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2010

LÁSZLÓ SIMON IS 70 YEARS OLD

By

ISTVÁN FARAGÓ and LÁSZLÓ LOVÁSZ

This year we celebrated the 70th birthday of László Simon, an outstanding scientist of the Hungarian an international mathematical life and an excellent teacher at the Institute of Mathematics, Eötvös Loránd University.

A short biography

László Simon was born on 13 April, 1940. He graduated from the Eötvös Loránd University in 1963 with a Diploma in Mathematics. He defended his PhD thesis, in which he studied boundary value problems for generalized functions, in 1966.

From September, 1968 he was correspondent fellow at the Department of Differential Equations of Moscow State University under the supervision of B. R. Vainberg. He became Candidate of Sciences in 1973 with his thesis “Approximation of elliptic boundary value problems on unbounded domains”. He became a Doctor of the Hungarian Academy of Sciences in March, 1990. The title of his thesis was “Elliptic equations of order $2m$ ”.

He has been teaching at the Department of Applied Analysis and Computational Mathematics (formerly Analysis II) of the Eötvös Loránd University since 1963.

He has participated in several international collaborations, among others with Moscow State University, the Institute of Applied Mathematics of Heidelberg University, the Department of Applied Mathematics of Complutense University, Madrid, the Institute of Mathematics at the University of Oulu and the Institute of Mathematics at the University of Strasbourg.

Research work, public activity

László Simon is an acknowledged scientist of the Hungarian and international mathematical life. His field of research is the theory of partial differential equations. His earlier research was mainly focussed on elliptic partial differential equations. At present he primarily studies nonlinear elliptic, parabolic and functional differential equations, and investigates the existence and qualitative behaviour of the solutions. To this aim, he applies the tools of functional analysis, mainly the theory of monotone operators. His results have been presented in many Journal articles, and also in the Handbook of Differential Equations published by Elsevier.

In the past 20 years, he was the principal investigator of several ministry and OTKA (Hungarian Scientific Research Fund) research projects. Since 1980, he has been member of the editorial board of the physics column of KöMaL (the Mathematical and Physical Journal for Secondary Schools). He was a member of the Mathematical Committee of the Hungarian Scientific Research Fund, the Mathematical Doctoral Committee of the Hungarian Academy of Sciences and the Committee of the Farkas Gyula Prize of the János Bolyai Mathematical Society for several years.

Educational activity, awards

László Simon has played an important role in the education of partial differential equations. He worked out the textbook of the partial differential equation course at the Eötvös Loránd University and he has been the Professor of this course for decades. He enriched the Hungarian literature of partial differential equations very remarkably. The classical theory is discussed in his book, written jointly with László Czák, which appeared in 1970. Another book he wrote with E. A. Baderko appeared in 1983. This book has become the fundamental Hungarian reference of the modern theory of partial differential equations (including the theory of distributions and Sobolev spaces). Generations of students have learned partial differential equations from this book.

He had four students who received the degree of Candidate of Sciences, and three students who obtained a Phd degree: Tamás Pfeil, Ferenc Izsák and Ádám Besenyei, who are recently members of the Institute of Mathematics.

In 1988 he was awarded the Prize of Natural Sciences of Eötvös Loránd University, and in 1993 the Pro Universitate Medal.

What is below the surface...

All those who have ever been in contact with László Simon, would certainly admit that he is a remarkably helpful and most reliable colleague. He does every job with the utmost care and devotion. Should anyone turn to him with any kind of problem, he always does his best to solve it. Laci's wide knowledge, along with his interest and willingness to work for the community, is fully acknowledged by everybody. It is no surprise that he is so respected and liked by his students as well as his colleagues.

Besides his work, he has several hobbies, in addition to which he is mostly engrossed by his 12 grandchildren, daughters and sons of his three children. According to the unanimous opinion of those who know him, he does this exactly as well, and with as much devotion, as his other engagements.

For an outsider it is almost incomprehensible how he has so much time and energy for all this!

The conference and the special issue

On the occasion of the round anniversary, the Institute of Mathematics organized a one-day conference on 20 May, 2010 in the honour of László Simon. Due to the limited time, there were relatively few talks. The invited speakers were not only outstanding specialists of the field of differential equations, but also well reflected the international reputation of our celebrated colleague. Speakers came from France, Russia and Germany, but colleagues from almost all leading Hungarian universities were also present.

This volume contains mainly the material of the talks presented in the conference. We hope that it offers a good illustration of the successful scientific and educational carrier of Laci.

Acknowledgements

Finally, we thank everybody who contributed to the celebration of László Simon's 70th birthday. However, the biggest gratitude is due to László himself: we thank him for all the great services he provided to the Institute and the Department during his career. We are grateful for his work, his educational activity, his unlimited patience, with which he helped and encouraged his students as well as us, his colleagues in our work. I would like to thank him

particularly for setting an example with his devotion to his work and with his lifestyle. We wish that all this will continue unchanged for a very-very long time!

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GENERALIZED SOLUTIONS OF THE CAUCHY PROBLEM FOR NON-DIVERGENCE PARABOLIC EQUATIONS

By

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(Received October 17, 2010)

Dedicated to Professor László Simon on the occasion of his 70th birthday

Abstract. A notion of generalized solution is introduced in this paper for the Cauchy problem for non-divergence parabolic equations in the half-space. The aim of the work is to prove existence of such a solution and to establish the character of its regularity in terms of estimates for the solution itself and its gradient.

1. Notations

Let $D = \{x \in \mathbb{R}^n, t > 0\}$. We denote by $C^0(\bar{D})$ the space of all functions f defined and continuous on \bar{D} for which

$$\|f\|_0 := \sup_D |f(x, t)| < \infty,$$

further, let $C^{1,0}(\bar{D})$ be the space of all functions f for which $\partial_x^l f$ are continuous in \bar{D} for all $l = (l_1, \dots, l_n)$ such that $0 \leq |l| \leq 1$, and $C^{2,1}(D)$ be the space of all functions f for which $\partial_t^k \partial_x^l f$ are continuous in D for all $k = 0, 1$ and l such that $0 \leq 2k + |l| \leq 2$.

We will also use the following notations:

$$\Delta_x f(x, t) = f(x + \Delta x, t) - f(x, t),$$

$$\Delta_x^2 f(x, t) = f(x + 2\Delta x, t) - 2f(x + \Delta x, t) + f(x, t).$$

The expressions $\Delta_l f(x, t)$ and $\Delta_t^2 f(x, t)$ are defined analogously.

Let $f \in C^0(\bar{D})$. We say that f is *Dini-continuous* on \bar{D} with respect to x if there exist a constant $C > 0$ and a function $\omega : [0, +\infty) \rightarrow \mathbb{R}$ such that for all $x, x + \Delta x \in \mathbb{R}^n$, $t \geq 0$,

$$|\Delta_x f(x, t)| \leq C\omega(|\Delta x|),$$

where the function ω is a modulus of continuity which satisfies the so-called Dini condition:

$$\tilde{\omega}(h) := \int_0^h \frac{\omega(z)}{z} dz < \infty, \quad h > 0.$$

(Cf. Dzjadyk [7, pp. 147–157] about the properties of the modulus of continuity). We denote

$$\|f\|_\omega := \|f\|_0 + \sup_D \frac{|\Delta_x f(x, t)|}{\omega(|\Delta x|)}.$$

2. Formulation of the Problem

We consider in the half-space D the linear parabolic operator

$$Lu = \partial_t u - \sum_{i,j=1}^n a_{ij} \partial_{ij} u + \sum_{i=1}^n b_i \partial_i u + cu,$$

where its coefficients are defined on \bar{D} and satisfy the following conditions:

- a) There exists $\delta_0 > 0$ such that $\sum_{i,j=1}^n a_{ij}(x, t)\sigma_i\sigma_j \geq \delta_0\sigma^2$ for all $(x, t) \in \bar{D}$ and $\sigma \in \mathbb{R}^n$;
- b) Functions $a_{ij} \in C^0(\bar{D})$ are Dini-continuous with respect to x in \bar{D} with the corresponding modulus of continuity ω_{ij} ($i, j = 1, \dots, n$);
- c) $b_i, c \in C^0(\bar{D})$ ($i = 1, \dots, n$).

Matichuk and Eidelman [7] have shown that there is a classical fundamental solution of operator L if, in addition to a), b), c), the following condition holds:

$$\int_0^h \frac{\tilde{\omega}_{ij}(z)}{z} dz < \infty, \quad h > 0, \quad i, j = 1, \dots, n.$$

The objective of this work is the study of the Cauchy problem

$$(1) \quad Lu = f \text{ in } D, \quad u|_{t=0} = 0 \text{ in } \mathbb{R}^n,$$

where $f \in C^0(\bar{D})$.

Il'in [3] has proved that there is, in general, no classical solution of (1) if f is not Dini-continuous with respect to x . We introduce the notion of a generalized solution of (1) in the following way.

DEFINITION 1. A function $u : \bar{D} \rightarrow \mathbb{R}$ is said to be a generalized solution of (1) if there is a sequence $v_k \subset C^{2,1}(D) \cap C^0(\bar{D})$, $k \in \mathbb{N}$ such that:

- (i) $v_k(x, t) \rightarrow u(x, t)$ as $k \rightarrow \infty$ for all $(x, t) \in \bar{D}$;
- (ii) $Lv_k = f_k$ in D ($k \in \mathbb{N}$), $v_k(x, t) \rightarrow 0$ as $t \rightarrow 0$ uniformly with respect to $x \in \mathbb{R}^n$, $k \in \mathbb{N}$;
- (iii) $f_k \rightarrow f$ as $k \rightarrow +\infty$ uniformly on $D \cap \{t_0 \leq t \leq T\}$ for all $t_0 > 0$, $T > t_0$.

REMARK 2. It follows from the maximum principle (cf., e.g., [6, p. 28]) that such a solution is unique.

THEOREM 2. *Let conditions a), b), c) be satisfied. If $f \in C^0(\bar{D})$ is uniformly continuous in $\bar{D} \cap \{t \leq T\}$ for all $T > 0$ then the Cauchy problem (1) has a (unique) generalized solution $u \in C^{1,0}(\bar{D})$ and the following estimates hold:*

$$(2) \quad |\partial_t u(x, t)| \leq Ce^{\lambda t} \|f\|_0,$$

$$(3) \quad |\Delta_x^2 \partial_t u(x, t)| \leq Ce^{\lambda t} \|f\|_0 |\Delta x|,$$

$$(4) \quad |\Delta_t \partial_t u(x, t)| \leq Ce^{\lambda(t+\Delta t)} \|f\|_0 (\Delta t)^{1/2},$$

$$(5) \quad |\Delta_t^2 u(x, t)| \leq Ce^{\lambda(t+\Delta t)} \|f\|_0 \Delta t,$$

for all $x \in \mathbb{R}^n$, $t + \Delta t \geq t > 0$; where C and $\lambda \geq 0$ depend on n , δ_0 , $\|a_{ij}\|_{\omega_{ij}}$, $\|b_i\|_0$ and $\|c\|_0$.

REMARK 4. Konenkov [5] has proved estimates (2)–(5) for the classical solution of (1) in the band $\bar{D} \cap \{t \leq T\}$ (under more restrictive conditions on a_{ij} , b_i , c and f).

3. Proof of Theorem 3

Let $Z(x - \xi, t; y, \tau)$ be the fundamental solution of the equation

$$\partial_t u(x, t) - \sum_{i,j=1}^n a_{ij}(y, t) \partial_{ij} u(x, t) = 0 \text{ in } D,$$

with the coefficients “frozen” at $x = y$ (cf. [2, p. 24]), i.e.,

$$Z(x, t; y, \tau) = \int_{\mathbb{R}^n} e^{ix\sigma} \exp \left\{ - \int_{\tau}^t \left(\sum_{i,j=1}^n a_{ij}(y, \eta) \sigma_i \sigma_j \right) d\eta \right\} d\sigma,$$

and let

$$Z^*(x_{n+1}, t) := \frac{1}{2\sqrt{\pi t}} \exp \left\{ - \frac{x_{n+1}^2}{4t} \right\}.$$

We will denote by C, c any positive constants depending on $n, \delta_0, \|a_{ij}\|_{\omega_{ij}}, \|b_i\|_0, \|c\|_0$, specific value of which is not important for the argument, further, let $\omega = \sum_{i,j=1}^n \omega_{ij}$.

For $\varphi \in C^0(\bar{D})$ we introduce the integral

$$(6) \quad v_{\varphi}(x, t; x_{n+1}) := 2 \int_0^t \int_0^{\infty} Z^*(x_{n+1} + y_{n+1}, t - \tau) dy_{n+1} \times \\ \times \int_{\mathbb{R}^n} Z(x - \xi, t; \xi, \tau) \varphi(\xi, \tau) d\xi d\tau,$$

where $(x, t) \in \bar{D}, x_{n+1} \geq 0$.

We will first prove the following two lemmas.

LEMMA 5. *Let $\varphi \in C^0(\bar{D})$ be uniformly continuous on $\bar{D} \cap \{t \leq T\}$ for all $T > 0$. Then*

$$\partial_{n+1}^2 v_{\varphi}(x, t; x_{n+1}) \rightarrow \varphi(x, t)$$

as $x_{n+1} \rightarrow +0$ uniformly on $D \cap \{t_0 \leq t \leq T\}$ for all $0 < t_0 < T$.

PROOF. Using the facts that for all $x \in \mathbb{R}^n, t > \tau \geq 0$,

$$\int_{\mathbb{R}^n} Z(x - \xi, t; x, \tau) d\xi = 1,$$

and for all $t > 0, x_{n+1} > 0$,

$$\int_{-\infty}^t \partial_{n+1} Z^*(x_{n+1}, t - \tau) d\tau = -\frac{1}{2},$$

we can write

$$\partial_{n+1}^2 v_{\varphi}(x, t; x_{n+1}) =$$

$$\begin{aligned}
&= \varphi(x, t) + 2\varphi(x, t) \int_{-\infty}^0 \partial_{n+1} Z^*(x_{n+1}, t - \tau) d\tau - \\
&\quad - 2 \int_0^t \partial_{n+1} Z^*(x_{n+1}, t - \tau) \times \\
&\quad \times \int_{\mathbb{R}^n} \left\{ [Z(x - \xi, t; \xi, \tau) - Z(x - \xi, t; x, \tau)] \varphi(x, t) + \right. \\
&\quad \left. + Z(x - \xi, t; \xi, \tau) [\varphi(\xi, \tau) - \varphi(x, t)] \right\} d\xi d\tau =: \\
&=: \varphi(x, t) + 2\varphi(x, t) V_1(x_{n+1}, t) - 2(V_2 + V_3)(x, t; x_{n+1}).
\end{aligned}$$

First we estimate V_1 :

$$|V_1| \leq C \frac{x_{n+1}}{t^{1/2}}$$

and thus $\varphi(x, t) V_1(x_{n+1}, t) \rightarrow 0$ as $x_{n+1} \rightarrow +0$ uniformly on $D \cap \{t \geq t_0\}$.

Now by using assumptions a), b) we obtain

$$|V_2| \leq C_\varphi \int_0^t \frac{x_{n+1}}{\tau^{3/2}} \exp\left(-\frac{x_{n+1}^2}{4\tau}\right) \omega(\tau^{1/2}) d\tau,$$

where $C_\varphi := C \|\varphi\|_0$.

Consequently, if $0 \leq t \leq x_{n+1}^2$, we have

$$|V_2(x, t; x_{n+1})| \leq C_\varphi \omega(x_{n+1}).$$

If $t > x_{n+1}^2$, then

$$\begin{aligned}
|V_2| &\leq C_\varphi \left(\int_0^{x_{n+1}^2} + \int_{x_{n+1}^2}^t \right) \frac{x_{n+1}}{\tau^{3/2}} \exp\left(-\frac{x_{n+1}^2}{4\tau}\right) \omega(\tau^{1/2}) d\tau \leq \\
&\leq C_\varphi \omega(x_{n+1}) \left(1 + \int_{x_{n+1}^2}^t \frac{d\tau}{\tau} \right) = C_\varphi \omega(x_{n+1})(1 + \ln t - 2 \ln x_{n+1}).
\end{aligned}$$

Note that for $r \in (0, 1)$,

$$\omega(r) |\ln r| = 2\omega(r) \int_r^{\sqrt{r}} \frac{dz}{z} \leq 2 \int_r^{\sqrt{r}} \frac{\omega(z)}{z} dz < 2\tilde{\omega}(r^{1/2}).$$

Consequently, if $t > x_{n+1}^2$, we have for $x_{n+1} \in (0, 1)$ that

$$|V_2(x, t; x_{n+1})| \leq C_\varphi \{ \omega(x_{n+1})(1 + \ln t) + \tilde{\omega}(x_{n+1}^{1/2}) \}.$$

Therefore, $V_2(x, t; x_{n+1}) \rightarrow 0$ as $x_{n+1} \rightarrow +0$ uniformly on $D \cap \{t_0 \leq t \leq T\}$.

Finally we estimate V_3 . Let $\varepsilon \in (0, 1)$ be arbitrary. There is $\delta = \delta(\varepsilon, T) > 0$ such that

$$|\varphi(x, t) - \varphi(\xi, t)| < \varepsilon \quad \text{if } |x - \xi| < \delta, \quad 0 \leq t \leq T,$$

and

$$|\varphi(\xi, t) - \varphi(\xi, \tau)| < \varepsilon \quad \text{if } t - \tau < \delta^2, \quad 0 \leq \tau < t \leq T.$$

We can assume $\delta < \varepsilon$. If $0 < t \leq \delta^2$ then

$$|V_3| \leq C\omega_{\varphi, T}(\varepsilon),$$

where $\omega_{\varphi, T}$ is the modulus of continuity for φ on $\bar{D} \cap \{t \leq T\}$.

If $t > \delta^2$ then writing V_3 in the form

$$\begin{aligned} V_3 = & \left\{ \int_0^t d\tau \int_{|x-\xi|<\delta} + \int_0^t d\tau \int_{|x-\xi|>\delta} + \right. \\ & \left. + \int_{t-\delta^2}^t d\tau \int_{\mathbb{R}^n} + \int_0^{t-\delta^2} d\tau \int_{\mathbb{R}^n} \right\} \dots d\xi, \end{aligned}$$

we deduce the estimate

$$|V_3| \leq C \left(\varepsilon + \|\varphi\|_0 \frac{x_{n+1}}{\delta^2} T^{\frac{1}{2}} \right).$$

Therefore, $V_3(x, t; x_{n+1}) \rightarrow 0$ as $x_{n+1} \rightarrow 0$ uniformly on $\bar{D} \cap \{t \leq T\}$. ■

LEMMA 6. *If $f \in C^0(\bar{D})$ then the integral equation*

$$(7) \quad \varphi(x, t) + \int_0^t \int_{\mathbb{R}^n} K(x, t; \xi, \tau) \varphi(\xi, \tau) d\xi d\tau = f(x, t), \quad (x, t) \in \bar{D},$$

where $K(x, t; \xi, \tau) = L_{x,t} Z(x - \xi, t; \xi, \tau)$, has a unique solution $\varphi \in C^0(\bar{D})$ and the following estimate holds:

$$|\varphi(x, t)| \leq C e^{\lambda t} \|f\|_0, \quad \forall (x, t) \in \bar{D}.$$

Furthermore, iff is uniformly continuous on $\bar{D} \cap \{t \leq T\}$, then φ is uniformly continuous on $\bar{D} \cap \{t \leq T\}$, too.

PROOF. Let $\lambda > 0$ be fixed. Multiplying both sides of (7) by $e^{-\lambda t}$, we obtain the equivalent equation

$$(8) \quad \varphi^*(x, t) + \int_0^t d\tau \int_{\mathbb{R}^n} K^*(x, t; \xi, \tau) \varphi^*(\xi, \tau) d\xi = f^*(x, t)$$

with

$$(9) \quad \varphi^* = e^{-\lambda t} \varphi, \quad f^* = e^{-\lambda t} f, \quad K^* = e^{-\lambda(t-\tau)} K.$$

Let $A : C^0(\bar{D}) \rightarrow C^0(\bar{D})$ be the operator defined by

$$A\varphi^*(x, t) = \int_0^t \int_{\mathbb{R}^n} K^*(x, t; \xi, \tau) \varphi^*(\xi, \tau) d\xi.$$

It follows from assumptions a)–c) that

$$|K^*(x, t; \xi, \tau)| \leq Ce^{-\lambda(t-\tau)/2} \frac{\omega((t-\tau)^{1/2})}{(t-\tau)^{(n+2)/2}} \exp \left\{ -c \frac{(x-\xi)^2}{t-\tau} \right\},$$

so that

$$\int_0^t d\tau \int_{\mathbb{R}^n} |K^*(x, t; \xi, \tau)| d\xi \leq C_0 \tilde{\omega}(t^{1/2})$$

for some $C_0 > 0$ and all $(x, t) \in \bar{D}$. Let $\varepsilon > 0$ be such that $C_0 \tilde{\omega}(\varepsilon) \leq \frac{1}{4}$. If $0 \leq t \leq \varepsilon^2$ then for all $x \in \mathbb{R}^n$,

$$|A\varphi^*(x, t)| \leq \frac{1}{4} \|\varphi^*\|_0.$$

If $t > \varepsilon^2$ then writing

$$A\varphi^*(x, t) = \left\{ \int_0^{\varepsilon^2} \int_{\mathbb{R}^n} + \int_{\varepsilon^2}^t \int_{\mathbb{R}^n} \right\} \dots d\xi,$$

we obtain

$$\begin{aligned} |A\varphi^*(x, t)| &\leq \frac{1}{4} \|\varphi^*\|_0 + C_0 \|\varphi^*\|_0 \frac{\tilde{\omega}(\varepsilon)}{\varepsilon} \int_0^{+\infty} (t-\tau)^{-1/2} e^{-\lambda(t-\tau)/2} d\tau \leq \\ &\leq \|\varphi^*\|_0 \left(\frac{1}{4} + C_1 \frac{\tilde{\omega}(\varepsilon)}{\varepsilon} \frac{1}{\lambda^{1/2}} \right). \end{aligned}$$

Choosing $\lambda > 0$ in the way that

$$C_1 \frac{\tilde{\omega}(\varepsilon)}{\varepsilon} \frac{1}{\lambda^{1/2}} = \frac{1}{4},$$

we obtain

$$\|A\varphi^*\|_0 \leq \frac{1}{2} \|\varphi^*\|_0.$$

Therefore from the contraction mapping principle we conclude that equation (8) has a unique solution $\varphi^* \in C^0(\bar{D})$ and the following estimate holds

$$\|\varphi^*\| \leq C \|f^*\|_0.$$

Returning now to the function φ by (9), we obtain the assertion of the lemma. \blacksquare

PROOF OF THEOREM 3 Let $\varphi \in C^0(\bar{D})$ be the solution of the integral equation (7) and let $v_k(x, t) := v_\varphi(x, t; \frac{1}{k})$, where v_φ is given by (6). Then by Lemma 5,

$$Lv_k(x, t) \equiv f(x, t) + \left[\delta_{n+1}^2 v_\varphi(x, t, \frac{1}{k}) - \varphi(x, t) \right] \rightarrow f(x, t)$$

as $k \rightarrow \infty$ uniformly on $D \cap \{t_0 \leq t \leq T\}$. Furthermore, for all $(x, t) \in \bar{D}$,

$$|v_k(x, t)| \leq Ct \|\varphi\|_0,$$

thus it follows that the function

$$u(x, t) \equiv v_\varphi(x, t; 0) \equiv \int_0^t d\tau \int_{\mathbb{R}^n} Z(x - \xi, t; \xi, \tau) \varphi(\xi, \tau) d\xi$$

is the generalized solution of (1).

The estimates (2)–(5) are proved similarly to the corresponding estimates for the heat volume potential from the work of Konenkov [4]. \blacksquare

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A VECTORIAL INGHAM–BEURLING TYPE THEOREM

By

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(Received July 27, 2010)

Dedicated to Professor László Simon on the occasion of his 70th birthday

Abstract. Baiocchi et al. generalized a few years ago a classical theorem of Ingham and Beurling by means of divided differences. The optimality of their assumption has been proven by the third author of this note. The purpose of this note is to extend these results to vector coefficient sums.

1. Introduction

Let $\Omega := (\omega_k)_{k \in \mathbb{Z}}$ be a family of real numbers satisfying the gap condition

$$(1.1) \quad \gamma := \inf_{k \neq n} |\omega_k - \omega_n| > 0.$$

Let us denote by $D^+ = D^+(\Omega)$ its Pólya upper density, defined by the formula $D^+ := \lim_{r \rightarrow \infty} r^{-1} n^+(r)$, where $n^+(r) = n^+(\Omega, r)$ denotes the largest number of terms of the sequence $(\omega_k)_{k \in \mathbb{Z}}$ contained in an interval of length r .

Let $(U_k)_{k \in \mathbb{Z}}$ be a corresponding family of unit vectors in some complex Hilbert space H and consider the sums

$$x(t) = \sum_{k \in \mathbb{Z}} x_k U_k e^{i \omega_k t}$$

with square summable complex coefficients x_k . We are interested in the validity of the estimates

$$(1.2) \quad \int_I |x(t)|_H^2 dt \asymp \sum_{k \in \mathbb{Z}} |x_k|^2$$

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where I is a bounded interval of length denoted by $|I|$ and where we write $A \asymp B$ if there exist two positive constants c_1, c_2 satisfying $c_1 A \leq B \leq c_2 A$.

The following result generalizes a theorem of Ingham [4]; for $d = 1$ it reduces to a theorem of Beurling [2].

THEOREM 1.1. *Let $\Omega := (\omega_k)_{k \in \mathbb{Z}}$ be a family of real numbers satisfying (1.1).*

- (a) *If $|I| > 2\pi D^+$, then the estimates (1.2) hold.*
- (b) *If the estimates (1.2) hold true and H has a finite dimension d , then $|I| \geq 2\pi D^+ / d$.*

The optimality of Theorem 1.1 will be deduced from the following result:

THEOREM 1.2. *Let Ω be a set of real numbers with a finite upper density D^+ and let $\alpha_1, \alpha_2, \dots$ be a finite or infinite sequence of numbers in $[0, 1]$ satisfying $\alpha_1 + \alpha_2 + \dots \geq 1$. Then there exists a partition $\Omega = \Omega_1 \cup \Omega_2 \cup \dots$ of Ω such that the upper density D_j^+ of Ω_j is equal to $\alpha_j D^+$ for every j .*

REMARK. It follows from the definition of the upper density that if $\Omega = \Omega_1 \cup \Omega_2 \cup \dots$ is a finite or infinite partition of Ω , then

$$(1.3) \quad \max\{D^+(\Omega_1), D^+(\Omega_2), \dots\} \leq D^+(\Omega) \leq D^+(\Omega_1) + D^+(\Omega_2) + \dots$$

This implies the necessity of the conditions $\alpha_j \leq 1$ and $\alpha_1 + \alpha_2 + \dots \geq 1$ in the theorem.

Now we have the following corollary:

COROLLARY 1.3. *Let $\Omega := (\omega_k)_{k \in \mathbb{Z}}$ be a family of real numbers satisfying (1.1), and H a finite-dimensional Hilbert space. Given any real number $\frac{1}{d} \leq \alpha \leq 1$ where $d = \dim H$, there exists a family $(U_k)_{k \in \mathbb{Z}}$ of unit vectors in H such that the estimates (1.2) hold if $|I| > 2\pi\alpha D^+$, and they fail if $|I| < 2\pi\alpha D^+$.*

We prove Theorem 1.1 in the next section and we extend it to the case of a weakened gap condition in Section 3. Theorem 1.2 and Corollary 1.3 are proved in Sections 4–6.

We refer to [5] for various control theoretical applications of theorems of this type.

2. Proof of Theorem 1.1

Part (a) readily follows from the scalar case. Indeed, fixing an orthonormal basis $(E_n)_{n \in N}$ of the closed linear hull of $(U_k)_{k \in \mathbb{Z}}$ in H and developing the vectors U_k into Fourier series: $U_k = \sum_{n \in N} u_{kn} E_n$, for $|I| > 2\pi D^+$ we have

$$\begin{aligned} \int_I \left\| \sum_{k \in \mathbb{Z}} x_k U_k e^{i\omega_k t} \right\|_H^2 dt &= \sum_{n \in N} \int_I \left| \sum_{k \in \mathbb{Z}} x_k u_{kn} e^{i\omega_k t} \right|^2 dt \asymp \\ &\asymp \sum_{n \in N} \sum_{k \in \mathbb{Z}} |x_k u_{kn}|^2 = \\ &= \sum_{k \in \mathbb{Z}} |x_k|^2. \end{aligned}$$

For the proof of part (b) we adapt the approach developed in [3] and [6]. We set $\gamma_k := 2\pi|I|^{-1}k$ for brevity. Given three real numbers y, r, R with $r, R > 0$, we introduce the orthogonal projections

$$P_{y,r} : L^2(I, H) \rightarrow V_{y,r} \quad \text{and} \quad Q_{y,r+R} : L^2(I, H) \rightarrow W_{y,r+R}$$

onto the finite-dimensional linear subspaces

$$V_{y,r} := \text{Vect} \left\{ U_k e^{i\omega_k t} : |\omega_k - y| < r \right\}$$

and

$$W_{y,r+R} := \text{Vect} \left\{ U e^{i\gamma_k t} : |\gamma_k - y| < r + R \quad \text{and} \quad U \in H \right\}.$$

Note that

$$(2.1) \quad n^+(2r) = \sup_y \dim V_{y,r}$$

and

$$(2.2) \quad (2r + 2R)d \frac{|I|}{2\pi} \leq \dim W_{y,r+R} \leq (2r + 2R + 1)d \frac{|I|}{2\pi}.$$

Setting

$$S_{y,r,R} := P_{y,r} \circ Q_{y,r+R} \circ i$$

where i denotes the injection $i : V_{y,r} \rightarrow L^2(I, H)$, we obtain a linear map of $V_{y,r}$ into itself. We are going to study its trace.

LEMMA 2.1. *We have*

$$|\mathrm{tr}(S_{y,r,R})| \leq \dim W_{y,r+R}.$$

PROOF. We have

$$\|S_{y,r,R}\| \leq \|P_{y,r}\| \cdot \|Q_{y,r+R}\| \leq 1.$$

Hence the eigenvalues of $S_{y,r,R}$ have modulus ≤ 1 and therefore

$$|\mathrm{tr}(S_{y,r,R})| \leq \mathrm{rang}(S_{y,r,R}) \leq \dim(W_{y,r+R}). \quad \blacksquare$$

LEMMA 2.2. *Writing $e_k(t) := U_k e^{i\omega_k t}$ for brevity, there exists $(\varphi_k)_{k \in \mathbb{Z}}$ a bounded biorthogonal family to $(e_k)_{k \in \mathbb{Z}}$ in $L^2(I, H)$ and we have*

$$\mathrm{tr}(S_{y,r,R}) = \dim V_{y,r} + \sum_{|\omega_k - y| < r} ((Q_{y,r+R} - \mathrm{Id})e_k, P_{y,r}\varphi_k)_H.$$

PROOF. The existence of a bounded biorthogonal family comes from (1.2) (see [5] for a proof). We then write $S_{y,r,R}e_k = \sum_{|\omega_j - y| < r} S_{k,j}e_j$. Since (φ_k) is biorthogonal, we have $(S_{y,r,R}e_k, \varphi_k)_{L^2(I, H)} = S_{k,k}$ and thus

$$\mathrm{tr}(S_{y,r,R}) = \sum_{|\omega_j - y| < r} S_{k,k} = \sum_{|\omega_k - y| < r} (S_{y,r,R}e_k, \varphi_k)_{L^2(I, H)},$$

so that

$$\begin{aligned} \mathrm{tr}(S_{y,r,R}) &= \sum_{|\omega_k - y| < r} (P_{y,r}e_k, \varphi_k)_{L^2(I, H)} + \\ &+ \sum_{|\omega_k - y| < r} (P_{y,r}(Q_{y,r+R} - \mathrm{Id})e_k, \varphi_k)_{L^2(I, H)}. \end{aligned}$$

Since $P_{y,r}e_k = e_k$, we have $(P_{y,r}e_k, \varphi_k)_{L^2(I, H)} = 1$ and the result follows.

LEMMA 2.3. *For $R \rightarrow \infty$ we have*

$$\|(Q_{y,r+R} - \mathrm{Id})e_k\| = O(1/\sqrt{R})$$

uniformly for all $y \in \mathbb{R}$, $r > 0$ and k satisfying $|\omega_k - y| < r$.

