

ANNALES UNIVERSITATIS SCIENTIARUM BUDAPESTINENSIS DE ROLANDO EÖTVÖS NOMINATAE

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ANNALES

UNIVERSITATIS SCIENTIARUM BUDAPESTINENSIS DE ROLANDO EÖTVÖS NOMINATAE

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ANDRÁS GÁCS (1969–2009)

We report with great sadness the passing away of András Gács, associate professor of the Department of Computer Science. He died suddenly at home on 28 October 2009 at the age of 41.

Gács studied mathematics at the Loránd Eötvös University between 1988 and 1993. After graduation, he stayed at the university as a student of the PhD program in Pure Mathematics then newly created. He obtained his degree in 1997 with the thesis “The Rédei Method in Finite Geometry” (advisor: Tamás Szőnyi). He spent the years between 1997 and 2000 as a researcher of the Alfréd Rényi Institute of the Hungarian Academy of Sciences. Then he returned to the Department of Computer Science as assistant professor. Gács was promoted to associate professor in 2002. He received a Bolyai János Fellowship between 2006 and 2009. With Zoltán Porkoláb he coordinated the mathematics-informatics seminar at Bolyai College.

His research interest lied in finite geometry, algebraic methods in combinatorics, and polynomials over finite fields. His main skill was the clever use



of polynomials for solving combinatorial problems arising from geometry. He always liked problems which looked easy, but in reality, were difficult, sometimes very much so. We mention only two of Gács's several strong results. He established a fifteen year old conjecture of Blokhuis and Szőnyi on regular semiovals by showing that they are either ovals or unitals. In a series of papers he managed to prove an astonishing generalization of Rédei's theorem on directions determined by p points in an affine Galois plane. His results inspired and will continue to inspire many researchers in finite geometry.

András was an enthusiastic teacher who also wanted to entertain his audience. His students loved him as a teacher and as a person. He attracted a strong group of gifted PhD students (Péter Csikvári, Tamás Héger, Zoltán Lóránt Nagy) and worked on several problems with them. Gács was always ready to help students, colleagues, and friends. He was a good friend of many, and a good colleague of us all. We will sorely miss his warm-hearted personality and special vein of humour.

The Institute of Mathematics of Eötvös Loránd University, Budapest organized a memorial seminar on the 29th of October, 2010. Family members, former students, classmates, and colleagues came together to share their memories about András. The program of the seminar was coordinated by Péter Sziklai. It consisted of the following talks:

- László Lovász: Opening
- Simeon Ball: On functions over a field of prime order p , which determine less than $p - \sqrt{p}$ directions
- Zsuzsa Weiner: Arcs and codes
- Aart Blokhuis: History and future of the directions problem
- Zoltán L. Nagy: Permutations, hyperplanes and polynomials over finite fields
- László Szegő: Laughing is recommended
- Tamás Szőnyi: From buildings to directions
- Leo Storme: Spectrum results in finite geometry
- Tamás Héger: Regular configurations in $\text{PG}(2, p)$
- Péter Csikvári: From the density Turán-problem to integral trees

Department of Computer Science, Institute of Mathematics

List of publications of András Gács

- [1] A. GÁCS, P. SZIKLAI, T. SZÖNYI: Two remarks on blocking sets and nuclei in planes of prime order *Designs, Codes and Cryptography*, **10** (1997), 29–39.
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- [15] A. GÁCS, T. SZÖNYI: Random constructions and density results, *Designs, Codes and Cryptography*, **47** (2007), 267–287.
- [16] S. BALL, A. BLOKHUIS, A. GÁCS, P. SZIKLAI, Zs. WEINER: On linear codes whose weights and length have a common divisor, *Advances in Mathematics*, **211** (2007), 94–104.
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SPECTRAL CALCULATIONS IN RINGS

By

GYULA LAKOS

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Abstract. We examine the validity of certain spectral integral formulas in topological rings. We consider the sign and square-root functions in polymetric rings containing $\frac{1}{2}$. It turns out that formal analogues of classical transformation kernels and the resolvent identity can be used to understand the situation. In the lack of $\frac{1}{2}$, the functions $\frac{1}{2} - \frac{1}{2} \operatorname{sgn}\left(\frac{1}{2} - z\right)$ and $\frac{1}{2} - \sqrt{\frac{1}{4} - z}$ can be generalized, respectively.

0. Introduction

Spectral integrals like

$$(1) \quad \operatorname{sgn} Q = \int_{\{z \in \mathbb{C} : |z|=1\}} \frac{\frac{1-z}{2} + \frac{1+z}{2} Q}{\frac{1+z}{2} + \frac{1-z}{2} Q} \frac{|dz|}{2\pi}$$

and

$$(2) \quad \sqrt{S} = \int_{\{z \in \mathbb{C} : |z|=1\}} \frac{S}{\frac{1}{z} \left(\left(\frac{1+z}{2} \right)^2 - \left(\frac{1-z}{2} \right)^2 S \right)} \frac{|dz|}{2\pi}$$

are often useful. They extend the complex functions $\operatorname{sgn} Q$, which is the sign function of (the real part of) Q , and \sqrt{S} , which is the square root function cut along the negative real axis, respectively. Definitions like above are justified if they are supported by appropriate algebraic identities and spectral

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properties. This is the situation in linear analysis, where the formulas above can be established for elements with appropriate spectral properties in great generality, even if the resolvent terms are not necessarily continuous, cf. HAASE [1], MARTÍNEZ CARRACEDO–SANZ ALIX [3]. We refer to this case as the “analytic case”.

However, in the analytic case, if the resolvent terms are continuous, then they are also smooth, and one can expand everything in terms of Fourier series, or rather Laurent series in z . One can naturally ask if similar computations can be done in more general rings, in particular, in rings without a natural \mathbb{R} -action. It is natural to check these ideas for formal Laurent series on polymetric rings. We refer to this case as the “algebraic case”.

We call a ring \mathfrak{A} polymetric if

- (a) its topology is induced by a family of “seminorms” $p : \mathfrak{A} \rightarrow [0, +\infty)$ such that $p(0) = 0$, $p(-X) = p(X)$, $p(X + Y) \leq p(X) + p(Y)$,
- (b) for each “seminorm” p there exists a “seminorm” \tilde{p} such that $p(XY) \leq \tilde{p}(X)\tilde{p}(Y)$;

i. e., if it is a polymetric space whose multiplication is compatible with the topology. Dealing with Laurent series in z , integration over the unit circle becomes a formal process. It is nothing else but detecting the coefficient of z^0 . On the other hand, for the multiplication of Laurent series, some sort of convergence control is required. Primarily, we will be interested in Laurent series with rapidly decreasing coefficients. A sequence is rapidly decreasing if it is rapidly decreasing in each seminorm. Furthermore, we consider only sequentially complete, Hausdorff polymetric rings. We also assume that $\frac{1}{2} \in \mathfrak{A}$. Then (1) and (2) are meaningful.

Indeed, applied to elements with appropriate “spectral” properties, the expressions $\operatorname{sgn} Q$ and \sqrt{S} will have good properties justifying the notation. This can be proved by an analysis of the coefficients. The algebraic approach is particularly manageable in the case of (1), it is essentially shown in KAROUBI [2], by a direct analysis of coefficients, that the expression $\operatorname{sgn} Q$ yields an involution compatible with factorization of affine loops. Nevertheless, such computations are not necessarily very enlightening. The objective of this paper is to prove our statements regarding the algebraic case and to do this in a manner which brings the algebraic and analytic cases together, at least formally. It turns out that the basic tool of the analytic case, the resolvent identity, works generally. Another natural question is what happens in the

lack of $\frac{1}{2}$. Then the sign and square root functions are not really appropriate. Instead, we can generalize

$$(3) \quad \text{idem } P = \int_{\{z \in \mathbb{C} : |z|=1\}} \frac{Pz}{1 - P + Pz} \frac{|dz|}{2\pi}$$

and

$$(4) \quad \sqrt[3]{T} = \int_{\{z \in \mathbb{C} : |z|=1\}} \frac{(1+z)T}{1 + (z - 2 + z^{-1})T} \frac{|dz|}{2\pi},$$

extending the functions $\text{idem } P = \frac{1}{2} - \frac{1}{2} \operatorname{sgn} \left(\frac{1}{2} - P \right)$ and $\sqrt[3]{T} = \frac{1}{2} - \sqrt{\frac{1}{4} - T}$, respectively.

1. Laurent series

1.1 If \mathfrak{A} is a polymetric ring, then we may consider formal Laurent series $a = \sum_{n \in \mathbb{Z}} a_n z^n$. If p is a seminorm on \mathfrak{A} and $\alpha : \mathbb{Z} \rightarrow \mathbb{R}_+$ is a non-negative function, then we may define $p_\alpha(a) = \sum_{n \in \mathbb{Z}} \alpha(n)p(a_n)$. Then we may consider the polymetric spaces

- (a) $\mathfrak{A}[z^{-1}, z]^f$ of essentially finite Laurent series,
- (b) $\mathfrak{A}[z^{-1}, z]^\infty$ of rapidly decreasing Laurent series,
- (c) $\mathfrak{A}[z^{-1}, z]^b$ of “summable” Laurent series,
- (d) $\mathfrak{A}[[z^{-1}, z]]^b$ of bounded Laurent series,
- (e) $\mathfrak{A}[[z^{-1}, z]]^\infty$ of polynomially growing Laurent series,
- (f) $\mathfrak{A}[[z^{-1}, z]]^f$ of formal Laurent series, as the spaces which contain series bounded for each p_α such that

- (a) α is unrestricted,
- (b) α is polynomially growing,
- (c) α is bounded,
- (d) α is summable,
- (e) α is rapidly decreasing,
- (f) α is vanishing except at finitely many places, respectively.

We have continuous inclusions

$$\begin{aligned} \mathfrak{A}[z^{-1}, z]^f &\hookrightarrow \mathfrak{A}[z^{-1}, z]^\infty \hookrightarrow \mathfrak{A}[z^{-1}, z]^b \hookrightarrow \mathfrak{A}[[z^{-1}, z]]^b \hookrightarrow \\ &\hookrightarrow \mathfrak{A}[[z^{-1}, z]]^\infty \hookrightarrow \mathfrak{A}[[z^{-1}, z]]^f. \end{aligned}$$

Of these spaces, $\mathfrak{A}[z^{-1}, z]^f$, $\mathfrak{A}[z^{-1}, z]^\infty$, $\mathfrak{A}[z^{-1}, z]^b$ will remain polymetric rings. Indeed, (\widetilde{p}_α) can be chosen as $\tilde{p}_{\check{\alpha}}$, where

$$\check{\alpha}(n) = 1 \vee \max_{-2|n| \leq m \leq 2|n|} |\alpha(m)|.$$

We have compatible continuous module actions $\mathfrak{A}[z^{-1}, z]^\bullet \times \mathfrak{A}[[z^{-1}, z]]^\bullet \rightarrow \mathfrak{A}[[z^{-1}, z]]^\bullet$. Essentially the same applies to the spaces of power series $\mathfrak{A}[z]^f$, $\mathfrak{A}[z]^\infty$, $\mathfrak{A}[z]^b$, $\mathfrak{A}[[z]]^b$, $\mathfrak{A}[[z]]^\infty$, $\mathfrak{A}[[z]]^f$, except here even $\mathfrak{A}[[z]]^\infty$, $\mathfrak{A}[[z]]^f$ are polymetric rings. Indeed, for them, (\widetilde{p}_α) can be chosen as $\tilde{p}_{\check{\alpha}}$, where $\check{\alpha}(n) = \sqrt{\max_{n \leq m} |\alpha(m)|}$. An element like $\frac{1+Q}{2} + \frac{1-Q}{2}z$ may be considered either as an element of $\mathfrak{A}[z^{-1}, z]^f$, $\mathfrak{A}[z^{-1}, z]^\infty$, or $\mathfrak{A}[z^{-1}, z]^b$, etc. Practically, the difference is that the larger the ring is the easier is to find a multiplicative inverse of the element given.

1.2 If $a(z) = \sum_{n \in \mathbb{Z}} a_n z^n \in \mathfrak{A}[[z^{-1}, z]]^f$, then we define formally

$$\int a(z) \frac{|dz|}{2\pi} = a_0.$$

The Hilbert kernel (“up to multiplication by i”) is defined as

$$\left[\frac{1+z}{1-z} \right] = \sum_{s \in \mathbb{Z}} (\operatorname{sgn} s) z^s \in \mathfrak{A}[[z^{-1}, z]]^b.$$

1.3 Some further terminology is as follows. For $a(z) = \sum_{n \in \mathbb{Z}} a_n z^n \in \mathfrak{A}[z^{-1}, z]^b$ we let

$$\lim_{z \nearrow 1} a(z) = \sum_{n \in \mathbb{Z}} a_n.$$

Naturally, this notation also applies to power series.

In what follows, we let $\mathfrak{A}[z^{-1}, z]$, $\mathfrak{A}[[z^{-1}, z]]$, $\mathfrak{A}[z]$, $\mathfrak{A}[[z]]$ denote $\mathfrak{A}[z^{-1}, z]^\infty$, $\mathfrak{A}[[z^{-1}, z]]^\infty$, $\mathfrak{A}[z]^\infty$, $\mathfrak{A}[[z]]^\infty$, respectively, but similar statements hold for f and b , too.

PROPOSITION 1.4. *For $a(t, z) \in \mathfrak{A}[t][[z^{-1}, z]]$ and $b(z) \in \mathfrak{A}[z^{-1}, z]$,*

$$\int \left(\lim_{t \nearrow 1} a(t, z) \right) b(z) \frac{|dz|}{2\pi} = \lim_{t \nearrow 1} \int (a(t, z)) b(z) \frac{|dz|}{2\pi}.$$

PROOF. This is just the generalized associativity of the rapidly decreasing (hence absolute convergent) sum $\sum_{n \in \mathbb{N}, s \in \mathbb{Z}} a_{n,s} b_{-s}$. \blacksquare

1.5. For the sake of brevity, we call the elements of $\mathfrak{A}[t][[z^{-1}, z]]$ as transformation kernels. Practically, the convenient thing is to consider those elements of the *ring* $\mathfrak{A}[[t]]^f[z^{-1}, z]^b$ which can be thought to be transformation kernels. (This is advantageous from computational viewpoint, because the product of $a(t, z)$ and $b(z) \in \mathfrak{A}[z^{-1}, z]$ formally yields the same element of $\mathfrak{A}[[t]]^f[[z^{-1}, z]]^b$ either we interpret $a(t, z) \in \mathfrak{A}[[t]]^f[z^{-1}, z]^b$ or $a(t, z) \in \mathfrak{A}[t][[z^{-1}, z]]$, but the first case is often easier to compute with.)

Such elements are the Poisson kernel

$$\mathcal{P}(t, z) = \frac{1 - t^2}{(1 - tz)(1 - tz^{-1})} = \sum_{s \in \mathbb{Z}} t^{|s|} z^s,$$

the Hilbert-Poisson kernel

$$\mathcal{H}(t, z) = \frac{t(z - z^{-1})}{(1 - tz)(1 - tz^{-1})} = \sum_{s \in \mathbb{Z}} (\operatorname{sgn} s) t^{|s|} z^s,$$

the $\frac{1}{2}$ -shifted odd Poisson kernel

$$\mathcal{L}(t, z) = \frac{(1 - t)(1 + z)}{(1 - tz)(1 - tz^{-1})} = \sum_{s \in \mathbb{N}} t^s z^{-s} + \sum_{s \in \mathbb{N}} t^s z^{s+1},$$

and the variant regularization kernel

$$\begin{aligned} \tilde{\mathcal{R}}(t, z) &= \frac{(1 + t)t(1 - z)(1 - z^{-1})}{2(1 - tz)(1 - tz^{-1})} = \\ &= \sum_{s \in \mathbb{N}, s > 0} \frac{t^{s+1} - t^s}{2} z^{-s} + t + \sum_{s \in \mathbb{N}, s > 0} \frac{t^{s+1} - t^s}{2} z^s. \end{aligned}$$

This latter one has the property

$$\lim_{t \nearrow 1} \tilde{\mathcal{R}}(t, z) = 1$$

(here we think of $\tilde{\mathcal{R}}(t, z)$ as an element of $\mathfrak{A}[t][[z^{-1}, z]]$). The ordinary regularization kernel

$$\mathcal{R}(t, z) = \frac{t(1 - z)(1 - z^{-1})}{(1 - tz)(1 - tz^{-1})}$$

is just an element of $\mathfrak{A}[[t]]^f[z^{-1}, z]^b$, hence it is not so convenient algebraically. On the other hand, the use of variant regularization makes the variant Hilbert-Poisson kernel

$$\tilde{\mathcal{H}}(t, z) = \frac{1+t}{2} \mathcal{H}(t, z) = \frac{(1+t)t(z - z^{-1})}{2(1-tz)(1-tz^{-1})} = \sum_{s \in \mathbb{Z}} (\operatorname{sgn} s) \frac{1+t}{2} t^{|s|} z^s$$

useful.

It is not hard to see that the definitions and the proposition above can be formulated in the case when we have many variables z, w, \dots instead of just z . For example, we may consider the Hilbert kernel

$$\left[\frac{z+w}{z-w} \right] = \left[\frac{1+wz^{-1}}{1-wz^{-1}} \right] = \sum_{s \in \mathbb{Z}} (\operatorname{sgn} s) w^s z^{-s}.$$

2. Spectral classes

2.1. In order to save some space we use the short-hand notation

$$\begin{aligned} \Lambda(a) &= \frac{1}{2}(1+a), \\ \Lambda(a, b) &= \frac{1}{2}(1+a+b-ab), \\ \Lambda(a, b, c) &= \frac{1}{4}(1+a+b+c-ab+ac-bc+abc), \\ \Lambda(a, b, c, d) &= \frac{1}{4}(1+a+b+c+d-ab-bc-cd+ac+ad+bd \\ &\quad + abc-acd-abd+bcd-abcd), \end{aligned}$$

etc., following the scheme

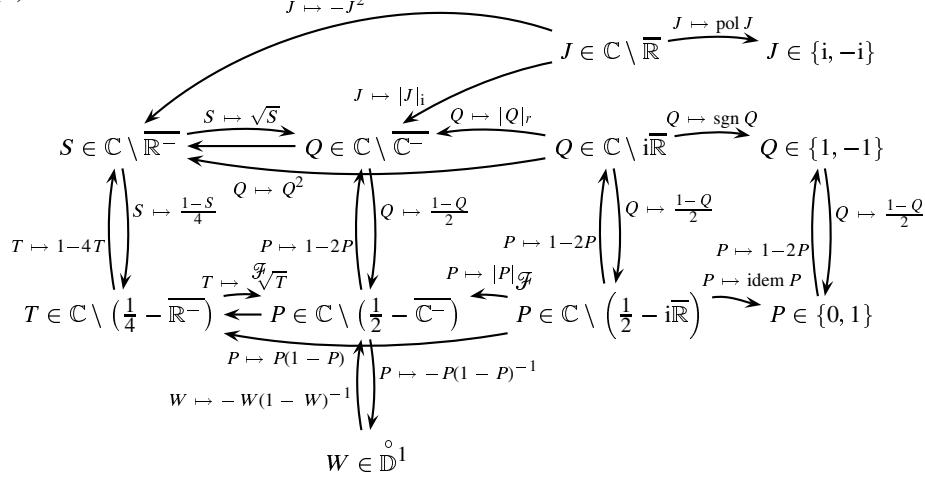
$$\Lambda(c_1, c_2, c_3, \dots, c_n) = 2^{-\lceil \frac{n}{2} \rceil} \sum_{\varepsilon \in \{0,1\}^n} \left(\prod_{1 \leq j < n} (-1)^{\varepsilon_j \varepsilon_{j+1}} \right) \left(\prod_{1 \leq k \leq n} c_k^{\varepsilon_k} \right)$$

such that the order of the symbols c_k is preserved in the products.

2.2. Let $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ denote the Riemann sphere. Some subsets are: $i\overline{\mathbb{R}} = i\mathbb{R} \cup \{\infty\}$, $\overline{\mathbb{R}^-} = (-\infty, 0] \cup \{\infty\}$, $\overline{\mathbb{C}^-} = \{s \in \mathbb{C} : \operatorname{Re} s \leq 0\} \cup \{\infty\}$, $\overset{\circ}{\mathbb{D}}^1 = \{z \in \mathbb{C} : |z| < 1\}$. We define the functions $\operatorname{pol} J = -i \operatorname{sgn} iJ$,

$|J|_i = \sqrt{-J^2}$, $|Q|_r = \sqrt{Q^2}$, $|P|_{\mathcal{F}} = \frac{1}{2} - \left| \frac{1}{2} - P \right|_r$. We have the following commutative diagram on certain subsets of the complex plane:

(5)



such that $\text{pol } J$, $\text{sgn } Q$, $\text{idem } P$, $|J|_i$, $|Q|_r$, $|P|_{\mathcal{F}}$ yield idempotent operations and they yield decompositions

$$J = |J|_i \text{pol } J, \quad Q = |Q|_r \text{sgn } Q, \quad P = \text{idem } P + |P|_{\mathcal{F}} - 2|P|_{\mathcal{F}} \text{idem } P.$$

DEFINITION 3.2. Suppose that \mathfrak{A} is a locally convex algebra and R is a compact subset of $\overline{\mathbb{C}}$. We define $\text{Spec}_R(\mathfrak{A})$ as the set containing all elements $X \in \mathfrak{A}$ such that the functions

$$f_R : z \in R \setminus \{\infty\} \mapsto (z - X)^{-1} \quad \text{and} \quad g_R : z \in R \setminus \{0\} \mapsto X(1 - z^{-1}X)^{-1}$$

are well-defined and continuous. The topology of $\text{Spec}_R(\mathfrak{A})$ is induced from the compact-open topology of the continuous functions $(f_R)|_{\overset{\circ}{D^1} \cap R}$ and $(g_R)|_{R \setminus \overset{\circ}{D^1}}$. (If R is symmetric for conjugation then we may use the functions $f_R(z)f_R(\bar{z})$ and $g_R(z)g_R(\bar{z})$ in order to get a formally real characterization.)

Classes of interest are like $\text{Spec}_{\overline{iR}}(\mathfrak{A})$, etc., i. e. the elements spectrally avoiding $i\overline{R}$, etc. Another way to specify spectral conditions is to ask for skew-involutions, involutions, or idempotents. The main spectral classes correspond to the sets in (5) for $\mathfrak{A} = \mathbb{C}$.

DEFINITION 2.4. We define the corresponding formal spectral classes by

$$J \in \widetilde{\text{Spec}}_{\overline{\mathbb{R}}}(\mathfrak{A}) \Leftrightarrow \frac{1}{z} \left(\left(\frac{1+z}{2} \right)^2 + \left(\frac{1-z}{2} \right)^2 J^2 \right) \text{ is invertible in } \mathfrak{A}[z, z^{-1}],$$

$$J \in \text{Skvol}(\mathfrak{A}) \Leftrightarrow J^2 = -1,$$

$$Q \in \widetilde{\text{Spec}}_{i\overline{\mathbb{R}}}(\mathfrak{A}) \Leftrightarrow \frac{1+z}{2} + \frac{1-z}{2} Q = \Lambda(z, Q) \text{ is invertible in } \mathfrak{A}[z, z^{-1}],$$

$$Q \in \widetilde{\text{Spec}}_{\overline{\mathbb{C}}^-}(\mathfrak{A}) \Leftrightarrow \frac{1+z}{2} + \frac{1-z}{2} Q = \Lambda(z, Q) \text{ is invertible in } \mathfrak{A}[z],$$

$$Q \in \text{Invol}(\mathfrak{A}) \Leftrightarrow Q^2 = 1,$$

$$S \in \widetilde{\text{Spec}}_{\overline{\mathbb{R}}^-}(\mathfrak{A}) \Leftrightarrow \frac{1}{z} \left(\left(\frac{1+z}{2} \right)^2 - \left(\frac{1-z}{2} \right)^2 S \right) = \Lambda(z, S, z^{-1})$$

$$\text{is invertible in } \mathfrak{A}[z, z^{-1}]$$

$$P \in \widetilde{\text{Spec}}_{\frac{1}{2}+i\overline{\mathbb{R}}}(\mathfrak{A}) \Leftrightarrow (1-P) + Pz \text{ is invertible in } \mathfrak{A}[z, z^{-1}],$$

$$P \in \widetilde{\text{Spec}}_{\frac{1}{2}-\overline{\mathbb{C}}^-}(\mathfrak{A}) \Leftrightarrow (1-P) + Pz \text{ is invertible in } \mathfrak{A}[z],$$

$$P \in \text{Idem}(\mathfrak{A}) \Leftrightarrow P^2 = P,$$

$$T \in \widetilde{\text{Spec}}_{\frac{1}{4}-\overline{\mathbb{R}}^-}(\mathfrak{A}) \Leftrightarrow 1 - (1-z)(1-z^{-1})T \text{ is invertible in } \mathfrak{A}[z, z^{-1}],$$

$$W \in \widetilde{\text{Spec}}_{\overline{\mathbb{C}} \setminus \overline{\mathbb{D}}^1}(\mathfrak{A}) \Leftrightarrow 1 + z W \text{ is invertible in } \mathfrak{A}[z].$$

2.5. If \mathfrak{A} is a locally convex algebra, then the formal spectral classes and their ordinary counterparts are the same. Indeed, the continuity of the resolvent terms implies smoothness by the resolvent identity, hence the existence of the appropriate Fourier series, and, conversely, the existence of the expansions implies continuity.

OBJECTIVE 2.6. We want to establish the spectral correspondences and decompositions as in point 2.2 for the formal spectral classes.

3. Calculations with $\frac{1}{2}$

A. Sign and square root

DEFINITION 3.1. For $Q \in \widetilde{\text{Spec}}_{i\overline{\mathbb{R}}}(\mathfrak{A})$, we define

$$\text{sgn } Q = \int \frac{\frac{1-z}{2} + \frac{1+z}{2}Q}{\frac{1+z}{2} + \frac{1-z}{2}Q} \frac{|dz|}{2\pi} = \int \frac{\Lambda(-z, Q)}{\Lambda(z, Q)} \frac{|dz|}{2\pi}.$$

PROPOSITION 3.2. If $Q \in \widetilde{\text{Spec}}_{i\overline{\mathbb{R}}}(\mathfrak{A})$, then $-Q, Q^{-1} \in \widetilde{\text{Spec}}_{i\overline{\mathbb{R}}}(\mathfrak{A})$. Q commutes with $\text{sgn } Q$. $\text{sgn } -Q = -\text{sgn } Q$ and $\text{sgn } Q^{-1} = \text{sgn } Q$. Moreover,

$$(\text{sgn } Q)^2 = 1.$$

PROOF. Substituting $z = -1$ we see that Q^{-1} exists. The first statement follows from the identities $\Lambda(z, -Q) = z \Lambda(z^{-1}, Q)$ and $\Lambda(z, Q^{-1}) = Q^{-1} \Lambda(-z, Q)$. Furthermore, Q and $\text{sgn } Q$ commute, because Q commutes with the integrand in $\text{sgn } Q$. The identities

$$\frac{\Lambda(-z, -Q)}{\Lambda(z, -Q)} \frac{|dz|}{2\pi} = -\frac{\Lambda(-z^{-1}, Q)}{\Lambda(z^{-1}, Q)} \frac{|d(z^{-1})|}{2\pi}$$

and

$$\frac{\Lambda(-z, Q^{-1})}{\Lambda(z, Q^{-1})} \frac{|dz|}{2\pi} = \frac{\Lambda(z, Q)}{\Lambda(-z, Q)} \frac{|d(-z)|}{2\pi}$$

integrated prove the first and second equalities, respectively.

The critical one is the involution property. We give several proofs.

“Matrix algebraic” proof. Let $H_{1/2} = \sum_{s \in \mathbb{Z} + \frac{1}{2}} (\text{sgn } s) \mathbf{e}_{s,s}$ be the

$$\left(\mathbb{Z} + \frac{1}{2} \right) \times \left(\mathbb{Z} + \frac{1}{2} \right)$$

matrix of the odd Hilbert transform. Let

$$U \left(\frac{1+Q}{2} z^{-1/2} + \frac{1-Q}{2} z^{1/2} \right) = \sum_{s \in \mathbb{Z} + \frac{1}{2}} \frac{1+Q}{2} \mathbf{e}_{s-\frac{1}{2},s} - \frac{1-Q}{2} \mathbf{e}_{s+\frac{1}{2},s}$$

be the $\mathbb{Z} \times \left(\mathbb{Z} + \frac{1}{2} \right)$ matrix of the action of multiplication by

$$\left(\frac{1+Q}{2} + \frac{1-Q}{2} z \right) z^{-1/2}.$$

According to our assumption, this has a $\left(\mathbb{Z} + \frac{1}{2}\right) \times \mathbb{Z}$ inverse matrix representing the action of multiplication by $z^{1/2} \left(\frac{1+Q}{2} + \frac{1-Q}{2}z\right)^{-1}$. So, we can consider the matrix

$$\begin{aligned} & \mathbf{B} \left(\frac{1+Q}{2} z^{-1/2} + \frac{1-Q}{2} z^{1/2} \right) = \\ & = \mathbf{U} \left(\frac{1+Q}{2} z^{-1/2} + \frac{1-Q}{2} z^{1/2} \right) \mathbf{H}_{1/2} \mathbf{U} \left(\frac{1+Q}{2} z^{-1/2} + \frac{1-Q}{2} z^{1/2} \right)^{-1}. \end{aligned}$$

Due to the special shape of the matrices involved, it is easy to see that this is an involution which is the same as the even Hilbert transform $\mathbf{H} = \sum_{s \in \mathbb{Z}} (\operatorname{sgn} s) \mathbf{e}_{s,s}$, except in the 0th column. This special shape implies that the diagonal element in the 0th column is an involution. On the other hand, it is easy to see that this diagonal element is exactly $\operatorname{sgn} Q$.

“Resolvent algebraic” proof. As

$$(6) \quad 1 - (\operatorname{sgn} Q)^2 = \iint \left(1 - \frac{\Lambda(-z, Q)}{\Lambda(z, Q)} \frac{\Lambda(-w, Q)}{\Lambda(w, Q)} \right) \frac{|dz|}{2\pi} \frac{|dw|}{2\pi},$$

we should show that this integral is 0. This, however, follows from the key identity

$$\begin{aligned} 1 - \frac{\Lambda(-z, Q)}{\Lambda(z, Q)} \frac{\Lambda(-w, Q)}{\Lambda(w, Q)} &= \left(\frac{z+w}{2} \right) \frac{1-Q^2}{\Lambda(z, Q)\Lambda(w, Q)} = \\ &= \frac{1}{2} \left[\frac{z+w}{z-w} \right] \frac{(z-w)(1-Q^2)}{\Lambda(z, Q)\Lambda(w, Q)} = \\ (7) \quad &= \frac{1}{2} \left[\frac{z+w}{z-w} \right] \left(\frac{(z-1)(1-Q^2)}{\Lambda(z, Q)} - \frac{(w-1)(1-Q^2)}{\Lambda(w, Q)} \right), \end{aligned}$$

which does make sense in $\mathfrak{A}[[z^{-1}, z]][[w^{-1}, w]]$. Indeed, (6) can be continued as

$$\begin{aligned} &= \iint \left[\frac{z+w}{z-w} \right] \frac{(z-1)(1-Q^2)}{2\Lambda(z, Q)} \frac{|dz|}{2\pi} \frac{|dw|}{2\pi} - \\ &\quad - \iint \left[\frac{z+w}{z-w} \right] \frac{(w-1)(1-Q^2)}{2\Lambda(w, Q)} \frac{|dz|}{2\pi} \frac{|dw|}{2\pi}. \end{aligned}$$

Evaluating the integrals we find

$$= \iint 0 \cdot \frac{(z-1)(1-Q^2)}{2\Lambda(z, Q)} \frac{|dz|}{2\pi} - \iint 0 \cdot \frac{(w-1)(1-Q^2)}{2\Lambda(w, Q)} \frac{|dw|}{2\pi} = 0 - 0 = 0.$$

This proof, like the previous one, relies heavily on the nature of Laurent series. Nevertheless the argument can be modified so that formally it makes sense in the analytical and the algebraic cases as well.

“Resolvent analytic” proof. According to the discussion about transformation kernels, (6) can be continued as follows:

$$= \lim_{t \nearrow 1} \iint \tilde{\mathcal{R}}(t, wz^{-1}) \left(1 - \frac{\Lambda(-z, Q)}{\Lambda(z, Q)} \frac{\Lambda(-w, Q)}{\Lambda(w, Q)} \right) \frac{|dz|}{2\pi} \frac{|dw|}{2\pi}.$$

By simple arithmetic in the integrand, this yields

$$\begin{aligned} &= \lim_{t \nearrow 1} \left(\iint \tilde{\mathcal{H}}(t, wz^{-1}) \frac{(z-1)(1-Q^2)}{2\Lambda(z, Q)} \frac{|dz|}{2\pi} \frac{|dw|}{2\pi} - \right. \\ &\quad \left. - \iint \tilde{\mathcal{H}}(t, wz^{-1}) \frac{(w-1)(1-Q^2)}{2\Lambda(w, Q)} \frac{|dz|}{2\pi} \frac{|dw|}{2\pi} \right). \end{aligned}$$

Executing the integrals we find

$$= \lim_{t \nearrow 1} \left(\int 0 \cdot \frac{(z-1)(1-Q^2)}{\Lambda(z, Q)} \frac{|dz|}{2\pi} - \int 0 \cdot \frac{(w-1)(1-Q^2)}{\Lambda(w, Q)} \frac{|dw|}{2\pi} \right) = 0+0=0,$$

yielding, ultimately, the identity. We remark that in the analytic case, it would actually be simpler to use the kernel $\mathcal{R}(t, wz^{-1})$. ■

DEFINITION 3.3. If $S \in \widetilde{\text{Spec}}_{\mathbb{R}^-}(\mathfrak{A})$, then we define the inverse square root operation as

$$\sqrt{S} = \int \frac{z S}{\left(\frac{1+z}{2}\right)^2 - \left(\frac{1-z}{2}\right)^2 S} \frac{|dz|}{2\pi} = \int \frac{S}{\Lambda(z, S, z^{-1})} \frac{|dz|}{2\pi}.$$

PROPOSITION 3.4. Suppose that

$$S \in \widetilde{\text{Spec}}_{\mathbb{R}^-}(\mathfrak{A}).$$

Then $S^{-1} \in \widetilde{\text{Spec}}_{\mathbb{R}^-}(\mathfrak{A})$. The elements \sqrt{S} and S commute with each other.

$\sqrt{S^{-1}} = \sqrt{S}^{-1}$. Furthermore,

$$(8) \quad (\sqrt{S})^2 = S.$$

PROOF. Substituting $z = -1$ into the resolvent term, we see that S^{-1} exists. The identity $\frac{1}{\Lambda(z, S^{-1}, z^{-1})} = \frac{S}{\Lambda(-z, S, (-z)^{-1})}$ shows that $S^{-1} \in \widetilde{\text{Spec}}_{\overline{\mathbb{R}}^-}(\mathfrak{A})$. Integrated, it yields $S\sqrt{S^{-1}} = \sqrt{S}$. If the square-root identity (8) holds, then this implies $\sqrt{S^{-1}} = \sqrt{S}^{-1}$. So, it remains to prove (8). As

$$(9) \quad (\sqrt{S})^2 - S = \iint \left(\frac{S}{\Lambda(z, S, z^{-1})} \frac{S}{\Lambda(w, S, w^{-1})} - S \right) \frac{|dz|}{2\pi} \frac{|dw|}{2\pi},$$

we have to show that this integral is 0. This follows using the key identities

$$(10) \quad \frac{S}{\Lambda(z, S, z^{-1})} \frac{S}{\Lambda(w, S, w^{-1})} - S = S \frac{S - \Lambda(z, S, z^{-1}, 1, w, S, w^{-1})}{2\Lambda(z, S, z^{-1})\Lambda(w, S, w^{-1})} + S \frac{S - \Lambda(z, S, z^{-1}, 1, w^{-1}, S, w)}{2\Lambda(z, S, z^{-1})\Lambda(w, S, w^{-1})};$$

$$(11) \quad \frac{S - \Lambda(z, S, z^{-1}, 1, w, S, w^{-1})}{2\Lambda(z, S, z^{-1})\Lambda(w, S, w^{-1})} = \frac{1}{2} \left[\frac{zw^{-1} + 1}{zw^{-1} - 1} \right] \left(\frac{\Lambda(-z, S, z^{-1})}{\Lambda(z, S, z^{-1})} - \frac{\Lambda(-w, S, w^{-1})}{\Lambda(w, S, w^{-1})} \right);$$

$$(12) \quad \frac{S - \Lambda(z, S, z^{-1}, 1, w^{-1}, S, w)}{2\Lambda(z, S, z^{-1})\Lambda(w, S, w^{-1})} = \frac{1}{2} \left[\frac{zw + 1}{zw - 1} \right] \left(\frac{\Lambda(-z, S, z^{-1})}{\Lambda(z, S, z^{-1})} - \frac{\Lambda(w, S, -w^{-1})}{\Lambda(w, S, w^{-1})} \right).$$

Indeed, after we decomposed the integrand in (9) according to (10–12), we can show that both parts are 0 as we did in the previous proof. ■

PROPOSITION 3.5. $Q \in \widetilde{\text{Spec}}_{\overline{\mathbb{R}}}(\mathfrak{A})$ if and only if $Q^2 \in \widetilde{\text{Spec}}_{\overline{\mathbb{R}}^-}(\mathfrak{A})$. In this case

$$\text{sgn } Q = Q^{-1} \sqrt{Q^2}.$$

PROOF. The first statement follows from the equality $\Lambda(z, Q^2, z^{-1}) = \Lambda(z, Q)\Lambda(z^{-1}, Q)$. The identity statement follows from

$$\text{sgn } Q = \frac{\text{sgn } Q}{2} + \frac{\text{sgn } Q}{2} =$$

$$\begin{aligned}
&= \frac{1}{2} \int \frac{\Lambda(-z, Q)}{\Lambda(z, Q)} \frac{|dz|}{2\pi} + \frac{1}{2} \int \frac{\Lambda(-z^{-1}, Q)}{\Lambda(z^{-1}, Q)} \frac{|d(z^{-1})|}{2\pi} = \\
&= \int \frac{1}{2} \left(\frac{\Lambda(-z, Q)}{\Lambda(z, Q)} + \frac{\Lambda(-z^{-1}, Q)}{\Lambda(z^{-1}, Q)} \right) \frac{|dz|}{2\pi} = \\
&= \int \frac{Q}{\Lambda(z, Q^2, z^{-1})} \frac{|dz|}{2\pi} = Q^{-1} \sqrt{Q^2}. \quad \blacksquare
\end{aligned}$$

B. Finer analysis of the resolvent terms

DEFINITION 3.6. (a) We define $|Q|_r = \sqrt{Q^2} = Q \operatorname{sgn} Q$.

(b) If $F \in \mathfrak{A}$ is an involution, then an element $A \in \mathfrak{A}$ can be written in matrix form

$$\begin{bmatrix} \frac{1-F}{2} A \frac{1-F}{2} & \frac{1-F}{2} A \frac{1+F}{2} \\ \frac{1+F}{2} A \frac{1-F}{2} & \frac{1+F}{2} A \frac{1+F}{2} \end{bmatrix}.$$

Suppose that $Q \in \widetilde{\operatorname{Spec}}_{i\overline{\mathbb{R}}}(\mathfrak{A})$. In the decomposition of \mathfrak{A} along the involution $\operatorname{sgn} Q$, the various components are denoted according to

$$\operatorname{sgn} Q = \begin{bmatrix} -1_{-\operatorname{sgn} Q} & 1_{\operatorname{sgn} Q} \end{bmatrix}, \quad Q = \begin{bmatrix} -Q^- & Q^+ \end{bmatrix}, \quad |Q|_r = \begin{bmatrix} Q^- & Q^+ \end{bmatrix}.$$

PROPOSITION 3.7. $Q^\pm \in \widetilde{\operatorname{Spec}}_{i\overline{\mathbb{R}}}(\mathfrak{A} 1_{\pm \operatorname{sgn} Q})$, $\operatorname{sgn} Q^\pm = 1_{\pm \operatorname{sgn} Q}$, and $\operatorname{sgn} |Q|_r = 1$.

