

# ANNALES UNIVERSITATIS SCIENTIARUM BUDAPESTINENSIS DE ROLANDO EÖTVÖS NOMINATAE

SECTIO MATHEMATICA

TOMUS LII.

REDIGIT  
Á. CSÁSZÁR

ADIUVANTIBUS

L. BABAI, A. BENCZÚR, K. BEZDEK., M. BOGNÁR, K. BÖRÖCZKY,  
I. CSISZÁR, J. DEMETROVICS, [GY. ELEKÉS], A. FRANK, E. FRIED,  
J. FRITZ, V. GROLMUSZ, A. HAJNAL, G. HALASZ, A. IVÁNYI, A. JÁRAI,  
P. KACSUK, I. KÁTAI, E. KISS, P. KOMJÁTH, M. LACZKOVICH, L. LOVÁSZ,  
GY. MICHALETZKY, J. MOLNÁR, P. P. PÁLFY, A. PRÉKOPOA, A. RECSKI,  
A. SÁRKÖZY, F. SCHIPP, Z. SEBESTYÉN, L. SIMON, P. SIMON, GY. SOÓS,  
L. SZEIDL, T. SZŐNYI, G. STOYAN, J. SZENTHE, G. SZÉKELY, A. SZŰCS,  
L. VARGA, F. WEISZ



2009

# ANNALES

## UNIVERSITATIS SCIENTIARUM BUDAPESTINENSIS

### DE ROLANDO EÖTVÖS NOMINATAE

SECTIO BIOLOGICA	
	incipit anno MCMLVII
SECTIO CHIMICA	
	incipit anno MCMLIX
SECTIO CLASSICA	
	incipit anno MCMXXIV
SECTIO COMPUTATORICA	
	incipit anno MCMLXXVIII
SECTIO GEOGRAPHICA	
	incipit anno MCMLXVI
SECTIO GEOLOGICA	
	incipit anno MCMLVII
SECTIO GEOPHYSICA ET METEOROLOGICA	
	incipit anno MCMLXXV
SECTIO HISTORICA	
	incipit anno MCMLVII
SECTIO IURIDICA	
	incipit anno MCMLIX
SECTIO LINGUISTICA	
	incipit anno MCMLXX
SECTIO MATHEMATICA	
	incipit anno MCMLVIII
SECTIO PAEDAGOGICA ET PSYCHOLOGICA	
	incipit anno MCMLXX
SECTIO PHILOLOGICA	
	incipit anno MCMLVII
SECTIO PHILOLOGICA HUNGARICA	
	incipit anno MCMLXX
SECTIO PHILOLOGICA MODERNA	
	incipit anno MCMLXX
SECTIO PHILOSOPHICA ET SOCIOLOGICA	
	incipit anno MCMLXII

We report with great sadness the passing away of György Elekes, professor of the Department of Computer Science.



The mathematical talent of Elekes showed early. Between 1963 and 1967 he was a student of Károly Kőváry, the fine mathematics teacher of the renowned Fazekas Highschool, Budapest. In 1965 and 1967, he won third, and then first prize at the International Mathematical Olimpiade. He won second prize at the József Kürschák Memorial Contest in 1966.

Between 1967 and 1972 Elekes studied mathematics at the Roland Eötvös University. After graduation, he stayed at the university, earlier at the Analysis I Department (until 1980 as a Lecturer, then as an Assistant Professor), in 1983 he became one of the founding members of newly created Department of Computer Science (until 1995 as an Assistant Professor, then till 2005 as an Associate Professor, after that as a Full Professor). He obtained the Dr. Rer. Nat. degree in 1978, the Candidate of Mathematical Sciences degree in 1994, and the Doctor of Mathematical Sciences title in 2001. He received a Széchenyi Fellowship between 1999–2002.

Elekes started doing research in set theory. He solved a problem of Erdős and Hajnal on partitions of infinite sets, which led to the development (in part with Erdős and Hajnal) of a new branch of combinatorial set theory. (Many of their results are still unpublished.)

Another important result is his theorem with György Hoffmann stating that there are almost-disjoint set systems with arbitrarily large chromatic number.

In all his life, Elekes was deeply interested in geometry, and from the 1970s, in the newly developed complexity theory. He proved a major result

in the theory of geometric algorithms. With the application of an elegant new inequality, he proved that the volume of a convex body in general dimension can only be estimated with very large error in polynomial time. This paper initiated several branches of important research.

His interest in geometric algorithms motivated his teaching as well: he created the curriculum of this topic at the Eötvös University. He did, however, change his research profile in the 1980s, and became interested in combinatorial geometry—another topic invented by Erdős. He discovered important connections to other branches of mathematics, in particular number theory, and his research area became what is now known as additive combinatorics. He gained significant fame in this very active topic where several Fields Medalists and other leading researchers in number theory and combinatorics work. Elekes not only proved hard theorems, but successfully discovered the algebraic structures hidden in the background of some combinatorial phenomena.

György Elekes was an outstanding teacher, who considered teaching his most noble task. His students loved him as teacher and as a person. He spent a lot of time writing and rewriting teaching resources, trying to find the best presentation of thoughts, examples, details.

During his fight with his illness, almost one year long, he closely followed the issues of the Institute, of teaching, and of research, and tried to participate in them. He worked on his papers as long as his strength allowed him.

He was good friend of many, and a good colleague of us all. We will sorely miss his knowledge and his deep devotion to teaching.

Department of Computer Science, Mathematical Institute

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## **Eulogy at the funeral of György Elekes**

**András Frank**

A broken body in the coffin, an unbreakable, pure soul in our hearts: Gyuri Elekes, this is what you became.

I recall your broad smile, I see your long steps, the inside warmth and radiating serenity, which were characteristic of you ever since we met more than forty years ago.

The noble dignity, with which you retained this serenity even in the days of the darkest earthly hell, set an example to all who were around you. Despite the fact that not all days of your youth were sunny. Perhaps your grandfather, whom you liked most as a child, and whose footsteps you tried to follow, was like this. It is good to know that as a mature man, your love was returned by your beautiful children—Vera and Marci, whom I have known since the moments of their births, and Stella, whom I met only when you entered eternity. I recall how many times you told me about the happy moments of the holidays spent at the lake Balaton with your children. I know, you told me a few days ago, that their love was the most important thing for you, and that of Panni, who represented for you intimate happiness for many long years, and who ensured for you the most any human can wish for—the possibility to die at home, holding the hands of your beloved ones.



Apart from love towards the world and serenity, there is one more word that comes to mind, characterizing Gyuri. He was strong. Strong in body. When we started the university, his athletic body distinguished him from the

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Translated by Anna Lovász and Péter Komjáth.

many puny fellow students learning mathematics. He was strong in mathematics. He was the best among the best as the winner of the high school contest many years ago. He was strong in will: he thought over everything thoroughly, and then he hardly ever changed his final decision, even when it was very hard to handle for those around him. This obstinate determinedness made it possible, that after those lost years following the bitterness of his youth, he could return to do research and produce a series of results which have became fundamental by now.

But he was strongest of all in his soul. It is astonishing how he could, overshadowed by the threatening monster, set about finishing his unfinished business. This spring, he drew no less than five papers to completion. Uncle Paul Erdős, who was his mentor, and to whom so many of his articles were dedicated, can pause in reading the Book and look down in satisfaction: Gyuri will not arrive empty-handed to the Final Meeting. This may be why he wanted to remain close to Erdős' final resting place, this time, for eternity.

Perhaps it will not be foreign to his spirit if I close my parting words by recalling a bit of mathematics. The theorem of Menger is one of the cornerstones of combinatorial optimization. Gyuri was the first one to tell me about it. We must have been in our third year when he explained Gallai's alternating path proof. He had a fantastic sense for explaining concepts, it is not surprising that he became one of the most popular instructors. The proof had such an effect on me that it has remained in the center of my mathematical interests to this day. A quarter of a century later I had the opportunity to return the favor, when I got the chance to tell Gyuri about the infinitely elegant Nagamochi–Ibaraki algorithm, which pertains to a close relative of the Menger theorem, and for which I was able to come up with a very simple proof. This in turn won Gyuri's liking so much that he soon included it in his lectures for first years students in finite mathematics. The other day I was paging through his university notes for the subject, and there they were one after the other: the Menger theorem, the Nagamochi–Ibaraki algorithm.

God bless you, Gyuri! It was great to have known you!

October 6, 2008.

# DENSITY VERSION OF A THEOREM OF BECK AND SZEMERÉDI–TROTTER IN COMBINATORIAL GEOMETRY

By

**GYÖRGY ELEKES**

*(Received March 1, 2010)*



The study of combinatorial properties of finite point configurations in the (Euclidean) plane is a vast area of research in geometry, whose origins go back (at least) to the ancient Greeks. (In this paper *plane* always means the usual Euclidean plane, and *line* means a straight line in the Euclidean plane.) The revival of the subject is mainly due to (who else?) Paul Erdős starting in the 1940s. This field of research is particularly rich in innocent-looking problems that a good highschool student can easily understand, but which have resisted the attacks of the best mathematicians for decades. A good example is Dirac’s problem from 1951; see Dirac [2]. (We are talking about Gabriel Dirac; his father was

the famous English physicist Paul Dirac, and his mother was the sister of the famous Hungarian-American physicist Jenő, or Eugene, Wigner.) Dirac’s problem goes as follows: Does there exist an absolute constant  $c$  such that any set of  $n$  points in the plane, not all on a line, has an element incident to at least  $\frac{n}{2} - c$  distinct connecting lines? (A connecting line is a straight line containing at least two of the given points.) Note that basically the same question was asked independently and around the same time by Motzkin [4].

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A draft of this paper was left behind by the late György Elekes. His family is greatly indebted to József Beck for writing up this final version. The family is also very grateful for the editor of this issue for his work.

If true, this conjecture is best possible, as is shown by the example of  $n$  points distributed evenly on two lines. Note that Felsner exhibited an infinite series of point configurations, showing that  $c \geq 3/2$ .

Somewhat later Erdős [3] suggested the difficulty of establishing *any* lower bound of the form  $c \cdot n$  in Dirac's problem (instead of the conjectured  $\frac{n}{2} - O(1)$ ). The independent efforts of Beck [1] and Szemerédi and Trotter [5] settled this long-standing open problem (the two proofs were very different), but the stronger problem of Dirac remains unsolved even today.

**THEOREM A.** *If  $n$  points in the plane are not collinear, then one of them must lie on at least  $c_1 n$  distinct connecting lines, where  $c_1 > 0$  is an absolute constant independent of  $n$ .*

Unfortunately, the original proofs of Theorem A provided extremely weak constants (less than  $10^{-1000}$ ). However, about 15 years later, L. Székely [6] succeeded to find an amazingly short and elegant new proof, which yields a much better constant  $c_1 > \frac{1}{10}$ .

A useful and more-or-less equivalent formulation of Theorem A, due to Erdős, goes as follows.

**THEOREM B.** *There is an absolute constant  $c_2 > 0$  such that, any set of  $n$  points in the plane, no more than  $n - k$  of which are on the same line, determine at least  $c_2 k n$  distinct connecting lines. (Here  $0 \leq k < n$  are arbitrary integers.)*

Again Székely's new proof provides a very good constant  $c_2$  (independent of  $n$  and  $k$ ). The choice  $k = \text{const} \cdot n$  in Theorem B implies the following particularly interesting special case.

**THEOREM C.** *There is an absolute constant  $c_3 > 0$  such that, given any set of  $n$  points in the plane, at least one of the following two options always holds:*

- (i) *there is a line containing at least  $c_3 n$  given points;*
- (ii) *there are at least  $c_3 n^2$  distinct connecting lines.*

The objective of this note is to prove a generalization of Theorem C; in fact, we prove a *density* version.

**THEOREM 1.** *Let  $\mathcal{P}$  be an arbitrary set of  $n$  points in the plane, and let  $G = G(\mathcal{P}, E)$  be an arbitrary (simple) graph with vertex set  $\mathcal{P}$  and edge set  $E$*

where  $|E| \geq cn^2$  and  $c > 0$  is a given constant. Every edge in  $E$  represents a pair of points  $P, P' \in \mathcal{P}$ ; the complete list of these  $PP'$ -lines is denoted by  $L_1, L_2, \dots, L_m$  where  $m = |E|$  — we call them  $G$ -lines.

Then there is a positive constant  $c' = c'(c) > 0$  depending only on  $c > 0$  but independent of  $n$  and  $G$  such that at least one of the following two options always holds:

- (i) at least  $c'n^2$   $G$ -lines are the same;
- (ii) there are at least  $c'n^2$  distinct  $G$ -lines.

Notice that, in the special case when  $G$  is the complete graph on  $n$  vertices, we obtain Theorem C.

Our proof of Theorem 1 uses the following fundamental result in combinatorial geometry.

**THEOREM D. (SZEMERÉDI–TROTTER [5])** *For  $n$  points and  $\ell$  lines in the plane, the number of incidences among the points and lines is*

$$(1) \quad O(n^{2/3} \ell^{2/3} + n + \ell).$$

(Note that Beck [1] proved a weaker version of Theorem D; it was sufficient to derive Theorems A, B and C.)

The upper bound in (1) cannot be improved (as is demonstrated by the  $\sqrt{n} \times \sqrt{n}$  square lattice). Unfortunately, the original proof of Theorem D provided an extremely poor implicit constant in (1), but again the new method of Székely [6] yields reasonable constants:

$$(2) \quad \text{number of PointLine incidences} < 4n^{2/3} \ell^{2/3} + 4n + \ell.$$

Now we are ready to discuss the

**PROOF OF THEOREM 1.** Recall that  $L_1, L_2, \dots, L_m$ , where  $m = |E| \geq cn^2$ , denote the  $G$ -lines. Let  $k$  denote the number of *distinct*  $G$ -lines; clearly  $1 \leq k \leq m$ . For notational convenience we may assume that  $L_1, L_2, \dots, L_k$  are all distinct, and each one of the remaining  $G$ -lines  $L_{k+1}, \dots, L_m$  coincides with one of  $L_1, L_2, \dots, L_k$ . For any  $G$ -line  $L_i$  ( $1 \leq i \leq k$ ), let  $p_i$  denote the number of elements of  $\mathcal{P}$  on  $L_i$ , and let  $e_i$  denote the number of edges of  $E$  where both endpoints are in  $L_i$ . Clearly  $e_i \leq \binom{p_i}{2}$ , which implies the inequality  $p_i \geq \sqrt{2e_i}$  for every  $i = 1, 2, \dots, k$ .

By definition,

$$(3) \quad \sum_{i=1}^k e_i = |E| = cn^2.$$

We divide the total sum in (3) into 4 parts:

$$(4) \quad cn^2 = |E| = \sum_{i=1}^k e_i = \sum_1 + \sum_2 + \sum_3 + \sum_4,$$

where

$$\sum_1 = \sum_{\substack{1 \leq i \leq k: \\ 1 \leq e_i < c_0}} e_i, \quad \sum_2 = \sum_{\substack{1 \leq i \leq k: \\ c_0 \leq e_i < n}} e_i,$$

$$\sum_3 = \sum_{\substack{1 \leq i \leq k: \\ n \leq e_i < c^* n^2}} e_i, \quad \sum_4 = \sum_{\substack{1 \leq i \leq k: \\ c^* n^2 \leq e_i}} e_i,$$

and the value of the constants  $c_0 = c_0(c) > 0$  and  $c^* = c^*(c) > 0$  will be specified later. (Note that there is a similar decomposition in Beck [1]; we follow closely that argument.)

The idea of the proof is the following: we show that, by an appropriate choice of the constants  $c_0$  and  $c^*$  (namely, by choosing a “large”  $c_0$  and a “small”  $c^*$ ), the subsums  $\sum_2$  and  $\sum_3$  become “negligible” in the quantitative sense

$$(5) \quad \sum_2 < \frac{c}{4}n^2 \text{ and } \sum_3 < \frac{c}{4}n^2.$$

Comparing (4) and (5), it follows that

$$\sum_1 + \sum_4 > \frac{c}{2}n^2.$$

If

$$\sum_1 = \sum_{\substack{1 \leq i \leq k: \\ 1 \leq e_i < c_0}} e_i > \frac{c}{4}n^2,$$

then there are at least  $c'n^2$  distinct  $G$ -lines with  $c' = \frac{c}{4c_0}$ , proving case (ii) in Theorem 1.

Finally, if

$$\sum_4 = \sum_{\substack{1 \leq i \leq k: \\ c^* n^2 \leq e_i}} e_i > \frac{c}{4} n^2,$$

then of course case (i) in Theorem 1 follows with  $c' = c^*$ .

Therefore, it suffices to prove (5). First we deal with  $\sum_2$ : we split it into two parts:

$$(6) \quad \sum_2 = \sum'_2 + \sum''_2,$$

where

$$\sum'_2 = \sum_{\substack{c_0 \leq e_i < n: \\ p_i \leq \sqrt{2n}}} e_i \quad \text{and} \quad \sum''_2 = \sum_{\substack{c_0 \leq e_i < n: \\ p_i > \sqrt{2n}}} e_i.$$

To estimate  $\sum''_2$  from above, we apply Theorem D (Szemerédi–Trotter) with Székely’s good constants (2). Every  $G$ -line  $L_i$  ( $1 \leq i \leq k$ ) that contributes to  $\sum''_2$  contains more than  $\sqrt{2n}$  points of  $\mathcal{P}$ , so the number of point-line incidences is more than  $\ell \sqrt{2n}$  where  $\ell$  denotes the number of distinct  $G$ -lines involved in  $\sum''_2$ . Applying (2), we have

$$(7) \quad \ell \sqrt{2n} < \text{number of PointLine incidences} < 4n^{2/3} \ell^{2/3} + 4n + \ell.$$

Routine calculations show that the choice  $\ell = 64\sqrt{n}$  and  $n > 10$  contradicts (7), thus we have  $\ell < 64\sqrt{n}$  (we assume  $n > 10$ ). Using this in the definition of  $\sum''_2$ , we obtain

$$(8) \quad \sum''_2 = \sum_{\substack{c_0 \leq e_i < n: \\ p_i > \sqrt{2n}}} e_i < 64\sqrt{n} \cdot n = 64n^{3/2},$$

which is obviously less than  $cn^2/4$  if  $n$  is sufficiently large.

Next we estimate  $\sum'_2$ . Again we apply (2) (“Szemerédi–Trotter”), and this time we combine it with a power-of-four decomposition. We prefer to work with 4 because the square-root of 4 is also a simple integer; let  $\log_4$  denote the base 4 logarithm. We have

$$(9) \quad \sum'_2 = \sum_{\substack{c_0 \leq e_i < n: \\ p_i \leq \sqrt{2n}}} e_i = \sum_{0 \leq j \leq \log_4(n/c_0)} \sum_{\substack{4^j c_0 \leq e_i < 4^{j+1} c_0: \\ p_i \leq \sqrt{2n}}} e_i.$$

We want to estimate the number of  $G$ -lines involved in the last sum in (9) — let  $l_j$  denote this number. Since  $p_i \geq \sqrt{2e_i}$ , each one of the  $l_j$   $G$ -lines in question contains at least  $2^j \sqrt{2c_0}$  points of  $\mathcal{P}$ , so the number of point-line incidences is more than  $\ell_j 2^j \sqrt{2c_0}$ . Again we apply (2):

$$(10) \quad \ell_j 2^j \sqrt{2c_0} < \text{number of PointLine incidences} < 4n^{2/3} \ell_j^{2/3} + 4n + \ell_j.$$

Routine calculations show that the choice

$$\ell_j = 125 \frac{n^2}{(2^j \sqrt{2c_0})^3} \quad \text{and} \quad c_0 > 100$$

contradicts (10), thus we have

$$\ell_j < 125 \frac{n^2}{(2^j \sqrt{2c_0})^3}, \quad \text{assuming } c_0 > 100.$$

Using this in (9), we have

$$\begin{aligned} \sum'_2 &= \sum_{0 \leq j \leq \log_4(n/c_0)} \sum_{\substack{4^j c_0 \leq e_i < 4^{j+1} c_0: \\ p_i \leq \sqrt{2n}}} e_i < \\ &< \sum_{0 \leq j \leq \log_4(n/c_0)} 125 \frac{n^2}{(2^j \sqrt{2c_0})^3} \cdot 4^{j+1} c_0 < \\ (11) \quad &< \frac{600}{2\sqrt{2c_0}} n^2 \sum_{j \geq 0} 2^{-j} = \frac{600}{\sqrt{2c_0}} n^2. \end{aligned}$$

Finally, we estimate  $\sum'_3$  in a similar way. We have

$$(12) \quad \sum'_3 = \sum_{n \leq e_i < c^* n^2} e_i = \sum_{0 \leq j \leq \log_4(c^* n)} \sum_{4^j n \leq e_i < 4^{j+1} n} e_i.$$

Again we want to estimate the number of  $G$ -lines involved in the last sum in (12) — again let  $l_j$  denote this number. Since  $p_i \geq \sqrt{2e_i}$ , each one of the  $l_j$   $G$ -lines in question contains at least  $2^j \sqrt{2n}$  points of  $\mathcal{P}$ , so the number of point-line incidences is more than  $\ell_j 2^j \sqrt{2n}$ . Again we apply (2):

$$(13) \quad \ell_j 2^j \sqrt{2n} < \text{number of PointLine incidences} < 4n^{2/3} \ell_j^{2/3} + 4n + \ell_j.$$

Routine calculations show that the choice  $\ell_j = \sqrt{n}2^{6-j}$  contradicts (13), thus we have  $\ell_j < \sqrt{n}2^{6-j}$ . Using this in (12), we have

$$\begin{aligned} \sum_3 &< \sum_{0 \leq j \leq \log_4(c^*n)} \sqrt{n}2^{6-j} \cdot 4^{j+1}n = \\ (14) \quad &= 2^8 n^{3/2} \sum_{0 \leq j \leq \log_4(c^*n)} 2^j < 2^9 n^{3/2} \sqrt{c^*n} = 2^9 \sqrt{c^*} n^2. \end{aligned}$$

Now we are ready to prove (5). By (6), (8), (11) and (14),

$$(15) \quad \sum_2 < 64n^{3/2} + \frac{600}{\sqrt{2c_0}}n^2 \quad \text{and} \quad \sum_3 < 2^9 \sqrt{c^*} n^2,$$

assuming  $c_0 > 100$  and  $n > 10$ . By choosing the constant  $c_0 = c_0(c) > 0$  sufficiently large and choosing  $c^* = c^*(c) > 0$  sufficiently small, (15) implies

$$\sum_2 < \frac{c}{4}n^2 \quad \text{and} \quad \sum_3 < \frac{c}{4}n^2,$$

assuming  $n$  is large enough depending only on the given positive constant  $c$ . This proves (5), and so the proof of Theorem 1 is complete. ■

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## GYURI ELEKES AND SET THEORY

By

ANDRÁS HAJNAL

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### 1. First results

I call my memories of colleague and friend, the mathematician György (Gyuri) Elekes (Budapest, May 19, 1949 – Fót, September 29, 2008).

Gyuri graduated in mathematics at the Eötvös University in 1972. I just returned from my year-long visit to Canada, and having left the university, worked at the mathematical institute of the Academy. Erdős, who recently lost his mother, also happened to be in Hungary. We were sitting all day long in the institute, usually in the director’s office, rarely used by its owner. A large flow of mathematicians kept visiting Erdős, telling him their mathematical problems or learning new ones from him. I used up the time of those visits to work on our problems. These were set theory’s years of prosperity following Cohen’s invention of forcing. And this was the golden age of combinatorial set theory, in no small part because EP with his characteristically sharp eyes raised a large number of problems, most of them could not be solved by the then existing methods.

We had just started the intense research of the chromatic number of set systems consisting of infinite sets. One of our questions was the following. For every infinite cardinality  $\kappa$  there exists an almost disjoint system of countably infinite sets, which has chromatic number greater than  $\kappa$ . We had already heard the news that two graduating students, Elekes and Hoffmann had proved this statement. (This is the content of their first paper, [1]). The proof itself was not very hard, it was even simplified by Komjáth ([7]) in 1981. Still, the topic survives—not long ago, Komjáth ([8]) found an interesting application of the result.

One of the authors of the Elekes–Hoffmann paper showed up one morning, with a modest, but knowledgeable smile. He told us that he could prove the conjecture that professor Erdős had mentioned in his talk a few days earlier:

If the set of all subsets of  $\omega$ , the set of nonnegative integers is decomposed into countably many parts, i.e., if

$$\mathcal{P}(\omega) = \bigcup\{S_n : n \in \omega\},$$

then there are a color  $n \in \omega$  and sets  $A, B$  such that

$$A, B, A \cup B \text{ are distinct and } \{A, B, A \cup B\} \subseteq S_n.$$

This Ramsey type question of Erdős was answered by Gyuri with the following results (for  $\omega$ ):

**THEOREM A.** ([3]) *Assume that  $\mathcal{P}(\omega) = \bigcup\{S_n : n \in \omega\}$ . Then there exist  $n \in \omega$  and an infinite  $Z \subset S_n$ , such that all nonempty finite unions formed from the elements of  $Z$  are distinct and in  $S_n$ .*

**DEFINITION.** For any set  $S \subset \mathcal{P}(\omega)$  let  $U(S)$  denote the set

$$\left\{ \bigcup Z : Z \subseteq S \text{ and } Z \neq 0 \right\}.$$

**THEOREM B.** ([3]) *There is a partition  $\mathcal{P}(\omega) = S_0 \cup S_1$  of the set  $\mathcal{P}(\omega)$ , such that for every  $Z \subset \mathcal{P}(\omega)$  if  $U(Z)$  is uncountable then  $U(Z) \cap S_0, U(Z) \cap S_1$  are both uncountable.*

It turned out that Gyuri found a general method for proving positive results similar to Theorem A. In topological spaces appropriately constructed from set systems he applied arguments similar to the proof of the Baire category theorem to obtain homogeneous sets.

## 2. Working together

In the years to come Erdős, Gyuri, and myself spent a lot of time with generalizations of Gyuri's results. Hindman published his basic result in 1974.

**THEOREM.** (Hindman, [6]) *If the finite subsets of  $\omega$  are decomposed into finitely many classes, that is, if*

$$[\omega]^{<\omega} = \bigcup\{S_n : n < r\}$$

for some  $r < \omega$ , then there are an infinite  $Z \subset [\omega]^{<\omega}$  consisting of pairwise disjoint sets, and a color  $n < r$  such that all finite subunions of  $Z$  are in  $S_n$ .