PROOF. Fixing an orthonormal basis E_1, \dots, E_d of H and setting

$$f_{n,j}(t) := |I|^{-1/2} E_j e^{i\gamma_n t}$$

we have

$$e_k = \sum_{n \in \mathbb{Z}} \sum_{j=1}^d (e_k, f_{n,j})_{L^2(I,H)} f_{n,j}$$

and

$$Q_{y,r+R} e_k = \sum_{|\gamma_n - y| < r+R} \sum_{j=1}^d (e_k, f_{n,j})_{L^2(I,H)} f_{n,j}.$$

Applying Parseval's equality it follows that

$$\|(Q_{y,r+R} - \text{Id})e_k\|^2 = \sum_{|\gamma_n - y| \geq r+R} \sum_{j=1}^d |(e_k, f_{n,j})_{L^2(I,H)}|^2.$$

Since

$$(2.3) \quad \left| (e_k, f_{n,j})_{L^2(I,H)} \right| = |I|^{-1/2} \left| \int_I (U_k, E_j)_H e^{i(\omega_k - \gamma_n)t} dt \right| \leq \frac{2|I|^{-1/2}}{|\omega_k - \gamma_n|},$$

and $|\omega_k - y| < r$, then we obtain that

$$\|(Q_{y,r+R} - \text{Id})e_k\|^2 \leq 4d|I|^{-1} \sum_{|\gamma_n - y| \geq r+R} \frac{1}{|\omega_k - \gamma_n|^2}$$

Note that from $|\gamma_n - y| \geq r + R$ and $|\omega_k - y| < r$, we get $|\gamma_n - \omega_k| > R$, and thus

$$\|(Q_{y,r+R} - \text{Id})e_k\|^2 \leq 8d|I|^{-1} \sum_{n=0}^{\infty} \frac{1}{[2\pi|I|^{-1}n + R]^2}.$$

Since the last expression doesn't depend on r, y and is $O(1/R)$ as $R \rightarrow \infty$, the lemma follows. \blacksquare

Now the proof of part (b) of Theorem 1.1 can be completed as follows. By the above lemmas we have

$$\begin{aligned} \dim W_{y,r+R} &\geq |\text{tr}(S)| = \\ &= \left| \dim V_{y,r} + \sum_{|\omega_k - y| < r} ((Q_{y,r+R} - \text{Id})e_k, P_{y,r}\varphi_k)_H \right| \geq \\ &\geq \dim V_{y,r} - O(1/\sqrt{R}) \dim V_{y,r}. \end{aligned}$$

Hence

$$\dim V_{y,r} \leq (1 + O(1/\sqrt{R})) \dim W_{y,r+R}, \quad R \rightarrow \infty,$$

and using (2.1)–(2.2) we conclude that

$$n^+(2r) \leq (1 + O(1/\sqrt{R}))d \frac{|I|}{2\pi} (2r + 2R + 1), \quad R \rightarrow \infty.$$

It follows that

$$\begin{aligned} D^+ &= \lim_{r \rightarrow \infty} \frac{n^+(2r)}{2r} \leq \\ &\leq (1 + O(1/\sqrt{R}))d \frac{|I|}{2\pi} \lim_{r \rightarrow \infty} \frac{2r + 2R + 1}{2r} = \\ &= (1 + O(1/\sqrt{R}))d \frac{|I|}{2\pi} \end{aligned}$$

for all $R > 0$. Letting $R \rightarrow \infty$ we conclude that $|I| \geq 2\pi D^+/d$.

3. The case of the divided differences

The gap condition (1.1) of the theorem may be weakened. Following [1] let $(\omega_k)_{k \in \mathbb{Z}}$ be a nondecreasing sequence of real numbers satisfying for some positive integer M and for some positive real number γ' the weakened gap condition

$$(3.1) \quad \omega_{k+M} - \omega_k \geq M\gamma' \quad \text{for all } k \in \mathbb{Z}.$$

This implies that $D^+ < \infty$. We say that $\omega_m, \dots, \omega_{m+j-1}$ is a γ' -close exponent chain ($m \in \mathbb{Z}, j = 1, \dots, M$) if

$$\begin{cases} \omega_m - \omega_{m-1} \geq \gamma', \\ \omega_k - \omega_{k-1} < \gamma' \quad \text{for } k = m+1, \dots, m+j-1, \\ \omega_{m+j} - \omega_{m+j-1} \geq \gamma'. \end{cases}$$

Then we define the divided differences $f_\ell = [\omega_m, \dots, \omega_\ell]$ by the formula

$$[\omega_m](t) := \exp(i\omega_m t), \quad [\omega_m, \omega_{m+1}](t) := it \int_0^1 \exp(i[s_m(\omega_{m+1} - \omega_m) + \omega_m]t) ds_m,$$

and for $\ell = m+2, \dots, m+j-1$,

$$\begin{aligned} [\omega_m, \dots, \omega_\ell](t) &:= (it)^{\ell-m} \int_0^1 \int_0^{s_m} \dots \int_0^{s_{\ell-2}} \\ &\quad \exp(i[s_{\ell-1}(\omega_\ell - \omega_{\ell-1}) + \dots + s_m(\omega_{m+1} - \omega_m) + \omega_m]t) ds_{\ell-1} \dots ds_m. \end{aligned}$$

We can now state a generalization of Theorem 1.1:

THEOREM 3.1. *Theorem 1.1 holds true if (1.1) is replaced by (1.3) and $e^{i\omega_k t}$ is replaced by $f_k(t)$.*

PROOF. Most of the proof of Theorem 1.1 may be easily adapted. For part (b) we have to replace the estimate (2.3) by the following:

$$(3.2) \quad \left| \int_I (U_k, E_j)_H f_k(t) e^{-i\gamma_n t} dt \right| \leq \left| \int_I f_k(t) e^{-i\gamma_n t} dt \right| \leq \frac{C}{|\omega_k - \gamma_n|},$$

with a constant C depending only on γ' , M and I . This is shown by arguing similarly as in [6]. We have

$$A := \int_I f_k(t) e^{-i\gamma_n t} dt = \int_I g(t) e^{i\omega_k t} e^{-i\gamma_n t} dt$$

with

$$g(t) = [\omega_m - \omega_k, \dots, \omega_k - \omega_k](t).$$

Integrating by parts in $I = (a, b)$ we obtain that

$$A = \left[\frac{1}{i\omega_k - i\gamma_n} g(t) e^{i\omega_k t} e^{-i\gamma_n t} \right]_a^b - \int_I \frac{1}{i\omega_k - i\gamma_n} g'(t) e^{i\omega_k t} e^{-i\gamma_n t} dt.$$

Now a direct computation shows that for any real numbers μ_1, \dots, μ_r the divided differences satisfy the inequality

$$[\mu_1, \dots, \mu_r]'(t) \leq \frac{(r-1)t^{r-2}}{(r-1)!} + (|\mu_r - \mu_{r-1}| + \dots + |\mu_2 - \mu_1| + |\mu_1|) \frac{t^{r-1}}{(r-1)!}.$$

Thus, in our case, thanks to the γ' -close exponent property, we have

$$|g'(t)| \leq (k-m) \frac{t^{k-m-1}}{(k-m)!} + (k-m)\gamma' \frac{t^{k-m}}{(k-m)!}$$

and this yields (3.2). ■

4. Proof of Theorem 1.2 for $\alpha_1 + \alpha_2 + \dots = 1$

Assume that the theorem holds for sets Ω which are bounded from below. Then by changing Ω to $-\Omega$ we obtain that the theorem also holds if Ω is bounded from above. Finally, if $\inf \Omega = -\infty$ and $\sup \Omega = \infty$, then applying these two special cases of the theorem to

$$\Omega^- := \Omega \cap (-\infty, 0) \quad \text{and} \quad \Omega^+ := \Omega \cap [0, \infty)$$

we obtain two partitions

$$\Omega^- = \Omega_{-1}^- \cup \Omega_{-2}^- \cup \dots \quad \text{and} \quad \Omega^+ = \Omega_{+1}^+ \cup \Omega_{+2}^+ \cup \dots$$

satisfying

$$D^+(\Omega_j^-) = \alpha_j D^+(\Omega^-) \quad \text{and} \quad D^+(\Omega_j^+) = \alpha_j D^+(\Omega^+)$$

for all j . Then setting $\Omega_j := \Omega_j^- \cup \Omega_j^+$ we obtain a partition of Ω with the required properties. This follows by applying the following lemma for the partitions $\Omega := \Omega^- \cup \Omega^+$ and $\Omega_j := \Omega_j^- \cup \Omega_j^+$.

LEMMA 4.1. *For any set A of real numbers, setting $A^- := A \cap (-\infty, 0)$ and $A^+ := A \cap [0, \infty)$ we have*

$$D^+(A) = \max\{D^+(A^-), D^+(A^+)\}.$$

PROOF. The easy inequality \geq follows from (1.3). Setting

$$M := \max\{D^+(A^-), D^+(A^+)\}$$

for brevity, for the converse inequality it is sufficient to show that

$$\limsup_{r,s \geq 0, r+s \rightarrow \infty} \frac{\text{Card}(A \cap [-r, s])}{r+s} \leq M.$$

The case $M = \infty$ is obvious. Assume henceforth that $M < \infty$. For any fixed $\varepsilon > 0$ we may fix two positive numbers $r_\varepsilon, s_\varepsilon$ satisfying

$$\text{Card}(A^- \cap [-r, 0]) \leq (D^+(A^-) + \varepsilon)r \quad \text{for all } r \geq r_\varepsilon$$

and

$$\text{Card}(A^+ \cap [0, s]) \leq (D^+(A^+) + \varepsilon)s \quad \text{for all } s \geq s_\varepsilon.$$

Adding the two inequalities and setting $K := \text{Card}(A \cap [-r_\varepsilon, s_\varepsilon])$ it follows that

$$\text{Card}(A \cap [-r, s]) \leq (M + \varepsilon)(r + s) + K$$

for all $r, s \geq 0$. Dividing by $r + s$ and letting $r + s \rightarrow \infty$ we conclude that

$$\limsup_{r,s, \geq 0, r+s \rightarrow \infty} \frac{\text{Card}(A \cap [-r, s])}{r + s} \leq M + \varepsilon$$

for all $\varepsilon > 0$, and the lemma follows. \blacksquare

Henceforth we assume that Ω is bounded from below. If $D^+ = 0$, then every partition of Ω has the required property because all upper densities are equal to zero. Henceforth we assume also that $0 < D^+ < \infty$; then Ω is an unbounded set and we may enumerate the elements of Ω into a strictly increasing infinite sequence $\omega_1 < \omega_2 < \dots$. Finally, by choosing $\Omega_j = \emptyset$ whenever $\alpha_j = 0$ we may assume without loss of generality that $0 < \alpha_j \leq 1$ for all j .

In order to explain the idea of the proof first we consider the special case of a finite sequence $\alpha_1, \dots, \alpha_d$ consisting of rational numbers. We fix a positive integer N such that $N\alpha_1, \dots, N\alpha_d$ are all integers, and we represent Ω as the union of the disjoint blocks

$$(4.1) \quad B_n := \{\omega_k \in \Omega : k = nN + 1, nN + 2, \dots, (n + 1)N\}, \quad n = 0, 1, \dots$$

Notice that each B_n has N elements. Therefore, since $N\alpha_1 + \dots + N\alpha_d = N$, we may define a partition $\Omega = \Omega_1 \cup \dots \cup \Omega_d$ of Ω such that

$$(4.2) \quad \text{Card}(B_n \cap \Omega_j) = N\alpha_j \quad \text{for all } n \text{ and } j.$$

We claim that $D^+_j \leq \alpha_j D^+$ for each j . This is obvious if $D^+_j = 0$. If $D^+_j > 0$ for some j , then we choose a sequence of bounded intervals (I_m^j) satisfying $|I_m^j| \rightarrow \infty$ and

$$\frac{\text{Card}(\Omega_j \cap I_m^j)}{|I_m^j|} \rightarrow D^+_j.$$

Since $D^+_j > 0$, hence $\text{Card}(\Omega_j \cap I_m^j) \rightarrow \infty$ and therefore $k_m \rightarrow \infty$ where k_m denotes the number of (consecutive) blocks B_n contained in I_m^j . Since $I_m^j \cap \Omega$ is contained in the union of at most $k_m + 2$ blocks B_n , using (4.2) it follows

that

$$\begin{aligned} \frac{\text{Card}(\Omega_j \cap I_m^j)}{|I_m^j|} &\leq \frac{2N + k_m N \alpha_j}{|I_m^j|} \leq \\ &\leq \frac{2N + k_m N \alpha_j}{|I_m^j|} \cdot \frac{\text{Card}(\Omega \cap I_m^j)}{k_m N} \leq \\ &\leq \frac{2 + k_m \alpha_j}{k_m} \cdot \frac{n^+(\Omega, |I_m^j|)}{|I_m^j|} \end{aligned}$$

for every m . Letting $m \rightarrow \infty$ we conclude that $D^+ j \leq \alpha_j D^+$.

In fact $D^+ j = \alpha_j D^+$ for all j . Indeed, if the inequalities $D^+ j \leq \alpha_j D^+$ were not all equalities, then using (1.3) we would obtain a contradiction:

$$D^+ \leq D^+_1 + \cdots + D^+_d < \alpha_1 D^+ + \cdots + \alpha_d D^+ = D^+.$$

Now we turn to the general case. We write $J = \{1, \dots, d\}$ in the finite case $\alpha_1 + \cdots + \alpha_d = 1$ and $J = \{1, 2, \dots\}$ in the infinite case.

Let

$$(k, j) : \{1, 2, \dots\} \rightarrow \{1, 2, \dots\} \times J$$

be the lexicographically strictly increasing enumeration of the pairs

$$(k, j) \in \{1, 2, \dots\} \times J \quad \text{satisfying } [k\alpha_j] > [(k-1)\alpha_j],$$

where $[x]$ denotes the integer part of x . This enumeration is possible because for each fixed k only finitely many indices $j \in J$ may satisfy this inequality. Indeed, by the convergence of the series $\sum_{j \in J} \alpha_j$ we have $k\alpha_j < 1$ and thus $[k\alpha_j] = [(k-1)\alpha_j] = 0$ for all sufficiently large indices j .

Observe that

$$[k\alpha_j] - [(k-1)\alpha_j] = \begin{cases} 1 & \text{if } (k, j) \text{ belongs to the sequence;} \\ 0 & \text{otherwise.} \end{cases}$$

For each $j \in J$ we set

$$\Omega_j := \{\omega_s : j(s) = j\}.$$

We claim that $D^+ j = \alpha_j D^+$ for all $j \in J$.

First we prove that $D^+ j \leq \alpha_j D^+$ for each fixed $j \in J$. The case of $D^+ j = 0$ is obvious. Let $D^+ j > 0$ and choose a sequence (I_m^j) of bounded intervals such that

$$|I_m^j| \rightarrow \infty \quad \text{and} \quad \frac{\text{Card}(\Omega_j \cap I_m^j)}{|I_m^j|} \rightarrow D^+ j.$$

Writing $\Omega \cap I_m^j = \{\omega_{s_m}, \dots, \omega_{t_m}\}$ we have

$$\ell_m := k(t_m) - k(s_m) \geq \text{Card}(\Omega_j \cap I_m^j) \rightarrow \infty;$$

the first inequality follows from the definition of Ω_j , while the second follows from our assumptions $D^+_j > 0$ and $|I_m^j| \rightarrow \infty$.

Now we have

$$\begin{aligned} \text{Card}(\Omega_j \cap I_m^j) &\leq \sum_{k=k(s_m)}^{k(t_m)} ([k\alpha_j] - [(k-1)\alpha_j]) = \\ &= [k(t_m)\alpha_j] - [(k(s_m)-1)\alpha_j] \leq \\ &\leq \ell_m \alpha_j + 1 \end{aligned}$$

by the definition of Ω_j . Furthermore, we have

$$\begin{aligned} \text{Card}(\Omega \cap I_m^j) &\geq \sum_{n \leq \sqrt{\ell_m}} \sum_{k=k(s_m)+1}^{k(t_m)-1} ([k\alpha_n] - [(k-1)\alpha_n]) = \\ &= \sum_{n \leq \sqrt{\ell_m}} ([(k(t_m)-1)\alpha_n] - [k(s_m)\alpha_n]) \geq \\ &\geq \sum_{n \leq \sqrt{\ell_m}} (\ell_m \alpha_n - 2) \geq \\ &\geq \ell_m \left(\sum_{n \leq \sqrt{\ell_m}} \alpha_n \right) - 2\sqrt{\ell_m}. \end{aligned}$$

Since $\ell_m \rightarrow \infty$ and $\sum \alpha_n = 1$, it follows from the above two estimates that

$$\text{Card}(\Omega_j \cap I_m^j) \leq \ell_m (\alpha_j + o(1))$$

and

$$\text{Card}(\Omega \cap I_m^j) \geq \ell_m (1 - o(1))$$

as $m \rightarrow \infty$. Hence

$$\begin{aligned} \frac{\text{Card}(\Omega_j \cap I_m^j)}{|I_m^j|} &\leq \frac{\ell_m (\alpha_j + o(1))}{|I_m^j|} \leq \\ &\leq \frac{\ell_m (\alpha_j + o(1))}{|I_m^j|} \cdot \frac{\text{Card}(\Omega \cap I_m^j)}{\ell_m (1 - o(1))} = \\ &= \frac{\alpha_j + o(1)}{1 - o(1)} \cdot \frac{\text{Card}(\Omega \cap I_m^j)}{|I_m^j|}. \end{aligned}$$

Letting $m \rightarrow \infty$ we conclude that

$$D^+_j \leq \alpha_j \limsup \frac{\text{Card}(\Omega \cap I_m^j)}{|I_m^j|} \leq \alpha_j D^+.$$

It remains to show that none of the inequalities $D^+_j \leq \alpha_j D^+$ is strict. However, in this case using (1.3) we would obtain the contradiction

$$D^+ \leq D^+_1 + D^+_2 + \dots < \alpha_1 D^+ + \alpha_2 D^+ + \dots = D^+.$$

5. Proof of Corollary 1.3

Fix $1/d \leq \alpha \leq 1$ arbitrarily and then choose $\alpha_1, \dots, \alpha_d \geq 0$ such that

$$\alpha_1 + \dots + \alpha_d = 1 \quad \text{and} \quad \max\{\alpha_1, \dots, \alpha_d\} = \alpha.$$

Applying the already proved part of Theorem 1.2 we obtain a partition $\Omega = \Omega_1 \cup \dots \cup \Omega_d$ of Ω such that $D^+(\Omega_j) = \alpha_j D^+$ for all j . Fix an orthonormal basis E_1, \dots, E_d of H and set $U_k = E_j$ if $\omega_k \in \Omega_j$. Then using the identity

$$\int_I \left\| \sum_{k \in \mathbb{Z}} x_k U_k e^{i\omega_k t} \right\|_H^2 dt = \sum_{j=1}^d \int_I \left| \sum_{\omega_k \in \Omega_j} x_k e^{i\omega_k t} \right|^2 dt$$

and applying the scalar case of the theorem we conclude that the estimates (1.2) hold if $|I| > 2\pi\alpha D^+$, and they do not hold if $|I| < 2\pi\alpha D^+$.

6. Proof of Theorem 1.2 for $\alpha_1 + \alpha_2 + \dots > 1$

By the same reasoning as at the beginning of Section 6, we may assume that $0 < D^+ < \infty$, $0 < \alpha_j \leq 1$ for all j , and that the elements of Ω form a strictly increasing infinite sequence $\omega_1 < \omega_2 < \dots$.

First we choose a positive integer N such that

$$(6.1) \quad \sum_j ([n+1]N\alpha_j) - [nN\alpha_j] \geq N \quad \text{for all } n = 0, 1, \dots$$

For this first we choose a positive integer k satisfying $\alpha_1 + \dots + \alpha_k > 1$, and then a positive integer N such that

$$\frac{k}{N} < \alpha_1 + \dots + \alpha_k - 1.$$

Then using the inequality $[x+y] \geq [x]+[y]$ we obtain the following estimate for all $n = 0, 1, \dots$:

$$\begin{aligned} \frac{1}{N} \sum_j ([n+1]N\alpha_j) - [nN\alpha_j] &\geq \frac{1}{N} \sum_{j=1}^k [N\alpha_j] > \\ &> \frac{1}{N} \sum_{j=1}^k (N\alpha_j - 1) = \\ &= \alpha_1 + \dots + \alpha_k - \frac{k}{N} > \\ &> 1. \end{aligned}$$

Note that

$$(6.2) \quad 0 \leq ([n+1]N\alpha_j) - [nN\alpha_j] \leq N$$

for all n and j because using the condition $\alpha_j \leq 1$ we have

$$[(n+1)N\alpha_j] = [nN\alpha_j + N\alpha_j] \leq [nN\alpha_j + N] = [nN\alpha_j] + N.$$

Next we represent Ω again as the union of the disjoint blocks B_n of N elements as in (4.1). Since the upper density of Ω does not change if we remove a finite number of initial terms, we may define by recurrence a sequence of bounded intervals (I_m) having the following four properties:

$$\sup I_m < \inf I_{m+1} \quad \text{for all } m;$$

no block B_n belongs to more than one interval I_m ;

$$(6.3) \quad \frac{\text{Card}(\Omega \cap I_m)}{|I_m|} \rightarrow D^+.$$

Let us also introduce a sequence of positive integers containing each index j infinitely many times, for example

$$(b_m) := 1 \dots d \ 1 \dots d \ 1 \dots d \ \dots$$

in case of a finite sequence $\alpha_1, \dots, \alpha_d$, and

$$(b_m) := 12 \ 123 \ 1234 \ 12345 \ 12 \dots$$

in case of an infinite sequence $\alpha_1, \alpha_2, \dots$.

Now, thanks to (6.1) and (6.2) we may define a partition $\Omega = \Omega_1 \cup \Omega_2 \cup \dots$ having the following properties, where we denote by B'_m the union of the k_m consecutive blocks $B_{n_m+1}, \dots, B_{n_m+k_m}$ contained in I_m :

$$(6.4) \text{Card}(B_n \cap \Omega_j) \leq [(n+1)N\alpha_j] - [nN\alpha_j] \quad \text{for all } n \text{ and } j;$$

$$(6.5) \text{Card}(B_n \cap \Omega_j) = [(n+1)N\alpha_j] - [nN\alpha_j] \quad \text{if } B_n \subset B'_m \text{ and } j = b_m.$$

We claim that $D^+_j \leq \alpha_j D^+$ for each j . The proof is similar to that in part (a). The case $D^+_j = 0$ is obvious. If $D^+_j > 0$ for some j , then let us choose a sequence of bounded intervals (I_m^j) satisfying $|I_m^j| \rightarrow \infty$ and

$$\frac{\text{Card}(\Omega_j \cap I_m^j)}{|I_m^j|} \rightarrow D^+_j.$$

Since $D^+_j > 0$, hence $\text{Card}(\Omega_j \cap I_m^j) \rightarrow \infty$ and therefore $k_m \rightarrow \infty$. Since $I_m^j \cap \Omega$ is contained in the union of the $k_m + 2$ blocks $B_{n_m}, \dots, B_{n_m+k_m+1}$, using (6.4) and the inequality

$$[x+y] - [x] \leq [y] + 1 \leq y + 1 \leq y + N$$

it follows that

$$\begin{aligned}
\frac{\text{Card}(\Omega_j \cap I_m^j)}{|I_m^j|} &\leq \frac{1}{|I_m^j|} \left(2N + \sum_{n=n_m+1}^{n_m+k_m} [(n+1)N\alpha_j] - [nN\alpha_j] \right) = \\
&= \frac{2N + [(n_m+k_m+1)N\alpha_j] - [(n_m+1)N\alpha_j]}{|I_m^j|} \leq \\
&\leq \frac{3N + k_m N\alpha_j}{|I_m^j|} \leq \\
&\leq \frac{3N + k_m N\alpha_j}{|I_m^j|} \cdot \frac{\text{Card}(\Omega \cap I_m^j)}{k_m N} \leq \\
&\leq \frac{3 + k_m \alpha_j}{k_m} \cdot \frac{n^+(\Omega, |I_m^j|)}{|I_m^j|}
\end{aligned}$$

for every m . Letting $m \rightarrow \infty$ we conclude that $D^+_j \leq \alpha_j D^+$.

We claim that $D^+_j \geq \alpha_j D^+$ and thus $D^+_j = \alpha_j D^+$ for each j . Indeed, for each fixed j there exist arbitrarily long intervals I_m for which $b_m = j$. For these intervals, using (6.5) and the inequality

$$[x+y] - [x] \geq [y] > y - 1 \geq y - N$$

we obtain

$$\begin{aligned}
\text{Card}(\Omega_j \cap I_m) &\geq \text{Card}(B'_m \cap \Omega_j) = \\
&= \sum_{n=n_m+1}^{n_m+k_m} [(n+1)N\alpha_j] - [nN\alpha_j] = \\
&= [(n_m+k_m+1)N\alpha_j] - [(n_m+1)N\alpha_j] \geq \\
&\geq k_m N\alpha_j - N
\end{aligned}$$

and therefore

$$\begin{aligned}
\frac{\text{Card}(\Omega_j \cap I_m)}{|I_m|} &\geq \frac{k_m N\alpha_j - N}{|I_m|} \geq \\
&\geq \frac{k_m N\alpha_j - N}{|I_m|} \cdot \frac{\text{Card}(\Omega \cap I_m)}{(k_m+2)N} = \\
&= \frac{k_m \alpha_j - 1}{k_m + 2} \cdot \frac{\text{Card}(\Omega \cap I_m)}{|I_m|}.
\end{aligned}$$

It follows that

$$\frac{n^+_j(\Omega_j, |I_m|)}{|I_m|} \geq \frac{k_m \alpha_j - 1}{k_m + 2} \cdot \frac{\text{Card}(\Omega \cap I_m)}{|I_m|}.$$

Since $D^+ > 0$ and $|I_m| \rightarrow \infty$, by (6.3) we have $\text{Card}(\Omega \cap I_m) \rightarrow \infty$ and therefore $k_m \rightarrow \infty$. Therefore, letting $|I_m| \rightarrow \infty$ we conclude that $D^+_j \geq \alpha_j D^+$.

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ON UNIFORMLY MONOTONE OPERATORS ARISING IN NONLINEAR ELLIPTIC AND PARABOLIC PROBLEMS

By

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Abstract. We show some examples for uniformly monotone operators arising in weak formulation of nonlinear elliptic and parabolic problems. Besides the classical p -Laplacian some less known examples are given which are of interest because of applications.

1. Introduction

The aim of the present paper is to show several examples for uniformly monotone operators arising in weak formulation of nonlinear elliptic and parabolic problems. Let X be a normed space and denote by X^* its dual, further, by $\langle \cdot, \cdot \rangle$ the pairing between X^* and X . Then, an operator $A: X \rightarrow X^*$ is called uniformly monotone (following the terminology of [7]) if there exist $p \geq 2$, $\gamma > 0$ such that

$$(1.1) \quad \langle A(u_1) - A(u_2), u_1 - u_2 \rangle \geq \gamma \cdot \|u_1 - u_2\|_X^p$$

for all $u_1, u_2 \in X$. In what follows, we study operators which are obtained by considering the weak formulation of an elliptic or parabolic equation or system with some boundary conditions, see, e.g., [6]. Namely, in the elliptic case let X be a linear subspace of $W^{1,p}(\Omega)$, where $\Omega \subset \mathbb{R}^n$ is bounded (with

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sufficiently smooth boundary), $p \geq 2$, and consider operator $A: X \rightarrow X^*$ defined by

(1.2)

$$\langle A(u), v \rangle = \int_{\Omega} \left(\sum_{i=1}^n a_i(x, u(x), Du(x)) D_i v(x) + a_0(x, u(x), Du(x)) v(x) \right) dx,$$

where D_i denotes the distributional derivative with respect to the i -th variable and $D = (D_1, \dots, D_n)$ is the gradient. The space X depends on the boundary conditions, for instance, $X = W^{1,p}(\Omega)$ in case of homogeneous Neumann type and $X = W_0^{1,p}(\Omega)$ (i.e. the closure of $C_0^\infty(\Omega)$ in $W^{1,p}(\Omega)$) in case of homogeneous Dirichlet type condition (one can also have mixed boundary conditions, see [2]).

A weak form of an elliptic problem may be written as $A(u) = F$ where $F \in X^*$. In the simplest case

$$\langle F, v \rangle = \int_{\Omega} f(x) v(x) dx$$

with some $f \in L^q(\Omega)$ where $\frac{1}{p} + \frac{1}{q} = 1$.

The parabolic case is a minor modification of the elliptic, X may be chosen as $L^p(0, T; V)$ (i.e., the set of measurable functions $u: (0, T) \rightarrow V$, see, e.g., [7]), where V is a linear subspace of $W^{1,p}(\Omega)$ and $0 < T < \infty$, further, functions a_i may depend on variable t , and in (1.2) one integrates on $(0, T) \times \Omega$. The weak formulation of a parabolic problem may be written in the form $D_t u + Au = F$, where D_t denotes the distributional derivative with respect to the variable t .

Supposing the uniform monotonicity (and some other properties) of an operator of the form (1.2), one can prove uniqueness of solutions to the above abstract equations, continuous dependence of the solutions on data and for parabolic equations one can obtain results on asymptotic behavior as $t \rightarrow \infty$, see, e.g., [3, 6].

The well-known example for an operator having the form (1.2) is the following:

$$\begin{aligned} a_i(x, \xi) &= \xi_i |(\xi_1, \dots, \xi_n)|^{p-2} \quad (i = 1, \dots, n), \\ a_0(x, \xi) &= \xi_0 |\xi_0|^{p-2} \end{aligned}$$

where $\xi = (\xi_0, \xi_1, \dots, \xi_n)$ refers to $(u, D_1 u, \dots, D_n u)$. In this case, assuming homogeneous Dirichlet boundary condition, Gauss's theorem yields that

operator (1.2) may be considered as the weak form of the classical operator $u \mapsto \Delta_p u + u|u|^{p-2}$ where

$$\Delta_p u = \operatorname{div}(\operatorname{grad} u | \operatorname{grad} u|^{p-2})$$

is the p -Laplacian. Note that in case $X = W_0^{1,p}(\Omega)$ operator Δ_p is also uniformly monotone since, due to Poincaré's inequality, an equivalent norm can be introduced in $W_0^{1,p}(\Omega)$, see [1].

In [2, 3] we considered a nonlinear system consisting of three different types of differential equations: a first order ODE, a parabolic and an elliptic PDE. Such a system may occur, e.g., as a generalization of a model describing fluid flow in porous media. In that case operator (1.2) has a special form: functions a_i do not depend on (ξ_0, \dots, ξ_k) if $i > k$. The present paper was motivated by such operators. Because of the application it is of interest to have some efficient criterion for uniform monotonicity and show some concrete uniform monotone operators of that type. In what follows, we shall give a variety of examples for uniformly monotone operators, including functions a_i of the above mentioned special type. In the next section we shall formulate and prove a result of [4] which is a sufficient condition on functions a_i for the uniform monotonicity of operator (1.2) and this will be applied to examples in Section 3. For further details on operators of monotone type, see [5, 7], for applications to parabolic and elliptic partial differential equations, see, [6].

2. A sufficient condition

Let $\Omega \subset \mathbb{R}^n$ be a bounded open subset and X be a linear subspace of $W^{1,p}(\Omega)$ ($p \geq 2$) and let us use the notations introduced in the previous section. We define operator $A: X \rightarrow X^*$ by the formula (1.2). Consider the abstract equation $A(u) = F$ where $F \in X^*$ (which may be obtained as a weak formulation of an elliptic boundary-value problem). Problems of this type have an extended classical theory (see, e.g., [5, 7]). Existence and uniqueness of solutions can be guaranteed by supposing the following well-known conditions:

- (A1) The functions $a_i: \Omega \times \mathbb{R}^{n+1} \rightarrow \mathbb{R}$ ($i = 0, \dots, n$) are of Carathéodory type, i.e., $a_i(x, \xi)$ is measurable in $x \in \Omega$ for all $\xi \in \mathbb{R}^{n+1}$, and continuous in $\xi \in \mathbb{R}^{n+1}$ for a.a. $x \in \Omega$.

- (A2) There exist a constant $c > 0$ and a function $k \in L^q(\Omega)$ such that for a.a. $x \in \Omega$ and all $\xi \in \mathbb{R}^{n+1}$,

$$|a_i(x, \xi)| \leq c \cdot |\xi|^{p-1} + k(x) \quad (i = 0, \dots, n).$$

- (A3) There exists a constant $C > 0$ such that for a.a. $x \in \Omega$ and all $\xi, \tilde{\xi} \in \mathbb{R}^n$,

$$\sum_{i=0}^n (a_i(x, \xi) - a_i(x, \tilde{\xi}))(\xi_i - \tilde{\xi}_i) \geq C \cdot |\xi - \tilde{\xi}|^p.$$

Clearly, after integration on Ω condition (A3) yields (1.1) for all $u_1, u_2 \in X$ with $\gamma = C$ thus (A3) ensures the uniform monotonicity of operator A . This implies uniqueness and continuous dependence on data of the solutions to $A(u) = F$. We note that existence of solutions may be shown also if condition (A3) is weakened as follows: the monotonicity is restricted to the main part of the operator and the so-called coercivity (in other words ellipticity) condition is assumed. In this case one obtains the pseudomonotonicity of operator A and then existence of solutions to $A(u) = F$ also follows, see [7].

Now we recall a result of [4] which is a sufficient condition on functions a_i guaranteeing condition (A3).

PROPOSITION 2.1. *Suppose that $p \geq 2$ and a_i is continuously differentiable in variable ξ for all $i = 0, \dots, n$. Further, assume that there exists a constant $\delta > 0$ such that for a.a. $x \in \Omega$, all $\xi \in \mathbb{R}^{n+1}$ and all $(z_0, \dots, z_n) \in \mathbb{R}^{n+1}$,*

$$(2.1) \quad \sum_{j=0}^n \sum_{i=0}^n D_j a_i(x, \xi) z_i z_j \geq \delta \cdot \sum_{i=0}^n |\xi_i|^{p-2} z_i^2.$$

Then condition (A3) holds.

To prove this assertion we shall apply the following elementary inequality (2.2) from [4]. For convenience we present a proof of it below.

