h$. In this case the operator Q_S is defined as

$$(17)\quad \left\langle Q_S \begin{pmatrix} u \\ u\big|_{\Gamma_N} \end{pmatrix}, \begin{pmatrix} v \\ v\big|_{\Gamma_N} \end{pmatrix} \right\rangle_S = \int_\Omega \big((\mathbf{b}\cdot\nabla u)v + (c-\sigma)uv\big) + \int_{\Gamma_N} (\alpha-\beta)uv.$$

THEOREM 10. *With Assumptions 7–8 and $A = G$, the CGN algorithm for system* (16) *yields*

$$\left(\frac{\|r_k\|_{\mathbf{S}_h}}{\|r_0\|_{\mathbf{S}_h}}\right)^{1/k} \leq \frac{2}{m^2}\left(\frac{1}{k}\sum_{i=1}^{k}\big(|\lambda_i(Q_S^* + Q_S)| + \lambda_i(Q_S^* Q_S)\big)\right) \xrightarrow{k\to\infty} 0,$$

where $m > 0$ comes from the S-coercivity of L in Theorem 9.

4.3. Finite difference discretization for a model problem

Let us consider a special model problem which has been analysed in [16] in the context of linear convergence. The convection-diffusion problem

$$(18)\qquad \left.\begin{aligned} Lu \equiv -\Delta u + \mathbf{b}\cdot\nabla u + cu &= g \\ u\big|_{\Gamma_D} &= 0 \end{aligned}\right\}$$

is posed on the unit square $\Omega := [0,1]^2 \subset \mathbb{R}^2$ with constant coefficients $\mathbf{b} = (b_1, b_2) \in \mathbb{R}^2$ and $c \in \mathbb{R}$. We assume $c \geq 0$, then the usual coercivity condition $c - \frac{1}{2}\operatorname{div}\mathbf{b} \geq 0$ holds. The FDM discretization of (18) on a given grid ω_h leads to a linear algebraic system $L_h u_h = g_h$. Our goal is to solve this equation by iteration, applying the preconditioned GCG-LS method. The proposed preconditioner is obtained via a symmetric preconditioning operator

$$(19)\qquad Su := -\Delta u + \sigma u \quad \text{for} \quad u\big|_{\partial\Omega} = 0,$$

where $\sigma \in \mathbb{R}$, $\sigma \geq 0$: namely, the matrix S_h is defined as the FDM discretization of the operator S on the same grid ω_h. The preconditioned form of the discretized system is $S_h^{-1} L_h u_h = f_h \equiv S_h^{-1} g_h$. Here we are interested in the superlinear convergence property of the preconditioned GCG-LS, where the operators L, S are replaced by the matrices L_h, S_h, respectively.

Let ω_h be a uniform grid on $[0,1]^2$, $b_1, b_2 \geq 0$ and let us define upwind or centered differencing for the first order and centered differencing for the second order derivatives, respectively. Denote by n the number of interior gridpoints in each direction, and by $h = 1/(n+1)$ the grid parameter. Let L_h, S_h and Q_h denote the matrices corresponding to the discretizations of L, S and $Q = L - S$, respectively.

PROPOSITION 11. *Let us consider problem* (18) *with a convection term $\mathbf{b} = (b, b)$, where $b \in \mathbb{R}^+$ is arbitrary, and let $\sigma := c$ in* (19)*, i.e. S is the symmetric* *part of L. Then, using either centered or backward differencing, the eigenvalues $\lambda_{jm}(S_h^{-1}Q_h)$ satisfy*

$$\text{(20)}\quad \frac{1}{k^2}\sum_{j,m=1}^{k}\left|\lambda_{jm}(S_h^{-1}Q_h)\right| \le \frac{2\sqrt{2}b}{k^2}\sum_{j,m=1}^{\left\lfloor\frac{k+1}{2}\right\rfloor}\frac{1}{\sqrt{\sigma+4m^2+4j^2}}\ (k=1,\dots,n),$$

where the sequence on the right-hand side is independent of h and tends to 0 *as* $k\to\infty$.

Inequality (20) shows that the sequence of the error quotients $\left(\frac{\|e_k\|_{L_h}}{\|e_0\|_{L_h}}\right)^{1/k}$ can be estimated in a mesh uniform superlinear way, analogously to estimate (9) for the finite element case.

4.4. Extension of the theory to systems

Let us consider systems of the form

$$\text{(21)}\qquad \left.\begin{aligned} -\operatorname{div}(K_i\,\nabla u_i)+\mathbf{b}_i\cdot\nabla u_i+\sum_{j=1}^{\ell}V_{ij}u_j &= g_i\\ u_i\big|_{\partial\Omega} &= 0\end{aligned}\right\}\qquad (i=1,\dots,\ell)$$

satisfying the following assumptions.

ASSUMPTIONS 12.