Our questions were motivated by the search of generalizations of this result with some  $\kappa > \omega$  in place of  $\omega$  and with some  $r \geq \omega$ . This worldview, uniquely characteristic of Erdős, made possible to investigate similar finite and infinite problems, and it was extremely useful, even though the infinite version of some interesting finite problems turned out to be trivial, and the finite form of certain interesting infinite statements were uninteresting.

We generalized, for example, Hindman's theorem for some regular  $\kappa > \omega$  by replacing “finite” by “of cardinality less than  $\kappa$ ” but of the elements of  $Z$  we could not reach that they be pairwise disjoint, only that they form a  $\Delta$  system. (A system of sets  $Z$  is a  $\Delta$  system, if there is a set  $D$ , such that the intersection of any two different members of  $Z$  is  $D$ . The existence of a homogeneous  $\Delta$  system was first proved by the Czech-American mathematician K. Prikry. For  $\kappa > \omega$  we used a method invented by J. E. Baumgartner.)

Alas, we only published a preliminary short note on our results in the Studia, [4]. The numerous results with the definitions are enumerated on 6 pages, giving only short hints of the proofs. After this, for a while, we have not considered these problems. Gyuri temporarily left set theory. Erdős, however, became more and more interested in the question why we could never obtain *pairwise disjoint* sets. In almost every problem paper he mentioned one of the following problems.

PROBLEM A. *Does there exist a cardinal  $\kappa$  such that for every coloring*

$$\mathcal{P}(\kappa) = \bigcup \{S_n : n \in \omega\}$$

*there are infinitely many pairwise disjoint sets, all whose finite, nonempty subunions get the same color?*

PROBLEM B. *Does there exist a cardinal  $\kappa$  such that for every coloring*

$$\mathcal{P}(\kappa) = S_0 \cup S_1$$

*there exist infinitely many pairwise disjoint sets with all finite unions having the same color?*

The prevailing opinion was that the answer may be independent of the axioms of set theory and a cardinal  $\kappa$  as above, if exists, must be very large. There were consistency proofs of the nonexistence, for example, in the model of constructibility. It was about 1990, when Gyuri and Kópé (Komjáth)

came to me with a big smile: there are no such cardinals, and only a few complementary positive results must be proved. This is how the paper [5] was born.

### 3. A nice proof

We are going to show that no cardinal as in Problem A exists. Identify  $\mathcal{P}(\kappa)$  with the set  $K$  of characteristic functions defined on  $\kappa$ .

Construct the following partition  $\bigcup\{S_n : n \in \omega\}$  of  $K$ . Consider  $K$  a vector space over GF(2), the two-element field. Let  $B = \{b_i : i \in I\}$  be a basis of this vector space. Set  $b \in S_n$  iff  $b$  is the sum of  $n$  elements of  $B$  of  $K$ . Assume that  $A_0 \dots A_r \dots$  are pairwise disjoint subsets of  $\kappa$  all being in  $S_n$ , for some  $n < \omega$ . Then

$$\chi(A_r) = \sum\{b_i \in E_r\}$$

for some  $E_r \subset I$ ,  $|E_r| = n$ , for every  $r < \omega$ . Using the  $\Delta$  system lemma we can assume that the sets  $E_r$  form a  $\Delta$  system.

Now we can use that the characteristic function of the union of disjoint sets is the sum of their characteristic functions, and an easy calculation gives that even the double and triple sums cannot all be in the same class  $S_n$ . ■

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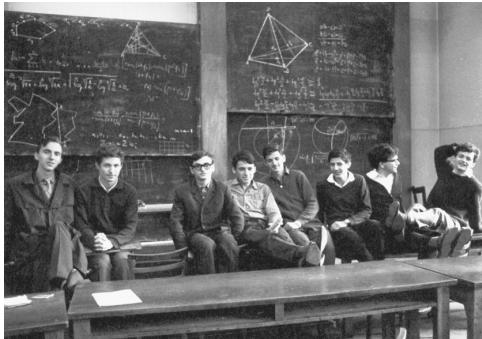
**PERSONAL REMINISCENCES, GYURI ELEKES**

By

MIKLÓS SIMONOVITS

(Received February 12, 2010)

Our friend, Gyuri Elekes<sup>1</sup> was already very ill when we, (Gyuri, Endre Szabó and I) have finished our paper [3]. He died within a month. This paper is written in his memory, and has a continuation [4] describing our joint work, partly published in [3] and also a basic ingredient of the proofs in [3] from a paper of Elekes and Szabó [5].



I have known Gyuri Elekes for years and have many kind memories of him. Here I start with a photo, back from his high school years. This shows Elekes during some preparation for the International Mathematical Olympiad.

This photo shows him and his mates: Z. Laborczi, A. Szűcs, L. Csirmaz, Gy. Elekes, L. Babai, J. Pintz, L. Surányi, and Gy. Hoffman. The left photo is the enlarged middle part of the right one. Elekes must have liked this photo, since a printed version of this could be seen on the wall in his office, at Eötvös University.

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<sup>1</sup> György Elekes, to be precise.

## 1. Old times



At the University he was one of the most talented students among his peers. When he finished his university studies, in 1972, Vera T. Sós “invited him” to become a lecturer at the Department of Analysis I, Eötvös University, Budapest. This is when we became colleagues,<sup>2</sup> friends, though I knew him already from his student years. Ákos Császár was the head of the department, Vera T. Sós, Kató Rényi, András Hajnal, Rózsa Péter, Lajos Pósa, Miklós Laczkovich were also working here.

The department was otherwise fairly small, fluctuating, but basically ten permanent persons.<sup>3</sup> We taught primarily Mathematical Analysis, Theory of Function, Combinatorics, Graph Theory, Set Theory, Mathematical Logic, Approximation Theory, Topology, etc. (There was also another department in Analysis, teaching the other Analysis-related subjects, e.g., Differential Equations, PDE, Functional Analysis, Fourier Series, etc.)

There were a lot of tensions, frictions between the distinct departments. However, our Department was extremely friendly. We enjoyed this friendly atmosphere very much, and also were proud to work at mathematically such an excellent department. And, beside being enthusiastic mathematicians, we also enjoyed very much the teaching, among others, the contact with outstanding young students. We worked together until 1985, here in the same group (of approximately 10 people). Then, for some strange reasons<sup>4</sup> I left the University, while he stayed.

Those days I had to think over, what had I felt sorry for to leave behind. I felt sorry to leave the teaching of excellent students, some of my dreams, to leave behind some of my best colleagues and friends, including Elekes. Let me describe here, how do I remember him.

- He was very talented, fast, clear-thinking;
- He was also very determined, sometimes tough, however, very kind at the same time;
- He was very kind not only because he was very often smiling, but because he was really a kind person.

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<sup>2</sup> I was working there since 1967.

<sup>3</sup> The “Mathematics Institute” consisted of 6, later of 7 such departments.

<sup>4</sup> tensions, fights, quarrels

## 2. Beginning of the end

When somebody learns that he will soon die, his reactions may characterize him. I do not know, how did Elekes learn that his time was up, but from that on he started finishing his papers.

Within a short period he has finished seven papers. One of them was the paper joint with Endre Szabó [5] and another joint with Endre and me [3].

Beside working very hard, Elekes, of course, tried to spend a lot of time with his family and friends as well. They were very important for him.

In January 2006 Elekes was doing something in the Rényi Institute, (he often came there) and we started a mathematical discussion. He got stuck in a problem and I suggested a possible approach to overcome this difficulty.

He wanted to prove that some functional relation of the form

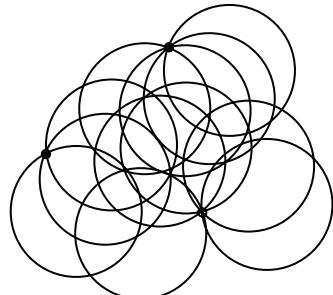
$$F(\varphi_1(u, v), \varphi_2(u, v), \varphi_3(u, v)) = 0$$

cannot be satisfied (see [5, 6, 7]) unless  $F$  has a very special form. I suggested to examine the singularities of the functions  $\varphi_i$  and derive a contradiction, using that these singularities cannot cancel out each other. (We had similar mathematical discussions several times even earlier, however, this was the first time we seriously started research together.) My idea came partly from a Number Theory book of Paul Turán, partly from my knowledge of function theory.

Within a few days Elekes came back with a ready to publish paper. Do not misunderstand it, the paper was far from being trivial. Its main result was

**THEOREM 1.** *Assume we have three distinct points in the plane, A, B, and C, and we have through each of them  $n$  unit circles.<sup>5</sup> Then the number of triple points, i.e. points belonging to three such circles, is at most  $cn^{2-\eta}$ , for some constants  $c > 0$  and  $\eta > 0$ .*

Elekes was interested in this question because if we take straight lines instead of unit circles, then we get a completely different answer:




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<sup>5</sup> From now on, we shall write “ $n + n + n$  curves” in such cases.

Assume we have a square grid arrangement in the plane and we also consider  $n$  horizontal,  $n$  vertical straight lines, and  $n$  of slope 1 (i.e. at  $45^\circ$ ). They will have  $cn^2$  triple points. This can be seen on Figure 1(b). Moreover, using, say, 100 slopes we obtain 100 families of straight lines for which the number of crossings of multiplicity 100 is  $c_{100}n^2$ .

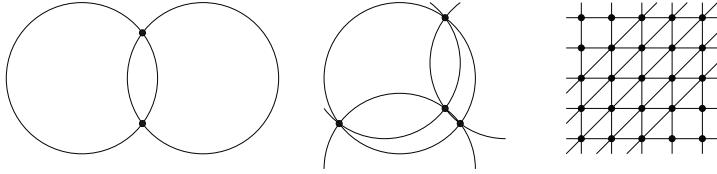


Figure 1.

(a) Circles

(b) and straight lines

So, this makes a combinatorial distinction between straight lines and unit circles. Of course, if we do not fix the radius of the circles, then such a distinction cannot be made, since any system of straight lines can be transformed into a system of circles, using an inversion.<sup>6</sup>

So the paper was finished, and it was far from being trivial. Yet, I felt the result too special, too narrow. So we have decided to try to find out, what is the more general situation, what is really going on in the background. We wanted to find out, when and why can such systems of curves have  $cn^2$  triple points. We shall see in the second part, that in some sense we, together with Endre Szabó, have succeeded, though this slowed down the publication of the paper by years. But this will be discussed in the next paper [4].

**REMARK.** Here I have to point out some slackness in the comment above. I wrote that Elekes wanted to find a combinatorial distinction between straight lines and unit circles. Several papers of Elekes were connected to this “distinction”. Yet, if we try to formulate, what does a combinatorial distinction really mean, then we have to be careful. We could say that whenever we have in  $\mathcal{R}^n$  some surfaces and some points, then we can attach to them a bipartite graph  $G[\mathbb{S}, \mathbb{P}]$ , where  $\mathbb{P}$  is the set of points,  $\mathbb{S}$  is the family of surfaces, and  $S \in \mathbb{S}$  is joined to  $P \in \mathbb{P}$  if they are incident. Now we speak about a “combinatorial property” of  $\mathbb{S}$  if it can be seen from  $G[\mathbb{S}, \mathbb{P}]$ .

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<sup>6</sup> Or, equivalently, the transformation  $w = \frac{1}{z}$ , assumed that all the straight lines avoid the origin. The opposite direction does not hold: two points can be contained in arbitrary many circles and this incidence pattern cannot be obtained using straight lines.

In this sense we can easily distinguish straight lines and unit circles in the following sense. For straight lines,  $G[\mathbb{S}, \mathbb{P}]$  does not contain  $C_4$ . For unit circles  $G[\mathbb{S}, \mathbb{P}]$  can contain  $C_4$  but cannot contain  $K(2, 3)$ . So, looking at  $G[\mathbb{S}, \mathbb{P}]$  we often can conclude that those curves in  $\mathbb{S}$  cannot be straight lines, . . . We have several interesting results and deep open problems in this field, however, we will not discussing this topic here.

## 2.1. Elekes and the Mathematics: “early influences”

Gyuri Elekes and I often discussed mathematics. Occasionally we had different views, and Gyuri was perhaps too modest in the sense that if I asked him, “Why don’t you learn this-and this, perhaps that helps in solving your problem”, he used to answer this question making the impression that that part of mathematics was to involved for him. The truth was just the opposite: he learnt a lot of new mathematics to proceed with his favorite problems. Then he applied these results in his research and teaching, too.



Let us stop at this point for a minute. I often felt that Elekes was in some positive sense self-certain. When he referred to the difficulty of learning something new, I often felt that he just wanted to go on with his original approach.

Trying to describe Elekes’ mathematical carrier, we see that he started with Erdős-Hajnal type combinatorial set theory, then he worked in theoretical computer science, above all in algorithms, and finally he worked in combinatorial geometry, on problems where deep algebraic methods had to be used.

He himself formulated this slightly differently. In one of his CV’s he wrote

**Field of interest:** Combinatorial Geometry and algorithms, combinatorial number theory, and combinatorial algebra.<sup>7</sup>

One of his related lectures on this topic had the title

**The interface between geometry, algebra and number theory**

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<sup>7</sup> I have to remark here that I have never heard of this last topic before.

Many of his papers could have this same title.

I close this part with just stating that for many people one of Elekes' result [1] became a crucial one. It is an extremely short and nice gem in mathematics with important consequences, saying that the volume of a high dimensional convex body, given by an oracle, cannot be approximated up to a factor of 2 in less than exponentially many steps. For a longer explanation see the paper of Lovász [2] here.

I finish these reminiscences with that he was a person who loved people and who was a person to be loved.

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## THE “LITTLE GEOMETER” AND THE DIFFICULTY OF COMPUTING THE VOLUME

By

LÁSZLÓ LOVÁSZ

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I met Gyuri Elekes in the math club of the Mihály Fazekas High School (a high school with a special program for mathematically talented students) some 47 years ago. He was a year younger, and excelled in solving geometry problems by particularly simple, elegant arguments. We (the old sophomores) called him among ourselves the “little geometer”. In this paper I would like to describe one of his results, which turned a “little geometer” into a “great geometer”.

Gyuri and I met and exchanged mathematical ideas many times, on both research and teaching. He was a PhD student of mine for a while (this is when I told him about the problem discussed below), and then we became colleagues at the Department of Computer Science of the Eötvös Loránd University. It is a very sad occasion to write this exposition of a real pearl of a result; but it is appropriate, since even during the last months of his struggle with illness, it was the beauty of mathematics that gave him consolation.

### 1. Background: convex sets and oracles

At the beginning of the 1980’s, Martin Grötschel, Alexander Schrijver and myself worked on an algorithmic theory of convex bodies. Our research was inspired by the Ellipsoid method, a polynomial time linear programming algorithm of Khachian. This algorithm worked for every reasonable convex body; but what did this mean in mathematical terms? A general convex body cannot be described by a finite number of data; only certain special classes, for example, convex polyhedra can. Do we have to describe and analyze the method separately for every such class?

Our solution was to describe the convex body by an *oracle*. There are several different ways to do so. The simplest and most natural one is a **MEMBERSHIP** oracle: a convex body (a compact, full-dimensional convex set)  $K \subseteq \mathbb{R}^n$  is specified by a black box, which accepts as its input a vector  $x \in \mathbb{R}^n$ , and tells whether or not  $x$  belongs to the body  $K$ . This can be strengthened to a **SEPARATION** oracle: if the answer is negative, then the oracle also returns a certificate of this in the form of the equation of a hyperplane separating  $x$  from  $K$ . Let me mention two further important oracles: **VALIDITY**, which tells you whether a halfspace (given by a linear inequality) contains  $K$ ; and **OPTIMIZATION**, which tells you the maximum value of a linear objective function over  $K$ .

My goal is not to go into the details of the theory of oracles for convex bodies. However, to describe the result of Elekes I have to make the above definition of oracles more precise through a few remarks.

— Together with the oracles, you have to specify a few more data (rational numbers, the number of which is polynomial in the dimension  $n$ ); without these, the oracles would not give sufficient information. For example, along with a **MEMBERSHIP** oracle, one has to specify the center and radius of a ball containing  $K$ , and also the center and radius of a ball contained in  $K$ . These additional data are called *guarantees*.

— Since we are interested in finite algorithms (in fact, mostly in algorithms whose running time is polynomial in  $n$  and in the total number of digits of the other input data), we assume that all inputs and outputs are rational. This is not always possible, for example, the maximum of a linear function with rational coefficients can be irrational for very “benign” convex bodies; this leads us to the next remark:

— We often use a “weak” version of the oracle. In this case, the output may be wrong, but must be within a prescribed error bound to the correct answer. For example, a weak **OPTIMIZATION** oracle accepts a rational vector  $a \in \mathbb{R}^n$  and a rational number  $\varepsilon > 0$  as its input. It returns a rational value  $c$  such that  $|\max\{a^\top x : x \in K\} - c| \leq \varepsilon$ . The input of a weak **MEMBERSHIP** oracle is a rational vector  $x \in \mathbb{R}^n$  and a rational number  $\varepsilon > 0$ . It returns YES if  $x$  is deep in  $K$  in the sense that  $x$  is farther from  $\mathbb{R}^n \setminus K$  than  $\varepsilon$ ; it returns NO if  $x$  is farther from  $K$  than  $\varepsilon$ . In the remaining case (when  $x$  is near the boundary of  $K$ ), it can return either YES or NO.

With all these additional features, it turned out that the weak forms of above oracles, with the right guarantees, are reducible to each other in

polynomial time (the most important ingredient of these reductions is the Ellipsoid Method of Khachian). For example, the polynomial time solvability of a linear program follows from the fact that for a convex body given by a weak SEPARATION oracle, we can realize a weak OPTIMIZATION oracle in polynomial time. Furthermore, many other algorithmic questions on convex bodies can be solved in polynomial time if the body is given by any one of these oracles.

One basic algorithmic question remained open however: can the volume of a convex body be computed in this model? The Ellipsoid Method gives something, but this is very weak. For every convex body  $K \subseteq \mathbb{R}^n$  (given, say, by a weak MEMBERSHIP oracle), we can compute in polynomial time an ellipsoid  $E$  with the following properties:  $E$  is contained in  $K$ , but if we blow it up by a factor of  $n^{3/2}$ , it will contain  $K$ . Hence

$$\text{vol}(E) \leq \text{vol}(K) \leq n^{3n/2} \text{vol}(E).$$

Since the volume of  $E$  is easy to compute, this yields an estimate of the volume of  $K$  with a relative error of  $n^{3n/2}$ .

This is a huge error, if  $n$  is large. Is there a better method, which estimates the volume of  $K$  with arbitrarily small relative error, or at least with a constant relative error?

## 2. The lower bound result of Elekes

Elekes [4] proved that the answer to this question is in the negative.

**THEOREM 2.1.** *Let  $\mathcal{A}$  be a polynomial algorithm that computes a number  $V(K)$  for every convex body  $K$  given by a SEPARATION oracle, such that  $V(K) \geq \text{vol}(K)$ . Then for every large enough dimension  $n$ , there is a convex body in  $\mathbb{R}^n$  for which  $V(K) > 2^{0.99n} \text{vol}(K)$ .*

We can replace the number 0.99 in the exponent by any number less than 1. The proof also shows that there is a trade-off between time and precision: if we want (say) a relative error less than 2, then we need exponential time.

The beautiful proof of this result can be given in full even in a short paper like this. The key is the following lemma.

**LEMMA 2.2.** *Let  $B$  be the unit ball in  $\mathbb{R}^n$ , and let  $P \subseteq B$  be a convex polytope with  $p$  vertices. Then*

$$\text{vol}(P) \leq \frac{p}{2^n} \text{vol}(B).$$

**PROOF OF THE LEMMA** For every vertex of the polytope  $P$ , consider the ball  $B_v$  for which the segment connecting  $v$  to the origin is a diameter (the “Thales ball” over the segment). We claim that these balls cover the whole polytope. Indeed, let  $u \in P$ , then the closed halfspace  $u^T x \geq u^T u$  contains at least one point of the polytope (namely  $u$ ), and hence it contains a vertex  $v$  of  $P$ . But the angle  $0uv$  is obtuse, and hence  $u \in B_v$ .

Since the diameter of  $B_v$  is at most half the diameter of  $B$ , we have  $\text{vol}(B_v) \leq 2^{-n} \text{vol}(B)$ . Using this, it is easy to estimate the volume of  $P$ :

$$\text{vol}(P) \leq \sum_v \text{vol}(B_v) \leq p 2^{-n} \text{vol}(B).$$

This proves the Lemma. ■

**PROOF OF THEOREM** Apply the algorithm  $\mathcal{A}$  with the unit ball  $B$  as its input. It returns a number  $V(B)$  which satisfies  $V(B) \geq \text{vol}(B)$ .

Next, let  $S$  be the set of points which were asked from the oracle and which it declared to be in the ball. Let  $Q = \{e_1, \dots, e_n, -e_1, \dots, -e_n\}$ , and let  $P$  be the convex hull of  $S \cup Q$ . (Throwing in the points of  $Q$  is a little technicality, which is needed since we could not guarantee an explicit ball contained in the convex hull of  $S$ .) Let  $p$  be the number of vertices of  $P$ .

If we apply the algorithm  $\mathcal{A}$  with input  $P$ , then comparing its run with the previous run step-by-step, we see that it asks the same points from the oracle, and to these questions the oracle gives the same answers. Hence the final result must be the same, and so  $V(P) = V(B)$ . By the Lemma,

$$V(P) = V(B) \geq \text{vol}(B) \geq \frac{2^n}{p} \text{vol}(P).$$

Since  $p$  is bounded by a polynomial in  $n$ , the theorem follows. ■

### 3. In the wake of the theorem

Bárány and Füredi [1] improved the result of Elekes: they proved that the error of  $2^{0.99n}$  in the theorem can be replaced by  $n^{0.99n}$ . This means that up to the coefficient of  $n$  in the exponent, the very rough bound obtained by the Ellipsoid Method cannot be improved.

This general negative result does not say anything about special convex bodies. Khachian proved that the exact computation of the volume of a polytope given by a system of linear inequalities is NP-hard (the result remains

valid if we allow an error which is exponentially small in the running time). It is not known whether the volume of a polytope given explicitly in this way can be estimated within a factor of 2 (say).

Since by the theorem of Elekes, the volume cannot be estimated by a polynomial time algorithm in a nontrivial way, the next question is, what happens if we allow randomized (Monte-Carlo type) algorithms. It was a real surprise when in 1989 Dyer, Frieze and Kannan [3] gave a polynomial time randomized algorithm which estimates the volume of a convex body (given by, say, a weak SEPARATION oracle) with arbitrarily small relative error. Together with the theorem of Elekes this shows that (at least in the oracle model) randomization can reduce the computational complexity of a problem (and in fact of a really basic algorithmic problem!) from exponential to polynomial. This is a very important fact, since randomization and its power are among the most fundamental issues in complexity theory (cf. [6]).

The algorithm of Dyer, Frieze and Kannan also shows that the argument of Elekes does not work for randomized volume algorithms (it is worth figuring out where it fails). But some modification of it can be applied to prove negative results even for randomized algorithms. To mention one such result, it can be proved [2] by a similar method that the diameter of a convex body given by, say, a weak MEMBERSHIP oracle cannot be estimated in polynomial time within an error of  $o(\sqrt{n \log n})$ , even if we allow randomization.

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# INCIDENCE GEOMETRY IN COMBINATORIAL ARITHMETIC IN MEMORIAM GYÖRGY ELEKES

By

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There is an ancient relation between arithmetic and geometry. Developed systematically by Minkowski, this connection has become fundamental to many areas of modern number theory. The aim of this writing is to give a rather informal tribute to a quite original and elegant idea of the late György Elekes that, since its appearance thirteen years ago, has been constantly shaping the emerging new area of additive combinatorics. I took the liberty of choosing three examples to illustrate this beautiful idea in its most striking simplicity.

I. Let  $A$  be a set of positive integers of cardinality  $n$ . Their Minkowski-sum  $A + A = \{a + b \mid a, b \in A\}$  clearly satisfies

$$2n - 1 \leq |A + A| \leq \frac{n(n + 1)}{2}.$$

A moment's reflection shows that  $|A + A| = 2n - 1$  if and only if the elements of  $A$  form an arithmetic progression. In general,  $|A + A| \leq 3n - 4$  can only happen if  $A$  is contained in an arithmetic progression of length  $2n - 3$ , and according to a deep result of Freiman [16], in case of  $|A + A| < cn$ , or, using Vinogradov's notation,  $|A + A| \ll n$ ,  $A$  is contained in a low-dimensional generalized arithmetic progression whose size is linear in  $n$ .

These results transfer naturally to the product set  $A \cdot A$ , where  $|A \cdot A| = 2n - 1$  happens if and only if the elements of  $A$  form an geometric progression. These two extremal structures, however, behave very differently; in case  $A$  is an arithmetic progression, one has  $|A \cdot A| \gg n^2 / \log^3 n$ , although no quadratic lower estimate holds in general. According to a conjecture of

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Erdős and Szemerédi [14], this is not a singular behavior. They suggested that

$$\max\{|A + A|, |A \cdot A|\} \gg n^{2-\epsilon}$$

must be true with arbitrarily small but positive epsilon. It is clear that the positivity condition on the elements of  $A$  does not make any difference in this latter context; and most likely the conjecture is valid even if  $A$  denotes a set of real numbers. The nontrivial lower bound [14] was improved upon by Nathanson [20] and Ford [15], but the real breakthrough was made by Elekes [12]. Using a tool from incidence geometry, he obtained a lower bound in the order of magnitude  $n^{5/4}$ .

The proof, which is valid for arbitrary sets of real numbers, depends on the following result of Szemerédi and Trotter [24]. Given in the plane a set of  $P$  points and a set of  $L$  lines, let  $I$  denote the number of point-line incidences among them. Then

$$I \ll \max\{L, P, (LP)^{2/3}\}.$$

If neither  $L$ , nor  $P$  is two small compared to the other one, then the dominating term here is  $(LP)^{2/3}$ , which is best possible.