LEMMA 2.2. *Let a, b be arbitrary and $s \geq 0$ real numbers. Then*

$$(2.2) \quad \int_0^1 |a + \tau b|^s d\tau \geq \frac{|b|^s}{2^s(s+1)}.$$

PROOF. The case $b = 0$ is obvious otherwise by homogeneity we may assume $b = 1$ and $a \leq -\frac{1}{2}$. Then by elementary transformations

$$\int_0^1 |a+\tau|^s d\tau = \int_0^1 \left| \tilde{\tau} - \frac{1}{2} \right|^s d\tilde{\tau} + \int_1^{\frac{1}{2}-a} \left| \tilde{\tau} - \frac{1}{2} \right|^s d\tilde{\tau} - \int_0^{-\frac{1}{2}-a} \left| \tilde{\tau} - \frac{1}{2} \right|^s d\tilde{\tau}.$$

Clearly,

$$\int_1^{\frac{1}{2}-a} \left| \tilde{\tau} - \frac{1}{2} \right|^s d\tilde{\tau} \geq \frac{1}{2^s} \geq \int_0^{-\frac{1}{2}-a} \left| \tilde{\tau} - \frac{1}{2} \right|^s d\tilde{\tau},$$

thus

$$\int_0^1 |a+\tau|^s d\tau \geq \int_0^1 \left| \tau - \frac{1}{2} \right|^s d\tau = \frac{1}{2^s(s+1)}.$$

We see that the inequality (2.2) is sharp, equality holds if and only if $a = -\frac{b}{2}$. ■

Now we prove Proposition 2.1. We follow the proof of [4].

PROOF OF PROPOSITION 2.1. For fixed $x \in \Omega$, $\xi, \tilde{\xi} \in \mathbb{R}^{n+1}$ we define the functions $f_i: [0, 1] \rightarrow \mathbb{R}$ by

$$f_i(\tau) = a_i(x, \tilde{\xi} + \tau(\xi - \tilde{\xi})), \quad i = 0, \dots, n.$$

Then by applying assumption (2.1) and inequality (2.2) we may deduce

$$\begin{aligned} \sum_{i=0}^n (a_i(x, \xi) - a_i(x, \tilde{\xi}))(\xi_i - \tilde{\xi}_i) &= \sum_{i=0}^n (f_i(1) - f_i(0))(\xi_i - \tilde{\xi}_i) = \\ &= \sum_{i=0}^n \int_0^1 \sum_{j=0}^n D_j a_i(\tilde{\xi} + \tau(\xi - \tilde{\xi}))(\xi_j - \tilde{\xi}_j)(\xi_i - \tilde{\xi}_i) d\tau \\ &\geq \delta \cdot \sum_{i=0}^n \int_0^1 |\tilde{\xi} + \tau(\xi - \tilde{\xi})|^{p-2} (\xi_i - \tilde{\xi}_i)^2 d\tau \geq \\ &\geq \frac{\delta}{2^{p-2}(p-1)} |\xi - \tilde{\xi}|^p. \end{aligned}$$

Whence after integration we conclude

$$\langle A(u_1) - A(u_2), u_1 - u_2 \rangle \geq \frac{\delta}{2^{p-2}(p-1)} \|u_1 - u_2\|_X^p.$$

Thus condition (A3) holds with $C = \frac{\delta}{2^{p-2}(p-1)}$. ■

3. Examples

Now we show some examples of uniformly monotone operators which fulfil also conditions (A1), (A2). For simplicity, we consider examples not depending on variable x . In the sequel we always suppose $p \geq 2$.

EXAMPLE 1. Let $a_i(\xi) = \xi_i |\xi_i|^{p-2}$ ($i = 0, \dots, n$). Then

$$\langle A(u), v \rangle = \int_{\Omega} \left(\sum_{i=1}^n D_i u D_i v |D_i u|^{p-2} + u v |u|^{p-2} \right) dx.$$

Note that functions a_i obviously fulfil conditions (A1), (A2). Now simple calculations yield $D_i a_i(\xi) = (p-1)|\xi_i|^{p-2}$ and $D_j a_i(\xi) = 0$ ($j \neq i$). Hence

$$\sum_{j=0}^n \sum_{i=0}^n D_j a_i(\xi) z_i z_j = (p-1) \sum_{i=0}^n |\xi_i|^{p-2} z_i^2,$$

thus by Proposition 2.1 condition (A3) holds as well.

EXAMPLE 2. Now let

$$\begin{aligned} a_i(\xi) &= \xi_i |(\xi_1, \dots, \xi_n)|^{p-2} \quad (i = 1, \dots, n), \\ a_0(\xi) &= \xi_0 |\xi_0|^{p-2}. \end{aligned}$$

In this case

$$\langle A(u), v \rangle = \int_{\Omega} \left(\sum_{i=1}^n D_i u D_i v |Du|^{p-2} + u v |u|^{p-2} \right) dx,$$

i.e., A is the weak form of operator $u \mapsto \Delta_p u + u|u|^{p-2}$ mentioned in the introduction. Obviously, functions a_i satisfy conditions (A1), (A2). It is easy to see that

$$\begin{cases} D_j a_i(\xi) = (p-2)\xi_j \xi_i |(\xi_1, \dots, \xi_n)|^{p-4}, & \text{for } i, j > 0, i \neq j; \\ D_i a_i(\xi) = |(\xi_1, \dots, \xi_n)|^{p-2} + (p-2)\xi_i^2 |(\xi_1, \dots, \xi_n)|^{p-4}, & \text{for } i > 0; \\ D_j a_0(\xi) = D_0 a_i(\xi) = 0, & \text{for } j > 0, i > 0; \\ D_0 a_0(\xi) = (p-1)|\xi_0|^{p-2}. & \end{cases}$$

Hence

$$\begin{aligned}
\sum_{j=0}^n \sum_{i=0}^n D_j a_i z_i z_j &= \sum_{i=1}^n |(\xi_1, \dots, \xi_n)|^{p-2} z_i^2 + (p-1) |\xi_0|^{p-2} z_0^2 + \\
&\quad + (p-2) |(\xi_1, \dots, \xi_n)|^{p-4} \cdot \sum_{j=1}^n \sum_{i=1}^n \xi_i \xi_j z_i z_j = \\
&= \sum_{i=1}^n |(\xi_1, \dots, \xi_n)|^{p-2} z_i^2 + (p-1) |\xi_0|^{p-2} z_0^2 + \\
&\quad + (p-2) |(\xi_1, \dots, \xi_n)|^{p-4} \cdot \left(\sum_{i=1}^n \xi_i z_i \right)^2 \geq \\
&\geq \sum_{i=0}^n |\xi_i|^{p-2} z_i^2,
\end{aligned}$$

thus from Proposition 2.1 it follows that operator A is uniformly monotone.

EXAMPLE 3. Let $a_i(\xi) = \xi_i |\xi|^{p-2} + g_i(\xi)$ ($i = 0, \dots, n$), where the functions $g_i: \mathbb{R}^{n+1} \rightarrow \mathbb{R}$ are continuous, further, there exist positive constants c, ε such that

$$(3.1) \quad |g_i(\xi)| \leq c \cdot |\xi|^{p-1} \quad \text{and} \quad |D_j g_i(\xi)| \leq \frac{1}{n+1+\varepsilon} \cdot |\xi|^{p-2}$$

for all $\xi = (\xi_0, \dots, \xi_n) \in \mathbb{R}^{n+1}$ ($i, j = 0, \dots, n$). It is clear that conditions (A1), (A2) hold. By using Example 2 and the inequality $|\alpha\beta| \leq \frac{1}{2}(\alpha^2 + \beta^2)$ one obtains

$$\begin{aligned}
\sum_{j=0}^n \sum_{i=0}^n D_j a_i(\xi) z_i z_j &\geq \sum_{i=0}^n |\xi|^{p-2} z_i^2 - \frac{1}{2} \sum_{j=0}^n \sum_{i=0}^n |D_j g_i(\xi)| (z_i^2 + z_j^2) \geq \\
&\geq \sum_{i=0}^n |\xi|^{p-2} z_i^2 - (n+1) \sum_{i=0}^n \frac{1}{n+1+\varepsilon} |\xi|^{p-2} z_i^2 = \\
&= \sum_{i=0}^n \frac{\varepsilon}{n+1+\varepsilon} |\xi|^{p-2} z_i^2,
\end{aligned}$$

which implies condition (A3). As an example for functions g_i with the properties (3.1), consider, e.g.,

$$g_i(\xi) = \frac{1}{(n+1+\varepsilon) \cdot \max\{\alpha_j, j=0, \dots, n\}} \prod_{j=0}^n |\xi_j|^{\alpha_j}$$

where $\alpha_j \geq 0$ for all $j = 0, \dots, n$ and $\sum_{j=0}^n \alpha_j = p - 1$.

EXAMPLE 4. Now we show example for the system considered in [2] (which was mentioned in the Introduction). Suppose $2 \leq p \leq 4$, $1 \leq k \leq n$ and let

$$\begin{aligned} a_i(\xi) &= \xi_i |\xi|^{p-2} \quad (0 \leq i \leq k \leq n), \\ a_i(\xi) &= \xi_i |(\xi_{k+1}, \dots, \xi_n)|^{p-2} \quad (k < i \leq n). \end{aligned}$$

We show that these functions fulfil condition (A3) ((A1) and (A2) obviously hold). Now for brevity let $\zeta = (\xi_{k+1}, \dots, \xi_n)$. Clearly,

$$\begin{cases} D_j a_i(\xi) = (p-2)\xi_i \xi_j |\xi|^{p-4}, & \text{for } 0 \leq i \leq k, 0 \leq j \leq n, j \neq i; \\ D_j a_i(\xi) = (p-2)\xi_i \xi_j |\zeta|^{p-4}, & \text{for } k < i \leq n, k < j < n, j \neq i; \\ D_j a_i(\xi) = 0, & \text{for } k < i \leq n, 0 \leq j \leq k \\ D_i a_i(\xi) = |\xi|^{p-2} + (p-2)\xi_i^2 |\xi|^{p-4}, & \text{for } 0 \leq i \leq k; \\ D_i a_i(\xi) = |\zeta|^{p-2} + (p-2)\xi_i^2 |\zeta|^{p-4}, & \text{for } k < i \leq n. \end{cases}$$

Then

$$\begin{aligned} &\sum_{j=0}^n \sum_{i=0}^n D_j a_i(\xi) z_i z_j = \\ &= \sum_{i=0}^k |\xi|^{p-2} z_i^2 + (p-2)|\xi|^{p-4} \sum_{j=0}^n \sum_{i=0}^k \xi_i \xi_j z_i z_j + \\ &+ \sum_{i=k+1}^n |\zeta|^{p-2} z_i^2 + (p-2)|\zeta|^{p-4} \sum_{j=k+1}^n \sum_{i=k+1}^n \xi_i \xi_j z_i z_j = \\ &= \sum_{i=0}^k |\xi|^{p-2} z_i^2 + \sum_{i=k+1}^n |\zeta|^{p-2} z_i^2 + (p-2)|\xi|^{p-4} \left(\sum_{i=0}^k \xi_i z_i \right)^2 + \end{aligned}$$

$$+ (p-2)|\xi|^{p-4} \left(\sum_{i=k+1}^n \xi_i z_i \right)^2 + (p-2)|\xi|^{p-4} \sum_{j=k+1}^n \sum_{i=0}^k \xi_i \xi_j z_i z_j.$$

By using the estimate

$$\begin{aligned} \sum_{j=k+1}^n \sum_{i=0}^k \xi_i \xi_j z_i z_j &= \left(\sum_{j=k+1}^n \xi_j z_j \right) \left(\sum_{i=0}^k \xi_i z_i \right) \geq \\ &\geq -\frac{1}{2} \left(\sum_{i=k+1}^n \xi_i z_i \right)^2 - \frac{1}{2} \left(\sum_{i=0}^k \xi_i z_i \right)^2. \end{aligned}$$

and the fact that $|\xi|^{p-4} \geq |\xi|^{p-4}$ (since $p \leq 4$) we conclude

$$\begin{aligned} \sum_{j=0}^n \sum_{i=0}^n D_j a_i(\xi) z_i z_j &= \\ &= \sum_{i=0}^k |\xi|^{p-2} z_i^2 + \sum_{i=k+1}^n |\xi|^{p-2} z_i^2 + \frac{1}{2}(p-2)|\xi|^{p-4} \left(\sum_{i=k+1}^n \xi_i z_i \right)^2 + \\ &\quad + \frac{1}{2}(p-2)|\xi|^{p-4} \left(\sum_{i=0}^k \xi_i z_i \right)^2 \geq \sum_{i=0}^n |\xi_i|^{p-2} z_i^2. \end{aligned}$$

Now Proposition 2.1 yields condition (A3).

In case $p > 4$ one may consider, e.g., the following functions:

$$a_i(\xi) = \xi_i |(\xi_0, \dots, \xi_k)|^{p-2} + \xi_i |\xi|^{r-2} \quad (0 \leq i \leq k \leq n),$$

$$a_i(\xi) = \xi_i |(\xi_{k+1}, \dots, \xi_n)|^{p-2} + \xi_i |(\xi_{k+1}, \dots, \xi_n)|^{r-2} \quad (k < i < n),$$

where $2 \leq r \leq 4$, $1 \leq k \leq n$. By using the previous examples it is not difficult to show that these functions satisfy condition (A3).

EXAMPLE 5. Now let

$$a_i(\xi) = \xi_i |\xi_i|^{p-2} + \prod_{k=0}^n \xi_k |\xi_k|^{p-2} \cdot h_i(\xi) \quad (i = 0, \dots, n)$$

where functions $h_i: \mathbb{R}^{n+1} \rightarrow \mathbb{R}$ ($i = 0, \dots, n$) are differentiable and have compact support S_i . Denote $S = \bigcup_{i=0}^n S_i$ and let

$$\alpha = p \max \left\{ \sup_{\xi \in S} |\xi|^{(n+1)(p-1)}, 1 \right\} \cdot \max \left\{ \sup_S |h|, \sup_S |Dh| \right\}.$$

We show that if α is sufficiently small then functions a_i fulfil condition (A3) ((A1) is obvious and due to the compact supports (A2) is also satisfied). Observe that

$$D_i a_i(\xi) =$$

$$= (p-1)|\xi_i|^{p-2} + (p-1)|\xi_i|^{p-2} \prod_{k \neq i} \xi_k |\xi_k|^{p-2} \cdot h_i(\xi) + \prod_{k=0}^n \xi_k |\xi_k|^{p-2} \cdot D_i h_i(\xi),$$

thus

$$D_i a_i(\xi) \geq (p-1)|\xi_i|^{p-2} - \alpha |\xi_i|^{p-2}.$$

In addition, for $j \neq i$,

$$D_j a_i(\xi) = (p-1)|\xi_j|^{p-2} \prod_{k \neq j} \xi_k |\xi_k|^{p-2} \cdot h_i + \prod_{k=0}^n \xi_k |\xi_k|^{p-2} \cdot D_j h_i(\xi)$$

so that $|D_j a_i(\xi)| \leq \alpha \cdot |\xi_i|^{p-2}$ and $|D_j a_i(\xi)| \leq \alpha \cdot |\xi_j|^{p-2}$ hence

$$|D_j a_i(\xi) z_i z_j| \leq \alpha \cdot \left(|\xi_i|^{p-2} z_i^2 + |\xi_j|^{p-2} z_j^2 \right).$$

Therefore,

$$\begin{aligned} \sum_{j=0}^n \sum_{i=0}^n D_j a_i(\xi) z_i z_j &\geq (p-1-\alpha) \sum_{i=0}^n |\xi_i|^{p-2} z_i^2 - \\ &\quad - \alpha \sum_{j=0}^n \sum_{i=0}^n \left(|\xi_i|^{p-2} z_i^2 + |\xi_j|^{p-2} z_j^2 \right) \geq \\ &\geq (p-1-\alpha) \sum_{i=0}^n |\xi_i|^{p-2} z_i^2 - 2n\alpha \sum_{i=0}^n |\xi_i|^{p-2} z_i^2 = \\ &= (p-1-(2n+1)\alpha) \sum_{i=0}^n |\xi_i|^{p-2} z_i^2. \end{aligned}$$

Hence (2.1) holds provided α is sufficiently small which implies condition (A3).

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BIFURCATIONS IN THE DIFFERENTIAL EQUATION MODEL OF A CHEMICAL REACTION

By

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Dedicated to Professor László Simon on the occasion of his 70th birthday

Abstract. In this paper a two-dimensional system of non-linear ordinary differential equations describing the oxygen reduction reaction on platinum surface is studied. The investigation is motivated by the fact that this reaction plays an important role in fuel cells. The mechanism of this reaction has been known for years, however, the detailed study of the mathematical model has not been carried out. The purpose of this paper is to reveal the dynamical behaviour of the ODE system, with emphasis on the number and type of the stationary points, the existence of periodic orbits and bifurcations. We point out that bistability occurs in the system, i.e. for certain values of the parameters two stable equilibria coexist that was not known before and is significant also from the chemical point of view. We also prove that the system has no periodic orbit. The saddle-node bifurcation curve is determined by using the Parametric Representation Method, and this enables us to determine numerically the parameter domain where bistability occurs.

1. Introduction

We will consider the following system of non-linear ordinary differential equations.

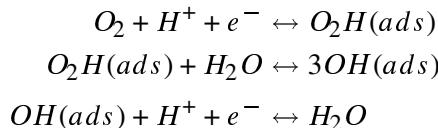
$$(1) \quad \dot{\theta}_1 = 3K_2\theta_2\theta_s^2 - 3L_2\theta_1^3 - K_3\theta_1 + L_3\theta_s,$$

$$(2) \quad \dot{\theta}_2 = K_1\theta_s - L_1\theta_2 - K_2\theta_2\theta_s^2 + L_2\theta_1^3,$$

where $\theta_s = 1 - \theta_1 - \theta_2$ and K_i, L_i are positive parameters for $i = 1, 2, 3$.

This system describes the oxygen reduction reaction on platinum, which is a reaction of fundamental importance in the modelling of fuel cells. The variables θ_1, θ_2 denote the relative coverage of the surface with OH and O_2H molecules, and θ_s denotes the number of free surface spaces per surface unit. The values of these variables are between zero and one.

Oxygen reduction reaction has been studied in many electrolytes, because of its high importance. The understanding of oxygen reduction reaction on platinum surface is a crucial point for the development of fuel cells, because this reaction is the rate determining step, and limits mostly the performance of fuel cells. The precise knowledge of the mechanism could help us to improve the performance of fuel cells and to decrease the amount of the catalyst used. A widely accepted mechanism to the oxygen reduction reaction in acidic media on platinum was created by Damjanovic et al [1, 2]. According to their theory the following mechanism occurs: after the first step, which is a fast oxygen adsorption with low surface coverage, an electrochemical reaction takes place, in which an adsorbed O_2H molecule is formed, as it can be seen below. After this, the molecule, under discussion, goes into a chemical reaction with a water molecule. The OH species, generated at this time reacts in the third, electrochemical step. Summarizing, the following reaction mechanism will be investigated.



According to this mechanism, one can create the following reaction rates:

$$\begin{aligned} v_1 &= k_1\theta_s \exp\left(-\frac{\beta_1(\eta - E_1)F}{RT}\right) - k_{-1}\theta_2 \exp\left(\frac{(1-\beta_1)(\eta - E_1)F}{RT}\right) \\ v_2 &= k_2\theta_2\theta_s^2 - k_{-2}\theta_1^3 \\ v_3 &= k_3\theta_1 \exp\left(-\frac{\beta_2(\eta - E_2)F}{RT}\right) - k_{-3}\theta_s \exp\left(\frac{(1-\beta_2)(\eta - E_2)F}{RT}\right) \end{aligned}$$

where η is the electrode potential, F is Faraday's constant, R is the universal gas constant, k_i are reaction rate constants and β_i, E_i are electro-chemical parameters. The kinetic equations of the above mechanism take the form:

$$(3) \quad \dot{\theta}_1 = 3v_2 - v_3$$

$$(4) \quad \dot{\theta}_2 = v_1 - v_2$$

Substituting the reaction rates v_i into this system we get system (1)–(2) with

$$K_1 = k_1 e^{-\frac{\beta_1(\eta - E_1)F}{RT}}, \quad K_2 = k_2, \quad K_3 = k_3 e^{-\frac{\beta_2(\eta - E_2)F}{RT}}$$

$$L_1 = k_{-1} e^{\frac{(1-\beta_1)(\eta - E_1)F}{RT}}, \quad L_2 = k_{-2}, \quad L_3 = k_{-3} e^{\frac{(1-\beta_2)(\eta - E_2)F}{RT}}$$

The aim of the paper is to study the dynamical behaviour of system (1)–(2). The structure of the paper is as follows. In Section 2 the global properties of the system are studied by means of index theory and the Poincaré-Bendixson theorem. In Section 3 the number of stationary points is determined, it will be proved that there is at least one equilibrium and there are at most three of them. In Section 4 the saddle-node bifurcation curve is investigated by using the Parametric Representation Method. We will show that for certain parameter values there are three equilibria and investigate how the parameter domain belonging to this case can be determined numerically.

2. Global results

In this Section we prove that there exists a bounded positively invariant set in the phase plane and show that there is no periodic orbit in it. Hence by the Poincaré-Bendixson theorem all trajectories in this set are converging to equilibrium points as time goes to infinity.

PROPOSITION 1. *The triangle in the phase plane with vertices $(0, 0)$, $(1, 0)$, $(0, 1)$ is positively invariant in system (1)–(2), and there is no periodic orbit in this triangle.*

PROOF. From equation (1) one can immediately see that $\theta_1 = 0$ and $\theta_2 \in [0, 1]$ imply $\dot{\theta}_1 > 0$. Similarly, if $\theta_2 = 0$ and $\theta_1 \in [0, 1]$, then $\dot{\theta}_2 > 0$. If $\theta_1 + \theta_2 = 1$ (so $\theta_s = 0$) and $\theta_1, \theta_2 \in [0, 1]$, then $\dot{\theta}_1 + \dot{\theta}_2 < 0$. Hence the trajectories cross the boundaries of the triangle in the direction of the interior of the triangle, therefore the triangle is positively invariant.

The Jacobian matrix of the system takes the form

$$J = \begin{pmatrix} -6K_2\theta_2\theta_s - 9L_2\theta_1^2 - K_3 - L_3 & 3K_2\theta_s^2 - 6K_2\theta_2\theta_s - L_3 \\ -K_1 + 2K_2\theta_2\theta_s + 3L_2\theta_1^2 & -K_1 - L_1 - K_2\theta_s^2 + 2K_2\theta_2\theta_s \end{pmatrix}.$$

It can be easily seen that the trace of this matrix is negative

$$\text{Tr}(J) = -4K_2\theta_2\theta_s - 9L_2\theta_1^2 - K_3 - L_3 - K_1 - L_1 - K_2\theta_s^2 < 0,$$

hence the divergence of the vector field is negative in the whole triangle. Therefore according to Bendixson's criterion there is no periodic orbit in the triangle. ■

An immediate consequence of this Proposition is that the trajectories converge to equilibrium. Since all trajectories are going into the triangle along its boundaries the index of the triangle is +1. Hence the sum of indices of the stationary points inside the triangle is 1. Therefore if the number of stationary points is three, then there is a saddle and there are two points that are nodes or focuses. At least one of those two points is stable and the focus is always stable, since the trace of the Jacobian is negative. If there is one stationary point, then it cannot be a saddle. We will show, that the number of stationary points is at most three. Hence one can expect two different types of phase portraits. In the first case there is one globally attracting stationary point in the triangle (all trajectories converge to that point). In the other case there are two stable stationary points and one saddle point. The stable manifold of the saddle (the two orbits converging into the saddle point) divides the triangle into two parts, which are the basins of attraction of the stable stationary points.

3. Number of stationary points

From system (3)–(4) the equations of the stationary points are $3v_2 = v_3$ and $v_1 = v_2$. Hence at the stationary points we have $3v_1 = v_3$ that can be written in the form $3K_1\theta_s - 3L_1\theta_2 = K_3\theta_1 - L_3\theta_s$. This equation contains θ_1 and in θ_2 linearly, hence from this equation and from equation $\theta_1 + \theta_2 = 1 - \theta_s$ we can express θ_1 and θ_2 in terms of θ_s as

$$\theta_1 = -\frac{3L_1}{D} + \theta_s \frac{3K_1 + L_3 + 3L_1}{D}, \quad \theta_2 = \frac{K_3}{D} - \theta_s \frac{3K_1 + L_3 + K_3}{D},$$

where $D = K_3 - 3L_1$. In the case $D = 0$ we cannot express θ_1 and θ_2 in terms of θ_s , instead we get $\theta_s = K_3/(3K_1 + L_3 + K_3)$. Let us assume first that $D \neq 0$. Then substituting these expressions of θ_1 and θ_2 into the equation $v_1 = v_2$ we get the third degree equation

$$K_1\theta_s - L_1(1 + a - \theta_s(1 + b)) = K_2(1 + a - \theta_s(1 + b))\theta_s^2 - L_2(b\theta_s - a)^3$$

for θ_s , where

$$(5) \quad a = \frac{3L_1}{D}, \quad b = \frac{3K_1 + L_3 + 3L_1}{D}.$$

Using these new parameters θ_1 and θ_2 can be expressed in terms of θ_s as

$$(6) \quad \theta_1 = b\theta_s - a, \quad \theta_2 = 1 + a - (1 + b)\theta_s.$$

Introducing

$$(7) \quad p(\theta_s) = v_1 - v_2 = K_1\theta_s - L_1\theta_2 - K_2\theta_2\theta_s^2 + L_2\theta_1^3,$$

where θ_1, θ_2 are given by (6), the above third degree equation can be written in the form

$$p(\theta_s) = 0, \quad p(\theta_s) = a_3\theta_s^3 + a_2\theta_s^2 + a_1\theta_s + a_0$$

where

$$a_3 = K_2(1 + b) + L_2b^3, \quad a_2 = -K_2(1 + a) - 3L_2ab^2,$$

$$a_1 = K_1 + L_1(1 + b) + 3L_2ba^2, \quad a_0 = -L_1(1 + a) - L_2a^3.$$

In the case $D = 0$ the value of θ_s is given, θ_2 can be expressed as $\theta_2 = 1 - \theta_s - \theta_1$, hence (7) is a third degree equation for θ_1 . This equation can be treated similarly to (7), hence in the rest of the paper we will consider the case $D \neq 0$.

Our goal is to determine the number of those roots of the polynomial p in the interval $[0, 1]$, for which $\theta_1 = b\theta_s - a \in [0, 1]$ and $\theta_2 = 1 + a - (1 + b)\theta_s \in [0, 1]$ also holds.

PROPOSITION 2. *The polynomial p has at least one root in the interval $(0, 1)$.*

PROOF. Let us consider first the case $D > 0$. In this case we have $a, b > 0$, $a_3, a_1 > 0$, and $a_2, a_0 < 0$. Hence $p(0) = a_0 < 0$, thus we have to prove that $p(1) > 0$. If $\theta_s = 1$, then $\theta_1 = b - a = \frac{3K_1 + L_3}{D} > 0$, and $\theta_2 = a - b < 0$, therefore using (7) we get $p(1) = K_1 - L_1\theta_2 - K_2\theta_2 + L_2\theta_1^3 > 0$, that we wanted to prove.

In the case $D < 0$ we have $0 < K_3 < 3L_1$, hence $a = \frac{3L_1}{K_3 - 3L_1} < -1$, moreover $b < a$ implies $b < -1$. Therefore $p(0) = a_0 = -L_1(1 + a) - L_2a^3 > 0$. Now we prove that $p(1) < 0$ holds. We have

$$\begin{aligned} p(1) &= K_1 - L_1\theta_2 - K_2\theta_2 + L_2\theta_1^3 = \\ &= K_1 - L_1 \left(-\frac{3K_1 + L_3}{D} \right) - K_2 \left(-\frac{3K_1 + L_3}{D} \right) + L_2 \frac{(3K_1 + L_3)^3}{D^3}. \end{aligned}$$

Multiplying this equation by D we can easily see that the right hand side is positive, i.e. $Dp(1) > 0$, hence $p(1) < 0$ holds. ■

A root θ_s of the polynomial p yields a stationary point (θ_1, θ_2) via (6) once $\theta_1, \theta_2 \in (0, 1)$ also holds. In the case $D > 0$, when we also have $a > 0$ and $b > 0$, one can easily see that $\theta_1 \in (0, 1)$ holds if and only if

$$\frac{a}{b} < \theta_s < \frac{1+a}{b}.$$

Similarly, using (6) one can easily show that $\theta_2 \in (0, 1)$ holds if and only if

$$\frac{a}{1+b} < \theta_s < \frac{1+a}{1+b}.$$

Thus $\theta_1, \theta_2 \in (0, 1)$ holds if and only if

$$\frac{a}{b} < \theta_s < \frac{1+a}{1+b}.$$

Because of $a < b$ we have $\frac{a}{b} \in (0, 1)$, $\frac{1+a}{1+b} \in (0, 1)$, and $\frac{a}{b} < \frac{1+a}{1+b}$.

Now, let us consider the case $D < 0$ when we have $a < -1$ and $b < -1$. Using (6) one can see that $\theta_1 \in (0, 1)$ holds if and only if

$$\frac{a}{b} > \theta_s > \frac{1+a}{b}.$$

Similarly, using (6) one can easily show that $\theta_2 \in (0, 1)$ holds if and only if

$$\frac{a}{1+b} > \theta_s > \frac{1+a}{1+b}.$$

Then it can be easily shown that $\theta_1, \theta_2 \in (0, 1)$ holds if and only if

$$\frac{1+a}{1+b} < \theta_s < \frac{a}{b}.$$

Thus we have proved the following Proposition.

PROPOSITION 3. *The values of θ_1 and θ_2 obtained from (6) are in the interval $[0, 1]$ if and only if $\theta_s \in I$, where I is the interval with endpoints $\frac{a}{b}$ and $\frac{1+a}{1+b}$. More precisely, $I = \left(\frac{a}{b}, \frac{1+a}{1+b}\right)$, when $D > 0$ and $I = \left(\frac{1+a}{1+b}, \frac{a}{b}\right)$, when $D < 0$.*

Now we can prove the existence of a stationary point.

PROPOSITION 4. *There exist at least one stationary point in the triangle.*

PROOF. According to the previous Proposition it is enough to prove that the polynomial p has a root in the interval I , that is it has different signs at the end points of the interval.

Using (7) we get

$$p\left(\frac{a}{b}\right) = \frac{a-b}{b} \left(K_2 \frac{a^2}{b^2} + L_1 \right) + K_1 \frac{a}{b} = \\ \frac{(a-b)L_1 + K_1 a}{b} + \frac{a-b}{b} K_2 \frac{a^2}{b^2} = -\frac{L_3 L_1}{b D} + \frac{a-b}{b} K_2 \frac{a^2}{b^2}.$$

Moreover,

$$p\left(\frac{1+a}{1+b}\right) = L_2 \left(\frac{b-a}{1+b}\right)^3 + K_1 \frac{1+a}{1+b}.$$

In the case $D > 0$, when $a > 0$, $b > 0$ and $b > a$ also hold, we get

$$p\left(\frac{a}{b}\right) < 0 < p\left(\frac{1+a}{1+b}\right).$$

In the case $D < 0$, we get the same signs since $a < -1$, $b < -1$ and $b < a$ hold. ■

Since the polynomial p is of degree three the number of roots can be at most three, hence we have the following Proposition.

PROPOSITION 5. *The number of stationary points in the triangle can be one, two or three.*

4. Study of the saddle-node bifurcation by using the parametric representation method

4.1. Parametric representation method

In this section we briefly introduce the parametric representation method (PRM), which is a systematic approach for constructing bifurcation diagrams [3, 4].

In the simplest case we have to determine the number of solutions of the equation $f(x, \lambda_1, \lambda_2) = 0$ as the parameter $\lambda = (\lambda_1, \lambda_2)$ is varied. Let $N(\lambda_1, \lambda_2) = |\{x \in \mathbb{R} : f(x, \lambda_1, \lambda_2) = 0\}|$ be the number of stationary points. The parameter pair $\lambda^0 = (\lambda_1^0, \lambda_2^0)$ is called regular, if there exists $\delta > 0$, such that $|\lambda - \lambda^0| < \delta$ implies $N(\lambda) = N(\lambda^0) < \infty$. At the parameter pair λ^0 there is a bifurcation, if it is not regular. The simplest bifurcation, the so-called saddle-node, or fold bifurcation may occur at the singularity set

$$S = \{\lambda \in \mathbb{R}^2 : \exists x, f(x, \lambda) = 0 \text{ and } \partial_x f(x, \lambda) = 0\}.$$

In chemical dynamical systems the parameter dependence is usually linear, that is f takes the form

$$(8) \quad f(x, \lambda_1, \lambda_2) = f_0(x) + \lambda_1 f_1(x) + \lambda_2 f_2(x),$$

where f_0, f_1, f_2 are given C^2 functions. In this case the equations defining the singularity set are linear in λ_1 and λ_2 , hence these equations can be solved for λ_1, λ_2 in terms of x . Hence the singularity set can be expressed as a curve parametrized by x . (This is why the method is called parametric representation method.) This curve will be referred to as *D*-curve.