PROOF. The decomposition of Q along $\operatorname{sgn} Q$ allows us to consider Q separately in the direct sum components of \mathfrak{A} . In particular, the sign integral splits, too, and it necessarily yields $\operatorname{sgn} Q^+ = 1_{\operatorname{sgn} Q}$ and $\operatorname{sgn} -Q^- = -1_{\operatorname{sgn} -Q}$. The statement follows from this immediately. \blacksquare

PROPOSITION 3.8. If $Q \in \widetilde{\operatorname{Spec}}_{i\overline{\mathbb{R}}}(\mathfrak{A})$, then

(a)

$$\frac{1}{\frac{1+z}{2} + \frac{1-z}{2} Q} = \frac{1}{\Lambda(z, Q)}$$

is given by

$$\begin{aligned} & \dots + \left[\begin{pmatrix} \left(\frac{Q^- - 1}{Q^- + 1} \right)^2 & \frac{2}{Q^- + 1} \\ 0 & 0 \end{pmatrix} z^{-3} + \begin{pmatrix} \left(\frac{Q^- - 1}{Q^- + 1} \right) \frac{2}{Q^- + 1} & 0 \\ 0 & 0 \end{pmatrix} z^{-2} + \right. \\ & \quad \left. + \begin{pmatrix} \frac{2}{Q^- + 1} & 0 \\ 0 & 0 \end{pmatrix} z^{-1} + \begin{pmatrix} 0 & \frac{2}{Q^+ + 1} \\ 0 & 0 \end{pmatrix} 1 + \begin{pmatrix} 0 & \frac{2}{Q^+ + 1} \left(\frac{Q^+ - 1}{Q^+ + 1} \right) \\ 0 & 0 \end{pmatrix} z + \right. \\ & \quad \left. + \begin{pmatrix} 0 & \frac{2}{Q^+ + 1} \left(\frac{Q^+ - 1}{Q^+ + 1} \right)^2 \\ 0 & 0 \end{pmatrix} z^2 + \dots \right] \end{aligned}$$

(b)

$$\frac{\frac{1-z}{2} + \frac{1+z}{2} Q}{\frac{1+z}{2} + \frac{1-z}{2} Q} = \frac{\Lambda(-z, Q)}{\Lambda(z, Q)}$$

is given by

$$\begin{aligned} & \dots + \left[\begin{pmatrix} -2 \left(\frac{Q^- - 1}{Q^- + 1} \right)^3 & 0 \\ 0 & 0 \end{pmatrix} z^{-3} + \begin{pmatrix} -2 \left(\frac{Q^- - 1}{Q^- + 1} \right)^2 & 0 \\ 0 & 0 \end{pmatrix} z^{-2} + \right. \\ & \quad \left. + \begin{pmatrix} -2 \left(\frac{Q^- - 1}{Q^- + 1} \right) & 0 \\ 0 & 0 \end{pmatrix} z^{-1} + \begin{pmatrix} -1 - \operatorname{sgn} Q & 1_{\operatorname{sgn} Q} \\ 1_{\operatorname{sgn} Q} & 0 \end{pmatrix} 1 + \right. \\ & \quad \left. + \begin{pmatrix} 0 & 2 \left(\frac{Q^+ - 1}{Q^+ + 1} \right) \\ 0 & 0 \end{pmatrix} z + \begin{pmatrix} 0 & 2 \left(\frac{Q^+ - 1}{Q^+ + 1} \right)^2 \\ 0 & 0 \end{pmatrix} z^2 + \right. \\ & \quad \left. + \begin{pmatrix} 0 & 2 \left(\frac{Q^+ - 1}{Q^+ + 1} \right)^3 \\ 0 & 0 \end{pmatrix} z^3 + \dots \right] \end{aligned}$$

(c) In particular,

$$\frac{1}{1 + |Q|_r} = \int \frac{1}{2} \frac{1+z}{\Lambda(z, Q)} \frac{|dz|}{2\pi}.$$

PROOF. (a) It is enough to consider the Q^+ part of the decomposition, because the other part follows from changing z to z^{-1} . So, we can suppose that $Q = |Q|$ and $\operatorname{sgn} Q = 1$.

We try to figure out the coefficients in the expansion

$$\frac{1}{\Lambda(z, Q)} = \sum_{n \in \mathbb{Z}} a_n z^n.$$

The equality $\operatorname{sgn} Q = 1$ means that

$$\begin{aligned} 0 &= \frac{1 - \operatorname{sgn} Q}{2} = \frac{1}{2} \int \left(1 - \frac{\Lambda(-z, Q)}{\Lambda(z, Q)} \right) \frac{|dz|}{2\pi} = \\ (13) \quad &= \int \frac{1 - Q}{2} \frac{z}{\Lambda(z, Q)} \frac{|dz|}{2\pi} = \frac{1 - Q}{2} a_{-1}. \end{aligned}$$

The product of $\Lambda(z, Q) = \frac{1+Q}{2} + \frac{1-Q}{2}z$ and $\sum_{n \in \mathbb{Z}} a_n z^n$ gives 1, so

$$(14) \quad \frac{1 - Q}{2} a_{n-1} + \frac{1 + Q}{2} a_n = \delta_{0,n} 1.$$

From (13) and the case $n = 0$ in (14), we obtain that $a_0 = \frac{2}{1+Q}$. After that, from (14), we find $a_{n+1} = \frac{Q-1}{Q+1} a_n$, yielding the positive-numbered coefficients. Then $F(z) = \sum_{n \in \mathbb{N}} a_n z^n$ already inverts $\Lambda(z, Q)$, hence, from the uniqueness of the inverse, $a(z) = F(z)$.

(b) and (c) follow from part (a) by simple algebra. ■

PROPOSITION 3.9. (a) If $S \in \widetilde{\operatorname{Spec}}_{\mathbb{R}^-}(\mathfrak{A})$, then

$$\frac{z}{\left(\frac{1+z}{2}\right)^2 - \left(\frac{1-z}{2}\right)^2 S} = \frac{1}{\Lambda(z, S, z^{-1})}$$

yields the expansion

$$\sqrt{S}^{-1} 1 + \sqrt{S}^{-1} \left(\frac{\sqrt{S} - 1}{\sqrt{S} + 1} \right) (z + z^{-1}) + \sqrt{S}^{-1} \left(\frac{\sqrt{S} - 1}{\sqrt{S} + 1} \right)^2 (z^2 + z^{-2}) + \dots$$

(b) In particular,

$$\frac{1}{\sqrt{S} + 1} = \int \frac{1}{2} \frac{1+z}{\Lambda(z, S, z^{-1})} \frac{|dz|}{2\pi}.$$

PROOF. (a) Take $Q = \sqrt{S}$. Proposition 3.5 yields $\operatorname{sgn} Q = 1$. Applying the identity $\frac{1}{\Lambda(z, S, z^{-1})} = \frac{1}{\Lambda(z, Q)\Lambda(z^{-1}, Q)}$ and Proposition 3.8, it follows that the coefficient of $z^{\pm n}$ ($n \geq 0$) in the expansion is

$$\left(\frac{2}{Q+1} \right)^2 \sum_{m \in \mathbb{N}} \left(\frac{Q-1}{Q+1} \right)^{n+2m} =$$

$$= \left(\frac{2}{Q+1} \right)^2 \left(\frac{Q-1}{Q+1} \right)^n \left(1 - \left(\frac{Q-1}{Q+1} \right)^2 \right)^{-1},$$

which simplifies as above. The rapid decrease of $\left(\frac{Q-1}{Q+1} \right)^s$ makes our computations legal.

(b) follows from part (a). ■

PROPOSITION 3.10. (a) $Q \in \widetilde{\text{Spec}}_{\overline{\mathbb{C}^-}}(\mathfrak{A})$ if and only if $Q \in \widetilde{\text{Spec}}_{\overline{i\mathbb{R}^-}}(\mathfrak{A})$ and $\text{sgn } Q = 1$.

(b) $W \in \widetilde{\text{Spec}}_{\overline{\mathbb{C} \setminus \mathbb{D}^1}}^\circ(\mathfrak{A})$ if and only if W^n is rapidly decreasing.

(c) If $Q \in \widetilde{\text{Spec}}_{\overline{\mathbb{C}^-}}(\mathfrak{A})$, then $\frac{1-Q}{1+Q} \in \widetilde{\text{Spec}}_{\overline{\mathbb{C} \setminus \mathbb{D}^1}}^\circ(\mathfrak{A})$. Conversely, if $W \in \widetilde{\text{Spec}}_{\overline{\mathbb{C} \setminus \mathbb{D}^1}}^\circ(\mathfrak{A})$, then $\frac{1-W}{1+W} \in \widetilde{\text{Spec}}_{\overline{\mathbb{C}^-}}(\mathfrak{A})$. This establishes a bijection.

PROOF. (a) follows from Proposition 3.8.a. (b) holds because in those cases $(1-Wz)^{-1} = \sum_{n \in \mathbb{N}} W^n z^n$ must hold. (c) follows from $\Lambda(z, \frac{1-W}{1+W}) = \frac{1+z}{1-W}$ and $1 + \frac{1-Q}{1+Q}z = \frac{2\Lambda(z, Q)}{1+Q}$. ■

C. On our objective

3.11. Now, it is easy to see that the propositions proven above are sufficient to establish all the spectral correspondences asked in 2.6. We merely define $|J|_i = \sqrt{-J^2}$, $\text{pol } J = |J| - J^2|_i^{-1}$, idem $P = \frac{1}{2} - \frac{1}{2} \text{sgn} \left(\frac{1}{2} - P \right)$, $|P|_{\mathcal{F}} = \frac{1}{2} - \left| \frac{1}{2} - P \right|_r$, and $\sqrt[3]{T} = \frac{1}{2} - \sqrt{\frac{1}{4} - T}$. Hence our objective is established.

This is, however, not to say that everything is just like for locally convex algebras:

D. Comparison to the case of locally convex algebras

PROPOSITION 3.12. *If \mathfrak{A} is a locally convex algebra, then the condition that $S \in \text{Spec}_{\overline{\mathbb{R}^-}}(\mathfrak{A})$ is equivalent to the condition that the function*

$$\frac{1+t}{2} + \frac{1-t}{2} S$$

has a continuous inverse on $[-1, 1]$. (This is the same thing as to say that the segment connecting 1 and S is continuously invertible.) The square root can be expressed as

$$\sqrt{S} = \int_{t \in [-1, 1]} \frac{S}{\frac{1+t}{2} + \frac{1-t}{2} S} \frac{dt}{\pi \sqrt{1-t^2}}.$$

PROOF. It follows by change of variables using $t = \frac{z+z^{-1}}{2}$. ■

3.13. If $S \in \text{Spec}_{\overline{\mathbb{C}^-}}(\mathfrak{A})$, then $\frac{1+t}{2} + \frac{1-t}{2} S$ ($t \in [-1, 1]$) is clearly invertible. As a rapidly decreasing power series in t , using $t = \frac{z+z^{-1}}{2}$ and considering the coefficients of z^k in $\left(\frac{z+z^{-1}}{2}\right)^n$, it follows that $S \in \text{Spec}_{\overline{\mathbb{R}^-}}(\mathfrak{A})$. Hence the inclusion $\text{Spec}_{\overline{\mathbb{C}^-}}(\mathfrak{A}) \subset \text{Spec}_{\overline{\mathbb{R}^-}}(\mathfrak{A})$ is true. In the general context, this cannot be done so, because the boundedness of the elements $\frac{1}{2^n} \binom{n}{k}$ is not always clear. Similar comment applies for $\text{Spec}_{\frac{1}{2}+\overline{\mathbb{C}^-}}(\mathfrak{A}) \subset \text{Spec}_{\frac{1}{2}+\overline{\mathbb{R}^-}}(\mathfrak{A})$.

4. Formal homotopies

4.1. One expects certain natural behaviour from the operations above. For example, one expects to have a homotopy from Q to $\text{sgn } Q$ inside $\widetilde{\text{Spec}}_{\overline{\mathbb{R}}}(\mathfrak{A})$. In general algebras, one cannot use continuous variables, but one can come up with homotopies using formal variables. Let us remind that an element $Q \in \widetilde{\text{Spec}}_{\overline{\mathbb{R}}}(\mathfrak{A})$ can be decomposed to a commuting pair, $\text{sgn } Q$ and a perturbation of 1 which is $|Q|_r$. But we may also consider this as a decomposition to the commuting pair $\text{sgn } Q$ and a perturbation of 0 which is

$$\text{pert}_r Q = \frac{|Q|_r - 1}{|Q|_r + 1} = - \int \frac{\frac{1+z}{2} - \frac{1-z}{2} Q}{\frac{1+z}{2} + \frac{1-z}{2} Q} \frac{|dz|}{2\pi} = - \int \frac{\Lambda(z, -Q)}{\Lambda(z, Q)} \frac{|dz|}{2\pi}.$$

If we replace $\text{pert}_r Q$ by $t \text{ pert}_r Q$ in the decomposition, then we obtain a homotopy, this appears as $K(t, -1, Q)$ in what follows.

DEFINITION 4.2. We define

$$\begin{aligned} K(t, z, Q) &= \frac{1 + \text{sgn } Q}{2} \frac{\Lambda(tz, |Q|_r)}{\Lambda(t, |Q|_r)} + \frac{1 - \text{sgn } Q}{2} z \frac{\Lambda(tz^{-1}, |Q|_r)}{\Lambda(t, |Q|_r)}, \\ H(t, z, Q) &= \frac{1 + \text{sgn } Q}{2} \frac{\Lambda(t, |Q|_r)}{\Lambda(tz, |Q|_r)} + \frac{1 - \text{sgn } Q}{2} z^{-1} \frac{\Lambda(t, |Q|_r)}{\Lambda(tz^{-1}, |Q|_r)}, \\ L(t, z, Q) &= \frac{1 + \text{sgn } Q}{2} \Lambda(tz, |Q|_r) + \frac{1 - \text{sgn } Q}{2} z \Lambda(tz^{-1}, |Q|_r), \\ G(t, z, Q) &= \frac{1 + \text{sgn } Q}{2} \frac{1}{\Lambda(tz, |Q|_r)} + \frac{1 - \text{sgn } Q}{2} z^{-1} \frac{1}{\Lambda(tz^{-1}, |Q|_r)}. \end{aligned}$$

PROPOSITION 4.3. *The expressions $K(t, z, Q)$ and $H(t, z, Q)$ are multiplicative inverses of each other.*

$$K(t, z, Q) = \frac{\Lambda(z, \text{sgn } Q, t, |Q|_r)}{\Lambda(t, |Q|_r)} = \Lambda\left(z, \frac{\Lambda(-t, |Q|_r)}{\Lambda(t, |Q|_r)} \text{sgn } Q\right).$$

$$\begin{aligned} K(t, 1, Q) &= 1, \quad K(1, -1, Q) = Q, \quad K(0, -1, Q) = \text{sgn } Q, \quad K(-1, -1, Q) = \\ &= Q^{-1}. \end{aligned}$$

$$H(t, z, Q) = \frac{\Lambda(t, |Q|_r) \Lambda(z^{-1}, \text{sgn } Q, t, |Q|_r)}{\Lambda(tz, |Q|_r) \Lambda(tz^{-1}, |Q|_r)}.$$

Similarly, the expressions $L(t, z, Q)$ and $G(t, z, Q)$ are inverses.

$$L(t, z, Q) = \Lambda(z, \text{sgn } Q, t, |Q|_r) = \Lambda(t, \text{sgn } Q, z, Q).$$

$$\begin{aligned} L(t, 1, Q) &= \Lambda(t, |Q|_r), \quad L(1, -1, Q) = Q, \quad L(0, -1, Q) = \frac{1}{2}(Q + \text{sgn } Q), \\ L(-1, -1, Q) &= \text{sgn } Q. \end{aligned}$$

$$G(t, z, Q) = \frac{\Lambda(z^{-1}, \text{sgn } Q, t, |Q|_r)}{\Lambda(tz, |Q|_r) \Lambda(tz^{-1}, |Q|_r)}.$$

PROOF. The computation is easy if we notice that in the defining formulas the coefficients of $\frac{1+\text{sgn } Q}{2}$ and $\frac{1-\text{sgn } Q}{2}$ live separate lives because $\text{sgn } Q$ is an involution. ■

The properties of K show that $\frac{\Lambda(-t, |Q|_r)}{\Lambda(t, |Q|_r)} \text{sgn } Q$ is a homotopy from Q ($t = 1$) to $\text{sgn } Q$ ($t = 0$) inside $\widetilde{\text{Spec}}_{i\mathbb{R}}(\mathfrak{A})$. Here the meaning of “inside” is that the whole expression satisfies the appropriate formal spectral condition.

PROPOSITION 4.4.

$$K(t, w, Q) = \int \mathcal{P}(t, z) \frac{\Lambda(zw, Q)}{\Lambda(z, Q)} \frac{|dz|}{2\pi},$$

$$H(t, w, Q) = \int \mathcal{P}(t, z) \frac{\Lambda(z, Q)}{\Lambda(zw, Q)} \frac{|dz|}{2\pi},$$

$$G(t, w, Q) = \int \mathcal{L}(t, z) \frac{1}{\Lambda(zw, Q)} \frac{|dz|}{2\pi}.$$

PROOF. This follows from the series expansion in Proposition 3.8. \blacksquare

REMARK 4.5. In locally convex algebras, the controllability of the powers of $\frac{z+z^{-1}}{2}$ makes possible to consider

$$\begin{aligned} & \Lambda(z, \Lambda(t, |Q|_r) \operatorname{sgn} Q) = \\ & = \frac{1 + \operatorname{sgn} Q}{2} \Lambda(\Lambda(z, t), |Q|_r) + \frac{1 - \operatorname{sgn} Q}{2} z \Lambda(\Lambda(z^{-1}, t), |Q|_r), \end{aligned}$$

whose inverse turns out to be

$$\frac{\Lambda(z^{-1}, \Lambda(t, |Q|_r) \operatorname{sgn} Q)}{\Lambda(\Lambda(z, t), |Q|_r) \Lambda(\Lambda(z^{-1}, t), |Q|_r)}.$$

This shows that $\Lambda(t, |Q|_r) \operatorname{sgn} Q = \frac{1+t}{2} \operatorname{sgn} Q + \frac{1-t}{2} Q$ is also a formal homotopy between Q ($t = -1$) and $\operatorname{sgn} Q$ ($t = 1$) inside $\widetilde{\operatorname{Spec}}_{\overline{\mathbb{R}}}(\mathfrak{A})$.

4.6. Similarly, we can contract elements inside $\widetilde{\operatorname{Spec}}_{\overline{\mathbb{R}}^-}(\mathfrak{A})$ to 1. For $S \in \widetilde{\operatorname{Spec}}_{\overline{\mathbb{R}}^-}(\mathfrak{A})$ consider

$$C(t, S) = \left(\frac{1 + t \frac{\sqrt{S}-1}{\sqrt{S}+1}}{1 - t \frac{\sqrt{S}-1}{\sqrt{S}+1}} \right)^2, \quad \sqrt{C(t, S)} = \frac{1 + t \frac{\sqrt{S}-1}{\sqrt{S}+1}}{1 - t \frac{\sqrt{S}-1}{\sqrt{S}+1}}.$$

The substitution $t \mapsto -t$ inverts them multiplicatively. In fact, the corresponding loops invert:

PROPOSITION 4.7. For $S \in \widetilde{\operatorname{Spec}}_{\overline{\mathbb{R}}^-}(\mathfrak{A})$, we have

$$\frac{1}{\Lambda(w, C(t, S), w^{-1})} = \frac{\sqrt{S}}{\sqrt{C(t, S)}} \int \mathcal{P}(t, z) \frac{1}{\Lambda(zw, S, (zw)^{-1})} \frac{|dz|}{2\pi}.$$

REMARK 4.8. In locally convex algebras, alternative contracting paths are rather trivial to find. It is more interesting to see that the class of loops of type

$$\frac{1}{\frac{1}{z} \left(\left(\frac{1+z}{2} \right)^2 A^{-1} - \left(\frac{1-z}{2} \right)^2 B^{-1} \right)}$$

remains invariant with respect to the Poisson kernel. For $t = 0$, they contract to the geometric mean

$$\sqrt{A \cdot B} = \int \frac{1}{\frac{1}{z} \left(\left(\frac{1+z}{2} \right)^2 A^{-1} - \left(\frac{1-z}{2} \right)^2 B^{-1} \right)} \frac{|dz|}{2\pi}.$$

5. Calculations without $\frac{1}{2}$

As we have seen, much can be generalized to the case $\frac{1}{2} \in \mathfrak{A}$. It is natural to ask what happens in the lack of $\frac{1}{2}$. Then only the lower portion of (5) can be generalized. Again, the idempotent and the \mathcal{F} -square-root identities are the key properties.

DEFINITION 5.1. For $P \in \text{Spec}_{\frac{1}{2} + i\mathbb{R}}(\mathfrak{A})$, we define

$$\text{idem } P = \int \frac{Pz}{1 - P + Pz} \frac{|dz|}{2\pi}.$$

PROPOSITION 5.2. If $P \in \text{Spec}_{\frac{1}{2} + i\mathbb{R}}(\mathfrak{A})$, then $1 - 2P$ is invertible, $1 - P, \frac{-P}{1-2P} \in \text{Spec}_{\frac{1}{2} + i\mathbb{R}}(\mathfrak{A})$, and $\text{idem}(1 - P) = 1 - \text{idem } P$, $\text{idem } \frac{-P}{1-2P} = \text{idem } P$. Furthermore,

$$(\text{idem } P)^2 = \text{idem } P.$$

PROOF. The invertibility statement follows from the substitution $z = -1$. The identities $P + (1 - P)z = (P + (1 - P)z^{-1})z$ and $(1 - \frac{-P}{1-2P}) + \frac{-P}{1-2P}z = (1 - 2P)^{-1}(1 - P + P(-z))$ imply the spectral statements. The identities

$$\frac{Pz}{(1 - P) + Pz} \frac{|dz|}{2\pi} = \left(1 - \frac{(1 - P)z^{-1}}{P + (1 - P)z^{-1}} \right) \frac{|d(z^{-1})|}{2\pi},$$

$$\frac{\frac{-P}{1-2Pz}}{(1-\frac{-P}{1-2P})+\frac{-P}{1-2Pz}} \frac{|dz|}{2\pi} = \frac{P(-z)}{1-P+P(-z)} \frac{|d(-z)|}{2\pi}$$

integrated prove the first and second equalities, respectively. We can prove the idempotent identity in several ways:

Matrix algebraic proof. We can proceed as before, but have to conjugate not the Hilbert transform involution but the idempotent $\sum_{s \in -\mathbb{N}-\frac{1}{2}} \mathbf{e}_{s,s}$.

A direct algebraic proof. See KAROUBI [2], Lemma III.1.23–24.

A resolvent algebraic proof. We should prove that

$$(15) \quad \text{idem } P(1 - \text{idem } P) = \iint \frac{Pz}{1 - P + Pz} \frac{1 - P}{1 - P + Pw} \frac{|dz|}{2\pi} \frac{|dw|}{2\pi}$$

is equal to 0. It is natural try the proof along the steps

$$(16) \quad \begin{aligned} & \frac{Pz}{1 - P + Pz} \frac{1 - P}{1 - P + Pw} \sim \left[\frac{z+w}{2} \right] \frac{P(1-P)}{(1-P+Pz)(1-P+Pw)} = \\ & = \left[\frac{1}{2} \frac{z+w}{z-w} \right] \left(\frac{(z-1)P(1-P)}{1-P+Pz} - \frac{(w-1)P(1-P)}{1-P+Pw} \right) \sim 0, \end{aligned}$$

except it seems to be plagued by $\frac{1}{2}$'s as before. We have to demonstrate that the use of division by 2 is of superficial nature in the proof. This can be done as follows.

For a Laurent series $a(z, w)$, we define : $a(z, w) :_{z,w}$ by linear extension from

$$: z^n w^m :_{z,w} = z^{\max(n,m)} w^{\min(n,m)}.$$

LEMMA 5.3. *For any Laurent series $a(z, w)$, we have*

$$\iint a(z, w) \frac{|dz|}{2\pi} \frac{|dw|}{2\pi} = \iint : a(z, w) :_{z,w} \frac{|dz|}{2\pi} \frac{|dw|}{2\pi}.$$

For a symmetric Laurent series $a(z, w)$, i. e. such that $a(z, w) = a(w, z)$, we define formally

$$: \left[\frac{z+w}{2} \right] a(z, w) :_{z,w} =: z a(z, w) :_{z,w} .$$

Suppose that $a(z, w)$ is anti-symmetric in its variables, included that the coefficient of $z^n w^n$ is always 0. We define : $\left[\frac{1}{2} \frac{z+w}{z-w} \right] a(z, w) :_{z,w}$ formally to be as it should be according to our natural expectations. We have to check

that the resulting expression is integral in terms of the coefficients. In the present case, the definition yields by linear extension from

$$\begin{aligned} & : \left[\frac{1}{2} \frac{z+w}{z-w} \right] (z^n w^m - z^m w^n) :_{z,w} = \\ & = z^n w^m + 2 \sum_{0 < k < \frac{n-m}{2}} z^{n-k} w^{m+k} + \delta_{\frac{n+m}{2} \in \mathbb{Z}} z^{\frac{n+m}{2}} w^{\frac{n+m}{2}}, \end{aligned}$$

where $n > m$. Checking for elements of suitable bases it is easy to see the following lemmas:

LEMMA 5.4. *For any symmetric Laurent series $a(z, w)$,*

$$: \left[\frac{z+w}{2} \right] a(z, w) :_{z,w} =: \left[\frac{1}{2} \frac{z+w}{z-w} \right] (z-w)a(z, w) :_{z,w}.$$

LEMMA 5.5 *For any Laurent series $a(z)$, we have*

$$\iint : \left[\frac{1}{2} \frac{z+w}{z-w} \right] (a(z) - a(w)) :_{z,w} \frac{|dz|}{2\pi} \frac{|dw|}{2\pi} = 0.$$

Now it is easy to carry out the proof. From (15) we should pass to the “normal ordered” form, after which the subsequent manipulations as in (16), leading to 0, make sense. ■

DEFINITION 5.6. For $T \in \text{Spec}_{\frac{1}{4}-\overline{\mathbb{R}^-}}(\mathfrak{A})$, we define

$$\sqrt[T]{T} = \int \frac{(1+z)T}{1+(z-2+z^{-1})T} \frac{|dz|}{2\pi}.$$

DEFINITION 5.7. Let $\mathfrak{A}\langle z \rangle^+$ be the space of formal Laurent series $a(z) = a_0 + \sum_{k=1}^{\infty} a_k \frac{z^k + z^{-k}}{2}$, and let $\mathfrak{A}[z]^+$ be the space of formal Laurent series $b(z) = b_0 + \sum_{k=1}^{\infty} b_k (z^k + z^{-k})$. Similarly, let $\mathfrak{A}\langle z \rangle^-$ be the space of formal Laurent series $a(z) = \sum_{k=1}^{\infty} a_{-k} \frac{z^k - z^{-k}}{2}$, and let $\mathfrak{A}[z]^-$ be the space of formal Laurent series $b(z) = \sum_{k=1}^{\infty} b_{-k} (z^k - z^{-k})$. It is easy to see that $\mathfrak{A}\langle z \rangle = \mathfrak{A}\langle z \rangle^+ \oplus \mathfrak{A}\langle z \rangle^-$ is a natural $\mathfrak{A}[z]^\pm = \mathfrak{A}[z]^+ \oplus \mathfrak{A}[z]^-$ -module, in fact, this action is \mathbf{Z}_2 -graded. Multiplication of $1 \in \mathfrak{A}\langle z \rangle$ yields a natural map from $\mathfrak{A}[z]^\pm$ into $\mathfrak{A}\langle z \rangle$:

$$1 \cdot \left(\sum_{k=1}^{\infty} b_{-k} (z^k - z^{-k}) + b_0 + \sum_{k=1}^{\infty} b_k (z^k + z^{-k}) \right) =$$

$$= \sum_{k=1}^{\infty} 2b_{-k} \frac{z^k - z^{-k}}{2} + b_0 + \sum_{k=1}^{\infty} 2b_k \frac{z^k + z^{-k}}{2}.$$

In fact, this notation can be extended to $b(z) \in \mathfrak{A}[z, z^{-1}]$ in a compatible way, by $1 \cdot z = \frac{z+z^{-1}}{2} + \frac{z-z^{-1}}{2}$, etc. Integration can be defined for elements of $\mathfrak{A}\langle z \rangle$ or $\mathfrak{A}[z]^\pm$. Again, it singles out the 0th coefficient. We see that if $a(z) \in \mathfrak{A}\langle z \rangle$ and $b(z) \in \mathfrak{A}[z]^\pm$ as above, then

$$\int a(z) \cdot b(z) \frac{|dz|}{2\pi} = \sum_{n \in \mathbb{Z}} a_n b_n.$$

For example, $\int \left(1 + \frac{z+z^{-1}}{2}\right) \cdot b(z) \frac{|dz|}{2\pi} = b_0 + b_1$. Furthermore, $\int b(z) \frac{|dz|}{2\pi} = \int 1 \cdot b(z) \frac{|dz|}{2\pi}$.

We can extend this formalism to multiple variables. The spaces $\mathfrak{A}\langle z \rangle\langle w \rangle$ and $\mathfrak{A}[z]^\pm[w]^\pm$ can be considered. In the case of $\mathfrak{A}\langle z \rangle\langle w \rangle$, colloquial notation like

$$\frac{zw^{-1} + wz^{-1}}{2} \equiv \frac{z + z^{-1}}{2} \frac{w + w^{-1}}{2} - \frac{z - z^{-1}}{2} \frac{w - w^{-1}}{2}$$

is allowed. However, another space between $\mathfrak{A}\langle z \rangle\langle w \rangle$ and $\mathfrak{A}[z]^\pm[w]^\pm$ can be considered. Indeed, let $\mathfrak{A}\{z, w\}$ be the space of the formal combinations of the basis elements

$$\begin{aligned} & 1, \frac{z^n + z^{-n}}{2}, \frac{z^n - z^{-n}}{2}, \frac{w^n + w^{-n}}{2}, \frac{w^n - w^{-n}}{2}, \\ & \frac{(z^n + z^{-n})(w^n + w^{-n})}{2}, \frac{(z^n + z^{-n})(w^n - w^{-n})}{2}, \\ & \frac{(z^n - z^{-n})(w^n + w^{-n})}{2}, \frac{(z^n - z^{-n})(w^n - w^{-n})}{2}, \end{aligned}$$

where $n, m \geq 1$. In this case, colloquial notation like

$$zw^{-1} - wz^{-1} \equiv \frac{(z - z^{-1})(w + w^{-1})}{2} - \frac{(z + z^{-1})(w - w^{-1})}{2}$$

is allowed. There are natural $\mathfrak{A}[z]^\pm[w]^\pm$ -module homomorphisms

$$\mathfrak{A}[z]^\pm[w]^\pm \rightarrow \mathfrak{A}\{z, w\} \rightarrow \mathfrak{A}\langle z \rangle\langle w \rangle$$

respecting the grading.

5.8. The advantage of the terminology above is that it allows us to rewrite the definition of the \mathcal{F} -square-root as the “manifestly real” expression

$$\sqrt[{\mathcal{F}}]{T} = \int \frac{(1 + \frac{z+z^{-1}}{2})T}{1 + (z - 2 + z^{-1})T} \frac{|dz|}{2\pi}.$$

PROPOSITION 5.9. If $T \in \text{Spec}_{\frac{1}{4}-\overline{\mathbb{R}^-}}(\mathfrak{A})$, then $\frac{-T}{1-4T}$ is invertible and $\sqrt[{\mathcal{F}}]{\frac{-T}{1-4T}} = \frac{-\sqrt[{\mathcal{F}}]{T}}{1-2\sqrt[{\mathcal{F}}]{T}}$. Furthermore,

$$(17) \quad \sqrt[{\mathcal{F}}]{T}(1 - \sqrt[{\mathcal{F}}]{T}) = T.$$

PROOF. Substituting $z = -1$ into the resolvent term, we see that $(1-4T)^{-1}$ exists. The identity

$$1 + (z - 2 + z^{-1}) \frac{-T}{1-4T} = (1-4T)^{-1}(1 + ((-z) - 2 + (-z)^{-1})T)$$

shows that $\frac{-T}{1-4T} \in \text{Spec}_{\frac{1}{4}-\overline{\mathbb{R}^-}}(\mathfrak{A})$. If (17) holds, then $(1-2\sqrt[{\mathcal{F}}]{T})^2 = 1-4T$,

and it is sufficient to prove that $\sqrt[{\mathcal{F}}]{\frac{-T}{1-4T}} = \frac{\sqrt[{\mathcal{F}}]{T}-2T}{1-4T}$. This, however, follows from the identity

$$\frac{(1 + \frac{z+z^{-1}}{2}) \frac{-T}{1-4T}}{1 + (z - 2 + z^{-1}) \frac{-T}{1-4T}} = \frac{\frac{(1 + \frac{(-z)+(-z)^{-1}}{2})T}{1+((-z)-2+(-z)^{-1})T} - 2T}{1-4T}$$

integrated. So, what we have to show is the \mathcal{F} -square-root identity (17). Now, as

$$(18) \quad \begin{aligned} & \sqrt[{\mathcal{F}}]{T}(1 - \sqrt[{\mathcal{F}}]{T}) - T = \\ &= \iint \frac{(1 + \frac{z+z^{-1}}{2})T}{1 + (z - 2 + z^{-1})T} \left(1 - \frac{(1 + \frac{w+w^{-1}}{2})T}{1 + (w - 2 + w^{-1})T} \right) - T \frac{|dz|}{2\pi} \frac{|dw|}{2\pi}, \end{aligned}$$

we should show that this latter term is 0. It would be natural to proceed along the steps

$$\frac{(1 + \frac{z+z^{-1}}{2})T}{1 + (z - 2 + z^{-1})T} \left(1 - \frac{(1 + \frac{w+w^{-1}}{2})T}{1 + (w - 2 + w^{-1})T} \right) - T =$$

$$\begin{aligned}
&= \frac{\left(\frac{z+z^{-1}}{2} - \frac{z+z^{-1}}{2}T - \frac{w+w^{-1}}{2}T + T + \frac{z+z^{-1}}{2}\frac{w+w^{-1}}{2}T\right)T(1-4T)}{(1+(z-2+z^{-1})T)(1+(w-2+w^{-1})T)} \sim \\
(19) \quad &\sim \frac{\left(\left[\frac{\frac{z+z^{-1}}{2}+\frac{w+w^{-1}}{2}}{2}\right](1-2T) + \left(1+\frac{z^{-1}w+zw^{-1}}{2}\right)T\right)T(1-4T)}{(1+(z-2+z^{-1})T)(1+(w-2+w^{-1})T)} \\
&= \left[\frac{1}{2}\frac{z+w}{z-w}\right] \frac{\left(\left(\frac{z-z^{-1}}{2}-\frac{w-w^{-1}}{2}\right)(1-2T) + (zw^{-1}-z^{-1}w)T\right)T(1-4T)}{(1+(z-2+z^{-1})T)(1+(w-2+w^{-1})T)} \\
&= \left[\frac{1}{2}\frac{z+w}{z-w}\right] \left(\frac{\frac{z-z^{-1}}{2}}{1+(z-2+z^{-1})T} - \frac{\frac{w-w^{-1}}{2}}{1+(w-2+w^{-1})T}\right)T(1-4T) \sim 0,
\end{aligned}$$

except we have to demonstrate that the use of $\frac{1}{2}$ is superficial.

Another reordering operation can be defined according to

$$\therefore z^n w^m ::_{z,w} = \frac{z^{\max(|n|,|m|)} + z^{-\max(|n|,|m|)}}{2} \frac{w^{\min(|n|,|m|)} + w^{-\min(|n|,|m|)}}{2}.$$

This applies to our standard $\mathfrak{A}[z]^{\pm}[w]^{\pm}$ -modules. In the context of $\mathfrak{A}\langle z \rangle \langle w \rangle$, it leaves only the (++)-graded parts and reorders them.

LEMMA 5.10. (a) *For any Laurent series $a(z, w) \in \mathfrak{A}\langle z \rangle \langle w \rangle$, we have*

$$\iint a(z, w) \frac{|dz|}{2\pi} \frac{|dw|}{2\pi} = \iint :: a(z, w) ::_{z,w} \frac{|dz|}{2\pi} \frac{|dw|}{2\pi}.$$

(b) *If $:: a_1(z, w) ::_{z,w} = :: a_2(z, w) ::_{z,w}$ and $b(z, w) \in \mathfrak{A}[z]^+[w]^+$ is symmetric, i. e. $b(z, w) = b(w, z)$, then*

$$\therefore a_1(z, w) b(z, w) ::_{z,w} = :: a_2(z, w) b(z, w) ::_{z,w}.$$

For any symmetric Laurent series $b(z, w) \in \mathfrak{A}[z]^+[w]^+$, i. e. such that $b(z, w) = b(w, z)$, we define formally

$$\therefore \left[\frac{\frac{z+z^{-1}}{2} + \frac{w+w^{-1}}{2}}{2} \right] b(z, w) ::_{z,w} = :: \frac{z+z^{-1}}{2} b(z, w) ::_{z,w}.$$

Let us consider a Laurent series $c(z, w) \in (\mathfrak{A}\{z, w\})^-$ such that it is antisymmetric, i. e. $c(z, w) = -c(w, z)$. Then $c(z, w)$ is a formal linear combination of the basis elements

$$c_{n,0} = \frac{z^n - z^{-n}}{2} - \frac{w^n - w^{-n}}{2}$$

and

$$c_{n,m} = \frac{(z^n - z^{-n})(w^m + w^{-m})}{2} - \frac{(w^n - w^{-n})(z^m + z^{-m})}{2},$$

where $n, m \geq 1$. For such $c(z, w)$, we can formally define

$$\therefore \left[\frac{1}{2} \frac{z+w}{z-w} \right] c(z, w) ::_{z,w}$$

according to our natural expectations. Again, we have to check that the result is integral in terms of coefficients of $c(z, w)$. We just give some samples in the table

$\therefore \left[\frac{1}{2} \frac{z+w}{z-w} \right] c_{n,m} ::_{z,w}$	$m = 0$	$m = 1$	$m = 2$	$m = 3$
$n = 1$	d_0^1	$d_1^1 + d_0^0$	$2d_0^1$	$-d_2^2 + d_1^1 + 2d_0^2$
$n = 2$	$d_0^2 + d_1^1$	$2d_1^2 + 2d_0^1$	$d_2^2 + 2d_1^1 + d_0^0$	$2d_1^2 + 2d_0^1$
$n = 3$	$d_0^3 + 2d_1^2$	$2d_1^3 + d_2^2 + d_1^1 + 2d_0^2$	$2d_2^3 + 2d_1^2 + 2d_0^1$	$d_3^3 + 2d_2^2 + 2d_1^1 + d_0^0$

where $d_m^n = \therefore z^n w^m ::_{z,w}$. In fact, at first sight, it looks more natural to choose $c(z, w)$ from the antisymmetric elements of $(\mathfrak{A}\langle z \rangle \langle w \rangle)^- = \mathfrak{A}\langle z \rangle^+ \langle w \rangle^- \oplus \mathfrak{A}\langle z \rangle^- \langle w \rangle^+$, but it turns out that the coefficients in $\therefore \left[\frac{1}{2} \frac{z+w}{z-w} \right] c(z, w) ::_{z,w}$ would fail to be integral.

LEMMA 5.11. *For any symmetric Laurent series $b(z, w) \in \mathfrak{A}[z]^+ [w]^+$, we have*

$$\begin{aligned} & \therefore \left[\frac{\frac{z+z^{-1}}{2} + \frac{w+w^{-1}}{2}}{2} \right] b(z, w) ::_{z,w} = \\ & = \therefore \left[\frac{1}{2} \frac{z+w}{z-w} \right] \left(\frac{z-z^{-1}}{2} - \frac{w-w^{-1}}{2} \right) b(z, w) ::_{z,w}, \end{aligned}$$

and

$$\therefore \left(1 + \frac{zw^{-1} + wz^{-1}}{2} \right) b(z, w) ::_{z,w} =$$

$$=:: \left[\frac{1}{2} \frac{z+w}{z-w} \right] (z w^{-1} - w z^{-1}) b(z, w) ::_{z,w} .$$

LEMMA 5.12. *For any Laurent series $a(z) \in \mathfrak{A}(z)^-$, we have*

$$\iint :: \left[\frac{1}{2} \frac{z+w}{z-w} \right] (a(z) - a(w)) ::_{z,w} \frac{|dz|}{2\pi} \frac{|dw|}{2\pi} = 0.$$

Now, it is easy to carry out the proof. From (18), we should pass to the “normal ordered” form, after which the subsequent manipulations leading to 0 make sense. \blacksquare

PROPOSITION 5.13. $P \in \text{Spec}_{\frac{1}{2} + i\overline{\mathbb{R}}}(\mathfrak{A})$ if and only if

$$P(1 - P) \in \text{Spec}_{\frac{1}{4} - \overline{\mathbb{R}}^-}(\mathfrak{A}).$$

Furthermore,

$$\sqrt[3]{P(1 - P)} = P + \text{idem } P - 2P \text{idem } P.$$

Consequently,

$$\text{idem } P = \frac{\sqrt[3]{P(1 - P)} - P}{1 - 2P}.$$

PROOF. The decomposition

$$1 + (z - 2 + z^{-1})P(1 - P) = (1 - P + Pz)(1 - P + Pz^{-1})$$

implies the spectral statement. The equality follows from the identity

$$\begin{aligned} & \frac{\left(1 + \frac{z+z^{-1}}{2}\right)P(1 - P)}{1 + (z - 2 + z^{-1})P(1 - P)} + \frac{\frac{z-z^{-1}}{2}P(1 - P)(1 - 2P)}{1 + (z - 2 + z^{-1})P(1 - P)} = \\ & = P + \frac{Pz}{1 - P + Pz}(1 - 2P) \end{aligned}$$

integrated. \blacksquare

Then we let $|P|_{\mathcal{G}} = \sqrt[3]{P(1 - P)}$. Further statements can be proven parallel to the case with $\frac{1}{2}$, except the formulas are less customary. E. g., the analogue of Proposition 3.9 is

PROPOSITION 5.14. For $T \in \text{Spec}_{\frac{1}{4}-\overline{\mathbb{R}^-}}(\mathfrak{A})$, we have

$$\begin{aligned} & \frac{1}{1 + (z - 2 + z^{-1})T} = \\ & = \frac{1}{1 - 2\sqrt[3]{T}} \left(1 + \frac{-\sqrt[3]{T}}{1 - \sqrt[3]{T}}(z + z^{-1}) + \left(\frac{-\sqrt[3]{T}}{1 - \sqrt[3]{T}} \right)^2 (z^2 + z^{-2}) + \dots \right) \end{aligned}$$

and

$$\frac{1}{1 - \sqrt[3]{T}} = \int \frac{1+z}{1 + (z - 2 + z^{-1})T} \frac{|dz|}{2\pi}.$$

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ON WEAKLY GENERALIZED RECURRENT MANIFOLDS

By

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Abstract. The object of the present paper is to introduce a non-flat Riemannian manifold called *weakly generalized recurrent manifold* and study its various geometric properties along with the existence by a proper example.