(i) *The bounded domain $\Omega\subset\mathbb{R}^d$ is C^2-diffeomorphic to a convex domain;*
(ii) *for all $i,j=1,\dots,\ell$ the functions $K_i\in C^1(\overline{\Omega})$, $V_{ij}\in L^\infty(\Omega)$ and $\mathbf{b}_i\in$ $\in C^1(\overline{\Omega})^d$;*
(iii) *there exists $m>0$ such that $K_i\ge m$ holds for all $i=1,\dots,\ell$;*
(iv) *letting $V=\left\{V_{ij}\right\}_{i,j=1}^{\ell}$, the coercivity property $\lambda_{min}(V+V^T)-$ $-\max_{1\le i\le\ell}\operatorname{div}\mathbf{b}_i\ge 0$ holds pointwise on Ω, where λ_{min} denotes the smallest eigenvalue;*
(v) *$g_i\in L^2(\Omega)$ for all $i=1,\dots,\ell$.*

For brevity, we write (21) – using vector notations – as

$$\text{(22)}\qquad \left.\begin{aligned} L\mathbf{u}\equiv-\operatorname{div}(\mathbf{K}\,\nabla\mathbf{u})+\mathbf{b}\cdot\nabla\mathbf{u}+V\mathbf{u} &= \mathbf{g}\\ \mathbf{u}\big|_{\partial\Omega} &= 0.\end{aligned}\right\}$$

For the numerical solution of system (22), one usually considers its FEM discretization, which leads to a linear algebraic system $\mathbf{L}_h\mathbf{c}=\mathbf{g}_h$. This can be solved by the CGM using some suitable preconditioner. Here we consider preconditioners based on the following preconditioning operator. Letting $\sigma_i \in L^\infty(\Omega)$, $\sigma_i \geq 0$ be suitable functions and

$$S_i u_i := -\operatorname{div}(K_i \nabla u_i) + \sigma_i u_i \qquad (i = 1, \ldots, \ell) \tag{23}$$

for $u_i\big|_{\partial\Omega} = 0$, and define the ℓ-tuple of independent elliptic operators

$$S\mathbf{u} = \begin{pmatrix} S_1 u_1 \\ \vdots \\ S_\ell u_\ell \end{pmatrix}. \tag{24}$$

We have proved mesh independent superlinear convergence of the preconditioned CGM in the framework of compact normal operators in Hilbert space. This has been achieved in two steps: on the theoretical level, the preconditioned form of system (22)

$$S^{-1} L\mathbf{u} = \mathbf{f} \equiv S^{-1}\mathbf{g} \tag{25}$$

has been considered and we have proved that the CGM converges superlinearly in the Sobolev space $H_0^1(\Omega)^\ell$. Based on this, on the practically relevant discrete level we have considered the preconditioned form

$$\mathbf{S}_h^{-1}\mathbf{L}_h\mathbf{c} = \mathbf{f}_h \equiv \mathbf{S}_h^{-1}\mathbf{g}_h \tag{26}$$

of the algebraic system $\mathbf{L}_h\mathbf{c} = \mathbf{g}_h$, where $\mathbf{S}_h$ denotes the discretization of S in the same FEM subspace as for $\mathbf{L}_h$, and we have proved that the superlinear convergence of the CGM is mesh independent, i.e. independent of the considered FEM subspace. On both levels the full GCG-LS and the truncated GCG-LS(0) algorithms has been considered, and the results have been proved under certain special assumptions that ensure the normality of the preconditioned operator in the corresponding Sobolev space. First we have considered symmetric part preconditioning. The symmetric part of L falls into the type (23) coordinatewise if and only if

$$V_{ij} = -V_{ji} \ \ (i \neq j), \ \text{ and } \sigma_i \text{ in (23) is chosen as } \sigma_i = V_{ii} - \frac{1}{2}\,(\operatorname{div}\mathbf{b}_i). \tag{27}$$

THEOREM 13. *Under Assumptions 12 and condition* (27)*, the preconditioned truncated GCG-LS(0) algorithm for system* (21) *with the preconditioning operator* (23)-(24) *converges superlinearly in the space $H_0^1(\Omega)^\ell$ according to the estimate* (7).

ASSUMPTIONS 14.

(i) For all $i = 1, \ldots, \ell$, $K_i \equiv K \in \mathbb{R}$, $\sigma_i \equiv \sigma \in \mathbb{R}$ and $\mathbf{b}_i \equiv \mathbf{b} \in \mathbb{R}^d$;
(ii) $V \in \mathbb{R}^{\ell\times\ell}$ is a normal matrix.

THEOREM 15. *Under Assumptions 12 and 14, the preconditioned full GCG-LS algorithm for system* (21) *with the preconditioning operator* (23)–(24) *converges superlinearly in the space* $H_0^1(\Omega)^\ell$ *according to the estimate* (7).