In order to have a unique solution of the linear system defining the singularity set, we will assume that $W(x) = f_1(x)f_2'(x) - f_1'(x)f_2(x) \neq 0$ for all $x \in \mathbb{R}$. In this case the *D*-curve (discriminant curve) is given by the parametrization $D : \mathbb{R} \rightarrow \mathbb{R}^2$

(9)

$$\lambda_1 = D_1(x) = \frac{f_2(x)f_0'(x) - f_0(x)f_2'(x)}{W(x)} \quad \lambda_2 = D_2(x) = \frac{f_0(x)f_1'(x) - f_1(x)f_0'(x)}{W(x)}.$$

Let $m(x) = \{(\lambda_1, \lambda_2) : f(x, \lambda_1, \lambda_2) = 0\}$ be the line in the parameter plane \mathbb{R}^2 along which x is a solution of the equation $f(x, \lambda_1, \lambda_2) = 0$. A simple calculation shows that $D'(x) = (B(x)/W(x))(f_2(x), -f_1(x))$, where $B(x) = f_0''(x) + f_1''(x)D_1(x) + f_2''(x)D_2(x)$. We assume that the roots of B are isolated.

It can be easily shown that the line $m(x)$ is the tangent of the D -curve at the point $D(x)$, that proves the following Lemma [3, 4].

LEMMA 1. (Tangential property) *Let us assume that f is given by (8) and the above conditions hold, then the number of solutions of $f(x, \lambda_1, \lambda_2) = 0$ belonging to a given parameter pair (λ_1, λ_2) is equal to number of tangents drawn to the D -curve from the point (λ_1, λ_2) , the values of the solutions can be read as the values x of tangent point on the D -curve.*

Moreover, the so-called convexity property helps to count the number of tangents easily [3, 4].

LEMMA 2. (Convexity property) *The D -curve consists of convex arcs, meaning that every arc lies on one side of its tangents. These arcs join at cusp points of the D -curve. There is a cusp point at x_0 , if the function $f_0''(x) + f_1''(x)D_1(x) + f_2''(x)D_2(x)$ changes sign at $x = x_0$.*

The use of the convexity property is based on the fact that to each convex arc there can be drawn at most two tangents. The exact number of tangents depends on the position of the point as it is shown in Figure 1.

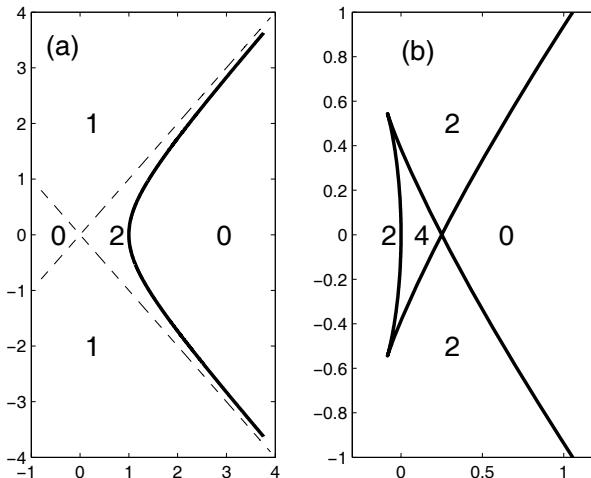


Fig. 1. The number of tangents that can be drawn from different domains to the given convex arc (a). The total number of tangents that can be drawn to the D -curve consisting of three convex arcs (b). (There are one or three tangents at the boundary points of the domains.)

Thus in order to use the PRM to determine the exact number of steady states we need the following characteristic properties of the D -curve, that help to determine the exact number of tangents from different points: the cusp points, the tangents at the endpoints and the position of the curve. These characteristic properties will be studied in the next Subsection.

4.2. Application of parametric representation method

We apply the PRM to the polynomial

$$p(\theta_s) = K_1\theta_s - L_1\theta_2 - K_2\theta_2\theta_s^2 + L_2\theta_1^3,$$

given by (7). We have seen that the number of steady states is equal to the number of roots of p in the interval I defined in Proposition 3. The parameters are $\lambda_1 = K_2$ and $\lambda_2 = L_2$. Hence our equation can be written as $f(x, \lambda_1, \lambda_2) = 0$, where $x = \theta_s$ and f can be given as in (8) with

$$f_0(\theta_s) = K_1\theta_s - L_1\theta_2 \quad f_1(\theta_s) = -\theta_2\theta_s^2 \quad f_2(\theta_s) = \theta_1^3,$$

and θ_1, θ_2 are given by (6). According to (9) the discriminant curve can be expressed as

$$K_2(\theta_s) = \frac{f'_0 f_2 - f_0 f'_2}{f_1 f'_2 - f'_1 f_2} \quad L_2(\theta_s) = \frac{f_0 f'_1 - f'_0 f_1}{f'_2 f_1 - f_2 f'_1}.$$

Calculating the derivatives of the functions f_0, f_1, f_2 we get after simple algebra that

$$\begin{aligned} K_2(\theta_s) &= \frac{K_1(\theta_1 - 3b\theta_s) + L_1((1+b)\theta_1 + 3b\theta_2)}{\theta_s(2\theta_2\theta_1 - (1+b)\theta_s\theta_1 - 3b\theta_2\theta_s)}, \\ L_2(\theta_s) &= \frac{K_1\theta_s((1+b)\theta_s - \theta_2) + 2L_1\theta_2^2}{\theta_1^2(2\theta_2\theta_1 - (1+b)\theta_s\theta_1 - 3b\theta_2\theta_s)}. \end{aligned}$$

Substituting θ_1 and θ_2 from (6) into these expressions we obtain

$$(10) \quad K_2 = \frac{A_1(\theta_s)}{\theta_s N(\theta_s)}, \quad L_2 = \frac{A_2(\theta_s)}{(b\theta_s - a)^2 N(\theta_s)},$$

where

$$A_1(\theta_s) = 2abL_1 + 3bL_1 - aL_1 - aK_1 - \theta_s(2b^2L_1 + 2bL_1 + 2bK_1),$$

$$N(\theta_s) = \theta_s(3a - b + 2ab) - 2a(a + 1),$$

$$A_2(\theta_s) = 2(1+b)(K_1 + (1+b)L_1)\theta_s^2 - (1+a)(K_1 + 4(1+b)L_1)\theta_s + 2(1+a)^2L_1.$$

Let us investigate the position of the D -curve. It has to be decided whether it enters the positive quadrant of the (K_2, L_2) parameter plane, i.e. saddle-node bifurcation occurs for positive parameter values. Hence the sign of the functions A_1, A_2, N has to be studied for $\theta_s \in I$. The functions A_1 and N are linear, hence it is enough to investigate their signs at the end points of the interval. Elementary calculation shows the following.

PROPOSITION 6.

- The linear function N does not change sign in the interval, since it has the same sign at the end points of the interval, namely

$$N\left(\frac{a}{b}\right) = \frac{3a(a-b)}{b}, \quad N\left(\frac{1+a}{1+b}\right) = \frac{(1+a)(a-b)}{1+b}.$$

More precisely, in the case $D > 0$ we have $N(\theta_s) < 0$, and in the case $D < 0$ we have $N(\theta_s) > 0$.

- We have $A_1\left(\frac{a}{b}\right) = \frac{3L_1L_3}{D}$, hence $K_2\left(\frac{a}{b}\right) < 0$. Thus in order to have positive K_2 values, the signs of $A_1\left(\frac{a}{b}\right)$ and $A_1\left(\frac{1+a}{1+b}\right)$ have to be opposite. This holds if and only if

$$(11) \quad L_1L_3(K_3 + 3K_1 + L_3) < 2K_1K_3(3K_1 + 3L_1 + L_3).$$

Now let us consider the sign of L_2 . Its denominator does not change sign according to the previous Proposition. The sign of A_2 is positive at the end points of the interval when $D > 0$. Hence if L_2 has positive values, then A_2 must have a root, i.e. its discriminant is positive, yielding

$$(12) \quad K_1 > 8L_1(1+b).$$

Hence we obtain the following Theorem.

THEOREM 1. Let us assume $D > 0$. If there is a domain in the positive quadrant of the (K_2, L_2) parameter plane where there are three stationary points, then (11) and (12) hold.

Our numerical studies show that the D -curve can have two different shapes. If conditions (11) and (12) hold, then the D -curve enters the positive quadrant and has a cusp point there as it is shown in Figure 2b. Hence for those parameter pairs lying inside the cusp domain there are three equilibria, and for those lying outside there is one. If one of these conditions does not hold, then the D -curve does not enter the positive quadrant as it is shown in

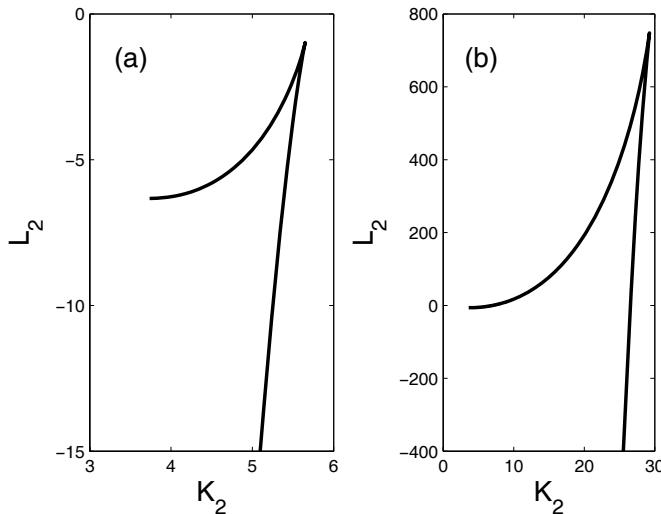


Fig. 2. The two possible positions of the D -curve: it lies outside the positive quadrant (a), and it enters the positive quadrant (b). The parameter values are $K_1 = 1$, $L_1 = 0.07$, $K_3 = 5$, $L_3 = 1$ (a), and $K_1 = 1$, $L_1 = 0.01$, $K_3 = 5$, $L_3 = 1$ (b).

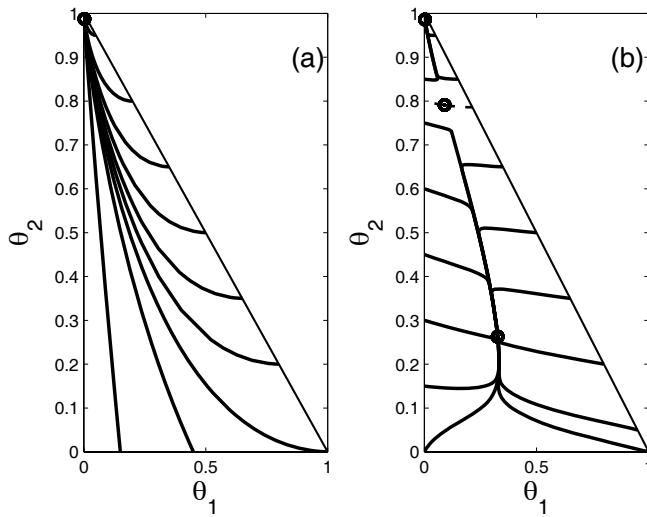


Fig. 3. The two possible phase portraits when the D -curve enters the positive quadrant. The parameter values are $K_1 = 1$, $L_1 = 0.01$, $K_2 = 5$, $L_2 = 1$, $K_3 = 5$, $L_3 = 1$ (a), and $K_1 = 1$, $L_1 = 0.01$, $K_2 = 10$, $L_2 = 1$, $K_3 = 5$, $L_3 = 1$ (b).

Figure 2a. Hence for all parameter pairs there there is one equilibrium point. Figure 3 shows the two possible phase portraits when the D -curve enters the positive quadrant. In this case bistability occurs when the (K_2, L_2) parameter pair lies inside the cusp domain. In Figure 3a there is a single stable stationary point (close to the upper vertex of the triangle). In Figure 3b there are three stationary points, two of them are stable and one is a saddle point. The stable manifold of the saddle point separates the basins of attraction of the stable points.

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THE EULER-POINCARÉ FORMULA FOR SYSTEMS WITH HYSTERESIS IN TWO DIMENSION

By

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Dedicated to Professor László Simon on the occasion of his 70th birthday

Abstract. The present paper is devoted to constructing two-dimensional continuous-time systems with hysteresis having a pair of parallel switching lines and k sources, ℓ saddles, and m sinks inside a given periodic orbit. Here k , ℓ , and m are nonnegative integers arbitrarily given. Still, when counterbalanced with the discontinuity effects on the switching lines, the famous Euler-Poincaré formula remains valid.

1. Motivation and a ‘natural’ counterexample

The piecewise affine two-parameter pair of ordinary differential equations

$$(1) \quad \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} \sigma & 1 \\ -1 & \sigma \end{pmatrix} \begin{pmatrix} x - 1 \\ y - \rho \end{pmatrix} \quad \text{if } x \geq -1 \text{ and } t \geq 0$$

$$(2) \quad \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} \sigma & 1 \\ -1 & \sigma \end{pmatrix} \begin{pmatrix} x + 1 \\ y + \rho \end{pmatrix} \quad \text{if } x \leq 1 \text{ and } t \geq 0$$

is known to exhibit a great variety of dynamical properties [10], [8], [3], [5]. System (1)–(2) is sometimes called Saito chaos generator and can be considered as a simplified model of Chua circuit [11]. Both the emergence of chaotic attractors/repellors and the transition between single-spiral and double-spiral chaotic attractors are well understood.

Two-valued Poincaré mappings for (1)–(2) were introduced and analyzed in our previous papers [4], [6]. The reduction to single-valued interval maps and thus the applicability of Lasota-Yorke results on absolutely continuous

invariant measures were made possible by what we called concatenated arclength transformation. Our main technical task was the search for maximal compact intervals positively invariant with respect to the associated single-valued interval maps. The endpoints of such intervals turned out to be determined by periodic orbits of system (1)–(2) without self-intersection.

Of course, at least in the chaotic parameter regime, system (1)–(2) has an abundance of periodic orbits. Periodic orbits without self-intersection are less numerous but, due to their role in the Lasota-Yorke probabilistic description of chaos [2], much more important.

For brevity, we say that a periodic orbit Γ of system (1)–(2) is *large* if Γ has no self-intersection points and it is not a subset of the vertical strip $\mathcal{S} := \{(x, y) \in \mathbb{R}^2 \mid |x| \leq 1\}$. Similarly, we say that a periodic orbit γ of system (1)–(2) is *small* if γ has no self-intersection points and it is a subset of \mathcal{S} .

The main results of our previous paper [5] on periodic orbits can be summarized as follows.

THEOREM 1. *There exist continuous functions $\kappa_1 : \mathbb{R} \rightarrow \mathbb{R}$ and $\kappa_2 : (-\infty, 0] \rightarrow \mathbb{R}$ with the properties that $\kappa_1(\sigma) < \kappa_2(\sigma) \leq 0$ for $\sigma \leq 0$, $\kappa_1(\sigma) < -\sigma$ for $\sigma > 0$, $\kappa_1(\sigma) + \sigma \rightarrow 0$ as $\sigma \rightarrow \infty$, and*

$$\begin{aligned} \#\Gamma = 2 &\Leftrightarrow \sigma < 0, \kappa_1(\sigma) \leq \rho < \kappa_2(\sigma), \\ \#\Gamma = 1 &\Leftrightarrow \begin{cases} \sigma < 0, \rho = \kappa_2(\sigma) \text{ or } \rho < \kappa_1(\sigma) \\ \sigma = 0, -1 = \kappa_1(0) \leq \rho < \kappa_2(0) = 0 \\ \sigma > 0, \rho \geq \kappa_1(\sigma), \end{cases} \\ \#\Gamma = 0 &\Leftrightarrow \begin{cases} \sigma < 0, \rho > \kappa_2(\sigma) \\ \sigma = 0, \rho < \kappa_1(0) \text{ or } \rho \geq \kappa_2(0) \\ \sigma > 0, \rho < \kappa_1(\sigma). \end{cases} \end{aligned}$$

Similarly, there exists a continuous function $\kappa_3 : \mathbb{R} \rightarrow \mathbb{R}$ with the properties that $\kappa_3(\sigma) > -(e-1)\sigma$ for $\sigma < 0$, $\kappa_3(\sigma) + (e-1)\sigma \rightarrow 0$ as $\sigma \rightarrow -\infty$, $\kappa_3(0) = 1$, $\kappa_3(\sigma) > -\sigma$ for $\sigma > 0$, $\kappa_3(\sigma) + \sigma \rightarrow 0$ as $\sigma \rightarrow \infty$, and

$$\begin{aligned} \#\gamma = 1 &\Leftrightarrow \rho \geq \kappa_3(\sigma), \\ \#\gamma = 0 &\Leftrightarrow \rho < \kappa_3(\sigma). \end{aligned}$$

Here $\#\Gamma$ and $\#\gamma$ stay for the number of large and small periodic orbits, respectively. See Figure 1 portraying the bifurcation curves $\mathcal{C}_0 = \{(\sigma, \rho) \in \mathbb{R}^2 \mid \sigma = 0\}$ and $\mathcal{C}_i = \text{graph}(\kappa_i)$, $i = 1, 2, 3$. Together with system (1)–(2), both large and small periodic orbits are symmetric with respect to the origin.

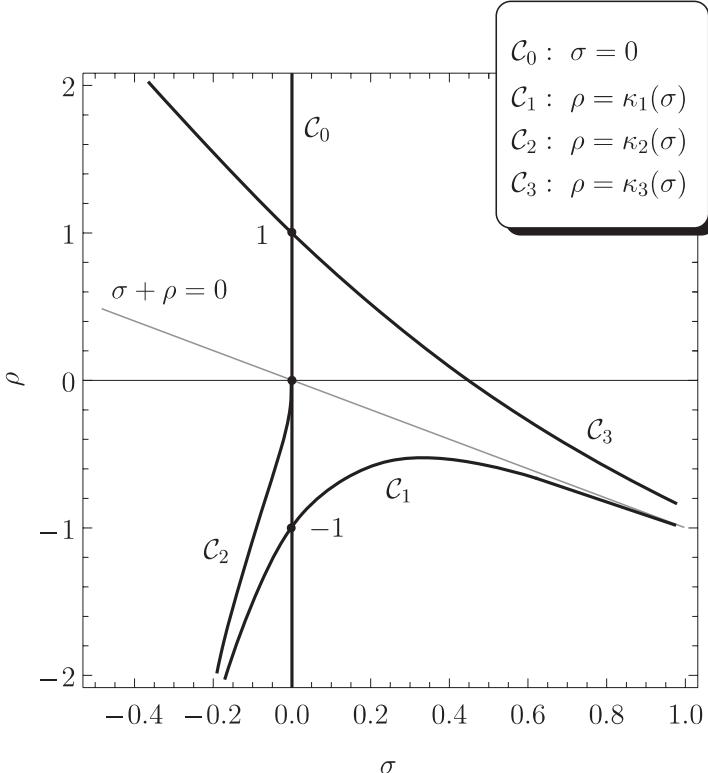


Fig. 1. The bifurcation curves in Theorem 1

Large periodic orbits contain the equilibria $E^+ = (1, \rho)$ and $E^- = (-1, -\rho)$ in their interior.

Fix $\rho = \rho_0 < -1$. At $(\sigma_0^L, \rho_0) \in \mathcal{C}_2$, system (1)–(2) undergoes a saddle-node bifurcation for periodic orbits. The emerging unstable and stable large periodic orbits disappear at $(\sigma_0^L, \rho_0) \in \mathcal{C}_1$, $\sigma_0^L < \sigma_-^L < 0$ [discontinuity-based grazing bifurcation at the switching lines with symmetry] and $(0, \rho_0)$ [coalescence with the point at infinity], respectively. Increasing parameter σ a bit further, the unstable large periodic orbit is reborn at $(\sigma_+^L, \rho_0) \in \mathcal{C}_1$, $\sigma_+^L > 0$. Small periodic orbits are born at $(\sigma_0^S, \rho_0) \in \mathcal{C}_3$, $\sigma_0^S > \sigma_+^L$ [discontinuity-based grazing bifurcation at the switching lines with symmetry]. Small periodic orbits are always unstable.

Figure 2 displays the phase portrait of system (1)–(2) for parameters $(\sigma_-^L, \rho_0) \in \mathcal{C}_1$ in a schematic way. (In comparison with Γ_u , the stable periodic

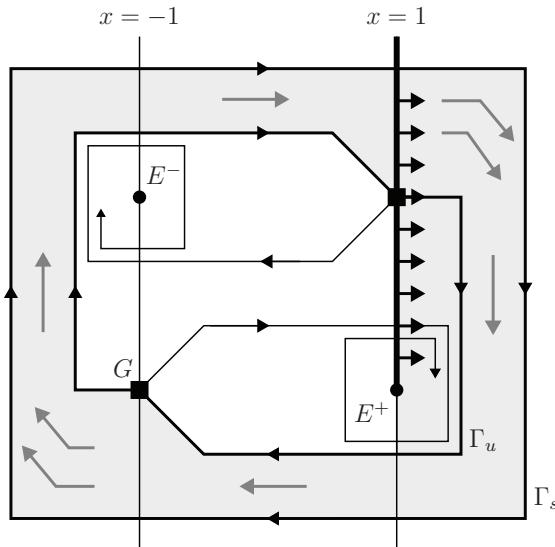


Fig. 2. The phase portrait of system (1)–(2) at $(\sigma_-^L, \rho_0) \in \mathcal{C}_1, \rho_0 < -1$

orbit Γ_s would be truly enormous on the natural scale.) The unstable periodic orbit Γ_u through the point of tangency G on the switching line $x = -1$ contains two stable equilibria in its interior. There are no other equilibria at all. This is somewhat ridiculous and seems to contradict the famous Euler-Poincaré formula

$$(3) \quad \# \text{sources} - \# \text{saddles} + \# \text{sinks} = 1$$

valid for equilibria inside periodic orbits of planar dynamical systems. (It is to assume that the dynamical system is smooth and that all equilibria inside the periodic orbit are hyperbolic.) Of course there is no contradiction at all. The pair of ordinary differential equations (1)–(2) with hysteresis does not induce any dynamical system. Solution dynamics induced by system (1)–(2) is a so-called switching dynamics – recall the particular role of the switching lines $x = \pm 1$ in system (1)–(2) and also the forward branching of the solution trajectory at the grazing point G in Figure 2. For the general theory of switched systems and their bifurcations, see the monograph [1].

2. A family of ‘artificial’ counterexamples

The phenomenon observed in system (1)–(2) occurs in two-dimensional systems with hysteresis in the greatest possible generality.

THEOREM 2. *For nonnegative integers k , ℓ , m arbitrarily given, there exists a pair of bounded and smooth functions $F = F(k, \ell, m)$, $f = f(k, \ell, m) : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ with the properties as follows. System*

$$(4) \quad \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = F \begin{pmatrix} x \\ y \end{pmatrix} \quad \text{if } x \geq -1 \quad \text{and} \quad t \geq 0$$

$$(5) \quad \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = f \begin{pmatrix} x \\ y \end{pmatrix} \quad \text{if } x \leq 1 \quad \text{and} \quad t \geq 0$$

admits a periodic orbit Γ without self-intersection, the number of equilibria inside Γ equals $N = k + \ell + m$, all equilibria are hyperbolic and belong to the half-plane $\{(x, y) \in \mathbb{R}^2 \mid x > 1\}$ and, last but not least,

$$(6) \quad \#\text{sources} = k, \quad \#\text{saddles} = \ell, \quad \#\text{sinks} = m.$$

PROOF. It is not hard to construct function F in such a way that the ordinary differential equation $\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = F \begin{pmatrix} x \\ y \end{pmatrix}$ has $N = k + \ell + m$ equilibria, all hyperbolic, belong to the half-plane $\{(x, y) \in \mathbb{R}^2 \mid x > 1\}$, and also condition (6) is satisfied. Function f is constructed in such a way that the ordinary differential equation $\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = f \begin{pmatrix} x \\ y \end{pmatrix}$ has no equilibria at all. With some additional care, it can be guaranteed that all the equilibria above belong to the interior of a periodic orbit of system (4)–(5).

In order to simplify the technicalities, only the respective phase portraits are constructed and these only in a not-entirely smooth category.

The idea is to build a planar ‘chest of drawers’. The ‘outer frame’ is an unbounded rectangle determined by the vertices $(-\infty, 0)$, $(3, 0)$, $(3, -2N - 1)$, $(-\infty, -2N - 1)$. The number of ‘drawers’ is N , and the i -th ‘drawer’ is determined by the vertices $(-\infty, -2i + 1)$, $(2, -2i + 1)$, $(2, -2i)$, $(-\infty, -2i)$, $i = 1, 2, \dots, N$. The ‘drawers’ are indistinguishable from the outside and can be replaced one by the other. Each ‘drawer’ contains a single equilibrium point, a source, a saddle or a sink. The solution dynamics of equation $\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = F \begin{pmatrix} x \\ y \end{pmatrix}$ within the ‘entire frame’ and within the three types of ‘drawers’ is

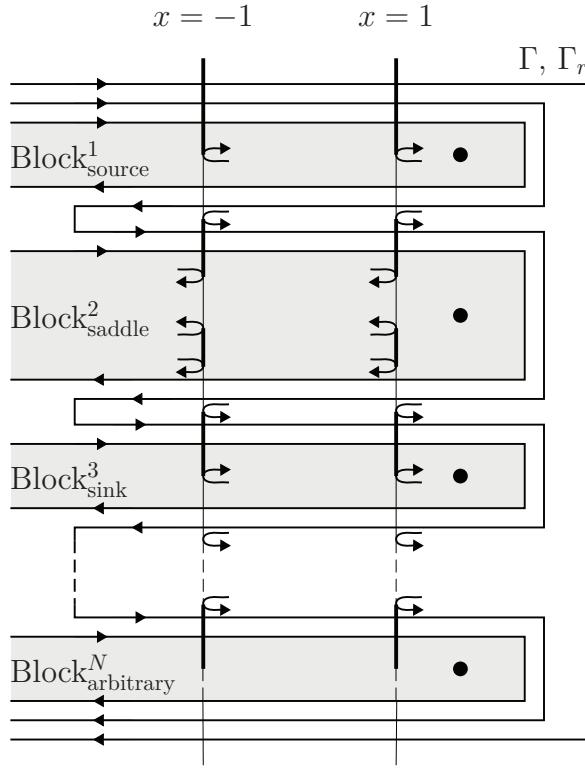


Fig. 3. The ‘entire frame’ for equation $\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = F \begin{pmatrix} x \\ y \end{pmatrix}$

indicated in Figures 3 and 4, respectively. In both Figures, boldface segments on the switching lines consist of entry points to the right. Hooked arrows show the bouncing of solution trajectories.

Then a second ‘chest of drawers’ is constructed. The ‘entire frame’ and the reversed dynamics within are obtained by reflection to the $x = 0$ axis. Now the ‘outer frame’ is determined by vertices $(\infty, -2N - 1)$, $(-3, -2N - 1)$, $(-3, 0)$, $(\infty, 0)$ whereas the ‘drawers’ are the unbounded rectangles with vertices $(\infty, -2i)$, $(-2, -2i)$, $(-2, -2i + 1)$, $(\infty, -2i + 1)$, $i = 1, 2, \dots, N$. Equipped with the solution dynamics of equation $\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = f \begin{pmatrix} x \\ y \end{pmatrix}$, now all ‘drawers’ are exactly the same: invariant and free of equilibria.

The desired result follows by hysteretic merging. Periodic orbit Γ is represented by the rectangle with vertices $(-3, 0)$, $(3, 0)$, $(3, -2N - 1)$,

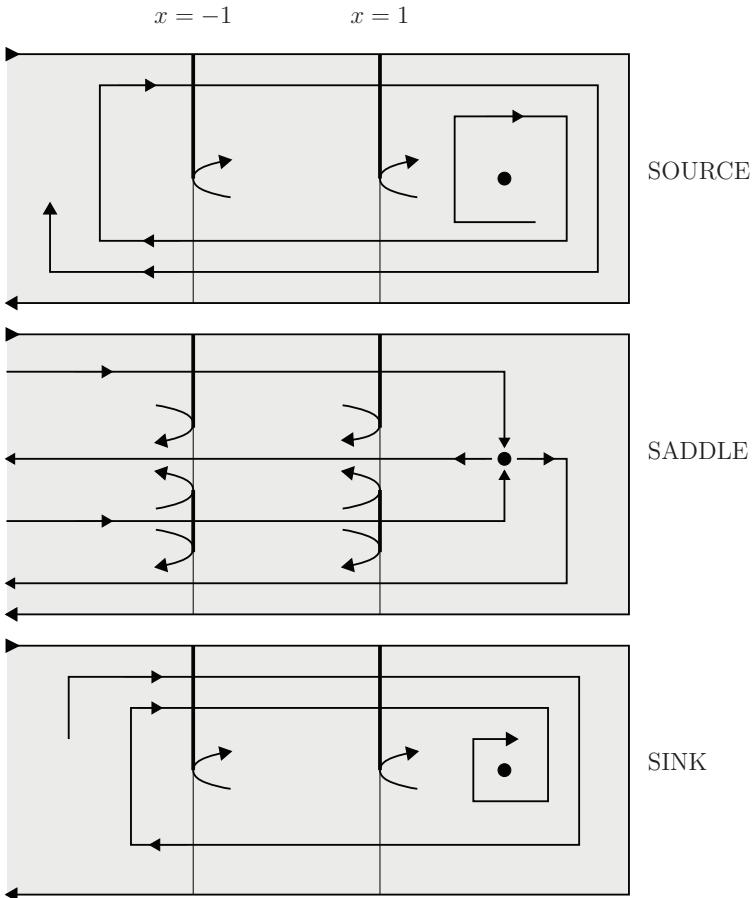


Fig. 4. The three types of ‘drawers’/blocks for equation $\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = F\begin{pmatrix} x \\ y \end{pmatrix}$

$(-3, -2N - 1)$. The orientation of the compound dynamics on Γ is clockwise.

In spite of the loose terminology, no confusion should arise. In a less informal language, ‘drawers’ can be called blocks (source blocks, saddle blocks, sink blocks, equilibrium-free blocks) etc. but we could not find any good terminus technicus for the ‘chest’ of a ‘chest of drawers’ (termed ‘entire frame’ above). ■

REMARK 1. Let Γ_r and Γ_ℓ denote the rectangles with vertices $(-1, 0)$, $(3, 0)$, $(3, -2N - 1)$, $(-1, -2N - 1)$ and $(-3, 0)$, $(1, 0)$, $(1, -2N - 1)$,

$(-3, -2N - 1)$, respectively. Since $F|_{\Gamma_r \cap \Gamma_\ell} = f|_{\Gamma_r \cap \Gamma_\ell}$, F and f together define a continuous function $\varphi : \Gamma \rightarrow \mathbb{R}^2$, the vector field belonging to the periodic orbit Γ of system (4)–(5). By the construction, the Poincaré index [9] $\text{ind}(\varphi, \Gamma)$ equals 1. On the other hand, the alternating sum $\#\text{sources} - \#\text{saddles} + \#\text{sinks}$ in formula (3) equals $k - \ell + m$ which need not be equal to 1. The way out of the situation is to define

$$\text{ind}(\text{hyster}(F, f), \Gamma) = \text{ind}(F, \Gamma_r) + \text{ind}(f, \Gamma_\ell).$$

This makes sense because, both for F and f , the infinite strip $\{(x, y) \in \mathbb{R}^2 \mid |x| \leq 1\}$ is free of equilibria and

$$\text{ind}(F, \Gamma_r) = k - \ell + m, \quad \text{ind}(f, \Gamma_\ell) = 0.$$

The problem of defining an appropriate index for general systems with hysteresis (switching lines intersecting each other in two, switching hypersurfaces in higher dimension etc.) remains open. It is worth mentioning here that several questions on hysteresis can be converted to an abstract setting where classical degree/index theories (both in the single-valued and in the multivalued case) apply. Various results of this type are cited in the monograph of Fečkan [7].

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HIGHER ORDER SMOOTHNESS OF RADIAL SOLUTIONS TO A NONLINEAR ELLIPTIC PROBLEM

By

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(Received October 20, 2010)

Dedicated to Professor László Simon on the occasion of his 70th birthday

Abstract. Radially symmetric solutions $u(x) = v(|x|) \equiv v(\rho)$ to the Dirichlet problems

$$\Delta u(x) + f(|x|, u(x), |\nabla u(x)|) = 0 \quad (x \in B), \quad u|_{\Gamma} = a \in \mathbb{R}$$

are considered, where B is the unit ball in \mathbb{R}^n centered at the origin ($n \geq 2$), $\Gamma := \partial B$; $f \in C(G_{a_0}; (0, \infty))$, where $G_{a_0} := [0, 1] \times (a_0, \infty) \times [0, \infty)$, a is arbitrary $a > a_0 \geq -\infty$. Sufficient conditions are presented that guarantee the properties $u \in C^4(\overline{B})$ or $u \in C^5(\overline{B})$ for all solutions $u = u(x)$ where $a > a_0$. The results are generalizations of the author's ones, published in EJQTDE, 2002, No. 18, pp. 1–28.

1. Introduction

Radially symmetric solutions of homogeneous Dirichlet problems or problems in the whole space \mathbb{R}^n (with a condition at infinity) for the nonlinearly perturbed Laplace operator or m -Laplacian have been investigated by many authors (see e.g. [1]–[4], [6] and their references). In [1]–[4] very general existence and uniqueness results are presented, [6] contains an analiticity result outside the origin, in [5] a bifurcation result is given for a semilinear elliptic equation. On the other hand C^1 , C^2 or higher order smoothness of solutions in the neighbourhoods of the origin or in the closure of the domain has not been investigated.

AMS Subject Classification (2000): 35B30, 35B65, 35J60, 35J65.