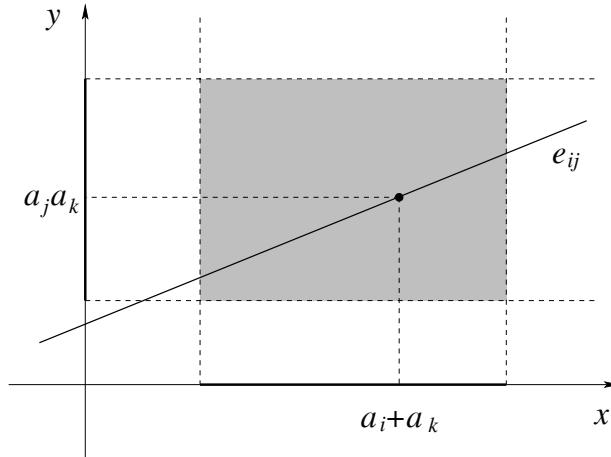


Fig. 1

For a set of real numbers  $A = \{a_1, a_2, \dots, a_n\}$ , consider the set  $(A + A) \times (A \cdot A)$  in the real affine plane. This is a set of  $P = |A + A| \cdot |A \cdot A|$  points

whose two coordinates belong to  $A + A$  and  $A \cdot A$ , respectively. In addition, consider the  $L = n^2$  lines defined by the equations

$$e_{ij} : \quad y = a_j(x - a_i).$$

Since the point  $(a_i + a_k, a_j a_k)$  is incident to the line  $e_{ij}$ , we find  $I \geq n^3$  point-line incidences. In view of  $L \ll P \ll L^2$ , an application of the Szemerédi–Trotter theorem gives  $n^3 \leq I \ll (LP)^{2/3} = (n^2 P)^{2/3}$ , that is,

$$\max\{|A + A|, |A \cdot A|\} \geq \sqrt{P} \gg n^{5/4}.$$

This approach of Gyuri Elekes revealed a so far hidden connection between geometry and number theory, more combinatorial in nature, that led to unexpected new developments in modern mathematics from algebra and number theory to theoretical computer science. First, Elekes and Ruzsa [13] verified the Erdős–Szemerédi conjecture for sets  $A$  of real numbers whose sumset  $A + A$  is small. Shortly afterwards a dual result for the case of small product sets was obtained by Chang [6]. This proof, which heavily uses the arithmetic properties of integers, introduced the elements of harmonic analysis into the study of the question, thus opening a way to attack the problem over finite fields and other algebraic structures. A most exciting example is a deep paper of Bourgain, Katz, and Tao [2] that includes among other applications the first Szemerédi–Trotter type theorem over prime fields, and a discrete analogue of the famous Kakeya-problem. Helfgott [17, 18] verified uniform exponential growth for the Cayley graphs of the groups  $\mathrm{SL}_2(\mathbb{Z}/p\mathbb{Z})$  and  $\mathrm{SL}_3(\mathbb{Z}/p\mathbb{Z})$ , making the first step towards the solution of a long-standing problem of Babai. Further developments in the theory of expander graphs and the spectral theory of Hecke operators are landmarked by the work of Bourgain and Gamburd [3, 4]. The pioneering work of Elekes thus triggered an unparalleled boom in research on the topmost international level, signified by papers in such journals of the highest standard as the Annals of Mathematics or the Inventiones Mathematicae. A new contribution related to Gyuri’s result appears now every week on the Internet. It can be said that his paper ‘On the number of sums and products’ has been of the greatest influence.

II. Let us turn back to the original paper. The closest pursuer of this geometric idea has been József Solymosi [21] who, apart from a logarithmic factor, improved Elekes’s lower bound to  $n^{14/11}$ . An additional value of this incidence geometry based argument is that it also works over the complex

number field, where no true analogue of the Szemerédi–Trotter theorem is known. In the following we sketch Solymosi’s latest idea [22] that improved the exponent to  $4/3$ . To emphasize the beauty of this surprising argument, the technical details will be omitted.

The first ingredient of the proof is the use of the quotient set  $A/A$  instead of the product set  $A \cdot A$ . Heuristically, if an element  $ab$  of the product set can also be written in the form  $cd$ , then the element  $a/d$  in the quotient set has also a representation in the form  $c/b$ . The transfer from  $A/A$  to  $A \cdot A$  can be made precise with the notion of multiplicative energy introduced by Tao [25]. Let  $|A/A| = \ell \gg 1$ , and for further simplification assume that each element of  $A/A$  is represented roughly the same number of times, that is,  $\approx n^2/\ell$  times as the quotient of two elements of  $A$ . Geometrically, we assume that the points of the Cartesian product  $A \times A$  are uniformly distributed on  $\ell$  half-lines that go through the origin.

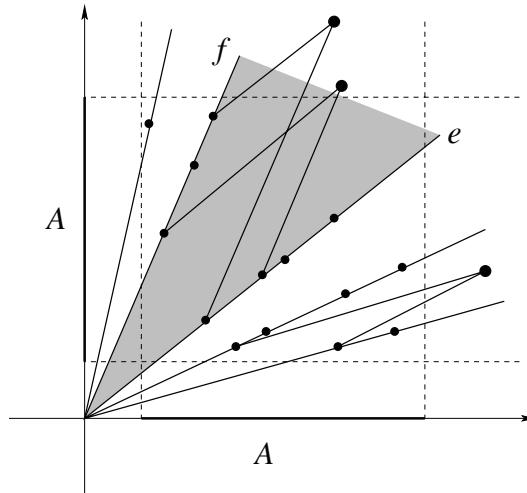


Fig. 2

Consider two neighbouring half-lines  $e$  and  $f$ . Let  $a, b, c, d$  be elements of  $A$  such that the points  $(a, b)$  and  $(c, d)$  are incident to the rays  $e$  and  $f$ , respectively. The point  $(a + c, b + d)$  then lies in the angular section bounded by  $e$  and  $f$ . Due to the linear independence of the directions of the two rays, another similar pair of points leads to a new element of  $(A+A) \times (A+A)$ . The above angular section therefore contains  $\gg (n^2/\ell)^2$  different points from the

set  $(A + A) \times (A + A)$ . Given that there are  $\ell - 1$  such angular sections with pairwise disjoint interiors, one finds that

$$|(A + A) \times (A + A)| \gg n^4/\ell,$$

implying  $|A + A|^2 \cdot |A/A| \gg n^4$ . It follows that  $\max\{|A + A|, |A/A|\} \gg n^{4/3}$ .

III. Let us complete this recollection with a modest personal contribution in the very spirit of Elekes's original proof. Let  $\mathbb{F}$  be an arbitrary field. Denote by  $p = p(\mathbb{F})$  the cardinality of the prime field of  $\mathbb{F}$ . Thus  $p(\mathbb{F}) = \infty$  if  $\mathbb{F}$  has characteristic 0, otherwise  $p$  equals the characteristic of the field. For a polynomial  $f \in \mathbb{F}[x_1, x_2, \dots, x_m]$  and nonempty finite subsets  $A_1, A_2, \dots, A_m$  of  $\mathbb{F}$ ,

$$V_f(A_1, A_2, \dots, A_m) = \{f(a_1, a_2, \dots, a_m) \mid a_i \in A_i\}$$

is the value set of the polynomial  $f$  restricted to  $A_1 \times A_2 \times \dots \times A_m$ . In the case  $A_1 = A_2 = \dots = A_m = A$  we simply write  $V_f(A)$ . When  $f(x_1, x_2, \dots, x_m) = x_1 + x_2 + \dots + x_m$ , the classical Cauchy–Davenport theorem [5, 7, 8] claims that

$$|V_f(A_1, A_2, \dots, A_m)| \geq \min\{p(\mathbb{F}), \sum_{i=1}^m |A_i| - m + 1\},$$

which is best possible. This has been generalized by Dias da Silva and Godinho [9, 10] for elementary symmetric functions, who under certain conditions obtained a linear lower bound using the exterior algebra method of [11]. Applying the polynomial method of Alon [1], Sun [23] found similar lower bounds for another family of polynomials. Once again, geometric incidence theorems proved to be strong enough to yield much better lower bounds [19].

To illustrate this, let  $A$  be an  $n$ -element subset of the real number field, and consider  $f(x, y, z) = xy + yz + zx \in \mathbb{R}[x, y, z]$ . One can readily check that

$$3n - 2 \leq |V_f(A)| \leq \frac{n^3 + 3n^2 + 2n}{6},$$

and that  $|V_f(A)| \ll n^2$  for  $A = \{1, 2, \dots, n\}$ . Here we prove the general estimate  $|V_f(A)| \gg n^{3/2}$ . Consider the set  $A \times V_f(A)$ ; this is a set of  $P = n|V_f(A)|$  points in the plane. For  $a, b, c \in A$ , the point  $(c, ab + bc + ca)$  of this set is incident to the line of equation

$$e_{a,b} : \quad y = (a + b)x + ab.$$

Since  $e_{a,b} = e_{b,a}$ , and the pair  $(a+b, ab)$  uniquely determines the set  $\{a, b\}$ , the number of lines we consider is  $L = \frac{n(n+1)}{2}$ , and  $L \ll P \ll L^2$ . As we counted  $n$  incidences on each line, the Szemerédi–Trotter theorem gives

$$n^3 \ll nL \leq I \ll (LP)^{2/3} \ll n^2 |V_f(A)|^{2/3}.$$

It follows that  $|V_f(A)| \gg n^{3/2}$ .

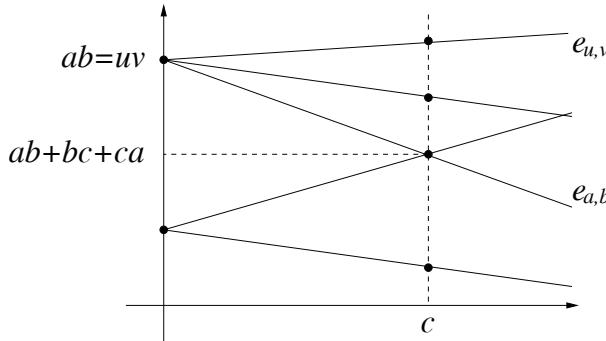


Fig. 3

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# THE WORK OF GYÖRGY ELEKES ON SOME COMBINATORIAL PROPERTIES OF POLYNOMIALS

By

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My active mathematical collaboration with Gyuri Elekes dates back to 1996, when he raised to me some of his problems of algebraic nature related to restricted polynomials. By that time we had a friendship for about fifteen years, we had played soccer together and had many interesting discussions on various mathematical subjects, but had no joint research work.

To be more specific about the problem he was then thinking about, let  $C > 1$  be a real number,  $n$  a positive integer, and  $F \in \mathbb{R}[x, y]$  be a bivariate polynomial with real coefficients. The polynomial  $F$  is called  *$(C, n)$ -restricted*, if there exist sets of reals  $X, Y \subset \mathbb{R}$  with  $|X| = |Y| = n$ , such that  $|F(X \times Y)| \leq Cn$ .

The following polynomials are restricted (with  $C = 2$ , and  $n$  arbitrary):  $F(x, y) = x + y$  and  $G(x, y) = xy$ . For  $F$ , arithmetic progressions such as  $\{0, 1, \dots, n - 1\}$  give suitable sets  $X, Y$ . As for  $G$ , geometric progressions like  $X = Y = \{1, 2, \dots, 2^{n-1}\}$  may serve as witnesses for restrictedness. The examples given here tend to be typical in a suitable sense. In fact, one of the main results of our joint paper [7] states:

**THEOREM 1.** *For every real number  $C \geq 1$  and for every positive integer  $d$  there exists a positive integer  $n_0 = n_0(C, d)$  with the following property: if the polynomial  $F \in \mathbb{R}[x, y]$  of degree  $d$  is  $(C, n)$ -restricted for some  $n > n_0$ , then there exist univariate polynomials  $f, g, h \in \mathbb{R}[z]$  such that either*

1.  $F(x, y) = f(g(x) + h(y)), \quad \text{or}$
2.  $F(x, y) = f(g(x) \cdot h(y)).$

There is a variant of the theorem in [7] which offers a similar characterization of restricted rational functions  $F \in \mathbb{R}(x, y)$  instead of polynomials.

The requirement of restrictedness can be somewhat weakened. It is sufficient to require that there exists a constant  $c > 0$  such that for a suitably large  $n$  there are sets  $X, Y, Z \subset \mathbb{R}$ ,  $|X| = |Y| = |Z| = n$ , such that the graph of  $F$  has at least  $cn^2$  points on the grid  $X \times Y \times Z$ .

The restrictedness of  $F$  can be checked in the following simple way. Put

$$q_1(x, y) := \frac{\partial F}{\partial x} / \frac{\partial F}{\partial y}.$$

By using Theorem 1 it can be shown without much difficulty that  $F$  can be written as in 1 or 2 if and only if

$$q_2(x, y) := \frac{\partial^2(\log |q_1(x, y)|)}{\partial x \partial y} = 0$$

holds at every point in the  $(x, y)$ -plane where  $q_2$  is defined.

This test for restrictedness allowed us to prove the following conjecture of G. Purdy:

**CONJECTURE 2.** *Let  $s$  and  $t$  be two lines in the Euclidean plane,  $\mathcal{U} \subset s$ ,  $\mathcal{V} \subset t$  be sets of points with  $|\mathcal{U}| = |\mathcal{V}| = n$ , and  $C > 0$  a constant. Suppose that among the  $n^2$  pairwise distances  $d(U_i, V_j)$  ( $U_i \in \mathcal{U}$ ,  $V_j \in \mathcal{V}$ ) only  $\leq Cn$  different values occur, and  $n > n_0(C)$ . Then the two lines are either parallel or perpendicular.*

Indeed, an application of the Cosine Theorem gives that we need to check the restrictedness of a polynomial of the form  $F(x, y) = x^2 - 2\lambda xy + y^2$ . Now simple calculation yields that  $q_2 \equiv 0$  is possible only if  $\lambda = 0$  or  $\pm 1$ . These values give precisely the alternatives in the conjecture. In [3] Gyuri Elekes, answering a question of P. Brass and J. Matoušek, obtained a stronger statement: if the two lines  $s$  and  $t$  are not perpendicular or parallel, then at least  $cn^{5/4}$  different values occur among the pairwise distances of the points of  $\mathcal{U}$  to the points of  $\mathcal{V}$ , where  $c > 0$  is a suitable constant.

The tools used in the proof of Theorem 1 are from several areas: the theorem of Pach and Sharir [9] on the number of incidences between points and curves from combinatorial geometry, the theorem of Lüroth from the theory of fields, and combinatorial arguments. Much of these have already appeared in earlier papers of Gyuri Elekes on related topics [1], [2], [4].

One of the algebraic tools used in the proof is the following Lüroth type statement. In fact, our collaboration started with thinking on a question asked by him, which has eventually led to this lemma. The lemma gives a handy necessary and sufficient condition for two rational curves to coincide.

LEMMA 3. *Let  $\mathbb{F}$  be a field and  $p_1(t), q_1(t), p_2(t), q_2(t) \in \mathbb{F}(t)$  rational functions in one variable each, such that the parameterized curves  $\gamma_1 : (p_1(t), q_1(t))$  and  $\gamma_2 : (p_2(t), q_2(t))$  coincide in the sense that they both satisfy the same irreducible polynomial  $h(x, y) \in \mathbb{F}[x, y]$ .*

*Then there are rational functions  $f, g, \phi_1, \phi_2 \in \mathbb{F}(t)$  for which  $p_i = f(\phi_i)$  and  $q_i = g(\phi_i)$  for  $i = 1, 2$ .*

*Moreover, if  $p_1(t), q_1(t), p_2(t), q_2(t)$  are all polynomials (from  $\mathbb{F}[t]$ ), then  $f, g, \phi_1, \phi_2$  can also be assumed to be polynomials.*

The incidence bound of Pach and Sharir [9] was used through its simple but amazingly useful consequence:

LEMMA 4. *For every  $c > 0$  and positive integer  $d$ , there are positive constants  $n_0 = n_0(c, d)$  and  $C' = C'(c, d)$  with the following property.*

*Let  $\mathcal{A} \subset \mathbb{R}^2$  with  $|\mathcal{A}| \leq N^2$  and assume that a set  $\Gamma$  of irreducible real algebraic curves of degree not exceeding  $d$  has the property that every  $\gamma \in \Gamma$  intersects  $\mathcal{A}$  in many points:*

$$|\gamma \cap \mathcal{A}| \geq cN.$$

*Then  $|\Gamma| \leq C'N$ , provided that  $N > n_0$ .*

Next we discuss in a bit more detail a result which plays an important role in the proof of Theorem 1. It provides a way to obtain a qualitative statement (on the shape of the polynomials involved) from the quantitative information at hand (on the number of incidences of curves and points).

THEOREM 5. *Let  $C \geq 1$  be a real number, and  $d$  a positive integer. Then there exists a real number  $c^* = c^*(C, d)$  for which the following holds: let  $X, Z \subset \mathbb{R}$  be two sets of real numbers,  $n \leq |X|, |Z| \leq Cn$  and  $\mathcal{F} \subset \mathbb{R}[t]$  be a subset of polynomials of degree at most  $d$  with  $|\mathcal{F}| \geq n$ . Suppose further that the graph of each  $f_i \in \mathcal{F}$  contains at least  $n$  points from the set  $X \times Z$ . Then from the polynomials  $f_i \in \mathcal{F}$  at least  $c^*n$  is of the form*

1.  $f_i(x) = f(g(x) + s_i)$  or
2.  $f_i(x) = f(g(x) \cdot s_i)$ ,

*where  $s_i \in \mathbb{R}$  and the polynomials  $f, g \in \mathbb{R}[t]$  are independent of  $i$ .*

Even the case  $d = 1$  of the preceding Theorem is interesting. In that case the polynomials  $f_i$  are linear, hence they determine lines. This special case appeared in [1] under the name of Linear Theorem. The Linear Theorem has a quite appealing geometric formulation. If the assumptions of the theorem hold, then from the lines in  $\mathcal{F}$  either we can select many parallel lines (alternative 1 in Theorem 5), or there are many which are concurrent (alternative 2).

Similar tools, including a projective version of the Linear Theorem, were used in the following result of Elekes and Király in [6], which extends a central research direction of additive number theory to non commutative groups. Freiman and Ruzsa studied subsets  $A, B$  of  $\mathbb{R}$ ,  $|A| = |B| = n$ , such that  $|A + B| \leq Cn$ , and described their structure in terms of some natural generalizations of arithmetic progressions. Generalizations to non Abelian groups were initiated by Elekes in [1] and [2] where the subsets of the one-dimensional affine group were considered. In [6] this was extended to the projective case.

Let  $\mathcal{P}$  stand for the group of automorphisms of the real projective line  $\mathbb{R} \cup \{\infty\}$

$$\mathcal{P} = \{x \mapsto \frac{ax + b}{cx + d}; \text{ where } a, b, c, d \in \mathbb{R} \text{ and } ad - bc \neq 0\}.$$

For  $\Phi, \Psi \subset \mathcal{P}$  we put

$$\Phi \circ \Psi = \{\varphi \circ \psi : \varphi \in \Phi, \psi \in \Psi\}.$$

**THEOREM 6.** *Let  $C > 0$  be a constant and  $\Phi, \Psi \subset \mathcal{P}$  be finite subsets. Suppose that  $|\Phi|, |\Psi| \geq n$  and  $|\Phi \circ \Psi| \leq Cn$ .*

*Then there exists an Abelian subgroup  $S$  of  $\mathcal{P}$  such that  $\Phi$  ( $\Psi$ , respectively) can be covered by a constant number of left cosets (right cosets, resp.) of the subgroup  $S < \mathcal{P}$ .*

Note that in the above result  $\Phi \circ \Psi$  is the product of the complexes  $\Phi$  and  $\Psi$  in the group  $\mathcal{P}$ . In this setting the cosets of  $S$  serve as generalized arithmetic progressions.

In their paper [8] Gyuri Elekes and Endre Szabó obtained far reaching extensions and generalizations of Theorem 1. Next I would like to highlight one of the appealing and powerful results from [8]. Let  $D \subset \mathbb{C}$  denote the unit disk on the complex plane  $\mathbb{C}$ .

**THEOREM 7.** *Let  $d$  be a positive integer. There exist positive constants  $\eta = \eta(d)$ , and  $n_0 = n_0(d)$  with the following property: if  $V \subset \mathbb{C}^3$  is an algebraic surface of degree  $d$ , then the following three statements are equivalent:*

(a) *There exists an integer  $n > n_0(d)$  and sets  $X, Y, Z \subset \mathbb{C}$  with  $|X| = |Y| = |Z| = n$  for which*

$$|V \cap (X \times Y \times Z)| \geq n^{2-\eta}.$$

(b)  *$V$  either contains a cylinder over a curve of the form  $F(x, y) = 0$  or  $F(x, z) = 0$ , or  $F(y, z) = 0$ ; or, otherwise, there are one-to-one complex analytic functions  $f, g, h : D \rightarrow \mathbb{C}$  with analytic inverses such that*

$$V \supset \{(f(x), g(x), h(x)) \in \mathbb{C}^3; x, y, z \in D, x + y + z = 0\}.$$

(c) *For every positive integer  $n$  there exist subsets  $X, Y, Z \subset \mathbb{C}$ ,  $|X| = |Y| = |Z| = n$ , such that*

$$|V \cap (X \times Y \times Z)| \geq (n - 2)^2/8.$$

The theorem makes a significant advance in that it holds over  $\mathbb{C}$ , while the earlier methods work over  $\mathbb{R}$  only. At the same time, the theorem has a real version dealing with surfaces  $V \subset \mathbb{R}^3$ . This involves real analytic functions defined on the interval  $(-1, 1)$ .

It is worth pointing out, that in Theorem 7  $V$  can be an arbitrary surface in the three-space, not just the graph of a function  $F(x, y)$ . In fact, Theorem 7 can be considered as a three dimensional special case of a very general result, namely the Main Theorem in [8] on bounded degree algebraic sets from higher dimensional ambient spaces.

A further important virtue of the theorem is that merely  $n^{2-\eta}$  incidences are sufficient to prove the implication (a) $\Rightarrow$ (b), instead of the  $cn^2$  incidences which were needed in earlier results of similar character.

The strength of their results is amply demonstrated by beautiful new applications to geometric problems. They give a strong answer to a question of Hirzebruch on the number of tangencies among  $n$  non-degenerate conic sections on the projective plane over a field of characteristic greater than 2, such that no three of the conic sections are tangent to each other at the same point. They improve the earlier bound  $O(n^{2-\frac{1}{777}})$  to  $O(n^{\frac{9}{5}})$ . As a direct application of Theorem 7, they obtain an interesting sub-quadratic upper bound on the number of triple points of circle grids with non collinear centers

$P_1, P_2, P_3$ . The strength and generality of the results in [8] make it very likely that further substantial applications will be found in the future.

A major innovation of the paper is an extension of former results of Elekes and his collaborators on graphs of polynomial functions to *algebraic multi-functions*. Instead of giving here the long and technical definition, we provide a motivating example. Let  $a, b, c \in \mathbb{C}$  be complex numbers and consider the curve

$$cx^2 + bx y + ay^2 = 0$$

on the complex  $(x, y)$ -plane. As long as  $c \neq 0$ , the familiar formula for solving quadratic equations tells us that this algebraic set can be considered as the graph of the multi-function (multiple valued map)

$$F_{a,b,c} : x \mapsto y = x \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

This multi-function is generically two valued in the sense that for most selection of the parameters  $(a, b, c) \in \mathbb{C}^3$  the map  $F_{a,b,c}$  sends  $x$  to two different values  $y$ . One can form the composition of multi-functions by viewing them as binary relations. For example, the composition of the two valued  $F_{a,b,c}$  and  $F_{a',b',c'}$  will be the (generically) four valued multi-function

$$F_{a',b',c'} \circ F_{a,b,c} : x \mapsto y = x \cdot \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \cdot \frac{-b' \pm \sqrt{b'^2 - 4a'c'}}{2a'}.$$

One of the central results of [8] is the Composition Lemma, which describes the most degenerate case of compositions of algebraic multi-function families by proving that they can be derived from an algebraic group structure.

As an interesting and perhaps surprising cultural connection of the paper, we mention that a famous and difficult result from model theory, E. Hrusovski's Group Configuration Theorem also enters the scene. In their Composition Lemma, Elekes and Szabó give an independent proof for a major special case of Hrusovski's theorem which is based on algebraic geometry. Hrusovski's theorem is stated in the language of model theory (see for example [10]). The Composition Lemma is given in a language much more accessible for those interested in combinatorial geometry, hence, taking also into consideration its generality, it is much more likely to lead to further applications to specific geometric problems.

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**GYURI ELEKES AND THE INCIDENCES**

By

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Our old friend, Gyuri Elekes<sup>1</sup> was already very ill when we have finished one of *his* last paper<sup>2</sup> [3]. He died within a month. *This* paper is a continuation of [5], that contains some personal memories of the first author. Here we describe our joint work, partly published in [3] and also a basic ingredient of it, a joint paper of Elekes and Szabó [4]. The papers [3] and [4] seemed to be very important for Gyuri Elekes.

When Elekes learnt that he was soon to die, in a very short period he has finished seven papers. Two of them were these papers [4] and [3].

**1. The beginning**

In January 2006 Elekes wanted to prove Theorem 1 below and for this he needed that some functional relation of the form

$$F(\varphi_1(u, v), \varphi_2(u, v), \varphi_3(u, v)) = 0$$

cannot be satisfied (see [4, 3] unless  $F$  has a very special form. Investigating the singularities of the functions  $\varphi_i$ , Elekes and Simonovits proved that<sup>3</sup>

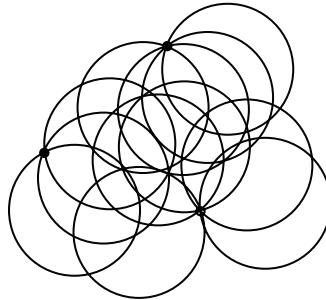
**THEOREM 1.** *There exist two constants  $c > 0$  and  $\eta > 0$  such that if we consider three non-collinear points in the plane,  $A$ ,  $B$ , and  $C$ , and we have*

<sup>1</sup> György Elekes, to be precise.

<sup>2</sup> More precisely, this three-author paper was written up by Gyuri.

<sup>3</sup> A more detailed story is described in [5].

through each of them  $n$  unit circles,<sup>4</sup> then the number of triple points, i.e. points belonging to three such circles, is at most  $cn^{2-\eta}$ .



The motivation of this was the following.

Consider a square grid arrangement in the plane and  $n$  horizontal,  $n$  vertical straight lines, and  $n$  of slope 1 (i.e. at  $45^\circ$ ). They will have  $cn^2$  triple points. This can be seen in Figure 1(b). Moreover, using  $k \geq 3$  appropriate slopes we obtain  $k$  families of straight lines for which the number of crossings of multiplicity  $k$  is  $c_k n^2$ .

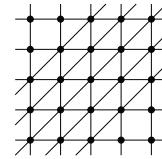
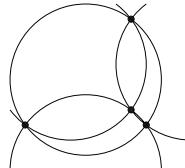
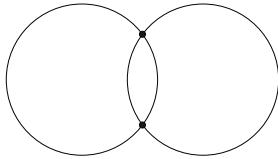


Figure 1.