1. Introduction

An n -dimensional Riemannian manifold M is said to be locally symmetric according to E. CARTAN if its curvature tensor R satisfies $\nabla R = 0$, where ∇ denotes the Levi-Civita connection. During the last five decades the notion of locally symmetric manifolds has been weakened by many authors in several ways to a different extent such as recurrent manifolds by A. G. WALKER [13], 2-recurrent manifolds by A. LICHNEROWICZ [7], Ricci recurrent manifolds by E. M. PATTERSON [9], concircular recurrent manifolds by T. MIYAZAWA [8], weakly symmetric manifolds by L. TAMÁSSY and T. Q. BINH [11], weakly Ricci symmetric manifolds by L. TAMÁSSY and T. Q. BINH [12], conformally recurrent manifolds [1], projectively recurrent manifolds [2], generalized recurrent manifolds ([3], [4]), generalized Ricci recurrent manifolds [5].

We denote by $\nabla^i T$ the covariant differential of the i th order, $i \geq 1$, of a $(0, k)$ tensor field T , $k \geq 1$, defined on a Riemannian manifold (M, g) with Levi-Civita connection ∇ . The tensor field T is said to be recurrent, respectively, 2-recurrent [10], if the following condition holds on M

(1.1)

$$(\nabla T)(X_1, \dots, X_k; X)T(Y_1, \dots, Y_k) = (\nabla T)(Y_1, \dots, Y_k; X)T(X_1, \dots, X_k),$$

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respectively,

$$(1.2) \quad \begin{aligned} (\nabla^2 T)(X_1, \dots, X_k; X, Y)T(Y_1, \dots, Y_k) &= \\ &= (\nabla T^2)(Y_1, \dots, Y_k; X, Y)T(X_1, \dots, X_k), \end{aligned}$$

where $X, Y, X_1, Y_1, \dots, X_k, Y_k \in TM$. From (1.1), (1.2) it follows that at a point $x \in M$ if the tensor T is non-zero, then there exists a unique 1-form ϕ , respectively, a $(0, 2)$ -tensor ψ , defined on a neighborhood U of x , such that

$$\nabla T = T \otimes \phi, \quad \phi = d(\log \|T\|),$$

respectively,

$$\nabla^2 T = T \otimes \psi,$$

holds on U , where $\|T\|$ denotes the norm of T , $\|T\|^2 = g(T, T)$.

Let (M, g) be a non-flat n -dimensional ($n \geq 2$) Riemannian manifold. Then (M, g) is said to be recurrent [13] if its curvature tensor R of type $(0, 4)$ satisfies the condition $\nabla R = A \otimes R$ where A is a non-zero 1-form. Such an n -dimensional manifold is denoted by K_n . Let $U_R = \{x \in M : \nabla R \neq A \otimes R \text{ at } x\}$. Then the manifold (M, g) is said to be generalized recurrent [4] if on $U_R \subset M$, we have $\nabla R = A \otimes R + B \otimes G$, where B is an 1-form on U_R and G is a tensor of type $(0, 4)$ given by

$$G(X, Y, Z, U) = g(X, U)g(Y, Z) - g(X, Z)g(Y, U)$$

for all $X, Y, Z, U \in \chi(M)$, $\chi(M)$ being the Lie algebra of smooth vector fields on M . Such an n -dimensional manifold is denoted by GK_n . It is clear that the 1-form B is non-zero at every point on U_R . From the above definition, it follows that every K_n is GK_n but not conversely.

The object of the present paper is to introduce a generalized class of recurrent manifolds called *weakly generalized recurrent manifolds*. A non-flat n -dimensional Riemannian manifold (M^n, g) ($n > 2$) [This condition is assumed throughout the paper] is said to be *weakly generalized recurrent manifold* if on $U_R \subset M$ the condition

$$(1.3) \quad \nabla R = A \otimes R + B \otimes \frac{1}{2}(S \wedge S)$$

holds, where S is the Ricci tensor of type $(0, 2)$, A, B are 1-forms such that $A(\cdot) = g(\cdot, \sigma)$ and $B(\cdot) = g(\cdot, \rho)$, and the Kulkarni-Nomizu product $E \wedge F$ of two $(0, 2)$ tensors E and F is defined by

$$\begin{aligned} (E \wedge F)(X_1, X_2, X_3, X_4) &= E(X_1, X_4)F(X_2, X_3) + E(X_2, X_3)F(X_1, X_4) - \\ &- E(X_1, X_3)F(X_2, X_4) - E(X_2, X_4)F(X_1, X_3), \end{aligned}$$

$X_i \in \chi(M)$, $i = 1, 2, 3, 4$. Such an n -dimensional manifold is denoted by WGK_n . It is clear that in a WGK_n the 1-forms A, B are non-zero and are called the associated 1-forms of the manifold.

Especially, if the manifold is Einstein with vanishing scalar curvature, then a WGK_n reduces to a K_n . And if the manifold is Einstein with non-vanishing scalar curvature, then manifold reduces to a GK_n ([3], [4]). Again, if a WGK_n is non-Einstein, then the manifold is neither K_n nor GK_n and the existence of such manifold is given by an example in section 3. Section 2 deals with some geometric properties of WGK_n .

A Riemannian manifold (M^n, g) ($n > 2$) is said to be Ricci recurrent [9] if its Ricci tensor is not identically zero and satisfies $\nabla S = A \otimes S$, where A is a non-zero 1-form. Such an n -dimensional manifold is denoted by RK_n . Let $U_S = \{x \in M : \nabla S \neq A \otimes S \text{ at } x\}$. Then the manifold (M, g) is said to be generalized Ricci recurrent [5] if on $U_S \subset M$, the condition $\nabla S = A \otimes S + B \otimes g$ holds where B is an 1-form on U_S . It is clear that the 1-form B is non-zero at every point of U_S . Such an n -dimensional manifold is denoted by GRK_n . In section 2 it is shown that a non-Einstein WGK_n satisfying the relation (2.2) is a GRK_n .

A projective transformation on a Riemannian manifold is a transformation under which geodesics transform into geodesics. The projective curvature tensor P of type $(0, 4)$ on a Riemannian manifold (M, g) is defined by ([6], [14])

$$(1.4) \quad P = R - \frac{1}{n-1}D,$$

where D is a tensor of type $(0, 4)$ is given by

$$D(X, Y, Z, U) = g(X, U)S(Y, Z) - g(Y, U)S(X, Z)$$

$\forall X, Y, Z, U \in \chi(M)$.

Let $U_P = \{x \in M : \nabla P \neq A \otimes P \text{ at } x\}$. Then the Riemannian manifold (M, g) is said to be *weakly projectively recurrent* (briefly, WPK_n) if on $U_P \subset M$, the relation

$$(1.5) \quad \nabla P = A \otimes P + B \otimes \frac{1}{2}(S \wedge S)$$

holds, where B is an 1-form on U_P . It is obvious that the 1-form B is non-zero at every point of U_P . In section 2 of the paper it is proved that if a non-Einstein WGK_n is a WPK_n , then the relation (2.2) holds.

A non-flat Riemannian manifold (M^n, g) ($n > 3$) is said to be generalized 2-recurrent [7] if its curvature tensor R satisfies

$$(1.6) \quad (\nabla \nabla R) = \alpha \otimes R + \beta \otimes G,$$

where α, β are tensors of type $(0, 2)$. Again M is said to be generalized 2-Ricci recurrent [5] if its Ricci tensor S is not identically zero and satisfies the following:

$$(1.7) \quad (\nabla \nabla S) = \alpha \otimes S + \beta \otimes g,$$

where α, β are tensors of type $(0, 2)$.

In section 2 it is shown that a non-Einstein WGK_n with non-zero constant scalar curvature satisfying the relation(2.2) is a generalized 2-Ricci recurrent manifold.

2. Some geometric properties of WGK_n

THEOREM 2.1. *In a Riemannian manifold (M^n, g) ($n \geq 3$) the following results hold:*

(i) *In a WGK_n the relation*

$$(2.1) \quad (r^2 - s^2)B(X) + rA(X) = 2[A(QX) + rB(QX) - g(QX, Q\rho)]$$

holds for all X , where r is the scalar curvature, s is the length of the Ricci tensor and Q being the symmetric endomorphism corresponding to the Ricci tensor S .

(ii) *In a WGK_n with non-zero constant scalar curvature the associated 1-forms A and B are related by $rA + (r^2 - s^2)B = 0$, and the relation $B(Q^2X) = A(QX) + rB(QX)$ holds for all X .*

(iii) *In a non-Einstein WGK_n with vanishing scalar curvature the relations*

$$B(Q^2X) = A(QX),$$

$$(\nabla_X S)(Z, \rho) = A(X)B(QZ) - B(X)A(QZ),$$

$$A(R(Z, X)\rho) = 0$$

and $A(X)B(QZ) - A(QZ)B(X) = A(Z)B(QX) - A(QX)B(Z)$.

hold for all $X, Y, Z, V \in \chi(M)$.

(iv) *In a non-Einstein Ricci symmetric WGK_n with vanishing scalar curvature, the relation*

$$A(X)B(QZ) = A(QZ)B(X)$$

holds $\forall X, Z$.

(v) In a WGK_n , with non-vanishing constant scalar curvature such that the length of the Ricci tensor is not equal to the scalar curvature, the associated 1-forms A and B are closed if and only if ρ and σ are codirectional.

(vi) Let (M, g) be a non-Einstein WGK_n such that the length of the Ricci tensor is not equal to the scalar curvature. Then the following holds:

(a) If (M, g) is a WPK_n , then the relation

$$(2.2) \quad |Q|^2 + (r^2 - s^2)I - r|Q| = 0$$

holds,

(b) If (M, g) satisfies the relation (2.2), then it is a GRK_n .

(c) If (M, g) is of non-zero constant scalar curvature such that ρ and σ are codirectional and satisfy the relation (2.2), then it is a generalized 2-Ricci recurrent manifold.

PROOF OF (I). After suitable contraction (1.3) yields

$$(2.3) \quad (\nabla_X S)(Z, U) = A_1(X)S(Z, U) - B(X)S(QU, Z),$$

where A_1 is a 1-form given by $A_1(X) = A(X) + rB(X)$.

From (2.3) it follows that

$$(2.4) \quad dr(X) = rA(X) + (r^2 - s^2)B(X),$$

and

$$(2.5) \quad dr(X) = 2[A(QX) + rB(QX) - g(QX, Q\rho)].$$

From (2.4) and (2.5), it follows (2.1). This proves (i).

PROOF OF (II). If r is a non-zero constant, then (2.4) and (2.5) implies that

$$(2.6) \quad rA + (r^2 - s^2)B = 0,$$

and

$$(2.7) \quad B(Q^2X) = A(QX) + rB(QX) \quad \forall X.$$

which proves (ii).

PROOF OF (III). If $r = 0$, then from (2.7) implies that

$$(2.8) \quad B(Q^2X) = A(QX).$$

By virtue of (2.3) and (2.8), it follows that

$$(2.9) \quad (\nabla_X S)(Z, U) = A(X)S(Z, U) - B(X)S(QU, Z).$$

Putting $U = \rho$ in (2.9) we get

$$(2.10) \quad (\nabla_X S)(Z, \rho) = A(X)B(QZ) - B(X)A(QZ)$$

By virtue of second Bianchi identity, (1.3) yields

$$\begin{aligned} & A(X)R(Y, Z, U, V) + B(X)[S(Y, V)S(Z, U) - S(Y, U)S(Z, V)] + \\ & + A(Y)R(Z, X, U, V) + B(Y)[S(Z, V)S(X, U) - S(Z, U)S(X, V)] + \\ & + A(Z)R(X, Y, U, V) + B(Z)[S(X, V)S(Y, U) - S(X, U)S(Y, V)] = \\ (2.11) \quad & = 0. \end{aligned}$$

Taking contraction over Y and V in (2.11), we obtain

$$\begin{aligned} & A(R(Z, X)U) + [A(X) + rB(X) - B(QX)]S(Z, U) + \\ & + [B(QZ) - A(Z) - rB(Z)]S(X, U) + B(Z)g(QX, QU) - \\ (2.12) \quad & - g(QZ, QU)B(X) = 0. \end{aligned}$$

If r is a non-zero constant, then we have (2.7). Setting $U = \rho$ in (2.12) we get

$$\begin{aligned} & A(R(Z, X)\rho) + A(X)B(QZ) + rB(QZ)B(X) - B(QX)A(Z) - \\ (2.13) \quad & - rB(Z)B(QX) + B(Z)g(QX, Q\rho) - g(QZ, Q\rho)B(X) = 0. \end{aligned}$$

Using (2.6) and (2.7) in (2.13), we get

$$(2.14) \quad A(R(Z, X)\rho) = 0.$$

Using (2.7) and (2.14) in (2.13), we obtain

$$A(X)B(QZ) - A(QZ)B(X) = A(Z)B(QX) - A(QX)B(Z).$$

This proves (iii).

PROOF OF (IV). If the manifold is Ricci symmetric, then from (2.10) we get

$$(2.15) \quad A(X)B(QZ) = A(QZ)B(X),$$

which proves (iv).

PROOF OF (V). Differentiating (1.3) covariantly, and then using (2.3) we obtain

$$\begin{aligned} (2.16) \quad & (\nabla_Y \nabla_X R)(Z, W, U, V) = [(\nabla_Y A)(X) + A(X)A(Y)]R(Z, W, U, V) + \\ & + (\nabla_Y B)(X)[S(Z, V)S(W, U) - S(Z, U)S(W, V)] + \\ & + [S(Z, V)S(W, U) - S(Z, U)S(W, V)][A(X)B(Y) + \\ & + 2B(X)A(Y) + 2rB(X)B(Y)] + B(X)B(Y)[S(W, V)S(QZ, U) + \\ & + S(Z, U)S(QW, V) - S(W, U)S(QZ, V) - S(Z, V)S(QW, U)]. \end{aligned}$$

Interchanging X and Y and then subtracting the result, we obtain
(2.17)

$$\begin{aligned} & (\nabla_Y \nabla_X R)(Z, W, U, V) - (\nabla_X \nabla_Y R)(Z, W, U, V) = \\ &= [(\nabla_Y A)(X) - (\nabla_X A)(Y)]R(Z, W, U, V) + \\ &+ [(\nabla_Y B)(X) - (\nabla_X B)(Y)][S(Z, V)S(W, U) - S(Z, U)S(W, V)] + \\ &+ [B(X)A(Y) - A(X)B(Y)][S(Z, V)S(W, U) - S(Z, U)S(W, V)]. \end{aligned}$$

We first suppose that σ and ρ are codirectional.

Then $B(X)A(Y) - A(X)B(Y) = 0$ for all X . Hence (2.17) reduces to
(2.18)

$$\begin{aligned} & (\nabla_Y \nabla_X R)(Z, W, U, V) - (\nabla_X \nabla_Y R)(Z, W, U, V) = \\ &= [(\nabla_Y A)(X) - (\nabla_X A)(Y)]R(Z, W, U, V) + \\ &+ [(\nabla_Y B)(X) - (\nabla_X B)(Y)][S(Z, V)S(W, U) - S(Z, U)S(W, V)]. \end{aligned}$$

From Walker's lemma ([13], equation (26)) we have

$$\begin{aligned} & (\nabla_X \nabla_Y R)(Z, W, U, V) - (\nabla_Y \nabla_X R)(Z, W, U, V) + \\ (2.19) \quad & + (\nabla_Z \nabla_W R)(X, Y, U, V) - (\nabla_W \nabla_Z R)(X, Y, U, V) + \\ & + (\nabla_U \nabla_V R)(Z, W, X, Y) - (\nabla_V \nabla_U R)(Z, W, X, Y) = 0. \end{aligned}$$

By virtue of (2.18), (2.19) yields

$$\begin{aligned} & M(X, Y)R(Z, W, U, V) + L(X, Y)\frac{1}{2}(S \wedge S)(Z, W, U, V) + \\ & + M(Z, W)R(X, Y, U, V) + L(Z, W)\frac{1}{2}(S \wedge S)(X, Y, U, V) + \\ & + M(U, V)R(Z, W, X, Y) + L(U, V)\frac{1}{2}(S \wedge S)(Z, W, X, Y) = 0, \end{aligned}$$

where

$$M(X, Y) = (\nabla_X A)(Y) - (\nabla_Y A)(X)$$

and

$$L(X, Y) = (\nabla_X B)(Y) - (\nabla_Y B)(X).$$

If the scalar curvature is a non-zero constant, then we have the relation (2.6). Using (2.6) in (2.20) we obtain

$$\begin{aligned} & M(X, Y)H(Z, W, U, V) + M(Z, W)H(X, Y, U, V) + \\ (2.21) \quad & + M(U, V)H(Z, W, X, Y) = 0, \end{aligned}$$

where $H = R - \frac{r}{2(r^2-s^2)}(S \wedge S)$, from which it follows that H is a symmetric $(0, 4)$ tensor with respect to the first pair of two indices and the last pair of

two indices. Consequently by virtue of Walker's lemma ([13], equation (27)), we obtain

$$M(X, Y) = L(X, Y) = 0,$$

for all X, Y . And hence

$$(\nabla_X A)(Y) - (\nabla_Y A)(X) = 0,$$

$$(\nabla_X B)(Y) - (\nabla_Y B)(X) = 0.$$

Therefore $dA(X, Y) = 0, dB(X, Y) = 0$.

Conversely, we suppose that associated 1-forms A and B are closed. Then from (2.17) we get

$$(2.22) \quad \begin{aligned} & (\nabla_Y \nabla_X R)(Z, W, U, V) - (\nabla_X \nabla_Y R)(Z, W, U, V) = \\ & = [B(X)A(Y) - A(X)B(Y)][S(Z, V)S(W, U) - S(Z, U)S(W, V)]. \end{aligned}$$

By virtue of (2.22), (2.19) yields

$$(2.23) \quad \begin{aligned} & [B(X)A(Y) - A(X)B(Y)][S(Z, V)S(W, U) - S(Z, U)S(W, V)] + \\ & + [B(W)A(Z) - A(W)B(Z)][S(X, V)S(Y, U) - S(X, U)S(Y, V)] + \\ & + [B(V)A(U) - A(V)B(U)][S(Z, Y)S(W, X) - S(Z, X)S(W, Y)] = 0. \end{aligned}$$

After suitable contraction, it follows from (2.23) that

$$B(X)A(Y) - A(X)B(Y) = 0 \quad \text{provided } r \neq s.$$

This proves (v).

PROOF OF (VI). From (2.3), (1.3) and (1.4) we obtain

$$(2.24) \quad \begin{aligned} & (\nabla_W P)R(X, Y, Z, U) = \\ & = A(W)P(X, Y, Z, U) + B(W)[S(X, U)S(Y, Z) - S(X, Z)S(Y, U)] - \\ & - \frac{1}{n-1}B(W)[r\{g(X, U)S(Y, Z) - S(X, Z)g(Y, U)\} + \\ & + g(Y, U)S(QX, Z) - g(X, U)S(QY, Z)]. \end{aligned}$$

Suppose that the manifold under consideration is a WPK_n . Then (2.24) yields

$$(2.25) \quad \begin{aligned} & r[g(X, U)S(Y, Z) - S(X, Z)g(Y, U)] + \\ & + [g(Y, U)S(QX, Z) - g(X, U)S(QY, Z)] = 0. \end{aligned}$$

Taking contraction over Y, Z , we obtain (2.2).

Using (2.2) in (2.3) we obtain

$$(2.26) \quad (\nabla_X S)(Z, U) = A(X)S(Z, U) + B(X)(r^2 - s^2)g(Z, U),$$

which proves (b) of (vi).

Suppose that the manifold is of non-zero constant scalar curvature and the length of the Ricci tensor is not equal to the scalar curvature. Then from (2.3) it follows that

$$(2.27) \quad (\nabla_Y \nabla_X S)(Z, W) = S(Z, W)[(\nabla_Y A)(X) + r(\nabla_Y B)(X) +$$

$$\begin{aligned} & + A(Y)A(X) + rB(X)A(Y) + rA(X)B(Y) + r^2B(X)B(Y)] - \\ & - [B(Y)A(X) + rB(Y)B(X)]S(QZ, W) - \\ & - [(\nabla_Y B)(X)S(QZ, W) + B(X)(\nabla_Y S)(QZ, W)]. \end{aligned}$$

Interchanging X, Y and subtracting the result, we obtain

$$(2.28) \quad \begin{aligned} (\nabla_X \nabla_Y S)(Z, W) - (\nabla_Y \nabla_X S)(Z, W) &= [M(X, Y) + \\ & + rL(X, Y)]S(Z, W) - L(X, Y)S(QZ, W) + \\ & + S(QZ, W)[B(Y)A(X) - B(X)A(Y)] + \\ & + [B(X)(\nabla_Y S)(QZ, W) - B(Y)(\nabla_X S)(QZ, W)]. \end{aligned}$$

If $A(X)B(Y) - A(Y)B(X) = 0$, then

$$B(X)(\nabla_Y S)(QZ, W) - B(Y)(\nabla_X S)(QZ, W) = 0.$$

Using above relations in (2.28) we obtain

$$(2.29) \quad \begin{aligned} (\nabla_X \nabla_Y S)(Z, W) - (\nabla_Y \nabla_X S)(Z, W) &= \\ & = [M(X, Y) + rL(X, Y)]S(Z, W) - L(X, Y)S(QZ, W). \end{aligned}$$

In view of (2.2), (2.3) and (2.29) we obtain

$$(2.30) \quad (R(X, Y).S)(Z, W) = K(X, Y)g(Z, W) + N(X, Y)S(Z, W),$$

where $K(X, Y) = (r^2 - s^2)[XB(Y) - YB(X) - 2B([X, Y])]$ and

$$N(X, Y) = XA(Y) - YA(X) - 2A([X, Y]).$$

The relation (2.30) implies that the manifold is a generalized 2-Ricci recurrent. This proves (c) of (vi).

3. An Example of WGK_4

In this section the existence of WGK_4 is ensured by a proper example.

EXAMPLE 3.1. We consider a Riemannian manifold \mathbb{R}^4 equipped with the metric given by

$$(3.1) \quad ds^2 = g_{ij}dx^i dx^j = (1+2p)[(dx^1)^2 + (dx^2)^2 + (dx^3)^2] + (dx^4)^2, \quad (i, j = 1, 2, \dots, 4),$$

where $p = \frac{e^x}{\rho^2}$ and ρ is a non-zero constant; x^1, \dots, x^4 are the standard coordinates of \mathbb{R}^4 .

The only non-vanishing components of the Christoffel symbols of second kind, the curvature tensor and their covariant derivatives are

$$(3.2) \quad \begin{aligned} \Gamma_{22}^1 &= \Gamma_{33}^1 = -\frac{p}{1+2p}, & \Gamma_{11}^1 &= \Gamma_{12}^2 = \Gamma_{13}^3 = \frac{p}{1+2p}. \\ R_{1221} &= R_{1331} = \frac{p}{1+2p}, & R_{2332} &= \frac{p^2}{1+2p}, \\ R_{1221,1} &= R_{1331,1} = \frac{p(1-4p)}{(1+2p)^2}, & R_{2332,1} &= \frac{2p^2(1-p)}{(1+2p)^2}, \end{aligned}$$

the other components can be obtained from these by the symmetry property of R . Using the above relations, we can find the non-vanishing components of Ricci tensor as follows:

$$(3.3) \quad S_{11} = \frac{2p}{(1+2p)^2}, \quad S_{22} = \frac{p+p^2}{(1+2p)^2}, \quad S_{33} = \frac{p+p^2}{(1+2p)^2}.$$

Also it can be easily shown that the scalar curvature of the manifold is non-vanishing and non-constant. We shall now show that M is a WGK_4 , i.e., it satisfies the defining condition (1.3). In terms of local coordinates we consider the components of the associated 1-forms as follows:

$$\begin{cases} A_i(x) = \begin{cases} \frac{7p-1}{(1+2p)(p-1)} & \text{for } i = 1 \\ 0 & \text{otherwise,} \end{cases} \\ B_i(x) = \begin{cases} -\frac{(1+2p)^3}{p^2-1} & \text{for } i = 1 \\ 0 & \text{otherwise,} \end{cases} \end{cases}$$

at any point $x \in M$. In terms of local coordinates the defining equation (1.3) of WGK_n can be written as

$$(3.5) \quad R_{hijk,l} = A_l R_{hijk} + B_l [S_{hk} S_{ij} - S_{hj} S_{ik}].$$

By virtue of (3.2)–(3.4), it follows that (3.5) holds for all $i, j, h, k, l = 1, 2, 3, 4$. Thus we can state the following:

THEOREM 3.1. *Let (\mathbb{R}^4, g) be a Riemannian manifold equipped with the metric given by (3.1). Then (\mathbb{R}^4, g) is a WGK_4 with non-vanishing and non-constant scalar curvature which is neither K_4 nor GK_4 .*

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**RAMSEY AND TURÁN-TYPE PROBLEMS FOR NON-CROSSING
SUBGRAPHS OF BIPARTITE GEOMETRIC GRAPHS**

By

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Abstract. Geometric versions of Ramsey-type and Turán-type problems are studied in a special but natural representation of bipartite graphs and similar questions are asked for general representations. A bipartite geometric graph $G(m, n) = [A, B]$ is *simple* if the vertex classes A, B of $G(m, n)$ are represented in R^2 as

$$A = \{(1, 0), (2, 0), \dots, (m, 0)\}, B = \{(1, 1), (2, 1), \dots, (n, 1)\}$$

and the edge ab is the line segment joining $a \in A$ and $b \in B$ in R^2 . This and similar representations (two-layer representations) are studied earlier, and from the point of view of edge crossings, this representation is equivalent to others already in the literature, for example to *cyclic bipartite graphs* or to *ordered bipartite graphs* and certainly almost all textbook figures represent bipartite graphs this way.

Subgraphs — paths, trees, double stars, matchings — are called non-crossing if they do not contain edges with common interior point. The choice of these subgraphs are explained by the fact that connected components of non-crossing subgraphs of simple bipartite geometric graphs must be special trees (caterpillars). We concentrate on *balanced* bipartite graphs, where $m = n$.

The maximum number of edges is determined in a simple bipartite geometric graph $G(n, n)$ that does not contain

- non-crossing matchings with $k + 1$ edges
- matchings with $k + 1$ pairwise crossing edges
- non-crossing trees with $k + 1$ vertices

and in the last case it is shown that any graph with more edges than the extremal value contains a non-crossing double star with $k + 1$ vertices. The Ramsey number of non-crossing double stars is also determined: in every 2-coloring of a geometric $K_{n,n}$ there is a non-crossing monochromatic double star with at least $\frac{4n}{5}$ vertices and this is best possible in asymptotic sense.

Finding the Turán number of non-crossing paths and the Ramsey number of non-crossing subtrees and paths remain open together with many similar problems where the position of the vertex set of the bipartite graph is less restricted, either in convex or in general position.

1. Introduction

This paper expands a short abstract [10]. Following [22], a *geometric graph* is a graph whose vertices are in the plane in general position and whose edges are straight-line segments joining the vertices. A geometric graph is *convex*, if its vertices form a convex polygon. A subgraph of a geometric graph is *non-crossing* if no two edges have a common interior point.

Analogues of Turán and Ramsey theories have been considered for geometric graphs and for convex geometric graphs, see [22], [3], [1], [16], [14], [15] and its references.

To describe a specific example, an old remark of Erdős and Rado says that in any 2-coloring of the edges of K_n there is a monochromatic spanning tree. It was proved by BIALOSTOCKI, DIERKER and VOXMAN in [2] that there is a monochromatic non-crossing spanning tree in every 2-coloring of the convex geometric graph K_n . They conjectured that this remains true for geometric complete graphs in general and their conjecture was proved by KÁROLYI, PACH and TÓTH in [14]. There are several Ramsey-type and Turán-type results for geometric graphs, see [22] chapter 14, [14], [15]. These results show that Ramsey numbers change significantly for paths, cycles by imposing the non-crossing condition. However, an unpublished result of PERLES (a proof is in [15]) states that the maximum number of edges in a graph of n vertices that does not contain a path of length k (determined by ERDŐS and GALLAI [7]) remains the same for non-crossing paths in convex geometric graphs.

In this note we consider geometric versions of Ramsey-type and Turán-type problems for *geometric balanced bipartite graphs*, $G(n, n)$, defined as a geometric graph, whose $2n$ vertices are in two disjoint n -element sets A, B ,

and its edges are some segments ab with $a \in A, b \in B$. The concept is studied earlier, for example in the form of considering A, B as red and blue sets and investigating the properties of red-blue segments, a survey is [18].

As convex geometric graphs form a natural subclass of geometric graphs, the following representation, apparently studied first in [6], seems to be a most natural subclass of balanced geometric bipartite graphs $G(n, n)$ (in fact a standard way of drawing bipartite graphs). The partite sets of G in R^2 are $A = \{a_1 = (1, 0), a_2 = (2, 0), \dots, a_n = (n, 0)\}$ and $B = \{b_1 = (1, 1), b_2 = (2, 1), \dots, b_n = (n, 1)\}$ and the edge $a_i b_j$ is the line segment joining $a_i \in A$ and $b_j \in B$. This representation is equivalent (from the point of view of crossings) to *cyclic bipartite graphs* (BRASS, KÁROLYI, VALTR [3]), i. e. convex geometric graphs with $2n$ vertices whose partite classes A, B form two ‘intervals’. It is also equivalent to *ordered bipartite graphs* where each element of A precedes each elements of B (FÜREDI and HAJNAL [8], PACH and TARDOS [23]). For easier reference we call this representation a *simple $G(n, n)$* in this paper.

Notice the following characterization of non-crossing subgraphs of simple $G(n, n)$ -s, sometimes referred as *biplanar graphs*, apparently first discovered by HARARY and SCHWENK, [13]. A *caterpillar* is a special tree in which the vertices of degree larger than one form a path.

PROPOSITION 1. ([13], [5]) *Every connected component of a non-crossing subgraph of simple $G(n, n)$ is a caterpillar.*

Simple geometric graphs are very similar to the ‘cyclic bipartite’ graphs considered by BRASS, KÁROLYI and VALTR in [3] and the ‘ordered bipartite’ graphs considered by FÜREDI and HAJNAL [8], and by PACH and TARDOS [23]. However, here we investigate only non-crossing subgraphs and do not go into finer details of ordered subgraphs.

It follows from counting arguments of MUBAYI [21] and LIU, MORRIS, PRINCE [20] that in every 2-coloring of the edges of the complete bipartite graph $K_{n,n}$ there is a monochromatic double star with at least n vertices and this is a sharp result. (A *double star* is a tree obtained by joining the centers of two disjoint stars by an edge, the base edge of the double star). Here we prove that in the geometric version the situation is different.

THEOREM 1. *In every 2-coloring of the simple $K_{n,n}$ there is a non-crossing monochromatic double star with at least $\frac{4n}{5}$ vertices. This bound is asymptotically best possible.*

THEOREM 2. *Suppose that $2n \geq k$ and a simple bipartite graph $G = G(n, n)$ does not contain non-crossing double stars with $k+1$ vertices. Then $|E(G)| \leq n(k-1) - \lfloor \frac{(k-1)^2}{4} \rfloor$. This bound is sharp for each pair of integers satisfying $2n \geq k$.*

It is worth noting that in Theorem 2 the maximum number of edges is approximately $\frac{n^2}{2}$ when k is approximately $(2 - \sqrt{2})n$. Thus Theorem 1 does not follow from Theorem 2. The construction, showing that Theorem 2 is sharp, does not contain *any* non-crossing subgraph with $k+1$ vertices.

Turán and Ramsey problems are also studied for matchings in geometric graphs. KUPITZ ([16], see also Theorem 14.4 in [22]) determined the maximal number of edges in a convex geometric graph with n vertices that does not contain non-crossing matchings with $k+1$ edges. Similar result is proved for matchings with pairwise crossing edges ([4], see also Theorem 14.14 in [22]). These results have a unified form for simple bipartite graphs.

THEOREM 3. *Assume that $G = G(n, n)$ is a simple bipartite graph that does not contain a non-crossing (crossing) matching with $k+1$ edges. Then $|E(G)| \leq 2kn - k^2$ and this bound is sharp for both cases.*

For the Ramsey problem, it is known ([14]) that in any 2-coloring of the edges of a geometric complete graph with n vertices there is a monochromatic non-crossing matching with $\lfloor \frac{n+1}{3} \rfloor$ edges and this is sharp. The bipartite version here follows immediately from the well-known result that every geometric $K_{n,n}$ contains a perfect matching [19] so the proof of the next proposition is left to the reader.

PROPOSITION 2. *In every r -coloring of the edges of a geometric graph $K_{n,n}$ there is a monochromatic non-crossing matching with at least $\lceil \frac{n}{r} \rceil$ edges. This bound is best possible.*

2. Open problems

2.1. Problems for simple geometric bipartite graphs.

The 2-color Ramsey problem for paths in balanced bipartite graphs have been solved independently in [11] and in [9]. Concerning the geometric version we ask

PROBLEM 1. *What is the length of the largest monochromatic non-crossing path that exists in every 2-coloring of a simple geometric $K_{n,n}$? The bounds $\frac{n}{2}$ and $\frac{2n}{3}$ follow from the cited result of Perles and from using the graph $H(n,p)$ defined below with suitable p .*

The Turán number of paths in bipartite graphs have been determined in [12]. Concerning the geometric version the straightforward question is

PROBLEM 2. *What is the maximum number of edges in a simple geometric graph $G(n,n)$ that does not contain a non-crossing path of length k ? The upper bound is $n(k-1)$ from the cited result of Perles and the lower bound is $n(k-1) - \left\lfloor \frac{(k-1)^2}{4} \right\rfloor$, according to the construction given in the proof of Theorem 2.*

Presently we can not separate the Ramsey number of non-crossing trees (caterpillars) from the Ramsey number of non-crossing double stars.

PROBLEM 3. *What is the order of the largest monochromatic non-crossing subtree (caterpillar) that exists in every 2-coloring of the edges of a simple geometric $K_{n,n}$? The bounds are $\frac{4n}{5}$ and n or $n+1$ (depending of the parity of n). Note that the largest monochromatic non-crossing subforest F has at least $n+1$ vertices because a simple geometric $K_{n,n}$ has non-crossing spanning trees (this remark is valid for any geometric $K_{n,n}$).*

2.2. Problems for general geometric bipartite graphs.

It looks as the ‘really geometric’ problems arise when the vertex set of $G(n,n)$ is allowed to be in general position or in convex position (with no restriction on the sets A, B). Of course, Proposition 1 limits the non-crossing subgraphs to caterpillars. There are some results in the literature, for example it is known that every geometric $K_{n,n}$ contains a perfect matching [19] and a

spanning tree of maximum degree three [17]. It seems that from the following (twelve) problems only one can be answered easily (Proposition 2).

PROBLEM 4. *What is the maximum number of edges in a geometric (convex geometric) $G(n, n)$ that does not contain a non-crossing matching (path, caterpillar) with k edges?*

PROBLEM 5. *What is the maximum number of edges in a non-crossing monochromatic matching (path, caterpillar) contained in every 2-coloring of the edges of a geometric (convex geometric) $K_{n,n}$?*

3. A construction

We shall often use a simple balanced geometric graph $H = H(n, p)$ defined as follows. Fix $1 \leq p \leq n$, and the vertices of H are $a_i = (i, 0), b_i = (i, 1)$ for $i = 1, 2, \dots, n$. The edge set of H is $\{a_i b_j : p + 2 \leq i + j \leq 2n - p\}$.

LEMMA 1. *Suppose that F is a non-crossing subgraph of H without isolated vertices. Then $|V(F)| \leq 2(n - p)$.*

PROOF. Let q be the smallest integer such that $a_q \in V(F)$ and let r be the smallest integer for which $a_q b_r \in E(F)$. Similarly, s is the largest integer such that $a_s \in V(F)$ and t is the largest integer for which $a_s b_t \in E(F)$. Since F has no isolates, q, s, t, r are well defined. Set

$$X = \{a_i : 1 \leq i < q\} \cup \{b_j : 1 \leq j < r\},$$

$$Y = \{a_i : s < i \leq n\} \cup \{b_j : t < j \leq n\}.$$

We claim that at least p vertices of X and at least p vertices of Y are not in $V(F)$. This is obvious for $q > p$ and for $s < n - p + 1$. Suppose $q \leq p$, now $q - 1$ vertices of $A \cap X$ are not in $V(F)$ from the definition of q . From the definition of H , $a_q b_j \notin E(H)$ for $j = 1, 2, \dots, p - q + 1$, thus any edge of F incident to b_j would cross the edge $a_q b_r$. Therefore none of the b_j -s are in $V(F)$ for $j = 1, 2, \dots, p - q + 1$. Thus we have at least $q - 1 + p - q + 1 = p$ vertices not in $V(F)$. A symmetric argument shows that at least p vertices of Y are not in $V(F)$. This proves the claim and the lemma follows from it. ■

In Ramsey problems we may consider H and its (bipartite) complement as a 2-colored simple geometrical graph $K_{n,n}$. In H the order of largest non-crossing tree, path, double star is the same. This is not the case for \overline{H} . For example, it is easy to see that for $2p \geq n$, \overline{H} always contains a non-crossing tree with $n + 1$ vertices. This is not true for double stars as the following lemma shows (we need that to show that Theorem 1 is asymptotically sharp).

LEMMA 2. Assume that $n = 5k + 2, p = 3k + 1$. Then the largest non-crossing double star of both H and \overline{H} has $4k + 2$ vertices.

PROOF. Any non-crossing subgraph of H without isolated vertices has at most $2(n - p) = 4k + 2$ vertices from Lemma 1. On the other hand, the complete geometric subgraph of H spanned by $\{a_1, \dots, a_{n-p}, b_{p+1}, \dots, b_n\}$ clearly has a spanning double star. Thus the statement of the lemma follows for H . To prove it for \overline{H} , we need to look at three cases only, according to the indices of the base edge $e = a_i b_j$ of a double star T . If i, j are both at most $n - p$ then T has at most $p + 1 = 3k + 2 < 4k + 2$ vertices. If $1 \leq i \leq n - p, n - p + 1 \leq j$ then $j \leq p + 1 - i$ follows from the definition of H . There are two non-crossing maximal double stars with base e . One is taking the j ‘left’ neighbors of a_i and the $2p - n + 2 - i + j$ ‘right’ neighbors of b_j . Now

$$\begin{aligned} 2p - n + 2 - i + j &\leq 2p - n + 2 - 1 + p + 1 - i \leq 3p - n + 1 = \\ &= 3(3k + 1) - (5k + 2) + 1 = 4k + 2 \end{aligned}$$

proving what we want. The other maximal non-crossing double star on e has $p - j + 2$ vertices since there are $p - j - i + 2$ ‘right’ neighbors of i and i ‘left’ neighbors of j . Clearly, $p - j + 2 < 4k + 2$ thus this double star is small. Finally, if $n - p + 1 \leq j \leq p + 1 - i, n - p + 1 \leq i \leq p + 1 - j$, then the two maximal non-crossing double stars on e have $2p - n + 2 - i + j$ and $2p - n + 2 - j + i$ vertices. Assume by symmetry that the first is the maximum, then

$$2p - n + 2 - i + j \leq 2p - n + 2 - i + p + 1 - i \leq 3p - n + 2 = 4k + 2. \blacksquare$$

4. Proof of Theorems 1,2,3

PROOF OF THEOREM 1. Consider an arbitrary red-blue coloring of the edges of a balanced geometric bipartite graph $G = [A, B]$. Let G_R, G_B denote the red and blue subgraphs of G . Set

$$D = \max\{d_{G_R}(a_1), d_{G_R}(b_n), d_{G_B}(a_1), d_{G_B}(b_n)\}.$$

Assume first that $D \geq (1/2 + 1/10)n$, without loss of generality the maximum is attained at b_n in the red color. Let i denote the smallest index for which $a_i b_n$ is red. If a_i has at least $(1/4 - 1/20)n$ red neighbors in B , we have a red non-crossing double star on $a_i b_n$ spanning at least $(1/4 + 1/2 + 1/20)n = \frac{4n}{5}$ vertices. Otherwise a_i has at least $(3/4 + 1/20)n = \frac{4n}{5}$ blue

neighbors in B giving a red star (a special non-crossing double star) that is as large as required.

In the case when $D < (1/2 + 1/10)n$, assume (w.l.o.g.) that edge a_1b_n is red. Now - from the definition of D - both $d_{G_R}(a_1)$ and $d_{G_R}(b_n)$ are at least $(1/2 - 1/10)n$ therefore we have a non-crossing red double star on a_1b_n with at least $2(1/2 - 1/10)n = \frac{4n}{5}$ vertices.

It follows from Lemma 2 that the bound is asymptotically sharp. ■

PROOF OF THEOREM 2. We claim first that a geometric $G(n, n)$ that contains no non-crossing double star with $k + 1$ vertices, has at most the claimed number of edges. The proof is by induction n , keeping k fixed. The cases $2n = k$ and $2n - 1 = k$ are obvious. For the induction step, assume that there is no non-crossing double star with $k + 1$ vertices in a geometric graph $G = G(n, n)$ for $2n \geq k + 2$. Select an edge $e = a_i b_j \in E(G)$ with $|i - j|$ as large as possible. From the choice of e , the edges of G incident to e form a non-crossing double star, thus, from the assumption on G , we have $d_G(a_i) + d_G(b_j) \leq k$. Deleting the vertices a_i, b_j with its incident edges we delete at most $k - 1$ edges and get a balanced geometric graph $F = G(n - 1, n - 1)$. By the inductive hypothesis

$$\begin{aligned} |E(G)| &\leq k - 1 + |E(F)| \leq k - 1 + (n - 1)(k - 1) - \left\lfloor \frac{(k - 1)^2}{4} \right\rfloor = \\ &= n(k - 1) - \left\lfloor \frac{(k - 1)^2}{4} \right\rfloor \end{aligned}$$

proving the claim.