The present paper serves as a continuation of the paper [8]. Here and in the next we will use the following notations and assumptions: $a_0 \geq -\infty$ is fixed, $a > a_0$ is arbitrary,

$$G_{a_0} := [0, 1] \times (a_0, \infty) \times [0, \infty), \quad G_a := [0, 1] \times [a, \infty] \times [0, \infty),$$

$f = f(\alpha_1, \alpha_2, \alpha_3)$ is a given function, $f \in C(G_{a_0}; (0, \infty))$; B is the unit ball in \mathbb{R}^n ($n \geq 2$) centered at the origin, $\Gamma := \partial B$.

In the paper [8] the following problem (Problem A) was considered: find a radially symmetric function $u = u(x)$ ($x \in \overline{B}$), i.e. a function for which there exists

$$v: [0, 1] \rightarrow \mathbb{R}, \quad v(\rho) \equiv v(|x|) = u(x) \quad (x \in \overline{B})$$

and for which

$$\begin{aligned} \Delta u(x) + f(|x|, u(x), |\nabla u(x)|) &= 0 \quad (x \in B), \\ u &\in C^2(B) \cap C(\overline{B}), \\ u|_\Gamma &= a. \end{aligned}$$

It is supposed that for any a there exists a constant $K_a > 0$ such, that

$$(1.1) \quad f \in C(G_a; (0, K_a]) \cap C^1(G_a).$$

Under these assumptions the following results have been proved among others: Problem A has a solution u and any of the solutions belongs to $C^3(\overline{B})$ if the additional condition

$$(1.2) \quad f_{\alpha_1}(0, \alpha_2, 0) = -\frac{1}{n} f_{\alpha_3}(0, \alpha_2, 0) f(0, \alpha_2, 0) \quad (\alpha_2 > a_0)$$

is fulfilled.

Now our purpose is to find conditions similar to (1.1) and (1.2) that guarantee C^4 , C^5 smoothness in \overline{B} for all solutions u of Problem A. It is reasonable to emphasize Theorem 1.2 of the paper [8] which says:

THEOREM 1.2 ([8]). *Let $a > a_0$ appearing in Problem A be arbitrarily fixed, and suppose that for any $b > a$ there exists a constant $K_{a,b} > 0$ such that*

$$f \in C(G_{a,b}; (0, K_{a,b})) \quad \text{where} \quad G_{a,b} := [0, 1] \times [a, b] \times [0, \infty).$$

If in addition

$$f \in C(G_a; (0, \infty)) \cap C^m((0, 1] \times [a, \infty) \times [0, \infty))$$

holds, where $1 \leq m \leq \infty$, then any solution u of Problem A has the smoothness property

$$u \in C^2(\overline{B}) \cap C^{m+2}(\overline{B} \setminus \{0\}).$$

This is why the main attention in the present paper is devoted to the study of the smoothness of solutions u of Problem A in the neighbourhoods of the origin.

THEOREM 1.1. Consider Problem A with arbitrarily fixed $a > a_0$, and suppose that

$$(1.3) \quad f \in C^2(G_a; (0, K_a])$$

and condition (1.2) is fulfilled. Then for any solution u of Problem A

$$u \in C^4(\overline{B})$$

holds.

THEOREM 1.2. If function f satisfies the following conditions

$$(1.4) \quad f \in C^3(G_a; (0, K_a]),$$

where $a > a_0$, and

$$(1.5) \quad f_{\alpha_1}(0, \alpha_2, 0) = -\frac{1}{n}f_{\alpha_3}(0, \alpha_2, 0)f(0, \alpha_2, 0) \quad (\alpha_2 > a_0),$$

$$\begin{aligned} & f_{\alpha_1\alpha_1\alpha_1}(0, \alpha_2, 0) + \frac{3}{n}f_{\alpha_1\alpha_1\alpha_3}(0, \alpha_2, 0)f(0, \alpha_2, 0) + \\ & + \frac{3}{n^2}f_{\alpha_1\alpha_3\alpha_3}(0, \alpha_2, 0)[f(0, \alpha_2, 0)]^2 + \frac{1}{n^3}f_{\alpha_3\alpha_3\alpha_3}(0, \alpha_2, 0)[f(0, \alpha_2, 0)]^3 - \\ & - \frac{3}{n}f_{\alpha_1\alpha_2}(0, \alpha_2, 0)f(0, \alpha_2, 0) - \frac{3}{n^2}f_{\alpha_2\alpha_3}(0, \alpha_2, 0)[f(0, \alpha_2, 0)]^2 + \\ (1.6) \quad & + \frac{3}{n+2}f_{\alpha_3}(0, \alpha_2, 0) \left\{ f_{\alpha_1\alpha_1}(0, \alpha_2, 0) + \frac{2}{n}f_{\alpha_1\alpha_3}(0, \alpha_2, 0)f(0, \alpha_2, 0) + \right. \\ & \left. + \frac{1}{n^2}f_{\alpha_3\alpha_3}(0, \alpha_2, 0)[f(0, \alpha_2, 0)]^2 - \frac{1}{n}f_{\alpha_2}(0, \alpha_2, 0)f(0, \alpha_2, 0) \right\} = 0 \end{aligned}$$

$$(\alpha_2 > a_0)$$

then for any solution u of Problem A

$$u \in C^5(\overline{B})$$

holds.

The set of admissible functions f in Theorem 1.2 is rich enough. For example:

PROPOSITION 1. *If function f satisfies condition (1.4) and*

$$(1.7) \quad \begin{aligned} f_{\alpha_1}(w) &= f_{\alpha_3}(w) = f_{\alpha_1\alpha_2}(w) = f_{\alpha_2\alpha_3}(w) = f_{\alpha_1\alpha_1\alpha_1}(w) = f_{\alpha_1\alpha_1\alpha_3}(w) = \\ &= f_{\alpha_1\alpha_3\alpha_3}(w) = f_{\alpha_3\alpha_3\alpha_3}(w) = 0 \quad \forall w := (0, \alpha_2, 0) \quad (\alpha_2 > a_0), \end{aligned}$$

then conditions (1.5), (1.6) are also fulfilled.

2. The Proof of Theorem 1.1

1. The existence of a solution $v(\rho) \equiv v(|x|) := u(x)$ to Problem A for any $a > a_0$ follows from Theorem 2.1 of the paper [8]. Using the radial symmetry of u we have (see (3.14) in [8])

$$(2.1) \quad v''(t) = -\alpha(t) + \left[\int_0^t \rho^{n-1} \alpha(\rho) d\rho \right] (n-1)t^{-n} \quad (0 < t \equiv |x| < \delta < 1),$$

where

$$(2.2) \quad \alpha(\rho) := f(\rho, v(\rho), -v'(\rho)) \quad (0 < \rho < \delta < 1).$$

Now, repeating the ideas of the proof of Theorem 1.2 ([8]) for $m = 2$ (see formulae of (1.97) in [8]) we get that both sides of (2.1) allow differentiation twice by t , moreover

$$(2.3) \quad v \in C^4(\overline{B} \setminus \{0\}) \cap C^2(\overline{B}).$$

Calculating the second order derivatives of (2.1) we get the equality for $t \in \in (0, 1]$

$$(2.4) \quad v^{(iv)}(t) = -\alpha''(t) + (n-1) \times$$

$$\times \left(\alpha'(t)t^{n+1} - (n+1)\alpha(t)t^n + n(n+1) \left[\int_0^t \rho^{n-1} \alpha(\rho) d\rho \right] \right) t^{-(n+2)}.$$

Now using condition (1.3), relation (2.3) and equalities $v'(0) = v'''(0) = 0$ (see (1.12), (3.22) in [8]) we can to prove the existence of the limit $\lim v^{(iv)}(t)$ as $t \rightarrow 0+0$, that implies in virtue of the Lagrange's theorem the existence of the derivative $v^{(iv)}(0)$ and equality

$$(2.5) \quad v^{(iv)}(0) = \lim_{t \rightarrow 0+0} v^{(iv)}(t).$$

In order to prove (2.5) we observe that the first term on the right hand side of (2.4): $-\alpha''(t)$ has a limit as $t \rightarrow 0+0$. Namely, from (2.2) we have

$$\begin{aligned} \alpha''(t) &= [f_{\alpha_1\alpha_1}(w) + f_{\alpha_1\alpha_2}(w)v'(t) - f_{\alpha_1\alpha_3}(w)v''(t)] + \\ &\quad + [f_{\alpha_2\alpha_1}(w)v'(t) + f_{\alpha_2\alpha_2}(w)(v'(t))^2 - f_{\alpha_2\alpha_3}(w)v'(t)v''(t)] + \\ &\quad + f_{\alpha_2}(w)v''(t) - f_{\alpha_3\alpha_1}(w)v''(t) - f_{\alpha_3\alpha_2}(w)v''(t)v'(t) + \\ &\quad + f_{\alpha_3\alpha_3}(w)(v''(t))^2 - f_{\alpha_3}(w)v'''(t) \\ &(0 < t < \delta < 1), \quad w := (t, v(t), -v'(t)). \end{aligned}$$

Now using, that

$$(2.6) \quad v'(t), v'''(t) \rightarrow 0 \quad \text{as } t \rightarrow 0+0, \quad v''(t) \rightarrow -\frac{1}{n}f(0, v(0), 0) \quad \text{as } t \rightarrow 0+0$$

(for the last relation see (1.17) in [8]) we get

$$\begin{aligned} (2.7) \quad \exists \lim_{t \rightarrow 0+0} \alpha''(t) &= f_{\alpha_1\alpha_1}(z) - \frac{1}{n}f(z)[f_{\alpha_2}(z) - 2f_{\alpha_1\alpha_3}(z) - \\ &\quad - \frac{1}{n}f(z)f_{\alpha_3\alpha_3}(z)] = \alpha''(0), \quad z := (0, v(0), 0). \end{aligned}$$

Further for the second term on the right hand side of (2.4) the l'Hospital's rule gives (using the existence of the limit in (2.7)) the limit $\alpha''(0)/(n+2)$, consequently

$$(2.8) \quad v^{(iv)}(t) \xrightarrow[t \rightarrow 0+0]{} -\alpha''(0) + \frac{n-1}{n+2}\alpha''(0) = -\frac{3}{n+2}\alpha''(0) = v^{(iv)}(0),$$

so the property

$$(2.9) \quad v \in C^4([0, 1])$$

is proved.

We emphasize that the equality $v^{(iv)}(0) = 0$ is not required.

2. Now we shall prove that (2.9) combined with (2.6) imply the property

$$u \in C^4(\overline{B}).$$

It is enough to prove that all partial derivatives of the fourth order of u belong to the class $C(\overline{B})$. These derivatives belong to $C(\overline{B} \setminus \{0\})$ in virtue of Theorem 1.2 of [8], therefore using the Lagrange's theorem it is sufficient to

prove that they have finite limits as $\rho := |x| \rightarrow 0 + 0$. For this reason, using the equality

$$u = u(x) := v(|x|) \equiv v(\rho) \quad (\rho \in (0, 1))$$

and calculating all of the possible derivatives of fourth order of u using the chain rule we derive the following representations (where i, j, k, ℓ denote four different indices):

$$(2.10) \quad u_{x_i x_j x_k x_\ell}(x) = A(\rho) \frac{x_i x_j x_k x_\ell}{\rho^4} \quad (x \in B \setminus \{0\}, \quad n \geq 4, \quad \rho \in (0, 1)),$$

$$(2.11) \quad u_{x_i x_i x_j x_k}(x) = A(\rho) \left(\frac{x_i}{\rho} \right)^2 \frac{x_j x_k}{\rho^2} + B(\rho) \frac{x_j x_k}{\rho^2}$$

$$(x \in B \setminus \{0\}), \quad n \geq 3, \quad (\rho \in (0, 1)),$$

$$(2.12) \quad u_{x_i x_i x_j x_j}(x) = A(\rho) \left(\frac{x_i}{\rho} \right)^2 \left(\frac{x_j}{\rho} \right)^2 + B(\rho) \left[\left(\frac{x_i}{\rho} \right)^2 + \left(\frac{x_j}{\rho} \right)^2 \right] + C(\rho)$$

$$(x \in B \setminus \{0\}), \quad n \geq 2, \quad (\rho \in (0, 1)),$$

$$(2.13) \quad u_{x_i x_i x_i x_j}(x) = A(\rho) \left(\frac{x_i}{\rho} \right)^3 \frac{x_j}{\rho} + 3B(\rho) \frac{x_i x_j}{\rho^2}$$

$$(x \in B \setminus \{0\}), \quad n \geq 2, \quad (\rho \in (0, 1)),$$

$$(2.14) \quad u_{x_i x_i x_i x_i}(x) = A(\rho) \left(\frac{x_i}{\rho} \right)^4 + 6B(\rho) \left(\frac{x_i}{\rho} \right)^2 + 3C(\rho)$$

$$(x \in B \setminus \{0\}), \quad n \geq 2, \quad (\rho \in (0, 1)),$$

where

$$(2.15) \quad \begin{aligned} A(\rho) &:= \{[v'(\rho)\rho^{-1}]'\rho^{-1}\}'\rho^3, \\ B(\rho) &:= [(v'(\rho)\rho^{-1})'\rho^{-1}]'\rho, \\ C(\rho) &:= (v'(\rho)\rho^{-1})'\rho^{-1} \quad (\rho \in (0, 1)). \end{aligned}$$

We have to prove that all of the right hand sides of (2.10)–(2.14) have finite limits as $x \rightarrow 0$ ($\rho \rightarrow 0 + 0$), but some auxiliary ideas may be derived only from representations (2.12)–(2.14) appearing in the smallest dimension $n = 2$. Namely for the existence of the right hand side-limits in (2.12)–(2.14) the

following relations are necessary:

$$(2.16) \quad \begin{aligned} \exists \lim_{\rho \rightarrow 0+0} C(\rho) &\equiv C_0 = \frac{1}{3} v^{(iv)}(0), \\ \exists \lim_{\rho \rightarrow 0+0} A(\rho) &\equiv A_0 = 0, \\ \exists \lim_{\rho \rightarrow 0+0} B(\rho) &\equiv B_0 = 0. \end{aligned}$$

In order to prove (2.16) let us substitute $x_i = 0$ in (2.14). We get

$$(2.17) \quad \begin{aligned} \exists \lim_{\rho \rightarrow 0+0} 3C(\rho) &= u_{x_i x_i x_i x_i}(0) = v^{(iv)}(0+0); \\ C_0 &= \frac{1}{3} v^{(iv)}(0). \end{aligned}$$

Further, substitution $x_i = 0$, $|x_j| = \rho > 0$ in (2.12) yields:

$$(2.18) \quad \exists \lim_{\rho \rightarrow 0+0} [B(\rho) + C(\rho)] = \lim_{x \rightarrow 0} u_{x_i x_i x_j x_j}(x)$$

which combined with (2.17) gives

$$\exists \lim_{\rho \rightarrow 0+0} B(\rho) \equiv B_0.$$

The same substitution in (2.13) yields:

$$\exists \lim_{x \rightarrow 0} u_{x_i x_i x_i x_j} = 0,$$

therefore substituting in (2.13)

$$(2.19) \quad x_i = x_j = \rho \frac{\sqrt{2}}{2}$$

we get

$$(2.20) \quad \exists \lim_{\rho \rightarrow 0+0} \left[\frac{1}{4} A(\rho) + \frac{3}{2} B(\rho) \right] = 0.$$

The same substitution (2.19) in (2.12) gives (combined with (2.18))

$$\exists \lim_{\rho \rightarrow 0+0} \left[\frac{1}{4} A(\rho) + B(\rho) + C(\rho) \right] = \lim_{\rho \rightarrow 0+0} [B(\rho) + C(\rho)],$$

consequently

$$(2.21) \quad \exists \lim_{\rho \rightarrow 0+0} A(\rho) \equiv A_0 = 0.$$

Finally (2.21) and (2.20) imply

$$\exists \lim_{\rho \rightarrow 0+0} B(\rho) \equiv B_0 = 0.$$

Summarising, (2.16) is proved.

On the other hand, relations (2.16) are (also) sufficient for the existence of the limits (as $x \rightarrow 0$) of the left hand sides in (2.10)–(2.14). In fact using also the boundedness of the factors $\frac{x_i}{\rho}$ ($i = \overline{1, n}$) we get

$$u_{x_i x_j x_k x_\ell}(x) \rightarrow 0 \quad \text{as } x \rightarrow 0 \ (\rho \rightarrow 0+0), \quad n \geq 4,$$

$$u_{x_i x_i x_j x_k}(x) \rightarrow 0 \quad \text{as } x \rightarrow 0 \ (\rho \rightarrow 0+0), \quad n \geq 3,$$

$$u_{x_i x_i x_j x_j} \rightarrow \frac{1}{3} v^{(iv)}(0) \quad \text{as } x \rightarrow 0 \ (\rho \rightarrow 0+0), \quad n \geq 2.$$

$$u_{x_i x_i x_i x_j}(x) \rightarrow 0 \quad \text{as } x \rightarrow 0 \ (\rho \rightarrow 0+0), \quad n \geq 2,$$

$$u_{x_i x_i x_i x_i}(x) \rightarrow v^{(iv)}(0) \quad \text{as } x \rightarrow 0 \ (\rho \rightarrow 0+0), \quad i = \overline{1, n}, \quad n \geq 2.$$

It remains to prove the following:

LEMMA 2.1. $C(\rho) \rightarrow \frac{1}{3} v^{(iv)}(0)$, $B(\rho) \rightarrow 0$, $A(\rho) \rightarrow 0$ as $\rho \rightarrow 0+0$.

PROOF. According to the definition of $C(\rho)$ we have

$$(2.22) \quad C(\rho) = \frac{\rho v''(\rho) - v'(\rho)}{\rho^3}.$$

Now using (2.6), (2.8) and l'Hospital's rule we arrive at the fraction

(2.23)

$$\frac{v'''(\rho)}{3\rho} = \frac{1}{3} \frac{v'''(\rho) - 0}{\rho - 0} = \frac{1}{3} \frac{v'''(\rho) - v'''(0)}{\rho - 0} \rightarrow \frac{1}{3} v^{(iv)}(0) \quad \text{as } \rho \rightarrow 0+0.$$

Similarly, calculating the derivatives in the definition of $B(\rho)$ we get

$$B(\rho) = \frac{v'''(\rho)\rho^2 - 3v''(\rho)\rho + 3v'(\rho)}{\rho^3} = \frac{v'''(\rho)}{\rho} - 3 \frac{v''(\rho)\rho - v'(\rho)}{\rho^3} \rightarrow 0,$$

as $\rho \rightarrow 0+0$ in virtue of (2.22), (2.23).

The proof of the last relation of Lemma 2.1, $A(\rho) \rightarrow 0$ ($\rho \rightarrow 0+0$) is also simple because after calculations of the derivatives in the definition of $A(\rho)$ we get the representation

$$A(\rho) = v^{(iv)}(\rho) - 6 \frac{v'''(\rho)}{\rho} + 15 \frac{v''(\rho)\rho - v'(\rho)}{\rho^3} \rightarrow 0 \text{ as } \rho \rightarrow 0+0$$

using (2.8), (2.22), (2.23). ■

Lemma 2.1 and Theorem 1.1 are proved.

3. The Proof of Theorem 1.2

We shall prove first the property $v \in C^5([0, 1])$, especially the equality $v^{(v)}(0) = 0$; and second, the property $u(x) \in C^5(\overline{B})$ for every solution u of Problem A for any $a > a_0$.

1. Using the results of Theorem 1.1 we can state that the right hand side terms in (2.4) allow differentiation by t for $0 < t \leq 1$, and we arrive at the property $v \in C^5((0, 1])$ and at the equality

(3.1)

$$\begin{aligned} v^{(v)}(t) = & -\alpha'''(t) + \frac{n-1}{t^{n+3}} \left\{ t^{n+2}\alpha''(t) - (n+2)\alpha'(t)t^{n+1} + \right. \\ & \left. + (n+1)(n+2)t^n\alpha(t) - n(n+1)(n+2) \left(\int_0^t \rho^{n-1}\alpha(\rho)d\rho \right) \right\} \quad (0 < t \leq 1). \end{aligned}$$

Recall formula (2.2) which implies $\alpha \in C^3([0, 1])$ because $\alpha'''(t)$ depends only on derivatives of $v(t)$ of the order less than or equal to four. Therefore it remains to prove for the relation $v \in C^5([0, 1])$ that the second term on the right hand side of (3.1) has a limit as $t \rightarrow 0+0$. Indeed using l'Hospital's rule we get for $t > 0$

$$\begin{aligned} & \frac{n-1}{(n+3)t^{n+2}} \{ t^{n+2}\alpha'''(t) + (n+2)t^{n+1}\alpha''(t) - (n+2)\alpha''(t)t^{n+1} - \\ & - (n+2)\alpha'(t)(n+1)t^n + (n+1)(n+2)t^n\alpha'(t) + (n+1)(n+2)nt^{n-1}\alpha(t) - \\ & - n(n+1)(n+2)t^{n-1}\alpha(t) \} = \frac{n-1}{n+3}\alpha'''(t) \xrightarrow[t \rightarrow 0+0]{} \frac{n-1}{n+3}\alpha'''(0), \end{aligned}$$

therefore

$$v^{(v)}(t) \xrightarrow[t \rightarrow 0+0]{} -\alpha'''(0) + \frac{n-1}{n+3}\alpha'''(0) = -\frac{4}{n+3}\alpha'''(0),$$

and the property $v \in C^5([0, 1])$ is proved. Also, considering the even functions

$$u(x_1, \dots, x_1) \quad (-1 \leq x_1 \leq 1),$$

that is $u|_{x_1 \geq 0} = v(x_1)$, we get that a necessary condition for the property

$$u_{x_1 x_1 x_1 x_1 x_1}(x) \in C(B)$$

is the equality

$$v^{(v)}(0) = 0 \Leftrightarrow \alpha'''(0) = 0 \Leftrightarrow \frac{d^3}{dt^3}f(t, v(t), -v'(t))|_{t=0} = 0.$$

The last equality is fulfilled because its detailed expression coincides with condition (1.6).

2. We shall prove, using the properties $v \in C^5([0, 1]); v'(0) = v'''(0) = v^{(v)}(0) = 0$, that all partial derivatives of u of the fifth order has (zero) limits as $|x| = \rho \rightarrow 0+0$. For this reason we present some expressions for them via derivatives of function $v(\rho)$ ($\rho > 0$) in coefficients A, B, C (see (2.15)), using also equalities

$$(3.2) \quad \begin{aligned} B\rho^{-1} &= C', \quad [B\rho^{-2}]'\rho^2 = A\rho^{-1}: \\ u_{x_i x_j x_k x_\ell x_m} &= [A\rho^{-4}]'\rho^4 \frac{x_i x_j x_k x_\ell x_m}{\rho^5}, \\ u_{x_i x_j x_k x_\ell x_\ell} &= [A\rho^{-4}]'\rho^4 \frac{x_i x_j x_k x_\ell^2}{\rho^5} + A \frac{x_i x_j x_k}{\rho^3}, \\ u_{x_i x_i x_j x_j x_k} &= [A\rho^{-4}]'\rho^4 \frac{x_i^2 x_j^2 x_k}{\rho^5} + [B\rho^{-2}]'\rho^2 \frac{x_i^2 x_k}{\rho^3} + \\ &\quad + [B\rho^{-2}]'\rho^2 \frac{x_j^2 x_k}{\rho^3} + C' \frac{x_k}{\rho}, \\ u_{x_i x_i x_i x_j x_j} &= [A\rho^{-4}]'\rho^4 \frac{x_i^3 x_j^2}{\rho^5} + 3[A\rho^{-1}] \frac{x_i x_j^2}{\rho^3} + [B\rho^{-2}]'\rho^2 \frac{x_i^3}{\rho^3} + 3C' \frac{x_i}{\rho}, \\ u_{x_i x_i x_i x_j x_k} &= [A\rho^{-4}]'\rho^4 \frac{x_i^3 x_j x_k}{\rho^5} + 3[B\rho^{-2}]'\rho^2 \frac{x_i x_j x_k}{\rho^3}, \end{aligned}$$

$$u_{x_i x_j x_i x_i x_j} = [A\rho^{-4}]' \rho^4 \frac{x_i^4 x_j}{\rho^5} + 6[B\rho^{-2}]' \rho^2 \frac{x_i^2 x_j}{\rho^3} + 3C' \frac{x_j}{\rho},$$

$$u_{x_i x_j x_i x_i x_i} = [A\rho^{-4}]' \rho^4 \frac{x_i^5}{\rho^5} + 10[A\rho^{-1}]' \frac{x_i^3}{\rho^3} + 15C' \frac{x_i}{\rho}.$$

It follows similarly as in Theorem 1.1 that for the existence of the limits (when $\rho \rightarrow 0$) of all these fifth order derivatives the following limit-relations are sufficient and necessary (for the second one see (3.2)):

(3.3)

$$C' \rightarrow 0, A\rho^{-1} = [B\rho^{-2}]' \rho^2 \rightarrow 0, A \rightarrow 0, [A\rho^{-4}]' \rho^4 \rightarrow 0 \quad \text{as } \rho \rightarrow 0.$$

The relation $A \rightarrow 0$ follows from $A\rho^{-1} \rightarrow 0$ as $\rho \rightarrow 0$. Let us prove the other three relations in (3.3). First:

$$C' = \frac{1}{\rho^4} [v''' \rho^2 - 3\rho v'' + 3v'],$$

where using twice l'Hospital's rule we get the expressions

$$\begin{aligned} & \frac{1}{12\rho^2} [v^{(v)} \rho^2 + \rho v^{(iv)} - v'''] = \\ & = \frac{1}{12} v^{(v)}(\rho) + \frac{1}{12\rho^2} [\rho v^{(iv)} - v'''] \rightarrow 0 \quad \text{as } \rho \rightarrow 0, \end{aligned}$$

because

$$\frac{1}{24\rho} [\rho v^{(v)} + v^{(iv)} - v^{(iv)}] = \frac{1}{24} v^{(v)}(\rho) \rightarrow 0 \quad \text{as } \rho \rightarrow 0.$$

Summarising, the relation $C' \rightarrow 0$ ($\rho \rightarrow 0$) is proved. Prove the second relation of (3.3). We have the representation

$$A\rho^{-1} = \frac{1}{\rho^4} [v^{(iv)} \rho^3 - 6\rho^2 v''' + 15\rho v'' - 15v'] \quad (\rho > 0),$$

from which using l'Hospital's rule we get the expressions

$$\begin{aligned} (3.4) \quad & \frac{1}{4\rho^3} [v^{(v)} \rho^3 - 3\rho^2 v^{(iv)} + 3\rho v'''] = \frac{1}{4} v^{(v)}(\rho) + \\ & + \frac{1}{4\rho^3} [-3\rho^2 v^{(iv)} + 3\rho v'''] \rightarrow 0 \quad \text{as } \rho \rightarrow 0 \end{aligned}$$

because $v^{(v)}(\rho) \rightarrow 0$ as $\rho \rightarrow 0$ and, for the last term of (3.4), l'Hospital's rule gives the following expressions:

$$(3.5) \quad \begin{aligned} \frac{1}{12\rho^2}[-3\rho^2v^{(v)} - 3\rho v^{(iv)} + 3v'''] &= -\frac{1}{4}v^{(v)} + \\ &+ \frac{1}{12\rho^2}[-3\rho v^{(iv)} + 3v'''] \rightarrow -\frac{1}{4}v^{(v)}(0) - \frac{1}{8}v^{(v)}(0) = 0 \quad \text{as } \rho \rightarrow 0. \end{aligned}$$

Therefore the relation $A\rho^{-1} \rightarrow 0$ ($\rho \rightarrow 0$) is also proved. It remains to prove the last of the relations in (3.3). From (2.15) we get the representation:

$$(3.6) \quad \begin{aligned} [A\rho^{-4}]\rho^4 &= \frac{1}{\rho^4}[v^{(v)}\rho^4 - 10\rho^3v^{(iv)} + 45\rho^2v''' - 105\rho v'' + 105v'] = \\ &= v^{(v)}(\rho) + \frac{1}{\rho^4}[-10\rho^3v^{(iv)} + 45\rho^2v''' - 105\rho v'' + 105v'] \quad (\rho > 0), \end{aligned}$$

where $v^{(v)}(\rho) \rightarrow 0$ as $\rho \rightarrow 0$ and for the last term of (3.6) by the l'Hospital's rule we get the following expressions:

$$(3.7) \quad \begin{aligned} \frac{1}{4\rho^3}[-10\rho^3v^{(v)} + 15\rho^2v^{(iv)} - 15\rho v'''] &= -\frac{5}{2}v^{(v)}(\rho) + \\ &+ \frac{15}{4}\rho^2[\rho v^{(iv)} - v'''] \rightarrow -\frac{5}{2}v^{(v)}(0) + \frac{15}{8}v^{(v)}(0) = 0 \quad \text{as } \rho \rightarrow 0. \end{aligned}$$

Consequently, the relation

$$(3.8) \quad [A\rho^{-4}]\rho^4 \rightarrow 0 \quad \text{as } \rho \rightarrow 0$$

and therefore Theorem 1.2 is also proved.

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ON A VASCULAR BIFURCATION PROBLEM OF MURRAY

By

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Abstract. Murray [8] studied a bifurcating artery which supplies two organs with a required amount of blood. She determined the location of the bifurcation when the total power dissipation in this system of arteries is minimal.

The problem is a special Fermat-Torricelli problem. In this paper we apply calculus to examine the local minima of the minimized function.

1. Introduction

Murray [7] examined the flow of blood in the arterial system of mammals. She considered the total energy needed to transport blood through a straight vessel and maintain the blood.

Poisseuille's law describes the motion of fluid in rigid pipes. If the length of the pipe is L and its constant radius is r , one end is at pressure p_1 , the other end is at pressure p_2 ($p_1 > p_2$) then the flux f of the flow of fluid with constant viscosity μ is

$$f = \frac{\pi}{8} \frac{p_1 - p_2}{L\mu} r^4.$$

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The power dissipated due to friction is

$$(p_1 - p_2)f = \frac{8\mu f^2 L}{r^4 \pi}.$$

Murray used the above formula for the power dissipated due to friction in case of the flow of blood in a straight vessel. The organism consumes energy to maintain the circulating blood and the vessel, which is proportional to the volume, $K\pi r^2 L$ for a positive constant K . Let $k := \frac{8\mu}{\pi}$, then the power dissipation is

$$P := k \frac{f^2 L}{r^4} + K\pi r^2 L.$$

For a given flux f the minimum of the power dissipation is obtained by differentiating P as a function of r . The minimum is attained at $r = \sqrt[6]{\frac{2f^2 k}{K\pi}}$, then the minimum value is

$$(1) \quad P_{\min} = \frac{3}{2} K\pi r^2 L.$$

Murray assumed that every vessel is optimal relative to power dissipation, then its value is given by (1).

Next Murray [8] considered the bifurcation of an artery. The author searched the optimal shape of the bifurcation when the total power dissipation in the system of three straight vessels is minimal. Murray applied the principle of minimum work to determine the angles of the optimum. Later Rosen ([9], Chapter 3) discussed this vascular bifurcation problem, he used the idea of Murray and perturbations to get the optimum. Neither of them concerned vessels with zero length.

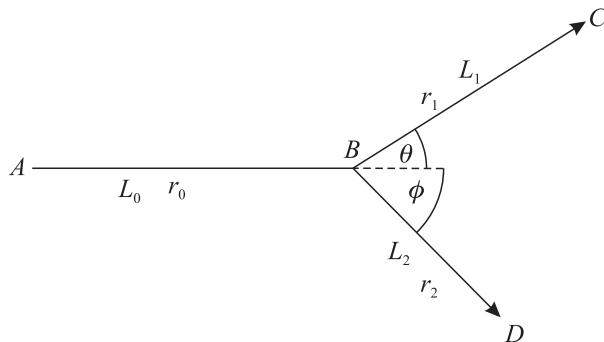


Fig. 1

Let B denote the bifurcation point, AB the main artery, BC and BD the branches. Let the lengths and the radii of the main artery and its branches be denoted by r_i and L_i , $i = 0, 1, 2$, respectively. Conservation of mass yields $r_0^3 = r_1^3 + r_2^3$. Suppose that all three vessels are optimal relative to power dissipation, then the total power dissipation of this arterial system is

$$P_t(B) = \frac{3}{2}K\pi \left(r_0^2 L_0 + r_1^2 L_1 + r_2^2 L_2 \right).$$

Finding the optimal branching point B means minimizing the function P_t , which is a weighted Fermat-Torricelli problem.

The weighted Fermat-Torricelli problem for given points A, C, D and weights $\alpha, \beta, \gamma \in \mathbb{R}^+$ is to find a point B such that

$$\alpha|AB| + \beta|CB| + \gamma|DB|$$

is minimal. The solution of the problem exists and is unique ([4] Theorem 8.5), it is called the weighted Fermat-Torricelli point. A good summary of the problem and the different solutions is given by Gueron and Tessler [3], and that of the geometric aspects of the problem by Kupitz and Martini [4]. The weighted problem is solved in [2], [3], [4], [5], [11] and [14]. Calculus is used to solve the original problem in [1], [6] and [10].

2. Calculus based proofs for the local minimum

First we recite Murray's result [8], and prove the theorem applying calculus.

THEOREM 1. [8] *Suppose that A, C, D are given non-collinear points and B is a point in their plane. Let $L_0(B)$, $L_1(B)$, $L_2(B)$ denote the lengths of the segments AB , BC and BD , respectively. If $r_0, r_1, r_2 \in \mathbb{R}^+$ then there exists a point B in the triangle ACD such that $r_0^2 L_0(B) + r_1^2 L_1(B) + r_2^2 L_2(B)$ is minimal.*

Let θ denote the angle of the lines AB and BC , ϕ denote the angle of the lines AB and BD for a local minimum point B different from the vertices.