(a) Circles

(b) and straight lines

This makes a combinatorial distinction between straight lines and *unit* circles. For arbitrary circles such a distinction cannot be made, since the straight lines can be transformed into circles by an inversion that keeps the incidences.<sup>5</sup>

Having this, we wanted to find out, when and why can such systems of curves have  $cn^2$  triple points. We wished to show that the number of triple points is small in the typical cases: if we have many triple points then we have a “strongly degenerate” case.

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<sup>4</sup> From now on, we shall write “ $n + n + n$  curves” in such cases.

<sup>5</sup> The opposite direction does not hold: two points can be contained in arbitrary many circles and this incidence pattern cannot be obtained using straight lines.

The proof of Theorem 1 and some of its generalizations heavily used (and also influenced?) a result of Elekes and Szabó [4]. This is why three of us decided to finish our work together. We had also another reason to join our forces: we three had three different ways of looking at the subject, and the various approaches helped each other.

We have already mentioned that Elekes worked a lot and hard in his last weeks on these two papers (too). Writing that “we have finished our paper” was a slight cheating. Actually, we have finished and published its first version [3], where we avoided using analytic branches of inverse functions of polynomials. This paper has already appeared. However, we still have to finish a more general, deeper version of our paper. Technically speaking, the difference between these versions is that in the first version we restrict our consideration to *explicitly parameterized* 1-parameter families of curves, while the second version will settle the problem of *implicitly parameterized* families.<sup>6</sup> Elekes felt that whenever one can apply the theorem to implicitly parameterized families of curves, then the curves can be cut into some subsegments, so the families can be replaced by explicitly parameterized families, and applying the special version (on explicitly parameterized curves) gives the same conclusion. This would mean that the practical difference between the “explicit” and “implicit” versions is not that much, at least from the point of view of applicability. The first author of this paper felt that the “implicit” version is much more natural and nicer.

The main question is as follows.

Given a 1-parameter family of “nice” curves, it may happen that they have  $cn^2$  triple points. We have seen this in case of straight lines. Applying inversion, we saw that this may happen for circles, too. We have stated that this cannot occur in case of unit circles, pierced by three distinct given points.

So the main question asks:

Given three 1-parameter families of nice curves, when can we have  $cn^2$  triple crossings and why?

**REMARK 2.** The properties we investigate are — in some sense — invariant under “nice” transformations. “Nice” could mean continuously differentiable, or  $C^\infty$ , however, for us “nice” means analytic, or algebraic. If we

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<sup>6</sup> Some definitions will follow only afterwards. Here we yield the definitions in this footnote but later we return to this. A family  $\{\gamma_t\}$  of plane curves given in the form  $F(x, y, t) = 0$  is implicitly parameterized, but if this family is parameterized in the form  $t = f(x, y)$  then it is explicitly parameterized.

have three nice families of curves with many triple points, we may apply any (nice) transformation<sup>7</sup>

$$(1) \quad u := f(x, y), \quad v := g(x, y)$$

to these families to get “nice” curves with the same number of triple points: If the transforming functions are nice then the new families will also be 1-parameter families, smooth (nice) and appropriate  $n + n + n$  curves of the new families will again have  $\approx cn^2$  triple crossings.<sup>8</sup>

## 2. Incidences

Elekes was interested above all, — at least in the area described here — in the following.

If we have a general family  $\mathcal{A}$  of plane curves and we select  $n$  curves from  $\mathcal{A}$ , then the number of double-points can easily be quadratic, (i.e.  $> cn^2$ ), however, the number of triple points is (mostly) relatively small. We have seen that in case of straight lines we may have  $cn^2$  triple points. However, this is a degeneracy, some kind of a strange coincidence. Which type of degeneracies should be excluded to ensure that the number of triple points be small?

Investigations of this type, on the incidence structures of straight lines and points, — or more general families of curves — belong to the fundamental questions of combinatorial geometry and perhaps the famous English mathematician, Sylvester was the first to investigate them, approximately 140 years ago. He proved that

**THEOREM 3. (Sylvester)** *Let  $\mathcal{L}$  be a system of  $n$  straight lines in  $\mathbb{R}^2$ . Then the number of triple points is at most  $n^2/6 + O(n)$ , and this is sharp.*

If we ask for the maximum number of triple points in case of unit circles, then not only the number of triple points but the number of double points (intersections) is at most  $n(n - 1)$ . Moreover, if we restrict ourselves to curves where any two intersect in at most  $B$  points, then the number of intersections is still at most  $O(n^2)$ . Answering a question of Paul Erdős, Elekes proved that

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<sup>7</sup> and not just the “inversion”

<sup>8</sup> Of course, we may use any — not so nice — transformations as well. However, here we are interested only in nice families.

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**THEOREM 4. (Elekes [2])** *There exist  $n$  unit circles with at least  $cn^{3/2}$  triple points.*

The upper and lower bounds are rather far from each other.

We could conjecture that if we consider “nice” families of curves that are not straight lines, then we may have only  $o(n^2)$  triple points. This is not so: the family of straight lines can be transformed into families of *congruent parabolas* with  $cn^2$  triple crossings. (Or — as we have mentioned — it can be transformed into families of circles.)

On the other hand, if we consider nice families of curves and exclude certain degeneracies, then we can prove that the number of triple points is only at most  $cn^{2-\eta}$ , for some suitable constants  $c > 0$  and  $\eta > 0$ .

Here we shall explain our results, and the background, without trying to formulate them in their most general form. We should clarify — among others — two questions:

- Which are the nice curves and nice families of curves?
- Which degenerate cases should be excluded to get only  $o(n^2)$  triple crossings.

## 2.1. Nice families of curves and their Enveloping curves

Below we wish to explain the most important phenomena. Therefore occasionally our formulations will be slightly loose, often heuristic. In our next paper, on the implicitly parameterized families, we shall take the effort to be 100% precise. We start with defining the curves, families of curves and the main content of our results. Here a *curve*  $\gamma$  is always the 0-set of some polynomial  $P$  of two variables, in the plane:

$$(2) \quad \gamma = \left\{ (x, y) \in \mathbb{R}^2 \mid P(x, y) = 0 \right\}$$

Of course, the questions we consider can be considered above any field, primarily above the reals and complex numbers. However, we assume that most of our readers prefer the real curves. Thus here we shall restrict ourselves (with two exceptions) to the real case. The curves will be denoted by script capital letters:  $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{E}$ , or sometimes by  $\gamma$ .

If we start changing smoothly some parameter in the equation (2) describing a curve, then the curve itself often starts changing smoothly. This

way we get a family of curves. If the parameter is  $t$ , then we mostly consider families of curves described by polynomials depending on  $t$ :

$$(3) \quad \gamma_t = \left\{ (x, y) \in \mathbb{R}^2 \mid P_t(x, y) = 0 \right\}.$$

Here we are interested only in families of curves where the coefficients are polynomials of a parameter  $t$ :  $P_t(x, y) = \tilde{P}(x, y, t)$ , where  $\tilde{P}$  is a polynomial of the three variables. The families usually will be denoted by script capital letters, like  $\{\mathcal{A}_t\}$ ,  $\{\mathcal{B}_t\}$ , where the index is the parameter: If  $t_1 \in \mathbb{R}$  is a parameter value, then  $\mathcal{A}_{t_1}$  denotes the member of the family  $\{\mathcal{A}_t\}$  corresponding to  $t = t_1$ .

**EXAMPLE 5.** Whatever we write here is valid for arbitrary families of curves. However, it is worth keeping in mind the following case that is slightly more general than the unit circles in Theorem 1. We are given a point  $A$  in the plane and a convex closed curve  $\mathcal{A}_0$ , defined by a polynomial. We rotate  $\mathcal{A}_0$  around  $A$ , getting a family of curves, see Figure 2. We do not assume  $A \in \mathcal{A}_0$ . Denote by  $\mathcal{A}_t$  the curve rotated by  $\alpha$ , for  $t = \tan \frac{\alpha}{2}$ .

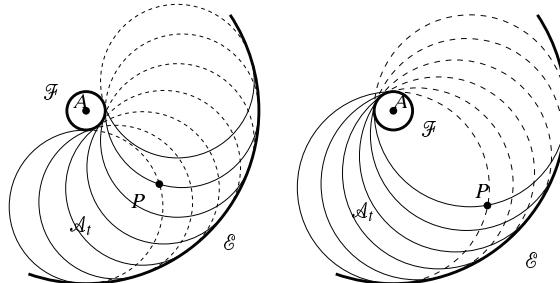


Figure 2. Families of convex curves, two situations

The reader could ask, why did we use  $t = \tan \frac{\alpha}{2}$ , to parametrize the curves, instead of using  $\alpha$  itself. The answer is that in the equations of the rotated curves we have  $\sin \alpha$  and  $\cos \alpha$ . They are not polynomials of  $\alpha$ . However, if we use  $t = \tan \frac{\alpha}{2}$ , then we shall obtain polynomials of  $t$ , after some work. Here we emphasize that the actual form of these polynomials is completely irrelevant for us, we need only the existence of these polynomials. Moreover, the proofs will neither use these polynomials.

The Reader can see two situations in Figure 2: On the right the center  $A$  (of the rotation) is inside of  $\mathcal{A}_t$ , on the left it is outside. The curves cover a ring type domain in both cases. If  $A$  happened to be on  $\mathcal{A}_0$ , then the inside

circle of this ring would shrink to a point, so the rotated curves would cover a disk.

One can see two special curves on this figure:  $\mathcal{E}$  and  $\mathcal{F}$ . They are the enveloping curves of these two families. In general,

**DEFINITION 6.** We call a curve  $\mathcal{E}$  the *enveloping curve* of  $\{\mathcal{A}_t\}$  if in each point of  $\mathcal{E}$  some curve  $\mathcal{A}_t$  of the family is tangent to  $\mathcal{E}$ , however, no  $\mathcal{A}_t$  has an arc common with  $\mathcal{E}$ .

So nearby points of  $\mathcal{E}$  are tangent-points of distinct curves  $\mathcal{A}_t$ , hence infinitely many curves  $\mathcal{A}_t$  are tangent with  $\mathcal{E}$ .<sup>9</sup><sup>10</sup>

We have subdivided each curve  $\mathcal{A}_t$  of Figure 2 into two arcs, one indicated by a thinner line the other by a thicker but broken line. Each point in the domain covered by the whole curves is covered by two such half-curves: a thin continuous line and a thick broken line.

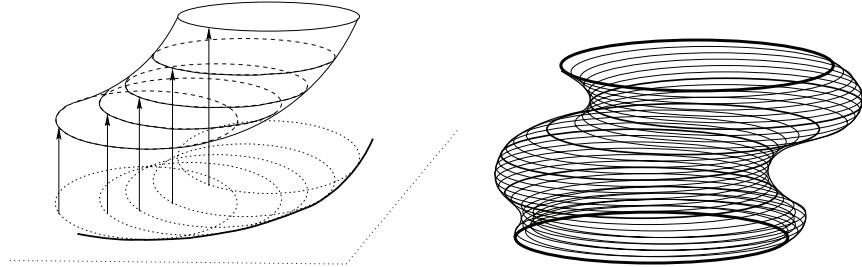


Figure 3. The lifted curves define a “spiral column”

It would simplify the situation if we could achieve that the points on a curve determine the parameter of that curve: each point is only on (at most) one curve  $\mathcal{A}_t$ . Staying in  $\mathbb{R}^2$ , we cannot achieve this. Therefore we apply a “lifting”: the curve  $A_t$  (see Figure 3) is lifted from the plane into the space,

<sup>9</sup> Observe that we have formulated our definition so that a non-degenerate sub-arc of an enveloping curve is also an enveloping curve, where degenerate means a point or an empty arc.

<sup>10</sup> If the parameterized family is given in form of  $F(x, y, t) = 0$ , then the enveloping curves are (basically) described by

$$F(x, u, t) = 0 \quad \text{and} \quad \frac{\partial}{\partial t} F(x, y, t) = 0.$$

We have to remark here that there are several ways to define the enveloping curves and they are not completely equivalent.

into height  $t$ . These space-curves constitute a spiral tube (similar to the barock columns). Let

$$V = \left\{ (P, t) \in \mathbb{R}^2 \times \mathbb{R} \mid P \in \mathcal{A}_t \right\}.$$

Clearly, if we draw our curves onto the surface  $V$ , then they become non-intersecting: each point of the surface belongs only to one curve. Let us return to the problem of triple points.

**DEFINITION 7.** Given three families of curves:  $\{\mathcal{A}_r\}$ ,  $\{\mathcal{B}_s\}$ , and  $\{\mathcal{C}_t\}$ , we shall say that they are “**in special position**”, if for some constant  $c > 0$ , for infinitely many integers  $n$  we can find three  $n$ -tuples of parameters,  $X, Y, Z \subset \mathbb{R}$  for which “the generalized grid”  $X \times Y \times Z \subset \mathbb{R}^3$  has at least  $cn^2$  parameter-triples  $(r, s, t)$  (out of the  $n^3$  possible ones) that are triple points:

$$\mathcal{A}_r \cap \mathcal{B}_s \cap \mathcal{C}_t \neq \emptyset.$$

Actually, a point can correspond to many triple intersections and we are counting the points, not the parameter triples.

If our curves are not “in special position”, then we say that they are “**in typical position**”.

These definitions are motivated by that mostly (?) the number of triple points is small. Thus, e.g., the families of circles in Theorem 1 are “in typical position”. Observe this fairly surprising jump in the number of triple points: three such families of curves either have (at least)  $cn^2$  triple points, for suitably chosen parameters, or the families are “in typical position”, and then — whichever way we choose the three  $n$ -tuples  $X$ ,  $Y$ , and  $Z$  above, — the generalized grid  $X \times Y \times Z \subset \mathbb{R}^3$  will give at most  $cn^{2-\eta}$  parameter-triples  $(r, s, t)$  corresponding to triple intersections.<sup>11</sup> We shall see both behaviours below. In Example 8 we shall see families of parallel straight lines “in special position”. On the other hand, in Example 9 we see families of circles “in typical position”. This will lead us to the generalization of Theorem 1.

**EXAMPLE 8.** Let  $\mathcal{A}_r, \mathcal{B}_s, \mathcal{C}_t$  denote the horizontal line  $y = r$ , the vertical line  $x = s$  and the skew line  $y = x + t$ , respectively. They form three families:  $\{\mathcal{A}_r\}$ ,  $\{\mathcal{B}_s\}$  and  $\{\mathcal{C}_t\}$ . In Figure 1(b) one can see this three families

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<sup>11</sup> If  $n$  is large, then  $n^2$  is much larger than  $n^{2-\eta}$ . So either we have very many triple points, or very few: there are no in-between situations. Similar phenomena occur in some other combinatorial situations as well.

of lines, and it is easy to count the triple-crossings. It is clearly a “special arrangement of curves”: for example if  $X = Y = Z = \{a, 2a, 3a, \dots, na\}$  for an arbitrary value  $a > 0$  then in the “generalized grid”  $X \times Y \times Z$  there are  $cn^2$  triple-crossings.

These three families of lines will play an important role. We shall see in Theorem 16 that each “special arrangement of curves” can be reconstructed<sup>12</sup> via clever transformations from this single configuration.

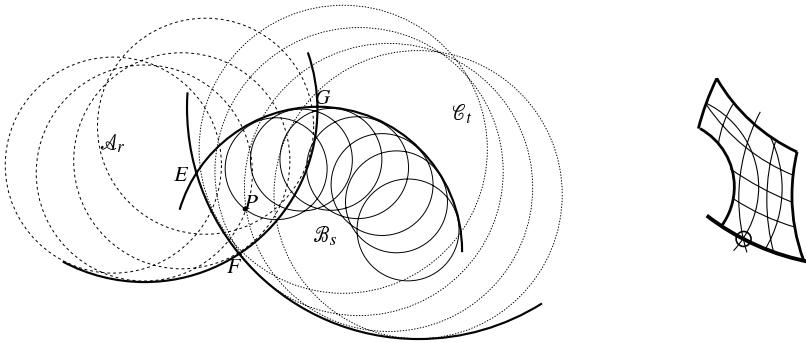


Figure 4. (a) Three families of circles

(b) circular pentagon

EXAMPLE 9. In Figure 4(a) one can see three copies of the families of curves of Example 5. Given three centers of rotation and three closed convex curves, we rotate each of the three curves around the center assigned to it. This way we get three families of curves. Denote them by  $\{\mathcal{A}_r\}$ ,  $\{\mathcal{B}_s\}$  and  $\{\mathcal{C}_t\}$ .

In Figure 4(a) the letter  $P$  marks a triple-crossing of three curves, one from each family: one curve is drawn continuously, the other one is dashed and the third one is dotted. Each of the three families sweeps through an annulus, the domain of triple-crossing points is the intersection of the three annuli. On the left it is just the “curvy triangle”  $EFG$ . In general, the situation can be more complicated. E.g., if we slightly shrink  $\mathcal{B}_0$ , the family  $\{\mathcal{B}_s\}$  will not reach  $F$ . Hence the triple-crossing points will not cover this corner of the domain  $EFG$ . There are cases where the domain of triple-crossing points is even more complicated, see Figure 4(b).

<sup>12</sup> Basically it contains an image of such a configuration.

**THEOREM 10.** Suppose that in Example 9 the three families of curves are arranged according to Figure 4(a), i.e. the set of triple-crossings is precisely the “curvy triangle”  $EFG$ . Then our configuration is a “typical arrangement of curves”, hence  $n - n - n$  curves can have at most  $cn^{2-\eta}$  triple-crossings for some suitable constants  $c > 0$  and  $\eta > 0$ .

Naturally there are many other “special arrangements of curves” besides the one in Example 8. We have seen in Remark 2, how to get, with the help of continuous transformations, numerous new “special arrangements of curves” from a single one. Presently we use only transformations that can be given via polynomial functions, and their image set is two dimensional. (This later condition excludes, e.g., the projection to a line.) They are called *polynomial transformations*.

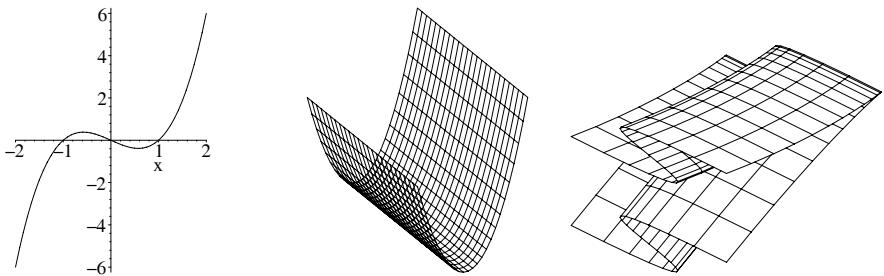


Figure 5. Folding the plane

**EXAMPLE 11.** Let  $f(x, y) = x^2$  and  $g(x, y) = y^3 - 3y$  in (1) in Remark 2. This is an extremely special situation, since the transforming functions depend only on one variable: we may study separately the horizontal and vertical behaviour of this transformation. On the one hand, the function  $x^2$  maps  $\mathbb{R}$  into itself “folding” the negative side back onto the positive side. Therefore our transformation also “folds the plane in half” along the vertical line  $x = 0$ . On the other hand, the function  $y^3 - 3y$  is increasing on the half-line  $(1, \infty)$ , then “turns back” and decreases on the interval  $(-1, 1)$ , then “turns” again and increases on the remaining half-line  $(1, \infty)$ . Therefore our transformation also “folds” the plane twice in the  $y$  direction: along the horizontal lines  $y = -1$  and  $y = 1$ . One can imagine a sheet first folded in half along a vertical line, then “wrinkled” along two (nearby) horizontal lines, and finally “ironed” into the plane. Our transformation works in the plane in much the same way, but there is a subtle difference. If one cuts the sheet into rectangles along the fold-lines, then the ironing smoothes each rectangle congruently into the

plane. However, our transformation “distorts” the rectangles both horizontally and vertically: some parts get stretched out, others are shrunk.

This example illustrates well the behaviour of other polynomial transformations — the most important difference being that the fold-lines are no longer straight lines. Let  $\phi : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  be any polynomial transformation. The locus of critical points, where the Jacobian determinant of  $\phi$  vanishes, consists of finitely many curve components and finitely many isolated points. The curve part is the correct generalization of the fold-lines of Example 11,  $\phi$  will actually “fold” the plane along some of these curves. One can add a few more curves to the picture so that these curves together “cut up” the plane into “curvy polygonal domains”, and  $\phi$  is one-to-one on each of these domains. The addition of extra curves is necessary as shown in the example  $(x, y) \rightarrow (x^2 - y^2, 2xy)$ , which is simply the map  $z \rightarrow z^2$  of the complex plane, and has a single critical point at the origin.

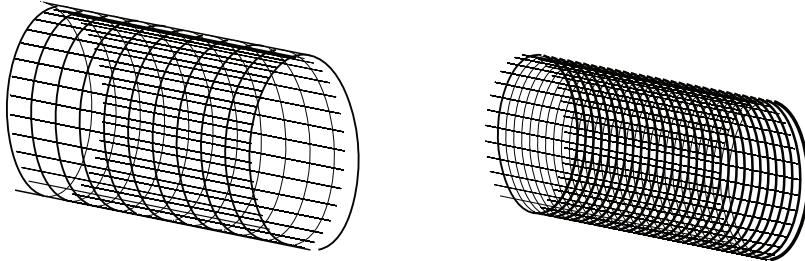
Univariate polynomials usually have no inverse function, hence we often use “multi-valued inverse functions”. Good examples are the formulas for solving quadric, cubic or quartic equations which are two, three and four-valued. More generally, the inverse of a polynomial of degree  $d$  is (at most)  $d$ -valued. Analogously, we can invert polynomial transformations. Their inverses are “multiple-valued transformations” which send each point to multiple images<sup>13</sup>. One can easily estimate the number of images. There can be finitely many exceptional points in the plane whose image is an entire curve (hence infinitely many points), but all other points can have at most  $\deg(f) \cdot \deg(g)$  images,<sup>14</sup> where  $f$  and  $g$  denote the polynomials defining the transformation to be inverted (see formula (1) of Remark 2).

There is another possibility that we haven’t used so far: one can draw not only in the plane, but on a sphere, on a cylinder, or on any other smooth enough surface. We met such a drawing, e.g., in Figure 3. In this paper we use *smooth algebraic surfaces*, that is, surfaces defined via polynomial equations which have a tangent plane at each of their points.

EXAMPLE 12. Let us consider again Example 8 and the corresponding Figure 1(b). If we “roll up” this drawing vertically then we get a horizontally

<sup>13</sup> More precisely: a single point can have zero, one, or many images.

<sup>14</sup> A system of two bivariate polynomial equations have either infinitely many solution or at most as many as the product of the degrees of the equations. (Bezout Theorem?)



(a) Spiral and “parallel lines”

(b) Grid on the cylinder

lying cylinder and three families of curves drawn on it: The horizontal lines turn into a horizontal ruling of the cylinder, the vertical lines turn into vertical circles (the orthogonal cross-sections of the cylinder), and the skew lines become spirals running round-and-round. They together form another “special arrangement of curves”.

If we “roll up” the cylinder again around a vertical axis, then we get a *torus*. The horizontal lines turn into horizontal circles, the vertical circles remain vertical circles (but they are no longer parallel, they revolve around the new axis) and the spirals on the cylinder will turn into spirals on the torus. This is again a “special arrangement of curves”.

The spirals drawn onto the torus can close into themselves, or can run round-and-round the torus indefinitely — it depends on the relationship between the two “rolling ups”. For us only the self-closing spirals are important, since they can be defined with polynomial equations. (The actual form of these polynomials is, as usual, irrelevant.)

It is worth contemplating for a moment: if a spiral drawn on a torus does not close into itself, then it travels all over the torus uniformly, and its image is a dense subset. These dense spirals are impossible to define with polynomial equations. Indeed, if a polynomial vanishes along a dense subset, then it is zero on the entire torus.

### 3. Reasons to be special

**DEFINITION 13.** We shall use the name *line-like arrangement of curves* for the three kinds of curve configurations we met in Examples 8 and 13. Let us note that none of the families in these arrangements has enveloping curve.

Soon we shall see that all “special arrangement of curves” can be obtained from one of the line-like arrangements of curves via polynomial transformations and their inverses: this is the meaning of Theorem 16. As a matter of fact, we constructed each line-like arrangement of curves via a continuous transformation from the line families of Example 8, hence all “special arrangement of curves” originates eventually from this single one. But the transformation used in Example 12, the “rolling up”, is not a polynomial transformation. (A periodic function cannot be a polynomial.) In this paper we prefer to use polynomials only, this is why we need three “basic” arrangements instead of just one.

EXAMPLE 14. Given a line-like arrangement of curves on a surface  $F$ . (Hence  $F$  is either the plane, or a cylinder, or a torus.) We would like to build from it as many “special arrangements of curves” as we can. Let us choose an arbitrary smooth algebraic surface  $W$  and two polynomial transformations:  $\phi : W \rightarrow \mathbb{R}^2$  and  $\psi : W \rightarrow F$ . Let us denote by  $\phi(\psi^{-1}(\_))$  that “multiple-valued transformation” that we get by first applying the inverse of  $\psi$  (this is multiple valued, denoted by  $\psi^{-1}$ ), and then applying  $\phi$ . This composed transformation maps the surface  $F$  first into  $W$  and than transforms it further into the plane. Naturally, the line-like arrangement of curves drawn on  $F$  gets also transformed into a “special arrangement of curves” in the plane.

In an ideal world one would hope that the main reason for a configuration to be special is that it contains some kind of image of the three line families of Example 8. This is essentially true, as we shall see, apart from a few exceptions.

Even if a configuration is a “special arrangement of curves”, we cannot expect that all of its portions be “nice”. It may happen that someone adds a few “unnecessary” arcs<sup>15</sup> to an already nice configuration — this will certainly not decrease the number of triple-crossings. Therefore the most we can hope for is that from each “special arrangement of curves” one can select the “essential” arcs, and this essential part behaves already “truly well”.

DEFINITION 15. Let  $\{\mathcal{A}_r\}$  and  $\{\mathcal{B}_s\}$  be two families of plane curves. We say that *they have a common component*, if for all values of  $r$  there is a value of  $s$  such that the curves  $\mathcal{A}_r$  and  $\mathcal{B}_s$  have a common arc.<sup>16</sup>

<sup>15</sup> The extra arcs are sometimes the continuations of the original arcs, but one can add brand new components as well.

<sup>16</sup> The definition suggests that this relation is symmetric, however, one has to prove this, under some additional conditions.