To see that the maximum can be attained, we use the graph $H(n, p)$ for even k . Set $p = n - \frac{k}{2}$, from Lemma 1, the largest non-crossing double star in H has at most $2(n - p) = k$ vertices. On the other hand, $H(n, p)$ has $n^2 - 2\binom{p+1}{2} = n(k - 1) - \lfloor \frac{(k-1)^2}{4} \rfloor$ vertices.

For odd k we modify $H(n, p)$ to $H'(n, p)$ so that its edge set is $\{a_i b_j : p + 2 \leq i + j \leq 2n - p - 1\}$. Set $p = n - \frac{k-1}{2}$, now the largest non-crossing double star of $H'(n, p)$ has at most $2(n - p) + 1 = k$ vertices and $H'(n, p)$ has $n^2 - \binom{p+1}{2} - \binom{p}{2} = n(k - 1) - \frac{(k-1)^2}{4}$ edges. ■

PROOF OF THEOREM 3. To prepare the proof, we define matchings in the geometric $K_{n,n}$ having pairwise non-crossing (pairwise crossing)

edges. For $i = 0, 1, 2, \dots, n - k - 1$ let M_i denote the matching with edge set $\{a_1, b_{1+i}, a_2 b_{2+i} \dots, a_{n-i} b_n\}$ and let N_i denote the edge set $\{a_n, b_{n-i}, a_{n-1} b_{n-1-i} \dots, a_{i+1} b_1\}$. Since $N_0 = M_0$, we have $2(n - k - 1) + 1$ edge-disjoint non-crossing matchings. Assume that $G = G(n, n)$ is a geometric graph containing no non-crossing matching with $k + 1$ edges. Then at most k edges of G can be selected from each of the matchings above, thus $|E(G)| \leq k(2n - 2k - 1) + m$ where m is the number of edges of $K_{n,n}$ not covered by the union of the matchings M_i, N_i . Since $m = 2\binom{k+1}{2}$ we get that

$$|E(G)| \leq k(2n - 2k - 1) + (k + 1)k = 2kn - k^2$$

as desired. This proves the extremal result for non-crossing matchings. The proof for the crossing matching is similar, since one can replace the non-crossing matchings M_i, N_i by matchings containing pairwise crossing edges. To see that both results are sharp, consider k -element sets $X \subset A, Y \subset B$ and define the graph whose edges are incident to $X \cup Y$. This graph has $2kn - k^2$ edges and for $X = \{a_1, \dots, a_k\}, Y = \{b_1, \dots, b_k\}$ it does not contain a non-crossing matching with $k + 1$ edges; for $X = \{a_1, \dots, a_k\}, Y = \{b_{n-k+1}, \dots, b_n\}$ it does not contain a matching with $k + 1$ pairwise crossing edges. ■

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**REMARKS TO ARSOVSKI'S PROOF OF
SNEVILY'S CONJECTURE**

By

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Dedicated to our late friend, András Gács

Abstract. Let G denote a finite Abelian group, and \mathbb{F} a field whose multiplicative group \mathbb{F}^\times contains an element whose order equals the exponent of G . For any pair $A = \{a_1, \dots, a_k\}$, $B = \{b_1, \dots, b_k\}$ of k -element subsets of G there exist homomorphisms $\chi_1, \dots, \chi_k : G \rightarrow \mathbb{F}^\times$ such that neither of the two matrices $(\chi_i(a_j))$ and $(\chi_i(b_j))$ is singular. This confirms a conjecture of Feng, Sun, and Xiang. We also give a shortened proof of Snevily's conjecture.

1. Introduction

Let G denote a finite Abelian group of order m and exponent n . We say that G is *fully representable* over a field \mathbb{F} if the multiplicative group \mathbb{F}^\times contains an element of order n . This happens if and only if the characteristic of the field \mathbb{F} does not divide n , and \mathbb{F} itself contains the splitting field of the polynomial $x^n - 1$ over its prime field. In this case \mathbb{F}^\times contains a unique cyclic subgroup H of order n that may be identified for every such field of the same characteristic, and every homomorphism from G to \mathbb{F}^\times maps into H . Such group characters with respect to pointwise multiplication form the character group $\widehat{G} \cong G$. It follows from the orthogonality relations

$$\sum_{g \in G} \chi_1(g) \chi_2^{-1}(g) = \begin{cases} |G| & \text{if } \chi_1 = \chi_2 \\ 0 & \text{otherwise} \end{cases}$$

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that the $m \times m$ matrix $(\chi(g))_{g \in G, \chi \in \widehat{G}}$ is nonsingular. Thus, the characters are linearly independent over \mathbb{F} and form a basis in the vector space of all $G \rightarrow \mathbb{F}$ functions over \mathbb{F} .

REMARK. The independence of the columns of the character table can be interpreted as follows: For any subset $A = \{a_1, \dots, a_k\}$ of G , those sets of characters χ for which the vectors $\chi(A) = \chi(a_i)_{1 \leq i \leq k}$ are independent over \mathbb{F} , form a rank k matroid \mathcal{M}_A over the ground set \widehat{G} . Here we prove that for any two sets $A, B \subseteq G$ of the same cardinality, the matroids \mathcal{M}_A and \mathcal{M}_B have a common basis.

THEOREM 1. *Assume that the finite Abelian group G is fully representable over the field \mathbb{F} . For any two subsets $A = \{a_1, \dots, a_k\}$ and $B = \{b_1, \dots, b_k\}$ of G there exist characters $\chi_1, \dots, \chi_k \in \widehat{G}$ such that both $\text{Det}(\chi_i(a_j))$ and $\text{Det}(\chi_i(b_j))$ are different from zero.*

This confirms a conjecture of FENG, SUN, and XIANG [4]. Applying the natural isomorphism between G and $\widehat{\widehat{G}}$, one obtains the following dual version.

THEOREM 2. *Under the conditions of the previous theorem, let $X = \{\chi_1, \dots, \chi_k\}$ and $\Psi = \{\psi_1, \dots, \psi_k\}$ be two subsets of \widehat{G} . Then there exist elements $a_1, \dots, a_k \in G$ such that both $\text{Det}(\chi_i(a_j))$ and $\text{Det}(\psi_i(a_j))$ are different from zero.*

Using the exterior algebra method, FENG, SUN, and XIANG [4] pointed out that a weaker form of Theorem 1 would imply Snevily's conjecture [6], which after a series of partially successful attempts [1,3,5], see also [7,8], was recently proved by ARSOVSKI [2]. Thus, one obtains the following affirmative answer for Snevily's problem.

THEOREM 3. *Let G be an Abelian group of odd order. For any two subsets $A = \{a_1, \dots, a_k\}$ and $B = \{b_1, \dots, b_k\}$ of G there exists a permutation $\pi \in S_k$ such that the elements $a_1 + b_{\pi(1)}, \dots, a_k + b_{\pi(k)}$ are pairwise different.*

The proof of Theorem 1, at least for finite fields \mathbb{F} of characteristic 2, is implicit in Arsovski's paper. Here we present a variant of his argument with considerable simplifications, which completely settles the conjecture of Feng, Sun, and Xiang.

2. The Proofs

Assume that, for a given finite Abelian group G , the statement of Theorem 1 fails for a certain field \mathbb{F} of characteristic c , then it also fails for every field of characteristic c over which G is fully representable. In particular, it fails for the purely transcendental extension $\mathbb{F}' = \mathbb{F}(t_1, \dots, t_m)$. Accordingly, let $A = \{a_1, \dots, a_k\}$ and $B = \{b_1, \dots, b_k\}$ be two subsets of G such that

$$\text{Det}(\chi_i(a_j)) \text{Det}(\chi_i(b_j)) = 0$$

holds for every k -tuple of characters $\chi_1, \dots, \chi_k \in \widehat{G}$. Write $\widehat{G} = \{\chi_1, \dots, \chi_m\}$. Let φ denote an arbitrary function from G to \mathbb{F}' ; it can be uniquely expressed as $\varphi = \sum_{u=1}^m \lambda_u \chi_u$ with Fourier-coefficients $\lambda_u \in \mathbb{F}'$. Consider the $k \times m$ matrices $M = (m_{i,u})$ and $N = (n_{i,u})$ with $m_{i,u} = \lambda_u \chi_u(a_i)$, resp. $n_{i,u} = \chi_u(b_i)$. In view of the Cauchy–Binet formula and the multilinearity of the determinant, for the $k \times k$ matrix L with (i,j) entry $\varphi(a_i + b_j)$ we obtain

$$\begin{aligned} \text{Det } L &= \text{Det} \left(\sum_{u=1}^m \lambda_u \chi_u(a_i + b_j) \right) = \text{Det} \left(\sum_{u=1}^m \lambda_u \chi_u(a_i) \chi_u(b_j) \right) = \\ &= \text{Det}(MN^\top) = \sum_{1 \leq u_1 < \dots < u_k \leq m} \text{Det}(m_{i,u_j}) \text{Det}(n_{i,u_j}) = \\ &= \sum_{1 \leq u_1 < \dots < u_k \leq m} (\lambda_{u_1} \cdots \lambda_{u_k}) \text{Det}(\chi_{u_i}(a_j)) \text{Det}(\chi_{u_i}(b_j)) = \\ &= 0. \end{aligned}$$

Enumerate the elements of G as g_1, \dots, g_m , and apply the above formula for the function φ that maps each g_i to the corresponding t_i . Then $\text{Det } L$ is the alternating sum of $k!$ monomial terms in t_1, \dots, t_m , each of degree k . Because of the algebraic independence of the elements t_i , $\text{Det } L$ can only vanish if each monomial term cancels out, either because it appears with both $+$ and $-$ signs, or because it appears at least c times with the same sign. Anyway, for any permutation $\pi \in S_k$ there exists a permutation $\sigma \neq \pi \in S_k$ such that the elements $a_1 + b_{\sigma(1)}, \dots, a_k + b_{\sigma(k)}$, in some order, coincide with the elements $a_1 + b_{\pi(1)}, \dots, a_k + b_{\pi(k)}$. According to the following simple combinatorial lemma inherent in [2], this is impossible.

LEMMA 4. *Let $A = \{a_1, \dots, a_k\}$ and $B = \{b_1, \dots, b_k\}$ be subsets of an arbitrary Abelian group G . There exists a permutation $\pi \in S_k$ such that for*

any permutation $\sigma \neq \pi \in S_k$, the multisets $\{a_1 + b_{\pi(1)}, \dots, a_k + b_{\pi(k)}\}$ and $\{a_1 + b_{\sigma(1)}, \dots, a_k + b_{\sigma(k)}\}$ are different.

PROOF. Fix the positive integer k , and assume that the lemma has already been verified for smaller values of k . Write $a_1 + b_1 = g$, and consider the set I of all indices i for which there exists an index j with $a_i + b_j = g$. We may assume that $I = \{1, \dots, \ell\}$. In the case $\ell = k$ there is a unique permutation π with $a_1 + b_{\pi(1)} = \dots = a_k + b_{\pi(k)} = g$. If $1 \leq \ell < k$, then fix the first ℓ values of π by $a_1 + b_{\pi(1)} = \dots = a_\ell + b_{\pi(\ell)} = g$, and apply the induction hypothesis for the subsets $A' = \{a_{\ell+1}, \dots, a_k\}$, $B' = \{b_i \mid i \neq \pi(1), \dots, \pi(\ell)\}$ to extend it to a permutation $\pi \in S_k$. ■

This contradiction proves Theorem 1. Theorem 3 follows by the sophisticated argument of [4] or by the elegant reasoning of ARSOVSKI [2]. In retrospect, the proof only relies on the identity (valid in characteristic 2)

$$\text{Det}(\varphi(a_i + b_j)) = \sum_{1 \leq u_1 < \dots < u_k \leq m} \sum_{\pi \in S_k} \text{Det}(\lambda_{u_j} \chi_{u_j}(a_i + b_{\pi(i)}))$$

and the existence of $\varphi = \sum_{u=1}^m \lambda_u \chi_u : G \rightarrow \mathbb{F}'$, guaranteed by Lemma 4, for which the left hand side is nonzero. Indeed, if Theorem 3 fails then each determinant on the right hand side is zero because the underlying matrix has two equal rows. The identity can be proved directly using the multilinearity of the determinant and the multiplicativity of the characters χ_u :

$$\begin{aligned} \text{Det}(\varphi(a_i + b_j)) &= \sum_{1 \leq u_1, \dots, u_k \leq m} \text{Det}(\lambda_{u_i} \chi_{u_i}(a_i + b_j)) = \\ &= \sum_{\substack{1 \leq u_1, \dots, u_k \leq m \\ \text{distinct}}} \text{Det}(\lambda_{u_i} \chi_{u_i}(a_i + b_j)) = \\ &= \sum_{\substack{1 \leq u_1, \dots, u_k \leq m \\ \text{distinct}}} \sum_{\pi \in S_k} \prod_{i=1}^k (\lambda_{u_i} \chi_{u_i}(a_i + b_{\pi(i)})) = \\ &= \sum_{1 \leq u_1 < \dots < u_k \leq m} \sum_{\pi \in S_k} \text{Det}(\lambda_{u_j} \chi_{u_j}(a_i + b_{\pi(i)})). \end{aligned}$$

It would be very interesting to find a purely combinatorial proof.

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**NEAR m -CONTINUITY FOR MULTIFUNCTIONS IN
BITOPOLOGICAL SPACES**

By

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Abstract. EKICI [4] introduced the notion of upper/lower nearly continuous multifunctions between topological spaces. In this paper, we obtain the unified form of several generalizations of upper/lower nearly continuous multifunctions between bitopological spaces.

1. Introduction

The notion of N -closed sets in a topological space is introduced in [3] and is studied in [12], [13] and other papers. Recently, EKICI [4] introduced and studied upper/lower nearly continuous multifunctions.

The present authors [17], [18] introduced the notions of minimal structures, m -spaces and m -continuous functions. The notion of upper/lower m -continuous multifunctions are introduced in [19]. As a generalization of upper/lower m -continuous multifunctions and upper/lower nearly continuous multifunctions, the present authors [16] introduced and studied the notion of upper/lower nearly m -continuous multifunctions as multifunctions from a set satisfying some minimal condition into a topological space. Quite recently, the present authors [15] investigated a new viewpoint in the study of continuity forms in bitopological spaces.

In this paper, analogously to [15], we introduce and study the notion of upper/lower nearly m -continuous multifunctions as multifunctions from a set satisfying some minimal conditions into a bitopological space. Then it turns out that we obtain the unified form of several generalizations of upper/lower nearly continuous multifunctions between bitopological spaces. Furthermore,

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similarly to the method in [14], we characterize the set of all such points at which a multifunction is not upper/lower nearly m -continuous.

2. Preliminaries

Throughout the present paper, (X, τ_1, τ_2) (resp. (X, τ)) is a bitopological (resp. topological) space. Let (X, τ) be a topological space and A a subset of X . The closure of A and the interior of A are denoted by $\text{Cl}(A)$ and $\text{Int}(A)$, respectively. Let (X, τ_1, τ_2) be a bitopological space and A a subset of X . The closure of A and the interior of A with respect to τ_i for $i = 1, 2$ are denoted by $i\text{Cl}(A)$ and $i\text{Int}(A)$, respectively. A subset A of (X, τ) is said to be regular open (resp. regular closed) if $\text{Int}(\text{Cl}(A)) = A$ (resp. $\text{Cl}(\text{Int}(A)) = A$).

DEFINITION 2.1. A subset A of a topological space (X, τ) is said to be *N-closed relative to X* (briefly *N-closed*) [3] if every cover of A by regular open sets of X has a finite subcover.

DEFINITION 2.2. A subset A of a bitopological space (X, τ_1, τ_2) is said to be

- (1) *(i, j) -semi-open* [2] if $A \subset j\text{Cl}(i\text{Int}(A))$, where $i \neq j$, $i, j = 1, 2$,
- (2) *(i, j) -preopen* [5] if $A \subset i\text{Int}(j\text{Cl}(A))$, where $i \neq j$, $i, j = 1, 2$,
- (3) *(i, j) - α -open* [6] if $A \subset i\text{Int}(j\text{Cl}(i\text{Int}(A)))$, where $i \neq j$, $i, j = 1, 2$,
- (4) *(i, j) -semi-preopen* [8] if there exists an (i, j) -preopen set U such that $U \subset A \subset j\text{Cl}(U)$, where $i \neq j$, $i, j = 1, 2$,
- (5) *(i, j) -regular open* [1] if $A = i\text{Int}(j\text{Cl}(A))$, where $i \neq j$, $i, j = 1, 2$.

The family of (i, j) -semi-open (resp. (i, j) -preopen, (i, j) - α -open, (i, j) -semi-preopen) sets of (X, τ_1, τ_2) is denoted by $(i, j)\text{SO}(X)$ (resp. $(i, j)\text{PO}(X)$, $(i, j)\alpha(X)$, $(i, j)\text{SPO}(X)$).

DEFINITION 2.3. The complement of an (i, j) -semi-open (resp. (i, j) -preopen, (i, j) - α -open, (i, j) -semi-preopen) set is said to be *(i, j) -semi-closed* (resp. *(i, j) -preclosed*, *(i, j) - α -closed*, *(i, j) -semi-preclosed*).

DEFINITION 2.4. The intersection of all (i, j) -semi-closed (resp. (i, j) -preclosed, (i, j) - α -closed, (i, j) -semi-preclosed) sets of X containing A is called the *(i, j) -semi-closure* [9] (resp. *(i, j) -preclosure* [8], *(i, j) - α -closure* [11], *(i, j) -semi-preclosure* [8]) of A and is denoted by $(i, j)\text{sCl}(A)$ (resp. $(i, j)\text{pCl}(A)$, $(i, j)\alpha\text{Cl}(A)$, $(i, j)\text{spCl}(A)$).

DEFINITION 2.5. The union of all (i, j) -semi-open (resp. (i, j) -preopen, (i, j) - α -open, (i, j) -semi-preopen) sets of X contained in A is called the (i, j) -*semi-interior* (resp. (i, j) -*preinterior*, (i, j) - α -*interior*, (i, j) -*semi-preinterior*) of A and is denoted by

$$(i, j)\text{sInt}(A) \text{ (resp. } (i, j)\text{pInt}(A), (i, j)\alpha\text{Int}(A), (i, j)\text{spInt}(A)).$$

DEFINITION 2.6. A subset A of a bitopological space (X, τ_1, τ_2) is said to be (i, j) -*N-closed* [20] if every cover of A by (i, j) -regular open sets of X has a finite subcover.

For a multifunction $F: X \rightarrow Y$, we shall denote the upper and lower inverse of a subset B of a space Y by $F^+(B)$ and $F^-(B)$, respectively, that is

$$F^+(B) = \{x \in X : F(x) \subset B\} \quad \text{and} \quad F^-(B) = \{x \in X : F(x) \cap B \neq \emptyset\}.$$

DEFINITION 2.7. A multifunction $F: (X, \tau) \rightarrow (Y, \sigma)$ is said to be

- (1) *upper nearly continuous* at a point $x \in X$ [4] if for each open set V containing $F(x)$ and having N -closed complement, there exists an open set U of X containing x such that $F(U) \subset V$,
- (2) *lower nearly continuous* at a point $x \in X$ [4] if for each open set V meeting $F(x)$ and having N -closed complement, there exists an open set U of X containing x such that $F(u) \cap V \neq \emptyset$ for each $u \in U$,
- (3) *upper/lower nearly continuous* on X if it has this property at each point of X .

3. Minimal structures and bitopological spaces

DEFINITION 3.1. A subfamily m_X of the power set $\mathcal{P}(X)$ of a nonempty set X is called a *minimal structure* (briefly m -structure) [17], [18] on X if $\emptyset \in m_X$ and $X \in m_X$.

By (X, m_X) (briefly (X, m)), we denote a nonempty set X with a minimal structure m_X on X and call it an m -space. Each member of m_X is said to be m_X -open (briefly m -open) and the complement of an m_X -open set is said to be m_X -closed (briefly m -closed).

DEFINITION 3.2. Let (X, m_X) be an m -space. For a subset A of X , the m_X -closure of A and the m_X -interior of A are defined in [10] as follows:

- (1) $m\text{Cl}(A) = \bigcap\{F: A \subset F, X - F \in m_X\}$,
- (2) $m\text{Int}(A) = \bigcup\{U: U \subset A, U \in m_X\}$.

REMARK 3.1. Let (X, τ_1, τ_2) be a bitopological space and A a subset of X .

- (1) The families $(i, j)\text{SO}(X)$, $(i, j)\text{PO}(X)$, $(i, j)\alpha(X)$, and $(i, j)\text{SPO}(X)$ are all m -structures on X .
- (2) If $m_X = (i, j)\text{SO}(X)$ (resp. $(i, j)\text{PO}(X)$, $(i, j)\alpha(X)$, $(i, j)\text{SPO}(X)$), then we have
 - (a) $\text{mCl}(A) = (i, j)\text{sCl}(A)$ (resp. $(i, j)\text{pCl}(A)$, $(i, j)\alpha\text{Cl}(A)$, $(i, j)\text{spCl}(A)$),
 - (b) $\text{mInt}(A) = (i, j)\text{sInt}(A)$ (resp. $(i, j)\text{pInt}(A)$, $(i, j)\alpha\text{Int}(A)$, $(i, j)\text{spInt}(A)$).

LEMMA 3.1. (MAKI ET AL. [10]). *Let (X, m_X) be an m -space. For subsets A and B of X , the following properties hold:*

- (1) $\text{mCl}(X - A) = X - \text{mInt}(A)$ and $\text{mInt}(X - A) = X - \text{mCl}(A)$,
- (2) If $(X - A) \in m_X$, then $\text{mCl}(A) = A$ and if $A \in m_X$, then $\text{mInt}(A) = A$,
- (3) $\text{mCl}(\emptyset) = \emptyset$, $\text{mCl}(X) = X$, $\text{mInt}(\emptyset) = \emptyset$ and $\text{mInt}(X) = X$,
- (4) If $A \subset B$, then $\text{mCl}(A) \subset \text{mCl}(B)$ and $\text{mInt}(A) \subset \text{mInt}(B)$,
- (5) $A \subset \text{mCl}(A)$ and $\text{mInt}(A) \subset A$,
- (6) $\text{mCl}(\text{mCl}(A)) = \text{mCl}(A)$ and $\text{mInt}(\text{mInt}(A)) = \text{mInt}(A)$.

DEFINITION 3.3. An m -structure m_X on a nonempty set X is said to have property \mathcal{B} [10] if the union of any family of subsets belonging to m_X belongs to m_X .

REMARK 3.2. Let (X, τ_1, τ_2) be a bitopological space. Then the family $(i, j)\text{SO}(X)$ (resp. $(i, j)\text{PO}(X)$, $(i, j)\alpha(X)$, $(i, j)\text{SPO}(X)$) is an m -structure on X having property \mathcal{B} by Theorem 2 of [9] (resp. Theorem 4.2 of [7] or Theorem 3.2 of [8], Theorem 1.5 of [11], Theorem 3.2 of [8]).

LEMMA 3.2. (POPA and NOIRI [17]). *For an m -structure m_X on a non-empty set X , the following properties are equivalent:*

- (1) m_X has property \mathcal{B} ;
- (2) If $\text{mInt}(A) = A$, then $A \in m_X$;
- (3) If $\text{mCl}(A) = A$, then A is m_X -closed.

DEFINITION 3.4. Let (X, m_X) be an m -space and (Y, σ) a topological space. A multifunction $F: (X, m_X) \rightarrow (Y, \sigma)$ is said to be

- (1) *upper m -continuous* [19] at a point $x \in X$ if for each open set V containing $F(x)$, there exists an m_X -open set U containing x such that $F(U) \subset V$,

- (2) *lower m -continuous* [19] at a point $x \in X$ if for each open set V meeting $F(x)$, there exists an m_X -open set U containing x such that $F(u) \cap V \neq \emptyset$ for each $u \in U$,
- (3) *upper/lower m -continuous* on X if it has this property at every point of X .

DEFINITION 3.5. Let (X, m_X) be an m -space and (Y, σ) a topological space. A multifunction $F: (X, m_X) \rightarrow (Y, \sigma)$ is said to be

- (1) *upper nearly m -continuous* [16] at a point $x \in X$ if for each open set V containing $F(x)$ and having N -closed complement, there exists an m_X -open set U containing x such that $F(U) \subset V$,
- (2) *lower nearly m -continuous* [16] at a point $x \in X$ if for each open set V meeting $F(x)$ and having N -closed complement, there exists an m_X -open set U containing x such that $F(u) \cap V \neq \emptyset$ for each $u \in U$,
- (3) *upper/lower nearly m -continuous* on X if it has this property at every point of X .

REMARK 3.3. Every upper/lower m -continuous multifunction is upper/lower nearly m -continuous. The converse is not true by Example 4 of [4].

4. (i, j) -upper/lower near m -continuity

DEFINITION 4.1. Let (X, m_X) be an m -space and (Y, σ_1, σ_2) a bitopological space. A multifunction $F: (X, m_X) \rightarrow (Y, \sigma_1, \sigma_2)$ is said to be

- (1) *(i, j) -upper nearly m -continuous* at a point $x \in X$ if for each σ_i -open set V containing $F(x)$ and having (i, j) - N -closed complement, there exists an m_X -open set U containing x such that $F(U) \subset V$,
- (2) *(i, j) -lower nearly m -continuous* at a point $x \in X$ if for each σ_i -open set V meeting $F(x)$ and having (i, j) - N -closed complement, there exists an m_X -open set U containing x such that $F(u) \cap V \neq \emptyset$ for each $u \in U$,
- (3) *(i, j) -upper/ (i, j) -lower nearly m -continuous* on X if it has this property at every point of X .

THEOREM 4.1. For a multifunction $F: (X, m_X) \rightarrow (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:

- (1) F is (i, j) -upper nearly m -continuous at $x \in X$;
- (2) $x \in \text{mInt}(F^+(V))$ for every σ_i -open set V of Y containing $F(x)$ and having (i, j) - N -closed complement;

- (3) $x \in F^-(i\text{Cl}(B))$ for every subset B of Y having (i,j) - N -closed σ_i -closure such that $x \in m\text{Cl}(F^-(B))$;
- (4) $x \in m\text{Int}(F^+(B))$ for every subset B of Y such that $x \in F^+(i\text{Int}(B))$ and $Y - i\text{Int}(B)$ is (i,j) - N -closed.

PROOF. (1) \Rightarrow (2): Let V be any σ_i -open set of Y containing $F(x)$ and having (i,j) - N -closed complement. There exists an m_X -open set U containing x such that $F(U) \subset V$. Thus $x \in U \subset F^+(V)$. Since U is m_X -open, we have $x \in m\text{Int}(F^+(V))$.

(2) \Rightarrow (3): Suppose that B is any subset of Y having (i,j) - N -closed σ_i -closure. Then $i\text{Cl}(B)$ is σ_i -closed and $Y - i\text{Cl}(B)$ is a σ_i -open set having (i,j) - N -closed complement. Let $x \notin F^-(i\text{Cl}(B))$, then $x \in X - F^-(i\text{Cl}(B)) = F^+(Y - i\text{Cl}(B))$. This implies that $F(x) \subset Y - i\text{Cl}(B)$. Since $Y - i\text{Cl}(B)$ is a σ_i -open set having (i,j) - N -closed complement, by (2) we have $x \in m\text{Int}(F^+(Y - i\text{Cl}(B))) = m\text{Int}(X - F^-(i\text{Cl}(B))) = X - m\text{Cl}(F^-(i\text{Cl}(B))) \subset X - m\text{Cl}(F^-(B))$. Hence $x \notin m\text{Cl}(F^-(B))$.

(3) \Rightarrow (4): Let B be any subset of Y such that $x \notin m\text{Int}(F^+(B))$ and $Y - i\text{Int}(B)$ is (i,j) - N -closed. Then we have $x \in X - m\text{Int}(F^+(B)) = m\text{Cl}(X - F^+(B)) = m\text{Cl}(F^-(Y - B))$. By (3) we have $x \in F^-(i\text{Cl}(Y - B)) = F^-(Y - i\text{Int}(B)) = X - F^+(i\text{Int}(B))$. Hence $x \notin F^+(i\text{Int}(B))$.

(4) \Rightarrow (1): Let V be any σ_i -open set of Y containing $F(x)$ and having (i,j) - N -closed complement. We have $x \in F^+(V) = F^+(i\text{Int}(V))$. Then, by (4) $x \in m\text{Int}(F^+(V))$. Therefore, there exists $U \in m_X$ such that $x \in U \subset F^+(V)$. Thus $F(U) \subset V$. This shows that F is (i,j) -upper nearly m -continuous at $x \in X$. ■

THEOREM 4.2. *For a multifunction $F: (X, m_X) \rightarrow (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

- (1) F is (i,j) -lower nearly m -continuous at $x \in X$;
- (2) $x \in m\text{Int}(F^-(V))$ for every σ_i -open set V of Y having (i,j) - N -closed complement such that $F(x) \cap V \neq \emptyset$;
- (3) $x \in F^+(i\text{Cl}(B))$ for every subset B of Y having (i,j) - N -closed σ_i -closure such that $x \in m\text{Cl}(F^+(B))$;
- (4) $x \in m\text{Int}(F^-(B))$ for every subset B of Y such that $x \in F^-(i\text{Int}(B))$ and $Y - i\text{Int}(B)$ is (i,j) - N -closed.

PROOF. The proof is similar to that of Theorem 4.1. ■

THEOREM 4.3. *For a multifunction $F: (X, m_X) \rightarrow (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

- (1) F is (i,j) -upper nearly m -continuous;
- (2) $F^+(V) = \text{mInt}(F^+(V))$ for each σ_i -open set V of Y having (i,j) - N -closed complement;
- (3) $F^-(K) = \text{mCl}(F^-(K))$ for every (i,j) - N -closed and σ_i -closed set K of Y ;
- (4) $\text{mCl}(F^-(B)) \subset F^-(i\text{Cl}(B))$ for every subset B of Y having (i,j) - N -closed σ_i -closure;
- (5) $F^+(i\text{Int}(B)) \subset \text{mInt}(F^+(B))$ for every subset B of Y such that $Y - i\text{Int}(B)$ is (i,j) - N -closed.

PROOF. (1) \Rightarrow (2): Let V be any σ_i -open set of Y having (i,j) - N -closed complement and $x \in F^+(V)$. Then $F(x) \subset V$ and there exists $U \in m_X$ containing x such that $F(U) \subset V$. Therefore, $x \in U \subset F^+(V)$ and hence $x \in \text{mInt}(F^+(V))$. This shows that $F^+(V) \subset \text{mInt}(F^+(V))$. Therefore, by Lemma 3.1 we obtain $F^+(V) = \text{mInt}(F^+(V))$.

(2) \Rightarrow (3): Let K be any (i,j) - N -closed and σ_i -closed set of Y . Then, by (2) and Lemma 3.1 we have $X - F^-(K) = F^+(Y - K) = \text{mInt}(F^+(Y - K)) = \text{mInt}(X - F^-(K)) = X - \text{mCl}(F^-(K))$. Therefore, we obtain $F^-(K) = \text{mCl}(F^-(K))$.

(3) \Rightarrow (4): Let B be any subset of Y having (i,j) - N -closed σ_i -closure. By (3) and Lemma 3.1, we have $F^-(B) \subset F^-(i\text{Cl}(B)) = \text{mCl}(F^-(i\text{Cl}(B)))$. Hence $\text{mCl}(F^-(B)) \subset \text{mCl}(F^-(i\text{Cl}(B))) = F^-(i\text{Cl}(B))$.

(4) \Rightarrow (5): Let B be a subset of Y such that $Y - i\text{Int}(B)$ is (i,j) - N -closed. Then by (4) and Lemma 3.1 we have

$$\begin{aligned} X - \text{mInt}(F^+(B)) &= \text{mCl}(X - F^+(B)) = \text{mCl}(F^-(Y - B)) \subset \\ &\subset F^-(i\text{Cl}(Y - B)) = F^-(Y - i\text{Int}(B)) = X - F^+(i\text{Int}(B)). \end{aligned}$$

Therefore, we obtain $F^+(i\text{Int}(B)) \subset \text{mInt}(F^+(B))$.

(5) \Rightarrow (1): Let $x \in X$ and V be any σ_i -open set of Y containing $F(x)$ and having (i,j) - N -closed complement. Then by (5) $x \in F^+(V) = F^+(i\text{Int}(V)) \subset \text{mInt}(F^+(V))$. There exists $U \in m_X$ containing x such that $U \subset F^+(V)$; hence $F(U) \subset V$. This shows that F is (i,j) -upper nearly m -continuous. ■

THEOREM 4.4. *For a multifunction $F: (X, m_X) \rightarrow (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

- (1) F is (i,j) -lower nearly m -continuous;
- (2) $F^-(V) = \text{mInt}(F^-(V))$ for each σ_i -open set V of Y having (i,j) - N -closed complement;

- (3) $F^+(K) = \text{mCl}(F^+(K))$ for every (i,j) - N -closed and σ_i -closed set K of Y ;
- (4) $\text{mCl}(F^+(B)) \subset F^+(\text{mCl}(B))$ for every subset B of Y having (i,j) - N -closed σ_i -closure;
- (5) $F^-(i\text{Int}(B)) \subset \text{mInt}(F^-(B))$ for every subset B of Y such that $Y - i\text{Int}(B)$ is (i,j) - N -closed.

PROOF. The proof is similar to that of Theorem 4.3. ■

COROLLARY 4.1. Let (X, m_X) be an m -space and m_X have property \mathcal{B} . For a multifunction $F: (X, m_X) \rightarrow (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:

- (1) F is (i,j) -upper nearly m -continuous (resp. (i,j) -lower nearly m -continuous);
- (2) $F^+(V)$ (resp. $F^-(V)$) is m_X -open for each σ_i -open set V of Y having (i,j) - N -closed complement;
- (3) $F^-(K)$ (resp. $F^+(K)$) is m_X -closed for every (i,j) - N -closed and σ_i -closed set K of Y .

PROOF. This is an immediate consequence of Theorems 4.3 and 4.4 and Lemma 3.2. ■

COROLLARY 4.2. A multifunction $F: (X, m_X) \rightarrow (Y, \sigma_1, \sigma_2)$ is (i,j) -upper nearly m -continuous (resp. (i,j) -lower nearly m -continuous) if $F^-(K) = \text{mCl}(F^-(K))$ (resp. $F^+(K) = \text{mCl}(F^+(K))$) for every (i,j) - N -closed set K of Y .

PROOF. Let G be any σ_i -open set of Y having (i,j) - N -closed complement. Then $Y - G$ is (i,j) - N -closed. By the hypothesis, $X - F^+(G) = F^-(Y - G) = \text{mCl}(F^-(Y - G)) = \text{mCl}(X - F^+(G)) = X - \text{mInt}(F^+(G))$ and hence, $F^+(G) = \text{mInt}(F^+(G))$. It follows from Theorem 4.3 that F is (i,j) -upper nearly m -continuous. The proof of (i,j) -lower near m -continuity is entirely similar. ■

DEFINITION 4.2. A function $f: (X, m_X) \rightarrow (Y, \sigma_1, \sigma_2)$ is said to be (i,j) nearly m -continuous if for each point $x \in X$ and each σ_i -open set V containing $f(x)$ and having (i,j) - N -closed complement, there exists an m_X -open set U containing x such that $f(U) \subset V$.

COROLLARY 4.3. For a function $f: (X, m_X) \rightarrow (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:

- (1) f is (i,j) nearly m -continuous;
- (2) $f^{-1}(V) = \text{mInt}(f^{-1}(V))$ for each σ_i -open set V of Y having (i,j) - N -closed complement;
- (3) $f^{-1}(K) = \text{mCl}(f^{-1}(K))$ for every (i,j) - N -closed and σ_i -closed set K of Y ;
- (4) $\text{mCl}(f^{-1}(B)) \subset f^{-1}(\text{iCl}(B))$ for every subset B of Y having (i,j) - N -closed σ_i -closure;
- (5) $f^{-1}(\text{iInt}(B)) \subset \text{mInt}(f^{-1}(B))$ for every subset B of Y such that $Y - i\text{Int}(B)$ is (i,j) - N -closed.

COROLLARY 4.4. For a function $f: (X, m_X) \rightarrow (Y, \sigma_1, \sigma_2)$, where m_X has property \mathcal{B} , the following properties are equivalent:

- (1) f is (i,j) nearly m -continuous;
- (2) $f^{-1}(V)$ is m_X -open for each σ_i -open set V of Y having (i,j) - N -closed complement;
- (3) $f^{-1}(K)$ is m_X -closed for every (i,j) - N -closed and σ_i -closed set K of Y .

For a multifunction $F: (X, m_X) \rightarrow (Y, \sigma_1, \sigma_2)$, we define $D^+_{nm}(F)$ and $D^-_{nm}(F)$ as follows:

$$D^+_{nm}(F) = \{x \in X : F \text{ is not } (i,j)\text{-upper } m\text{-continuous at } x\},$$

$$D^-_{nm}(F) = \{x \in X : F \text{ is not } (i,j)\text{-lower } m\text{-continuous at } x\}.$$

THEOREM 4.5. For a multifunction $F: (X, m_X) \rightarrow (Y, \sigma_1, \sigma_2)$, the following properties hold:

$$\begin{aligned} D^+_{nm}(F) &= \bigcup_{G \in \sigma_i NC} \{F^+(G) - \text{mInt}(F^+(G))\} = \\ &= \bigcup_{B \in i NC} \{F^+(\text{iInt}(B)) - \text{mInt}(F^+(B))\} = \\ &= \bigcup_{B \in NC} \{\text{mCl}(F^-(B)) - F^-(\text{iCl}(B))\} = \\ &= \bigcup_{H \in \mathcal{F}} \{\text{mCl}(F^-(H)) - F^-(H)\}, \end{aligned}$$

where $\sigma_i NC$ is the family of all σ_i -open sets of Y having (i,j) - N -closed complement,

iNC is the family of all subsets B of Y such that $Y - i\text{Int}(B)$ is (i,j) - N -closed,

NC is the family of all subsets of Y having (i,j) - N -closed σ_i -closure,

\mathcal{F} is the family of all subsets H of Y which are (i,j) - N -closed and σ_i -closed.

PROOF. We shall show only the first equality and the last since the proofs of other are similar to the first.

Let $x \in D^+_{nm}(F)$. By Theorem 4.1, there exists a σ_i -open set V of Y having (i,j) - N -closed complement such that $x \in F^+(V)$ and $x \notin m\text{Int}(F^+(V))$. Therefore, we have

$$x \in F^+(V) - m\text{Int}(F^+(V)) \subset \bigcup_{G \in \sigma_i NC} \{F^+(G) - m\text{Int}(F^+(G))\}.$$

Conversely, let $x \in \bigcup_{G \in \sigma_i NC} \{F^+(G) - m\text{Int}(F^+(G))\}$. There exists a σ_i -open set V of Y having (i,j) - N -closed complement such that $x \in F^+(V) - m\text{Int}(F^+(V))$. By Theorem 4.1, we obtain $x \in D^+_{nm}(F)$.

We prove the last equality.

$$\begin{aligned} & \bigcup_{H \in \mathcal{F}} \{m\text{Cl}(F^-(H)) - F^-(H)\} \subset \\ & \subset \bigcup_{B \in NC} \{m\text{Cl}(F^-(B)) - F^-(i\text{Cl}(B))\} = D^+_{nm}(F). \end{aligned}$$

Conversely, by Lemma 3.1 we have

$$\begin{aligned} D^+_{nm}(F) &= \bigcup_{B \in NC} \{m\text{Cl}(F^-(B)) - F^-(i\text{Cl}(B))\} \subset \\ &\subset \bigcup_{H \in \mathcal{F}} \{m\text{Cl}(F^-(H)) - F^-(H)\}. \end{aligned}$$

THEOREM 4.6. For a multifunction $F: (X, m_X) \rightarrow (Y, \sigma_1, \sigma_2)$, the following properties hold:

$$\begin{aligned} D^-_{nm}(F) &= \bigcup_{G \in \sigma_i NC} \{F^-(G) - m\text{Int}(F^-(G))\} = \\ &= \bigcup_{B \in i NC} \{F^-(i\text{Int}(B)) - m\text{Int}(F^-(B))\} = \end{aligned}$$

$$\begin{aligned}
&= \bigcup_{B \in NC} \{mCl(F^+(B)) - F^+(iCl(B))\} = \\
&= \bigcup_{H \in \mathcal{F}} \{mCl(F^+(H)) - F^+(H)\}.
\end{aligned}$$

PROOF. The proof is similar to that of Theorem 4.5. ■

5. (i,j) near m -continuity in bitopological spaces

DEFINITION 5.1. Let (X, τ_1, τ_2) be a bitopological space and $m_{ij} = m(\tau_1, \tau_2)$ an m -structure on X determined by τ_1 and τ_2 . A multifunction $F: (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ is said to be (i,j) -upper/(i,j)-lower nearly m -continuous at $x \in X$ (resp. on X) if $F: (X, m_{ij}) \rightarrow (Y, \sigma_1, \sigma_2)$ is (i,j) -upper/(i,j)-lower nearly m -continuous at $x \in X$ (resp. on X). Hence

- (1) A multifunction $F: (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ is (i,j) -upper nearly m -continuous at a point $x \in X$ if for each σ_i -open set V containing $F(x)$ and having (i,j) - N -closed complement, there exists $U \in m_{ij}$ containing x such that $F(U) \subset V$,
- (2) A multifunction $F: (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ is (i,j) -lower nearly m -continuous at a point $x \in X$ if for each σ_i -open set V meeting $F(x)$ and having (i,j) - N -closed complement, there exists $U \in m_{ij}$ containing x such that $F(u) \cap V \neq \emptyset$ for each $u \in U$,
- (3) (i,j) -upper/(i,j)-lower nearly m -continuous on X if $F: (X, m_{ij}) \rightarrow (Y, \sigma_1, \sigma_2)$ has this property at every point of X .