Then

$$(2) \quad \begin{aligned} \cos \theta &= \frac{r_0^4 + r_1^4 - r_2^4}{2r_0^2 r_1^2}, \\ \cos \phi &= \frac{r_0^4 + r_2^4 - r_1^4}{2r_0^2 r_2^2}, \\ \cos(\theta + \phi) &= \frac{r_0^4 - r_1^4 - r_2^4}{2r_1^2 r_2^2}. \end{aligned}$$

PROOF. The function

$$f(B) := r_0^2 L_0(B) + r_1^2 L_1(B) + r_2^2 L_2(B)$$

defined in the closed circle with center A and radius $3 \max\{|AC|, |AD|\}$ does not attain its minimum on the boundary of its domain. During projections to the line of a side the value of f does not increase, therefore the minimum is taken only in the triangle ACD . Let B^* denote a local minimum point of f different from the vertices.

First we examine the function f restricted to the line AB . $L_1(B)$ and $L_2(B)$ can be rewritten applying the cosine theorem and Taylor's formula to derive

$$\begin{aligned} f(B) - f(B^*) &= \\ &= \delta(r_0^2 - r_1^2 \cos \theta - r_2^2 \cos \phi) + \delta^2 F(\delta, r_1, r_2, L_1(B^*), L_2(B^*), \theta, \phi) \end{aligned}$$

for B close to B^* , where δ denotes the length of BB^* with sign and F is a bounded function of δ . Then $r_0^2 - r_1^2 \cos \theta - r_2^2 \cos \phi = 0$. Restrictions to the line BC or BD and arguments similar to the first case give the other two equations. Together the system

$$\left. \begin{aligned} r_0^2 &= r_1^2 \cos \theta + r_2^2 \cos \phi \\ r_1^2 &= r_0^2 \cos \theta - r_2^2 \cos(\theta + \phi) \\ r_2^2 &= r_0^2 \cos \phi - r_1^2 \cos(\theta + \phi) \end{aligned} \right\}$$

is equivalent to (2). ■

REMARK. The first and the second equation of (2) implies the third one. If r_0^2, r_1^2, r_2^2 fullfill the weak triangle inequalities $r_0^2 \leq r_1^2 + r_2^2$, $r_1^2 \leq r_2^2 + r_0^2$ and $r_2^2 \leq r_0^2 + r_1^2$ then system (2) has a unique solution $(\theta, \phi) \in [0, \pi]^2$.

Otherwise the minimum point is the vertex with the largest weight (see [2], [3]).

In the vascular bifurcation problem $r_0^3 = r_1^3 + r_2^3$ holds, thus the solution is in $(0, \frac{\pi}{2})^2$.

Theorem 1 determines the angles for a local minimum point, which is different from the vertices. Now we examine the vertices with respect to local minimum.

THEOREM 2. *Let A, C, D be given non-collinear points and $r_0, r_1, r_2 \in \mathbb{R}^+$. Then the function f attains a local minimum at A if and only if we have $\theta + \phi \leq CAD^\triangleleft$ for the solution $(\theta, \phi) \in [0, \pi]^2$ of (2).*

The function f attains a local minimum at C if and only if $\pi - \phi \leq ACD^\triangleleft$.

The function f attains a local minimum at D if and only if $\pi - \theta \leq ADC^\triangleleft$.

PROOF. Let B be located in the triangle ACD , different form A . Let $\alpha := CAD^\triangleleft$, $\alpha_1 := CAB^\triangleleft$ and $\alpha_2 := BAD^\triangleleft$.

We can rewrite $f(B)$ with the cosine theorem and Taylor's formula to get

$$\begin{aligned} f(B) - f(A) &= \\ &= \delta(r_0^2 - r_1^2 \cos \alpha_1 - r_2^2 \cos \alpha_2) + \delta^2 G(\delta, r_1, r_2, L_1(A), L_2(A), \alpha_1, \alpha_2) \end{aligned}$$

for B close to A , where $\delta := |AB|$ and G is a bounded positive function of δ . Thus an equivalent condition for f to have a local minimum at A is

$$g(\alpha_1) := r_0^2 - r_1^2 \cos \alpha_1 - r_2^2 \cos(\alpha - \alpha_1) \geq 0, \quad \alpha_1 \in [0, \alpha].$$

The derivative g' vanishes at $\alpha_1 \in (0, \alpha)$ if

$$(3) \quad \frac{\sin(\alpha - \alpha_1)}{\sin \alpha_1} = \frac{r_1^2}{r_2^2}.$$

This equation has a unique solution $\alpha_1^* \in (0, \alpha)$ which is a global minimum point of f .

Equation (3) and some trigonometry implies

$$\sin \alpha_1^* = \frac{r_2^2 \sin \alpha}{\sqrt{r_1^4 + 2r_1^2 r_2^2 \cos \alpha + r_2^4}}, \quad |\cos \alpha_1^*| = \frac{r_1^2 + r_2^2 \cos \alpha}{\sqrt{r_1^4 + 2r_1^2 r_2^2 \cos \alpha + r_2^4}}$$

which lead to

$$g(\alpha_1^*) = r_0^2 - \sqrt{r_1^4 + 2r_1^2 r_2^2 \cos \alpha + r_2^4}.$$

This result and the last equation of (2) shows that $\theta + \phi \leq \alpha$ is fulfilled when g has a non-negative minimum.

The problem is symmetric in the vertices, which proves the rest of the theorem. ■

REMARK. The characterization (2) of the weighted Fermat-Torricelli point was also given by Gueron and Tessler [3]. They show that the conditions in Therorem 2 ensure global minimum at the vertex if all three triangle inequalities are valid for r_0^2, r_1^2, r_2^2 .

The same problem occurs when oil or gas is transported through pipelines from two locations to a common destination [5], here the direction of the flow is the opposite to the one in Murray's problem. The problem is similar, when three factories are supplied from a single warehouse [2]. In the latter case there are no conditions for the weights.

In the location science algorithms are given to determine the weighted Fermat-Torricelli point of a finite set of points [13]. The problem is called the Weber problem.

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A NOTE ON NONNEGATIVE SOLUTIONS OF A PERTURBED SYSTEM OF ORDINARY DIFFERENTIAL EQUATIONS

By

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Abstract. According to a result due to Perron and Lettenmeyer, the Liapunov exponent of a nonvanishing solution of a quasilinear system of ordinary differential equations is equal to the real part of one of the eigenvalues of the unperturbed linear equation. In this note, we show that if the solution is nonnegative, then its Liapunov exponent is an eigenvalue of the unperturbed equation and there exists a corresponding nonnegative eigenvector.

Let \mathbb{R} and \mathbb{N} denote the set of real numbers and the set of nonnegative integers, respectively. For $d \in \mathbb{N}$, $d \geq 1$, \mathbb{R}^d denotes the d -dimensional space of real column vectors with any given norm $|\cdot|$. Let $M_d(\mathbb{R})$ be the space of $d \times d$ matrices with real entries. The norm of a matrix $A \in M_d(\mathbb{R})$ is defined by

$$\|A\| = \sup_{0 \neq x \in \mathbb{R}^d} \frac{|Ax|}{|x|}.$$

Consider the system of ordinary differential equations

$$x' = Bx + f(t, x) \tag{1}$$

as a perturbation of the linear system

$$x' = Bx, \tag{2}$$

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where $B \in M_d(\mathbb{R})$ and $f: [0, \infty) \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ is a continuous function. One of the key results in the asymptotic theory of perturbed systems is the following theorem (see [1, Chap. IV, Theorem 5]).

THEOREM 1. *Let $x: [0, \infty) \rightarrow \mathbb{R}^d$ be a solution of Eq. (1) such that $x(t) \neq 0$ for $t \geq 0$. Suppose that*

$$|f(t, x(t))| \leq \gamma(t)|x(t)|, \quad t \geq 0, \quad (3)$$

where $\gamma: [0, \infty) \rightarrow [0, \infty)$ is a continuous function satisfying

$$\int_t^{t+1} \gamma(u) du \rightarrow 0, \quad t \rightarrow \infty. \quad (4)$$

Then the limit

$$\mu(x) = \lim_{t \rightarrow \infty} \frac{\log |x(t)|}{t} \quad (5)$$

exists and is equal to the real part of one of the eigenvalues of B .

Theorem 1 is essentially due to Perron [3] and Lettenmeyer [2]. The limit (5) (if it exists) is sometimes called the *strict Liapunov exponent of the solution x* .

In this note, we will study those solutions of (1) which are nonnegative. A solution $x = (x_1, x_2, \dots, x_d)^T: [0, \infty) \rightarrow \mathbb{R}^d$ of (1) is said to be *nonnegative* if $x_j(t) \geq 0$ for all $t \geq 0$ and $j = 1, 2, \dots, d$. We will show that if the solution x in Theorem 1 is nonnegative, then the following stronger result holds.

THEOREM 2. *Adopt the hypotheses and the notations of Theorem 1. If the solution x is nonnegative, then the strict Liapunov exponent $\mu(x)$ is an eigenvalue of B with an elementwise nonnegative eigenvector.*

The proof of Theorem 2 will be based on the following result about nonnegative solutions of linear difference equations with asymptotically constant coefficients (see [4, Theorem 1.3]).

PROPOSITION 3. *Let $A_n \in M_d(\mathbb{R})$ for $n \in \mathbb{N}$ and suppose that*

$$A_n \rightarrow A, \quad n \rightarrow \infty, \quad (6)$$

for some $A \in M_d(\mathbb{R})$. Let $y = (y_n)_{n \in \mathbb{N}}$ be a sequence of nonzero, (elementwise) nonnegative vectors in \mathbb{R}^d such that

$$y_{n+1} = A_n y_n, \quad n \in \mathbb{N}. \quad (7)$$

Then

$$\rho(y) = \lim_{n \rightarrow \infty} \sqrt[n]{|y_n|} \quad (8)$$

is an eigenvalue of A with a nonnegative eigenvector.

Note that the existence of the limit (8) is part of the conclusion of the proposition.

PROOF OF THEOREM 2. Since all norms on \mathbb{R}^d are equivalent, the limit $\mu(x)$ is independent of the norm used. Therefore, we may restrict ourselves to the Euclidean norm.

Let $\tau \in (0, 1)$ be fixed. By the variation-of-constants formula, we have for $n \in \mathbb{N}$ and $s \geq n\tau$,

$$x(s) = e^{B(s-n\tau)}x(n\tau) + \int_{n\tau}^s e^{B(s-u)}f(u, x(u))du, \quad (9)$$

and hence

$$|x(s)| \leq e^{\|B\|(s-n\tau)}|x(n\tau)| + \int_{n\tau}^s e^{\|B\|(s-u)}\gamma(u)|x(u)|du.$$

From this, we find for $n \in \mathbb{N}$ and $s \geq n\tau$,

$$e^{-\|B\|s}|x(s)| \leq e^{-\|B\|n\tau}|x(n\tau)| + \int_{n\tau}^s \gamma(u)e^{-\|B\|u}|x(u)|du.$$

By Gronwall's lemma, we have

$$|x(s)| \leq e^{\|B\|(s-n\tau)}|x(n\tau)| \exp\left(\int_{n\tau}^s \gamma(u)du\right)$$

for $n \in \mathbb{N}$ and $s \geq n\tau$. From this and (4), we obtain

$$|x(s)| \leq K|x(n\tau)| \quad \text{whenever } n \in \mathbb{N} \text{ and } s \in [n\tau, (n+1)\tau], \quad (10)$$

where

$$K = e^{\|B\|\tau} \exp\left(\sup_{t \geq 0} \int_t^{t+\tau} \gamma(u)du\right) < \infty.$$

Writing $s = (n+1)\tau$ in (9), we obtain

$$x((n+1)\tau) = e^{B\tau}x(n\tau) + h(n\tau), \quad n \in \mathbb{N}, \quad (11)$$

where

$$h(t) = \int_t^{t+\tau} e^{B(t+\tau-u)}f(u, x(u))du, \quad t \geq 0.$$

We have for $n \in \mathbb{N}$,

$$\begin{aligned} |h(n\tau)| &\leq \int_{n\tau}^{(n+1)\tau} e^{\|B\|((n+1)\tau-u)} \gamma(u) |x(u)| du \leq \\ &\leq \int_{n\tau}^{(n+1)\tau} e^{\|B\|\tau} \gamma(u) K |x(n\tau)| du, \end{aligned}$$

the last inequality being a consequence of (10). This, together with (4), yields

$$\frac{|h(n\tau)|}{|x(n\tau)|} \leq e^{\|B\|\tau} K \int_{n\tau}^{(n+1)\tau} \gamma(u) du \longrightarrow 0, \quad n \rightarrow \infty. \quad (12)$$

Define

$$y_n = x(n\tau), \quad n \in \mathbb{N}.$$

Eq. (11) can be written in the form (7) with

$$A_n = e^{B\tau} + \frac{h(n\tau)[x(n\tau)]^T}{|x(n\tau)|^2}, \quad n \in \mathbb{N},$$

T denoting the transpose. It is easily shown that

$$\|h(n\tau)[x(n\tau)]^T\| \leq |h(n\tau)||x(n\tau)|, \quad n \in \mathbb{N}.$$

This, together with (12), implies that hypothesis (6) of Proposition 3 is satisfied with

$$A = \lim_{n \rightarrow \infty} A_n = e^{B\tau}.$$

By the application of Proposition 3 to the nonvanishing, nonnegative solution $y = (y_n)_{n \in \mathbb{N}}$ of (7), we conclude that the limit

$$\rho(\tau) = \lim_{n \rightarrow \infty} \sqrt[n]{|x(n\tau)|} \quad (13)$$

is an eigenvalue of $A = e^{B\tau}$ and there exists a nonnegative vector $v(\tau) \in \mathbb{R}^d$ with $|v(\tau)| = 1$ such that

$$e^{B\tau} v(\tau) = \rho(\tau) v(\tau). \quad (14)$$

According to the Spectral Mapping Theorem, we have

$$\rho(\tau) = e^{\tau\lambda(\tau)} \quad \text{for some eigenvalue } \lambda(\tau) \text{ of } B. \quad (15)$$

Take a sequence $\tau_j \in (0, 1)$, $j \in \mathbb{N}$, such that $\tau_j \rightarrow 0$ as $j \rightarrow \infty$. Since both sequences $(\lambda(\tau_j))_{j \in \mathbb{N}}$ and $(v(\tau_j))_{j \in \mathbb{N}}$ are bounded, there exists a subsequence $(\tau_{j_k})_{k \in \mathbb{N}}$ of $(\tau_j)_{j \in \mathbb{N}}$ such that the (finite) limits

$$\lambda = \lim_{k \rightarrow \infty} \lambda(\tau_{j_k}) \quad (16)$$

and

$$v = \lim_{k \rightarrow \infty} v(\tau_{j_k}) \quad (17)$$

exist. The limit vector v in (17) is nonnegative and $|v| = 1$ as a consequence of the same properties of $v(\tau_{j_k})$, $k \in \mathbb{N}$. Since the eigenvalues of B form a finite set, (16) implies that $\lambda(\tau_{j_k}) = \lambda$ for all large k . This, combined with (14) and (15), yields

$$\rho(\tau_{j_k}) = e^{\lambda\tau_{j_k}} \quad (18)$$

and

$$e^{B\tau_{j_k}} v(\tau_{j_k}) = e^{\lambda\tau_{j_k}} v(\tau_{j_k}) \quad (19)$$

for all large k . From this, we find for all large k ,

$$\frac{e^{B\tau_{j_k}} - I}{\tau_{j_k}} v(\tau_{j_k}) = \frac{e^{\lambda\tau_{j_k}} - 1}{\tau_{j_k}} v(\tau_{j_k}).$$

Letting $k \rightarrow \infty$ and taking into account that $\tau_{j_k} \rightarrow 0$ as $k \rightarrow \infty$, we obtain

$$Bv = \lambda v. \quad (20)$$

Clearly, this implies that λ is a real eigenvalue of B . As noted before, if k is large enough, then (18) holds and we get

$$\begin{aligned} \mu(x) &= \lim_{t \rightarrow \infty} \frac{\log |x(t)|}{t} = \lim_{n \rightarrow \infty} \frac{\log |x(n\tau_{j_k})|}{n\tau_{j_k}} = \frac{1}{\tau_{j_k}} \lim_{n \rightarrow \infty} \log \sqrt[n]{|x(n\tau_{j_k})|} \\ &= \frac{1}{\tau_{j_k}} \log \rho(\tau_{j_k}) = \frac{1}{\tau_{j_k}} \log e^{\lambda\tau_{j_k}} = \lambda. \end{aligned}$$

This, together with (20), completes the proof. ■

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ON PARAMETRIZATION FOR SOME LINEAR THREE-POINT BOUNDARY VALUE PROBLEMS WITH ARGUMENT DEVIATION

By

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Abstract. We obtain some results concerning the investigation of the solutions of three-point boundary value problems for a certain class of linear functional differential equations. We show that it is useful to reduce the given problem to the parametrized two-point boundary value problem for a suitably perturbed system containing some artificially introduced parameters both in the constructed inhomogeneous two-point boundary conditions and in the modified functional differential equations.

To study the transformed parametrized two-point problem, we use a method which is based on a special type of successive approximations constructed in an analytic form. We prove the uniform convergence of these approximations to the parametrized limit function. Our techniques lead us to a certain system of algebraic equations with respect to the introduced parameters whose solutions provide those numerical values of the introduced parameters that correspond to the solution of the given three-point boundary value problem. We also establish some properties of the limit function and so-called determining functions.

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1. Introduction

Analysis of the literature devoted to the theory of regular boundary value problems for ordinary differential equations shows that various analytic, functional-analytic, numerical and numerical-analytic methods based on some types of successive approximations are now extensively used and studied. It is natural that each group of methods has certain advantages and disadvantages.

The analytic and functional-analytic methods in the theory of boundary value problems are generally used for the investigation of qualitative properties such as the existence, uniqueness, stability, dichotomy, branching of solutions (see, e.g., [12, 5, 30, 31, 23, 24, 13, 1, 32, 16, 49] and the references in [47]). For obtaining existence results, this group of methods widely uses techniques of functional analysis, topological degree theory, and the theory of approximate methods for solving operator equations [25, 9, 29, 17, 15, 7, 8, 26, 2, 35, 36, 34, 56, 6, 38, 55]. The group of numerical methods, under the assumption on the existence of solutions, gives practical numerical algorithms for their approximate construction [3, 22]. The numerical construction of approximate solutions is usually based on the idea of the shooting method and may face certain difficulties because the regularity conditions for the right-hand side function of the boundary value problem (e.g., the Lipschitz condition), as a rule, should be assumed globally, i.e., fulfilled for all the values of space variables, which is quite often not the case.

In studies of solutions of various types of boundary value problems for ordinary differential equations, side by side with the methods mentioned above, one often uses appropriate techniques belonging to the group of the so-called numerical-analytic methods which are based upon some types of successive approximations constructed in an analytic form. Such an approach belongs to the few of them that offer constructive possibilities both for the investigation of the existence of a solution and for its approximate construction. In the theory of nonlinear oscillations, such types of numerical-analytic methods based upon successive approximations for the investigation of periodic boundary value problems were apparently first developed in [10, 17, 50, 51]. Appropriate versions were later developed for handling more general types of nonlinear boundary value problems for ordinary and functional-differential equations. We refer, e.g., to the books [52, 53, 54, 47], the handbook [41], the papers [4, 27, 28, 40, 45, 44, 42, 18, 19, 20, 21, 37, 11, 33, 43], and the series of survey papers [48] for the related references.

According to the basic idea, the boundary value problem posed in $D \subset \mathbb{R}^n$ is formally replaced by the Cauchy problem for a suitably modified system of integro-differential equations containing some artificially introduced vector parameter $z \in D \subset \mathbb{R}^n$, whose numerical value is to be determined later. This parameter z usually has the meaning of the initial value that the solution of the given boundary value problem under consideration takes at a certain given point. The solution of Cauchy problem for the constructed “perturbed” equation is sought for in an analytic form by successive iterations. The “perturbation term,” which depends on the introduced parameter z and on the boundary conditions and whose form is determined by the right-hand side of the differential equation, yields a system of algebraic or transcendental “determining equations.” The numerical solutions of this “determining system” give the values of the parameter z that correspond to the solution of the given boundary value problem. By studying the solvability of some “approximate determining systems,” one can obtain conditions free of unknown functions and, in this way, establish existence results for the original boundary value problem. It is clear that both the form and complexity of the given equations and boundary conditions have an essential influence both on the possibility of efficient construction of approximate solutions and the solvability analysis of the given boundary value problem.

The aim of this paper is to extend, in a certain way, the techniques used in [42] for the system of n linear functional differential equations of the form

$$(1.1) \quad x'(t) = P_0(t)x(t) + P_1(t)x(\beta(t)) + f(t), \quad t \in [0, T],$$

subjected to the inhomogeneous three-point Cauchy–Nicoletti boundary conditions

$$\begin{aligned} x_1(0) &= x_{10}, \quad \dots, \quad x_p(0) = x_{p0}, \\ x_{p+1}(\xi) &= d_{p+1}, \quad \dots, \quad x_{p+q}(\xi) = d_{p+q}, \\ x_{p+q+1}(T) &= d_{p+q+1}, \quad \dots, \quad x_n(T) = d_n, \end{aligned}$$

where $\xi \in (0, T)$, to the case where the three-point boundary conditions have the nonseparated matrix-vector form

$$(1.2) \quad Ax(0) + Bx(\xi) + \bar{C}x(T) = d,$$

where A , B , and \bar{C} are singular matrices, $d = \text{col}(d_1, \dots, d_n)$. The difficulties related to this type of boundary conditions are due to the singularity of the matrices that determine them.

The following notation is used in the sequel: $C([0, T], \mathbb{R}^n)$ is the Banach space of the continuous functions $[0, T] \rightarrow \mathbb{R}^n$ with the standard uniform norm; $L_1([0, T], \mathbb{R}^n)$ is the usual Banach space of the vector functions

$[0, T] \rightarrow \mathbb{R}^n$ with Lebesgue integrable components; $\mathcal{L}(\mathbb{R}^n)$ is the algebra of all the square matrices of dimension n with real elements; $r(Q)$ is the maximal in module eigenvalue of the matrix $Q \in \mathcal{L}(\mathbb{R}^n)$; I_n is the unit matrix of dimension n ; $B_{i,j}$ is the matrix of dimension $i \times j$; $0_{i,j}$ is the zero matrix of dimension $i \times j$; $0_k = 0_{k,k}$.

2. Problem setting and reduction to two-point parametrized conditions

We consider the system of n linear functional differential equations with argument deviations (1.1) subjected to the nonseparated inhomogeneous three-point boundary conditions (1.2). In the problem (1.1), (1.2), we suppose that $T \in [0, +\infty)$, the elements of the matrix-valued functions $P_j: [0, T] \rightarrow \mathcal{L}(\mathbb{R}^n)$, $j = 0, 1$, are Lebesgue integrable, $f \in L_1([0, T], \mathbb{R}^n)$, $\beta: [0, T] \rightarrow [0, T]$ is a Lebesgue measurable function, $d \in \mathbb{R}^n$, the matrices $\{A, B, \bar{C}\} \subset \mathcal{L}(\mathbb{R}^n)$ are singular and, furthermore, \bar{C} has the form:

$$(2.1) \quad \bar{C} = \begin{pmatrix} \bar{C}_{q,q} & W_{q,n-q} \\ 0_{n-q,q} & 0_{n-q} \end{pmatrix},$$

where $\bar{C}_{q,q}$ is non-singular square matrix of dimension $q < n$ and $W_{q,n-q}$ is some arbitrary matrix of dimension $q \times (n-q)$. The singularity of the matrices determining the boundary conditions (1.2) causes certain difficulties. To avoid dealing with singular matrices and simplify the construction of a solution in an analytic form, we use a certain parametrization technique on two levels. The first level allows one to replace the three-point boundary conditions by a suitable parametrized family of two-point inhomogeneous conditions. The second level of parametrization is then used for the construction and subsequent investigation of an auxiliary perturbed differential system. Finally, the study of certain algebraic determining equations gives one those numerical values of the unknown parameters that correspond to the solutions of the given three-point boundary value problem.

We construct the auxiliary family of two-point problems by “freezing” the values of certain components of x at ξ and T as follows:

$$(2.2) \quad \text{col}(x_1(\xi), \dots, x_n(\xi)) = \lambda,$$

$$(2.3) \quad \text{col}(x_{q+1}(T), \dots, x_n(T)) = \eta,$$

where

$$(2.4) \quad \lambda = \text{col}(\lambda_1, \dots, \lambda_n), \quad \eta = \text{col}(\eta_1, \dots, \eta_{n-q})$$

are vector parameters of the appropriate dimensions. Instead of problem (1.1), (1.2), we consider the parametrized two-point problem (1.1), (2.5)

$$(2.5) \quad Ax(0) + Cx(T) = d(\lambda, \eta),$$

where

$$(2.6) \quad d(\lambda, \eta) := d - B\lambda + \text{col}(\underbrace{0, \dots, 0}_q, \eta_1, \dots, \eta_{n-q})$$

for all $\lambda \in \mathbb{R}^n$, $\eta \in \mathbb{R}^{n-q}$, and the matrix C is given by the equality

$$(2.7) \quad C = \begin{pmatrix} \bar{C}_{q,q} & W_{q,n-q} \\ 0_{n-q,q} & I_{n-q} \end{pmatrix}.$$

Note that, in contrast to the three-point condition (1.2), the matrix C appearing in (2.5) is non-singular.

REMARK 1. It is clear, that the solutions of the original three-point boundary value problem (1.1), (1.2) coincide with that solutions of two-point boundary value problem (1.1), (2.5), which fulfil the additional conditions (2.2), (2.3).

REMARK 2. The matrices A and B in the boundary conditions (1.2) may be either singular or not. If the number of boundary conditions in (1.2) is $l < n$, i.e., the matrices A , B , \bar{C} and the vector d have $n-l$ last zero rows and the rank of the $(n \times 3n)$ -dimensional matrix $[A \ B \ \bar{C}]$ is l , then the boundary value problem (1.1), (1.2) may have an $(n-l)$ -parametric family of solutions.

3. Subsidiary statements

Let us formulate two lemmas that are used in what follows. Define the sequence of functions $\{\alpha_m\}_{m=0}^\infty \subset C([0, T], \mathbb{R})$ by the recurrence relation

$$(3.1) \quad \alpha_{m+1}(t) = \left(1 - \frac{t}{T}\right) \int_0^t \alpha_m(s) ds + \frac{t}{T} \int_t^T \alpha_m(s) ds, \quad m = 0, 1, 2, \dots,$$

where $\alpha_0(t) := 1$, $t \in [0, T]$. It is obvious that, in particular,

$$(3.2) \quad \alpha_1(t) = 2t \left(1 - \frac{t}{T}\right), \quad t \in [0, T],$$

and

$$(3.3) \quad \max_{t \in [0, T]} \alpha_1(t) = \frac{T}{2}$$

LEMMA 1. *For an arbitrary essentially bounded function $u:[0, T] \rightarrow \mathbb{R}$, the estimate*

$$(3.4) \quad \left| \int_0^t \left[u(\tau) - \frac{1}{T} \int_0^T u(s) ds \right] d\tau \right| \leq \frac{\alpha_1(t)}{2} \left(\underset{s \in [0, T]}{\text{ess sup}} u(s) - \underset{s \in [0, T]}{\text{ess inf}} u(s) \right),$$

for all $t \in [0, T]$ is true, where α_1 is the function defined by equality (3.2).

Inequality (3.4) is established similarly to the [46, Lemma 3] or [47, Lemma 2.3].

LEMMA 2. *The following inequalities are true:*

$$(3.5) \quad \begin{aligned} \alpha_{m+1}(t) &\leq \frac{3T}{10} \alpha_m(t), \quad m \geq 2, \\ \alpha_{m+1}(t) &\leq \frac{10}{9} \left(\frac{3T}{10} \right)^m \alpha_1(t), \quad m \geq 0, \end{aligned}$$

where α_1 is given by (3.2).

For the proof, see [46, Lemma 4] or [47, Lemma 2.4].

4. Successive approximations and convergence analysis for general type of argument deviation

To study the solution of the auxiliary two-point parametrized boundary value problem (1.1), (2.5) let us introduce the sequence of functions

$$(4.1) \quad \begin{aligned} x_{m+1}(t, z, \lambda, \eta) &:= \\ &:= z + \int_0^t [P_0(s)x_m(s, z, \lambda, \eta) + P_1(s)x_m(\beta(s), z, \lambda, \eta) + f(s)] ds \\ &- \frac{t}{T} \int_0^T [P_0(s)x_m(s, z, \lambda, \eta) + P_1(s)x_m(\beta(s), z, \lambda, \eta) + f(s)] ds \\ &+ \frac{t}{T} \left[C^{-1}d(\lambda, \eta) - (C^{-1}A + I_n)z \right], \quad m = 0, 1, 2, \dots, \end{aligned}$$

where $x_0(t, z, \lambda, \eta) = z$, the vectors $z \in \mathbb{R}^n$, $\lambda \in \mathbb{R}^n$, $\eta \in \mathbb{R}^{n-q}$ are considered as parameters, and $d(\lambda, \eta)$ is given by (2.6). Let us establish the convergence of the sequence (4.1) for an arbitrary deviation function $\beta:[0, T] \rightarrow [0, T]$.

THEOREM 1. Let us suppose that in (1.1) the elements of matrix-valued functions $P_i: [0, T] \rightarrow \mathcal{L}(\mathbb{R}^n)$, $i = 0, 1$, are Lebesgue integrable, $f \in L_1([0, T], \mathbb{R}^n)$ and $\beta: [0, T] \rightarrow [0, T]$ is a Lebesgue measurable function. Moreover, if

$$(4.2) \quad r(K_0 + K_1) < \frac{2}{T},$$

where

$$(4.3) \quad K_i := \operatorname{ess\,sup}_{s \in [0, T]} |P_i(s)|, \quad i = 0, 1,$$

then, for arbitrary fixed $z \in \mathbb{R}^n$, $\lambda \in \mathbb{R}^n$, $\eta \in \mathbb{R}^{n-q}$:

1. All the members of the sequence (4.1), starting from the first one, are absolutely continuous functions satisfying the two-point parametrized boundary conditions (2.5).

$$(4.4) \quad Ax_m(0, z, \lambda, \eta) + Cx_m(T, z, \lambda, \eta) = d(\lambda, \eta)$$

and the initial condition $x_m(0, z, \lambda, \eta) = z$, $m = 1, 2, \dots$

2. The sequence of functions (4.1) converges uniformly in $t \in [0, T]$ to a limit function $x^*(t, z, \lambda, \eta)$

$$(4.5) \quad x^*(t, z, \lambda, \eta) = \lim_{m \rightarrow \infty} x_m(t, z, \lambda, \eta),$$

as $m \rightarrow \infty$.

3. The limit function $x^*(t, z, \lambda, \eta)$ satisfies the initial condition $x^*(0, z, \lambda, \eta) = z$ and the boundary condition (2.5): $Ax^*(0, z, \lambda, \eta) + Cx^*(T, z, \lambda, \eta) = d(\lambda, \eta)$.
4. The limit function (4.5) is a unique absolutely continuous solution of the integro-functional equation

$$(4.6) \quad \begin{aligned} x(t) := z &+ \int_0^t [P_0(s)x(s) + P_1(s)x(\beta(s)) + f(s)] ds \\ &- \frac{t}{T} \int_0^T [P_0(s)x(s) + P_1(s)x(\beta(s)) + f(s)] ds \\ &+ \frac{t}{T} \left[C^{-1}d(\lambda, \eta) - \left(C^{-1}A + I_n \right) z \right], \quad t \in [0, T]. \end{aligned}$$

5. The error estimate

$$(4.7) \quad |x^*(t, z, \lambda) - x_m(t, z, \lambda)| \leq G^m (I_n - G)^{-1} \gamma(z, \lambda, \eta)$$

holds, where $G := \frac{T}{2}(K_0 + K_1)$,

$$(4.8) \quad \gamma(z, \lambda, \eta) := \frac{T}{2}\delta(z) + \left| d(\lambda, \eta) - (C^{-1}A + I_n)z \right|,$$

$$(4.9) \quad \begin{aligned} \delta(z) := \frac{1}{2} & \left[\underset{s \in [0, T]}{\text{ess sup}} (P_0(s)z + P_1(s)z + f(s)) - \right. \\ & \left. - \underset{s \in [0, T]}{\text{ess inf}} (P_0(s)z + P_1(s)z + f(s)) \right]. \end{aligned}$$

In (4.3), (4.7), (4.8) and similar relations below, the signs $|\cdot|$, \leq , \geq , ess sup , ess inf are understood componentwise.

PROOF. The validity of assertion 1 is an immediate consequence of the formula (4.1). To obtain the other required properties, we shall show that, under the conditions assumed, sequence (4.1) is a Cauchy sequence in the Banach space $C([0, T], \mathbb{R}^n)$ equipped with the standard uniform norm. Due to estimate (3.4) of Lemma 1, it follows from (4.1) that for $m = 0$ and for arbitrary fixed $z \in \mathbb{R}^n$, $\lambda \in \mathbb{R}^n$, $\eta \in \mathbb{R}^{n-q}$:

$$\begin{aligned} & |x_1(t, z, \lambda, \eta) - z| = \\ & = \left| \int_0^t \left[(P_0(s)z + P_1(s)z + f(s)) - \frac{1}{T} \int_0^T (P_0(\tau)z + P_1(\tau)z + f(\tau)) d\tau \right] ds + \right. \\ & \quad \left. + \frac{t}{T} \left[d(\lambda, \eta) - (C^{-1}A + I_n)z \right] \right| \leq \\ (4.10) \quad & \leq \alpha_1(t)\delta(z) + \left| d(\lambda, \eta) - (C^{-1}A + I_n)z \right|, \end{aligned}$$

where α_1 is the function (3.2) and $\delta(z)$ is defined by (4.9).