**THEOREM 16.** Suppose that the families  $\{\mathcal{A}_r\}$ ,  $\{\mathcal{B}_s\}$ ,  $\{\mathcal{C}_t\}$  form a “special arrangement of curves”. Then, as in the Example 14, there exists

- a line-like arrangement of curves  $\{\overline{\mathcal{A}}_{\bar{r}}\}$ ,  $\{\overline{\mathcal{B}}_{\bar{s}}\}$  and  $\{\overline{\mathcal{C}}_{\bar{t}}\}$  drawn on a surface  $F$  (so  $F$  is either the plane, or a cylinder, or a torus),<sup>17</sup>
- a smooth algebraic surface  $W$  and two polynomial transformations  $\phi: W \rightarrow \mathbb{R}^2$  and  $\psi: W \rightarrow F$

such that the “multiple-valued transformation”  $\phi(\psi^{-1}(\_))$  sends the family  $\{\overline{\mathcal{A}}_{\bar{r}}\}$  to a family of plane curves that have a common component with the family  $\{\mathcal{A}_r\}$ , and similarly, the transformed image of  $\{\overline{\mathcal{B}}_{\bar{s}}\}$  and  $\{\overline{\mathcal{C}}_{\bar{t}}\}$  have a common component with  $\{\mathcal{B}_s\}$  and  $\{\mathcal{C}_t\}$  respectively. (One can choose the arcs of these common components in such a way that they too form a “special arrangement of curves”.)

We shall close this paper with a variant of this theorem, (namely Theorem 20) as it was used in [3]. Here we shall continue with some topological explanations. Much of this — as we stated — is on the level of heuristic arguments that can be turned into precise arguments.

#### 4. Reasons to be typical

It turns out that there is a very general principle, a good geometric explanation for the “typical” behaviour of most curve configurations. On the one hand, a family of plane curves usually has plenty of envelopes. On the other hand, envelopes are scarce inside a “special arrangement of curves”. We shall use an irreducibility condition to formulate this principle precisely. A priori an enveloping curve can have information only about those arcs in the family members which are tangent to it — but the irreducibility will ensure us that this information automatically “spreads all over”: in an irreducible family almost all the arcs behave uniformly.

**DEFINITION 17. (Concurrency)** Given three families of curves in the plane. We say that “**the graph of triple-crossings is irreducible**”, if there is an irreducible trivariate polynomial  $\Psi(r, s, t)$  that vanishes in all the

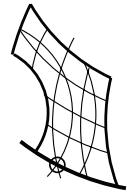
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<sup>17</sup> We use the notation  $\bar{r}$ ,  $\bar{s}$  and  $\bar{t}$  for these parameters to distinguish them from the original  $r$ ,  $s$  and  $t$  parameters. The theorem states that each curve  $\mathcal{A}_r$  is related in a certain way to some other curve  $\overline{\mathcal{A}}_{\bar{r}}$ , but it does not tell anything about the relationship between the two parameter values  $r$  and  $\bar{r}$ .

parameter-points  $(r, s, t)$  corresponding to the triple-crossings.  $\Psi$  is called the **concurrency function**.

At the moment, this definition may seem too algebraic to be useful. To remedy the situation, we shall describe a simple geometric test for irreducibility. The spatial surface given by the equation  $\Psi(r, s, t) = 0$  could be called the graph of triple-crossings, but here we use slightly different language instead.

**THEOREM 18.** *Given three families of curves:  $\{\mathcal{A}_r\}$ ,  $\{\mathcal{B}_s\}$  and  $\{\mathcal{C}_t\}$ . Suppose that one of them, say  $\{\mathcal{C}_t\}$  has an envelope  $\mathcal{E}$ , and this envelope meets in a single point two members, say  $\mathcal{A}_{r_0}$  and  $\mathcal{B}_{s_0}$  from both of the other families (here  $r_0$  and  $s_0$  denote appropriate parameter values). If no two of the three curves  $\mathcal{E}$ ,  $\mathcal{A}_{r_0}$  and  $\mathcal{B}_{s_0}$  are tangent to each other, and the graph of triple-crossings is irreducible, then the three families form a “typical arrangement of curves”, i.e. three times  $n$  curves can have at most  $cn^{2-\eta}$  triple-crossings.*



This theorem, again, has several forms and variants. Below we formulate a variant of this theorem, as stated in [3] (without much explanation, in its complex version). Denote  $\text{cl}(X)$  the closure of  $X$ .

**THEOREM 19. (Main Theorem)** *Let  $\{\mathcal{A}_r\}$ ,  $\{\mathcal{B}_s\}$  and  $\{\mathcal{C}_t\}$  be families explicitly parameterized by the functions  $f_1, f_2, f_3$ , analytic<sup>18</sup> on some open domains  $G_1$ ,  $G_2$ ,  $G_3$  and continuous on the closed domains  $\text{cl}(G_1)$ ,  $\text{cl}(G_2)$ ,  $\text{cl}(G_3)$ , respectively. Assume that  $\mathbf{G} = G_1 \cap G_2 \cap G_3$  is connected. Assume that any two curves intersect in at most  $B$  points, and the concurrency of three curves is described by a polynomial  $\Psi$  (see Definition 17). Moreover, assume that*

- (a)  $\{\mathcal{C}_t\}$  has an envelope  $\mathcal{E}$ ;
- (b)  $\mathcal{E} \subseteq G_1 \cap G_2$ ;
- (c) No  $f_i$  ( $i = 1, 2, 3$ ) is constant on any non-empty open sub-arc of  $\mathcal{E}$ .<sup>19</sup>

Then the number of triple points is at most  $B \cdot n^{2-\eta}$ , for a suitable  $\eta = \eta(\deg(\Psi))$  — provided that  $n > n_0 = n_0(\deg(\Psi))$ .

<sup>18</sup> Here we could write polynomial instead of analytic, since we promised to restrict ourselves to polynomially defined curves.

<sup>19</sup> Intuitively: no non-empty open sub-arc of  $\mathcal{E}$  is contained in any of the considered curves of  $\{\mathcal{A}_r\}$ ,  $\{\mathcal{B}_s\}$  and  $\{\mathcal{C}_t\}$ .

Let us return to Theorem 10. It is a special case of Theorem 18. We shall see later that in this situation the graph of triple-crossings is irreducible. In Figure 4 we can see three envelopes: the fattened arcs  $EF$ ,  $FG$  and  $GH$ . Let us consider, say, the arc  $EG$ . One can see well in the drawing that both the dashed curves (the members of the family  $\{\mathcal{A}_r\}$ ) and the dotted curves (the members of the family  $\{\mathcal{B}_s\}$ ) intersect the arc  $EG$ , but none of them are tangent to it. Moreover, those dashed and dotted curves which meet each other just on the arc  $EG$ , cross each-other transversally. Therefore one can apply Theorem 18 to this enveloping arc  $EG$ , proving Theorem 10. (Needless to say that one could apply it to the other two envelopes as well.)

#### 4.1. Some remarks about the proof of Theorem 18

Suppose, to the contrary, that the three families form a “special arrangement of curves”. According to Theorem 16, there is a “multiple-valued transformation”  $\phi(\psi^{-1}(\_))$  which rebuilds<sup>20</sup> our three families from a line-like arrangement of curves. (A pleasant consequence of irreducibility is that the transformation rebuilds our entire configuration, not just a small portion of it.) Of course, the line-like arrangement of curves contains no envelopes at all, hence our envelope  $\mathcal{E}$  “was born” during the transformation. We have discussed on page 63 after Example 11 that a polynomial transformation behaves very nicely in the complement of a certain curve  $F$  (which contains the critical curves, among them all the fold-curves, and may have further components). Each connected component  $K$  of the complement of  $F$  is mapped one-to-one into the plane. If a family of curves has no envelopes inside  $K$ , then after transformation, this portion of the family looks “essentially the same”, cannot “develop” an envelope. The same description remains valid for “multiple-valued transformations”. Therefore envelopes can “be born” only along the curve  $F$  corresponding to  $\phi(\psi^{-1}(\_))$ . Let us pretend for a moment that  $F$  consists of fold-curves only. The image of a fold-curve should be an envelope for all three families, as all curves “turn back” at the fold-curve (in fact the entire “sheet of paper” turns back there). But this possibility is excluded by the other condition in Theorem 18: the curves  $\mathcal{A}_{r_0}$  and  $\mathcal{B}_{s_0}$  cross transversally our envelope  $\mathcal{E}$ .<sup>21</sup> We ran into a contradiction, hence the

<sup>20</sup> Rebuilds? This heuristic description means that one of the configurations of three families of lines is “embedded” into our families and therefore provides the many triple points, proving that our families are “in special position”, as explained in Examples 8, 12 and Theorem 16.

<sup>21</sup> Here the transversality of one of them would be enough, but later we need both.

three families do not form a “special arrangement of curves”. In general,  $F$  may have components which are not fold-curves. It turns out that only those components can produce envelopes which are critical (i.e. where the Jacobian of the transformation is not invertible, the fold-curves are among them), and a slight modification of the above argument applies to all such components.<sup>22</sup>

Next we introduce a very general geometric technique, the construction of ramified coverings. We shall use it to show (as promised) that the graph of triple-crossings is irreducible in Theorem 10. General techniques like this one have many applications. For example, they play a central role in the proof of Theorem 16 (that we cannot reproduce here because of its length).

Let us return to Figure 4 and the families of Theorem 10. According to our assumptions, all triple-crossings live in the “curvy triangle”  $EFG$ . Hence we shall restrict our attention to this domain only and to the portion of the arcs running within that. In Figure 4 we see three arcs passing through the point  $P$ , but the drawing is not complete: each of the three families has two arcs (altogether six arcs) passing through  $P$  and through any other point of the “curvy triangle”  $EFG$ . As a contrast, in Figure 1(b), at each point there is only a single line passing through from each of the three families (altogether three lines). With our ramified coverings, we shall try to eliminate this striking difference.

We have seen in Figure 3 how “lifting” helps to separate in the space the members of a family of plane curves. Let us repeat the same trickery, say, with the family  $\{\mathcal{B}_s\}$ . Originally, in Figure 3 we have “unfolded” a planar domain (an annulus) along one of its boundary arcs (the darkened one) and obtained a domain on a spatial surface (a tube-like thing in that case) which covers doubly the original domain, and the new family drawn onto the new surface has only one curve passing through any of the points. From the whole annulus right now we are interested in the “curvy triangle”  $EFG$  only (see that in Figure 4). Therefore, instead of the whole darkened fold-curve of Figure 3, only the arc  $EG$  will play a role now. (The rest of that picture, though remains valid, is simply ignored.)




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<sup>22</sup> This is the point where we need the transversality of both curves  $\mathcal{A}_{r_0}$  and  $\mathcal{B}_{s_0}$ .

*Figure 7.* Folding up a flattened ball

Similarly, now we unfold (i.e. double) our “curvy triangle” along the arc  $EG$ , and the two copies smooth out into a spherical slice. This unfolding, among other things, is illustrated on the last two images of Figure 7. We have photographed the three-step folding up of a flattened ball. There are four phases in Figure 7, with a single fold taking place between any two consecutive images. In the last step we have folded in half a spherical slice, and obtained a spherical triangle — quite a bit thickened, since the sheets of a real-world ball do not usually squeeze tightly together. The final spherical triangle is very much like our “curvy triangle”  $EFG$ , so our unfolding should look much the same as the unfolding of the folded-up ball (doing backward the last step of Figure 7). One can clearly see on the photo that, after doubling, the two copies of the front side  $EF$  of the spherical triangle smooth together into the circular arc (the front edge on the third photo), and the same thing happens with the right hand side  $FG$  (which turns into the back side edge on the third photo).

This trick separates only the curves of the family  $\{\mathcal{B}_s\}$ , they will disjointly rule the ball-slice on the third photo (similarly to Figure 3). Imagine now that we copy also the other two families of curves,  $\{\mathcal{A}_r\}$  and  $\{\mathcal{C}_t\}$  on our folded-up ball (fourth photo), the same way as they can be seen in the “curvy triangle”  $EFG$  in Figure 4. Press the pencil hard to make the curves appear on all the eight sheets. After the unfolding, each arc appears twice on the ball-slice (on the third photo), but the two copies of the arcs in the families  $\{\mathcal{A}_r\}$  and  $\{\mathcal{C}_t\}$  do not smooth out, they stay separate. Our goal is not completely achieved yet: through each point of the ball-slice there are still two arcs passing through from both the families  $\{\mathcal{A}_r\}$  and  $\{\mathcal{C}_t\}$ .

We repeat the process two more times, we “unfold completely” the ball as it is shown on the series of images (from right to left) in Figure 7. In the first step, as we have already discussed it, we have separated the arcs of the family  $\{\mathcal{B}_s\}$ . It is easy to check that with the second step we separate the arcs of the family  $\{\mathcal{A}_r\}$ , and the last step separates the arcs of the family  $\{\mathcal{C}_t\}$  as well. Hence on the fully unfolded ball, through each point there is only one arc passing through from each of the three families.

One can easily verify, that along the way each arc is quadrupled — since they are not doubled in that step when we “separated” them from their own family mates. The unfolded arcs are not necessarily smooth, they can have angles along the fold-curves.

Why did we work so hard? Of course, we had no chance to get the same arrangement as the one in Figure 1(b): after all, our configuration is not supposed to be a “special arrangement of curves”. But we had quite a different goal in mind. Within our new configuration drawn on the surface of the ball, any triple-crossing can be moved continuously into any other (the sphere is a connected surface). In fact, as we shall explain it soon, this indicates that the graph of triple-crossings is irreducible.

In general, we can play our unfolding game on any configuration along each enveloping arc. If we duly go through the whole process, finally we arrive to a surface  $W$  which is “ruled only one-fold” by each of the three families of curves. Of course, there is no guarantee that we get a sphere again. It may very well happen that our new  $W$  consists of, say, two spheres and a torus. It turns out, that if  $W$  has only a single component, then the graph of triple-crossings is irreducible. It is a much more involved task to decide the shape of the components of  $W$ , but luckily we are not concerned now with that problem.

We have completed this unfolding for the configuration of Theorem 10 and we have obtained a single component, a sphere. Hence the graph of triple-crossings is irreducible, one can apply Theorem 18. In fact, this is the standard way to apply our methods. We always unfold our surface and study the unfolded  $W$  instead. Finally, let us reveal one more secret of the trade: even if the irreducibility condition fails, and one cannot apply Theorem 18 directly, one can still study the components of  $W$  separately and try to find for each of them an appropriate enveloping curve. With this method, most of the time it is possible to decide whether one has a “special arrangement of curves” or not.

## 5. Appendix: The Surface theorem

With this we have finished the mathematical discussions. As an “Appendix” we include that very version of Theorem which we used in [3], from [4]. Again, we skip most of the explanation. The meaning of the next theorem is that if we have many triple points then the surface  $\Psi(r, s, t) = 0$  goes through many “generalized grid points”  $(r, s, t) \in X \times Y \times Z$  and this can happen only if  $\Psi$  is basically  $r + s + t$ , apart from some coordinatewise transformation.

**THEOREM 20.** (“**Surface Theorem**”, see [4], **Theorem 3.**) *For any positive integer  $d$  there exist positive constants  $\eta = \eta(d) \in (0, 1)$  and  $n_0 = n_0(d)$  with the following property.*

*If  $V \subset \mathbb{C}^3$  is an algebraic surface (i.e. each component is two dimensional) of degree  $\leq d$  then the following are equivalent:*

- (a) *For at least one  $n > n_0(d)$  there exist  $X, Y, Z \subset \mathbb{C}$  such that  $|X| = |Y| = |Z| = n$  and*

$$|V \cap (X \times Y \times Z)| \geq n^{2-\eta};$$

- (b) *Let  $\mathbb{D} \subset \mathbb{C}$  denote the open unit disc. Then either  $V$  contains a cylinder over a curve  $F(x, y) = 0$  or  $F(x, z) = 0$  or  $F(y, z) = 0$  or, otherwise, there are one-to-one analytic functions  $g_1, g_2, g_3 : \mathbb{D} \rightarrow \mathbb{C}$  with analytic inverses such that  $V$  contains the  $g_1 \times g_2 \times g_3$ -image of a part of the plane  $x + y + z = 0$  near the origin:*

$$V \supseteq \left\{ \left( g_1(x), g_2(y), g_3(z) \right) \in \mathbb{C}^3 : x, y, z \in \mathbb{D}, x + y + z = 0 \right\}.$$

- (c) *For all positive integers  $n$  there exist  $X, Y, Z \subset \mathbb{C}$  such that  $|X| = |Y| = |Z| = n$  and  $|V \cap (X \times Y \times Z)| \geq (n - 2)^2/8$ .*

- (d) *Both (b) and (c) can be localized in the following sense. There is a finite subset  $H \subset \mathbb{C}$  and an irreducible component  $V_0 \subseteq V$  such that whenever  $P \in V_0$  is a point whose coordinates are not in  $H$  and  $U \subseteq \mathbb{C}^3$  is any neighborhood of  $P$ , then one may require that  $\left( g_1(0), g_2(0), g_3(0) \right) = P$  in (b), and the Cartesian product  $X \times Y \times Z$  in (c) lies entirely inside  $U$ . Furthermore,  $P$  has a neighborhood  $U'$  such that each irreducible component  $W$  of the analytic set  $V_0 \cap U'$ , with appropriate  $g_1, g_2$  and  $g_3$ , can be written in the form*

$$W = \left\{ \left( g_1(x), g_2(y), g_3(z) \right) \in \mathbb{C}^3 : x, y, z \in \mathbb{D}, x + y + z = 0 \right\}.$$

*If  $V \subset \mathbb{R}^3$  then the equivalence of (a), (b), (c) and (d) still holds with real analytic functions  $g_1, g_2, g_3$  defined on the interval  $(-1, 1)$ .*

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**ON DISTINCT DISTANCES AND INCIDENCES: ELEKES'S  
TRANSFORMATION AND THE NEW ALGEBRAIC  
DEVELOPMENTS\***

By

MICHA SHARIR

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**Abstract.** We first present a transformation that Gyuri Elekes has devised, about a decade ago, from the celebrated problem of Erdős of lower-bounding the number of distinct distances determined by a set  $S$  of  $s$  points in the plane to an incidence problem between points and a certain class of helices (or parabolas) in three dimensions. Elekes has offered conjectures involving the new setup, which, if correct, would imply that the number of distinct distances in an  $s$ -element point set in the plane is always  $\Omega(s/\log s)$ . Unfortunately, these conjectures are still not fully resolved. We then review the recent progress made on the transformed incidence problem, based on a new algebraic approach, originally introduced by Guth and Katz. Full details of the results reviewed in this note are given in a joint work with ELEKES [8].

## 1. Introduction

The motivation for the study reported in this paper comes from the celebrated and long-standing problem, originally posed by ERDŐS [9] in 1946, of obtaining a sharp lower bound for the number of distinct distances guaranteed to exist in any set  $S$  of  $s$  points in the plane. Erdős has shown that a section of the integer lattice determines only  $O(s/\sqrt{\log s})$  distinct distances, and conjectured this to be a lower bound for any planar point set. In spite of

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steady progress on this problem, reviewed next, Erdős's conjecture is still open.

L. MOSER [14], CHUNG [4], and CHUNG *et al.* [5] proved that the number of distinct distances determined by  $s$  points in the plane is  $\Omega(s^{2/3})$ ,  $\Omega(s^{5/7})$ , and  $\Omega(s^{4/5}/\text{polylog}(s))$ , respectively. SZÉKELY [22] managed to get rid of the polylogarithmic factor, while SOLYMOSI and TÓTH [20] improved this bound to  $\Omega(s^{6/7})$ . This was a real breakthrough. Their analysis was subsequently refined by TARDOS [25] and then by KATZ and TARDOS [13], who obtained the current record of  $\Omega(s^{(48-14e)/(55-16e)-\varepsilon})$ , for any  $\varepsilon > 0$ , which is  $\Omega(s^{0.8641})$ .

This was one of the problems that Gyuri Elekes has been thinking of for a long time. About a decade ago, he came up with an interesting transformation of the problem, which leads to an incidence problem between points and a special kind of curves in three dimensions (helices or parabolas with some special structure). The reduction is very unusual and rather surprising, but the new problem that it leads to is by no means an easy one. In fact, when Elekes communicated these ideas to me, around the turn of the millennium, the new incidence problem looked pretty hopeless, and the tex file that he has sent me has gathered dust, so to speak, for nearly a decade. In fact, Gyuri has passed away, in September 2008, before seeing any real progress on the problem.

In this note I will present Elekes's transformation in detail, and tell the story of the recent developments involving the transformed incidence problem and several related problems.

Trying to push his new ideas further, Elekes has proposed several simpler variants of the new problems, related to problems that I have been thinking of for a long time. Specifically, consider a set  $L$  of  $n$  lines in three dimensions. A point  $q$  is called a *joint* of  $L$  if it is incident to at least three non-coplanar lines of  $L$ . For example, if we take  $k$  planes (in general position) in  $\mathbb{R}^3$ , and let  $L$  be the set of their  $\binom{k}{2}$  intersection lines, then every vertex of the resulting arrangement (an intersection point of three of the planes) is a joint of  $L$ . We have  $n = |L| = \binom{k}{2}$  and the number of joints is  $\binom{k}{3} = \Theta(n^{3/2})$ . A long standing conjecture was that this is also an upper bound on the number of joints in any set of  $n$  lines in 3-space.

Work on resolving this conjecture has been going on for almost 20 years [3, 10, 18] (see also [2, Chapter 7.1, Problem 4]), and, until very recently, the best known upper bound, established by Sharir and Feldman in 2005 [10], was  $O(n^{1.6232})$ . The proof techniques were rather complicated, involving a battery of tools from combinatorial geometry, including forbidden subgraphs

in extremal graph theory, space decomposition techniques, and some basic results in the geometry of lines in space (e.g., Plücker coordinates).

An extension of the problem is to bound the number of incidences between  $n$  lines in 3-space and their joints. In the lower bound construction, each joint is incident to exactly three lines, so the number of incidences is just three times the number of joints. However, it is conceivable that the number of incidences is considerably larger than the number of joints. Still, with the lack of any larger lower bound, the prevailing conjecture has been that the number of incidences is also at most  $O(n^{3/2})$ . The best upper bound on this quantity, until the recent developments, was  $O(n^{5/3})$ , due to Sharir and Welzl [19].

Elekes has proposed to study a special case of the incidence problem, in which all the lines in  $L$  are *equally inclined*, i.e., they all make the same angle (say,  $45^\circ$ ) with the  $z$ -axis.<sup>1</sup> The lower bound construction can, with some care, be realized with equally inclined lines, so the goal was to establish the upper bound  $O(n^{3/2})$  for the number of incidences between  $n$  equally inclined lines in  $\mathbb{R}^3$  and their joints. Elekes has managed to establish the almost tight bound  $O(n^{3/2} \log n)$ . Although the proof was far from trivial, Elekes considered (probably justifiably so) this result as a rather minor development.

After Elekes's death, his son Márton has gone through his father's files and found the note containing this result. He has contacted me and asked if I could finish it up and get it published. I obliged, and even managed to tighten the bound to  $O(n^{3/2})$  (still, only for equally inclined lines), which made the result a little stronger. I turned it into a joint paper with Elekes, and submitted it, in January 2009, to János Pach, editor-in-chief of *Discrete and Computational Geometry*, for publication.

János's response was quick, merciless, and extremely valuable:

*Dear Micha:*

*Have you seen arXiv:0812.1043*

*Title: Algebraic Methods in Discrete Analogs of the Kakeya Problem*

*Authors: Larry Guth, Nets Hawk Katz*

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<sup>1</sup> This, by the way, is also a variant of the complex Szemerédi–Trotter problem, of bounding the number of incidences between points and lines in the complex plane; see the concluding section for more details.

*If the proof is correct, DCG is not a possibility for the Elekes–Sharir note.*

*Cheers, János*

What János was referring to was a rather dramatic development where, building on a recent result of Dvir [6] for a variant of the so-called Kakeya problem for finite fields, Guth and Katz [11] have settled the conjecture in the affirmative, showing that the number of joints (in three dimensions) is indeed  $O(n^{3/2})$ . Their proof technique is completely different from the traditional approaches, and uses fairly simple tools from algebraic geometry. This has grabbed me, so to speak, and for the next six month I did little else but work on the new approach and advance it as far as possible.

This work has culminated (so far) in three papers. In the first one, I managed, with Kaplan and Shustin [12], to obtain an extremely simple proof of the joints conjecture, following the new algebraic approach of Guth and Katz. As a matter of fact, we also extended the result to any dimension  $d \geq 2$ , showing that the maximum possible number of joints in a set of  $n$  lines in  $\mathbb{R}^d$  is  $\Theta(n^{d/(d-1)})$ ; here a joint is a point incident to at least  $d$  of the given lines, not all in a common hyperplane. (In another rather surprising turn of events, the same results were obtained independently and simultaneously<sup>2</sup> by R. Quilodrán [15], using a very similar approach.)

In a second paper [7], we have simplified and extended the analysis technique of Guth and Katz [11] to obtain tight bounds on the number of incidences between  $n$  lines in 3-space and their joints, showing that the number of such incidences is  $O(n^{3/2})$ . As mentioned above, the best previous bound on this quantity [19] was  $O(n^{5/3})$ . (This says that when the number of joints is near the upper bound, each joint is incident, on average, to only  $O(1)$  lines; as observed, this is indeed the case in the lower bound construction.) We have also shown that the maximum possible number of incidences between the lines of  $L$  and any number  $m \geq n$  of their joints is  $\Theta(m^{1/3}n)$ , and that in fact this bound also holds for the number of incidences between  $n$  lines and  $m \geq n$  arbitrary points, provided that each point is incident to at least three lines, and that no plane contains more than  $O(n)$  points; both conditions are easily seen to hold for joints.

It is however the third paper [8] that I want to highlight in this note. In this paper, co-authored with Elekes, I describe his ingenious transformation

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<sup>2</sup> Both papers, Quilodrán’s and ours, were posted on arXiv on the same day, June 2, 2009.

from the problem of distinct distances in the plane to an incidence problem between points and helices (or parabolas) in three dimensions. For this transformation to yield sharper bounds on the number of distinct distances, Elekes has posed a couple of (rather deep) conjectures, which are still open. I managed to obtain several partial results concerning these conjectures, but they are still far from what one needs for the motivating distinct distances problem.

What I find interesting and gratifying in the developments of the past year and a half is the coincidental confluence between Elekes's dormant incidence problem and the new machinery provided by the breakthrough of Guth and Katz. Before this breakthrough there seemed to be little hope to make any progress on Elekes's incidence problem, but the scene has now changed unexpectedly and completely, and hope is on the horizon. In fact, Elekes's problem now provides a strong motivation to study incidences between points and curves in three dimensions, and I hope that, with this strong motivation and with the powerful new machinery at hand, this topic will flourish in the coming years.