COROLLARY 16. *Let Assumptions 12 hold. Consider the FEM discretization of system* (21), *using the stiffness matrix of* (24) *as preconditioner, under one of the following conditions:*

(a) the requirements in (27) *hold,* $V_h \subset H_0^1(\Omega)^\ell$ *is an arbitrary FEM subspace and the truncated GCG-LS*(0) *algorithm is used (here the* $\mathbf{S}_h$*-normality of* $\mathbf{S}_h^{-1}\mathbf{Q}_h$ *automatically holds);*

(b) Assumptions 14 hold, $V_h \subset H_0^1(\Omega)^\ell$ *is a FEM subspace for which the matrix* $\mathbf{S}_h^{-1}\mathbf{Q}_h$ *is* $\mathbf{S}_h$*-normal, and the full GCG-LS is used.*

Then the mesh independent superlinear convergence estimate (9) *is valid.*

4.5. Systems with nonhomogeneous mixed boundary conditions

We have applied the operator pair approach for elliptic systems of the form

$$\left.\begin{aligned} -\operatorname{div}(A_i\,\nabla u_i) + \mathbf{b}_i\cdot\nabla u_i + \sum_{j=1}^{\ell} V_{ij}u_j &= g_i \\ u_i\big|_{\Gamma_D} = 0, \qquad \frac{\partial u_i}{\partial \nu_{A_i}} + \alpha_i u_i\big|_{\Gamma_N} &= \gamma_i \end{aligned}\right\} \qquad (i=1,\dots,\ell) \tag{28}$$

satisfying the combination of Assumptions 7 and 12, where the corresponding operator L and the preconditioner S are defined as an ℓ-tuple of operator pairs:

$$L = (L_1,\dots,L_\ell) = \left(\begin{pmatrix} M_1 \\ P_1\end{pmatrix},\dots,\begin{pmatrix} M_\ell \\ P_\ell\end{pmatrix}\right),$$

$$S = (S_1,\dots,S_\ell) = \left(\begin{pmatrix} N_1 \\ R_1\end{pmatrix},\dots,\begin{pmatrix} N_\ell \\ R_\ell\end{pmatrix}\right),$$

satisfying similar conditions as in Assumptions 7-8 and the operator pairs are defined analogously to (13)-(14). As in (15) for a single equation, we look for the weak solution of the operator equation

$$L\begin{pmatrix} \mathbf{u} \\ \mathbf{u}\big|_{\Gamma_N}\end{pmatrix} = \begin{pmatrix} \mathbf{g} \\ \gamma \end{pmatrix}. \tag{29}$$

Extending the results for equations to systems, it is easy to verify that for $G_i = A_i$ $(i=1,\dots,\ell)$ the operators L and S are compact-equivalent with $\mu = 1$, i.e. $L_S = I + Q_S$ holds in H_S with some compact operator Q_S. Now let us consider the discrete equation $\mathbf{L}_h\mathbf{c} = \mathbf{d}_h$ and its preconditioned form

$$\mathbf{S}_h^{-1}\mathbf{L}_h\mathbf{c} = \left(\mathbf{I}_h + \mathbf{S}_h^{-1}\mathbf{Q}_h\right)\mathbf{c} = \mathbf{S}_h^{-1}\mathbf{d}_h, \tag{30}$$

where $\mathbf{L}_h$ and $\mathbf{S}_h$ come from the elliptic operators L and S, $\mathbf{Q}_h = \mathbf{L}_h - \mathbf{S}_h$. Symmetric part preconditioning can also be considered, analogously to the previous subsection. When S is not the symmetric part of L, then $Q_S \in B(H_S)$ can be defined as the sum of similar operators corresponding to (17). Now the conditions of Theorem 3 are satisfied, thus the CGN algorithm provides a mesh independent superlinear convergence result.

COROLLARY 17. *With suitable combination of Assumptions 7-8 and* $A_i = G_i$ $(i = 1, \dots, \ell)$, *the CGN algorithm for the system* (30) *yields*

$$\left(\frac{\|r_k\|_{\mathbf{S}_h}}{\|r_0\|_{\mathbf{S}_h}}\right)^{1/k} \leq \frac{2}{m^2}\left(\frac{1}{k}\sum_{j=1}^{k}\left(\left|\lambda_j(Q_S^* + Q_S)\right| + \lambda_j(Q_S^* Q_S)\right)\right) \xrightarrow{k\to\infty} 0.$$

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CONSTRUCTIONAL METHODS IN FINITE PROJECTIVE GEOMETRY
Abstract of Ph.D. Thesis

By
CSABA MENGYÁN
ADVISOR: TAMÁS SZŐNYI

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1. Introduction

The thesis came about as a cooperation between the advisor, the author and András Gács. The author wishes to thank Tamás Szőnyi for insights into problems in finite geometry that could actually be solved, and András Gács for guiding him through the process all the way from research to publication. Without the help received from Tamás Szőnyi and András Gács this thesis would not have gone to completion. Below we sum the main results of the thesis.

In finite projective geometry several methods both from geometry and algebra can be used to attain new results. The geometric methods we discuss are embedding, partitioning, random choice, adding and deleting points and use of subsets [15]. We also consider algebraic tools in solving such problems, like Weil's estimate and its variants [22], a lemma of Turán and a lemma on bipartite graphs [11]. In the thesis, that is based on four articles of the author, we concentrate on using these methods in the construction of minimal blocking sets and a closely related notion, strong representative systems.