According to formulae (4.1), for all $t \in [0, T]$, $z \in \mathbb{R}^n$, $\lambda \in \mathbb{R}^n$, $\eta \in \mathbb{R}^{n-q}$, and $m = 1, 2, \dots$, for the difference of functions

$$(4.11) \quad r_{m+1}(t, z, \lambda, \eta) := x_{m+1}(t, z, \lambda, \eta) - x_m(t, z, \lambda, \eta),$$

we have

$$\begin{aligned} r_{m+1}(t, z, \lambda, \eta) &= \left(1 - \frac{t}{T}\right) \int_0^t [P_0(s)(x_m(s, z, \lambda, \eta) - x_{m-1}(s, z, \lambda, \eta)) + \\ &\quad + P_1(s)(x_m(\beta(s), z, \lambda, \eta) - x_{m-1}(\beta(s), z, \lambda, \eta))] ds \end{aligned}$$

$$(4.12) \quad -\frac{t}{T} \int_t^T [P_0(s) (x_m(s, z, \lambda, \eta) - x_{m-1}(s, z, \lambda, \eta)) + \\ + P_1(s) (x_m(\beta(s), z, \lambda, \eta) - x_{m-1}(\beta(s), z, \lambda, \eta))] ds.$$

Equality (4.12) implies that for all $m = 1, 2, \dots$, arbitrary $z \in \mathbb{R}^n$, $\lambda \in \mathbb{R}^n$, $\eta \in \mathbb{R}^{n-q}$ and $t \in [0, T]$,

$$(4.13) \quad |r_{m+1}(t, z, \lambda, \eta)| \leq \\ \leq K_0 \left[\left(1 - \frac{t}{T}\right) \int_0^t |r_m(s, z, \lambda, \eta)| ds + \frac{t}{T} \int_t^T |r_m(s, z, \lambda, \eta)| ds \right] + \\ + K_1 \left[\left(1 - \frac{t}{T}\right) \int_0^t |r_m(\beta(s), z, \lambda, \eta)| ds + \frac{t}{T} \int_t^T r_m(\beta(s), z, \lambda, \eta) ds \right],$$

where K_0 and K_1 are the non-negative matrices given by formula (4.3). Relation (4.10) yields

$$(4.14) \quad |r_1(t, z, \lambda, \eta)| \leq \alpha_1(t) \delta(z) + |d(\lambda, \eta) - (C^{-1}A + I_n)z|, \quad t \in [0, T],$$

where δ is given by (4.9). In view of property (3.3) of the function α_1 , estimate (4.14) gives

$$(4.15) \quad |r_1(t, z, \lambda, \eta)| \leq \frac{T}{2} \delta(z) + |d(\lambda, \eta) - (C^{-1}A + I_n)z| = \gamma(z, \lambda, \eta), \quad t \in [0, T].$$

Let us now estimate $|r_2(t, z, \lambda)|$ using (4.13) and (4.15):

$$\begin{aligned} & |r_2(t, z, \lambda, \eta)| \leq \\ & \leq K_0 \left[\left(1 - \frac{t}{T}\right) \int_0^t |r_1(s, z, \lambda, \eta)| ds + \frac{t}{T} \int_t^T |r_1(s, z, \lambda, \eta)| ds \right] + \\ & + K_1 \left[\left(1 - \frac{t}{T}\right) \int_0^t |r_1(\beta(s), z, \lambda, \eta)| ds + \frac{t}{T} \int_t^T |r_1(\beta(s), z, \lambda, \eta)| ds \right] \leq \\ (4.16) \quad & \leq K_0 \left[\left(1 - \frac{t}{T}\right) \int_0^t \gamma(z, \lambda, \eta) ds + \frac{t}{T} \int_t^T \gamma(z, \lambda, \eta) ds \right] + \end{aligned}$$

$$+K_1 \left[\left(1 - \frac{t}{T}\right) \int_0^t \gamma(z, \lambda, \eta) ds + \frac{t}{T} \int_t^T \gamma(z, \lambda, \eta) ds \right], \quad t \in [0, T].$$

Taking relations (3.1), (3.2) into account and using equality (3.3), from (4.16) we get

$$(4.17) \quad \begin{aligned} & |r_2(t, z, \lambda, \eta)| \leq \\ & \leq [K_0 + K_1] \gamma(z, \lambda, \eta) \alpha_1(t) \leq \frac{T}{2} [K_0 + K_1] \gamma(z, \lambda, \eta), \quad t \in [0, T]. \end{aligned}$$

Arguing by induction, we then obtain that, for all $t \in [0, T]$ and $m = 1, 2, \dots$, the estimates

$$(4.18) \quad \begin{aligned} & |r_m(t, z, \lambda, \eta)| \leq \\ & \leq \left(\frac{T}{2}\right)^{m-1} [K_0 + K_1]^{m-1} \gamma(z, \lambda, \eta) = G^{m-1} \gamma(z, \lambda, \eta) \end{aligned}$$

are true, where

$$(4.19) \quad G = \frac{T}{2} (K_0 + K_1).$$

Estimate (4.18), in view of (4.11), yields

$$\begin{aligned} 4.20 \quad & |x_{m+j}(t, z, \lambda, \eta) - x_m(t, z, \lambda, \eta)| = \left| \sum_{i=1}^j r_{m+i}(t, z, \lambda, \eta) \right| \leq \\ & \leq \sum_{i=1}^j |r_{m+i}(t, z, \lambda, \eta)| \leq \\ & \leq G^m \sum_{i=0}^{j-1} G^i \gamma(z, \lambda, \eta) \end{aligned}$$

for all $t \in [0, T]$, $m = 1, 2, \dots$, whence, by virtue of assumption (4.2) and notation (4.19), it follows that

$$(4.21) \quad \begin{aligned} & |x_{m+j}(t, z, \lambda, \eta) - x_m(t, z, \lambda, \eta)| \leq \\ & \leq G^m \sum_{i=0}^{\infty} G^i \gamma(z, \lambda, \eta) = G^m (I_h - G)^{-1} \gamma(z, \lambda, \eta) \end{aligned}$$

for all $t \in [0, T]$, $m = 1, 2, \dots$. Since, due to (4.2), $\lim_{m \rightarrow \infty} G^m \rightarrow 0_n$, it is clear from (4.21) that (4.1) is a Cauchy sequence in the Banach space

$C([0, T], \mathbb{R}^n)$ and, consequently, it is uniformly converges for all $t \in [0, T]$ and any $z \in \mathbb{R}^n$, $\lambda \in \mathbb{R}^n$, $\eta \in \mathbb{R}^{n-q}$, i.e., assertion 2 holds. Since all the functions $x_m(\cdot, z, \lambda, \eta)$ of the sequence (4.1) satisfy the boundary conditions (2.5), it follows that so does the limit function $x^*(\cdot, z, \lambda, \eta)$.

Passing to the limit as $m \rightarrow \infty$ in (4.1) and (4.4), we show that the limit function is a solution of the integro-functional equation (4.6). Passing to the limit as $j \rightarrow \infty$ in (4.21), we arrive at estimate (4.7). This completes the proof of Theorem 1. ■

REMARK 3. The iterative formula (4.1) can be replaced (cf. [37]) by the following one:

$$\begin{aligned}
 & x_{m+1}(t, z, \lambda, \eta) := \\
 & := z + \int_0^t [P_0(s)x_m(s, z, \lambda, \eta) + P_1(s)x_m(\beta(s), z, \lambda, \eta) + f(s)] ds - \\
 & - \frac{\omega(t)}{T} \int_0^T [P_0(s)x_m(s, z, \lambda, \eta) + P_1(s)x_m(\beta(s), z, \lambda, \eta) + f(s)] ds + \\
 (5.27) \quad & + \frac{\omega(t)}{T} \left[C^{-1}d(\lambda) - (C^{-1}A + I_n)z \right],
 \end{aligned}$$

where $\omega : [0, T] \rightarrow [0, T]$ is a continuous function with the properties $\omega(0) = 0$ and $\omega(T) = T$.

We have the following simple statement.

PROPOSITION 1. *If, under the assumptions of Theorem 1, the limit function $x^*(\cdot, z, \lambda, \eta)$ satisfies the condition*

$$\begin{aligned}
 & C^{-1}d(\lambda, \eta) - (C^{-1}A + I_n)z = \\
 & = \int_0^T [P_0(s)x^*(s, z, \lambda, \eta) + P_1(s)x^*(\beta(s), z, \lambda, \eta) + f(s)] ds
 \end{aligned}$$

for certain values of the vectors z, λ and η , then for these z, λ and η , it is also a solution of the boundary value problem (1.1), (2.5).

The proof is a straightforward application of the above theorem.

5. Some properties of the limit function

Let us establish at first the connection of the limit function $x^*(t, z, \lambda, \eta)$ of the form (4.5) with the solution of the auxiliary two-point parametrized boundary value problem (1.1), (2.5). We show that it can be chosen such values of the parameters $z = z^*$, $\lambda = \lambda^*$, $\eta = \eta^*$ for which the function $x^*(t, z^*, \lambda^*)$ will be the solution of the original three-point boundary value problem (1.1), (1.2).

Along with system (1.1), we also consider the system with the additive perturbation of the right-hand side

$$(5.1) \quad x'(t) = P_0(t)x(t) + P_1(t)x(\beta(t)) + f(t) + \mu,$$

with the initial condition

$$(5.2) \quad x(0) = z,$$

where $\mu = \text{col}(\mu_1, \dots, \mu_n)$ is a control parameter.

We shall show that, for any $z \in \mathbb{R}^n$, $\lambda \in \mathbb{R}^n$, $\eta \in \mathbb{R}^{n-q}$, the parameter μ can always be chosen so that the solution $x(\cdot, z, \lambda, \eta, \mu)$ of the initial value problem (5.1), (5.2) is, at the same time, a solution of the two point parametrized boundary value problem (5.1), (2.5).

PROPOSITION 2. *Assume that the system of differential equations (1.1) satisfies the conditions of Theorem 1. Then, for arbitrary $z \in \mathbb{R}^n$, $\lambda \in \mathbb{R}^n$, and $\eta \in \mathbb{R}^{n-q}$,*

$$(5.3) \quad \begin{aligned} \mu &= \frac{1}{T} \left[C^{-1}d(\lambda, \eta) - \left(C^{-1}A + I_n \right) z \right] - \\ &- \frac{1}{T} \int_0^T [P_0(s)x^*(s, z, \lambda, \eta) + P_1(s)x^*(\beta(s), z, \lambda, \eta) + f(s)] ds, \end{aligned}$$

is the unique value of the control parameter μ for which the solution $x(\cdot, z, \lambda, \eta, \mu)$ of the initial value problem (5.1), (5.2) with μ given by (5.3) is also a solution of the boundary value problem (5.1), (2.5). Moreover, with this values of μ

$$(5.4) \quad x(t, z, \lambda, \eta, \mu) = x^*(t, z, \lambda, \eta) = \lim_{m \rightarrow \infty} x_m(t, z, \lambda, \eta),$$

where $\{x_m(\cdot, z, \lambda, \eta)\}_{m=1}^\infty$ is the sequence of functions defined according (4.1).

PROOF. The assertion of Proposition 2 is obtained by analogy to the proof of Theorem 4.2 from [44]. Since for (1.1) the conditions of Theorem 1

hold, the sequence of functions (4.1) satisfying the boundary conditions (2.5) converges uniformly in $t \in [0, T]$ for arbitrary fixed $z \in \mathbb{R}^n$, $\lambda \in \mathbb{R}^n$, $\eta \in \mathbb{R}^{n-q}$ to the function $x = x^*(t, z, \lambda, \eta)$ that also satisfies the boundary conditions (2.5) and the initial conditions (5.2). Therefore, it follows from Theorem 1 that if (5.3) holds then we found the value of the parameter μ of the form (5.3) for which (5.4) holds.

This parameter value is unique because, for any other value $\bar{\mu}$ different from that given by (5.3), the solution $x = x(t, z, \lambda, \eta, \bar{\mu})$ of the initial value problem (5.1), (5.2) does not satisfy the two-point boundary conditions (2.5). Indeed, assume the contrary. Then there exist at least two values μ and $\bar{\mu} \neq \mu$ such that the solutions $x = x(t, z, \lambda, \eta, \mu) = x_\mu$ and $x = x(t, z, \lambda, \eta, \bar{\mu}) = x_{\bar{\mu}}$ of the Cauchy problems (5.1), (5.2) and (5.5), (5.2)

$$(5.5) \quad x'(t) = P_0(t)x(t) + P_1(t)x(\beta(t)) + f(t) + \bar{\mu},$$

also satisfy the two point boundary conditions (2.5). It is clear from Theorem 1 that every solution x_μ and $x_{\bar{\mu}}$ of the integro-functional equation (4.6) with the initial value $x(0) = z$ satisfies the two-point boundary conditions (2.5).

Obviously, that the functions x_μ and $x_{\bar{\mu}}$ satisfy the integral equations

$$(5.6) \quad x_\mu(t) = z + \int_0^t [P_0(s)x_\mu(s) + P_1(s)x_\mu(\beta(s)) + f(s)] ds + \mu t, \quad t \in [0, T],$$

and

$$(5.7) \quad x_{\bar{\mu}}(t) = z + \int_0^t [P_0(s)x_{\bar{\mu}}(s) + P_1(s)x_{\bar{\mu}}(\beta(s)) + f(s)] ds + \bar{\mu}t, \quad t \in [0, T],$$

From (5.6) and (5.7) for $t = T$ we obtain

$$(5.8) \quad T\mu = x_\mu(T) - z - \int_0^T [P_0(s)x_\mu(s) + P_1(s)x_\mu(\beta(s)) + f(s)] ds,$$

$$(5.9) \quad T\bar{\mu} = x_{\bar{\mu}}(T) - z - \int_0^T [P_0(s)x_{\bar{\mu}}(s) + P_1(s)x_{\bar{\mu}}(\beta(s)) + f(s)] ds.$$

According to our conditions, the function $x_\mu(t)$ satisfy the boundary condition

$$Ax_\mu(0) + Cx_\mu(T) = d(\lambda, \eta),$$

and the initial condition $x_\mu(0) = z$. Therefore,

$$(5.10) \quad x_\mu(T) = C^{-1}d(\lambda, \eta) - C^{-1}Az.$$

By analogy, we conclude that

$$(5.11) \quad x_{\bar{\mu}}(T) = C^{-1}d(\lambda, \eta) - C^{-1}Az.$$

Consequently from (5.8), (5.9), using (5.10), (5.11), we obtain

$$(5.12) \quad \begin{aligned} \mu &= \frac{1}{T} \left[C^{-1}d(\lambda, \eta) - \left(C^{-1}A + I_n \right) z - \right. \\ &\quad \left. - \int_0^T [P_0(s)x_\mu(s) + P_1(s)x_\mu(\beta(s)) + f(s)] ds \right], \end{aligned}$$

$$(5.13) \quad \begin{aligned} \bar{\mu} &= \frac{1}{T} \left[C^{-1}d(\lambda, \eta) - \left(C^{-1}A + I_n \right) z - \right. \\ &\quad \left. - \int_0^T [P_0(s)x_{\bar{\mu}}(s) + P_1(s)x_{\bar{\mu}}(\beta(s)) + f(s)] ds \right]. \end{aligned}$$

Put (5.12) and (5.13) into (5.6) and (5.7), we obtain

$$(5.14) \quad \begin{aligned} x_\mu(t) &= z + \int_0^t [P_0(s)x_\mu(s) + P_1(s)x_\mu(\beta(s)) + f(s)] ds - \\ &\quad - \frac{t}{T} \int_0^T [P_0(s)x_\mu(s) + P_1(s)x_\mu(\beta(s)) + f(s)] ds + \\ &\quad + \frac{t}{T} \left[C^{-1}d(\lambda, \eta) - \left(C^{-1}A + I_n \right) z \right] \end{aligned}$$

and

$$(5.15) \quad \begin{aligned} x_{\bar{\mu}}(t) &= z + \int_0^t [P_0(s)x_{\bar{\mu}}(s) + P_1(s)x_{\bar{\mu}}(\beta(s)) + f(s)] ds - \\ &\quad - \frac{t}{T} \int_0^T [P_0(s)x_{\bar{\mu}}(s) + P_1(s)x_{\bar{\mu}}(\beta(s)) + f(s)] ds + \end{aligned}$$

$$+\frac{t}{T} \left[C^{-1}d(\lambda, \eta) - \left(C^{-1}A + I_n \right) z \right],$$

for all $t \in [0, T]$ and $z \in \mathbb{R}^n$, $\lambda \in \mathbb{R}^n$, $\eta \in \mathbb{R}^{n-q}$. From (5.14), (5.15) we have

$$\begin{aligned} x_\mu(t) - x_{\bar{\mu}}(t) &= \int_0^t [P_0(s)(x_\mu(s) - x_{\bar{\mu}}(s)) + P_1(s)(x_\mu(\beta(s)) - x_{\bar{\mu}}(\beta(s)))] ds \\ &\quad - \frac{t}{T} \int_0^T [P_0(s)(x_\mu(s) - x_{\bar{\mu}}(s)) + P_1(s)(x_\mu(\beta(s)) - x_{\bar{\mu}}(\beta(s)))] ds, \end{aligned}$$

and therefore, due to (4.3), the function

$$(5.16) \quad r(t) = |x_\mu(t) - x_{\bar{\mu}}(t)|$$

satisfies the inequality

$$(5.17) \quad r(t) \leq (K_0 + K_1) \left[\left(1 - \frac{t}{T}\right) \int_0^t \tilde{r} ds + \frac{t}{T} \int_t^T \tilde{r} ds \right] \leq (K_0 + K_1) \alpha_1(t) \tilde{r},$$

where

$$\tilde{r} := \text{col} \left(\max_{t \in [0, T]} r_1(t), \dots, \max_{t \in [0, T]} r_n(t) \right)$$

and the function α_1 is given by (3.2). Using (5.17) by iteration, we have

$$(5.18) \quad r(t) \leq (K_0 + K_1)^m \alpha_m(t) \tilde{r},$$

where α_m is the function (3.1). Using (3.5), from (5.18) for all $t \in [0, T]$ and $z \in \mathbb{R}^n$, $\lambda \in \mathbb{R}^n$, $\eta \in \mathbb{R}^{n-q}$, we obtain

$$(5.19) \quad r(t) \leq \frac{10}{9} \left(\frac{3T}{10} (K_0 + K_1) \right)^{m-1} (K_0 + K_1) \alpha_1(t) \tilde{r}.$$

In view of the fact that, componentwise,

$$G_1 = \frac{3T}{10} (K_0 + K_1) < G = \frac{T}{2} (K_1 + K_2)$$

and all eigenvalues of the matrix (4.19) lie inside the unit disk, passing to the limit as $m \rightarrow \infty$ in (5.19), we conclude that the inequality (5.19) on the interval $[0, T]$ is possible only for $r(t) \equiv 0$, i.e., for $x_\mu \equiv x_{\bar{\mu}}$ or, alternatively, for $\mu = \bar{\mu}$. The contradiction obtained proves the uniqueness of the parameter μ . ■

The following statement shows the relation of the limit function (4.5) to the solution of the original three-point boundary value problem (1.1), (1.2).

DEFINITION 1. For any $k = 1, 2, \dots, n$, we define the n -dimensional row-vector e_k by putting

$$e_k = (\underbrace{0, 0, \dots, 0}_{k-1}, 1, 0, \dots, 0).$$

Let us consider the function $\Delta: \mathbb{R}^{3n-q} \rightarrow \mathbb{R}^n$ given by the formula

$$(5.20) \quad \begin{aligned} \Delta(z, \lambda, \eta) &:= \frac{1}{T} \left[C^{-1}d(\lambda, \eta) - \left(C^{-1}A + I_n \right) z \right] \\ &\quad - \frac{1}{T} \int_0^T [P_0(s)x^*(s, z, \lambda, \eta) + P_1(s)x^*(\beta(s), z, \lambda, \eta) + f(s)] ds \end{aligned}$$

for all $z \in \mathbb{R}^n, \lambda \in \mathbb{R}^n, \eta \in \mathbb{R}^{n-q}$, where x^* is the limit function (4.5).

THEOREM 2. *Assume the conditions of Theorem 1. Then the function $x^*(\cdot, z, \lambda, \eta)$ is a solution of the three-point boundary value problem (1.1), (1.2) if and only if the triplet z, λ, η satisfies the system of $3n - q$ algebraic equations*

$$(5.21) \quad \Delta(z, \lambda, \eta) = 0,$$

$$(5.22)$$

$$e_1 x^*(\xi, z, \lambda, \eta) = \lambda_1, e_2 x^*(\xi, z, \lambda, \eta) = \lambda_2, \dots, e_n x^*(\xi, z, \lambda, \eta) = \lambda_n,$$

$$(5.23) \quad e_{q+1} x^*(T, z, \lambda, \eta) = \eta_1, \dots, e_n x^*(T, z, \lambda, \eta) = \eta_{n-q}.$$

PROOF. It is sufficient to apply Proposition 2 and notice that the differential equation in (5.1) coincides with (1.1) if and only if the triplet z, λ, η satisfies (5.21). On the other hand, equations (5.22) and (5.23) bring us from the auxiliary two-point parametrized conditions to the three-point conditions (1.2). ■

Let us define the matrix R by putting

$$R := \sup_{t \in [0, T]} \left| I_n - \frac{t}{T} \left(C^{-1}A + I_n \right) \right|,$$

and set

$$h(\eta, \lambda) := C^{-1} \text{col}(0, \dots, 0, \eta_1, \dots, \eta_{n-q}) - C^{-1}B \text{col}(\lambda_1, \dots, \lambda_n)$$

for all $\eta \in \mathbb{R}^{n-q}$ and $\lambda \in \mathbb{R}^n$.

PROPOSITION 3. Under the assumptions of Theorem 1, the limit function (4.5) of sequence (4.1) satisfies, in the variables z, λ, η , the Lipschitz condition

$$(5.24) \quad \begin{aligned} & \left| x^* \left(t, z^0, \lambda^0, \eta^0 \right) - x^* \left(t, z^1, \lambda^1, \eta^1 \right) \right| \leq \\ & \leq (I_n - G)^{-1} \left[R \left| z^0 - z^1 \right| + \left| h(\eta^0 - \eta^1, \lambda^0 - \lambda^1) \right| \right], \end{aligned}$$

$$(5.25) \quad \begin{aligned} & \left| x^* \left(\beta(t), z^0, \lambda, \eta \right) - x^* \left(\beta(t), z^1, \lambda, \eta \right) \right| \leq \\ & \leq (I_n - G)^{-1} \left[R \left| z^0 - z^1 \right| + \left| h(\eta^0 - \eta^1, \lambda^0 - \lambda^1) \right| \right], \end{aligned}$$

where $\{z^0, \lambda^0, \eta^0\}, \{z^1, \lambda^1, \eta^1\}$ and $t \in [0, T]$ are arbitrary and the matrix G is given by formula (4.19).

PROOF. It follows immediately from (4.1) that

$$\begin{aligned} & x_1 \left(t, z^0, \lambda^0, \eta^0 \right) - x_1 \left(t, z^1, \lambda^1, \eta^1 \right) = \\ & = z^0 - z^1 + \left(1 - \frac{t}{T} \right) \int_0^t \left[P_0(s) (z^0 - z^1) + P_1(s) (z^0 - z^1) \right] ds - \\ & - \frac{t}{T} \int_t^T \left[P_0(s) (z^0 - z^1) + P_1(s) (z^0 - z^1) \right] ds - \\ & - \frac{t}{T} \left(C^{-1} A + I_n \right) (z^0 - z^1) + \frac{t}{T} h(\eta^0 - \eta^1, \lambda^0 - \lambda^1). \end{aligned}$$

Using identity (3.1) and taking relations (4.3), (3.2), and (3.3) into account, we obtain

$$(5.26) \quad \begin{aligned} & \left| x_1 \left(t, z^0, \lambda^0, \eta^0 \right) - x_1 \left(t, z^1, \lambda^1, \eta^1 \right) \right| \leq \\ & \leq R \left| z^0 - z^1 \right| + \left| h(\eta^0 - \eta^1, \lambda^0 - \lambda^1) \right| + \\ & + \left[\left(1 - \frac{t}{T} \right) \int_0^t ds + \frac{t}{T} \int_t^T ds \right] [K_0 + K_1] \left| z^0 - z^1 \right| = \\ & = [R + \alpha_1(t)(K_0 + K_1)] \left| z^0 - z^1 \right| + \left| h(\eta^0 - \eta^1, \lambda^0 - \lambda^1) \right| \leq \\ & \leq \left[R + \frac{T}{2}(K_0 + K_1) \right] \left| z^0 - z^1 \right| + \left| h(\eta^0 - \eta^1, \lambda^0 - \lambda^1) \right|. \end{aligned}$$

for all for $t \in [0, T]$. It follows from (5.26) that, in particular,

$$(5.27) \quad \begin{aligned} & \left| x_1 \left(\beta(t), z^0, \lambda^0, \eta^0 \right) - x_1 \left(\beta(t), z^1, \lambda^1, \eta^1 \right) \right| \leq \\ & \leq \left[R + \frac{T}{2} (K_0 + K_1) \right] \left| z^0 - z^1 \right| + \left| h(\eta^0 - \eta^1, \lambda^0 - \lambda^1) \right| \end{aligned}$$

for a.e. $t \in [0, T]$. Similarly, using (4.1), (3.1), (3.3), (5.26), and (5.27), we get

$$\begin{aligned} & \left| x_2 \left(t, z^0, \lambda^0, \eta \right) - x_2 \left(t, z^1, \lambda^1, \eta^1 \right) \right| \leq R \left| z^0 - z^1 \right| + \left| h(\eta^0 - \eta^1, \lambda^0 - \lambda^1) \right| + \\ & + [K_0 + K_1] \left[\left(1 - \frac{t}{T} \right) \int_0^t \left\{ \left[R + \frac{T}{2} (K_0 + K_1) \right] \left| z^0 - z^1 \right| + \right. \right. \\ & \quad \left. \left. + \left| h(\eta^0 - \eta^1, \lambda^0 - \lambda^1) \right| \right\} ds + \right. \\ & \quad \left. + \frac{t}{T} \int_t^T \left\{ \left[R + \frac{T}{2} (K_0 + K_1) \right] \left| z^0 - z^1 \right| + \left| h(\eta^0 - \eta^1, \lambda^0 - \lambda^1) \right| \right\} ds \right] \leq \\ & \leq R \left| z^0 - z \right| + \left| h(\eta^0 - \eta^1, \lambda^0 - \lambda^1) \right| \\ & + (K_0 + K_1) \left\{ \left[R + \frac{T}{2} (K_0 + K_1) \right] \alpha_1(t) \left| z^0 - z^1 \right| + \right. \\ & \quad \left. + \left| h(\eta^0 - \eta^1, \lambda^0 - \lambda^1) \right| \alpha_1(t) \right\} \leq \\ & \leq \left[R + \frac{T}{2} (K_0 + K_1) R + \left[\frac{T}{2} (K_0 + K_1) \right]^2 \right] \left| z^0 - z^1 \right| + \\ & + \frac{T}{2} (K_0 + K_1) \left| h(\eta^0 - \eta^1, \lambda^0 - \lambda^1) \right| + \left| h(\eta^0 - \eta^1, \lambda^0 - \lambda^1) \right|. \end{aligned}$$

From these relations we conclude by induction that

$$\begin{aligned} & \left| x_m \left(t, z^0, \lambda^0, \eta^0 \right) - x_m \left(t, z^1, \lambda^1, \eta^1 \right) \right| \leq \\ & \leq \left[R + \sum_{i=1}^{m-1} \left[\frac{T}{2} (K_0 + K_1) \right]^i R + \left[\frac{T}{2} (K_0 + K_1) \right]^m \right] \left| z^0 - z^1 \right| + \\ & + \sum_{i=0}^{m-1} \left[\frac{T}{2} (K_0 + K_1) \right]^i \left| h(\eta^0 - \eta^1, \lambda^0 - \lambda^1) \right| \leq \end{aligned}$$

$$(5.28) \quad \begin{aligned} & \leq \left[\sum_{i=0}^{m-1} \left[\frac{T}{2} (K_0 + K_1) \right]^i R + \left[\frac{T}{2} (K_0 + K_1) \right]^m \right] |z^0 - z^1| + \\ & + \sum_{i=0}^{m-1} \left[\frac{T}{2} (K_0 + K_1) \right]^i |h(\eta^0 - \eta^1, \lambda^0 - \lambda^1)| \end{aligned}$$

for any $t \in [0, T]$. Passing to the limit as $m \rightarrow \infty$ in (5.28) and using the estimate (4.2) we obtain the inequalities

$$\begin{aligned} & |x^*(t, z^0, \lambda^0, \eta^0) - x^*(t, z^1, \lambda^1, \eta^1)| \leq \\ & \leq \left[\sum_{i=0}^{\infty} \left[\frac{T}{2} (K_0 + K_1) \right]^i R + \left[\frac{T}{2} (K_0 + K_1) \right]^{\infty} \right] |z^0 - z^1| + \\ & + \sum_{i=0}^{\infty} \left[\frac{T}{2} (K_0 + K_1) \right]^i |h(\eta^0 - \eta^1, \lambda^0 - \lambda^1)| \leq \\ & \leq (I_n - G)^{-1} R |z^0 - z^1| + (I_n - G)^{-1} |h(\eta^0 - \eta^1, \lambda^0 - \lambda^1)|, \end{aligned}$$

whence the inequalities (5.24) and (5.25) follow. ■

Now we establish some properties of the so called determining function $\Delta: \mathbb{R}^{3n-q} \rightarrow \mathbb{R}^n$ given by (5.20).

PROPOSITION 4. *Under the conditions of Theorem 1, formula (5.20) determines a well-defined function $\Delta: \mathbb{R}^{3n-q} \rightarrow \mathbb{R}^n$, which satisfies the estimate*

$$\begin{aligned} & |\Delta(z^0, \lambda^0, \eta^0) - \Delta(z^1, \lambda^1, \eta^1)| \leq \\ & \leq \left[\left| \frac{1}{T} (C^{-1} A + I_n) \right| + (K_0 + K_1) [(I_n - G)^{-1} R] \right] |z^0 - z^1| + \\ & + \frac{1}{T} |h(\eta^0 - \eta^1, \lambda^0 - \lambda^1)| + (K_0 + K_1) (I_n - G)^{-1} |h(\eta^0 - \eta^1, \lambda^0 - \lambda^1)|. \end{aligned}$$

PROOF. According (5.20) and (4.3)

$$\begin{aligned} & \Delta(z^0, \lambda^0, \eta^0) - \Delta(z^1, \lambda^1, \eta^1) = \\ & = \frac{1}{T} h(\eta^0 - \eta^1, \lambda^0 - \lambda^1) - \frac{1}{T} (C^{-1} A + I_n) (z^0 - z^1) + \end{aligned}$$

$$\begin{aligned}
& + \frac{1}{T} \int_0^T [P_0(s) \left(x^* \left(s, z^1, \lambda^1, \eta^1 \right) - x^* \left(s, z^0, \lambda^0, \eta^0 \right) \right) + \\
& + P_1(s) \left(x^* \left(\beta(s), z^1, \lambda^1, \eta^1 \right) - x^* \left(\beta(s), z^0, \lambda^0, \eta^0 \right) \right)] ds,
\end{aligned}$$

and, therefore,

$$\begin{aligned}
& \left| \Delta \left(z^0, \lambda^0, \eta^0 \right) - \Delta \left(z^1, \lambda^1, \eta^1 \right) \right| \leq \\
& \leq \left| \frac{1}{T} \left(C^{-1} A + I_n \right) \right| \left| z^0 - z^1 \right| + \frac{1}{T} \left| h(\eta^0 - \eta^1, \lambda^0 - \lambda^1) \right| + \\
& + \frac{1}{T} \int_0^T [K_0 \left| x^* \left(s, z^1, \lambda^1, \eta^1 \right) - x^* \left(s, z^0, \lambda^0, \eta^0 \right) \right| + \\
& + K_2 \left| x^* \left(\beta(s), z^1, \lambda^1, \eta^1 \right) - x^* \left(\beta(s), z^0, \lambda^0, \eta^0 \right) \right|] ds,
\end{aligned} \tag{5.29}$$

where K_0 and K_1 are given by (4.3). Substituting (5.24), (5.25) into (5.29), we get

$$\begin{aligned}
& \left| \Delta(z^0, \lambda, \eta) - \Delta(z^1, \lambda, \eta) \right| \leq \\
& \leq \left| \frac{1}{T} \left(C^{-1} A + I_n \right) \right| \left| z^0 - z^1 \right| + \frac{1}{T} \left| h(\eta^0 - \eta^1, \lambda^0 - \lambda^1) \right| + \\
& + \frac{1}{T} \int_0^T (K_0 + K_1) (I_n - G)^{-1} \left[R \left| z^0 - z^1 \right| + \left| h(\eta^0 - \eta^1, \lambda^0 - \lambda^1) \right| \right] ds = \\
& = \left[\left| \frac{1}{T} \left(C^{-1} A + I_n \right) \right| + (K_0 + K_1) (I_n - G)^{-1} R \right] \left| z^0 - z^1 \right| + \\
& + \frac{1}{T} \left| h(\eta^0 - \eta^1, \lambda^0 - \lambda^1) \right| + (K_0 + K_1) (I_n - G)^{-1} \left| h(\eta^0 - \eta^1, \lambda^0 - \lambda^1) \right|,
\end{aligned}$$

which completes the proof. ■

Theorems 1 and 2 suggest the following numerical-analytic algorithm for the construction of the solution of the three-point boundary value problem (1.1), (1.2):

1. We analytically construct the sequence of functions $x_m(\cdot, z, \lambda, \eta)$ depending on the parameters z, λ, η and satysfying the auxiliary two-point boundary condition (2.5).