Before closing the introduction, I would like to share with the reader some more personal notes concerning the interaction with Elekes many years ago. When he sent me his note on the number of incidences between equally inclined lines and their joints, he added the following letter.

*Dear Micha,*

*The summer is over (hope you had a nice one) and I have long been planning to write you about what I could (not) do. In a nutshell: I could not improve on your bounds. (You may not be too surprised :)*

*I could not even prove the  $O(n^{4/3})$  bound on the number of 45 degree lines determined by  $n$  points. You certainly know that this is equivalent to the statement that in the plane,  $n$  circles can only have  $n^{4/3}$  points of tangencies. Moreover, even this problem can be re-phrased in terms of helices — which all start from the same direction (e.g., they all start at North).*

*I have just observed that your conjecture on “cutting  $n$  circles into  $n^{4/3}$  pseudo-segments” is very strong; it would immediately imply the previous bound.*

*By the way, how about parabolas? You mentioned at the Elbe sandstone Geometry Workshop that you could prove my conjecture*

*on the number of incidences if all pairs intersect. Have you written it up and if so, could you please send me a copy?*

*And now about the only minor fact I have observed. I did not even consider it interesting until I read your JCTA 94 paper on joints. Let me tell the details.*

Apparently, I have managed to misplace the file, so I wrote to Elekes a few years later, asking him for a fresh copy. He sent me the file again, and added:

*Dear Micha,*

*I also had to dig back for the proof and could only find a TeX file which I included in my e-mail (pls find it enclosed, together with some remarks just added). As already mentioned, I do NOT want to publish it on my own.*

*If I knew for sure that during the next thirty years — which is a loose upper bound for my life span — no new method would be developed to completely solve the  $n^{4/3}$  problem, then I would immediately suggest that we publish all we have in a joint paper.*

*However, at the moment, I think we had better wait for the big fish (à la Wiles :)*

*By the way, in case of something unexpected happens to me (car accident, plane crash, a brick on the top of my skull) I definitely ask you to publish anything we have, at your will.*

*Gyuri*

I find this “scientific will” very touching; it has made me reflect a lot about the fragility of our life and work. At the risk of sounding too sentimental, let me close this personal part by saying that I hope that, in mathematicians’ heaven, Gyuri Elekes is looking with satisfaction at the recent developments, even though his conjectures are still unresolved.

Before proceeding to describe Elekes’s transformation, let me comment that problems involving incidences between points and curves are related to, and are regarded as discrete analogs of the celebrated Kakeya problem. This relation was first noted by Wolff [26], who observed a connection between the problem of counting joints to the Kakeya problem. Bennett et al. [1] exploited this connection and proved an upper bound on the number of so-called  $\theta$ -transverse joints in  $\mathbb{R}^3$ , namely, joints incident to at least one triple of lines

for which the volume of the parallelepiped generated by the three unit vectors along these lines is at least  $\theta$ . This bound is  $O(n^{3/2+\varepsilon}/\theta^{1/2+\varepsilon})$ , for any  $\varepsilon > 0$ , where the constant of proportionality depends on  $\varepsilon$ . See Tao [24] for a review of the Kakeya problem and its connections to combinatorial geometry (and to many other fields of mathematics).

## 2. Distinct distances and incidences with helices

In this section we present Elekes's transformation from the problem of distinct distances in the plane to a three-dimensional incidence problem. The material presented here is taken from [8] (and a significant portion of it is taken almost verbatim from the notes that Elekes has sent me long time ago).

The transformation proceeds through the following steps.

**(H1)** Let  $S$  be a set of  $s$  points in the plane with  $x$  distinct distances. Let  $K$  denote the set of all quadruples  $(a, b, a', b') \in S^4$ , such that the pairs  $(a, b)$  and  $(a', b')$  are distinct (although the points themselves need not be) and  $|ab| = |a'b'| > 0$ .

Let  $\delta_1, \dots, \delta_x$  denote the  $x$  distinct distances in  $S$ , and let  $E_i = \{(a, b) \in S^2 \mid |ab| = \delta_i\}$ . We have

$$|K| = 2 \sum_{i=1}^x \binom{|E_i|}{2} \geq \sum_{i=1}^x (|E_i| - 1)^2 \geq \frac{1}{x} \left[ \sum_{i=1}^x (|E_i| - 1) \right]^2 = \frac{[s(s-1) - x]^2}{x}.$$

**(H2)** We associate each  $(a, b, a', b') \in K$  with a (unique) *rotation* (or, rather, a rigid, orientation-preserving transformation of the plane)  $\tau$ , which maps  $a$  to  $a'$  and  $b$  to  $b'$ . A rotation  $\tau$ , in complex notation, can be written as the transformation  $z \mapsto pz + q$ , where  $p, q \in \mathbb{C}$  and  $|p| = 1$ . Putting  $p = e^{i\theta}$ ,  $q = \xi + i\eta$ , we can represent  $\tau$  by the point  $\tau^* = (\xi, \eta, \theta) \in \mathbb{R}^3$ . In the planar context,  $\theta$  is the counterclockwise angle of the rotation, and the center of rotation is  $c = q/(1 - e^{i\theta})$ , which is defined for  $\theta \neq 0$ ; for  $\theta = 0$ ,  $\tau$  is a pure translation.

The *multiplicity*  $\mu(\tau)$  of a rotation  $\tau$  (with respect to  $S$ ) is defined as  $|\tau(S) \cap S|$  = the number of pairs  $(a, b) \in S^2$  such that  $\tau(a) = b$ . Clearly, one always has  $\mu(\tau) \leq s$ , and we will mostly consider only rotations satisfying  $\mu(\tau) \geq 2$ . As a matter of fact, the bulk of the analysis will only consider rotations with multiplicity at least 3. Rotations with multiplicity 2 are harder to analyze.

If  $\mu(\tau) = k$  then  $S$  contains two congruent and equally oriented copies  $A, B$  of some  $k$ -element set, such that  $\tau(A) = B$ . Thus, studying multiplicities of rotations is closely related to analyzing repeated (congruent and equally oriented) patterns in a planar point set; see [2] for a review of many problems of this kind.

**(H3)** If  $\mu(\tau) = k$  then  $\tau$  contributes  $\binom{k}{2}$  quadruples to  $K$ . Let  $N_k$  (resp.,  $N_{\geq k}$ ) denote the number of rotations with multiplicity exactly  $k$  (resp., at least  $k$ ), for  $k \geq 2$ . Then

$$|K| = \sum_{k=2}^s \binom{k}{2} N_k = \sum_{k=2}^s \binom{k}{2} (N_{\geq k} - N_{\geq k+1}) = N_{\geq 2} + \sum_{k \geq 3} (k-1) N_{\geq k}.$$

**(H4)** The main conjecture posed by Elekes is:

CONJECTURE 1. *For any  $2 \leq k \leq s$ , we have*

$$N_{\geq k} = O\left(s^3/k^2\right).$$

Suppose that the conjecture were true. Then we would have

$$\frac{[s(s-1)-x]^2}{x} \leq |K| = O(s^3) \cdot \left[ 1 + \sum_{k \geq 3} \frac{1}{k} \right] = O(s^3 \log s),$$

which would have implied that  $x = \Omega(s/\log s)$ . This would have almost settled the problem of obtaining a tight bound for the minimum number of distinct distances guaranteed to exist in any set of  $s$  points in the plane, since, as mentioned above, the upper bound for this quantity is  $O(s/\sqrt{\log s})$  [9].

We note that Conjecture 1 is rather deep; even the simple instance  $k = 2$ , asserting that there are only  $O(s^3)$  rotations which map (at least) two points of  $S$  to two other points of  $S$  (at the same distance apart), seems quite difficult.

In the paper reviewed in this note, a variety of upper bounds on the number of rotations and on the sum of their multiplicities are derived. In particular, these results provide a partial positive answer to the above conjecture, showing that  $N_{\geq 3} = O(s^3)$ ; that is, the number of rotations which map a (degenerate or non-degenerate) triangle determined by  $S$  to another congruent (and equally oriented) such triangle, is  $O(s^3)$ . Bounding  $N_2$  by  $O(s^3)$  is still an open problem.

**Lower bound.** It is interesting to note the following lower bound construction.

LEMMA 2. *There exist sets  $S$  in the plane of arbitrarily large cardinality, which determine  $\Theta(|S|^3)$  distinct rotations, each mapping a triple of points of  $S$  to another triple of points of  $S$ .*

PROOF. Consider the set  $S = S_1 \cup S_2 \cup S_3$ , where

$$\begin{aligned} S_1 &= \{(i, 0) \mid i = 1, \dots, s\}, \\ S_2 &= \{(i, 1) \mid i = 1, \dots, s\}, \\ S_3 &= \{(i/2, 1/2) \mid i = 1, \dots, 2s\}. \end{aligned}$$

See Figure 1.

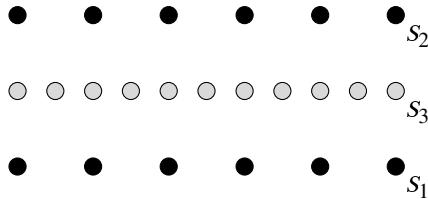


Figure 1. A lower bound construction of  $\Theta(|S|^3)$  rotations with multiplicity 3

For each triple  $a, b, c \in \{1, \dots, s\}$  such that  $a + b - c$  also belongs to  $\{1, \dots, s\}$ , construct the rotation  $\tau_{a,b,c}$  which maps  $(a, 0)$  to  $(b, 0)$  and  $(c, 1)$  to  $(a + b - c, 1)$ . Since the distance between the two source points is equal to the distance between their images,  $\tau_{a,b,c}$  is well (and uniquely) defined. Moreover,  $\tau_{a,b,c}$  maps the midpoint  $((a + c)/2, 1/2)$  to the midpoint  $((a + 2b - c)/2, 1/2)$ . It is fairly easy to show that the rotations  $\tau_{a,b,c}$  are all distinct (see [8] for details). Since there are  $\Theta(s^3)$  triples  $(a, b, c)$  with the above properties, the claim follows. ■

**Remark.** A “weakness” of this construction is that each of the rotations  $\tau_{a,b,c}$  maps a *collinear* triple of points of  $S$  to another collinear triple. (In the terminology to follow, these will be called *flat* rotations.) We do not know whether the number of rotations which map a *non-collinear* triple of points of  $S$  to another non-collinear triple can be  $\Omega(|S|^3)$ . We tend to conjecture that this is indeed the case.

**(H5)** To estimate  $N_{\geq k}$ , we reduce the problem of analyzing rotations and their interaction with  $S$  to an incidence problem in three dimensions, as follows.

With each pair  $(a, b) \in S^2$ , we associate the curve  $h_{a,b}$ , in a 3-dimensional space parametrized by  $(\xi, \eta, \theta)$ , which is the locus of all rotations which map  $a$  to  $b$ . That is, the equation of  $h_{a,b}$  is given by

$$h_{a,b} = \{(\xi, \eta, \theta) \mid b = a e^{i\theta} + (\xi, \eta)\}.$$

Putting  $a = (a_1, a_2)$ ,  $b = (b_1, b_2)$ , this becomes

$$(1) \quad \begin{aligned} \xi &= b_1 - (a_1 \cos \theta - a_2 \sin \theta), \\ \eta &= b_2 - (a_1 \sin \theta + a_2 \cos \theta). \end{aligned}$$

This is a *helix* in  $\mathbb{R}^3$ , having four degrees of freedom, parametrized by  $(a_1, a_2, b_1, b_2)$ . It extends from the plane  $\theta = 0$  to the plane  $\theta = 2\pi$ ; its two endpoints lie vertically above each other, and it completes exactly one revolution between them.

**(H6)** Let  $P$  be a set of rotations, represented by points in  $\mathbb{R}^3$ , as above, and let  $H$  denote the set of all  $s^2$  helices  $h_{a,b}$ , for  $(a, b) \in S^2$  (note that  $a = b$  is permitted). Let  $I(P, H)$  denote the number of incidences between  $P$  and  $H$ . Then we have

$$I(P, H) = \sum_{\tau \in P} \mu(\tau).$$

Rotations  $\tau$  with  $\mu(\tau) = 1$  are not interesting, because each of them only contributes 1 to the count  $I(P, H)$ , and we will mostly ignore them. For the same reason, rotations with  $\mu(\tau) = 2$  are also not interesting for estimating  $I(P, H)$ , but they need to be included in the analysis of  $N_{\geq 2}$ . Unfortunately, as already noted, we do not yet have a good upper bound (i.e., cubic in  $s$ ) on the number of such rotations.

**(H7)** Another conjecture that Elekes has offered is

CONJECTURE 3. *For any  $P$  and  $H$  as above, we have*

$$I(P, H) = O(|P|^{1/2}|H|^{3/4} + |P| + |H|).$$

Suppose that Conjecture 3 were true. Let  $P_{\geq k}$  denote the set of all rotations with multiplicity at least  $k$  (with respect to  $S$ ). We then have

$$k N_{\geq k} = k |P_{\geq k}| \leq I(P_{\geq k}, H) = O(N_{\geq k}^{1/2}|H|^{3/4} + N_{\geq k} + |H|),$$

from which we obtain

$$N_{\geq k} = O\left(\frac{s^3}{k^2} + \frac{s^2}{k}\right) = O\left(\frac{s^3}{k^2}\right),$$

thus establishing Conjecture 1, and therefore also the lower bound for  $x$  (the number of distinct distances in  $S$ ) derived above from this conjecture.

Note that two helices  $h_{a,b}$  and  $h_{c,d}$  intersect in at most one point—this is the unique rotation which maps  $a$  to  $b$  and  $c$  to  $d$  (if it exists at all, namely if  $|ac| = |bd|$ ). Hence, combining this fact with a standard cutting-based decomposition technique, similar to what has been noted in [19], say, yields the weaker bound

$$(2) \quad I(P, H) = O(|P|^{2/3}|H|^{2/3} + |P| + |H|),$$

which, alas, only yields the much weaker bound

$$N_{\geq k} = O\left(\frac{s^4}{k^3}\right),$$

which is completely useless for deriving any lower bound on  $x$ .

**(H8) From helices to parabolas.** The helices  $h_{a,b}$  are non-algebraic curves, because of the use of the angle  $\theta$  as a parameter. This can be easily remedied, in the following standard manner. Assume that  $\theta$  ranges from  $-\pi$  to  $\pi$ , and substitute, in the equations (1),  $Z = \tan(\theta/2)$ , to obtain

$$\begin{aligned} \xi &= b_1 - \left[ \frac{a_1(1-Z^2)}{1+Z^2} - \frac{2a_2Z}{1+Z^2} \right] \\ \eta &= b_2 - \left[ \frac{2a_1Z}{1+Z^2} + \frac{a_2(1-Z^2)}{1+Z^2} \right]. \end{aligned}$$

Next, substitute  $X = \xi(1+Z^2)$ ,  $Y = \eta(1+Z^2)$ , to obtain

$$(3) \quad \begin{aligned} X &= (a_1 + b_1)Z^2 + 2a_2Z + (b_1 - a_1) \\ Y &= (a_2 + b_2)Z^2 - 2a_1Z + (b_2 - a_2), \end{aligned}$$

which are the equations of a *planar parabola* in the  $(X, Y, Z)$ -space. We denote the parabola corresponding to the helix  $h_{a,b}$  as  $h_{a,b}^*$ , and refer to it as an *h-parabola*.

**(H9) Joint and flat rotations.** A rotation  $\tau \in P$  is called a *joint* of  $H$  if  $\tau$  is incident to at least three helices of  $H$  whose tangent lines at  $\tau$  are non-coplanar. Otherwise, still assuming that  $\tau$  is incident to at least three helices of  $H$ ,  $\tau$  is called *flat*.

A somewhat puzzling feature of the analysis, which is carried over from the study of standard joints and their incidences in [7, 11, 12], is that it can

only handle rotations incident to at least three helices/parabolas, i.e., rotations of multiplicity at least 3, and is (at the moment) helpless in dealing with rotations of multiplicity 2.

Using a rather simple analysis, it is shown in [8] that three helices  $h_{a,b}$ ,  $h_{c,d}$ ,  $h_{e,f}$  form a joint at a rotation  $\tau$  if and only if the three points  $a, c, e$  are non-collinear. Since  $\tau$  maps  $a$  to  $b$ ,  $c$  to  $d$ , and  $e$  to  $f$ , it follows that  $b, d, f$  are also non-collinear. That is, we have:

**CLAIM 4.** *A rotation  $\tau$  is a joint of  $H$  if and only if  $\tau$  maps a non-degenerate triangle determined by  $S$  to another (congruent and equally oriented) non-degenerate triangle determined by  $S$ . A rotation  $\tau$  is a flat rotation if and only if  $\tau$  maps at least three collinear points of  $S$  to another collinear triple of points of  $S$ , but does not map any point of  $S$  outside the line containing the triple to another point of  $S$ .*

**REMARKS.** (1) Note that if  $\tau$  is a flat rotation, it maps the entire line containing the three source points to the line containing their images. Specifically (see also below), we can respectively parametrize points on these lines as  $a_0 + tu$ ,  $b_0 + tv$ , for  $t \in \mathbb{R}$ , such that  $\tau$  maps  $a_0 + tu$  to  $b_0 + tv$  for every  $t$ .  
(2) For flat rotations, the geometry of our helices ensures that the three (or more) helices incident to a flat rotation  $\tau$  are such that their tangents at  $\tau$  are all distinct (see [8]).

### 3. Incidences between rotations and helices/parabolas

The preceding analysis leads to the following main problem. We are given a collection  $H$  of  $n \leq s^2$   $h$ -parabolas in  $\mathbb{R}^3$  (of the form (3)), and a set  $P$  of  $m$  rotations, represented as points in  $\mathbb{R}^3$ , and our goal is to estimate the number of incidences between the rotations of  $P$  and the parabolas of  $H$ , which we denote by  $I(P, H)$ . Ideally, we would like to prove Conjecture 3, but at the moment we are still far away from that.

Nevertheless, the recent developments, reviewed in the introduction, provide the (algebraic) machinery for obtaining nontrivial bounds on  $I(P, H)$ . This part of the analysis is rather technical and somewhat involved. Full details are provided in [8], and derivation of analogous bounds for point-line incidences in  $\mathbb{R}^3$  can be found in [7]. Here we only sketch the analysis, leaving out most of the details.

First, as already noted, because of some technical steps in the algebraic analysis, we can only handle joint or flat rotations incident to at least three parabolas; the same phenomenon occurs in the analysis of point-line incidences.

**The algebraic approach in a nutshell.** The basic idea of the new technique is as follows. We have a set  $P$  of  $m$  rotations (points in  $\mathbb{R}^3$ ). We construct a (nontrivial) trivariate polynomial  $p$  which vanishes at all the points of  $P$ . A simple linear-algebra argument (see Proposition 7 below) shows that there exists such a polynomial whose degree is  $d = O(m^{1/3})$ . Now if an  $h$ -parabola  $h_{a,b}^*$  contains more than  $2d$  rotations then  $p$  has to vanish identically on  $h_{a,b}^*$ , (a simple application of Bézout’s theorem; see below). Assume that  $p \equiv 0$  on all  $h$ -parabolas. Then, intuitively (and informally), the zero set of  $p$  has a very complicated shape. In particular, since each rotation  $\tau$  is incident to at least three  $h$ -parabolas, we can infer certain properties of the local structure of  $p$  in the vicinity of  $\tau$ . Specifically, if  $\tau$  is a joint rotation then it must be a critical (i.e., singular) point of  $p$ . If  $\tau$  is a flat rotation then some other polynomial, depending on  $p$ , has to vanish at  $\tau$ . These constraints are then exploited to derive upper bounds on  $m$  and on the number of incidences between the rotations and  $h$ -parabolas.

This high-level approach faces however several technical complications. The main one is that the fact that  $p$  vanishes on many  $h$ -parabolas is in itself not that significant, because all these parabolas could lie on a common surface  $\Sigma$ , which is the zero set of some polynomial factor of  $p$ . Understanding what happens on such a “special surface” occupies a large portion of the analysis. (In the analogous study of point-line incidences [7, 11], the corresponding “special surfaces” were planes, arising from possible linear factors of  $p$ .)

The first step in the analysis is therefore to study the structure of those special surfaces which may contain many  $h$ -parabolas. As it turns out, there is a lot of geometric beauty in the structure of these surfaces, which we will only be able to sketch briefly. Full details are given in [8].

**(H10) Special surfaces.** Let  $\tau$  be a flat rotation, with multiplicity  $k \geq 3$ , and let  $\ell$  and  $\ell'$  be the corresponding lines in the plane, such that there exist  $k$  points  $a_1, \dots, a_k \in S \cap \ell$  and  $k$  points  $b_1, \dots, b_k \in S \cap \ell'$ , such that  $\tau$  maps  $a_i$  to  $b_i$  for each  $i$  (and in particular maps  $\ell$  to  $\ell'$ ). By definition,  $\tau$  is incident to the  $k$  helices  $h_{a_i, b_i}$ , for  $i = 1, \dots, k$ .

Let  $u$  and  $v$  denote unit vectors in the direction of  $\ell$  and  $\ell'$ , respectively. Clearly, there exist two reference points  $a \in \ell$  and  $b \in \ell'$ , such that for each  $i$  there is a real number  $t_i$  such that  $a_i = a + t_i u$  and  $b_i = b + t_i v$ . As a matter

of fact, for each real  $t$ ,  $\tau$  maps  $a + tu$  to  $b + tv$ , so it is incident to  $h_{a+tu, b+tv}$ . Note that  $a$  and  $b$ , which can “slide” along their respective lines (by equal distances), are not uniquely defined.

Let  $H(a, b; u, v)$  denote the set of these helices. Since a pair of helices can meet in at most one point, all the helices in  $H(a, b; u, v)$  pass through  $\tau$  but are otherwise pairwise disjoint. Using the re-parametrization  $(\xi, \eta, \theta) \mapsto (X, Y, Z)$ , we denote by  $\Sigma = \Sigma(a, b; u, v)$  the surface which is the union of all the  $h$ -parabolas that are the images of the helices in  $H(a, b; u, v)$ . We refer to such a surface  $\Sigma$  as a *special surface*.

An important comment is that most of the ongoing analysis also applies when only two helices are incident to  $\tau$ ; they suffice to determine the four parameters  $a, b, u, v$  that define the surface  $\Sigma$ .

We also remark that, although we started the definition of  $\Sigma(a, b; u, v)$  with a flat rotation  $\tau$ , the definition only depends on the parameters  $a, b, u$ , and  $v$  (and even there we have, as just noted, one degree of freedom in choosing  $a$  and  $b$ ). If  $\tau$  is not flat it may determine many special surfaces, one for each line that contains two or more points of  $S$  which  $\tau$  maps to other (also collinear) points of  $S$ . Also, as we will shortly see, the same surface can be obtained from a different set (in fact, many such sets) of parameters  $a', b', u'$ , and  $v'$  (or, alternatively, from different (flat) rotations  $\tau'$ ).

**The equation of a special surface.** Routine, though somewhat tedious calculations, detailed in [8], show that the surface  $\Sigma$  is a cubic algebraic surface, whose equation is given by

$$(4) \quad E_2(Z)X - E_1(Z)Y + K(Z) = 0,$$

where

$$\begin{aligned} E_1(Z) &= (u_1 + v_1)Z + (u_2 + v_2) \\ E_2(Z) &= (u_2 + v_2)Z - (u_1 + v_1), \end{aligned}$$

and

$$\begin{aligned} K(Z) &= \left( (u_1 + v_1)Z + (u_2 + v_2) \right) \left( (a_2 + b_2)Z^2 - 2a_1Z + (b_2 - a_2) \right) - \\ &\quad - \left( (u_2 + v_2)Z - (u_1 + v_1) \right) \left( (a_1 + b_1)Z^2 + 2a_2Z + (b_1 - a_1) \right). \end{aligned}$$

We refer to the cubic polynomial in the left-hand side of (4) as a *special polynomial*. Thus a special surface is the zero set of a special polynomial. Note that special polynomials are cubic in  $Z$  but are only linear in  $X$  and  $Y$ .

**(H11)** Special surfaces pose a technical challenge to the analysis. Specifically, each special surface  $\Sigma$  captures a certain underlying pattern in the ground set  $S$ , which may result in many incidences between rotations and  $h$ -parabolas, all contained in  $\Sigma$ .

Consider first a simple instance of this situation, in which two special surfaces  $\Sigma, \Sigma'$ , generated by two distinct flat rotations  $\tau, \tau'$ , coincide. More precisely, there exist four parameters  $a, b, u, v$  such that  $\tau$  maps the line  $\ell_1 = a + tu$  to the line  $\ell_2 = b + tv$  (so that points with the same parameter  $t$  are mapped to one another), and four other parameters  $a', b', u', v'$  such that  $\tau'$  maps (in a similar manner) the line  $\ell'_1 = a' + tu'$  to the line  $\ell'_2 = b' + tv'$ , and  $\Sigma(a, b; u, v) = \Sigma(a', b'; u', v')$ . Denote this common surface by  $\Sigma$ . Let  $a_0$  be the intersection point of  $\ell_1$  and  $\ell'_1$ , and let  $b_0$  be the intersection point of  $\ell_2$  and  $\ell'_2$ . Then it is easy to show that both  $\tau$  and  $\tau'$  map  $a_0$  to  $b_0$ , and  $h_{a_0, b_0}^*$  is contained in  $\Sigma$ . See Figure 2.

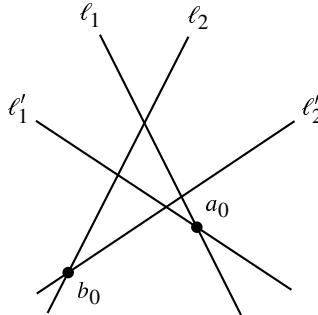


Figure 2. The structure of  $\tau$  and  $\tau'$  on a common special surface  $\Sigma$

Since the preceding analysis applies to any pair of distinct rotations on a common special surface  $\Sigma$ , it follows that we can associate with  $\Sigma$  a common direction  $w$  and a common shift  $\delta$ , so that for each  $\tau \in \Sigma$  there exist two lines  $\ell, \ell'$ , where  $\tau$  maps  $\ell$  to  $\ell'$ , so that the angle bisector between these lines is in direction  $w$ , and  $\tau$  is the unique rigid motion, obtained by rotating  $\ell$  to  $\ell'$  around their intersection point  $\ell \cap \ell'$ , and then shifting  $\ell'$  along itself by a distance whose projection in direction  $w$  is  $\delta$ . Again, refer to Figure 2.