A *blocking set* in a projective plane is a set of points which intersects every line. A blocking set is said to be *minimal*, when no proper subset of it is a blocking set. A *flag* of $\mathrm{PG}(2,q)$ is an incident point-line pair (P,r). A set of flags $B=\{(P_1,r_1),\dots,(P_k,r_k)\}$ is a *strong representative system* if and only if $P_i \in r_j$ means $i=j$. B is *maximal* if it is maximal subject to inclusion [16, 7].

Firstly, an important question is on the existence of minimal blocking sets of given size (the spectrum problem), in particular lower and upper bounds on the sizes. For $q = 2$ we have the Fano plane, and as Neumann and Morgenstern already observed in the 1940's there are no minimal blocking sets in this case. Therefore, we always consider q to be greater than 2 when considering minimal blocking sets. A lower bound for the size of minimal blocking sets was given by BRUEN [8] and PELIKÁN, proving that *non-trivial minimal blocking sets of* $\mathrm{PG}(2,q)$ *contain at least* $q+\sqrt{q}+1$ *points*. If there is equality, then the minimal blocking set is a Baer subplane. Improvements on this bound were obtained by several authors, for example Blokhuis has given improvements for the case when q is prime [6], SZŐNYI for minimal blocking sets of the Desarguesian plane [19]. Looking at the other end of the spectrum, an upper bound for the size of minimal blocking sets was given by BRUEN and THAS [9]: *If B is a minimal blocking set of* $\mathrm{PG}(2,q)$ *then* $|B| \leq q\sqrt{q}+1$. *In case of equality B is a unital (and q is square).*

An improvement on the Bruen-Thas upper bound is possible, as we show.

THEOREM 1.1. (SZŐNYI, COSSIDENTE, GÁCS, MENGYÁN, SICILIANO, WEINER, [4]) *Suppose B is a minimal blocking set in* $\mathrm{PG}(2,q)$, $q \neq 5$, *and denote by s the fractional part of* $\sqrt{q}$. *Then* $|B| \leq q\sqrt{q} + 1 - \frac{1}{4}s(1-s)q$.

Note that this always implies at least a $1/8\sqrt{q}$ improvement on the Bruen–Thas upper bound. On the other hand, it is easy to see that if q is not too close to a square, then this implies a cq improvement. The proof uses a lemma that was also used in the proof of Turán's theorem on graphs containing no K_r, and it is purely combinatorial, so the result is true for any projective plane (except possibly for $q = 26$).

2. Constructions in space

In relation to graphs ANDRÁS GYÁRFÁS asked for the chromatic index of the bipartite graph corresponding to $\mathrm{PG}(2,q)$ in [10]. Recall that a *strong colour class* in a graph G is a set of independent edges with the extra property that this set of edges is an induced subgraph of G. For the point-line incidence graph G of $\mathrm{PG}(2,q)$ a strong colour class is just a strong representative system. The *strong chromatic index* of a graph G is the minimum number of colours in an edge-colouring with the property that the edges having the same colour form a strong colour class. Geometrically, for G this is the minimum number of strong representative systems covering the flags of $\mathrm{PG}(2,q)$. In [16] it is implicitly proved that (trivially) for a general q this number is at most q^2+2q.

ILLÉS, SZŐNYI and WETTL [16] extended the Bruen–Thas upper bound and proved that the maximal size of a strong representative system is $q\sqrt{q}+1$. From this it follows that at least roughly $q\sqrt{q}$ strong representative systems are needed to partition all flags. They proved that this is indeed the case for q an odd square: *the flags of* $\mathrm{PG}(2,q)$, *q an odd square, can be partitioned into* $(q-1)\sqrt{q}+$ $+3q$ *strong representative systems.* It is straightforward to show that substituting Hermitian curves for the parabolas in their proof (in [16]) can be used for q an even square, too.

We investigate the more general case when q^h is not a prime. For this, we consider the generalized Buekenhout construction from [4] which produces minimal blocking sets of size $q^{h+1}+1$ in $\mathrm{PG}(2,q^h)$. Hoping that we can repeat the trick of partitioning the affine plane with copies of the affine part of this blocking set, from this size (and since the number of flags is approximately q^{3h}) it is natural to expect that something of order $q^{2h-1}+E(q)$ holds in $\mathrm{PG}(2,q^h)$ with $E(q)o(q^{2h-1})$. To show that this is the case, we use the embedding method connected to the generalized Buekenhout construction.

THEOREM 2.1. (MENGYÁN, [3]) *The flags of* $\mathrm{PG}(2,q^h)$, $h\geq 2$, *can be partitioned into* $q^{2h-1}+2q^h$ *strong representative systems.*

Theorem 2.1. gives approximately a factor of q improvement on the trivial estimate. When $h=2$ and q is a square, it is also sligthly better than the result obtained from the ILLÉS, SZŐNYI and WETTL method.