REMARK 5.1. Let (X, τ_1, τ_2) be a bitopological space and $m_{ij} = m(\tau_1, \tau_2)$ an m -structure on X determined by τ_1 and τ_2 . If $m_{ij} = (i,j)SO(X)$ (resp. $(i,j)PO(X)$, $(i,j)\alpha(X)$, $(i,j)SPO(X)$) and $F: (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ is (i,j) -upper/(i,j)-lower nearly m -continuous, then F is (i,j) -upper/(i,j)-lower nearly semi-continuous (resp. (i,j) -upper/(i,j)-lower nearly precontinuous, (i,j) -upper/(i,j)-lower nearly α -continuous, (i,j) -upper/(i,j)-lower nearly semi-precontinuous).

By Definition 5.1 and Theorems 4.3 and 4.4, we have the following two theorems.

THEOREM 5.1. Let (X, τ_1, τ_2) be a bitopological space and $m_{ij} = m(\tau_1, \tau_2)$ an m -structure on X determined by τ_1 and τ_2 . For a multifunction $F: (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:

- (1) F is (i,j) -upper nearly m -continuous;
- (2) $F^+(V) = m_{ij} \text{Int}(F^+(V))$ for each σ_i -open set V of Y having (i,j) - N -closed complement;
- (3) $F^-(K) = m_{ij} \text{Cl}(F^-(K))$ for every (i,j) - N -closed and σ_i -closed set K of Y ;
- (4) $m_{ij} \text{Cl}(F^-(B)) \subset F^-(i\text{Cl}(B))$ for every subset B of Y having (i,j) - N -closed σ_i -closure;
- (5) $F^+(i\text{Int}(B)) \subset m_{ij} \text{Int}(F^+(B))$ for every subset B of Y such that $Y - i\text{Int}(B)$ is (i,j) - N -closed.

THEOREM 5.2. Let (X, τ_1, τ_2) be a bitopological space and $m_{ij} = m(\tau_1, \tau_2)$ an m -structure on X determined by τ_1 and τ_2 . For a multifunction $F: (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:

- (1) F is (i,j) -lower nearly m -continuous;
- (2) $F^-(V) = m_{ij} \text{Int}(F^-(V))$ for each σ_i -open set V of Y having (i,j) - N -closed complement;
- (3) $F^+(K) = m_{ij} \text{Cl}(F^+(K))$ for every (i,j) - N -closed and σ_i -closed set K of Y ;
- (4) $m_{ij} \text{Cl}(F^+(B)) \subset F^+(i\text{Cl}(B))$ for every subset B of Y having (i,j) - N -closed σ_i -closure;
- (5) $F^-(i\text{Int}(B)) \subset m_{ij} \text{Int}(F^-(B))$ for every subset B of Y such that $Y - i\text{Int}(B)$ is (i,j) - N -closed.

COROLLARY 5.1. Let (X, τ_1, τ_2) be a bitopological space and $m_{ij} = m(\tau_1, \tau_2)$ an m -structure on X determined by τ_1 and τ_2 having property \mathcal{B} . For a multifunction $F: (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:

- (1) F is (i,j) -upper nearly m -continuous (resp. (i,j) -lower nearly m -continuous);
- (2) $F^+(V)$ (resp. $F^-(V)$) is m_{ij} -open for each σ_i -open set V of Y having (i,j) - N -closed complement;
- (3) $F^-(K)$ (resp. $F^+(K)$) is m_{ij} -closed for every (i,j) - N -closed and σ_i -closed set K of Y .

REMARK 5.2. If, for example, $m_{ij} = (i,j)\text{SO}(X)$, then by Theorems 5.1 and 5.2 and Corollary 5.1 we obtain Theorems 5.3 and 5.4 and Corollary 5.2 (below), respectively.

THEOREM 5.3. Let (X, τ_1, τ_2) be a bitopological space and $m_{ij} = (i, j)\text{SO}(X)$. Then, for a multifunction $F: (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:

- (1) F is (i, j) -upper nearly semi-continuous;
- (2) $F^+(V) = (i, j)\text{sInt}(F^+(V))$ for each σ_i -open set V of Y having (i, j) - N -closed complement;
- (3) $F^-(K) = (i, j)\text{sCl}(F^-(K))$ for every (i, j) - N -closed and σ_i -closed set K of Y ;
- (4) $(i, j)\text{sCl}(F^-(B)) \subset F^-(i\text{Cl}(B))$ for every subset B of Y having (i, j) - N -closed σ_i -closure;
- (5) $F^+(i\text{Int}(B)) \subset (i, j)\text{sInt}(F^+(B))$ for every subset B of Y such that $Y - i\text{Int}(B)$ is (i, j) - N -closed.

THEOREM 5.4. Let (X, τ_1, τ_2) be a bitopological space and $m_{ij} = (i, j)\text{SO}(X)$. Then, for a multifunction $F: (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:

- (1) F is (i, j) -lower nearly semi-continuous;
- (2) $F^-(V) = (i, j)\text{sInt}(F^-(V))$ for each σ_i -open set V of Y having (i, j) - N -closed complement;
- (3) $F^+(K) = (i, j)\text{sCl}(F^+(K))$ is for every (i, j) - N -closed and σ_i -closed set K of Y ;
- (4) $(i, j)\text{sCl}(F^+(B)) \subset F^+(i\text{Cl}(B))$ for every subset B of Y having (i, j) - N -closed σ_i -closure;
- (5) $F^-(i\text{Int}(B)) \subset (i, j)\text{sInt}(F^-(B))$ for every subset B of Y such that $Y - i\text{Int}(B)$ is (i, j) - N -closed.

COROLLARY 5.2. Let (X, τ_1, τ_2) be a bitopological space and $m_{ij} = (i, j)\text{SO}(X)$. Then, for a multifunction $F: (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:

- (1) F is (i, j) -upper nearly semi-continuous (resp. (i, j) -lower nearly semi-continuous);
- (2) $F^+(V)$ (resp. $F^-(V)$) is (i, j) -semi-open for each σ_i -open set V of Y having (i, j) - N -closed complement;
- (3) $F^-(K)$ (resp. $F^+(K)$) is (i, j) -semi-closed for every (i, j) - N -closed and σ_i -closed set K of Y .

Let $F: (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ be a multifunction and $m_{ij} = m(\tau_1, \tau_2)$ an m -structure on X determined by τ_1 and τ_2 . Then we define $D^+_{nm_{ij}}(F)$ and $D^-_{nm_{ij}}(F)$ as follows:

$$D^+_{nm_{ij}}(F) = \{x \in X : F \text{ is not } (i, j)\text{-upper nearly } m\text{-continuous at } x\},$$

$$D^-_{nm_{ij}}(F) = \{x \in X : F \text{ is not } (i, j)\text{-lower nearly } m\text{-continuous at } x\}.$$

By Theorems 4.5 and 4.6, we obtain the following two theorems.

THEOREM 5.5. *Let (X, τ_1, τ_2) be a bitopological space and $m_{ij} = m(\tau_1, \tau_2)$ an m -structure on X determined by τ_1 and τ_2 . For a multifunction $F: (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$, the following equalities hold:*

$$\begin{aligned} D^+_{nm_{ij}}(F) &= \bigcup_{G \in \sigma_i NC} \{F^+(G) - m_{ij} \text{Int}(F^+(G))\} = \\ &= \bigcup_{B \in i NC} \{F^+(i \text{Int}(B)) - m_{ij} \text{Int}(F^+(B))\} = \\ &= \bigcup_{B \in NC} \{m_{ij} \text{Cl}(F^-(B)) - F^-(i \text{Cl}(B))\} = \\ &= \bigcup_{H \in \mathcal{F}} \{m_{ij} \text{Cl}(F^-(H)) - F^-(H)\}. \end{aligned}$$

THEOREM 5.6. *For a multifunction $F: (X, m_X) \rightarrow (Y, \sigma_1, \sigma_2)$, the following equalities hold:*

$$\begin{aligned} D^-_{nm_{ij}}(F) &= \bigcup_{G \in \sigma_i NC} \{F^-(G) - m_{ij} \text{Int}(F^-(G))\} = \\ &= \bigcup_{B \in i NC} \{F^-(i \text{Int}(B)) - m_{ij} \text{Int}(F^-(B))\} = \\ &= \bigcup_{B \in NC} \{m_{ij} \text{Cl}(F^+(B)) - F^+(i \text{Cl}(B))\} = \\ &= \bigcup_{H \in \mathcal{F}} \{m_{ij} \text{Cl}(F^+(H)) - F^+(H)\}. \end{aligned}$$

REMARK 5.3. If, for example, $m_{ij} = (i, j)\text{SO}(X)$, then by Theorems 5.5 and 5.6 we obtain Corollaries 5.3 and 5.4 (below), respectively.

COROLLARY 5.3. Let (X, τ_1, τ_2) be a bitopological space and $m_{ij} = (i, j)\text{SO}(X)$. Then, for a multifunction $F: (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$, the following equalities hold:

$$\begin{aligned} D^+_{nm_{ij}}(F) &= \bigcup_{G \in \sigma_i NC} \{F^+(G) - (i, j)\text{sInt}(F^+(G))\} = \\ &= \bigcup_{B \in i NC} \{F^+(i\text{Int}(B)) - (i, j)\text{sInt}(F^+(B))\} = \\ &= \bigcup_{B \in NC} \{(i, j)\text{sCl}(F^-(B)) - F^-(i\text{Cl}(B))\} = \\ &= \bigcup_{H \in \mathcal{F}} \{(i, j)\text{sCl}(F^-(H)) - F^-(H)\}. \end{aligned}$$

COROLLARY 5.4. Let (X, τ_1, τ_2) be a bitopological space and $m_{ij} = (i, j)\text{SO}(X)$. Then, for a multifunction $F: (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$, the following equalities hold:

$$\begin{aligned} D^-_{nm_{ij}}(F) &= \bigcup_{G \in \sigma_i NC} \{F^-(G) - (i, j)\text{sInt}(F^-(G))\} = \\ &= \bigcup_{B \in i NC} \{F^-(i\text{Int}(B)) - (i, j)\text{sInt}(F^-(B))\} = \\ &= \bigcup_{B \in NC} \{(i, j)\text{sCl}(F^+(B)) - F^+(i\text{Cl}(B))\} = \\ &= \bigcup_{H \in \mathcal{F}} \{(i, j)\text{sCl}(F^+(H)) - F^+(H)\}. \end{aligned}$$

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**A UNIFIED THEORY FOR
GENERALIZATIONS OF COMPACT SPACES**

By

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Abstract. We introduce certain operations on an m -structure with property \mathcal{B} (the generalized topology in the sense of LUGOJAN [20]). This operation enables us to unify several modifications of α -compactness due to KASAHARA [17].

1. Introduction

Let (X, τ) be a topological space and $\mathcal{P}(X)$ the power set of X . In 1979, KASAHARA [17] initiated the study of operations on the topology τ . An operation α on τ is a function $\alpha : \tau \rightarrow \mathcal{P}(X)$ such that $U \subset \alpha(U) = U^\alpha$ for each $U \in \tau$. By using the operation, he obtained the unified characterizations of some generalizations of compact spaces. In 1991, OGATA [30] used the term *operation γ* instead of the *operation α* due to KASAHARA. OGATA introduced the notion of γ -open sets and investigated the associated topology τ_γ and weak separation axioms γ - T_i ($i = 0, 1/2, 1, 2$). Recently, KRISHNAN ET AL. [18] (resp. AN ET AL. [3]) have defined an operation $\gamma : SO(X) \rightarrow \mathcal{P}(X)$ (resp. $\gamma_p : PO(X) \rightarrow \mathcal{P}(X)$) and the notion of semi- γ -open (resp. pre γ_p -open) sets and investigated weak separation axioms analogous to γ - T_i due to OGATA [30]. On the other hand, CSÁSZÁR [9] introduced the notion of γ -open sets which are different from γ -open sets due to OGATA. Moreover, in [10], γ -compact spaces are defined as the unified definition of some modifications of compact spaces. In this paper, we define an operation on an m -structure with property \mathcal{B} (the generalized topology in the sense of LUGOJAN [20]). The

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operation is defined as a function $m\gamma : m_X \rightarrow \mathcal{P}(X)$ such that $U \subset m\gamma(U) = U^{m\gamma}$ for each $U \in m_X$ and is called an operation $m\gamma$ on m_X . Then, it turns out that the operation is a unified form of several operations defined on the family of generalized open sets. Moreover, we can obtain a unified theory for generalizations of compact spaces. In § 5, we obtain some characterizations of $m\gamma$ -compactness. In § 6, we show several examples for $m\gamma$ -compactness which are analogous to some results established by CsÁSZÁR [10]. In the last section, we suggest some generalizations of compact spaces by using recent modifications of open sets in a topological space.

2. Generalized open sets

Let (X, τ) be a topological space and A a subset of X . The closure of A and the interior of A are denoted by $\text{Cl}(A)$ and $\text{Int}(A)$, respectively. We recall some generalizations of open sets in topological spaces.

DEFINITION 2.1. Let (X, τ) be a topological space. A subset A of X is said to be (1) *α -open* [28] if $A \subset \text{Int}(\text{Cl}(\text{Int}(A)))$, (2) *semi-open* [19] if $A \subset \text{Cl}(\text{Int}(A))$, (3) *preopen* [23] if $A \subset \text{Int}(\text{Cl}(A))$, (4) *b -open* [5] if $A \subset \text{Int}(\text{Cl}(A)) \cup \text{Cl}(\text{Int}(A))$, (5) *semi-preopen* [4] or *β -open* [2] if $A \subset \text{Cl}(\text{Int}(\text{Cl}(A)))$.

The family of all α -open (resp. semi-open, preopen, b -open, semi-preopen) sets in (X, τ) is denoted by $\alpha(X)$ (resp. $\text{SO}(X)$, $\text{PO}(X)$, $\text{BO}(X)$, $\text{SPO}(X)$).

For generalizations of open sets defined above, the following relationships are known:

$$\begin{array}{c} \text{open} \Rightarrow \alpha\text{-open} \Rightarrow \text{preopen} \\ \Downarrow \qquad \qquad \Downarrow \\ \text{semi-open} \Rightarrow b\text{-open} \Rightarrow \text{semi-preopen} \end{array}$$

DEFINITION 2.2. Let (X, τ) be a topological space. A subset A of X is said to be *α -closed* [26] (resp. *semi-closed* [8], *preclosed* [23], *semi-preclosed* [4], *b -closed* [5]) if the complement of A is α -open (resp. semi-open, preopen, semi-preopen, b -open).

DEFINITION 2.3. Let (X, τ) be a topological space and A a subset of X . The intersection of all α -closed (resp. semi-closed, preclosed, semi-preclosed,

b-closed) sets of X containing A is called the *α -closure* [26] (resp. *semi-closure* [8], *preclosure* [15], *semi-preclosure* [4], *b-closure* [5]) of A and is denoted by $\alpha\text{Cl}(A)$ (resp. $s\text{Cl}(A)$, $p\text{Cl}(A)$, $sp\text{Cl}(A)$, $b\text{Cl}(A)$).

3. Minimal structures

DEFINITION 3.1. Let X be a nonempty set and $\mathcal{P}(X)$ the power set of X . A subfamily m_X of $\mathcal{P}(X)$ is called a *minimal structure* (briefly *m-structure*) on X if m_X satisfies the following conditions:(1) $\emptyset \in m_X$ and $X \in m_X$,(2) The union of any family of subsets belonging to m_X belongs to m_X .

A set X with an *m-structure* is called an *m-space* and is denoted by (X, m_X) . Each member of m_X is said to be *m_X* -open (briefly *m-open*) and the complement of an *m_X -open* set is said to be *m_X -closed* (briefly *m-closed*).

REMARK 3.1. (1) In [33], m_X is called an *m-structure* if it satisfies the condition (1). The condition (2) is called *property \mathcal{B}* in [22].(2) The *m-structure* m_X in Definition 3.1 coincides with *generalized topology* in the sense of Lugojan [20].

(3) Császár [9], [11] calls the subfamily m_X *generalized topology* if m_X satisfies (1) $\emptyset \in m_X$ and (2) the union of any family of subsets belonging to m_X belongs to m_X .

(4) MASHHOUR ET AL. [25] call m_X *supratopology* if m_X satisfies (1) $X \in m_X$ and (2) the union of any family of subsets belonging to m_X belongs to m_X .

(5) Let (X, τ) be a topological space. Then the family $\alpha(X)$ is a topology for X . The families $SO(X)$, $PO(X)$, $SPO(X)$ and $BO(X)$ are all *m-structures*.

DEFINITION 3.2. Let X be a nonempty set and m_X an *m-structure* on X . For a subset A of X , the *m_X -closure* of A is defined in [22] as follows:

$$m\text{Cl}(A) = \cap\{F : A \subset F, X - F \in m_X\}.$$

REMARK 3.2. Let (X, τ) be a topological space and A a subset of X . If $m_X = \tau$ (resp. $SO(X)$, $PO(X)$, $\alpha(X)$, $SPO(X)$, $BO(X)$), then we have

$$m\text{Cl}(A) = \text{Cl}(A) \text{ (resp. } s\text{Cl}(A), p\text{Cl}(A), \alpha\text{Cl}(A), sp\text{Cl}(A), b\text{Cl}(A)).$$

LEMMA 3.1. (MAKI ET AL. [22]). *Let X be a nonempty set and m_X an m -structure on X . For the m_X -closure, the following properties hold, where A and B are subsets of X :*

- (1) $A \subset m\text{Cl}(A)$,
- (2) $m\text{Cl}(\emptyset) = \emptyset$, $m\text{Cl}(X) = X$,
- (3) If $A \subset B$, then $m\text{Cl}(A) \subset m\text{Cl}(B)$,
- (4) $m\text{Cl}(m\text{Cl}(A)) = m\text{Cl}(A)$.

LEMMA 3.2. (POPA and NOIRI [33]). *Let (X, m_X) be an m -space and A a subset of X . Then $x \in m\text{Cl}(A)$ if and only if $U \cap A \neq \emptyset$ for every $U \in m_X$ containing x .*

LEMMA 3.3. (POPA and NOIRI [34]). *Let (X, m_X) be an m -space and A a subset of X . Then, the following properties hold:*

- (1) A is m_X -closed if and only if $m\text{Cl}(A) = A$,
- (2) $m\text{Cl}(A)$ is m_X -closed.

REMARK 3.3. Lemmas 3.1 and 3.2 hold without the condition (2) (property \mathcal{B}) in Definition 3.1.

4. $m\gamma$ -open sets

DEFINITION 4.1. Let (X, m_X) be an m -space. Let $m\gamma : m_X \rightarrow \mathcal{P}(X)$ be a function from m_X into $\mathcal{P}(X)$ such that $U \subset m\gamma(U)$ for each $U \in m_X$. The function $m\gamma$ is called an operation on m_X and the image $m\gamma(U)$ is denoted by $U^{m\gamma}$.

DEFINITION 4.2. Let (X, m_X) be an m -space and $m\gamma$ an operation on m_X . A subset A of X is said to be $m\gamma$ -open if for each $x \in A$ there exists $U \in m_X$ such that $x \in U \subset U^{m\gamma} \subset A$. The family of all $m\gamma$ -open sets of (X, m_X) is denoted by $m\gamma(X)$.

REMARK 4.1. We assume that the empty set \emptyset is an $m\gamma$ -open set, that is, $\emptyset \in m\gamma(X)$.

THEOREM 4.1. *Let (X, m_X) be an m -space. For $m\gamma(X)$, the following properties hold:*

- (1) $\emptyset, X \in m\gamma(X)$,
- (2) If $A_\alpha \in m\gamma(X)$ for each $\alpha \in \Lambda$, then $\cup_{\alpha \in \Lambda} A_\alpha \in m\gamma(X)$,
- (3) $m\gamma(X) \subset m_X$.

PROOF. (1) We assumed that $\emptyset \in m\gamma(X)$. Next, for each $x \in X$, there exists $X \in m_X$ such that $x \in X \subset X^{m\gamma}$ and hence $x \in X \subset X^{m\gamma} \subset X$. Therefore, we have $X \in m\gamma(X)$.

(2) For each $x \in \cup_{\alpha \in \Lambda} A_\alpha$, there exists an $\alpha_0 \in \Lambda$ such that $x \in A_{\alpha_0}$. Since A_{α_0} is $m\gamma$ -open, there exists $U \in m_X$ such that $x \in U \subset U^{m\gamma} \subset A_{\alpha_0}$ and hence $x \in U \subset U^{m\gamma} \subset \cup_{\alpha \in \Lambda} A_\alpha$. This shows that $\cup_{\alpha \in \Lambda} A_\alpha \in m\gamma(X)$.

(3) Let $A \in m\gamma(X)$. For each $x \in A$, there exists $U_x \in m_X$ such that $x \in U_x \subset U_x^{m\gamma} \subset A$. Hence, we have $A = \cup_{x \in A} U_x$ and hence $A \in m_X$. Therefore, we obtain that $m\gamma(X) \subset m_X$. ■

REMARK 4.2. (1) By (1) and (2) of Theorem 4.1, it turns out that $m\gamma(X)$ is an m -structure. However, in general, $m\gamma(X)$ is not closed under a finite intersection. It is shown in Example 2.8 of [30] that the intersection of two γ -open sets is not always γ -open.

(2) Let (X, τ) be a topological space. If $m_X = \tau$ (resp. $SO(X)$, $PO(X)$), then an $m\gamma$ -open set is said to be γ -open [30] (resp. *semi*- γ -open [18], *pre*- γ_p -open [3]). Moreover, by Theorem 4.1 we obtain the results established in Proposition 2.3 of [30] (resp. Theorem 2.13 of [18], Theorem 3.3 of [3]).

DEFINITION 4.3. An m -space (X, m_X) is said to be *$m\gamma$ -regular* if for each $x \in X$ and each $U \in m_X$ containing x , there exists $V \in m_X$ such that $x \in V \subset V^{m\gamma} \subset U$.

THEOREM 4.2. For an m -space (X, m_X) , the following properties are equivalent:

- (1) $m_X = m\gamma(X)$;
- (2) (X, m_X) is $m\gamma$ -regular;
- (3) For each $x \in X$ and each $U \in m_X$ containing x , there exists $W \in m\gamma(X)$ such that $x \in W \subset W^{m\gamma} \subset U$.

PROOF. (1) \Leftrightarrow (2): This follows immediately from Definition 4.3.

(2) \Rightarrow (3): For each $x \in X$ and each $U \in m_X$ containing x , by (2) there exists $W \in m_X$ such that $x \in W \subset W^{m\gamma} \subset U$. By (1), $m_X = m\gamma(X)$ and hence $W \in m\gamma(X)$ and $x \in W \subset W^{m\gamma} \subset U$. (3) \Rightarrow (1): By Theorem 4.1, we have $m\gamma(X) \subset m_X$ and we show that $m_X \subset m\gamma(X)$. Let $U \in m_X$. For any $x \in U$, by (3) there exists $W_x \in m\gamma(X)$ such that $x \in W_x \subset U$. Therefore, by Theorem 4.1 we have $U = \cup_{x \in U} W_x \in m\gamma(X)$. ■

REMARK 4.3. Let (X, τ) be a topological space. If $m_X = \tau$ (resp. $\text{SO}(X)$, $\text{PO}(X)$), then (X, τ) is said to be γ -regular [30] (resp. semi- γ -regular [18], pre- γ_p -regular [3]). Moreover, by Theorem 4.2 we obtain the results established in Proposition 2.4 of [30] (resp. Theorem 2.16 of [18], Theorem 3.6(i) of [3]).

DEFINITION 4.4. Let (X, m_X) be an m -space. An operation $m\gamma$ is said to be m -regular if for each $x \in X$ and each $U, V \in m_X$ containing x , there exists $W \in m_X$ such that $x \in W \subset W^{m\gamma} \subset U^{m\gamma} \cap V^{m\gamma}$.

THEOREM 4.3. Let (X, m_X) be an m -space. Then $m\gamma(X)$ is a topology for X if the operation $m\gamma$ is m -regular.

PROOF. By Theorem 4.1, $m\gamma(X)$ is an m -structure and we show that $m\gamma(X)$ is closed under a finite intersection. Let $A, B \in m\gamma(X)$ and $x \in A \cap B$. Then there exist $U, V \in m_X$ such that $x \in U \subset U^{m\gamma} \subset A$ and $x \in V \subset V^{m\gamma} \subset B$. Since $m\gamma$ is m -regular, there exists $W \in m_X$ such that $x \in W \subset W^{m\gamma} \subset U^{m\gamma} \cap V^{m\gamma} \subset A \cap B$. This shows that $A \cap B \in m\gamma(X)$. ■

REMARK 4.4. Let (X, τ) be a topological space. If $m_X = \tau$ (resp. $\text{SO}(X)$, $\text{PO}(X)$), then by Theorem 4.3 we obtain the results established in Proposition 2.9 of [30] (resp. Theorem 2.20 of [18], Theorem 3.8(iv) of [3]).

DEFINITION 4.5. Let (X, m_X) be an m -space. A subset A of X is said to be $m\gamma$ -closed if the complement of A is $m\gamma$ -open. The $m\gamma$ -closure of A , $m\gamma \text{Cl}(A)$, is defined as follows:

$$m\gamma \text{Cl}(A) = \bigcap \{F : A \subset F, X - F \in m\gamma(X)\}.$$

THEOREM 4.4. Let (X, m_X) be an m -structure on X . For the $m\gamma$ -closure, the following properties hold, where A and B are subsets of X : (1) $A \subset m\gamma \text{Cl}(A)$, (2) $m\gamma \text{Cl}(\emptyset) = \emptyset$, $m\gamma \text{Cl}(X) = X$, (3) If $A \subset B$, then $m\gamma \text{Cl}(A) \subset m\gamma \text{Cl}(B)$, (4) $m\gamma \text{Cl}(m\gamma \text{Cl}(A)) = m\gamma \text{Cl}(A)$, (5) A is $m\gamma$ -closed if and only if $m\gamma \text{Cl}(A) = A$, (6) $m\gamma \text{Cl}(A)$ is $m\gamma$ -closed, (7) $x \in m\gamma \text{Cl}(A)$ if and only if $U \cap A \neq \emptyset$ for every $m\gamma$ -open set U containing x .

PROOF. By Theorem 4.1, $m\gamma(X)$ is an m -structure and Theorem 4.4 is an immediate consequence of Lemmas 3.1, 3.2 and 3.3. ■

REMARK 4.5. Let (X, τ) be a topological space and A a subset of X . If $m_X = \tau$ (resp. $\text{SO}(X)$, $\text{PO}(X)$), then $m\gamma \text{Cl}(A)$ is denoted by $\tau_\gamma\text{-Cl}(A)$ [30] (resp. $\text{SO}(X)_\gamma\text{-Cl}(A)$ [18], $\text{PO}(X)_{\gamma_p}\text{-Cl}(A)$ [3]). Moreover, by Theorem 4.4

we obtain the results established in Proposition 3.3 of [30] (resp. Theorem 2.25 of [18], Theorem 3.11 and 3.13 of [3]).

REMARK 4.6. If we put $m_X = \alpha(X)$ (resp. $\text{BO}(X)$, $\text{SPO}(X)$), then we can obtain the corresponding operation $m\gamma$ and $m\gamma$ -open set and the results in Section 4 are valid for such operations and sets.

5. $m\gamma$ -compact spaces

First, we recall some generalizations of compact spaces used in the sequel.

DEFINITION 5.1. A topological space (X, τ) is said to be

- (1) *α -compact* [21] (resp. *semi-compact* [7], *strongly compact* [24], *b-compact* [14], *β -compact* [1]) if every cover of X by α -open (resp. semi-open, preopen, b-open, β -open) sets of X admits a finite subcover,
- (2) *quasi H-closed* [35] (resp. *s-closed* [12], *p-closed* [13], *b-closed* [31], *β -closed*) if for every cover of $\{V_\alpha : \alpha \in \Lambda\}$ of X by open (resp. semi-open, preopen, b-open, β -open) sets of X , there exists a finite subset Λ_0 of Λ such that $\cup_{\alpha \in \Lambda_0} \text{Cl}(V_\alpha) = X$ (resp. $\cup_{\alpha \in \Lambda_0} \text{sCl}(V_\alpha) = X$, $\cup_{\alpha \in \Lambda_0} \text{pCl}(V_\alpha) = X$, $\cup_{\alpha \in \Lambda_0} \text{bCl}(V_\alpha) = X$, $\cup_{\alpha \in \Lambda_0} \text{spCl}(V_\alpha) = X$).

DEFINITION 5.2. An m -space (X, m_X) is said to be *m -compact* [34] (resp. *m -closed* [34]) if for each cover $\{U_\alpha : \alpha \in \Lambda\}$ of X by m -open sets of X , there exists a finite subset Λ_0 of Λ such that $\cup_{\alpha \in \Lambda_0} U_\alpha = X$ (resp. $\cup_{\alpha \in \Lambda_0} \text{mCl}(U_\alpha) = X$).

DEFINITION 5.3. Let (X, m_X) be an m -space and $m\gamma$ an operation on m_X . Then (X, m_X) is said to be *$m\gamma$ -compact* if for each cover $\{U_\alpha : \alpha \in \Lambda\}$ of X by m -open sets of X , there exists a finite subset Λ_0 of Λ such that $\cup\{U_\alpha^{m\gamma} : \alpha \in \Lambda_0\} = X$.

REMARK 5.1. Let (X, m_X) be an m -space and $m\gamma$ an operation on m_X . If $m\gamma$ is the identity (resp. m_X -closure) operation, then an $m\gamma$ -compactness coincides with m -compactness (resp. m -closedness).

REMARK 5.2. Let (X, τ) be a topological space and $m_X = \tau$ (resp. $\text{SO}(X)$, $\text{PO}(X)$, $\text{BO}(X)$, $\text{SPO}(X)$).

(1) If $m\gamma$ is the identity operation, then “ $m\gamma$ -compact” coincides with compact (resp. semi-compact, strongly compact, *b*-compact, β -compact),

(2) If $m\gamma$ is the m_X -closure operation, then “ $m\gamma$ -compact” coincides with quasi H -closed (resp. s -closed, p -closed, b -closed, β -closed).

THEOREM 5.1. *For an $m\gamma$ -regular m -space (X, m_X) , the following properties are equivalent. The implications $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4)$ hold without the assumption “ $m\gamma$ -regular” on (X, m_X) .*

- (1) (X, m_X) is m -compact;
- (2) (X, m_X) is $m\gamma$ -compact;
- (3) $(X, m\gamma(X))$ is m -compact;
- (4) $(X, m\gamma(X))$ is $m\gamma$ -compact.

PROOF. (1) \Rightarrow (2): Let (X, m_X) be m -compact. For any cover $\{U_\alpha : \alpha \in \Lambda\}$ of X by m -open sets of X , there exists a finite subset Λ_0 of Λ such that $X = \cup\{U_\alpha : \alpha \in \Lambda_0\} \subset \cup\{U_\alpha^{m\gamma} : \alpha \in \Lambda_0\}$. Therefore, (X, m_X) is $m\gamma$ -compact.

(2) \Rightarrow (3): Let (X, m_X) be $m\gamma$ -compact and $\{U_\alpha : \alpha \in \Lambda\}$ a cover of X by $m\gamma$ -open sets of X . For each $x \in X$, there exists $\alpha(x) \in \Lambda$ such that $x \in U_{\alpha(x)}$. Since $U_{\alpha(x)}$ is $m\gamma$ -open, there exists $V_{\alpha(x)} \in m_X$ such that $x \in V_{\alpha(x)} \subset V_{\alpha(x)}^{m\gamma} \subset U_{\alpha(x)}$. The family $\{V_{\alpha(x)} : x \in X\}$ is an m -open cover of X and (X, m_X) is $m\gamma$ -compact, there exist a finite number of points, say, $x_1, x_2, \dots, x_n \in X$ such that $\cup_{i=1}^n V_{\alpha(x_i)}^{m\gamma} = X$; hence $\cup_{i=1}^n U_{\alpha(x_i)} = X$. This shows that $(X, m\gamma(X))$ is m -compact.

(3) \Rightarrow (4): By Theorem 4.1, $m\gamma(X)$ is an m -structure and it follows from the same argument as (1) \Rightarrow (2) that $(X, m\gamma(X))$ is $m\gamma$ -compact.

(4) \Rightarrow (1): Suppose that (X, m_X) is $m\gamma$ -regular. Let $(X, m\gamma(X))$ be $m\gamma$ -compact. By Theorem 4.2, $m_X = m\gamma(X)$ and (X, m_X) is $m\gamma$ -compact. Let $\{U_\alpha : \alpha \in \Lambda\}$ be any m -open cover of X . For each $x \in X$, there exists $\alpha(x) \in \Lambda$ such that $x \in U_{\alpha(x)}$. Since (X, m_X) is $m\gamma$ -regular, there exists $V_{\alpha(x)} \in m_X$ such that $x \in V_{\alpha(x)} \subset V_{\alpha(x)}^{m\gamma} \subset U_{\alpha(x)}$. Since $\{V_{\alpha(x)} : x \in X\}$ is an m -open cover of X and (X, m_X) is $m\gamma$ -compact, there exist a finite number of points, say, $x_1, x_2, \dots, x_n \in X$ such that $\cup_{i=1}^n V_{\alpha(x_i)}^{m\gamma} = X$; hence $\cup_{i=1}^n U_{\alpha(x_i)} = X$. This shows that (X, m_X) is m -compact. ■

DEFINITION 5.4. Let (X, m_X) be an m -space and $m\gamma$ an operation on m_X . A filterbase \mathcal{F} on X is said to be (1) $m\gamma$ -converge to a point $x_0 \in X$ if for each m -open set U containing x_0 , there exists $F \in \mathcal{F}$ such that $F \subset U^{m\gamma}$, (2)

$m\gamma$ -accumulate at $x_0 \in X$ if $F \cap U^{m\gamma} \neq \emptyset$ for every $F \in \mathcal{F}$ and every m -open set U containing x_0 .

LEMMA 5.1. *If a maximal filterbase \mathcal{F} $m\gamma$ -accumulates at $x_0 \in X$, then \mathcal{F} $m\gamma$ -converges to x_0 .*

PROOF. Let \mathcal{F} be a maximal filterbase which $m\gamma$ -accumulates at x_0 . If \mathcal{F} does not $m\gamma$ -converge to x_0 , then there exists $U_0 \in m_X$ containing x_0 such that $F \cap U_0^{m\gamma} \neq \emptyset$ and $F \cap (X - U_0^{m\gamma}) \neq \emptyset$ for every $F \in \mathcal{F}$. Then $\mathcal{F} \cup \{F \cap U_0^{m\gamma} : F \in \mathcal{F}\}$ is a filterbase on X which strictly contains \mathcal{F} . This is contrary that \mathcal{F} is a maximal filterbase. ■

THEOREM 5.2. *Let (X, m_X) be an m -space and $m\gamma$ an operation on m_X . Then the following properties are equivalent:*

- (1) (X, m_X) is $m\gamma$ -compact;
- (2) Every maximal filterbase $m\gamma$ -converges to some point of X ;
- (3) Every filterbase $m\gamma$ -accumulates at some point of X .

PROOF. (1) \Rightarrow (2): Let (X, m_X) be $m\gamma$ -compact and \mathcal{F}_0 a maximal filterbase on X . Suppose that \mathcal{F}_0 does not $m\gamma$ -converge to any point of X . By Lemma 5.1, \mathcal{F}_0 does not $m\gamma$ -accumulate at any point of X . For each $x \in X$, there exists $F_x \in \mathcal{F}_0$ and $U_x \in m_X$ containing x such that $F_x \cap U_x^{m\gamma} = \emptyset$. The family $\{U_x : x \in X\}$ is a cover of X by m -open sets of X . By (1), there exists a finite number of points, says, $x_1, x_2, \dots, x_n \in X$ such that $X = \cup_{i=1}^n U_{x_i}^{m\gamma}$. Since \mathcal{F}_0 is a filterbase on X , there exists $F_0 \in \mathcal{F}_0$ such that $F_0 \subset \cap_{i=1}^n F_{x_i}$. Then, we have $F_0 = F_0 \cap (\cup_{i=1}^n U_{x_i}^{m\gamma}) = \cup_{i=1}^n (F_0 \cap U_{x_i}^{m\gamma}) \subset \cup_{i=1}^n (F_{x_i} \cap U_{x_i}^{m\gamma}) = \emptyset$. This is contrary that $F_0 \in \mathcal{F}_0$. Therefore, \mathcal{F}_0 $m\gamma$ -converges to some point of X .

(2) \Rightarrow (3): Let \mathcal{F} a filterbase on X . There exists a maximal filterbase \mathcal{F}_0 such that $\mathcal{F} \subset \mathcal{F}_0$. By (2) \mathcal{F}_0 $m\gamma$ -converges to some point $x_0 \in X$. For any $U \in m_X$ containing x_0 , there exists $F_0 \in \mathcal{F}_0$ such that $F_0 \subset U^{m\gamma}$. For any $F \in \mathcal{F}$, $F \in \mathcal{F}_0$ and $\emptyset \neq F \cap F_0 \subset F \cap U^{m\gamma}$. This shows that \mathcal{F} $m\gamma$ -accumulates at $x_0 \in X$.

(3) \Rightarrow (1): Suppose that (3) holds and (X, m_X) is not $m\gamma$ -compact. Then there exists a cover $\{U_\alpha : \alpha \in \Lambda\}$ of X by m -open sets of X such that $X \neq \cup\{U_\alpha^{m\gamma} : \alpha \in \Gamma\}$ for every finite subset Γ of Λ . Now, by $\Gamma(\Lambda)$ we denote the family of all finite subsets of Λ and let

$$\mathcal{F} = \{X - \cup_{\alpha \in \Lambda_\lambda} U_\alpha^{m\gamma} : \Lambda_\lambda \in \Gamma(\Lambda)\}.$$

Then \mathcal{F} is a filterbase on X and by (3) \mathcal{F} $m\gamma$ -accumulates at some point $x_0 \in X$. Since $\{U_\alpha : \alpha \in \Lambda\}$ is a cover X by m -open sets of X , there exists $\alpha(x_0) \in \Lambda$ such that $x_0 \in U_{\alpha(x_0)}$. Then, we have $X - U_{\alpha(x_0)}^{m\gamma} \in \mathcal{F}$ and $(X - U_{\alpha(x_0)}^{m\gamma}) \cap U_{\alpha(x_0)}^{m\gamma} = \emptyset$. This contradicts that \mathcal{F} $m\gamma$ -accumulates at x_0 . Therefore, (X, m_X) is $m\gamma$ -compact. ■

REMARK 5.3. (1) Let (X, m_X) be an m -space and $m\gamma$ an operation on m_X . If $m\gamma$ is the identity (resp. m_X -closure) operation, then as a corollary of Theorem 5.2, we obtain the results established in Theorem 3.1 (resp. Theorem 5.2) of [27].

(2) Let (X, τ) be a topological space, $m_X = \tau$ and $m\gamma$ be an operation α due to Kasahara. As a corollary of Theorem 5.2, we obtain the results established in Theorem 8 of [17].

6. Examples

Let (X, τ) be a topological space, $m_X = \tau$, $\alpha(X)$, $SO(X)$, $PO(X)$, $BO(X)$, or $SPO(X)$ and $m\gamma = Cl$, αCl , sCl , pCl , bCl , or $spCl$. For each m_X , we take an $m\gamma$ -operation and investigate the space which corresponds to the $m\gamma$ -compact space. If $m_X = \tau$ (resp. $\alpha(X)$, $SO(X)$, $PO(X)$, $BO(X)$, $SPO(X)$) and $m\gamma$ is the identity operation, then the corresponding space is compact (resp. α -compact, semi-compact, strongly compact, b -compact, β -compact), as we pointed out in Remark 5.2.

A subset A of a topological space (X, τ) is said to be *regular open* (resp. *regular closed*) if $Int(Cl(A)) = A$ (resp. $Cl(Int(A)) = A$).

DEFINITION 6.1. A topological space (X, τ) is said to be (1) *S-closed* [38] if for every semi-open cover $\{V_\alpha : \alpha \in \Lambda\}$ of X , there exists a finite subset Λ_0 of Λ such that $\cup\{Cl(V_\alpha) : \alpha \in \Lambda_0\} = X$, equivalently if every regular closed cover of X admits a finite subcover, (2) *nearly-compact* [37] if for every open cover $\{V_\alpha : \alpha \in \Lambda\}$ of X , there exists a finite subset Λ_0 of Λ such that $\cup\{Int(Cl(V_\alpha)) : \alpha \in \Lambda_0\} = X$, equivalently if every regular open cover of X admits a finite subcover.