Let  $\Sigma$  be a special surface, generated by  $H(a, b; u, v)$ ; that is,  $\Sigma$  is the union of all parabolas of the form  $h_{a+tu, b+tv}^*$ , for  $t \in \mathbb{R}$ . Let  $\tau_0$  be the common rotation to all these parabolas, so it maps the line  $\ell_0 = \{a + tu \mid t \in \mathbb{R}\}$  to the line  $\ell'_0 = \{b + tv \mid t \in \mathbb{R}\}$ , so that every point  $a + tu$  is mapped to  $b + tv$ .

Let  $h_{c,d}^*$  be a parabola contained in  $\Sigma$  but not passing through  $\tau_0$ . Take any pair of distinct rotations  $\tau_1, \tau_2$  on  $h_{c,d}^*$ . Then there exist two respective real numbers  $t_1, t_2$ , such that  $\tau_i \in h_{a+t_i u, b+t_i v}^*$ , for  $i = 1, 2$ . Thus  $\tau_i$  is the unique rotation which maps  $c$  to  $d$  and  $a_i = a + t_i u$  to  $b_i = b + t_i v$ . In particular, we have  $|a + t_i u - c| = |b + t_i v - d|$ . This in turn implies that the triangles  $a_1 a_2 c$  and  $b_1 b_2 d$  are congruent; see Figure 3.

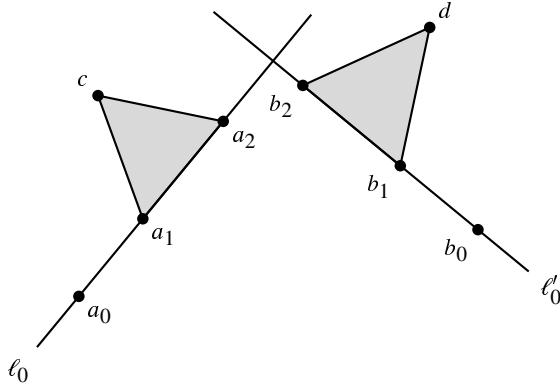


Figure 3. The geometric configuration corresponding to a parabola  $h_{c,d}^*$  contained in  $\Sigma$

Given  $c$ , this determines  $d$ , up to a reflection about  $\ell'_0$ . We claim that  $d$  has to be on the “other side” of  $\ell'_0$ , namely, be such that the triangles  $a_1 a_2 c$  and  $b_1 b_2 d$  are oppositely oriented. Indeed, if they were equally oriented, then  $\tau_0$  would have mapped  $c$  to  $d$ , and then  $h_{c,d}^*$  would have passed through  $\tau_0$ , contrary to assumption.

Now form the two sets

$$(5) \quad \begin{aligned} A &= \{p \mid \text{there exists } q \in S \text{ such that } h_{p,q}^* \subset \Sigma\} \\ B &= \{q \mid \text{there exists } p \in S \text{ such that } h_{p,q}^* \subset \Sigma\}. \end{aligned}$$

The preceding discussion implies that  $A$  and  $B$  are congruent and oppositely oriented.

To recap, each rotation  $\tau \in \Sigma$ , incident to  $k \geq 2$  parabolas contained in  $\Sigma$  corresponds to a pair of lines  $\ell, \ell'$  with the above properties, so that  $\tau$  maps  $k$  points of  $S \cap \ell$  (rather, of  $A \cap \ell$ ) to  $k$  points of  $S \cap \ell'$  (that is, of  $B \cap \ell'$ ). If  $\tau$  is flat, its entire multiplicity comes from points of  $S$  on  $\ell$  (these are the points of  $A \cap \ell$ ) which are mapped by  $\tau$  to points of  $S$  on  $\ell'$  (these are points of  $B \cap \ell'$ ), and all the corresponding parabolas are contained in  $\Sigma$ . If  $\tau$  is a

joint then, for any other point  $p \in S$  outside  $\ell$  which is mapped by  $\tau$  to a point  $q \in S$  outside  $\ell'$ , the parabola  $h_{p,q}^*$  is not contained in  $\Sigma$ , and crosses it transversally at the unique rotation  $\tau$ .

Note also that any pair of parabolas  $h_{c_1,d_1}^*$  and  $h_{c_2,d_2}^*$  which are contained in  $\Sigma$  intersect, necessarily at the unique rotation which maps  $c_1$  to  $d_1$  and  $c_2$  to  $d_2$ . This holds because  $|c_1 c_2| = |d_1 d_2|$ , as follows from the preceding discussion.

**Special surfaces and repeated patterns in  $S$ .** As just noted, a special surface  $\Sigma$  corresponds to two (maximal) subsets  $A, B \subseteq S$ , which are congruent and oppositely oriented, so that the number of  $h$ -parabolas contained in  $\Sigma$  is equal to  $|A| = |B|$ . Hence a natural interesting problem is to analyze such repeated patterns in  $S$ . For example, how many such *maximal* repeated patterns can  $S$  contain, for which  $|A| = |B| \geq k$ ? Note that one has to insist on maximal patterns, because one can always take  $S$  to be the union of two congruent and oppositely oriented sets  $S^+, S^-$ , and then every subset  $A^+$  of  $S^+$  and its image  $A^-$  in  $S^-$  form such a repeated pattern (but there is only one maximal repeated pattern, namely  $S^+$  and  $S^-$ ).

As a matter of fact, a special surface is nothing but an “anti-rotation”, namely a rigid motion that reverses the orientation of the plane; the multiplicity of this anti-rotation is the size of the subsets  $A, B$  in the corresponding repeated pattern. Hence, bounding the number of “rich” special surfaces is nothing but a variant of the problem we started with, namely of bounding the number of “rich” rotations (see Conjecture 1).

### 3.1. Tools from algebraic geometry

We review in this subsection (without proofs) the basic tools from algebraic geometry that have been used in [7, 8, 11]. We state here the variants that arise in the context of incidences between points and our  $h$ -parabolas.

So let  $C$  be a set of  $n \leq s^2$   $h$ -parabolas in  $\mathbb{R}^3$ . Recalling the definitions in (H9), we say that a point (rotation)  $a$  is a *joint* of  $C$  if it is incident to three parabolas of  $C$  whose tangents at  $a$  are non-coplanar. Let  $J = J_C$  denote the set of joints of  $C$ . We will also consider points  $a$  that are incident to three or more parabolas of  $C$ , so that the tangents to all these parabolas are coplanar, and refer to such points as *flat* points of  $C$ . We recall (see (H9)) that any pair of distinct  $h$ -parabolas which meet at a point have there distinct tangents.

First, we note that, using a trivial application of Bézout's theorem [17], a trivariate polynomial  $p$  of degree  $d$  which vanishes at  $2d + 1$  points that lie on a common  $h$ -parabola  $h^* \in C$  must vanish identically on  $h^*$ .

**Critical points and parabolas.** A point  $a$  is *critical* (or *singular*) for a trivariate polynomial  $p$  if  $p(a) = 0$  and  $\nabla p(a) = 0$ ; any other point  $a$  in the zero set of  $p$  is called *regular*. A parabola  $h^*$  is *critical* if all its points are critical.

Another application of Bézout's theorem implies the following.

**PROPOSITION 5.** *Let  $C$  be as above. Then any trivariate square-free polynomial  $p$  of degree  $d$  can have at most  $d(d - 1)$  critical parabolas in  $C$ .*

For regular points of  $p$ , we have the following easy observation.

**PROPOSITION 6.** *Let  $a$  be a regular point of  $p$ , so that  $p \equiv 0$  on three parabolas of  $C$  passing through  $a$ . Then these parabolas must have coplanar tangents at  $a$ .*

Hence, a point  $a$  incident to three parabolas of  $C$  whose tangent lines at  $a$  are non-coplanar, so that  $p \equiv 0$  on each of these parabolas, must be a critical point of  $p$ .

The main ingredient in the algebraic approach to incidence problems is the following, fairly easy (and rather well-known) result.

**PROPOSITION 7.** *Given a set  $S$  of  $m$  points in 3-space, there exists a trivariate polynomial  $p(x, y, z)$  which vanishes at all the points of  $S$ , of degree  $d$ , for any  $d$  satisfying  $\binom{d+3}{3} > m$ .*

**PROOF.** (See [7, 8, 11].) A trivariate polynomial of degree  $d$  has  $\binom{d+3}{3}$  monomials, and requiring it to vanish at  $m$  points yields these many homogeneous equations in the coefficients of these monomials. Such an underdetermined system always has a nontrivial solution. ■

**Flat points and parabolas.** Call a regular point  $\tau$  of a trivariate polynomial  $p$  *geometrically flat* if it is incident to three distinct parabolas of  $C$  (with necessarily coplanar tangent lines at  $\tau$ , no pair of which are collinear) on which  $p$  vanishes identically.

Handling geometrically flat points in our analysis is somewhat trickier than handling critical points, and involves the second-order partial derivatives of  $p$ . The analysis, detailed in [8], leads to the following properties.

**PROPOSITION 8.** *Let  $p$  be a trivariate polynomial, and define*

$$\Pi(p) = p_Y^2 p_{XX} - 2p_X p_Y p_{XY} + p_X^2 p_{YY}.$$

*Then, if  $\tau$  is a regular geometrically flat point of  $p$  (with respect to three parabolas of  $C$ ) then  $\Pi(p)(\tau) = 0$ .*

**Remark.**  $\Pi(p)$  is one of the polynomials that form the *second fundamental form* of  $p$ ; see [7, 8, 11, 16] for details.

In particular, if the degree of  $p$  is  $d$  then the degree of  $\Pi(p)$  is at most  $(d-1) + (d-1) + (d-2) = 3d-4$ .

In what follows, we call a point  $\tau$  *flat* for  $p$  if  $\Pi(p)(\tau) = 0$ . Call an  $h$ -parabola  $h^* \in C$  *flat* for  $p$  if all the points of  $h^*$  are flat points of  $p$  (with the possible exception of a discrete subset). Arguing as in the case of critical points, if  $h^*$  contains more than  $2(3d-4)$  flat points then  $h^*$  is a flat parabola.

The next proposition shows that, in general, trivariate polynomials do not have too many flat parabolas. The proof is based on Bézout's theorem, as does the proof of Proposition 5.

**PROPOSITION 9.** *Let  $p$  be any trivariate square-free polynomial of degree  $d$  with no special polynomial factors. Then  $p$  can have at most  $d(3d-4)$  flat  $h$ -parabolas in  $C$ .*

### 3.2. Joint and flat rotations in a set of $h$ -parabolas in $\mathbb{R}^3$

In this subsection we extend the recent algebraic machinery of Guth and Katz [11], as further developed by Elekes et al. [7], using the algebraic tools set forth in the preceding subsection, to establish the bound  $O(n^{3/2}) = O(s^3)$  on the number of rotations with multiplicity at least 3 in a collection of  $n$   $h$ -parabolas. Specifically, we have:

**THEOREM 10.** *Let  $C$  be a set of at most  $n$   $h$ -parabolas in  $\mathbb{R}^3$ , and let  $P$  be a set of  $m$  rotations, each of which is incident to at least three parabolas of  $C$ . Suppose further that no special surface contains more than  $q$  parabolas of  $C$ . Then  $m = O(n^{3/2} + nq)$ .*

**Remarks.** (1) The recent results of [12, 15] imply that the number of joints in a set of  $n$   $h$ -parabolas is  $O(n^{3/2})$ . The proofs in [12, 15] are much simpler than the proof given below, but they do not apply to flat points as does Theorem 10.

- (2) One can show that we always have  $q \leq s$ , and we also have  $n^{1/2} \leq s$ , so the “worst-case” bound on  $m$  is  $O(ns)$ .
- (3) Note that the parameter  $n$  in the statement of the theorem is arbitrary, not necessarily the maximum number  $s^2$ . When  $n$  attains its maximum possible value  $s^2$ , the bound becomes  $m = O(n^{3/2}) = O(s^3)$ .

The proof of Theorem 10, whose full details can be found in [8], uses the proof technique of [7] (for incidences with lines), properly adapted to the present, somewhat more involved context of  $h$ -parabolas and rotations. Here we only give a very brief sketch of the main steps in the proof.

We first prove the theorem under the additional assumption that  $q = n^{1/2}$ . The proof proceeds by induction on  $n$ , and shows that  $m \leq An^{3/2}$ , where  $A$  is a sufficiently large constant. Let  $C$  and  $P$  be as in the statement of the theorem, with  $|C| = n$ , and suppose to the contrary that  $|P| > An^{3/2}$ .

We first apply the following iterative pruning process to  $C$ . As long as there exists a parabola  $h^* \in C$  incident to fewer than  $cn^{1/2}$  rotations of  $P$ , for some constant  $1 \leq c \ll A$  that we will fix later, we remove  $h^*$  from  $C$ , remove its incident rotations from  $P$ , and repeat this step with respect to the reduced set of rotations. In this process we delete at most  $cn^{3/2}$  rotations. We are thus left with a subset of at least  $(A - c)n^{3/2}$  of the original rotations, so that each surviving parabola is incident to at least  $cn^{1/2}$  surviving rotations, and each surviving rotation is still incident to at least three surviving parabolas. For simplicity, continue to denote these sets as  $C$  and  $P$ .

In the actual proof, the constants of proportionality play an important role. In this informal overview, we ignore this issue, making the presentation “slightly incorrect”, but hopefully making its main ideas easier to grasp.

We collect about  $n^{1/2}$  rotations from each surviving parabola, and obtain a set  $S$  of  $O(n^{3/2})$  rotations.

We next construct, using Proposition 7, a nontrivial trivariate polynomial  $p(X, Y, Z)$  which vanishes at all the rotations of  $S$ , whose degree is  $d = O(|S|^{1/3}) = O(n^{1/2})$ . Without loss of generality, we may assume that  $p$  is square-free—by removing repeated factors, we get a square-free polynomial which vanishes on the same set as the original  $p$ , with the same upper bound on its degree.

The polynomial  $p$  vanishes on  $\Theta(n^{1/2})$  points on each parabola. By playing with the constants of proportionality, we can ensure that this number is larger than  $2d$ . Hence  $p$  vanishes identically on all the surviving parabolas of  $C$ .

We can also ensure the property that each parabola of  $C$  contains at least  $9d$  points of  $P$ .

We note that  $p$  can have at most  $d/3$  special polynomial factors (since each of them is a cubic polynomial); i.e.,  $p$  can vanish identically on at most  $d/3$  respective special surfaces  $\Xi_1, \dots, \Xi_k$ , for  $k \leq d/3$ . We factor out all these special polynomial factors from  $p$ , and let  $\tilde{p}$  denote the resulting polynomial, which is a square-free polynomial without any special polynomial factors, of degree at most  $d$ .

Consider one of the special surfaces  $\Xi_i$ , and let  $t_i$  denote the number of parabolas contained in  $\Xi_i$ . Then any rotation on  $\Xi_i$  is either an intersection point of (at least) two of these parabolas, or it lies on at most one of them. The number of rotations of the first kind is  $O(t_i^2)$ . Any rotation  $\tau$  of the second kind is incident to at least one parabola of  $C$  which crosses  $\Xi_i$  transversally at  $\tau$ . A simple algebraic calculation shows that each  $h$ -parabola  $h^*$  can cross  $\Xi_i$  in at most three points. Hence, the number of rotations of the second kind is  $O(n)$ , and the overall number of rotations on  $\Xi_i$  is  $O(t_i^2 + n) = O(n)$ , since we have assumed in the present version of the proof that  $t_i = O(n^{1/2})$ .

Summing the bounds over all surfaces  $\Xi_i$ , we conclude that altogether they contain  $O(nd)$  rotations, which we bound by  $bn^{3/2}$ , for some absolute constant  $b$ .

We remove all these vanishing special surfaces, together with the rotations and the parabolas which are fully contained in them, and let  $C_1 \subseteq C$  and  $P_1 \subseteq P$  denote, respectively, the set of those parabolas of  $C$  (rotations of  $P$ ) which are not contained in any of the vanishing surfaces  $\Xi_i$ .

Note that there are still at least three parabolas of  $C_1$  incident to any remaining rotation in  $P_1$ , since none of the rotations of  $P_1$  lie in any surface  $\Xi_i$ , so all parabolas incident to such a rotation are still in  $C_1$ .

Clearly,  $\tilde{p}$  vanishes identically on every  $h^* \in C_1$ . Furthermore, every  $h^* \in C_1$  contains at most  $d$  points in the surfaces  $\Xi_i$ , because, as just argued, it crosses each surface  $\Xi_i$  in at most three points.

Note that this also holds for every parabola  $h^*$  in  $C \setminus C_1$ , if we only count intersections of  $h^*$  with surfaces  $\Xi_i$  which do not fully contain  $h^*$ .

Hence, each  $h^* \in C_1$  contains at least  $8d$  rotations of  $P_1$ . Since each of these rotations is incident to at least three parabolas in  $C_1$ , each of these rotations is either critical or geometrically flat for  $\tilde{p}$ .

Consider a parabola  $h^* \in C_1$ . If  $h^*$  contains more than  $2d$  critical rotations then  $h^*$  is a critical parabola for  $\tilde{p}$ . By Proposition 5, the number of such parabolas is at most  $d(d - 1)$ . Any other parabola  $h^* \in C_1$  contains more than  $6d$  geometrically flat points and hence  $h^*$  must be a flat parabola for  $\tilde{p}$ . By Proposition 9, the number of such parabolas is at most  $d(3d - 4)$ . Summing up we obtain

$$|C_1| \leq d(d - 1) + d(3d - 4) < 4d^2.$$

An appropriate choice of constants ensures that  $4d^2 < n/2$ .

We next want to apply the induction hypothesis to  $C_1$ , with the parameter  $4d^2$  (which dominates the size of  $C_1$ ). For this, we first argue that each special surface contains at most  $3d/2$  parabolas of  $C_1$  (proof omitted; see [8]). Since  $3d/2 \leq (4d^2)^{1/2}$ , we can apply the induction hypothesis, and conclude that the number of points in  $P_1$  is at most

$$A(4d^2)^{3/2} \leq \frac{A}{2^{3/2}}n^{3/2}.$$

Adding up the bounds on the number of points on parabolas removed during the pruning process and on the special surfaces  $\Xi_i$  (which correspond to the special polynomial factors of  $p$ ), we obtain

$$|P| \leq \frac{A}{2^{3/2}}n^{3/2} + (b + c)n^{3/2} \leq An^{3/2},$$

with an appropriate, final choice of the various constants. This contradicts the assumption that  $|P| > An^{3/2}$ , and thus establishes the induction step for  $n$ , and, consequently, completes the proof of the restricted version of the theorem. We omit the rather similar proof of the general version of the theorem. ■

**COROLLARY 11.** *Let  $S$  be a set of  $s$  points in the plane. Then there are at most  $O(s^3)$  rotations which map some (degenerate or non-degenerate) triangle spanned by  $S$  to another (congruent and equally oriented) such triangle. By Lemma 2, this bound is tight in the worst case.*

### 3.3. Incidences between parabolas and rotations

In this subsection we further adapt the machinery of [7] to derive an upper bound on the number of incidences between  $m$  rotations and  $n$   $h$ -parabolas in  $\mathbb{R}^3$ , where each rotation is incident to at least three parabolas (i.e., has multiplicity  $\geq 3$ ). We present the results and omit all proofs (which, as usual, can be found in [8]).

We begin with a bound which is independent of the number  $m$  of rotations.

**THEOREM 12.** *For an underlying ground set  $S$  of  $s$  points in the plane, let  $C$  be a set of at most  $n \leq s^2$   $h$ -parabolas defined on  $S$ , and let  $P$  be a set of rotations with multiplicity at least 3 with respect to  $S$ , such that no special surface contains more than  $n^{1/2}$  parabolas of  $C$ . Then the number of incidences between  $P$  and  $C$  is  $O(n^{3/2})$ .*

Theorem 12 is used to prove the following more general bound.

**THEOREM 13.** *For an underlying ground set  $S$  of  $s$  points in the plane, let  $C$  be a set of at most  $n \leq s^2$   $h$ -parabolas defined on  $S$ , and let  $P$  be a set of  $m$  rotations with multiplicity at least 3 (with respect to  $S$ ).*

(i) *Assuming further that no special surface contains more than  $n^{1/2}$  parabolas of  $C$ , we have*

$$I(P, C) = O(m^{1/3}n).$$

(ii) *Without the additional assumption in part (i), we have*

$$I(P, C) = O(m^{1/3}n + m^{2/3}n^{1/3}s^{1/3}).$$

**Remark.** As easily checked, the first term in (ii) dominates the second term when  $m \leq n^2/s$ , and the second term dominates when  $n^2/s < m \leq ns$ . In particular, the first term dominates when  $n = s^2$ , because we have  $m = O(s^3) = O(n^2/s)$ .

It is interesting to note that the proof technique also yields the following result.

COROLLARY 14. *Let  $C$  be a set of  $n$   $h$ -parabolas and  $P$  a set of points in 3-space which satisfy the conditions of Theorem 13(i). Then, for any  $k \geq 1$ , the number  $M_{\geq k}$  of points of  $P$  incident to at least  $k$  parabolas of  $C$  satisfies*

$$M_{\geq k} = \begin{cases} O\left(\frac{n^{3/2}}{k^{3/2}}\right) & \text{for } k \leq n^{1/3}, \\ O\left(\frac{n^2}{k^3} + \frac{n}{k}\right) & \text{for } k > n^{1/3}. \end{cases}$$

PROOF. Write  $m = M_{\geq k}$  for short. We clearly have  $I(P, C) \geq km$ . Theorem 13(i) then implies  $km = O(m^{1/3}n)$ , or  $m = O((n/k)^{3/2})$ . If  $k > n^{1/3}$  we use the other bound (in (2)), to obtain  $km = O(m^{2/3}n^{2/3} + m + n)$ , which implies that  $m = O(n^2/k^3 + n/k)$  (which is in fact an equivalent statement of the classical Szemerédi–Trotter bound). ■

We can also obtain more general bounds using Theorem 13(ii), but we do not state them, because we are going to improve them anyway in the next subsection.

### 3.4. Further improvements

In this subsection we further improve the bound in Theorem 13 (and Corollary 14) using more standard space decomposition techniques. Omitting all details, we obtain:

THEOREM 15. *The number of incidences between  $m$  arbitrary rotations and  $n$   $h$ -parabolas, defined for a planar ground set with  $s$  points, is*

$$O^*\left(m^{5/12}n^{5/6}s^{1/12} + m^{2/3}n^{1/3}s^{1/3} + n\right),$$

where the  $O^*(\cdot)$  notation hides polylogarithmic factors. In particular, when all  $n = s^2$   $h$ -parabolas are considered, the bound is

$$O^*\left(m^{5/12}s^{7/4} + s^2\right).$$

Using this bound, we can strengthen Corollary 14, as follows.

COROLLARY 16. *Let  $C$  be a set of  $n$   $h$ -parabolas and  $P$  a set of rotations, with respect to a planar ground set  $S$  of  $s$  points. Then, for any  $k \geq 1$ , the number  $M_{\geq k}$  of rotations of  $P$  incident to at least  $k$  parabolas of  $C$  satisfies*

$$M_{\geq k} = O\left(\frac{n^{10/7}s^{1/7}}{k^{12/7}} + \frac{ns}{k^3} + \frac{n}{k}\right).$$

For  $n = s^2$ , the bound becomes

$$M_{\geq k} = O\left(\frac{s^3}{k^{12/7}}\right).$$

PROOF. The proof is similar to the proof of Corollary 14, and we omit its routine details. ■

#### 4. Conclusion

In this paper we have reduced the problem of obtaining a near-linear lower bound for the number of distinct distances in the plane to a problem involving incidences between points and a special class of parabolas (or helices) in three dimensions. We have made significant progress in obtaining upper bounds for the number of such incidences, but we are still short of tightening these bounds to meet Elekes's conjectures on these bounds made in Section 2.

To see how far we still have to go, consider the bound in Corollary 16, for the case  $n = s^2$ , which then becomes  $O(s^3/k^{12/7})$ . Moreover, we also have the Szemerédi–Trotter bound  $O(s^4/k^3)$ , which is smaller than the previous bound for  $k \geq s^{7/9}$ . Substituting these bounds in the analysis of (H3) and (H4), we get

$$\begin{aligned} \frac{[s(s-1)-x]^2}{x} &\leq |K| = N_{\geq 2} + \sum_{k \geq 3} (k-1)N_{\geq k} = \\ &= N_{\geq 2} + O(s^3) \cdot \left[ 1 + \sum_{k=3}^{s^{7/9}} \frac{1}{k^{5/7}} + \sum_{k>s^{7/9}} \frac{s^4}{k^2} \right] = N_{\geq 2} + O(s^{29/9}). \end{aligned}$$

It is fairly easy to show that  $N_{\geq 2}$  is  $O(s^{10/3})$ , by noting that  $N_{\geq 2}$  can be upper bounded by  $O\left(\sum_i |E_i|^2\right)$ , where  $E_i$  is as defined in (H1). Using the upper bound  $|E_i| = O(s^{4/3})$  [21], we get

$$N_{\geq 2} = O\left(\sum_i |E_i|^2\right) = O(s^{4/3}) \cdot O\left(\sum_i |E_i|\right) = O(s^{10/3}).$$

Thus, at the moment,  $N_{\geq 2}$  is the bottleneck in the above bound, and we only get the (very weak) lower bound  $\Omega(s^{2/3})$  on the number of distinct distances. Showing that  $N_{\geq 2} = O(s^{29/9})$  too (hopefully, a rather modest goal) would improve the lower bound to  $\Omega(s^{7/9})$ , still a rather weak lower bound.

Nevertheless, we feel that the reduction to incidences in three dimensions is fruitful, because

- (i) It sheds new light on the geometry of planar point sets related to the distinct distances problem.
- (ii) It gave us a new, and considerably more involved setup in which the new algebraic technique of Guth and Katz could be applied. As such, the analysis reviewed in this note might prove useful for obtaining improved incidence bounds for points and other classes of curves in three dimensions. The case of points and circles is an immediate next challenge.