In graph theoretic terminology we thus answered the question of GYÁRFÁS [10] on the strong chromatic index of the point-line incidence graph of $\mathrm{PG}(2,q)$, for q not a prime.

3. Constructions in the plane

Connected to the spectrum of minimal blocking sets is the problem of determining the number of minimal blocking sets of a given size. In particular, GYÖRGY TURÁN asked whether for any given size k the number of non-isomorphic minimal blocking sets of size k is more than polynomial.

The term more than polynomial (or superpolynomial) refers to a function that grows faster than any polynomial of the variable, i.e. a function of the form $f(q)=q^{g(q)}$, where $\lim_{q\to\infty} g(q)=\infty$.

The answer is clearly negative for the low end of the spectrum: for $k=q+1$ we only have the lines as blocking sets and they are all equivalent; for $k=q+$ $+\sqrt{q}+1$ we only have Baer subplanes, which are all equivalent, too. For the largest possible value the question is open: the number of isomorphism classes for the known unitals in $\mathrm{PG}(2,q)$, q odd, is polynomial (see BAKER and EBERT

[5]) and some people conjecture that there are no more examples. On the other hand, the situation might turn out to be similar to the situation of maximal arcs, where recently some new examples were constructed by MATHON [18] giving more than polynomial of them for some sizes, see [14].

The main results in this direction are the following.

THEOREM 3.1. (MENGYÁN, [2]) *Let* log *denote the natural base logarithm.*

0) *There are constants* c *and* C *such that for* $q \equiv 1(4)$, *the number of non-isomorphic minimal blocking sets in* $\mathrm{PG}(2, q)$ *of size from the interval* $[cq \log q, Cq \log q]$ *is more than polynomial.*

1) *In* $\mathrm{PG}(2, q)$ *there are more than polynomial non-isomorphic minimal blocking sets for any size in the interval* $[5q \log q, q\sqrt{q} - 2q]$ *whenever* q *is a square.*

2) *In* $\mathrm{PG}(2, q)$ *there are more than polynomial non-isomorphic minimal blocking sets for any size in the interval* $[2q - 1, 3q - 4]$ *whenever* q *is odd or* q *is even and* $q - 1$ *is not a prime.*

3) *Let* d *be an arbitrary integer,* $2 \leq d \leq \sqrt{q}$, $d \mid q - 1$. *The number of minimal blocking sets* M *of size* $2q + 1 - d$ *is more than polynomial.*

The proofs are given in four sections. The idea behind the four results is similar: take a well-known construction (or density result) for minimal blocking sets and modify/generalize it in such a way that one has a lot of choices at a certain step of the construction. However, the techniques are different (depending on the original construction): for part 2) we use the method of subsets, for 3) we use Megyesi's construction (a subgroup and its cosets), for 1) we use random choice together with geometric arguments, while in the proof of part 0) besides random choice and geometric arguments we will also need an algebraic lemma based on Weil's estimate on the number of $\mathrm{GF}(q)$-rational points of an algebraic curve.

In all constructions we will try to find more than polynomial number of non-equivalent minimal blocking sets for a given size, that is, two blocking sets for which there is a collineation of $\mathrm{PG}(2, q)$ taking one to the other are counted once. On the other hand, since the order of the collineation group of $\mathrm{PG}(2, q)$ is definitely smaller than q^9, finding more than polynomial different minimal blocking sets will automatically imply the existence of more than polynomial non-equivalent examples.

4. A generalization of Megyesi's construction

A different example than Megyesi's construction, contained in the union of four lines, was constructed by Gács [12] giving an infinite series of examples determining $7q/9$ directions approximately, thus yielding a minimal blocking set of approximate size $2q-2q/9$: *Let* 3 *be a divisor of* $q-1$*, and let* $1,\alpha,\alpha^2$ *be coset represenatives of the multiplicative subgroup* G *of index* 3*. Let*

$$U_i=\{(0,0)\}\cup\{(x,0):x\in\alpha^iG\}\cup\{(x,x):x\in G\}\cup\{(0,x):x\in\alpha^iG\}.$$

Denote by $|D_i|$ *the number of directions determined by* U_i*. Then* $|D_1|+$ $+|D_2|+|D_3|=3q+1-2(q-1)/3$*, and* $|D_i|=7q/9+O(\sqrt{q})$*.*

In both the Megyesi and the Gács constructions a subgroup and its cosets were placed on lines. A natural question is whether a generalization would be possible when the number of cosets is larger than three, or when the number of lines from which points are taken is increased. We show that this problem can be reformulated using some trivial equations and solved by Weil's estimate. In some sense the technique we use here is similar to techniques used by Korchmáros in [17] and Szőnyi in [20], though our method seems distinct from theirs. Here again we suppose that q is large enough, as the remainder term in Weil's estimate may be too large for small values.