LEMMA 6.1. (ANDRIJEVIĆ [5]). *Let A be a subset of a topological space (X, τ) . Then the following properties hold:*

- (1) $\alpha Cl(A) = A \cup Cl(Int(Cl(A)))$,
- (2) $sCl(A) = A \cup Int(Cl(A))$,

- (3) $\text{spCl}(A) = A \cup \text{Int}(\text{Cl}(\text{Int}(A)))$,
(4) $\text{pCl}(A) = A \cup \text{Cl}(\text{Int}(A))$, (5) $\text{bCl}(A) = \text{sCl}(A) \cap \text{pCl}(A)$.

LEMMA 6.2. *For a subset A of a topological space (X, τ) , the following properties hold:*

$$\begin{array}{c} \text{Cl}(A) \supset \alpha \text{ Cl}(A) \supset \text{sCl}(A) \\ \cup \qquad \qquad \cup \\ \text{pCl}(A) \supset \text{bCl}(A) \supset \text{spCl}(A) \end{array}$$

PROOF. This follows from the relations among τ , $\alpha(X)$, $\text{SO}(X)$, $\text{PO}(X)$, $\text{BO}(X)$, and $\text{SPO}(X)$. \blacksquare

EXAMPLE 6.1. Let (X, τ) be a topological space and $m_X = \tau$. (1) If $m\gamma = \text{Cl}$, $\alpha \text{ Cl}$ or pCl , then (X, τ) is quasi H-closed. (2) If $m\gamma = \text{sCl}$, bCl or spCl , then (X, τ) is nearly-compact.

PROOF. (1) By Lemma 6.1, we have $\text{pCl}(U) = U \cup \text{Cl}(\text{Int}(U)) = \text{Cl}(U)$ for every open set U of X . Moreover, by Lemma 6.2, we have $\text{Cl}(U) \supset \alpha \text{ Cl}(U) \supset \text{pCl}(U)$ and hence $\text{Cl}(U) = \alpha \text{ Cl}(U) = \text{pCl}(U)$ for every $U \in \tau$. Therefore, (X, τ) is quasi H-closed.

(2) By Lemma 6.1, $\text{sCl}(U) = U \cup \text{Int}(\text{Cl}(U)) = \text{Int}(\text{Cl}(U)) = U \cup \cup \text{Int}(\text{Cl}(\text{Int}(U))) = \text{spCl}(U)$ for every open set U of X . Therefore, by Lemma 6.2 $\text{sCl}(U) = \text{bCl}(U) = \text{spCl}(U) = \text{Int}(\text{Cl}(U))$ for every $U \in \tau$. Therefore, (X, τ) is nearly-compact. \blacksquare

EXAMPLE 6.2. Let (X, τ) be a topological space and $m_X = \alpha(X)$.

- (1) If $m\gamma = \text{Cl}$, $\alpha \text{ Cl}$ or pCl , then (X, τ) is quasi H-closed.
(2) If $m\gamma = \text{sCl}$, bCl or spCl , then (X, τ) is nearly-compact.

PROOF. (1) First, for each $U \in \alpha(X)$, $U \subset \text{Int}(\text{Cl}(\text{Int}(U)))$ and $\text{Cl}(U) \subset \text{Cl}(\text{Int}(U))$ and hence $\text{Cl}(U) = \text{Cl}(\text{Int}(U))$. For each $U \in \alpha(X)$, $\alpha \text{ Cl}(U) = U \cup \text{Cl}(\text{Int}(\text{Cl}(U))) = U \cup \text{Cl}(\text{Int}(U)) = \text{pCl}(U)$ and $U \cup \text{Cl}(\text{Int}(U)) = \text{Cl}(U)$. Therefore, $\alpha \text{ Cl}(U) = \text{pCl}(U) = \text{Cl}(U) = \text{Cl}(\text{Int}(U))$ for every $U \in \alpha(X)$. Suppose that (X, m_X) is $m\gamma$ -compact, where $m_X = \alpha(X)$ and $m\gamma = \text{Cl}$, $\alpha \text{ Cl}$ or pCl . Let $\{V_\alpha : \alpha \in \Lambda\}$ be an open cover of X . Since $\tau \subset \alpha(X)$, $\{V_\alpha : \alpha \in \Lambda\}$ is an α -open cover of X and there exists a finite subset Λ_0 of Λ such that $\cup \{\text{Cl}(V_\alpha) : \alpha \in \Lambda_0\} = X$. Hence (X, τ) is quasi H-closed. Conversely, suppose that (X, τ) is quasi H-closed. Let $\{U_\alpha : \alpha \in \Lambda\}$ be an α -open cover of X . Then $\{\text{Int}(\text{Cl}(\text{Int}(U_\alpha))) : \alpha \in \Lambda\}$ is an open cover

of X and there exists a finite subset Λ_0 of Λ such that $\cup\{\text{Cl}(\text{Int}(U_\alpha)) : \alpha \in \Lambda_0\} = X$. This shows that (X, m_X) is $m\gamma$ -compact, where $m\gamma = \text{Cl}$, αCl or $p\text{Cl}$.

(2) First, for each $U \in \alpha(X)$, $\text{Cl}(U) = \text{Cl}(\text{Int}(U))$ and hence $\text{Int}(\text{Cl}(U)) = \text{Int}(\text{Cl}(\text{Int}(U)))$. For each $U \in \alpha(X)$, by Lemma 6.1 $s\text{Cl}(U) = U \cup \text{Int}(\text{Cl}(U)) = \text{Int}(\text{Cl}(U)) = \text{Int}(\text{Cl}(\text{Int}(U))) = U \cup \text{Int}(\text{Cl}(\text{Int}(U))) = s\text{Cl}(U)$. By Lemma 6.2, we have $s\text{Cl}(U) = b\text{Cl}(U) = sp\text{Cl}(U) = \text{Int}(\text{Cl}(U))$ for every $U \in \alpha(X)$. Suppose that (X, m_X) is $m\gamma$ -compact, where $m_X = \alpha(X)$ and $m\gamma = s\text{Cl}$, $b\text{Cl}$ or $sp\text{Cl}$. Let $\{V_\alpha : \alpha \in \Lambda\}$ be an open cover of X . Since $\tau \subset \alpha(X)$, $\{V_\alpha : \alpha \in \Lambda\}$ is an α -open cover of X and there exists a finite subset Λ_0 of Λ such that $\cup\{s\text{Cl}(V_\alpha) : \alpha \in \Lambda_0\} = X$; hence $\cup\{\text{Int}(\text{Cl}(V_\alpha)) : \alpha \in \Lambda_0\} = X$. Hence (X, τ) is nearly-compact. Conversely, suppose that (X, τ) is nearly-compact. Let $\{U_\alpha : \alpha \in \Lambda\}$ be an α -open cover of X . Since $U_\alpha \subset \text{Int}(\text{Cl}(\text{Int}(U_\alpha))) \subset \text{Int}(\text{Cl}(U_\alpha))$ for each $\alpha \in \Lambda$, $\{\text{Int}(\text{Cl}(U_\alpha)) : \alpha \in \Lambda\}$ is a regular open cover of X and there exists a finite subset Λ_0 of Λ such that $\cup\{\text{Int}(\text{Cl}(U_\alpha)) : \alpha \in \Lambda_0\} = X$. This shows that (X, m_X) is $m\gamma$ -compact, where $m\gamma = s\text{Cl}$, $b\text{Cl}$ or $sp\text{Cl}$. ■

EXAMPLE 6.3. Let (X, τ) be a topological space and $m_X = \text{PO}(X)$.

- (1) If $m\gamma = \text{Cl}$ or αCl , then (X, τ) is quasi H -closed.
- (2) If $m\gamma = p\text{Cl}$, then (X, τ) is p -closed.
- (3) If $m\gamma = s\text{Cl}$, then (X, τ) is nearly-compact.

PROOF. (1) For every $U \in \text{PO}(X)$, $U \subset \text{Int}(\text{Cl}(U))$ and $\text{Cl}(U) = \text{Cl}(\text{Int}(\text{Cl}(U)))$. By Lemma 6.1, we have $\alpha\text{Cl}(U) = U \cup \text{Cl}(\text{Int}(\text{Cl}(U))) = \text{Cl}(U)$. Suppose that (X, m_X) is $m\gamma$ -compact, where $m_X = \text{PO}(X)$ and $m\gamma = \text{Cl}$ or αCl . Let $\{V_\alpha : \alpha \in \Lambda\}$ be an open cover of X . Since $\tau \subset \text{PO}(X)$, $\{V_\alpha : \alpha \in \Lambda\}$ is a preopen cover of X and there exists a finite subset Λ_0 of Λ such that $\cup\{\text{Cl}(V_\alpha) : \alpha \in \Lambda_0\} = X$. Hence (X, τ) is quasi H -closed. Conversely, suppose that (X, τ) is quasi H -closed. Let $\{U_\alpha : \alpha \in \Lambda\}$ be a preopen cover of X . Then $\{\text{Int}(\text{Cl}(U_\alpha)) : \alpha \in \Lambda\}$ is an open cover of X and there exists a finite subset Λ_0 of Λ such that $\cup\{\text{Cl}(U_\alpha) : \alpha \in \Lambda_0\} = X$. This shows that (X, m_X) is $m\gamma$ -compact, where $m\gamma = \text{Cl}$ or αCl .

(2) This is obvious by the definition.

(3) For every $U \in \text{PO}(X)$, by Lemma 6.1, we have $s\text{Cl}(U) = U \cup \cup \text{Int}(\text{Cl}(U)) = \text{Int}(\text{Cl}(U))$. Suppose that (X, m_X) is $m\gamma$ -compact, where $m_X = \text{PO}(X)$ and $m\gamma = s\text{Cl}$. Let $\{V_\alpha : \alpha \in \Lambda\}$ be an open cover of X . Since $\tau \subset \text{PO}(X)$, $\{V_\alpha : \alpha \in \Lambda\}$ is a preopen cover of X and there

exists a finite subset Λ_0 of Λ such that $\cup\{\text{sCl}(V_\alpha) : \alpha \in \Lambda_0\} = X$; hence $\cup\{\text{Int}(\text{Cl}(V_\alpha)) : \alpha \in \Lambda_0\} = X$. Hence (X, τ) is nearly-compact. Conversely, suppose that (X, τ) is nearly-compact. Let $\{U_\alpha : \alpha \in \Lambda\}$ be a preopen cover of X . Since $U_\alpha \subset \text{Int}(\text{Cl}(U_\alpha))$ for each $\alpha \in \Lambda$, $\{\text{Int}(\text{Cl}(U_\alpha)) : \alpha \in \Lambda\}$ is a regular open cover of X and there exists a finite subset Λ_0 of Λ such that $\cup\{\text{Int}(\text{Cl}(U_\alpha)) : \alpha \in \Lambda_0\} = X$; hence $\cup\{\text{sCl}(U_\alpha) : \alpha \in \Lambda_0\} = X$. This shows that (X, m_X) is $m\gamma$ -compact, where $m\gamma = \text{sCl}$. ■

EXAMPLE 6.4. Let (X, τ) be a topological space and $m_X = \text{SO}(X)$.

- (1) If $m\gamma = \text{Cl}$, αCl or pCl , then (X, τ) is S -closed.
- (2) If $m\gamma = \text{sCl}$, bCl or spCl , then (X, τ) is s -closed.

PROOF. (1) For every $U \in \text{SO}(X)$, $U \subset \text{Cl}(\text{Int}(U))$ and $\text{Cl}(U) = \text{Cl}(\text{Int}(U))$. By Lemma 6.1, $\text{pCl}(U) = U \cup \text{Cl}(\text{Int}(U)) = \text{Cl}(\text{Int}(U)) = \text{Cl}(U)$. By Lemma 6.2, we have $\text{Cl}(U) = \alpha\text{Cl}(U) = \text{pCl}(U) = \text{Cl}(\text{Int}(U))$ for every $U \in \text{SO}(X)$. Suppose that (X, m_X) is $m\gamma$ -compact, where $m_X = \text{SO}(X)$ and $m\gamma = \text{Cl}$, αCl or pCl . Let $\{V_\alpha : \alpha \in \Lambda\}$ be a semi-open cover of X . There exists a finite subset Λ_0 of Λ such that $\cup\{\text{Cl}(V_\alpha) : \alpha \in \Lambda_0\} = X$. This shows that (X, τ) is S -closed. The converse is obvious.

(2) For every $U \in \text{SO}(X)$, $\text{Cl}(U) = \text{Cl}(\text{Int}(U))$ and hence by Lemma 6.1 $\text{sCl}(U) = U \cup \text{Int}(\text{Cl}(U)) = U \cup \text{Int}(\text{Cl}(\text{Int}(U))) = \text{spCl}(U)$. By Lemma 6.2, we have $\text{sCl}(U) = \text{bCl}(U) = \text{spCl}(U)$ for every $U \in \text{SO}(X)$. Therefore, if $m\gamma = \text{sCl}$, bCl or spCl , then (X, τ) is s -closed. Conversely, if (X, τ) is s -closed, then (X, m_X) is $m\gamma$ -compact with respect to $m\gamma = \text{sCl}$, bCl or spCl . ■

EXAMPLE 6.5. Let (X, τ) be a topological space and $m_X = \text{BO}(X)$.

- (1) If $m\gamma = \text{Cl}$ or αCl , then (X, τ) is S -closed.
- (2) If $m\gamma = \text{bCl}$, then (X, τ) is b -closed.

PROOF. (1) For every $U \in \text{BO}(X)$, $U \subset \text{Int}(\text{Cl}(U)) \cup \text{Cl}(\text{Int}(U)) \subset \text{Cl}(U)$ and $\text{Cl}(U) \subset \text{Cl}(\text{Int}(\text{Cl}(U)) \cup \text{Cl}(\text{Int}(U))) = \text{Cl}(\text{Int}(\text{Cl}(U))) \subset \text{Cl}(U)$. By Lemma 6.1, we have $\text{Cl}(U) = \text{Cl}(\text{Int}(\text{Cl}(U))) = \text{Cl}(\text{Int}(\text{Cl}(U))) \cup U = \alpha\text{Cl}(U)$ for every $U \in \text{BO}(X)$. Suppose that (X, m_X) is $m\gamma$ -compact, where $m_X = \text{BO}(X)$ and $m\gamma = \text{Cl}$ or αCl . Let $\{V_\alpha : \alpha \in \Lambda\}$ be a semi-open cover of X . Since $\text{SO}(X) \subset \text{BO}(X)$, $\{V_\alpha : \alpha \in \Lambda\}$ is a b -open cover of X and there exists a finite subset Λ_0 of Λ such that $\cup\{\text{Cl}(V_\alpha) : \alpha \in \Lambda_0\} = X$. Hence (X, τ) is S -closed. Conversely, suppose that (X, τ) is S -closed. Let $\{U_\alpha : \alpha \in \Lambda\}$ be a b -open cover of X . Since $U_\alpha \subset \text{Cl}(\text{Int}(\text{Cl}(U_\alpha)))$ for each $\alpha \in \Lambda$, $\{\text{Cl}(\text{Int}(\text{Cl}(U_\alpha))) : \alpha \in \Lambda\}$ is a regular closed cover of X and there

exists a finite subset Λ_0 of Λ such that $\cup\{\text{Cl}(\text{Int}(\text{Cl}(U_\alpha))) : \alpha \in \Lambda_0\} = \cup\{\text{Cl}(U_\alpha) : \alpha \in \Lambda_0\} = X$. This shows that (X, m_X) is $m\gamma$ -compact, where $m\gamma = \text{Cl}$ or $\alpha \text{ Cl}$.

(2) This follows from the definition. ■

EXAMPLE 6.6. Let (X, τ) be a topological space and $m_X = \text{SPO}(X)$.

- (1) If $m\gamma = \text{Cl}$ or $\alpha \text{ Cl}$, then (X, τ) is S -closed.
- (2) If $m\gamma = \text{spCl}$, then (X, τ) is β -closed.

PROOF. (1) For every $U \in \text{SPO}(X)$, $U \subset \text{Cl}(\text{Int}(\text{Cl}(U)))$ and $\text{Cl}(U) = \text{Cl}(\text{Int}(\text{Cl}(U))) = U \cup \text{Cl}(\text{Int}(\text{Cl}(U))) = \alpha \text{ Cl}(U)$. Suppose that (X, m_X) is $m\gamma$ -compact, where $m_X = \text{SPO}(X)$ and $m\gamma = \text{Cl}$ or $\alpha \text{ Cl}$. Let $\{V_\alpha : \alpha \in \Lambda\}$ be a semi-open cover of X . Since $\text{SO}(X) \subset \text{SPO}(X)$, $\{V_\alpha : \alpha \in \Lambda\}$ is a semi-preopen cover of X and there exists a finite subset Λ_0 of Λ such that $\cup\{\text{Cl}(V_\alpha) : \alpha \in \Lambda_0\} = X$. Hence (X, τ) is S -closed. Conversely, suppose that (X, τ) is S -closed. Let $\{U_\alpha : \alpha \in \Lambda\}$ be a semi-preopen cover of X . Since $U_\alpha \subset \text{Cl}(\text{Int}(\text{Cl}(U_\alpha)))$ for each $\alpha \in \Lambda$, $\{\text{Cl}(\text{Int}(\text{Cl}(U_\alpha))) : \alpha \in \Lambda\}$ is a regular closed cover of X and there exists a finite subset Λ_0 of Λ such that $\cup\{\text{Cl}(\text{Int}(\text{Cl}(U_\alpha))) : \alpha \in \Lambda_0\} = \cup\{\text{Cl}(U_\alpha) : \alpha \in \Lambda_0\} = X$. This shows that (X, m_X) is $m\gamma$ -compact, where $m\gamma = \text{Cl}$ or $\alpha \text{ Cl}$.

(2) This follows from the definition. ■

QUESTIONS. Whether $m\gamma$ -compact spaces corresponding to the following cases are identical (or equivalent) to certain known spaces or not?

- (1) In Example 6.3, $m_X = \text{PO}(X)$ and $m\gamma = \text{bCl}$ and spCl .
- (2) In Example 6.5, $m_X = \text{BO}(X)$ and $m\gamma = \text{pCl}$, sCl and spCl .
- (3) In Example 6.6, $m_X = \text{SPO}(X)$ and $m\gamma = \text{pCl}$, sCl and bCl .

7. New forms of $m\gamma$ -compactness

Let (X, τ) be a topological space and A a subset of X . A point $x \in X$ is called a θ -cluster (resp. δ -cluster) point of A if $\text{Cl}(V) \cap A \neq \emptyset$ (resp. $\text{Int}(\text{Cl}(V)) \cap A \neq \emptyset$) for every open set V containing x . The set of all θ -cluster (resp. δ -cluster) points of A is called the θ -closure (resp. δ -closure) of A and is denoted by $\text{Cl}_\theta(A)$ (resp. $\text{Cl}_\delta(A)$) [39]. A subset A is said to be θ -closed (resp. δ -closed) if $\text{Cl}_\theta(A) = A$ (resp. $\text{Cl}_\delta(A) = A$). The complement of a θ -closed (resp. δ -closed) set is said to be θ -open (resp. δ -open). The union of all θ -open (resp. δ -open) sets contained in the subset A is called the θ -interior (resp. δ -interior) of A and is denoted by $\text{Int}_\theta(A)$ (resp. $\text{Int}_\delta(A)$).

DEFINITION 7.1. A subset A of a topological space (X, τ) is said to be(1) δ -semiopen [32] (resp. θ -semiopen [6]) if $A \subset \text{Cl}(\text{Int}_\delta(A))$ (resp. $A \subset \subset \text{Cl}(\text{Int}_\theta(A))$),(2) δ -preopen [36] (resp. θ -preopen [29]) if $A \subset \text{Int}(\text{Cl}_\delta(A))$ (resp. $A \subset \text{Int}(\text{Cl}_\theta(A))$),(3) δ -sp-open [16] (resp. θ -sp-open [29]) if $A \subset \subset \text{Cl}(\text{Int}(\text{Cl}_\delta(A)))$ (resp. $A \subset \text{Cl}(\text{Int}(\text{Cl}_\theta(A)))$).

By δ SO(X) (resp. δ PO(X), δ SPO(X), θ SO(X), θ PO(X), θ SPO(X)), we denote the collection of all δ -semiopen (resp. δ -preopen, δ -sp-open, θ -semiopen, θ -preopen, θ -sp-open) sets of a topological space (X, τ) . These six collections are all m -structures (with property \mathcal{B}). Actually, in [29] and [6], the following relationships are known:

$$\begin{array}{ccccccccc} \theta\text{-open} & \Rightarrow & \delta\text{-open} & \Rightarrow & \text{open} & \Rightarrow & \text{preopen} & \Rightarrow & \delta\text{-preopen} \Rightarrow \theta\text{-preopen} \\ \Downarrow & & \Downarrow & & \Downarrow & & \Downarrow & & \Downarrow \\ \theta\text{-semiopen} & \Rightarrow & \delta\text{-semiopen} & \Rightarrow & \text{semi-open} & \Rightarrow & \text{sp-open} & \Rightarrow & \delta\text{-sp-open} \Rightarrow \theta\text{-sp-open} \end{array}$$

DEFINITION 7.2. Let (X, τ) be a topological space. A subset A of X is said to be δ -semiclosed [32] (resp. δ -preclosed [36], δ -sp-closed [16], θ -semiclosed [6], θ -preclosed [29], θ -sp-closed [29]) if the complement of A is δ -semiopen (resp. δ -preopen, δ -sp-open, θ -semiopen, θ -preopen, θ -sp-open).

DEFINITION 7.3. Let (X, τ) be a topological space and A a subset of X . The intersection of all δ -semiclosed (resp. δ -preclosed, δ -sp-closed, θ -semiclosed, θ -preclosed, θ -sp-closed) sets of (X, τ) containing A is called the δ -semiclosure [32] (resp. δ -closure [36], δ -sp-closure, θ -semiclosure, θ -closure, θ -sp-closure) of A and is denoted by $s\text{Cl}_\delta(A)$ (resp. $p\text{Cl}_\delta(A)$, $s\text{Cl}_\theta(A)$, $p\text{Cl}_\theta(A)$, $s\text{Cl}_\theta(A)$, $p\text{Cl}_\theta(A)$).

For subsets of a topological space (X, τ) , we can define many new variations of $m\gamma$ -compactness. For example, in case $m_X = \delta$ SO(X), δ PO(X), δ SPO(X), θ SO(X), θ PO(X) or θ SPO(X) we can define new types of operations on each family of m_X as follows: $s\text{Cl}_\delta$, $p\text{Cl}_\delta$, $s\text{Cl}_\theta$, $p\text{Cl}_\theta$ or $s\text{Cl}_\theta$. Then, we can define new types of $m\gamma$ -compactness.

REMARK 7.1. Let (X, τ) be a topological space.

(1) We can define a new form of $m\gamma$ -compactness if we choose a family and an operation from the following m_X and $m\gamma$:

$$\begin{aligned} m_X &= \delta \text{ SO}(X), \delta \text{ PO}(X), \delta \text{ SPO}(X), \theta \text{ SO}(X), \theta \text{ PO}(X), \theta \text{ SPO}(X), \\ m\gamma &= s\text{Cl}_\delta, p\text{Cl}_\delta, s\text{Cl}_\theta, p\text{Cl}_\theta, s\text{Cl}_\theta. \end{aligned}$$

(2) Furthermore, by adding τ , $\alpha(X)$, $\text{SO}(X)$, $\text{PO}(X)$, $\text{BO}(X)$ and $\text{SPO}(X)$ to the above m_X and Cl, α Cl, sCl, pCl, bCl, spCl to the above operations, we can define the further new forms of $m\gamma$ -compactness.

CONCLUSION. We can apply the results established in Sections 4 and 5 to all $m\gamma$ -compactness suggested in Section 7.

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A RANDOM WALK ON THE PLANE

By

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1. Introduction and the main result

Consider the well-known symmetric random walk. That is a particle moves on the vertical axis starting from the origin. It moves at times $1, 2, \dots$ one unit step upward with probability $1/2$, one unit step downward with probability $1/2$. The steps are independent. Let S_n denote the position of the particle at time n (let $S_0 = 0$).

Another popular description of S_1, S_2, \dots is given by the coin tossing experiment. Consider independent tossings of a fair coin. Let $X_k = 1$ if the result of the k th trial is head and $X_k = -1$ otherwise. We have $\mathbf{P}(X_k = 1) = \mathbf{P}(X_k = -1) = 1/2$. Then $S_n = X_1 + \dots + X_n$ is the cumulative excess of heads over tails. (S_n can be interpreted as the gamblers accumulated net gain.)

Both of the above approaches have two-dimensional extensions. The extension of the first approach is the well-known symmetric random walk on the plane. It means that a particle starts at the origin and moves one unit in one of the four directions parallel to the x - and y -axes. Each direction has probability $1/4$. In this case the time is $n = 0, 1, 2, \dots$ but the phase space is two-dimensional.

On the other hand, a possible two-dimensional extension of the second approach leads to a multiindex process (a random field) with one-dimensional phase space. That is let $X_{(i,j)}$, $i, j = 1, 2, \dots$, be independent random variables

with $\mathbf{P}(X_{(i,j)} = 1) = \mathbf{P}(X_{(i,j)} = -1) = 1/2$. Let

$$S_{(m,n)} = \sum_{i=1}^m \sum_{j=1}^n X_{(i,j)}$$

be the usual partial sum. Consider the trajectory of this random field, i.e. the points

$$\{(m, n, S_{(m,n)}) : m, n = 1, 2, \dots\}$$

in the three-dimensional space. Let $I(x) = 1$, if $x \geq 0$ and $I(x) = 0$, if $x < 0$. Then

$$\sum_{i=1}^m \sum_{j=1}^n I(S_{(i,j)})$$

is the number of points of the trajectory being not below the horizontal coordinate plane.

For the one-dimensional random walk the asymptotic behaviour of the time that the particle spends on the positive side is known. Now we can study similar problems for the above defined quantity. More generally, we can deal with the d -dimensional case. The following question is due to PAUL RÉVÉSZ [3].

Let d be a given positive integer. Let $X_{\mathbf{n}}$, $\mathbf{n} \in \mathbb{N}^d$, be independent identically distributed random variables taking values $+1$ and -1 with probabilities $1/2 - 1/2$. Let $S_{\mathbf{n}} = \sum_{\mathbf{k} \leq \mathbf{n}} X_{\mathbf{k}}$, $\mathbf{n} \in \mathbb{N}^d$. The question is the following. What is the value of $\liminf_{n \rightarrow \infty} \frac{1}{n} \max_{\mathbf{m}: |\mathbf{m}|=n} \sum_{\mathbf{k} \leq \mathbf{m}} I(S_{\mathbf{k}})$? Here $|\mathbf{m}| = m_1 m_2 \dots m_d$ is the number of the integer lattice points in the rectangle $\{\mathbf{k} \in \mathbb{N}^d : \mathbf{1} \leq \mathbf{k} \leq \mathbf{m}\}$ where $\mathbf{1} = (1, 1, \dots, 1) \in \mathbb{N}^d$, $\mathbf{m} = (m_1, m_2, \dots, m_d) \in \mathbb{N}^d$. The relation $\mathbf{k} \leq \mathbf{m}$ is defined coordinatewise.

THEOREM 1.1.

$$\liminf_{n \rightarrow \infty} \frac{1}{n} \max_{\mathbf{m}: |\mathbf{m}|=n} \sum_{\mathbf{k} \leq \mathbf{m}} I(S_{\mathbf{k}}) = 0$$

with probability 1.

The proof is based on elementary properties of the one-dimensional symmetric random walk. The details will be given for $d = 2$ in Section 3. In Section 2 we list certain facts about the one-dimensional random walk and mention a surprising property of a modified random walk (Proposition 2.1).

2. On the probability of long leads

Let S_0, S_1, S_2, \dots denote the one-dimensional symmetric random walk, i.e. $S_n = X_1 + \dots + X_n$. We say that the particle spends the time from $k-1$ to k on the positive side if either $S_k > 0$ or $S_k = 0$ but $S_{k-1} > 0$ (it means that the k th side of the path lies above the x -axis). Let π_n denote the time that the particle spends on the positive side from the time interval $[0, n]$. Let $p_{2k,2n} = \mathbf{P}(\pi_{2n} = 2k)$. By the famous discrete arc sine law (see [2], III/5),

$$p_{2k,2n} = u_{2k} u_{2n-2k}, \quad k = 0, 1, \dots, n,$$

where $u_{2n} = \binom{2n}{n}/2^{2n}$, $n = 0, 1, 2, \dots$

Now substitute constant 1 for X_1 . That is let the first step be always +1. In this way we obtain a modified random walk. Denote by $\tilde{\pi}_n$ the time that the modified random walk spends on the positive side from the time interval $[0, n]$. Let $\tilde{p}_{2k,2n} = \mathbf{P}(\tilde{\pi}_{2n} = 2k)$, $k = 0, 1, \dots, n$ be its distribution. We shall prove that we obtain the value of $\tilde{p}_{2n,2n}$ from the original discrete arc sine law by moving the probability of the shortest lead to the probability of the longest lead. The other values of $p_{2k,2n}$ remain unaltered.

PROPOSITION 2.1. $\tilde{p}_{0,2n} = 0$, $\tilde{p}_{2k,2n} = p_{2k,2n}$ if $k = 1, 2, \dots, n-1$, $\tilde{p}_{2n,2n} = 2p_{2n,2n}$.

PROOF. It is obvious that $\tilde{p}_{0,2n} = 0$. Let A_{2n} denote the set of paths lying on the positive side. Then $\tilde{p}_{2n,2n} = \#(A_{2n})/2^{2n-1}$. Here $\#$ denotes the cardinality of a set. Furthermore, we have for the usual random walk that

$$\frac{\#(A_{2n})}{2^{2n}} = \mathbf{P}(S_1 \geq 0, \dots, S_{2n} \geq 0) = u_{2n}$$

(see [2], III/4). The above facts imply that $\tilde{p}_{2n,2n} = 2u_{2n} = 2p_{2n,2n} = p_{0,2n} + p_{2n,2n}$.

Now, let $0 < k < n$. Assume that the particle returns first to the origin at time $2r$, $r = 1, \dots, k$. Therefore

$$\tilde{p}_{2k,2n} = \sum_{r=1}^k f_{2r} p_{2k-2r,2n-2r}.$$

Here f_{2r} is the probability that the modified random walk returns first to the origin at time $2r$. However, it is easy to see that f_{2r} is equal to the probability

that the usual random walk returns first to the origin at time $2r$. Moreover, by the discrete arc sine law, $p_{2k-2r, 2n-2r} = u_{2k-2r} u_{2n-2k}$. So

$$\tilde{p}_{2k, 2n} = \left(\sum_{r=1}^k f_{2r} u_{2k-2r} \right) u_{2n-2k} = u_{2k} u_{2n-2k} = p_{2k, 2n}.$$

Here we applied that $u_{2k} = \mathbf{P}(S_{2k} = 0)$ is equal to $\sum_{r=1}^k f_{2r} u_{2k-2r}$. \blacksquare

By the famous arc sine law (see [2], III/5), we have

$$(2.1) \quad \lim_{n \rightarrow \infty} \mathbf{P} \left(\frac{\pi_n}{n} < x \right) = \frac{2}{\pi} \arcsin \sqrt{x}, \quad 0 \leq x \leq 1.$$

Now we have the same arc sine law for the modified random walk. That is the limiting distribution remains unaltered if the first step of the random walk is the constant 1.

PROPOSITION 2.2.

$$\lim_{n \rightarrow \infty} \mathbf{P} \left(\frac{\tilde{\pi}_n}{n} < x \right) = \frac{2}{\pi} \arcsin \sqrt{x}, \quad 0 \leq x \leq 1.$$

PROOF. By the Stirling formula, $p_{0, 2n} = u_{2n} \approx 1/\sqrt{\pi n} \rightarrow 0$, as $n \rightarrow \infty$. So the limiting distribution of $\tilde{\pi}_{2n}/(2n)$ is the same as that of $\pi_{2n}/(2n)$. As $\tilde{\pi}_{2n} \leq \tilde{\pi}_{2n+1} \leq \tilde{\pi}_{2n} + 1$, the limiting distribution of $\tilde{\pi}_{2n+1}/(2n+1)$ is the same as that of $\tilde{\pi}_{2n}/(2n)$. \blacksquare

Finally, we turn to the role of zeros in the sequence S_0, S_1, S_2, \dots .

REMARK 2.1. By Theorem 2' of [1], $\vartheta_n/n \rightarrow 0$ almost surely where ϑ_n is the time spending in the origin by the usual one-dimensional symmetric random walk $S_0, S_1, S_2, \dots, S_n$.

Therefore, if we redefine π_n as $\pi_n^* = \sum_{k=1}^n I(S_k)$, the limiting distribution (2.1) remains unaltered. That is

$$\lim_{n \rightarrow \infty} \mathbf{P} \left(\frac{\pi_n^*}{n} < x \right) = \frac{2}{\pi} \arcsin \sqrt{x}, \quad 0 \leq x \leq 1.$$

π_n^* is the time that the particle spends on the positive side or in the origin.

3. Proof of Theorem 1.1

Consider the sequence of the prime numbers (as a subsequence of the positive integers). For a prime p , relation $|\mathbf{m}| = m_1 m_2 = p$ is valid only for $\mathbf{m} = (p, 1)$ and $\mathbf{m} = (1, p)$. Therefore, introduce the following notation

$$\pi_p^{(1*)} = \sum_{\mathbf{k} \leq (1,p)} I(S_{\mathbf{k}}), \quad \pi_p^{(2*)} = \sum_{\mathbf{k} \leq (p,1)} I(S_{\mathbf{k}}).$$

Then $\pi_p^{(1*)}$ is the time spending on the positive side or in the origin by the usual one-dimensional symmetric random walk determined by $X_{(1,1)}, X_{(1,2)}, \dots, X_{(1,p)}$. On the other hand, $\pi_p^{(2*)}$ is the time spending on the positive side or in the origin by the usual random walk determined by $X_{(1,1)}, X_{(2,1)}, \dots, X_{(p,1)}$. The p th element of the sequence in question is

$$(3.1) \quad \frac{1}{p} \max_{\mathbf{m}: |\mathbf{m}|=p} \sum_{\mathbf{k} \leq \mathbf{m}} I(S_{\mathbf{k}}) = \max \left\{ \pi_p^{(1*)}/p, \pi_p^{(2*)}/p \right\}.$$

First consider $\pi_p^{(1*)}$. We have already seen that the time that the particle spends on the positive side is defined in a slightly different way. Denote by $\pi_p^{(1)}$ the usual time spending on the positive side by the one-dimensional symmetric random walk determined by $X_{(1,1)}, X_{(1,2)}, \dots, X_{(1,p)}$. Actually, by the definition of $\pi_p^{(1)}$, that part of the time that the particle spends in the origin but immediately before it the particle was on the positive side is considered as time being on the positive side. Instead of $\pi_p^{(1*)}$ we shall use $\pi_p^{(1)}$. Similarly for $\pi_p^{(2*)}$. By Remark 2.1,

$$\liminf_{p \rightarrow \infty} \max \left\{ \pi_p^{(1*)}/p, \pi_p^{(2*)}/p \right\} = \liminf_{p \rightarrow \infty} \max \left\{ \pi_p^{(1)}/p, \pi_p^{(2)}/p \right\}$$

with probability 1.

Now substitute constant 1 for $X_{(1,1)}$. Then, instead of $\pi_p^{(1)}$ we obtain $\tilde{\pi}_p^{(1)}$ which is greater than or equal to $\pi_p^{(1)}$. We obtain $\tilde{\pi}_p^{(2)}$ from $\pi_p^{(2)}$ in a similar way. Therefore

$$\max \left\{ \pi_p^{(1)}/p, \pi_p^{(2)}/p \right\} \leq \max \left\{ \tilde{\pi}_p^{(1)}/p, \tilde{\pi}_p^{(2)}/p \right\}.$$

Here $\tilde{\pi}_p^{(1)}$ and $\tilde{\pi}_p^{(2)}$ are independent. Therefore, for an arbitrary $0 < \varepsilon < 1$, we have

$$\begin{aligned} \mathbf{P} \left(\max \left\{ \frac{\tilde{\pi}_p^{(1)}}{p}, \frac{\tilde{\pi}_p^{(2)}}{p} \right\} < \varepsilon \right) &= \mathbf{P} \left(\frac{\tilde{\pi}_p^{(1)}}{p} < \varepsilon \right) \mathbf{P} \left(\frac{\tilde{\pi}_p^{(2)}}{p} < \varepsilon \right) \rightarrow \\ (3.2) \qquad \qquad \qquad &\rightarrow \left(\frac{2}{\pi} \arcsin \sqrt{\varepsilon} \right)^2, \end{aligned}$$

as $p \rightarrow \infty$. To obtain (3.2) we applied Proposition 2.2.

Now let

$$A_\varepsilon = \left\{ \liminf_{p \rightarrow \infty} \max \left\{ \tilde{\pi}_p^{(1)}/p, \tilde{\pi}_p^{(2)}/p \right\} < \varepsilon \right\}.$$

A_ε is invariant under finite permutations of the indices in the sequence

$$\left(\begin{array}{c} X_{(1,2)} \\ X_{(2,1)} \end{array} \right), \left(\begin{array}{c} X_{(1,3)} \\ X_{(3,1)} \end{array} \right), \dots$$

By the Hewitt-Savage zero-one law, the probability of A_ε can be 0 or 1. However, by (3.2), 0 is not possible.

Finally, $\mathbf{P}(A_\varepsilon) = 1$ implies that

$$\liminf_{n \rightarrow \infty} \frac{1}{n} \max_{\mathbf{m}: |\mathbf{m}|=n} \sum_{\mathbf{k} \leq \mathbf{m}} I(S_{\mathbf{k}}) = 0$$

with probability 1.

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**A NEW TYPE OF SETS BETWEEN
CLOSED SETS AND $g\mu$ -CLOSED SETS**

By

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Abstract. By using open (resp. semi-open, β -open, π -open, δ -open) sets in a topological space, the notions of g -closed (resp. ω -closed or \hat{g} -closed, βg^* -closed, πg -closed, $g\delta$ -closed) sets are introduced and investigated by different mathematicians. In this paper we introduce the notion of μg^* -closed sets and obtain a unified theory for above collection of subsets between closed sets and $g\mu$ -closed sets.

1. Introduction

In 1970, Levine introduced the concept of generalized closed (g -closed) set. Recently, many versions of g -closed sets are introduced and investigated by different mathematicians. As an application of these sets, many low separation axioms are introduced. On the other hand, the notions of generalized open sets, was introduced by A. Császár. Especially, the author defined some basic operators on generalized topological spaces. It is observed that a large number of papers are devoted to the study of generalized open sets like open sets of a topological space, containing the class of open sets and possessing properties more or less similar to those of open sets.

We recall some notions defined in [2]. Let X be a non-empty set and $\exp X$ denote the power set of X . We call a class $\mu \subseteq \exp X$ a generalized topology [2], (briefly, GT) if $\emptyset \in \mu$ and union of elements of μ belongs to μ . A set X with a GT μ on it is called a generalized topological space (briefly, GTS) and is denoted by (X, μ) . A subset A of a topological space (X, τ)

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is said to be preopen [9] (resp. semi-open [7], α -open [11], β -open [1]) if $A \subseteq \text{int}(\text{cl}(A))$ (resp. $A \subseteq \text{cl}(\text{int}(A))$, $A \subseteq \text{int}(\text{cl}(\text{int}(A)))$, $A \subseteq \text{cl}(\text{int}(\text{cl}(A)))$). A subset A is called regular open if $A = \text{intcl } A$. The finite union of regular open sets is said to be π -open [6]. The complement of a π -open set is said to be π -closed. The δ -interior [15] of a subset A of X is the union of all regular open sets of X contained in A and is denoted by $\text{int}_\delta(A)$. A subset A is δ -open if and only if $A = \text{int}_\delta(A)$. For any topological space (X, τ) , the collection of all open (resp. preopen, semi-open, δ -open, α -open, β -open, π -open) sets is denoted by τ (resp. $PO(X)$, $SO(X)$, $\delta O(X)$, $\alpha O(X)$, $\beta O(X)$, $\pi O(X)$). Each of these collections is a generalized topology on (X, τ) .

For a GTS (X, μ) , the elements of μ are called μ -open sets and the complement of μ -open sets are called μ -closed sets. For $A \subseteq X$, we denote by $c_\mu(A)$ the intersection of all μ -closed sets containing A , i.e., the smallest μ -closed set containing A ; and by $i_\mu(A)$ the union of all μ -open sets contained in A , i.e., the largest μ -open set contained in A (see [2, 3]).

It is easy to observe that i_μ and c_μ are idempotent and monotonic, where $\gamma: \exp X \rightarrow \exp X$ is said to be idempotent if $A \subseteq X$ implies $\gamma(\gamma(A)) = \gamma(A)$ and monotonic if $A \subseteq B \subseteq X$ implies $\gamma(A) \subseteq \gamma(B)$. It is also well known from [3, 4] that if μ is a GT on X , $x \in X$ and $A \subseteq X$, then $x \in c_\mu(A)$ iff $x \in M \in \mu \Rightarrow M \cap A \neq \emptyset$ and $c_\mu(X \setminus A) = X \setminus i_\mu(A)$.

We recall the following definitions to be used in sequel.

DEFINITION 1.1. Let (X, τ) be a topological space. A subset A is said to be g -closed [7] (resp. ω -closed [13] or \hat{g} -closed [14], βg^* -closed [10], πg -closed [6], $g\delta$ -closed [5]) if $\text{cl } A \subseteq U$ and U is open (resp. semi-open, β -open, π -open, δ -open) in (X, τ) .