2. We find the limit $x^*(\cdot, z, \lambda, \eta)$ of the sequence $x_m(\cdot, z, \lambda, \eta)$, $m \geq 1$, satisfying (2.5).
3. We construct the algebraic determining system of the form (5.21), (5.22), (5.23) with respect to the $3n - q$ scalar parameters $z_1, \dots, z_n, \lambda_1, \dots, \lambda_n, \eta_1, \dots, \eta_{n-q}$.
4. Using a suitable method for the numerical solution of system (5.21), (5.22), (5.23), we (approximately) find a solution

$$(5.30) \quad \begin{aligned} z^* &= (z_1^*, \dots, z_n^*) \in \mathbb{R}^n, \quad \lambda^* = \text{col}(\lambda_1^*, \dots, \lambda_n^*) \in \mathbb{R}^n, \\ \eta^* &= \text{col}(\eta_1^*, \dots, \eta_{n-q}^*) \in \mathbb{R}^{n-q} \end{aligned}$$

of the determining system (5.21), (5.22), (5.23).

5. Substituting values (5.30) into $x^*(\cdot, z, \lambda, \eta)$, we get the solution of the original three-point boundary value problem (1.1), (1.2) in the form

$$(5.31) \quad x = x^*(t, z^*, \lambda^*, \eta^*).$$

This solution (5.31) can also be obtained by solving the Cauchy problem $x(0) = z^*$ for equation (1.1).

The fundamental difficulty in the realization of this approach is related to the analytic construction of the limit function $x^*(\cdot, z, \lambda, \eta)$. However, in a number of cases, this problem can be overcome because, as can be shown, it is possible to prove the existence of a solution of the three-point boundary value problem (1.1), (1.2) based on the properties of a certain approximation $x_m(\cdot, z, \lambda, \eta)$ known in its analytic form.

For $m \geq 1$, define the function $\Delta_m: \mathbb{R}^{3n-q} \rightarrow \mathbb{R}^n$ according to the formula

$$\begin{aligned} \Delta_m(z, \lambda, \eta) := & \frac{1}{T} \left[C^{-1}d(\lambda) - \left(C^{-1}A + I_n \right) z \right] - \\ & - \frac{1}{T} \int_0^T [P_0(s)x_m(s, z, \lambda, \eta) + P_1(s)x_m(\beta(s), z, \lambda, \eta) + f(s)] ds, \end{aligned}$$

for arbitrary z , λ , and η . To investigate the solvability of the three-point boundary value problem (1.1), (1.2), in addition to determining system (5.21), (5.22), (5.23), we introduce the m th approximate determining system

$$(5.32) \quad \Delta_m(z, \lambda, \eta) = 0,$$

$$(5.33) \quad e_1 x_m(\xi, z, \lambda, \eta) = \lambda_1, \quad e_2 x_m(\xi, z, \lambda, \eta) = \lambda_2, \quad \dots, \quad e_n x_m(\xi, z, \lambda, \eta) = \lambda_n,$$

$$(5.34) \quad e_{q+1}x_m(T, z, \lambda, \eta) = \eta_1, \dots, e_n x_m(T, z, \lambda, \eta) = \eta_{n-q}.$$

where e_i , $i = 1, 2, \dots, n$, are the vectors given by (5.20) and the vector function $x_m(\cdot, z, \lambda, \eta)$ is defined by formula (4.1).

It is natural to expect that, under suitable conditions, the systems (5.21), (5.22), (5.23) and (5.32), (5.33), (5.34) are “close enough” to one another for m sufficiently large. The existence result the three-point boundary value problem (1.1), (1.2) can be obtained by studying the solutions of the approximate determining system (5.32)–(5.34), in the case of periodic boundary conditions (see, e.g., [41]).

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**PH. D. THESIS
ON NONLINEAR SYSTEMS CONTAINING NONLOCAL TERMS**

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Abstract. We give an overview of [7]. Two systems of differential equations containing nonlocal terms are considered. By using the theory of monotone operators, existence of weak solutions is proved in $(0, T)$ for $0 < T \leq \infty$, further, the qualitative properties of solutions are investigated.

1. Introduction

The aim of the present paper is to give an overview of the author's dissertation [7]. In the following we shall present mostly the motivation and the main points of our investigations. For rigorous explanation of the results we always refer to the dissertation and the references therein.

We study systems of nonlinear parabolic differential equations containing nonlocal terms, in other words, functional differential equations. By "nonlocal term" we mean terms which may depend not only on the value of the unknown at a certain point but also on values at other points, for example, it may contain a delay or an integral of the unknown on a domain etc. Such problems may occur in some physical models. For instance, in some diffusion processes the diffusion coefficient may depend on a nonlocal quantity, e.g., in population dynamics the growing rate of a population may depend on the size of the population, mathematically, on the integral of the density, see [12, 13]. We mention two other important applications. First, climatology, see, e.g., [15]. Second, modelling of fluid flow, especially in porous media, see [14, 18]. For other nonlocal models such as transmission problems, or nonlocal boundary conditions, see the references of the dissertation. We note

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that instead of equations one may consider nonlocal variational inequalities. That type of problems occur in elasticity theory, see [4, 16].

In the following, we consider two systems of differential equations containing nonlocal terms. The first one consists of parabolic functional equations of general divergence form. The second one is a generalization of a system describing fluid flow in porous media and consists of three different types of differential equations.

The main tool of our further investigations will be the theory of operators of monotone type. For a detailed introduction to this theory and its applications, see [10, 17, 24]. In particular, we shall apply some results of [9, 11] related to pseudomonotone operators. For details on applications of monotone type operators to parabolic and functional parabolic partial differential equations we refer to [21].

Existence of weak solutions in time interval $(0, T)$ ($0 < T \leq \infty$) is shown for both systems, further, asymptotic properties are studied such as the boundedness and stabilization (i.e., convergence to equilibrium) of solutions. For examples illustrating our results we refer to our dissertation. It is worth mentioning the monographs [19, 23] which consider functional differential equations by means of semigroups.

2. Notation

Throughout this paper, $\Omega \subset \mathbb{R}^n$ will be a bounded domain with smooth boundary (e.g., C^1 is sufficient) and $0 < T < \infty$. We briefly write $Q_T = (0, T) \times \Omega$ and $Q_\infty = (0, \infty) \times \Omega$. In the sequel, $W^{1,p}(\Omega)$, $W_0^{1,p}(\Omega)$ denotes the usual Sobolev spaces, further, $L^p(0, T; V)$ will be the set of measurable functions $u: (0, T) \rightarrow V$ such that $\|u\|_{L^p(0, T; V)} := \left(\int_0^T \|u\|_V^p \right)^{\frac{1}{p}} < \infty$ ($1 < p < \infty$). It is well-known that $(L^p(0, T; V))^* = L^q(0, T; V^*)$ where $\frac{1}{p} + \frac{1}{q} = 1$ and V^* is the dual space of V . In addition, $L_{\text{loc}}^p(0, \infty; V)$ is the set of measurable functions $u: (0, \infty) \rightarrow V$ such that $u|_{(0, T)} \in L^p(0, T; V)$ for every $0 < T < \infty$. The pairing between $(W^{1,p}(\Omega))^*$ and $W^{1,p}(\Omega)$, further, between $L^q(0, T; V^*)$ and $L^p(0, T; V)$ is denoted by $\langle \cdot, \cdot \rangle$, $[\cdot, \cdot]$, respectively. Finally, D_i , D_t stand for the distributional differentiation with respect to x_i and t , respectively, and $D = (D_1, \dots, D_n)$.

3. A system of parabolic equations

Consider the following parabolic differential equation containing nonlocal term:

$$(3.1) \quad D_t u(t, x) - \operatorname{div} \left(g \left(\int_{\Omega} u(t, x) dx \right) Du(t, x) \right) = f(t, x)$$

for $t > 0$, $x \in \mathbb{R}^n$ where functions $f: (0, \infty) \times \mathbb{R}^n \rightarrow \mathbb{R}$, $g: \mathbb{R} \rightarrow \mathbb{R}$ are given and $u: (0, \infty) \times \mathbb{R}^n \rightarrow \mathbb{R}$ is the unknown with initial condition $u(0, x) = \varphi(x)$ for $x \in \mathbb{R}^n$. Such equation is motivated, e.g., by diffusion processes for heat or population. In [12, 13] quasilinear equations similar to (3.1) were considered and existence of solutions, further, asymptotic properties of solutions were shown. In [20] quasilinear parabolic functional equations of general divergence form were investigated by means of monotone operators. Existence of weak solutions and some qualitative properties of weak solutions were proved. These results are extended to systems of nonlocal parabolic equations in the first part of the dissertation which will be now briefly summarized (see [1]).

Now consider the following system of N nonlocal parabolic differential equations:

$$(3.2) \quad \begin{aligned} & D_t u^{(l)}(\cdot) - \\ & - \sum_{i=1}^n D_i \left[a_i^{(l)}(\cdot, u^{(1)}(\cdot), \dots, u^{(N)}(\cdot), Du^{(1)}(\cdot), \dots, Du^{(N)}(\cdot); u^{(1)}, \dots, u^{(N)}) \right] + \\ & + a_0^{(l)}(\cdot, u^{(1)}(\cdot), \dots, u^{(N)}(\cdot), Du^{(1)}(\cdot), \dots, Du^{(N)}(\cdot); u^{(1)}, \dots, u^{(N)}) = f^{(l)}(\cdot) \end{aligned}$$

where (\cdot) refers to the variable $(t, x) \in Q_T$ and the terms after the symbol “;” represent the nonlocal variables ($l = 1, \dots, N$). We may pose, for simplicity, homogeneous initial condition, further, boundary conditions of homogeneous Dirichlet or Neumann type.

Fix $p \geq 2$ and denote by V a closed linear subspace of $(W^{1,p}(\Omega))^N$ (determined by the boundary condition, e.g., in case of homogeneous Neumann type $V = (W^{1,p}(\Omega))^N$, in case of homogeneous Dirichlet type $V = (W_0^{1,p}(\Omega))^N$). Let $X = L^p(0, T; V)$ which will be the space of weak solutions. A function $v \in X$ has its coordinate-functions $(v^{(1)}, \dots, v^{(N)})$, vectors $\xi \in \mathbb{R}^{(n+1)N}$ have the form $\xi = (\xi_0, \xi)$ where $\xi_0 = (\xi_0^{(1)}, \dots, \xi_0^{(N)}) \in \mathbb{R}^N$ and $\xi = (\xi_1^{(1)}, \dots, \xi_1^{(N)}, \dots, \xi_n^{(1)}, \dots, \xi_n^{(N)}) \in \mathbb{R}^{nN}$ (the sub-indeces indicate the variable of differentiation, the super-indeces the actual coordinate-function).

We pose some natural assumptions on the above functions $a_i^{(l)}$ to obtain existence of weak solutions in $(0, T)$ to system (3.2). For $i = 0, \dots, n$; $l = 1, \dots, N$, suppose:

- (A1) Function $a_i^{(l)}: Q_T \times \mathbb{R}^{(n+1)N} \times X \rightarrow \mathbb{R}$ has the Carathéodory property for every fixed $v \in X$, i.e., it is measurable in (t, x) for every $(\xi_0, \xi) \in \mathbb{R}^{(n+1)N}$ and continuous in (ξ_0, ξ) for a.e. $(t, x) \in Q_T$.
- (A2) There exist bounded operators $g_1: X \rightarrow \mathbb{R}^+$ and $k_1: X \rightarrow L^q(Q_T)$ such that $|a_i^{(l)}(t, x, \xi_0, \xi; v)| \leq g_1(v) (\|\xi_0\|^{p-1} + \|\xi\|^{p-1}) + [k_1(v)](t, x)$ holds for a.e. $(t, x) \in Q_T$, every $(\xi_0, \xi) \in \mathbb{R}^{(n+1)N}$ and $v \in X$.
- (A3) For a.e. $(t, x) \in Q_T$, every $\xi \neq \tilde{\xi} \in \mathbb{R}^{nN}$, $\xi_0 \in \mathbb{R}^N$ and $v \in X$,

$$\sum_{l=1}^N \sum_{i=1}^n \left(a_i^{(l)}(t, x, \xi_0, \xi; v) - a_i^{(l)}(t, x, \xi_0, \tilde{\xi}; v) \right) (\xi_i^{(l)} - \tilde{\xi}_i^{(l)}) > 0.$$

- (A4) There exist operators $g_2: X \rightarrow \mathbb{R}^+$ and $k_2: X \rightarrow L^1(Q_T)$ such that for a.e. $(t, x) \in Q_T$, every $(\xi_0, \xi) \in \mathbb{R}^{(n+1)N}$ and $v \in X$,

$$\sum_{l=1}^N \sum_{i=0}^n a_i^{(l)}(t, x, \xi_0, \xi; v) \xi_i^{(l)} \geq g_2(v) (\|\xi_0\|^p + \|\xi\|^p) - [k_2(v)](t, x),$$

$$\text{further, } \lim_{\|v\|_X \rightarrow \infty} \left(g_2(v) \|v\|_X^{p-1} - \|k_2(v)\|_{L^1(Q_T)} \|v\|_X^{-1} \right) = +\infty.$$

- (A5) If $u_k \rightarrow u$ weakly in X and strongly in $L^p(0, T; (L^p(\Omega))^N)$ then

$$\lim_{k \rightarrow \infty} \|a_i^{(l)}(\cdot, u_k(\cdot), Du_k(\cdot); u_k) - a_i^{(l)}(\cdot, u_k(\cdot), Du_k(\cdot); u)\|_{L^q(Q_T)} = 0.$$

Conditions (A1)–(A4) are similar to the classical case when there is no nonlocality, see [11, 17, 24], (A2)–(A4) represent growth, monotonicity, coercivity. Besides these, (A5) means a kind of “continuity” in the nonlocal variable.

Now let us introduce operator $A: X \rightarrow X^*$ as follows. For $u, v \in X$ define

$$[A(u), v] := \sum_{l=1}^N \int_{Q_T} \left(\sum_{i=1}^n a_i^{(l)}(u, Du; u) D_l v^{(l)} + a_0^{(l)}(u, Du; u) v^{(l)} \right).$$

Further, let $D(L) = \{u \in X : D_t u \in X^*, u(0) = 0\}$ and define $L: D(L) \rightarrow X^*$ as $Lu = D_t u$. Finally, let $F \in X^*$. By the operators above, the weak form of system (3.2) in $(0, T)$ is

$$(3.3) \quad Lu + A(u) = F.$$

By applying the theory of pseudomonotone operators (see [9]) we may prove

THEOREM 3.1. *Assume that conditions (A1)–(A5) hold. Then operator $A: X \rightarrow X^*$ is bounded, demicontinuous, coercive and pseudomonotone with respect to $D(L)$. Consequently, for every $F \in X^*$ there exists a solution $u \in X$ of problem (3.3).*

It is not so difficult to show existence of weak solutions to (3.2) in $(0, \infty)$. Let $X^\infty = L_{\text{loc}}^p(0, \infty; V)$ which will be the space of weak solutions in $(0, \infty)$ and assume

(Vol) The restrictions $a_i^{(l)}(t, x, \xi_0, \xi; v)|_{(0, T)}$ of functions $a_i^{(l)}: Q_\infty \times \mathbb{R}^{n+1} \times X^\infty \rightarrow \mathbb{R}$ ($i = 0, \dots, n$; $l = 1, \dots, N$) depend only on $v|_{(0, T)}$ for every $0 < T < \infty$.

The above condition is called Volterra property which means, roughly speaking, that the nonlocality does not depend on the future. Now by using a “diagonal method” one obtains

THEOREM 3.2. *Assume (Vol) and suppose that conditions (A1)–(A5) hold in $(0, \infty)$ in the sense that they are satisfied by the restrictions of functions $a_i^{(l)}$ to $(0, T)$ for all $0 < T < \infty$. Then there exists $u \in X^\infty$ which is a weak solution of (3.2) in $(0, \infty)$ in the sense that $u|_{(0, T)}$ is a solution of problem (3.3) for every $0 < T < \infty$.*

Boundedness of solutions in $(0, \infty)$ follows by posing extra coercivity conditions:

(A4*) There exist a constant $g_2 > 0$ and a Volterra operator $k_2: X^\infty \rightarrow L_{\text{loc}}^1(Q_\infty)$ such that for a.e. $(t, x) \in Q_\infty$, every $(\xi_0, \xi) \in \mathbb{R}^{(n+1)N}$ and $v \in X^\infty$,

$$\sum_{l=1}^N \sum_{i=0}^n a_i^{(l)}(t, x, \xi_0, \xi; v) \xi_i^{(l)} \geq g_2 (|\xi_0|^p + |\xi|^p) - [k_2(v)](t, x),$$

where k_2 satisfies the following conditions: there exist constants $c_4 > 0$, $0 \leq p_1 < p$ and a function $\varphi \in C(\mathbb{R}^+)$ such that $\lim_{\tau \rightarrow \infty} \varphi(\tau) = 0$, further, if $v \in X^\infty$ and $D_t v \in L_{\text{loc}}^q(0, \infty; V^*)$ then for a.e. $t > 0$,

$$\begin{aligned} & \int_{\Omega} |[k_2(v)](t, x)| dx \leq \\ & \leq c_4 \left(\sup_{\tau \in [0, t]} \|v(\tau)\|_{(L^2(\Omega))^N}^{p_1} + \varphi(t) \cdot \sup_{\tau \in [0, t]} \|v(\tau)\|_{(L^2(\Omega))^N}^p + 1 \right). \end{aligned}$$

THEOREM 3.3. *Assume (Vol), further, conditions (A1)–(A5) are satisfied in $(0, \infty)$ (in the same sense as in Theorem 3.2) with extra assumptions (A4*) and $F \in L^\infty(0, \infty; V^*)$. Then $u \in L^\infty(0, \infty; (L^2(\Omega))^N)$ for the solutions u formulated in Theorem 3.2.*

With some further assumptions stabilization of solutions as $t \rightarrow \infty$ follows.

(A2⁺) For every fixed $v \in X^\infty \cap L^\infty(0, \infty; (L^2(\Omega))^N)$ there exists $c_v > 0$ and $k_v \in L^q(\Omega)$ such that

$$|a_i^{(l)}(t, x, \xi_0, \xi; v)| \leq c_v \left(|\xi_0|^{p-1} + |\xi|^{p-1} \right) + k_v(x)$$

for a.e. $x \in \Omega$ and every $(\xi_0, \xi) \in \mathbb{R}^{(n+1)N}$ ($i = 0, \dots, n$; $l = 1, \dots, N$).

(A6) There exist Carathéodory functions $a_{i,\infty}^{(l)} : \Omega \times \mathbb{R}^{(n+1)N} \rightarrow \mathbb{R}$ such that for every fixed $v \in X^\infty \cap L^\infty(0, \infty; (L^2(\Omega))^N)$, for a.e. $x \in \Omega$ and every $(\xi_0, \xi) \in \mathbb{R}^{(n+1)N}$, $\lim_{t \rightarrow \infty} a_i^{(l)}(t, x, \xi_0, \xi; v) = a_{i,\infty}^{(l)}(x, \xi_0, \xi)$.

(A7) There exists $c_5 > 0$ such that for a.e. $x \in \Omega$, every $(\xi_0, \xi), (\tilde{\xi}_0, \tilde{\xi}) \in \mathbb{R}^{(n+1)N}$, $v \in X^\infty$,

$$\begin{aligned} & \sum_{l=1}^N \sum_{i=0}^n \left(a_i^{(l)}(t, x, \xi_0, \xi; v) - a_i^{(l)}(t, x, \tilde{\xi}_0, \tilde{\xi}; v) \right) (\xi_i^{(l)} - \tilde{\xi}_i^{(l)}) \\ & \geq c_5 \left(|\xi_0 - \tilde{\xi}_0|^p + |\xi - \tilde{\xi}|^p \right) - k_3(t, x, \xi_0, \tilde{\xi}_0; v) \end{aligned}$$

where

$$\lim_{t \rightarrow \infty} \int_{\Omega} k_3(t, x, u(t, x), \tilde{u}(t, x); v) dx = 0$$

if $u, \tilde{u}, v \in L^\infty(0, \infty; (L^2(\Omega))^N)$.

Note that (A6) is a natural assumption that means the stabilization of the functions $a_i^{(l)}$ as $t \rightarrow \infty$, further, (A7) is a uniform monotonicity condition which provides existence of a unique equilibrium state. Now define operator $A_\infty: V \rightarrow V^*$, for $v, w \in V$ let

$$\langle A_\infty(v), w \rangle := \sum_{l=1}^N \int_{\Omega} \left(\sum_{i=1}^n a_{i,\infty}^{(l)}(x, v, Dv) D_i w^{(l)} + a_{0,\infty}^{(l)}(x, v, Dv) w^{(l)} \right) dx.$$

THEOREM 3.4. *Assume (Vol), further, that (A1)–(A7) are satisfied in $(0, \infty)$ (in the same sense as in Theorem 3.2) and there exists $F_\infty \in V^*$ such that $\lim_{t \rightarrow \infty} \|F(t) - F_\infty\|_{V^*} = 0$. Then there exists a unique $u_\infty \in V$ such that $A_\infty(u_\infty) = F_\infty$. In addition, if u is a solution formulated in Theorem 3.2 then $\lim_{t \rightarrow \infty} \|u(t) - u_\infty\|_{(L^2(\Omega))^N} = 0$.*

By posing polynomial estimates on the “speed” of the convergences in condition (A6), one may obtain polynomial estimates for the “speed” of the convergences stated in Theorem 3.4. For some examples satisfying the conditions of the above theorems, see [1, 5].

For system (3.2) it is not quite clear how to define the notion of periodic solutions, however, for a modified system it is possible and one may show existence of them.

4. A system containing three types of equations

The second part of the dissertation is devoted to the investigation of a system which consists of three different types of differential equations: a first order ordinary, a parabolic and an elliptic one. This kind of problem is motivated by a fluid flow model in porous medium. A porous medium is a solid medium with lots of tiny holes (e.g., limestone). The flow of a fluid through the medium is determined by the large surface of the solid matrix and the closeness of the holes. If the fluid carries chemical species, chemical reactions can occur which can change the porosity (i.e. the proportion of the holes). In [18] this process was modelled by the following system in one dimension:

$$(4.1) \quad \omega D_t u = D_x (\alpha |v| D_x u) + K(\omega) D_x p \cdot D_x u - k u g(\omega)$$

$$(4.2) \quad D_t \omega = b u g(\omega)$$

$$(4.3) \quad D_x (K(\omega) D_x p) = b u g(\omega),$$

$$(4.4) \quad v = -K(\omega) D_x p$$

in $(0, \infty) \times (0, 1)$ with initial and boundary conditions $u(0, x) = u_0(x)$, $\omega(0, x) = \omega_0(x)$ for $x \in (0, 1)$, $u(t, 0) = u_1(t)$, $D_x u(t, 1) = 0$ for $t > 0$ and $p(t, 0) = 1$, $p(t, 1) = 0$ for $t > 0$ where ω is the porosity, u is the concentration of the dissolved chemical solute carried by the fluid, p is the pressure, v is the velocity, further, α , k , b are given constants, K and g are given real functions. Observe that v is explicitly given by ω and p in equation (4.4) thus we may eliminate equation (4.4) by substituting it into (4.1). Further, for fixed u equation (4.2) is an ordinary differential equation with respect to the function ω ; for fixed ω and p equation (4.1) is of parabolic type with respect to the function u ; and for fixed ω and u equation (4.3) is of elliptic type with respect to the function p . This argument shows that the above system is a hybrid evolutionary/elliptic problem. In [14] a similar model was considered by using the method of Rothe. We note that in [22] a system consisting of a parabolic and a first order ordinary differential equations was studied where the parabolic equation is not uniformly parabolic (in the sense that condition (A4) in Section 1 is satisfied with an operator g_2 depending also on time t).

The following generalization of the above system was considered in the second part of the dissertation (see [2, 3, 6]):

$$(4.5) \quad D_t \omega(t, x) = f(t, x, \omega(t, x), u(t, x), u), \quad \omega(0, x) = \omega_0(x),$$

$$D_t u(t, x) -$$

$$(4.6) \quad - \sum_{i=1}^n D_i [a_i(t, x, \omega(t, x), u(t, x), Du(t, x), \mathbf{p}(t, x), D\mathbf{p}(t, x); \omega, u, \mathbf{p})]$$

$$+ a_0(t, x, \omega(t, x), u(t, x), Du(t, x), \mathbf{p}(t, x), D\mathbf{p}(t, x); \omega, u, \mathbf{p}) = g(t, x),$$

$$(4.7) \quad \sum_{i=1}^n D_i [b_i(t, x, \omega(t, x), u(t, x), \mathbf{p}(t, x), D\mathbf{p}(t, x); \omega, u, \mathbf{p})]$$

$$+ b_0(t, x, \omega(t, x), u(t, x), \mathbf{p}(t, x), D\mathbf{p}(t, x); \omega, u, \mathbf{p}) = h(t, x)$$

with initial conditions $\omega(0, x) = \omega_0(x)$, $u(0, x) = 0$ and boundary conditions of homogeneous Dirichlet or Neumann type (\mathbf{p} is written by boldface letter in order to distinguish it from exponents). Existence and some qualitative properties of weak solutions is proved by using the theory of operators of monotone type (see [2, 3, 6]).

Let $2 \leq p_1, p_2 < \infty$, V_i be a closed linear subspace of $W^{1,p_i}(\Omega)$ (depending on the boundary conditions), $X_i = L^{p_i}(0, T; V_i)$ ($i = 1, 2$) which will be the space of weak solutions u and \mathbf{p} , respectively. According to its original physical meaning, the space of ω will be $L^\infty(Q_T)$. Note that the choice of the space of weak solutions for equation (4.7) is $L^{p_2}(0, T; V_i)$ which

is not the conventional space for an elliptic problem since now we also have time dependence.

We sketch the assumptions made on functions a_i, b_i analogously to Section 1, further, we add the assumptions made on f .

- (A) Functions a_i : Carathéodory, growth, “monotonicity”, coercivity, “continuity” in the nonlocal variables; conditions determined by exponent p_1 .
- (B) Functions b_i : Carathéodory, growth, “monotonicity”, coercivity, “continuity” in the nonlocal variables; conditions determined by exponent p_2 .
- (F) Function f : Carathéodory, Lipschitz, “continuity” in the nonlocal variable, “sign” condition (attractive steady-state).

Now define operators

$$A: L^\infty(Q_T) \times X_1 \times X_2 \rightarrow X_1^*, \quad B: L^\infty(Q_T) \times X_1 \times X_2 \rightarrow X_2^* :$$

$$[A(\omega, u, \mathbf{p}), v_1] =$$

$$= \int_{Q_T} \left(\sum_{i=1}^n a_i(\omega, u, Du, \mathbf{p}, D\mathbf{p}; \omega, u, \mathbf{p}) D_i v_1 + a_0(\omega, u, Du, \mathbf{p}, D\mathbf{p}; \omega, u, \mathbf{p}) v_1 \right),$$

$$[B(\omega, u, \mathbf{p}), v_2] =$$

$$= \int_{Q_T} \left(\sum_{i=1}^n b_i(\omega, u, \mathbf{p}, D\mathbf{p}; \omega, u, \mathbf{p}) D_i v_2 + b_0(\omega, u, \mathbf{p}, D\mathbf{p}; \omega, u, \mathbf{p}) v_2 \right),$$

for $v_i \in X_i$ ($i = 1, 2$). In addition, let $L : D(L) \rightarrow X_1^*$ be the operator of differentiation $Lu = D_t u$ with its domain $D(L) = \{u \in X_1 : D_t u \in X_1^*, u(0) = 0\}$.

By the operators above, the weak form of (4.5)–(4.7) in $(0, T)$ is defined as

$$(4.8) \quad \omega(t, x) = \omega_0(x) + \int_0^t f(s, x, \omega(s, x), u(s, x); u) ds \quad \text{a.e. in } Q_T$$

$$(4.9) \quad Lu + A(\omega, u, \mathbf{p}) = G$$

$$(4.10) \quad B(\omega, u, \mathbf{p}) = H.$$

where $G \in X_1^*$, $H \in X_2^*$. Our result on existence of solutions is (see [2, 6])

THEOREM 4.1. *Suppose that conditions (A), (B), (F) hold. Then for every $\omega_0 \in L^\infty(\Omega)$, $G \in X_1^*$ and $H \in X_2^*$ there exists a solution $\omega \in L^\infty(Q_T)$, $u \in D(L)$, $\mathbf{p} \in X_2$ of (4.8)–(4.10).*

For a proof based on the Schauder fixed point theorem we refer to [8].

As in Section 1, it is not difficult to show existence of weak solutions in $(0, \infty)$ by supposing the Volterra property (see [3, 6]). Let $X_i^\infty := L_{\text{loc}}^{p_i}(0, \infty; V_i)$ ($i = 1, 2$).

THEOREM 4.2. *Assume that functions a_i, b_i, f have the Volterra property, further, the conditions of Theorem 4.1 are satisfied for all $0 < T < \infty$. Then for every $G \in L_{\text{loc}}^{q_1}(0, \infty; V_1^*)$, $H \in L_{\text{loc}}^{q_2}(0, \infty; V_2^*)$ there exist $\omega \in L^\infty(Q_\infty)$, $u \in X_1^\infty$, $\mathbf{p} \in X_2^\infty$ such that their restriction to $(0, T)$ satisfies (4.8)–(4.10) for every $0 < T < \infty$.*

One may obtain results also on the long-time behaviour of solutions (see [3, 6]). First, by posing extra coercivity assumptions on functions a_i, b_i (analogously to condition (A4*) in Section 1) boundedness of weak solutions follows.

THEOREM 4.3. *Assume that the conditions of Theorem 4.2 are satisfied with some additional assumptions on coercivity (see [3, 6]) and let $G \in L^\infty(0, \infty; V_1^*)$, $H \in L^\infty(0, \infty; V_2^*)$. Then for the solutions ω, u, \mathbf{p} formulated in Theorem 4.2,*

$$\omega \in L^\infty(Q_\infty), u \in L^\infty(0, \infty; L^2(\Omega)), \mathbf{p} \in L^\infty(0, \infty; V_2)$$

hold.

In case $p_1 = p_2 = p$ and $X_1^\infty = X_2^\infty = X^\infty$ we have results on stabilization of solutions as $t \rightarrow \infty$. By assuming the stabilization of functions a_i, b_i, f (analogously to (A6), (B6) in Section 1) we may introduce operators $A_\infty, B_\infty: L^\infty(\Omega) \times V \times V \rightarrow V^*$ as follows

$$\langle A_\infty(\omega, u, \mathbf{p}), v \rangle :=$$

$$:= \int_{\Omega} \left(\sum_{i=1}^n a_{i,\infty}(x, \omega, u, Du, \mathbf{p}, D\mathbf{p}) D_i v + a_{0,\infty}(x, \omega, u, Du, \mathbf{p}, D\mathbf{p}) v \right) dx,$$

$$\langle B_\infty(\omega, u, \mathbf{p}), v \rangle :=$$

$$:= \int_{\Omega} \left(\sum_{i=1}^n b_{i,\infty}(x, \omega, u, \mathbf{p}, D\mathbf{p}) D_i v + b_{x,0,\infty}(\omega, u, \mathbf{p}, D\mathbf{p}) v \right) dx.$$

By adding some extra conditions on uniformy monotonicity (similary to (A7)) and on the exponential stability of the steady-state of equation (4.8), one may verify

THEOREM 4.4. *Assume that the conditions of Theorem 4.3 are satisfied with some additional assumptions (see [3]). Further, suppose that*

$$\lim_{t \rightarrow \infty} \|G(t) - G_\infty\|_{V^*} = 0, \quad \lim_{t \rightarrow \infty} \|H(t) - H_\infty\|_{V^*} = 0$$

hold for some $G_\infty, H_\infty \in V^$. Then there exist unique $u_\infty, \mathbf{p}_\infty \in V$ such that $A_\infty(\omega^*, u_\infty, \mathbf{p}_\infty) = G_\infty$, $B_\infty(\omega^*, u_\infty, \mathbf{p}_\infty) = H_\infty$ where ω^* is the steady-state of (4.8). Further, if ω, u, \mathbf{p} are solutions formulated in Theorem 4.2 then $\omega(t, \cdot) \rightarrow \omega^*$ in $L^\infty(\Omega)$, $u(t) \rightarrow u_\infty$ in $L^2(\Omega)$, $\int_{t-1}^{t+1} \|u(s) - u_\infty\|_V^p ds \rightarrow 0$, $\int_{t-1}^{t+1} \|\mathbf{p}(s) - \mathbf{p}_\infty\|_V^p ds \rightarrow 0$ as $t \rightarrow \infty$.*

Similarly to Section 1 one may obtain estimates on the convergences stated in the above theorem. For some examples, see [2, 3, 5, 6].

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