We also describe a simple embedding method connected to the minimal blocking sets obtainable from the generalization of Megyesi's construction that can give under some light conditions non-Rédei minimal blocking sets; these are blocking sets B for which there is no line l such that $|B\setminus l|=q$. The following result is found in [1]: *Let* $x>1$ *be an integer. If there is a minimal blocking set of size* $2q-x$ *in* $\mathrm{PG}(2,q)$ *then there are minimal blocking sets of size* $2q^h-x$ *and* $2q^h-x+1$ *in* $\mathrm{PG}(2,q^h)$*.*

Our results indicate the existence of a large number of minimal blocking sets slightly under the size of $2q$ that are non-Rédei minimal blocking sets.

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COBORDISM OF SINGULAR MAPS
Abstract of Ph.D. Thesis

By
TAMÁS TERPAI
ADVISOR: ANDRÁS SZŰCS

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1. Introduction

Local singularity theory studies the properties of smooth $(\mathbb{R}^n, 0) \to \to (\mathbb{R}^{n+k}, 0)$ mapping germs. Two germs are called $\mathcal{A}$-equivalent if a smooth reparametrization of the source and the target space takes one to the other. The resulting equivalence classes are called (mono)singularities. Complete classification of singularities is currently a hopeless problem, here we will only investigate cases where the classification is already known.

Global singularity theory focuses on singular loci of mappings between smooth manifolds. From this point of view it becomes important to consider the local behaviour of a mapping on the full preimage of a point in the target space. Such a description, consisting of multiple monosingularities, is called a multisingularity.

The set of points at which a given sufficiently generic map has a certain monosingularity or multisingularity often carries information about other properties of the mapping itself as well as the source and target manifolds. The initial question of this thesis is the following problem: given a mapping, what is the obstruction to the existence of a mapping that is equivalent to it in some sense, but only has certain (multi)singularities?

In the study of mappings between manifolds we aim for classifications up to so-called singular bordism and singular cobordism. That is, we consider two mappings equivalent, if together they form the boundary of a mapping between manifolds with boundary that only has singularities from a set fixed in advance. The reason for this choice is that while the manifolds are well understood up to abstract cobordism (without restrictions on the involved mappings) thanks to the works of Thom and Wall, there is no such practically applicable classification for a finer equivalence relation, hence the investigation of mappings would have to be cumbersome.

2. Methods and results

The primary tool for investigation of bordism and cobordism groups is the generalized Pontryagin-Thom construction [9], which transforms the calculation of cobordism groups into a purely homotopy theoretical question. The investigation of this construction and the classifying spaces that play a fundamental role in it is the main goal of this thesis.

2.1. Cobordism groups

The general construction of classifying spaces [9] glues these spaces together from blocks corresponding to the allowed multisingularities. As a consequence, the homotopy groups of the resulting space are very hard to compute. Szűcs [16] proved that when the set of allowed multisingularities consists of all multisingularities composed from a fixed set of monosingularies (there are no global restrictions), there exists a so-called "key fibration" [16, Definition 109] between the classifying spaces that makes the calculation of their homotopy groups more approachable. We give a new, more geometric proof of the existence of the key fibration, which stays valid under more general conditions than the original one. For example, we can prove singularity removal theorems in the case of negative codimensional mappings analogous to the positive codimensional ones:

Theorem 1. *If M is a closed 4-manifold and P is a closed 3-manifold, then any smooth generic mapping $f : M \to P$ is cobordant to a generic mapping without definite or indefinite swallowtail singularities. Should M and P be given orientations, the source manifold of the cobordism can be chosen to be oriented as well.*

As another extension, we can handle a case with a global restriction (generalizing [16, Proposition 108]):

Theorem 2. *Let η be a monosingularity of $k > 0$ codimensional mappings, let τ' be the set of all multisingularities composed from a fixed set of monosingularities, and assume that while $\partial\eta \subseteq \tau'$, $\eta \notin \tau'$. Furthermore let τ_r for $r > 0$ be the set of all multisingularities composed from a singularity in τ' and at most r*

points with η singularity, and denote by $\tilde{\Gamma}_r$ the classifying space of immersions with a normal $\tilde{\xi}_\eta$-structure and at most r-tuple self-intersection points. Then the natural forgetful mapping $X_{\tau_r} \to \tilde{\Gamma}_r$ is a Serre fibration with fiber $X_{\tau'}$.

COROLLARY 3. *Under the assumptions of the previous theorem a τ_r-mapping is τ_r-cobordant to a τ'-mapping exactly when its singular set of η-points is null-cobordant as an immersion with a normal $\tilde{\xi}_\eta$-structure and at most r-tuple self-intersection points.*

With the help of the key fibration we compute the cobordism groups of fold maps (which may only have the simplest monosingularity apart from the regular point) in the first two cases when they cannot be trivially identified with the abstract cobordism groups.