DEFINITION 1.2. [12] Let μ be a GT on a topological space (X, τ) . Then $A \subseteq X$ is called generalized μ -closed set (or simply $g\mu$ -closed set) if $c_\mu(A) \subseteq U$ whenever $A \subseteq U \in \tau$. The complement of a $g\mu$ -closed set is called a generalized μ -open (or simply $g\mu$ -open) set.

2. μg^* -closed sets

DEFINITION 2.1. Let μ be a GT on a topological space (X, τ) . Then $A \subseteq X$ is called μg^* -closed set if $\text{cl } A \subseteq U$ whenever $A \subseteq U$ and $U \in \mu$. The complement of a μg^* -closed set is called a μg^* -open set.

REMARK 2.2. Let μ be a GT on a topological space (X, τ) . Then every μg^* -closed set reduces to g -closed [8] (resp. \hat{g} -closed or ω -closed [14, 13],

βg^* -closed [10], πg -closed [6], $g\delta$ -closed [5]) set if one takes μ to be τ (resp. $SO(X)$, $\beta O(X)$, $\pi O(X)$, $\delta O(X)$).

REMARK 2.3. Let μ be a GT on a topological space (X, τ) such that $\tau \subseteq \mu$. Then the following implications hold:

$$\text{closed set} \Rightarrow \mu g^*\text{-closed set} \Rightarrow g\mu\text{-closed set.}$$

In fact, it follows from Definition 2.1 that every closed is μg^* -closed set. Suppose that A is a μg^* -closed set. Let $A \subseteq U$ and $U \in \tau$. Since $\tau \subseteq \mu$, $\text{cl } A \subseteq U$ and hence A is $g\mu$ -closed.

EXAMPLE 2.4. (a) Let $X = \{a, b, c\}$ and $\mu = \{\emptyset, \{a\}, \{a, b\}\}$ and $\tau = \{\emptyset, X, \{a\}, \{a, c\}\}$. Then μ is a GT on the topological space (X, τ) . We note that $\{b\}$ is a $g\mu$ -closed set but not a μg^* -closed set.

(b) Consider the topological space (X, τ) where $X = \{a, b, c\}$ and $\tau = \{\emptyset, X, \{a\}, \{c\}, \{a, c\}, \{a, b\}\}$. Let $\mu = \{\emptyset, X, \{a, b\}, \{b, c\}\}$ be a GT on X . Then it is easy to verify that $\{a\}$ is μg^* -closed set but not a closed set.

PROPOSITION 2.5. Let μ be a GT on a topological space (X, τ) . If $\{A_\alpha : \alpha \in \Lambda\}$ is a locally finite family of sets in (X, τ) and A_α is μg^* -closed for each $\alpha \in \Lambda$ then $\cup\{A_\alpha : \alpha \in \Lambda\}$ is μg^* -closed.

PROOF. Let $\cup\{A_\alpha : \alpha \in \Lambda\} \subseteq U$ and $U \in \mu$. Then $A_\alpha \subseteq U$ for each $\alpha \in \Lambda$. Since each A_α is μg^* -closed for each $\alpha \in \Lambda$, we have $\text{cl } A_\alpha \subseteq U$ and hence $\text{cl}(\cup_{\alpha \in \Lambda} A_\alpha) = \cup_{\alpha \in \Lambda} \text{cl } A_\alpha \subseteq U$. Therefore $\cup\{A_\alpha : \alpha \in \Lambda\}$ is μg^* -closed. ■

COROLLARY 2.6. Let μ be a GT on a topological space (X, τ) . The union of two μg^* -closed set is again a μg^* -closed set.

EXAMPLE 2.7. Consider $X = \{a, b, c\}$, $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$ and $\mu = \{\emptyset, X, \{a\}, \{a, c\}, \{b, c\}\}$. Then μ is a GT on the topological space (X, τ) . It is easy to verify that $A = \{a, b\}$ and $B = \{a, c\}$ are two μg^* -closed set but $A \cap B = \{a\}$ is not a μg^* -closed set.

PROPOSITION 2.8. Let μ be a GT on a topological space (X, τ) . If $A (\subseteq X)$ is μg^* -closed and μ -open then A is closed.

PROOF. Trivial. ■

PROPOSITION 2.9. Let μ be a GT on a topological space (X, τ) such that $\mu \subseteq \tau$. Then every $g\mu$ -closed set is μg^* -closed set.

PROOF. Let A be a $g\mu$ -closed set and $A \subseteq U$ where $U \in \mu$. Since $\mu \subseteq \tau$, $\text{cl } A \subseteq U$. Therefore, A is μg^* -closed. ■

PROPOSITION 2.10. *Let μ be a GT on a topological space (X, τ) . If $A \subseteq B \subseteq \text{cl } A$ and A is μg^* -closed then B is μg^* -closed.*

PROOF. Let U be a μ -open set such that $B \subseteq U$. Then $A \subseteq U$ and hence by μg^* -closedness of A , $\text{cl } A \subseteq U$. Thus $\text{cl } B = \text{cl } A \subseteq U$. So B is a μg^* -closed set. ■

PROPOSITION 2.11. *Let μ be a GT on a topological space (X, τ) . A subset A of X is μg^* -open if and only if $F \subseteq \text{int } A$ whenever $F \subseteq A$ and F is μ -closed.*

PROOF. Let A be a μg^* -open set and $F \subseteq A$, where F is μ -closed. Then $X \setminus A$ is a μg^* -closed set contained in the μ -open set $X \setminus F$. Hence $\text{cl}(X \setminus A) \subseteq X \setminus F$, i.e., $X \setminus \text{int } A \subseteq X \setminus F$. So $F \subseteq \text{int } A$.

Conversely, suppose that $F \subseteq \text{int } A$ for any μ -closed set F whenever $F \subseteq A$. Let $X \setminus A \subseteq U$, where U is μ -open. Then $X \setminus U \subseteq A$ and $X \setminus U$ is μ -closed. By assumption, $X \setminus U \subseteq \text{int } A$ and hence $\text{cl}(X \setminus A) = X \setminus \text{int } A \subseteq U$. Hence $X \setminus A$ is μg^* -closed and hence A is μg^* -open. ■

THEOREM 2.12. *Let μ be a GT on a topological space (X, τ) . Then a subset A of X is μg^* -closed if and only if $\text{cl } A \cap F = \emptyset$ whenever $F \cap A = \emptyset$ and F is μ -closed.*

PROOF. Suppose that A is μg^* -closed. Let $A \cap F = \emptyset$ and F is μ -closed. Then $A \subseteq X \setminus F \in \mu$ and $\text{cl } A \subseteq X \setminus F$. Therefore, we have $\text{cl } A \cap F = \emptyset$.

Conversely, let $A \subseteq U$ and $U \in \mu$. Then $A \cap (X \setminus U) = \emptyset$ and $X \setminus U$ is μ -closed. Thus by hypothesis, $\text{cl } A \cap (X \setminus U) = \emptyset$ and hence $\text{cl } A \subseteq U$. Therefore, A is μg^* -closed. ■

THEOREM 2.13. *Let μ be a GT on a topological space (X, τ) such that $\tau \subseteq \mu$. Then a subset A is μg^* -closed if and only if $\text{cl } A \setminus A$ does not contain any non-empty μ -closed set.*

PROOF. Suppose that A is μg^* -closed set. Let $F \subseteq \text{cl } A \setminus A$ and F be μ -closed. Then $A \subseteq X \setminus F \in \mu$ and hence $\text{cl } A \subseteq X \setminus F$. Therefore, we have $F \subseteq X \setminus \text{cl } A$. On the other hand $F \subseteq \text{cl } A$ and hence $F \subseteq \text{cl } A \cap (X \setminus \text{cl } A) = \emptyset$.

Conversely, suppose that A is not μg^* -closed. Then $\emptyset \neq \text{cl } A \setminus U$ for some $U \in \mu$ containing A . Since $\tau \subseteq \mu$, $\text{cl } A \setminus U$ is μ -closed such that $\text{cl } A \setminus U \subseteq \text{cl } A \setminus A$. ■

THEOREM 2.14. Let μ be a GT on a topological space (X, τ) such that $\tau \subseteq \mu$. Then a subset A of X is μg^* -closed if and only if $\text{cl } A \setminus A$ is μg^* -open.

PROOF. Suppose that A is μg^* -closed set. Let $F \subseteq \text{cl } A \setminus A$ and F is μ -closed. Then by Theorem 2.13, we have $F = \emptyset$ and $F \subseteq \text{int}(\text{cl } A \setminus A)$. Thus by Proposition 2.11, $\text{cl } A \setminus A$ is μg^* -open.

Conversely, let $A \subseteq U$ and $U \in \mu$. Then $\text{cl } A \cap (X \setminus U) \subseteq \text{cl } A \setminus A$ and $\text{cl } A \setminus A$ is μg^* -open. Since $\tau \subseteq \mu$, $\text{cl } A \cap (X \setminus U)$ is μ -closed and by Proposition 2.11 we have $\text{cl } A \cap (X \setminus U) \subseteq \text{int}(\text{cl } A \setminus A)$. Now, $\text{int}(\text{cl } A \setminus A) \subseteq \subseteq \text{cl } A \cap \text{int}(X \setminus A) = \text{cl } A \cap (X \setminus \text{cl } A) = \emptyset$. Therefore, we have $\text{cl } A \cap (X \setminus U) = \emptyset$ and hence $\text{cl } A \subseteq U$. This shows that A is μg^* -closed. \blacksquare

THEOREM 2.15. Let μ be a GT on a topological space (X, τ) . A subset A of X is μg^* -closed if and only if $c_\mu(\{x\}) \cap A \neq \emptyset$ for each $x \in \text{cl } A$.

PROOF. Suppose A is μg^* -closed and $c_\mu(\{x\}) \cap A = \emptyset$ for some $x \in \text{cl } A$. Then $A \subseteq X \setminus c_\mu(\{x\}) \in \mu$. Since A is μg^* -closed, $\text{cl } A \subseteq X \setminus c_\mu(\{x\}) \subseteq \subseteq X \setminus \{x\}$. This is a contradiction to the fact that $x \in \text{cl } A$.

Conversely, suppose that A is not μg^* -closed. Then $\emptyset \neq \text{cl } A \setminus U$ for some $U \in \mu$ containing A . Let $x \in \text{cl } A \setminus U$. Then $c_\mu(\{x\}) \cap U = \emptyset$ (as $x \notin U$). Hence $c_\mu(\{x\}) \cap A \subseteq c_\mu(\{x\}) \cap U = \emptyset$. This shows that $c_\mu(\{x\}) \cap A = \emptyset$ for some $x \in \text{cl } A$. \blacksquare

3. Preservation theorems

DEFINITION 3.1. Let (X, μ) and (Y, λ) be two GTS. A mapping $f: (X, \mu) \rightarrow (Y, \lambda)$ is said to be

- (i) (μ, λ) -continuous [2] if $f^{-1}(U) \in \mu$ for each $U \in \lambda$.
- (ii) (μ, λ) -closed if $f(F)$ is λ -closed in Y for each μ -closed subset F of X .

LEMMA 3.2. Let (X, μ) and (Y, λ) be two GTS with the topologies τ and σ on X and Y respectively. A mapping $f: (X, \tau) \rightarrow (Y, \sigma)$ is (μ, λ) -closed iff for each subset B of Y and each μ -open set U in X containing $f^{-1}(B)$, there exists a λ -open set V in Y such that $B \subseteq V$, $f^{-1}(V) \subseteq U$.

PROOF. Let f be (μ, λ) -closed and $B \subseteq Y$ be such that $f^{-1}(B) \subseteq U$ where $U \in \mu$. Let $V = Y \setminus f(X \setminus U)$. Then V is a λ -open set such that $B \subseteq V$ and $f^{-1}(V) \subseteq U$.

Conversely, let F be any μ -closed subset of X . Set $f(F) = B$, then $F \subseteq f^{-1}(B)$ and $f^{-1}(Y \setminus B) \subseteq (X \setminus F) \in \mu$. Thus by the hypothesis, there exists a λ -open set V such that $Y \setminus B \subseteq V$ and $f^{-1}(V) \subseteq X \setminus F$. Therefore, we obtain $Y \setminus V \subseteq B = f(F) \subseteq Y \setminus V$ and hence $f(F) = Y \setminus V$ and hence $f(F)$ is λ -closed. This shows that f is (μ, λ) -closed. ■

THEOREM 3.3. *Let (X, τ) and (Y, σ) be two topological spaces with the GTS μ and λ on X and Y respectively. If $f: (X, \tau) \rightarrow (Y, \sigma)$ is closed and $f: (X, \mu) \rightarrow (Y, \lambda)$ is (μ, λ) -continuous, then for each μg^* -closed set A of X , $f(A)$ is λg^* -closed in Y .*

PROOF. Let A be any μg^* -closed set in X and $f(A) \subseteq V \in \lambda$. Then $A \subseteq f^{-1}(V) \in \mu$. Since A is μg^* -closed, $\text{cl } A \subseteq f^{-1}(V)$ and so $f(\text{cl } A) \subseteq \subseteq V$. Since f is closed, $\text{cl}(f(A)) \subseteq f(\text{cl } A) \subseteq V$. This shows that $f(A)$ is λg^* -closed in Y . ■

THEOREM 3.4. *Let (X, τ) and (Y, σ) be two topological spaces with the GTS μ and λ on X and Y respectively. If $f: (X, \tau) \rightarrow (Y, \sigma)$ is continuous and $f: (X, \mu) \rightarrow (Y, \lambda)$ is (μ, λ) -closed, $f^{-1}(B)$ is μg^* -closed in X , for each λg^* -closed set B of Y .*

PROOF. Let B be any λg^* -closed subset of Y and $f^{-1}(B) \subseteq U \in \mu$. Since f is (μ, λ) -closed, by Lemma 3.2 there exists $V \in \lambda$ such that $B \subseteq V$ and $f^{-1}(V) \subseteq U$. Since B is λg^* -closed, $\text{cl } B \subseteq V$ and hence $\text{cl}(f^{-1}(B)) \subseteq \subseteq f^{-1}(\text{cl}(B)) \subseteq f^{-1}(V) \subseteq U$ because f is continuous. This shows that $f^{-1}(B)$ is μg^* -closed in X . ■

CONCLUSION: The definition of many other similar types of generalized closed sets may be introduced on a topological space (X, τ) from the definition of μg^* -closed sets by replacing μ with the help of different GT on X .

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STATISTICAL ANALYSIS OF RANDOM PERMUTATIONS

Abstract of Ph.D. Thesis

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(Defended March 23, 2009)

1. Preliminaries

The dissertation deals with some aspects of the statistical analysis of random permutations. Random permutations most frequently appear as orderings of the elements of a finite set. In sociological studies, individuals may be asked to rank a number of choices according to importance or preference. In other cases, voters, judges, or exam boards rank candidates, tenders, or applicants, and make decisions based on these orderings. Moreover, a permutation can describe the pairing of the elements of two sets: one can pair jobs with employees or students with tutors. Finally, any real dataset may be analysed based on only the ranks.

My aim was to give an overview of the various parametric models applicable to permutation data, to study estimation of the parameters and assess of fit, and to develop new models. Part of my motivation for this research was the experience that we – my supervisor and I – could not find a well-fitting simple model for the 1980 election data of the American Psychological Association. This dataset is one of the most well-studied in the literature, see for example [1, 3, 10, 11, 13]. We could formulate a simple model, which did not provide a satisfactory fit for these data, but performed significantly better than the other models. This new model led me to consider conditional independence models and factorizing models for random permutations. In the dissertation, borrowing terminology from the theory of contingency table analysis, I call these models hierarchical models.

The book by MARDEN [10] gives a thorough overview of the existing models for random rankings. The author was partly inspired by the conference entitled “Probability Models and Statistical Analyses for Ranking

Data,” held in Amherst in 1990, the proceedings of which was edited by FLIGNER and VERDUCCI [5]. The paper by CRITCHLOW, FLIGNER and VERDUCCI [2] also deserves mentioning, which, while shorter, contains important results about the properties of the models (reversibility, label-invariance, L-decomposability, unimodality, complete consensus).

2. Methods

One of the tools I used is algebraic statistics. As the name suggests, this field is concerned with the application of algebraic tools in statistics. These tools are especially suited for studying closures of exponential families, which is of both theoretical and practical importance. The following specific results, used in the dissertation, can be found in the papers by DIACONIS and STURMFELS [4], GEIGER, MEEK and STURMFELS [6], and RAPALLO [12].

Let $\mathcal{X} = \{x_1, \dots, x_s\}$ denote a finite set, and let $M = (m_{ij})$ be a $t \times s$ matrix with nonnegative integer entries. The probability distribution $p = (p(x_1), \dots, p(x_s))$ is said to belong to the so-called *toric model* $\mathbf{F}(M)$, if there exist nonnegative parameters $\lambda_1, \dots, \lambda_t$, with which

$$p(x_i) = c(\lambda) \prod_{j=1}^t \lambda_j^{m_{ji}}, \quad 1 \leq i \leq s.$$

The set T is called *M-feasible*, if for each $i \notin T$, we have $\text{Supp}(m_i) \not\subseteq \cup_{j \in T} \text{Supp}(m_j)$, where m_i denotes the i th column vector of M , and $\text{Supp}(\cdot)$ denotes the support of the vector in the argument. The nonnegative *toric variety* associated with M is the set

$$X_M = \{x \in \mathbb{R}_{\geq 0}^s : x^u - x^v = 0 \quad \forall u, v \in \mathbb{N}^s \text{ such that } Mu = Mv\},$$

where $x^u = \prod_i x_i^{u_i}$. The toric ideal generated by the polynomials $x^u - x^v$ above is denoted by I_M .

THEOREM 2.1. (GEIGER ET AL. [6]) *cl($\mathbf{F}(M)$) = X_M , where $\text{cl}(\cdot)$ stands for closure. Moreover, for $p \in X_M$, we have $p \in \mathbf{F}(M)$ if and only if the support of p is *M*-feasible.*

THEOREM 2.2. (RAPALLO [12]) *For every M , there exists a maximal representation M_{\max} , for which $\text{cl}(\mathbf{F}(M)) = \mathbf{F}(M_{\max})$.*

The functions $f_1, \dots, f_L : \mathcal{X} \rightarrow \mathbb{Z}$ are said to be a *Markov basis* for the model $\mathbf{F}(M)$, if for every u , the steps f_i generate a strongly connected graph

on the frequency vectors $g : \mathcal{X} \rightarrow \mathbb{N}$ satisfying $Mg = u$ (where the step f_i takes g to $g + f_i$). Markov bases can be used to run Monte Carlo procedures for assessing goodness of fit in the case of small datasets, where the asymptotics of the χ^2 test is not applicable.

THEOREM 2.3. (DIACONIS and STURMFELS [4]) *The functions f_1, \dots, f_L constitute a Markov basis if and only if the polynomials $x^{f_i^+} - x^{f_i^-}$ are a generating set of the ideal I_M , where f_i^+ (f_i^-) is the positive (negative) part of f_i .*

I also used the theory of hierarchical and log-linear models, or more generally the theory of discrete exponential families. The relevant results can be found in the book by LAURITZEN [8]. Keeping the previous setting, let $A \in \mathcal{A}$ be partitions of the set \mathcal{X} , and let the rows of the matrix $M_{\mathcal{A}}$ be the indicator vectors of the classes of these partitions (so for a partition with k classes, we have k rows in $M_{\mathcal{A}}$). Then in the model $\text{cl}(\mathbf{F}(M_{\mathcal{A}}))$, the maximum likelihood estimate exists uniquely, and it can be found by e.g. *iterative proportional scaling* (IPS). For any $x \in \mathcal{X}$, let $x(A)$ be the class of partition A containing x , and for any probability distribution p on \mathcal{X} , let $p(x(A)) = \sum_{y \in \mathcal{X}: y(A)=x(A)} p(y)$ be the p -probability of this class. Suppose we have a sample with empirical distribution r , and we want to find the element of $\text{cl}(\mathbf{F}(M_{\mathcal{A}}))$, which maximizes the likelihood of the sample. Let $p^{(0)}$ be an arbitrary strictly positive element of the model $\mathbf{F}(M_{\mathcal{A}})$ (e.g. the uniform distribution). Then the $(t + 1)$ st iteration step of the IPS algorithm updates $p^{(t)}$ as

$$p^{(t+1)}(x) = \frac{r(x(A))}{p^{(t)}(x(A))} p^{(t)}(x), \quad x \in \mathcal{X},$$

where A runs cyclically over the set \mathcal{A} .

3. Results

3.1. The inversions model of McCullagh

The following model was introduced by PETER MCCULLAGH [11]. Let $C \subseteq [n]$ be a k -element subset. A permutation σ_C of the elements of C is called a $(k - 1)$ st order inversion, if none of its coordinates is in its own place with respect to the monotone increasing order (where $[n] = \{1, \dots, n\}$). We say that the permutation $\pi \in S_n$ contains the inversion σ_C (in notation

$\sigma_C \subseteq \pi$), if the elements of C appear in π in the order σ_C . Then the model defined by the inversions $\sigma_{C_1}^1, \dots, \sigma_{C_s}^s$ consists of distributions satisfying

$$(1) \quad \log p_\theta(\pi) = \sum_{i: \sigma_{C_i}^i \subseteq \pi} \theta_i, \quad \theta = (\theta_1, \dots, \theta_s) \in \mathbb{R}^s.$$

In the dissertation, I prove McCullagh's following conjecture.

THEOREM 3.1. *In the model (1), if $\theta \neq \tau$, then $p_\theta \neq p_\tau$.*

The theorem is equivalent to the following combinatorical reformulation. Define a graph G_H on the vertex set S_n as follows. From the permutation π , there is a directed edge to all permutations obtained from π by moving one element to its proper position (other elements are shifted, if necessary). As an example, for $n = 5$, from (24351) there are directed edges to the permutations (12435), (42351), (23541), (24315).

THEOREM 3.2. *There are no directed cycles in the graph G_H .*

3.2. EM algorithms for Plackett-Luce-type models

Take n players, the overall ability of the i th one is expressed by the parameter λ_i . According to the Plackett-Luce model, if the players $I \subseteq [n]$ take part in a competition, then the probability of the ordering π is

$$(2) \quad p(\pi) = \prod_{k=1}^{|I|} \frac{\lambda_{\pi(k)}}{\sum_{j=k}^{|I|} \lambda_{\pi(j)}},$$

where $\pi(1)$ is the overall winner, $\pi(|I|)$ is the overall loser. LUCE [9] derived this model from the ranking postulate and the choice axiom. It is easy to check that if Z_i ($i = 1, \dots, n$) are independent random variables, exponentially distributed with parameters λ_i , then the righthandside of (2) is just the probability $P(Z_{\pi(1)} < \dots < Z_{\pi(|I|)})$. This observation leads to the following EM algorithm, which iteratively finds the maximum likelihood estimate of the parameters λ_i . Suppose we have m observations, the r th of which consists of the ordering π_r of the players in I_r . For all $i \in I_r$, let $\alpha_r(i)$ be the rank of i

in the ordering π_r . Finally, let m_i denote the number of observed orderings containing player i . With these notations, one EM-step is given by

$$\lambda_i^{(t+1)} = m_i \left[\sum_{r:i \in I_r} \sum_{k=1}^{\alpha_r(i)} \frac{1}{\sum_{j=k}^{|I_r|} \lambda_{\pi_r(j)}^{(t)}} \right]^{-1} \quad 1 \leq i \leq n.$$

HUNTER [7] derived MM algorithms for this model and its generalizations. He also gave conditions under which the algorithms converge to the unique maximum likelihood estimate. I showed that EM algorithms are also a natural choice for this estimation problem, although, according to my simulation studies, their convergence is slower than the convergence of the MM algorithms.

3.3. L-decomposability

For a permutation $\pi = (\pi(1), \dots, \pi(n))$, let $\pi\{i..j\} = \{\pi(i), \dots, \pi(j)\}$ and $\pi(i..j) = (\pi(i), \dots, \pi(j))$. The random permutation Π (and its distribution) is *L-decomposable*, if the sets $\Pi\{1..k\}$, $k = 1, \dots, n$ form a Markov chain. “L” stands for Luce, since these are exactly the distributions satisfying Luce’s ranking postulate. Denote by (π, ρ) the concatenation of two partial permutations, and for a subset $C \subseteq [n]$, let S_C consist of all permutations of the elements of C .

THEOREM 3.3. *The L-decomposable distributions form a closed toric model. The toric ideal corresponding to the model is generated by all polynomials of form $x_{(\pi_1, \rho_1)}x_{(\pi_2, \rho_2)} - x_{(\pi_1, \rho_2)}x_{(\pi_2, \rho_1)}$, where $\pi_1, \pi_2 \in S_C$ and $\rho_1, \rho_2 \in S_{[n] \setminus C}$ for some C .*

THEREOM 3.4. *For $n = 4$ and $n = 5$, the L-decomposable model has a unique minimal Markov basis, which is equal to the one described in the previous theorem. For $n \geq 6$, the basis of the previous theorem is not minimal, and the minimal basis is not unique.*

The following theorem is about the properties of the maximum likelihood (ML) estimate.

THEOREM 3.5. *In the L-decomposable model, the ML estimate always exists uniquely, it has an explicite form, and its exact distribution can be calculated. Moreover, the following hyper Markov property holds: for all k , the random distributions $\{\hat{P}(\Pi(1..k) = u)\}_u$ and $\{\hat{P}(\Pi(k+1..n) = v)\}_v$*

are conditionally independent, given the random distribution $\{\hat{P}(\Pi\{1..k\} = C)\}_C$, where \hat{P} denotes the ML estimate.

3.4. Bi-L-decomposability

The following property was not studied in the literature before. Let us call the random permutation Π (and its distribution) *bi-L-decomposable*, if both Π and Π^{-1} are L-decomposable. I introduced hierarchical models for random permutations to study bi-L-decomposability. Let \mathcal{D} (resp. \mathcal{R}) be partitions of $[n]$ with d (resp. r) classes. The *coarsening* of π on the product partition $\mathcal{P} = \mathcal{D} \times \mathcal{R}$ is the $d \times r$ matrix

$$|\pi(\mathcal{P})| = (t_{ij}), \quad t_{ij} = |\{1 \leq s \leq n : s \in D_i, \pi(s) \in R_j\}|.$$

DEFINITION 3.6. Let $\mathcal{P}_1, \dots, \mathcal{P}_s$ be product partitions of $[n] \times [n]$. The strictly positive distribution p on S_n belongs to the hierarchical model with generators $\mathcal{P}_1, \dots, \mathcal{P}_s$, in notation $p \in \mathcal{L}(\mathcal{P}_1, \dots, \mathcal{P}_s)$, if there exist functions θ_i such that

$$\log p(\pi) = \sum_{i=1}^s \theta_i(|\pi(\mathcal{P}_i)|) \quad \forall \pi \in S_n.$$

We write $\mathcal{D}' \succeq \mathcal{D}$ if the partition \mathcal{D}' is finer than \mathcal{D} .

THEOREM 3.7. Let $\mathcal{L}(\mathcal{D}_i \times \mathcal{R} : i = 1, \dots, s)$ and $\mathcal{L}(\mathcal{D} \times \mathcal{R}_j : j = 1, \dots, t)$ be two hierarchical models, where $\mathcal{D} \succeq \mathcal{D}_i$ and $\mathcal{R} \succeq \mathcal{R}_j$ for all $1 \leq i \leq s$, $1 \leq j \leq t$. Then the intersection of the two models is the hierarchical model $\mathcal{L}(\mathcal{D}_i \times \mathcal{R}_j : i = 1, \dots, s, j = 1, \dots, t)$.

This theorem is applicable to the L-decomposable hierarchical model and its inverse, whose intersection is the bi-L-decomposable hierarchical model. Some further calculations yield the following.

THEOREM 3.8. The family of strictly positive bi-L-decomposable distributions has $\sum_{i=1}^{n-1} i^2$ free parameters.

In the dissertation, I give two parametrizations, which correspond to two bases of the subspace in $\mathbb{R}^{n!}$ spanned by the logarithms of bi-L-decomposable distributions. One basis is orthogonal, the other is 0 – 1.

Each hierarchical model has a matrix $M_{\mathcal{A}}$ of the type described in the Methods section. Denote by M_L the matrix of the L-decomposable model, and by M_B the matrix of the bi-L-decomposable model. I could determine the Markov basis of $\mathbf{F}(M_B)$ only for $n = 4$.

THEOREM 3.9. *The minimal Markov basis of $\mathbf{F}(M_B)$ for $n = 4$ consists of 10 degree-2 polynomials (from the Markov bases of $\mathbf{F}(M_L)$ and its inverse), and 8 degree-4 polynomials.*

This observation led to the following result, valid for all n .

THEOREM 3.10. *The model $\mathbf{F}(M_B)$ is not closed, and even its closure is a strict subset of all bi-L-decomposable distributions.*

3.5. S-decomposability

In the analysis of bi-L-decomposable distributions, a stronger property, S-decomposability played an important role. The random permutation Π and its distribution p is *S-decomposable*, if there exist parameters $\Lambda(C) \geq 0$ ($C \subseteq [n]$) such that $p(\pi) = \prod_{k=1}^n \Lambda(\pi\{1..k\})$. Π is *bi-S-decomposable*, if both Π and Π^{-1} are S-decomposable.

THEOREM 3.11. *A strictly positive distribution p is S-decomposable, if and only if it is L-decomposable, and there exist parameters $\Lambda'(C) > 0$ such that*

$$P(\Pi(k+1) = x \mid \Pi\{1..k\} = C) = \frac{\Lambda'(C \cup x)}{\sum_{y \notin C} \Lambda'(C \cup y)}.$$

Strictly positive S-decomposable distributions form a hierarchical model with model matrix M_S . The corresponding toric model $\mathbf{F}(M_S)$ is not closed, however, its Markov basis can be characterized.

THEOREM 3.12. *Let the sets $C_1, D_1, C_2, D_2, \dots, C_j, D_j \subseteq [n]$ satisfy $|C_i| = k$, $|D_i| = k + 1$, and $C_i, C_{i+1} \subset D_i$ (with $C_{j+1} := C_1$). Let $\pi_i \in S_{C_i}$ and $\rho_i \in S_{[n] \setminus D_i}$. For all such choices, create the polynomial*

$$\prod_{i=1}^j x_{(\pi_i, D_i \setminus C_i, \rho_i)} - \prod_{i=1}^j x_{(\pi_i, D_{i-1} \setminus C_i, \rho_{i-1})},$$

where $D_0 = D_j$ and $\rho_0 = \rho_j$. These polynomials, together with the ones in the Markov basis of $\mathbf{F}(M_L)$, form a Markov basis of the model $\mathbf{F}(M_S)$.

In summary, we found that the S- and bi-S-decomposable families are more complex algebraically than the L- and bi-L-decomposable families.

3.6. Label-invariance

Suppose the elements of a set are labelled with the integers $1, \dots, n$, and an ordering of these elements is given by π . If we relabel the elements, i.e. change label i to label $\sigma(i)$, then the same ordering is given by the permutation $\sigma\pi$ (the operation here is group multiplication). Similarly, if π^{-1} is a ranking expressed with the original labelling, then the same ranking with the new labelling becomes $\pi^{-1}\sigma^{-1}$. This motivates the question, whether a model for random permutations is invariant under multiplications from the left and right.

THEOREM 3.13. *Let $n \geq 4$. The family of L-decomposable distributions is invariant under left multiplications. It is invariant under right multiplication by σ , if and only if σ belongs to the eight-element subgroup of S_n generated by the permutations $(n\ n-1\dots 21)$ and $(2134\dots n)$.*

It is natural to ask which subfamily of the L-decomposable distributions is invariant under all right multiplications. More precisely, we are looking for those distributions on S_n , which remain L-decomposable after any right multiplication.

THEOREM 3.14. *Let $n \geq 4$. A strictly positive distribution p on S_n remains L-decomposable after all right multiplications, if and only if it is quasi-independent, i.e. there exist parameters $c_i(x)$, $1 \leq i, x \leq n$ such that $p(\pi) = \prod_{i=1}^n c_i(\pi(i))$.*

4. Conclusions

The research reported in the dissertation showed that while there are many simple, elegant, practical and realistic models to describe random permutations, there is still room for the development of new models. Such could be hierarchical models, some of which can be interpreted as conditional independence models. I would like to characterize “simple” hierarchical models,

which would be an analogue of decomposable graphical models in the classical theory. Greater insight could be gained by calculating the Markov basis of other hierarchical models, the main difficulty is that current algorithms quickly become infeasible as n grows. It would be useful to give general upper bounds for the degree of these Markov bases. A characterization of the intersection of hierarchical models in the general case is also open.

The dissertation is based on the following papers

- Conditional independence relations and log-linear models for random matchings. *Acta Math. Hungar.*, Online First (2008).
- (with Rejtő, L. and Tusnády, G.) Statistical Inference on Random Structures. In: *Horizons of Combinatorics*, Bolyai Society Mathematical Studies **17**, Springer (2008), 37-66.
- Markov bases of conditional independence models for permutations. To appear in *Kybernetika*.
- On L-decomposability of random permutations. Submitted to *J. Math. Psych.*, under revision.
- An acyclic operation on the symmetric group. Submitted.

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MULTIPLE-POINT FORMULAS AND THEIR APPLICATIONS

Abstract of Ph.D. Thesis

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(Defended January 30, 2009)

1. Introduction

Singularity theory studies the behaviour of smooth map germs $(\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^{n+k}, 0)$. Two germs f, g are said to be \mathcal{A} -equivalent if there is a smooth reparametrization of both the target and the source space which transforms f into g . The equivalence classes are the so called singularity types. In the case of $k > 0$ the simplest type is the case when the differential of f has rank n . All such germs are equivalent by the implicit function theorem. When the differential has rank $n - 1$ (so corank 1) then the picture becomes more complicated, but the complete classification is known. The set of corank 1 germs splits into countably many equivalence classes that are usually denoted by A_n and referred to as Morin singularities. As the corank of the germ increases the classification becomes more and more difficult. A general classification is completely unknown.

Global singularity theory considers smooth maps $f : M \rightarrow N$ between smooth manifolds, and classifies the points of M according to the local singularity type of f at the given point. This can be done in two ways. The mono-singularity type of $x \in M$ is the actual local singularity type of f at x . The multi-singularity type is the list of local singularity types of all the points $y \in M$ such that $f(x) = f(y)$. When this list consists of only the simplest (corank n) singularity type, the corresponding subsets of M are the so-called multiple-point manifolds.

Multiple-point manifolds have been long studied from different viewpoints. From the point of topology the question arises as follows. Given a

generic immersion $f : M^n \rightarrow N^{n+k}$, is it possible to express topological invariants of its multiple-point manifolds using invariants of M, N and f ?

The first such questions considered were the homology classes of the multiple-point manifolds. It turned out that indeed these homology classes are related to each other in a simple way that includes only little information from f (namely the Euler class of its normal bundle and f^* , the induced map in cohomology). The formula was first stated by LASHOF and SMALE [9] but it turned out to be partially false. It was corrected by HERBERT [5] and later RONGA [17] gave a simple and very geometric proof.

THEOREM 1. (Herbert's formula) *Let $f : M^n \rightarrow N^{n+k}$ be a generic immersion. Then the closures of the r -tuple point sets, $\bar{N}_r(f)$ and $\bar{M}_r(f)$ carry fundamental classes with \mathbb{Z}_2 coefficients. Denoting by n_r and m_r their Poincaré duals in N and M respectively and setting $e = e(v_f)$ we have:*

$$m_r = f^*(n_{r-1}) - e \cdot m_{r-1}$$

If both M and N are oriented and k is even, then the multiple-point manifolds can be given a natural orientation. Thus one can interpret m_r and n_r as cohomology classes with integer coefficients and the formula remains valid in its original form.

2. Results and methods

2.1. Multiple-point manifolds

Herbert's theorem is the starting point of our work. In the first main part of the thesis we give a generalization of this formula that allows us to move from homology classes to cobordism classes and is based on ideas of Ronga, Kamata and Szűcs.

SZŰCS [23] used the Herbert-Ronga formula in the oriented case for double-point manifolds in K -theory and translated it to ordinary cohomology via the Chern character to obtain a sequence of formulas involving push-forwards of Pontrjagin classes of the double-point manifold.

KAMATA [6] used the Herbert–Ronga formula in the unoriented cobordism cohomology and translated it via the Boardman homomorphism to obtain a sequence of formulas involving the push-forwards of the Stiefel–Whitney classes of the multiple-point manifolds.

Later in [24] SZŰCS investigated the case of oriented manifolds immersed in Euclidean space. Using a filtration on the multiple-point manifold he could calculate its Pontrjagin numbers without pushing them forward to M .

The result presented here contains all three above results at the same time and which avoids complicated homological calculations or the use of natural transformations between extraordinary cohomology theories and ordinary cohomology.

The method relies on the fact that the cobordism class of a manifold is determined by its characteristic numbers - certain cohomology classes evaluated on the fundamental class. This evaluation can be done by pushing forward the cohomology classes from the multiple-point manifold to M and evaluating them on the fundamental class of M . Hence we derive a formula for the pushforwards of these cohomology classes.

$$(1) \quad m_r = \gamma(v_f) \cdot (f^*n_{r-1} - e_h(v_f) \cdot m_{r-1})$$

Then under additional conditions we get useful special cases of the formula. From these special cases it is easy to derive the previously known results that had motivated us.

THEOREM 2. *Let $f : M^m \rightarrow N^{m+k}$ be a generic immersion of even codimension between oriented manifolds and choose a cohomology theory with coefficient ring $h^0(pt) \cong \mathbb{Q}$. Then we have*

$$m_r = \gamma(v_f) \cdot \left(\frac{f^*f_!(m_{r-1})}{r-1} - e_h(v_f)m_{r-1} \right)$$

THEOREM 3. *Let f and h^* be as in Theorem 2. Further assume that there is a bundle $\xi : E \rightarrow N$ such that $TM = f^*\xi$, and that $e(v_f) = f^*(y)$ for an $y \in h^*(N)$. Then for every $r \geq 1$ there is a cohomology class $k_r \in h^*(N)$ such that $m_r = f^*(k_r)$ and the following simple recursion formula holds:*

$$k_r = \gamma(TN)\beta(\xi) \left(\frac{c}{r-1} - y \right) \cdot k_{r-1}$$

where $c = f_!f^*(1) \in h^*(N)$.

Szűcs used his original formulas to show that there are cobordism classes of manifolds that do not contain double-point manifolds. Here we carry out similar calculations for multiple-point manifolds of arbitrary multiplicity.

As a direct application of the general formula we give a condition that an r -tuple point manifold of an immersion must necessarily fulfill. This shows that a typical manifold will not be an r -tuple point manifold.

THEOREM 4. *If a $4t$ -dimensional oriented manifold V^{4t} is the r -tuple-point manifold of an immersion $f : M^m \rightarrow N^{m+k}$ (ie. $V = \Delta_r(f)$) with $f^* = 0$ in positive dimension, then $\langle p_1^t(V), [V] \rangle$ is divisible by r^t .*

COROLLARY 1. *If $m - k(r - 1)$ is divisible by four then $|\text{coker}(\Delta_r)| \geq r^{\frac{m-k(r-1)}{4}}/3^\varepsilon$ where $\varepsilon = 1$ if $3 \mid r$ and $m - k(r - 1) \equiv 1 \pmod{3}$, else $\varepsilon = 0$.*

2.2. Projected immersions

There is a surprising relation between the multiple-points of an immersion $g : M \looparrowright N \times \mathbb{R}$ and the singularities of its projection $f : M \rightarrow N$ that was found by SZÜCS in [25] (see also [20]). Namely he showed that if N is a Euclidean space then the $r + 1$ -tuple-points of g are cobordant to the Σ^{1r} points of f . The proof of this result involved computing the characteristic numbers of the two manifolds and observing that they coincide.

It is very natural to ask whether this cobordism can be “seen” in an explicit way hidden in the geometry of f , not just as mere luck that all the characteristic numbers coincide.

We shall answer this question in the affirmative by constructing a cobordism that connects the two manifolds. This allows us to slightly extend the original theorem: instead of cobordism of manifolds we obtain singular bordism of maps, and we prove the theorem for any smooth target manifold N .

THEOREM 5. *Let $f : M^n \rightarrow N^{n+k}$ be a prim map, and let $g : M \looparrowright N \times \mathbb{R}$ be its lift to an immersion. Then for any $r \geq 1$ we have $g_r \sim \Sigma^{1r-1}(f)$, that is they represent the same element in the singular bordism group $\mathcal{N}(M)$.*

If M and N are oriented and the codimension k is odd, then $g_r \sim \sim_{SO} \Sigma^{1r-1}(f)$, that is they represent the same element in the singular oriented bordism group $\Omega(M)$.

2.3. Product maps

The results of this section are the first steps in understanding how the direct product operation affects the singularities of maps. They show that indeed there is some well controllable effect, at least in the simplest cases. There are two main difficulties. The first one is that the direct product of

generic maps will not be generic, so one has to take a small perturbation. This makes it hard to understand the singular strata geometrically. The second one is that generally the product of two singular maps even after a generic perturbation will have more complicated singularities than the original maps had.

We start with studying products of immersions. Here only the first type of problem arises, namely that the self intersections will not be transverse. This can be overcome by employing our general multiple-point formula that helps to compute the characteristic numbers of multiple-point manifolds. It turns out that the multiple-point manifolds multiply independently of each other:

THEOREM 6. *Let $g_i : M_i^{n_i} \rightarrow \mathbb{R}^{n_i+k_i}$; ($i = 1, 2$) be generic immersions. Then the r -tuple point manifold $\Delta_r(g_1 \times g_2) \sim (-1)^{r-1} \Delta_r(g_1) \times \Delta_r(g_2)$ where \sim stands for “unoriented-cobordant”.*

If the M_i are oriented and the k_i are even, then their r -tuple point manifolds are oriented cobordant.

Then we study Morin maps. In this case one has to deal with the second kind of problem. We get around this by increasing the dimension of the target space by one, and hence we are able to define a multiplication of Morin maps which preserves the Morin property and is distributive with respect to the disjoint union of maps.

DEFINITION 1. Let $\text{Mor}_{\mathbb{Q}}$ denote the group $\bigoplus_{n,k} \text{Morin}^{SO}(n, k) \otimes \mathbb{Q}$ with this ring structure. $\text{Mor}_{\mathbb{Q}}$ is a bigraded ring, the two grades being n and $k + 1$. Note that this implies that the direct sum $\bigoplus_{k \text{ odd}} \text{Morin}^{SO}(n, k) \otimes \mathbb{Q}$ is a subring or $\text{Mor}_{\mathbb{Q}}$.

First, we identify the components of $\text{Mor}_{\mathbb{Q}}$ as subgroups of the rational oriented cobordism ring $\Omega_* \otimes \mathbb{Q}$. Then combining the results of the previous sections we show that the singular strata behave nicely under the multiplication we defined. We can consider Σ^{1_r} as a map from $\text{Mor}_{\mathbb{Q}}$ to the rational oriented cobordism ring:

$$\Sigma^{1_r} : \bigoplus_{k,n} \text{Morin}^{SO}(n, k) \otimes \mathbb{Q} \rightarrow \Omega_* \otimes \mathbb{Q}.$$

THEOREM 7. *The map Σ^{1_r} is a ring homomorphism. In other words*

$$\Sigma^{1_r}(f * g) \sim \Sigma^{1_r}(f) \times \Sigma^{1_r}(g)$$

holds for any two Morin maps f, g to Euclidean spaces where \sim now stands for rationally cobordant (in the oriented sense).

It turns out that this information is actually enough to compute $\text{Mor}_{\mathbb{Q}}$.

Finally the last part deals with general singular maps. We show that a Cartan-type formula relates the homology class of Σ^1 points of two maps with that of their direct product.

THEOREM 8. *Let $f : M_1^{n_1} \rightarrow N_1^{n_1+k_1}, g : M_2^{n_2} \rightarrow N_2^{n_2+k_2}$ be two generic maps. Then for a generic perturbation of their product we have*

$$\begin{aligned} [\Sigma^1 f \times g] &= \\ &= \sum_{j \geq 1} \left([\Sigma^1 f_{j-1}] \times (\text{id}_{M_2}^j)^* [\Sigma^1 g_{(-j)}] + (\text{id}_{M_1}^j)^* [\Sigma^1 f_{(-j)}] \times [\Sigma^1 g_{j-1}] \right) \end{aligned}$$

We compute the oriented Thom polynomial of the Σ^2 singularity with \mathbb{Q} coefficients.

THEOREM 9. *Let $f : M^n \rightarrow N^{n+k}$ be a generic map where $k = 2t - 2$. Then the rational cohomology class dual to the closure of the set of Σ^2 -points of f (for short $[\Sigma^2 f]$) equals $p_t(v_f)$, where $p_t \in H^{4t}(M; \mathbb{Q})$ is the t^{th} Pontrjagin class.*

Finally we derive a Cartan-type formula for the Σ^2 points as well.

THEOREM 10. *Let $f : M_1^{n_1} \rightarrow N_1^{n_1+k_1}, g : M_2^{n_2} \rightarrow N_2^{n_2+k_2}$ be two generic maps of even codimension. Then for a generic perturbation of their product we have*

$$\begin{aligned} [\Sigma^2 f \times g] &= \\ &= \sum_{j \geq 1} ([\Sigma^2 f_{2j-2}] \times (\text{id}_{M_2}^{2j})^* [\Sigma^2 g_{(-2j)}] + (\text{id}_{M_1}^{2j})^* [\Sigma^2 f_{(-2j)}] \times [\Sigma^2 g_{2j-2}]) \end{aligned}$$

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**CROUZEIX-VELTE DECOMPOSITIONS
AND THE STOKES PROBLEM**

Abstract of Ph.D. Thesis

By

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ADVISOR: GISBERT STOYAN

(Defended: June 18, 2009)

1. Abstract

In our dissertation the staggered grid approximation of the Stokes problem on several special domains and with different boundary conditions is investigated, using finite difference and box methods. (for finite element method see: GIRAUT-RAVIART, 1986) We deal with methods where the discrete Crouzeix-Velte decomposition (see: STOYAN-STRUBER-BARAN, 2004; or Subsection 1.2.) exists: we prove the existence of this decomposition and show numerical results to prove the effectiveness of these discretizations.

1.1. The Stokes problem

In our dissertation we consider at first the following first-kind Stokes problem in a cartesian coordinate system:

$$(1.1) \quad -\Delta \vec{u} + \operatorname{grad} p = \vec{f}, \text{ in } \Omega,$$

$$(1.2) \quad \operatorname{div} \vec{u} = 0, \text{ in } \Omega,$$

where $\Omega \subset \mathbb{R}^n$, $n = 2, 3$ is a bounded, simply-connected open domain, with Lipschitz – continuous boundary $\partial\Omega$, and

$$\vec{u}(x) = (u_1(x), \dots, u_n(x))^T,$$

$$\vec{f}(x) = (f_1(x), \dots, f_n(x))^T,$$

defined for $x = (x_1, \dots, x_n) \in \Omega$. On the boundary, homogeneous Dirichlet boundary conditions are imposed:

$$(1.3) \quad \vec{u} = 0, \text{ on } \partial\Omega.$$

The problem consists in finding a vector-function $\vec{u}(x)$ and a scalar function $p(x)$ that satisfy the system of partial differential equations above. The function \vec{u} is the velocity of fluid and p is the (kinematic) pressure. For a constant also appearing in the first equation, the kinematic viscosity, we have chosen the value 1. Finally, f describes accelerations caused by an external force field, and the boundary conditions mean that the walls are impermeable and at rest.

A unique weak solution $\vec{u} \in V$ and $p \in P$ exists when, for example, $\vec{f} \in (L_2(\Omega))^n$, (see, e. g., VARNHORN, 1994), where $P := L_{2,0}(\Omega)$ is the subspace of $L_2(\Omega)$ of square integrable functions with zero integral over Ω , with the Hilbert space

$$L_2(\Omega) = \{\phi | (\phi, \phi) < \infty\}, (\phi, \psi) = \int_{\Omega} \phi \psi d\Omega$$

and where $V := (H_0^1(\Omega))^n$ is the well-known Sobolev space, with generalized derivatives in $(L_2(\Omega))^n$ and with zero boundary values in the sense of traces on the boundary $\partial\Omega$.

1.2. The Crouzeix–Velte decomposition

For an $n > 2$, let (\cdot, \cdot) denote the Euclidean scalar product in \mathbb{R}^n , moreover, let $A, B, C \in \mathbb{R}^{n \times n}$ be matrices satisfying

$$(1.4) \quad A = B + C,$$

$$(1.5) \quad A = A^T > 0,$$

$$(1.6) \quad B = B^T \geq 0, \quad C = C^T \geq 0,$$

$$(1.7) \quad \delta := \dim \ker B \geq 1, \quad \rho := \dim \ker C \geq 1.$$

Then, with a suitable subspace $W \subset \mathbb{R}^n$ (which may turn out to be empty) and with orthogonality to be understood in the sense of the scalar product $(A \cdot, \cdot)$, the following orthogonal decomposition of \mathbb{R}^n can be derived:

$$(1.8) \quad \mathbb{R}^n = \ker B \oplus \ker C \oplus W.$$

This decomposition is called the *algebraic* Crouzeix–Velte decomposition of \mathbb{R}^n . (The definition of the *analytical* Crouzeix–Velte decomposition is described in the dissertation.)

For the eigenvalues λ_i of the generalized eigenvalue problem

$$(1.9) \quad \lambda Ax = Bx$$

we have

$$(1.10) \quad \lambda_i \in [0, 1] \text{ for all } i$$

and also using the eigenvectors $x^{(i)}$, the subspaces in (1.8) can be characterized as follows:

$$\begin{aligned} \ker B &= \text{span}(x^{(i)}, \lambda_i = 0), \\ \ker C &= \text{span}(x^{(i)}, \lambda_i = 1), \\ W &= \text{span}(x^{(i)}, \lambda_i \in (0, 1)). \end{aligned}$$

We can use the following correspondences connected with the matrices in (1.4):

$$A \sim -\Delta, \quad B \sim -\text{grad div}, \quad C \sim \text{curl rot},$$

where Δ is the (vector) Laplace operator and where the sign \sim expresses only an analogy between a differential operator and a matrix, and is not necessarily a (good) approximation. In this sense (1.4) corresponds to the well known identity

$$(1.11) \quad -\Delta = -\text{grad div} + \text{curl rot}$$

of vector analysis.

The algebraic Crouzeix–Velte decomposition is *proper* in case $\dim W = n - \delta - \rho > 0$.

In the *discrete* case, the velocity space (which approximates $(H^1(\Omega))^n$ or a subspace of the latter) will be denoted by \tilde{V}_h , the pressure space will be denoted by P_h , and div_h and rot_h will be written for the discrete equivalents of the divergence and rotation operator, Δ_h will denote the discrete Laplace operator. The matrix corresponding to the mapping $-\text{div}_h$ from the velocity space into the pressure space is denoted by \tilde{B}_h and we introduce the following notations: \tilde{C}_h for the matrix of the operator rot_h and A_h for the matrix of the operator $-\Delta_h$. If A_h , \tilde{B}_h , \tilde{C}_h matrices satisfy the following:

$$(1.12) \quad \begin{aligned} A_h &= B_h + C_h, \\ A_h &= A_h^T > 0, \end{aligned}$$

where $B_h = \tilde{B}_h^T \tilde{B}_h$ and $C_h = \tilde{C}_h^T \tilde{C}_h$ and $\ker \tilde{B}_h \neq \emptyset$ and $\ker \tilde{C}_h \neq \emptyset$, then a discrete Crouzeix–Velte decomposition exists and (1.8) takes the form

$$V_h = \ker \text{div}_h \oplus \ker \text{rot}_h \oplus W = V_{h,0} \oplus V_{h,1} \oplus V_{h,\beta}.$$

P_h is decomposed similarly into three orthogonal subspaces:

$$P_h = \ker \operatorname{grad}_h \oplus \operatorname{div}_h \ker \operatorname{rot}_h \oplus \operatorname{div}_h V_{h,\beta} = P_{h,0} \oplus P_{h,1} \oplus P_{h,\beta}.$$

After discretization by the finite element or finite difference methods, the Stokes problem takes the following form:

$$(1.13) \quad \begin{pmatrix} A_h & \tilde{B}_h^T \\ \tilde{B}_h & 0 \end{pmatrix} \begin{pmatrix} u \\ p \end{pmatrix} = \begin{pmatrix} f \\ 0 \end{pmatrix}.$$

With these matrices (1.9) corresponds to

$$\lambda_h A_h x_h = B_h x_h,$$

which is transformed by $p_h := \tilde{B}_h x_h$ into

$$\lambda_h p_h = S_h p_h,$$

where S_h is the discrete Schur complement operator of the Stokes problem, $S_h := \tilde{B}_h A_h^{-1} \tilde{B}_h^T$. If the discrete Crouzeix-Velte decomposition exists then $\lambda_h \in [0, 1]$.

In our dissertation we deal with approximations of the Stokes problem on several domains, where the discrete Crouzeix-Velte decomposition exists. In this case in the spectrum of the discrete Schur complement operator, there is an eigenvalue 1 of high multiplicity (of the order of inner grid points). Then, the error components lying in the eigensubspace corresponding to this eigenvalue can be removed by one step of a simple damped Jacobi iteration or Uzawa-algorithm. The remaining error lies in an eigensubspace of much smaller dimension connected only with boundary effects. Moreover, the conjugate gradient iteration automatically takes advantage of such a spectrum and converges faster than for discrete spaces without a decomposition: as it is proved by the numerical results.

In our dissertation the following methods are used: finite difference and box methods for approximation of Stokes-problem, and Uzawa-algorithm (as outer iteration) in the numerical experiments, moreover the conjugate gradient method (as inner iteration) with an effective preconditioning matrix and FFT algorithm and alternatively multigrid method solving the discrete Poisson equations.

2. Results

2.1. First order staggered grid approximation on non-equidistant rectangular grids

First we consider the well-known staggered-grid approximation where Ω is a rectangle subdivided by a non-equidistant grid into $(n - 1)(m - 1)$ rectangular cells. The boundary conditions are homogeneous Dirichlet boundary conditions.

For pressure vectors p_h and velocity vectors $\vec{u}_h = (u_h, v_h)^T$ suitable discrete scalar products and the corresponding norms are introduced. Using these scalar products and norms the following Theorem is proved, with the notations $A_h, \tilde{B}_h, \tilde{C}_h$ for the matrices of the operators $-\Delta_h$, $-\operatorname{div}_h$ ill rot_h :

THEOREM 1.

$$(2.1.) \quad (A_h \vec{u}_h, \vec{u}_h)_{0,h} = \|\tilde{B}_h \vec{u}_h\|_{0,h}^2 + \|\tilde{C}_h \vec{u}_h\|_{0,\tilde{h}}^2$$

holds for all vectors $\vec{u}_h := (u_h, v_h)^T \in \vec{V}_h$ if and only if

$$h_{1,i+1/2} = \frac{h_{1,i-1/2} + h_{1,i+3/2}}{2} \quad \text{and} \quad h_{2,j+1/2} = \frac{h_{2,j-1/2} + h_{2,j+3/2}}{2}. \blacksquare$$

REMARK 1. It is proved that $\dim(V_{h0}) = (n - 2)^2$, $\dim(V_{h1}) = (n - 3)^2$ and $\dim(V_{h\beta}) = 4n - 9$, where n denotes the number of grid points (including corner points) along a side of the square. \blacksquare

REMARK 2. We prove that A_h is symmetric in the sense of the scalar product $(\vec{u}_h, \vec{v}_h)_{0,h}$ above, and the following equation holds:

$$(2.2) \quad D_A A_h = \tilde{B}_h^T D_B \tilde{B}_h + \tilde{C}_h^T D_C \tilde{C}_h,$$

where D_A, D_B, D_C are diagonal matrices corresponding to the adequate norms. That is

$$(2.3) \quad D_A A_h =: \hat{A}_h = B_h + C_h,$$

where $B_h = \hat{B}^T \hat{B}$ and $C_h = \hat{C}^T \hat{C}$ with the notations: $\hat{B} = D_B^{1/2} \tilde{B}_h$, $\hat{C} = D_C^{1/2} \tilde{C}_h$. \blacksquare

REMARK 3. We prove that $D_A A_h = \hat{A}_h$ matrix is positive definite, that is $\hat{A}_h > 0$. Together with remark 1 and 2, it means that a proper Crouzeix–Velte

decomposition of the velocity and the pressure space into three nontrivial parts exists, if $n > 3$. ■

2.2. First order staggered grid approximation based on the finite volume (box) method on non-equidistant rectangular grids

Then we apply finite volume method on the staggered grid, where Ω is also a rectangle subdivided by a non-equidistant grid into $(n - 1)(m - 1)$ rectangular cells. The boundary conditions are homogeneous Dirichlet boundary conditions.

We prove that using this approximation, the discrete Crouzeix–Velte decomposition exists, if $n > 3$, without any restriction of grid spacing.

REMARK. The results above (using either finite difference or finite volume methods) hold if Ω is a union of rectangles such that all the boundary lines of the different rectangles fit on the same global grid with grid spacing h_1 and h_2 . ■

2.3. Numerical results

We show with numerical experiments — using the staggered grid approximation based on the finite volume method — that both the Uzawa-type and the conjugate gradient-type methods are faster on such grids which are condensing in the center and coarser near the boundary of the domain. In the case of Uzawa-type methods we determine the optimal non-equidistant grid.

2.4. Second order staggered grid approximation based on the finite difference method on equidistant rectangular grids

Then we apply second order finite difference method on the staggered grid, where Ω is a unit square subdivided by an equidistant grid. The boundary conditions are homogeneous Dirichlet boundary conditions. After defining the suitable approximations, discrete scalar products and the corresponding norms, the following Theorem is proved:

THEOREM. *For the second order staggered grid approximation,*

$$(A_{\bar{h}} \vec{u}_h, \vec{u}_h)_{0,h} - \|\tilde{B}_{\bar{h}} \vec{u}_h\|_{0,h}^2 - \|\tilde{C}_{\bar{h}} \vec{u}_h\|_{0,h}^2 =$$

$$(2.4) \quad = - \sum_{i=2}^{n-1} 2(u_{i,1}^2 + u_{i,n-1}^2) - \sum_{j=2}^{n-1} 2(v_{1,j}^2 + v_{n-1,j}^2),$$

where $A_{\bar{h}} := -\Delta_{\bar{h}}$, $\tilde{B}_{\bar{h}} := -\operatorname{div}_{\bar{h}}$ and $\tilde{C}_{\bar{h}} := \operatorname{rot}_{\bar{h}}$. ■

REMARK. We introduce the notation $\tilde{A}_{\bar{h}}$, where

$$(\tilde{A}_{\bar{h}}\vec{u}_h, \vec{u}_h)_{0,h} = (A_{\bar{h}}\vec{u}_h, \vec{u}_h)_{0,h} + \sum_{i=2}^{n-1} 2(u_{i,1}^2 + u_{i,n-1}^2) + \sum_{j=2}^{n-1} 2(v_{1,j}^2 + v_{n-1,j}^2).$$

$\tilde{A}_{\bar{h}}$ is a symmetric positive definite matrix in the sense of the corresponding scalar product, together with $A_{\bar{h}}$. Since $\dim(V_{h0}) = (n-2)^2$, $\dim(V_{h1}) = (n-3)^2$ and $\dim(V_{h,\beta}) = 4n-9$, a proper Crouzeix–Velte decomposition exists in this case as well, for $n > 3$. In this case the algebraic decomposition exists not for the matrix $A_{\bar{h}}$, but for $\tilde{A}_{\bar{h}}$, hence $\tilde{A}_{\bar{h}}$ can be advantageous as a preconditioner matrix solving $A_{\bar{h}}\vec{u}_h = b_h$. ■

2.5. Finite difference approximation in the case of non-standard boundary conditions

We investigate the finite difference approximation of the Stokes problem on the staggered grid with the following non-standard boundary conditions:

$$(2.5) \quad u_{1,j} = u_{n,j} = v_{i,1} = v_{i,n} = 0,$$

for $1 \leq i \leq n-1$, $1 \leq j \leq n-1$ and

$$(2.6) \quad (\operatorname{rot}_h \vec{u}_h)_{0,j} = (\operatorname{rot}_h \vec{u}_h)_{n-1,j} = (\operatorname{rot}_h \vec{u}_h)_{i,1} = (\operatorname{rot}_h \vec{u}_h)_{i,n} = 0,$$

$0 \leq i \leq n-1$, $2 \leq j \leq n-1$, where rot_h is defined as usual. (This boundary condition satisfies the Lopatinski-condition.)

We show that a proper discrete Crouzeix–Velte decomposition exists in the case of this boundary condition as well, using either first or second order approximation. ■

2.6. Finite difference approximation in the case of periodical boundary conditions

As we show in this point, the results on the existence of an analytical Crouzeix–Velte decomposition for the Stokes problem along with Dirichlet and periodical boundary conditions carry over to the discrete case for the staggered grid approximation.

We apply first order finite difference approximation on the staggered grid, where Ω is a unit square subdivided by an equidistant grid. Periodical boundary conditions are assumed on the left and right sides of the unit square:

$$(2.7) \quad \begin{aligned} u_{1,j} &= u_{n-1,j}, u_{2,j} = u_{n,j}, & 0 \leq j \leq n, \\ v_{1,j} &= v_{n-1,j}, v_{2,j} = v_{n,j}, & 1 \leq j \leq n. \end{aligned}$$

On the upper and lower sides of the unit square we prescribe homogeneous Dirichlet conditions:

$$\begin{aligned} u_{i,0} &= u_{i,n} = 0, & 1 \leq i \leq n, \\ v_{i,1} &= v_{i,n} = 0, & 1 \leq i \leq n-1, \end{aligned}$$

where $u_{i,0}, u_{i,n}$ are values on two additional grid lines, which the grid has been supplemented with. We prove that a proper discrete Crouzeix–Velte decomposition exists in this case as well. ■

2.7. Approximation on a nonequidistant grid in 3D with homogeneous Dirichlet boundary conditions

Then the well-known difference approximation on a staggered grid is considered in 3D case. In our case Ω is a rectangular parallelepipedon subdivided by a rectangular grid into $(n-1)(m-1)(l-1)$ cells of volume $h_1 h_2 h_3$ each, $h_1 := 1/(n-1), h_2 := 1/(m-1), h_3 := 1/(l-1)$. We assume $n, m, l \geq 3$.

First we assume homogeneous Dirichlet boundary conditions:

$$u_{1,j,k} = u_{n,j,k} = v_{i,1,k} = v_{i,m,k} = w_{i,j,1} = w_{i,j,l} = 0,$$

where $\tilde{u}_h := (u_h, v_h, w_h)^T$ is the velocity vector and $1 \leq i \leq n-1, 1 \leq j \leq m-1, 1 \leq k \leq l-1$.

We prove the existence of the discrete Crouzeix–Velte decomposition. ■

2.8. Approximation on an equidistant grid in 3D with periodical boundary conditions

Then we consider the first order staggered grid approximation on an equidistant, cubic grid. On the front-back sides and on the north-south sides of the cube we prescribe homogeneous Dirichlet conditions, and periodical boundary conditions are assumed on the east-west sides. We prove the existence of the discrete Crouzeix–Velte decomposition in this case as well. ■

2.9. The Stokes problem in polar coordinates for the disk domain

Finally let Ω be the unit disk

$$\Omega = \{(r, \varphi) | 0 \leq r < 1, 0 \leq \varphi < 2\pi\},$$

and consider the following Stokes problem:

$$(2.8) \quad \Delta_{r\varphi} u - \frac{u}{r^2} - \frac{2}{r^2} \frac{\partial v}{\partial \varphi} - \frac{\partial p}{\partial r} = f_1,$$

$$(2.9) \quad \Delta_{r\varphi} v - \frac{v}{r^2} + \frac{2}{r^2} \frac{\partial u}{\partial \varphi} - \frac{1}{r} \frac{\partial p}{\partial \varphi} = f_2,$$

$$(2.10) \quad \operatorname{div} \vec{u} = \frac{1}{r} \left(\frac{\partial}{\partial r} (ru) + \frac{\partial v}{\partial \varphi} \right) = 0,$$

where $(u, v) = \vec{u}$ and $(f_1, f_2) = \vec{f}$ and

$$\Delta_{r\varphi} = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2}{\partial \varphi^2}.$$

On the boundary, homogeneous Dirichlet boundary conditions are imposed:

$$(2.11) \quad \vec{u} = 0, \text{ on } \partial\Omega.$$

The problem consists in finding a vector-function $\vec{u}(x)$ and a scalar function $p(x)$ that satisfy the system of partial differential equations above. In our dissertation we consider a suitable second order finite difference approximation and introduce the corresponding discrete scalar products and norms. The following Theorem — similarly to (2.1) — is proved:

THEOREM.

$$(2.12) \quad (\tilde{A}_h \vec{u}_h, \vec{u}_h)_{0,h} = \|\tilde{B}_h \vec{u}_h\|_{0,\bar{r},h}^2 + \|\tilde{C}_h \vec{u}_h\|_{0,r,h}^2$$

holds for all vectors $\vec{u}_h := (u_h, v_h)^T \in \vec{V}_h$, where \tilde{B}_h corresponds to the negative divergence operator, \tilde{C}_h corresponds to the rotation operator and \tilde{A}_h corresponds to the negative Laplace operator, which is in polar coordinates:

$$(2.13) \quad \Delta \vec{u} = \begin{pmatrix} \Delta_{r\varphi} u - \frac{u}{r^2} - \frac{2}{r^2} \frac{\partial v}{\partial \varphi} \\ \Delta_{r\varphi} v - \frac{v}{r^2} + \frac{2}{r^2} \frac{\partial u}{\partial \varphi} \end{pmatrix}. \blacksquare$$

REMARK. From the Theorem above we obtain that \tilde{A}_h can be written in the following form:

$$(2.14) \quad D_{\tilde{A}} \tilde{A}_h =: A_h = B_h + C_h,$$

where $B_h = \hat{B}_h^T \hat{B}_h$, $C_h = \hat{C}_h^T \hat{C}_h$ and $\hat{B}_h = D_{\tilde{B}}^{1/2} \tilde{B}_h$, $\hat{C}_h = D_{\tilde{C}}^{1/2} \tilde{C}_h$ and where $D_{\tilde{A}}$, $D_{\tilde{B}}$, $D_{\tilde{C}}$ are diagonal matrices corresponding to the adequate scalar products. \blacksquare

2.10. Numerical results for approximation on the unit disk

2.10.1. Uzawa-algorithm

Using the notations (2.14) our problem consists in finding the solution of the following algebraic system:

$$(2.15) \quad A_h \vec{u}_h + \hat{B}_h^T p_h = \vec{f}_h,$$

$$(2.16) \quad \hat{B}_h \vec{u}_h = g_h.$$

We use the Uzawa-algorithm to solve (2.15), (2.16):

$$(2.17) \quad \begin{aligned} p_h^{(0)} &:= 0, \\ p_h^{(i+1)} &:= p_h^{(i)} + \omega (\hat{B}_h \vec{u}_h^{(i)} - g_h) \\ \vec{u}_h^{(i)} &:= A_h^{-1} (\vec{f}_h - \hat{B}_h^T p_h^{(i)}) \\ i &= 0, 1, 2, \dots \end{aligned}$$

Since the discrete Crouzeix–Velte decomposition exists, using the Uzawa-algorithm we can reach the third Crouzeix–Velte subspace after 1 step (with $\omega = 1$). In this subspace the spectrum of the Schur complement is closer, and the algorithm shows effective convergence. The optimal iteration parameter is

also calculated using the smallest and the largest of the eigenvalues different from 0 and 1 of the discrete Schur complement. The numerical results show that the discretization obeying a discrete Crouzeix–Velte decomposition leads to effectively solvable systems of algebraic equations. The average number of the iterations using the Uzawa algorithm is 3-5, and the speed of convergence is growing together with the refinement of the grid.

2.10.2. Fourier transformation and conjugate gradient method

Instead of the calculation of A_h^{-1} in (2.17) in the first case the fast Fourier transformation is used in combination with the preconditioned conjugate gradient method. In the numerical experiments several preconditioning matrices are investigated: the number of inner iterations needed to reach the stopping criterion of the conjugate gradient method, using the best preconditioning matrix, is only 4-5, and the speed of convergence is growing together with the refinement of the grid.

2.10.3. Multigrid method

In the second case the multigrid method is used to calculate A_h^{-1} . We describe the restriction operator and optimize the prolongation operator with numerical experiments. We compare several pre- and post-smoothing iterations. The optimal iteration parameter of the damped Jacobi iteration — as smoothing iteration — is also calculated. The results show that the Gauss–Seidel iteration combined with block Gauss–Seidel iteration — as smoothing iteration — is the most effective: the multigrid method using this iteration is on average 5-8 times faster than using the Gauss–Seidel iteration and this advantage is growing with the refinement of the grid. The less effective smoothing iteration is the Jacobi iteration, it results on average three times slower convergence than the Gauss–Seidel iteration.

3. Publications

GY. STRAUBER: Discrete Crouzeix–Velte decompositions on nonequidistant rectangular grids, *Annales Univ. Sci. Budapest.*, **44** (2002), 63–82.

G. STOYAN, GY. STRAUBER, A., BARAN: Generalizations to discrete and analytical Crouzeix–Velte decompositions, *Numer. Lin. Algebra with Appl.*, **11** (2004), 565–590.

GY. STRAUBER: Discrete Crouzeix–Velte decomposition for the disk domain, *Miskolc Mathematical Notes*, **6** (2005), No. 1, 129–143.

**LEGENDRIAN AND TRANSVERSE KNOTS IN THE LIGHT OF
HEEGAARD FLOER HOMOLOGY**

Abstract of Ph.D. Thesis

By

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(*Defended: June 3, 2009*)

1. Introduction

Legendrian and transverse knot theory has been shaped by advances in convex surface theory [7] (showing that different looking objects are actually equivalent) and by the introduction of various invariants of these knots — proving that different looking objects are, in fact, different. Examples of such invariants are provided by Chekanov’s differential graded algebras and contact homology [2, 3]. More recently, Heegaard Floer homology provided various sets of invariants: for knots in the standard contact 3-sphere the combinatorial construction of knot Floer homology through grid diagrams [16, 24], for null-homologous knots in general contact 3-manifolds the Legendrian invariant of [13] and for general Legendrian knots the sutured invariant of the knot complement [10]. In this dissertation we study these invariants to get a better understanding of Legendrian and transverse knots.

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1.1. Heegaard Floer theories

Heegaard Floer homologies, (OZSVÁTH–SZABÓ, [19, 20, 22]) the recently-discovered invariants for 3- and 4-manifolds, come from an application of Lagrangian Floer homology to spaces associated to Heegaard diagrams. Although this theory is conjecturally isomorphic to Seiberg–Witten theory, it is more topological and combinatorial in its flavor and thus easier to work with in certain contexts. These homologies admit generalizations and refinements for knots (OZSVÁTH–SZABÓ [18] and RASMUSSEN [26]) and links (OZSVÁTH–SZABÓ [23]) in 3-manifolds and for non-closed 3-manifolds with certain boundary conditions (JUHÁSZ [11]), called sutured Floer homology. The tools used to define the link-version were later applied to define a completely combinatorial version of knot Floer homology in the 3-sphere.

1.2. Contact 3-manifolds

Although contact geometry was born in the late 19th century in the work of Sophus Lie, it has just recently started to develop rapidly, with the discovery of convex surface theory and by recognizing their role in other parts of topology. For example Property P for knots — a possible first step for resolving the Poincaré conjecture — was proved using contact 3-manifolds (KRONHEIMER–MROWKA [12]). Also, the fact that Heegaard Floer homology determines the Seifert genus of a knot was first proved with the help of contact 3-manifolds (OZSVÁTH–SZABÓ [17]). Being the natural boundaries of Stein domains, the use of contact 3-manifolds resulted in a topological description of Stein-manifolds. A *contact structure* on an oriented 3-manifold is a totally non-integrable plane field. In other words it is a plane distribution that is not everywhere tangent to any open embedded surface. Any 3-manifold admits a contact structure (MARTINET [14]). It is more subtle though to understand the set of all different contact structures on a given 3-manifold. One way to understand them is by examining lower dimension submanifolds that respect the structure in a way. The 2 dimensional such submanifolds are called *convex surfaces*. These are surfaces with a vectorfield in their neighborhood which is transverse to the surface and whose flow preserves the contact plane distribution. Contact structures in the neighborhood of a convex surface are determined by a set of closed curves (*dividing curves*) on the surface (GIROUX [9]). Thus convex surfaces became the right boundary conditions for contact 3-manifolds. In Heegaard Floer homology contact invariants were defined for contact 3-manifolds without (OZSVÁTH–SZABÓ [21]) or with (HONDA–KAZEZ–MATIC [10]) boundary. These invariants had many applications the

most recent is a new proof for the fact that a contact 3-manifold having Giroux torsion cannot be Stein-fillable (GHIGGINI–HONDA–VAN HORN–MORRIS [8]).

1.3. Legendrian and transverse knots

There are two ways for a one dimensional submanifold to respects the contact structure. Its tangents can entirely lie in the plane distribution, in which case the knot is called *Legendrian knot*, or if the tangents are transverse to the planes, the knot is then called a *transverse knot*. A Legendrian knot with a given knot type has two classical invariants: its Thurston-Bennequin number and its rotation number. While for transverse knots there is only one invariant; the self-linking number. The problem of classifying Legendrian (transverse) knots up to Legendrian (transverse) isotopy naturally leads to the question whether these invariants classify Legendrian (transverse) knots. A knot type is called *Legendrian (transverse) simple* if any two realizations of it with equal classical invariants are Legendrian (transverse) isotopic. The unknot (ELIASHBERG–FRASER [4]), torus knots and the figure-eight knot (ETNYRE–HONDA [7]) were proved to be both Legendrian and transversely simple. By constructing a new invariant for Legendrian knots, CHEKANOV [2] showed that not all knots are Legendrian simple, in particular he proved that the knot 5_2 is not Legendrian simple. Later many other Legendrian non-simple knots were found (EPSTEIN–FUCHS–MEYER [5] and NG [15]). The case for transverse knots turned out to be harder. BIRMAN and MENASCO [1], and ETNYRE and HONDA [6] constructed families of transversely non-simple knots using braid and convex surface theory. The Legendrian invariant in the combinatorial Floer homology provided another tool to construct transversely non-simple knots (NG–OZSVÁTH–THURSTON [16]).

Using the language of Heegaard Floer homology recently three different invariants were defined for Legendrian and transverse knots. One in the combinatorial settings of knot Floer homology for the 3-sphere [25]: $\hat{\lambda}$, one in knot Floer homology for a general contact 3-manifold [13]: $\hat{\mathcal{L}}$ and one defined as the contact invariant associated to the knot-complement: EH.

2. Main Results

In the dissertation I used Heegaard Floer homology to prove theorems about Legendrian and transverse knots.

Similarly to the smooth case there is a well defined notion of connect summing Legendrian or transverse knots.

THEOREM 2.1. VÉRTESI [29] *Let L_1 and L_2 be (oriented) Legendrian knots of topological type K_1 and K_2 . Then there is an isomorphism*

$$\text{HFK}^-(m(K_1)) \otimes_{\mathbb{F}_2[U]} \text{HFK}^-(m(K_2)) \rightarrow \text{HFK}^-(m(K_1 \# K_2))$$

which maps $\lambda_+(L_1) \otimes \lambda_+(L_2)$ to $\lambda_+(L_1 \# L_2)$. Similar statement holds for the λ_- -invariant.

COROLLARY 2.2. VÉRTESI [29] *Let L_1 and L_2 be (oriented) Legendrian knots of topological type K_1 and K_2 . Then there is an isomorphism*

$$\widehat{\text{HFK}}(m(K_1)) \otimes_{\mathbb{F}_2} \widehat{\text{HFK}}(m(K_2)) \rightarrow \widehat{\text{HFK}}(m(K_1 \# K_2))$$

which maps $\widehat{\lambda}_+(L_1) \otimes \widehat{\lambda}_+(L_2)$ to $\widehat{\lambda}_+(L_1 \# L_2)$. Similar statement holds for the $\widehat{\lambda}_-$ -invariant. ■

Similar results hold for the θ -invariant of transverse knots:

COROLLARY 2.3. VÉRTESI [29] *Let T_1 and T_2 be transverse knots of topological type K_1 and K_2 . Then there are isomorphisms*

$$\text{HFK}^-(m(K_1)) \otimes_{\mathbb{F}_2[U]} \text{HFK}^-(m(K_2)) \rightarrow \text{HFK}^-(m(K_1 \# K_2))$$

and

$$\widehat{\text{HFK}}(m(K_1)) \otimes_{\mathbb{F}_2} \widehat{\text{HFK}}(m(K_2)) \rightarrow \widehat{\text{HFK}}(m(K_1 \# K_2))$$

which map $\theta(T_1) \otimes \theta(T_2)$ to $\theta(T_1 \# T_2)$ and $\widehat{\theta}(T_1) \otimes \widehat{\theta}(T_2)$ to $\widehat{\theta}(T_1 \# T_2)$, respectively. ■

As an application of the above result we prove:

THEOREM 2.4. VÉRTESI [29] *There exist infinitely many transversely non-simple knots.*

The definition of the contact invariant in Heegaard Floer homology admits a generalization for Legendrian and transverse knots $\widehat{\mathcal{L}}$ in the knot Floer homology (LISCA–OZSVÁTH–STIPSICZ–SZABÓ [13]). The contact invariant of Honda, Kazez and Matic for the complement of a Legendrian knot gives rise to a Legendrian invariant: the EH-class. With Stipsicz we understood the relation between these two invariants:

THEOREM 2.5. STIPSICZ–VÉRTESI [27] *There is a map from the sutured Floer homology for the knot-complement to the knot Floer homology mapping $\widehat{\mathcal{L}}$ to EH.*

A nice consequence of this theorem, which was independently obtained by VELA-VICK [28], is the following:

THEOREM 2.6. STIPSICZ–VÉRTESI [27] *If the knot complement contains Giroux torsion, then $\widehat{\mathcal{L}}$ vanishes.*

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CONFERENCES

**Student Research Circle Conference
Eötvös Loránd University, December 3, 2010**

Gábor Csörgő: The comparison and bifurcational investigation of equations describing chemical reactions

Balázs Kovács: Comparison of efficient numerical methods on an elliptic problem

The aim of this paper is to compare and realize three efficient iterative methods which have mesh independent convergence.

There are several approaches for solving partial differential equations. The finite element method is widely used. The system of algebraic equations given by the FEM discretization can be solved in both iterative and direct ways. If the equation is nonlinear then we need an iterative method, as is the case in this paper where we discuss the radiative cooling problem:

$$\begin{cases} -\operatorname{div}(k \nabla u) + \sigma u^4 = 0 & \text{in } \Omega \\ k \partial_\nu u + \alpha(u - \tilde{u}) = 0 & \text{on } \partial\Omega. \end{cases}$$

We look for the solution of the above model problem using finite element discretization with gradient and Newton type methods. We show that there exists a unique weak solution, we state convergence theorems concerning the numerical methods and we also verify that they are applicable.

Three numerical methods have been carried out, namely, the gradient, Newton and quasi-Newton methods. We have solved the model problem

with these methods, we have investigated the differences between them and analyzed their efficiency and mesh independence.

Gergő Nemes: $\log 2$, π , infinite series, continued fractions and Laplace transforms

In this paper we investigate the behavior of the remainder of Leibniz's series:

$$\frac{\pi}{4} - \sum_{k=1}^N \frac{(-1)^{k-1}}{2k-1}.$$

We also examine the accuracy of the well-known series for $\log 2$:

$$\log 2 - \sum_{k=1}^N \frac{(-1)^{k-1}}{k}.$$

Using Laplace transforms we give asymptotic expansions and continued fraction representations for the remainders in both cases. Some of these are generalizations of previous results. We characterize the error terms by inverse factorial series as well. These turn out to be useful tools to prove some identities between various sequences.

László Sajtos: Modelling changes of stock exchange indices by copulas

We have investigated the structure of relationships of several American and European stock indices by copulas (we used daily log-return data from the past few years). The fundamental idea was the probability integral transformation, which allowed us to study the problem in one dimension, called Kendall transform. We studied and illustrated the theoretical and empirical Kendall transform of Gauss, t and Archimedean copulas, including the goodness of fit of these copulas. The temporal dynamics can be observed by shifted-windows, thus the time-dependence of the changes of the fitted parameters of the used copulas could be realized. Consequently, this method can be applied to study the changes of financial markets.

Dániel Soukup: The inconvenient D -property

We prove that there exists a 0-dimensional, scattered T_2 space X such that X is aD but not linearly D, answering a question of Arhangel'skii. The constructions are based on Shelah's club guessing principles.

András Szabó: Epidemic spread on contact networks: Comparison of the differential equation and the Monte Carlo simulation

The dynamics of disease transmission strongly depends on the properties of the population contact network. In this paper, using a continuous time Markov Chain, we start from the exact formulation of a simple epidemic model on an arbitrary contact network and compare with heuristic approximations and with Monte Carlo simulation results. We are looking for the expected value of the number of infected nodes. The main result is that the error of the approximations is large when we use a special graph, called modified cycle graph. This graph is similar to the well known "small world" graph, but it is not a random graph. Finally, we discuss the advantages and possible applications of the approximations.

Dorottya Sziráki: Applying Algebraic Logic to Vaught's Conjecture and Related Problems

Let Σ be a complete first order theory in a countable language with equality. Vaught conjectured that if Σ has more than \aleph_0 pairwise non-isomorphic countable models, then it has 2^{\aleph_0} such models. Since it was first published, Vaught's conjecture has become an important open problem and has been researched intensively.

In this paper, we investigate what happens when we replace the role of isomorphism with that of elementary embeddability, and, if for a submonoid S of injective functions on ω , we allow only elementary embeddings that are elements of S . We also consider the problem for first order logic without equality. Thus, we obtain the following variant of the original conjecture. Let Σ be a theory in a first order language with or without equality, and let S be a submonoid of the injective functions on ω . If Σ has more than countably many pairwise non S -elementarily embeddable countable models, then it has continuum many such models.

Our main result is proving the above variant for σ -compact monoids S . The proofs are based on the representation theories of certain cylindric and quasi-polyadic algebras and investigations of the Stone spaces of these algebras. A similar result had been proven in the case models are considered up to isomorphism, for first order logic without equality. Our techniques and results are generalizations of the ones there.

István Tomon: Covering point sets with monotone paths

Lilla Tóthmérész: The application of co-citation and SimRank to large biological networks

Co-citation and SimRank are functions on graphs, that assign a similarity value to each pairs of nodes. These measures were inspired by the analysis of the web and by scientific literature analysis. Our aim was to test how well these measures reflect the functional similarities of proteins, when applied to protein interaction networks. For testing the measures, we applied them to human and yeast protein interaction networks. We compared the results to the similarities indicated by the Pfam protein families using the Kruskal-Goodman Γ measure.

In this application we have quite large graphs, therefore the fast computability of these measures is an important question. In the paper we proposed a refined version of the SimRank computing algorithm.

Biological data may contain many faults, therefore we also examined the stability of these measures.

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