THEOREM 4 ([20]). *Denote by τ the set of all multisingularities composed from regular and fold monosingularities.*

(a) *For all $k \geq 1$ $Cob_\tau(2k+1, k) \cong \mathfrak{N}_{2k+1}$;*

(b$_1$) *$Cob_\tau^{SO}(5,2) \cong \Omega_5 \oplus \mathbb{Z}_2 \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$;*

(b$_*$) *For all $m \geq 2$ $Cob_\tau^{SO}(4m+1, 2m) \cong \Omega_{4m+1}$;*

(c) *For all $m \geq 1$ $Cob_\tau^{SO}(4m-1, 2m-1) \cong \Omega_{4m-1} \oplus \mathbb{Z}_{3^t}$, where $t = \min\{j \mid \alpha_3(2m+j) \leq 3j\}$ and $\alpha_3(x)$ denotes the sum of digits of the natural number x in base 3.*

THEOREM 5.

a) *$Cob_\tau(2k+2, k)$ is the kernel of the characteristic number $\bar{w}_{k+1}^2 + \bar{w}_{k+2}\bar{w}_k$, an index 2 subgroup of $\mathfrak{N}_{2k+2}$;*

b$_2$) *$Cob_\tau^{SO}(6,2) \cong 0$;*

b$_*$) *for all even $k \geq 4$ the group $Cob_\tau^{SO}(2k+2, k)$ is the kernel of the characteristic number $\bar{w}_{k+1}^2 + \bar{w}_{k+2}\bar{w}_k$, an index 2 subgroup of Ω_{2k+2};*

c) *for all odd k the group $Cob_\tau^{SO}(2k+2, k)$ is the kernel of the characteristic number $\bar{p}_{(k+1)/2}$, a nontrivial subgroup of Ω_{2k+2}.*

In the first case a small modification of the argument gives the bordism groups of fold maps in relation to the bordism groups of all (oriented) maps [13].

THEOREM 6.

(a) *For all $k \geq 0$ $Bord_\tau(2k+1, k) \cong \mathfrak{N}(2k+1, k)$;*

(b) *For all $m \geq 1$ $Bord_\tau^{SO}(4m+1, 2m) \cong \Omega(4m+1, 2m)$;*

(c) *For all $m \geq 1$ the following sequence is exact:*

$$0 \to \mathbb{Z}_{3^u} \to Bord_\tau^{SO}(4m-1, 2m-1) \to \Omega(4m-1, 2m-1) \to 0;$$

here u satisfies the inequality $0 \leq u \leq t$, $t = \min\{j \mid \alpha_3(2m+j) \leq 3j\}$, using the notation of Theorem 4.

In the course of the proof we give new examples of mappings that have a null-homologous cusp locus both in source and target (hence the Thom polynomial [17] cannot detect the cusp singularity), but no fold mapping is even abstractly cobordant to it.

Similar results can be proven in the case when τ is the set of all multi-singularities composed from regular, fold and cusp points, and the goal is the elimination from the cobordisms of the so-called $III_{2,2}$ singularity, which is the simplest corank 2 monosingularity.

If the dimension of the singularity to avoid is more than 1, the geometric methods employed in the proofs of the theorems above fail. For relatively small dimensions, however, we can still assign geometric meaning to the obstructions that need to be computed. These obstructions are elements of the homotopy groups of the form $\pi_*(\Gamma T\tilde{\xi})$, where $\tilde{\xi}$ denotes the universal target bundle [9] of the singularity; it describes the global behaviour of the singular points in the target manifold. For a fixed r denote by $\mathcal{C}$ the Serre class of finite abelian groups of odd order divisible only by primes dividing $r+1$ and let $\mathcal{C}^+$ be the class of finite abelian groups of order divisible only by primes dividing $2(r+1)$. Isomorphism up to groups in $\mathcal{C}$ and $\mathcal{C}^+$ will be denoted by $\cong_{\mathcal{C}}$ and $\cong_{\mathcal{C}^+}$ respectively.

THEOREM 7. *Let $\tilde{\xi}$ be the universal target bundle of the Morin singularity $\Sigma^{1_r,0}$ in either the unoriented or the oriented case. For a given mapping $\kappa : \mathbb{S}^{n+k} \to \to T\tilde{\xi}$, set $M = M(\kappa) = \kappa^{-1}(0_{\tilde{\xi}})$ the transverse preimage of the zero section of $\tilde{\xi}$ (with the induced orientation when $\tilde{\xi}$ is orientable). Then for all $0 \leq m < k$ the following statements hold.*

- *If $\tilde{\xi}$ is not orientable, $\pi_{m+rk+k+r}(\Gamma T\tilde{\xi}) \in \mathcal{C}^+$.*
- *If $\tilde{\xi}$ is orientable, $\pi_{m+rk+k+r}(\Gamma T\tilde{\xi}) \cong_{\mathcal{C}^+} \Omega_m$. The isomorphism is given by the abstract oriented bordism class of M.*
- *If r is even and $\tilde{\xi}$ is not orientable, $\pi_{m+rk+k+r}(\Gamma T\tilde{\xi}) \cong_{\mathcal{C}} \mathfrak{N}_m(\mathbb{R}P^\infty)$. The isomorphism is given by the cobordism class of M decorated with the kernel bundle of the mapping f classified by κ.*
- *If r is even and $\tilde{\xi}$ is orientable, $\pi_{m+rk+k+r}(\Gamma T\tilde{\xi}) \cong_{\mathcal{C}} \Omega_m(\mathbb{R}P^\infty)$. The isomorphism is given by the oriented cobordism class of M decorated with the kernel bundle.*

2.2. Bordism groups

The singular bordism groups are somewhat easier to compute than the singular cobordism groups, since the generalized Pontryagin-Thom construction identifies the former groups with the abstract bordism groups of the corresponding classifying spaces, and these can be handled even with the original block-construction. Additionally, when looking for cohomological obstructions, investigating the so-called Kazarian space instead of the much more complicated classifying space already gives results. For example, with the help of the Kazarian space we can determine the avoiding ideal of the cusp singularity. The avoiding ideal [4] (with $\mathbb{Z}_2$ coefficients) of the monosingularity η is the set of those Stiefel-Whitney characteristic classes that vanish on the virtual normal bundle of any smooth, fiberwise polynomial bundle mapping between vector bundles.

THEOREM 8. *The avoiding ideal of the singularity $\Sigma^{1,1}$ is generated as an $H^*(BO;\mathbb{Z}_2)$ ideal by the set*

$$\{w_{k+l}w_{k+m}+w_{k+q}w_{k+r}|l,m,q,r\geq 0 \text{ and } l+m=q+r\geq 2\}.$$

COROLLARY 9. *The avoiding ideal of the singularity $\Sigma^{1,1}$ consists exactly of the classes of the form $\sum_{I\in\mathcal{I}} w_I$ such that*

$$\sum_{I\in\mathcal{I}} c^S w_k^{|I^+|} w_{I\setminus I^+}=0,$$

where $\mathcal{I}$ contains only index sets I with $\max I>k$, I^+ denotes the subsequence $\cup\{J\subseteq I|\min J>k\}$ and $S=\sum_{i\in I^+}(i-k)$.

As an application of this result we can obtain nearly optimal bounds on the existence of fold mappings of real projective spaces into Euclidean spaces:

THEOREM 10. *Let $n=2^s+t$ with s and $t<2^s$ nonnegative integers. If there exists a fold mapping from $\mathbb{R}P^n$ to $\mathbb{R}^{n+k}$, then*

- *if $\frac{4}{3}2^s<n<2^{s+1}$, then $k\geq 2^{s+1}-n-2$.*
- *if $2^s<n<\frac{4}{3}2^s$, then*
  - *for $n=2^u(8a+3)+b$ with $0\leq b<2^u$ and maximal u, $k\geq 2^{u+2}a+$ $+2^u-b-2$.*
  - *if $\lfloor\frac{n}{2^u}\rfloor\not\equiv 3 \bmod 4$ for all $u\geq 0$, then*
    - *$k\geq\frac{n-3}{2}$ for odd n and*
    - *$k\geq\frac{n}{2}-2^{p-1}$ for $n=2^p m$ with an odd m.*

Using a deep result of [16], the so-called Kazarian conjecture, we compute the unoriented bordism groups of fold mappings from the homology groups of the Kazarian space of the fold mappings.

Finally we investigate the classifying space to obtain singular bordism groups in the case when allowed mappings have only single fold points and regular points with bounded multiplicity. For a fixed codimension $k > 0$ denote by I_2 the set containing the fold and the regular points with multiplicity at most 2; let q_m be the number of partitions of the nonnegative integer m that contain no elements greater than k, and set $d_m = \dim_{\mathbb{Z}_2} \mathfrak{N}_m$.

THEOREM 11.

$$\dim Bord_{I_2}(n) = \sum_{s=0}^{n+k} \sum_{j=0}^{\left\lfloor \frac{n-k-s-1}{2} \right\rfloor} q_j q_{n-k-s-j} d_s + \sum_{s=0}^{n+k} \sum_{r=\left\lceil \frac{n-k-s}{2} \right\rceil}^{n-k-s-1} q_r d_s + \\ + \sum_{s=0}^{n+k} q_{n-s} d_s + d_{n+k}$$

We get a similar explicit formula for the I_2-bordism group of cooriented mappings, and from these we determine the rest of the unoriented, respectively cooriented I-bordism groups for all such sets I that contain only single fold points and regular points with bounded multiplicity. In the case of mappings between oriented manifolds we compute the I-bordism groups rationally.

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Address:

MATHEMATICAL INSTITUTE
EÖTVÖS LORÁND UNIVERSITY
ELTE TTK KARI KÖNYVTÁR
MATEMATIKAI TUDOMÁNYÁGI SZAKGYŰJTEMÉNY
PÁZMÁNY PÉTER SÉTÁNY 1/c
1117 BUDAPEST, HUNGARY

Műszaki szerkesztő:

Fried Katalin PhD

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