Another comment is in order. Our work can be regarded as a special variant of the complex version of the Szemerédi–Trotter theorem on point-line incidences [23]. In the complex plane, the equation of a line (in complex notation) is  $w = pz + q$ . Interpreting this equation as a transformation of the real plane, we get a *homothetic map*, i.e., a rigid motion followed by a scaling. We can therefore rephrase the complex version of the Szemerédi–Trotter theorem as follows. We are given a set  $P$  of  $m$  pairs of points in the (real) plane, and a set  $M$  of  $n$  homothetic maps, and we seek an upper bound on the number of times a map  $\tau \in M$  and a pair  $(a, b) \in P$  “coincide”, in the sense that  $\tau(a) = b$ . In our work we only consider “complex lines” whose “slope”  $p$  has absolute value 1 (these are our rotations), and the set  $P$  is simply  $S \times S$ . This explains in part Elekes’s interest in incidences with equally inclined lines in  $\mathbb{R}^3$ , as mentioned in the introduction.

The main open problems raised by this work are:

- (a) Obtain a cubic upper bound for the number of rotations which map only two points of the given ground planar set  $S$  to another pair of points of  $S$ .

Any upper bound smaller than  $O(s^{3.1358})$  would already be a significant step towards improving the current lower bound of  $\Omega(s^{0.8641})$  on distinct distances [13].

(b) Improve further the upper bound on the number of incidences between rotations and  $h$ -parabolas. Ideally, establish Conjectures 1 and 3.

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**ROTER TYPE EQUATIONS  
FOR A CLASS OF ANTI-KÄHLER MANIFOLDS**

By

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*Dedicated to Professor Dr. Witold Roter on his 77th birthday*

**Abstract.** We investigate the anti-Kähler manifolds  $(M, g, f)$  of quasi-constant totally real sectional curvatures. We prove that such manifolds satisfy a Roter type equation. Conversely, every HC-flat manifold  $(M, g, f)$  satisfying that equation is of quasi-constant totally real sectional curvatures.

### 1. The object of the paper

Investigating birecurrent manifolds, i.e. semi-Riemannian manifolds whose the Riemann–Christoffel curvature tensor  $R$  satisfies the condition

$$\nabla^2 R = R \otimes a,$$

where  $a$  is some  $(0, 2)$ -tensor, W. Roter proved in [16] that some of them satisfy

$$(1) \quad R(X, Y, Z, W) = \phi(\rho(X, W)\rho(Y, Z) - \rho(X, Z)\rho(Y, W)),$$

for some scalar function  $\phi$  and any vector fields  $X, Y, Z, W$  on  $M$ . In other words, their Riemann–Christoffel tensor  $R$  is proportional to a  $(0, 4)$ -tensor formed from the Ricci tensor  $\rho$  only. The first named author of this paper in [3] examined semi-Riemannian manifolds  $(M, g)$ , of dimension  $\geq 4$ , for

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which the tensor  $R$  is expressed by a linear combination of  $(0, 4)$ -tensors formed from the metric tensor  $g$  and the Ricci tensor  $\rho$  as follows

$$\begin{aligned}
 R(X, Y, Z, W) = & \phi_1(\rho(X, W)\rho(Y, Z) - \rho(X, Z)\rho(Y, W)) + \\
 & + \phi_2(g(X, W)\rho(Y, Z) + g(Y, Z)\rho(X, W) - \\
 & - g(X, Z)\rho(Y, W) - g(Y, W)\rho(X, Z)) + \\
 (2) \quad & + \phi_3(g(X, W)g(Y, Z) - g(X, Z)g(Y, W)),
 \end{aligned}$$

where  $\phi_1, \phi_2$  and  $\phi_3$  are some functions on  $M$ . The equation (2) is called an equation of Roter type. We refer to [8] for details as a survey of results on manifolds satisfying (2).

In this paper we investigate anti-Kähler manifolds  $(M, g, f)$  of quasi-constant totally real sectional curvatures, i.e. anti-Kähler manifolds whose Riemannian curvature tensor has the form (6). In section 3 we prove that any of such manifolds  $(M, g, f)$  is HC-flat and satisfies the Roter type equation (7), and conversely, any HC-flat manifold  $(M, g, f)$  satisfying (7) is of quasi-constant totally real sectional curvature. In section 4 we investigate a special case of (7). That case was also considered in [15]. In the last section we prove that such manifolds are semi-symmetric. This fact gives rise to the conception of h-pseudosymmetry and Ricci h-pseudosymmetry for anti-Kähler manifolds.

## 2. Preliminaries

The anti-Kähler manifold  $(M, g, f)$  is a manifold  $M$  endowed with the metric  $g$  and the complex structure  $f$  such that

$$f^2 = -Id, \quad g(fX, fY) = -g(X, Y), \quad \nabla f = 0,$$

for all  $X, Y \in T_p M$ , where  $T_p M$  is the tangent space of  $M$  at  $p \in M$  and  $\nabla$  is the Levi-Civita connection. In some earlier papers the manifold  $(M, g, f)$  was named the B-manifold ([5], [6], [12], [13], [15]) and in another – Kähler manifolds with the Norden metric ([7]).

The manifold  $(M, g, f)$  is orientable and  $\dim M = 2n$ ,  $n \geq 2$ . The metric of such manifold is indefinite and the signature is  $(n, n)$ . Also,  $trf = 0$ . If we set  $F(X, Y) = g(fX, Y)$  then  $F(X, Y) = F(Y, X)$  because of which we can endow the manifold  $M$  with the metric  $F(X, Y)$  as well. Let  $R(X, Y, Z, W)$

and  $\overset{*}{R}(X, Y, Z, W)$  be the Riemann–Christoffel curvature tensor of  $(M, g, f)$  and  $(M, F, f)$ , respectively. Then

$$R(X, Y, fZ, fW) = -R(X, Y, Z, W),$$

$$\overset{*}{R}(X, Y, fZ, fW) = -\overset{*}{R}(X, Y, Z, W),$$

$$\overset{*}{R}(X, Y, Z, W) = R(fX, Y, Z, W),$$

for all  $X, Y, Z, W \in T_p M$ . If  $\rho$  and  $\overset{*}{\rho}$  are the Ricci tensors corresponding to  $R$  and  $\overset{*}{R}$ , respectively, then

$$\rho(fX, fY) = -\rho(X, Y), \quad \overset{*}{\rho}(X, Y) = \rho(fX, Y) = \overset{*}{\rho}(Y, X).$$

Further, the scalar curvatures  $\kappa$  and  $\overset{*}{\kappa}$  are defined by

$$\kappa = \sum_{i=1}^{2n} \rho(e_i, e_i), \quad \overset{*}{\kappa} = \sum_{i=1}^{2n} \overset{*}{\rho}(e_i, e_i),$$

respectively, where  $\{e_i\}$  is an orthonormal basis of  $T_p M$ . For every non-degenerate, with respect to the metric  $g$ , 2-plane  $\pi$  in  $T_p M$  we define its sectional curvatures  $K(\pi, p)$  and  $\overset{*}{K}(\pi, p)$  by

$$K(\pi, p) = \frac{R(X, Y, Y, X)}{\gamma(X, Y, Y, X)}, \quad \overset{*}{K}(\pi, p) = \frac{\overset{*}{R}(X, Y, Y, X)}{\gamma(X, Y, Y, X)},$$

respectively, where  $\{X, Y\}$  is a basis of  $\pi$  and

$$(3) \quad \gamma(X, Y, Z, W) = g(X, W)g(Y, Z) - g(X, Z)g(Y, W).$$

A 2-plane  $\pi$  in  $T_p M$  is said to be totally real if  $\pi \neq f\pi$  and  $\pi \perp f\pi$  with respect to  $g$ . The manifold  $(M, g, f)$  is said to be of pointwise constant totally real sectional curvatures if at  $p \in M$ ,  $K(\pi, p)$ , as well as  $\overset{*}{K}(\pi, p)$  are independent of non-degenerate totally real 2-plane  $\pi$  in  $T_p M$ .

It is known ([5], [6]) that the necessary and the sufficient condition for  $(M, g, f)$ ,  $n \geq 2$ , to be of pointwise constant totally real sectional curvature is

$$(4) \quad R(X, Y, Z, W) = \frac{1}{4n(n-1)}(\kappa G(X, Y, Z, W) - \overset{*}{\kappa} G(fX, Y, Z, W)),$$

where

$$G(X, Y, Z, W) = g(X, W)g(Y, Z) - g(X, Z)g(Y, W) - F(X, W)F(Y, Z) + F(X, Z)F(Y, W).$$

The holomorphically conformal curvature tensor  $HC$  of  $(M, g, f)$  is defined by ([13], [14])

$$\begin{aligned}
 (HC)(X, Y, Z, W) = & \\
 = R(X, Y, Z, W) - \frac{1}{2(n-2)}(g(X, W)\rho(Y, Z) + g(Y, Z)\rho(X, W) - & \\
 - g(X, Z)\rho(Y, W) - g(Y, W)\rho(X, Z) - F(X, W)\overset{*}{\rho}(Y, Z) - & \\
 - F(Y, Z)\overset{*}{\rho}(X, W) + F(X, Z)\overset{*}{\rho}(Y, W) + F(Y, W)\overset{*}{\rho}(X, Z)) + & \\
 (5) \quad + \frac{1}{4(n-1)(n-2)}(\kappa G(X, Y, Z, W) - \overset{*}{\kappa} G(f X, Y, Z, W)). &
 \end{aligned}$$

Tensor  $HC$  was also studied in [12].

In this paper we investigate the anti-Kähler manifolds  $(M, g, f)$  whose curvature tensor can be expressed in the following form

$$\begin{aligned}
 R(X, Y, Z, W) = & \lambda G(X, Y, Z, W) + \mu G(f X, Y, Z, W) + \\
 + g(X, W)(\alpha V(Y, Z) + \beta \overline{V}(Y, Z)) + & \\
 + g(Y, Z)(\alpha V(X, W) + \beta \overline{V}(X, W)) - & \\
 - g(X, Z)(\alpha V(Y, W) + \beta \overline{V}(Y, W)) - & \\
 - g(Y, W)(\alpha V(X, Z) + \beta \overline{V}(X, Z)) - & \\
 - F(X, W)(-\beta V(Y, Z) + \alpha \overline{V}(Y, Z)) - & \\
 - F(Y, Z)(-\beta V(X, W) + \alpha \overline{V}(X, W)) + & \\
 + F(X, Z)(-\beta V(Y, W) + \alpha \overline{V}(Y, W)) + & \\
 (6) \quad + F(Y, W)(-\beta V(X, Z) + \alpha \overline{V}(X, Z)), &
 \end{aligned}$$

where  $\lambda, \mu, \alpha$  and  $\beta$  are some scalar functions and

$$V(X, Y) = V(X)V(Y) - \overline{V}(X)\overline{V}(Y),$$

$$\overline{V}(X, Y) = V(f X, Y) = \overline{V}(X)V(Y) + V(X)\overline{V}(Y),$$

$V$  is a 1-form and  $\overline{V}(X) = V(f X)$ . Manifolds satisfying (6) arose during study of HC-flat holomorphic hypersurfaces of a HC-flat anti-Kähler manifold. Namely, in [4](Theorem 2) it was proved that if  $n > 3$ , then the HC-flat holomorphic hypersurface (i.e. the holomorphic hypersurface whose the HC-tensor defined by (5) vanishes) of a manifold of constant totally real sectional curvatures satisfies the condition (6). Conversely, any manifold satisfying (6) is HC-flat.

If  $\alpha = \beta = 0$  then (6) reduces to

$$R(X, Y, Z, W) = \lambda G(X, Y, Z, W) + \mu G(fX, Y, Z, W),$$

from which it follows that

$$\lambda = \frac{\kappa}{4n(n-1)}, \quad \mu = -\frac{\kappa^*}{4n(n-1)},$$

i.e. (6) reduces to (4). Therefore, if the Riemann–Christoffel curvature tensor of a manifold  $(M, g, f)$  can be expressed in the form (6) then we say that  $(M, g, f)$  is a *manifold of quasi-constant totally real sectional curvatures*. For an anti-Kähler manifold the condition corresponding to the Roter type equation (2) is

$$(7) \quad \begin{aligned} R(X, Y, Z, W) = & \mathcal{L}_1 \Gamma(X, Y, Z, W) + \mathcal{L}_2 \Gamma(fX, Y, Z, W) + \\ & + \mathcal{L}_3 G(X, Y, Z, W) + \mathcal{L}_4 G(fX, Y, Z, W), \end{aligned}$$

where

$$\begin{aligned} \Gamma(X, Y, Z, W) = & \rho(X, W)\rho(Y, Z) - \rho(X, Z)\rho(Y, W) - \\ & - \rho^*(X, W)\rho^*(Y, Z) + \rho^*(X, Z)\rho^*(Y, W), \end{aligned}$$

and  $\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3$  and  $\mathcal{L}_4$  are some scalar functions on the appropriate set  $U \subset M$ . We call (7) the *Roter type equation for an anti-Kähler manifold*. For the later use we present the following result.

LEMMA 2.1. ([4], Lemma 2) *Let  $A$  be a symmetric 2-form on  $(M, g, f)$  satisfying*

$$A(fX, fY) = -A(X, Y)$$

*and*

$$\begin{aligned} EG(X, Y, Z, W) + HG(fX, Y, Z, W) = & \\ = & A(X, W)A(Y, Z) - A(X, Z)A(Y, W) - \\ & - A(fX, W)A(fY, Z) + A(fX, Z)A(fY, W), \end{aligned}$$

*where  $E$  and  $H$  are some scalar functions. Then*

$$\begin{aligned} A(X, Y) = & \overline{C}g(X, Y) + \overline{B}f(X, Y) + \tau(V(X)V(Y) - \overline{V}(X)\overline{V}(Y)) + \\ & + \sigma(\overline{V}(X)V(Y) + V(X)\overline{V}(Y)), \end{aligned}$$

*where  $\overline{C}, \overline{B}, \tau$  and  $\sigma$  are some scalar functions,  $V$  is an 1-form and  $\overline{V}(X) = V(fX)$ .*

### 3. Manifolds of quasi-constant totally real sectional curvatures and the Roter type equation

First we shall show that an anti-Kähler manifold of quasi-constant totally real sectional curvatures satisfies the Roter type equation (7). We present (6) in the local coordinates

$$\begin{aligned}
 R_{ijlm} = & \lambda G_{ijlm} + \mu f_i^a G_{ajlm} + \\
 & + g_{im}(\alpha V_{jl} + \beta \bar{V}_{jl}) + g_{jl}(\alpha V_{im} + \beta \bar{V}_{im}) - \\
 & - g_{il}(\alpha V_{jm} + \beta \bar{V}_{jm}) - g_{jm}(\alpha V_{il} + \beta \bar{V}_{il}) - \\
 & - F_{im}(-\beta V_{jl} + \alpha \bar{V}_{jl}) - F_{jl}(-\beta V_{im} + \alpha \bar{V}_{im}) + \\
 (8) \quad & + F_{il}(-\beta V_{jm} + \alpha \bar{V}_{jm}) + F_{jm}(-\beta V_{il} + \alpha \bar{V}_{il}),
 \end{aligned}$$

where

$$V_{jl} = V_j V_l - \bar{V}_j \bar{V}_l, \bar{V}_{jl} = f_j^a V_{al} = \bar{V}_j V_l + V_j \bar{V}_l,$$

$V_j$  are the local components of the 1-form  $V$  and  $\bar{V}_j = f_j^a V_a$ . Without loss of generality we can suppose that

$$V_{ij} g^{ij} = 2 V_a V^a = 1, \bar{V}_{ij} g^{ij} = 2 \bar{V}_a V^a = 1.$$

Thus we have

$$\begin{aligned}
 \rho_{jl} = & (2(n-1)\lambda + (\alpha + \beta))g_{jl} + (2(n-1)\mu - (\alpha - \beta))F_{jl} + \\
 & + 2(n-2)(\alpha V_{jl} + \beta \bar{V}_{jl}), \\
 \rho_{jl}^* = & -(2(n-1)\mu - (\alpha - \beta))g_{jl} + (2(n-1)\lambda + (\alpha + \beta))F_{jl} + \\
 (9) \quad & + 2(n-2)(-\beta V_{jl} + \alpha \bar{V}_{jl}),
 \end{aligned}$$

$$(10) \quad \kappa = 4(n-1)(n\lambda + (\alpha + \beta)), \quad \kappa^* = 4(n-1)(-n\mu + (\alpha - \beta)).$$

If we set

$$(11) \quad L = 2(n-1)\lambda + (\alpha + \beta), \quad N = 2(n-1)\mu - (\alpha - \beta),$$

then (9) becomes

$$\begin{aligned}
 \rho_{jl} = & L g_{jl} + N F_{jl} + 2(n-2)(\alpha V_{jl} + \beta \bar{V}_{jl}), \\
 \rho_{jl}^* = & -N g_{jl} + L F_{jl} + 2(n-2)(-\beta V_{jl} + \alpha \bar{V}_{jl}),
 \end{aligned}$$

i.e.

$$(12) \quad \begin{aligned} \alpha V_{jl} + \beta \bar{V}_{jl} &= \frac{1}{2(n-2)}(\rho_{jl} - Lg_{jl} - NF_{jl}), \\ -\beta V_{jl} + \alpha \bar{V}_{jl} &= \frac{1}{2(n-2)}(\rho_{jl}^* + Ng_{jl} - LF_{jl}). \end{aligned}$$

Substituting this into (8) we find

$$(13) \quad R_{ijlm} = \left( \lambda - \frac{L}{n-2} \right) G_{ijlm} + \left( \mu - \frac{N}{n-2} \right) f_i^a G_{ajlm} + \frac{1}{2(n-2)} M_{ijlm},$$

where

$$\begin{aligned} M_{ijlm} = g_{im}\rho_{jl} + g_{jl}\rho_{im} - g_{il}\rho_{jm} - g_{jm}\rho_{il} - \\ - F_{im} \rho_{jl}^* - F_{jl} \rho_{im}^* + F_{il} \rho_{jm}^* + F_{jm} \rho_{il}^*. \end{aligned}$$

It is easy to see that

$$(14) \quad \begin{aligned} V_{im} V_{jl} - V_{il} V_{jm} - \bar{V}_{im} \bar{V}_{jl} + \bar{V}_{il} \bar{V}_{jm} &= 0, \\ V_{im} \bar{V}_{jl} + \bar{V}_{im} V_{jl} - \bar{V}_{il} V_{jm} - V_{il} \bar{V}_{jm} &= 0, \end{aligned}$$

which leads to

$$\begin{aligned} (\alpha V_{im} + \beta \bar{V}_{im})(\alpha V_{jl} + \beta \bar{V}_{jl}) - (\alpha V_{il} + \beta \bar{V}_{il})(\alpha V_{jm} + \beta \bar{V}_{jm}) - \\ - (-\beta V_{im} + \alpha \bar{V}_{im})(-\beta V_{jl} + \alpha \bar{V}_{jl}) + (-\beta V_{il} + \alpha \bar{V}_{il})(-\beta V_{jm} + \alpha \bar{V}_{jm}) = \\ = (\alpha^2 - \beta^2)(V_{im} V_{jl} - V_{il} V_{jm} - \bar{V}_{im} \bar{V}_{jl} + \bar{V}_{il} \bar{V}_{jm}) + \\ + 2\alpha\beta(V_{im} \bar{V}_{jl} + \bar{V}_{im} V_{jl} - \bar{V}_{il} V_{jm} - V_{il} \bar{V}_{jm}) = 0. \end{aligned}$$

By making use of (12), this means that

$$\begin{aligned} (\rho_{im} - Lg_{im} - NF_{im})(\rho_{jl} - Lg_{jl} - NF_{jl}) - \\ - (\rho_{il} - Lg_{il} - NF_{il})(\rho_{jm} - Lg_{jm} - NF_{jm}) - \\ - (\rho_{im}^* + Ng_{im} - LF_{im})(\rho_{jl}^* + Ng_{jl} - LF_{jl}) + \\ + (\rho_{il}^* + Ng_{il} - LF_{il})(\rho_{jm}^* + Ng_{jm} - LF_{jm}) = 0, \end{aligned}$$

i.e.

$$LM_{ijlm} + Nf_i^a M_{ajlm} = \Gamma_{ijlm} + (L^2 - N^2)G_{ijlm} + 2LNf_i^a G_{ajlm}.$$

The last equation, together with

$$-NM_{ijlm} + Lf_i^a M_{ajlm} = f_i^a \Gamma_{ajlm} - 2LN G_{ijlm} + (L^2 - N^2)f_i^a G_{ajlm},$$

yields

$$M_{ijlm} = \frac{1}{L^2 + N^2} (L\Gamma_{ijlm} - Nf_i^a \Gamma_{ajlm}) + LG_{ijlm} + Nf_i^a G_{ajlm}.$$

Substituting this into (13) we get

$$(15) \quad R_{ijlm} = \left( \lambda - \frac{L}{2(n-2)} \right) G_{ijlm} + \left( \mu - \frac{N}{2(n-2)} \right) f_i^a G_{ajlm} + \frac{1}{2(n-2)(L^2 + N^2)} (L\Gamma_{ijlm} - Nf_i^a \Gamma_{ajlm}).$$

Thus we see that (7) is an Roter type equation.

Now, let us suppose that, conversely, the Roter type equation (7) is satisfied. Then

$$\begin{aligned} \rho_{im} = & \mathcal{L}_1 (\kappa \rho_{im} - \overset{*}{\kappa} \rho_{im} - 2\rho_{ia} \rho_m^a) + \\ & + \mathcal{L}_2 (\overset{*}{\kappa} \rho_{im} + \kappa \overset{*}{\rho}_{im} - 2\overset{*}{\rho}_{ia} \rho_m^a) + \\ & + 2(n-1) (\mathcal{L}_3 g_{im} + \mathcal{L}_4 F_{im}), \end{aligned}$$

i.e.

$$\begin{aligned} \mathcal{L}_1 \rho_{ia} \rho_m^a + \mathcal{L}_2 \overset{*}{\rho}_{ia} \rho_m^a = & \frac{1}{2} (\mathcal{L}_1 \kappa + \mathcal{L}_2 \overset{*}{\kappa} - 1) \rho_{im} + \\ & + \frac{1}{2} (-\mathcal{L}_1 \overset{*}{\kappa} + \mathcal{L}_2 \kappa) \overset{*}{\rho}_{im} + (n-1) (\mathcal{L}_3 g_{im} + \mathcal{L}_4 F_{im}). \end{aligned}$$

This implies

$$\begin{aligned} -\mathcal{L}_2 \rho_{ia} \rho_m^a + \mathcal{L}_1 \overset{*}{\rho}_{ia} \rho_m^a = & \frac{1}{2} (\mathcal{L}_1 \overset{*}{\kappa} - \mathcal{L}_2 \kappa) \rho_{im} + \\ & + \frac{1}{2} (\mathcal{L}_1 \kappa + \mathcal{L}_2 \overset{*}{\kappa} - 1) \overset{*}{\rho}_{im} - (n-1) (\mathcal{L}_4 g_{im} - \mathcal{L}_3 F_{im}). \end{aligned}$$

Therefore

$$(16) \quad \rho_{ia} \rho_m^a = p \rho_{im} - q \overset{*}{\rho}_{im} + c g_{im} + d F_{im},$$

where

$$(17) \quad \begin{aligned} p = & \frac{1}{2} \left( \kappa - \frac{\mathcal{L}_1}{\mathcal{L}_1^2 + \mathcal{L}_2^2} \right), \quad q = \frac{1}{2} \left( \overset{*}{\kappa} - \frac{\mathcal{L}_2}{\mathcal{L}_1^2 + \mathcal{L}_2^2} \right), \\ c = & (n-1) \frac{\mathcal{L}_1 \mathcal{L}_3 + \mathcal{L}_2 \mathcal{L}_4}{\mathcal{L}_1^2 + \mathcal{L}_2^2}, \quad d = (n-1) \frac{\mathcal{L}_1 \mathcal{L}_4 - \mathcal{L}_2 \mathcal{L}_3}{\mathcal{L}_1^2 + \mathcal{L}_2^2}. \end{aligned}$$

The relation (16) yields

$$(18) \quad \rho_{ia}^* \rho_m^a = q \rho_{im} + p \rho_{im}^* - d g_{im} + c F_{im}.$$

Now using (7), (16) and (18) we get

$$\begin{aligned} R_{rsia} \rho_j^a + R_{rsja} \rho_i^a &= \\ &= -(n-2) \mathcal{L}_3 (g_{si} \rho_{rj} - g_{ri} \rho_{sj} - F_{si} \rho_{rj}^* + \\ &\quad + F_{ri} \rho_{sj}^* + g_{sj} \rho_{ri} - g_{rj} \rho_{si} - F_{sj} \rho_{ri}^* + F_{rj} \rho_{si}^*) - \\ &\quad - (n-2) \mathcal{L}_4 (F_{si} \rho_{rj} + g_{si} \rho_{rj}^* - \\ &\quad - g_{ri} \rho_{sj}^* - F_{ri} \rho_{sj} + F_{sj} \rho_{ri} + g_{sj} \rho_{ri}^* - g_{rj} \rho_{si}^* - F_{rj} \rho_{si}). \end{aligned} \quad (19)$$

If we suppose that the manifold is (HC)-flat and (7) is satisfied then

$$(20) \quad R_{rsia} = \frac{1}{2(n-2)} M_{rsia} - \frac{1}{4(n-1)(n-2)} \left( \kappa G_{rsia} - \kappa^* f_r^t G_{tsia} \right).$$

Substituting this into (19) and using (16) and (18) we get

$$(21) \quad a Q(g, \rho)_{rsij} + b f_r^a Q(g, \rho)_{asij} = 0,$$

where

$$\begin{aligned} a &= \frac{p}{2(n-2)} - \frac{\kappa}{4(n-1)(n-2)} + (n-2) \mathcal{L}_3, \\ (22) \quad b &= -\frac{q}{2(n-2)} + \frac{\kappa^*}{4(n-1)(n-2)} + (n-2) \mathcal{L}_4 \end{aligned}$$

and

$$\begin{aligned} Q(g, \rho)_{rsij} &= g_{si} \rho_{rj} - g_{ri} \rho_{sj} - F_{si} \rho_{rj}^* + F_{ri} \rho_{sj}^* + \\ &\quad + g_{sj} \rho_{ri} - g_{rj} \rho_{si} - F_{sj} \rho_{ri}^* + F_{rj} \rho_{si}^*. \end{aligned}$$

The relation (21), together with

$$-b Q(g, \rho)_{rsij} + a f_r^a Q(g, \rho)_{asij} = 0,$$

implies

$$(a^2 + b^2) Q(g, \rho)_{rsij} = 0.$$

Therefore, if  $(M, g, f)$  is (HC)-flat and satisfies (7) then either  $Q(g, \rho) = 0$  or  $a = b = 0$ . In the first case we have

$$(23) \quad \rho_{ij} = \frac{1}{2n} \left( \kappa g_{ij} - \kappa^* F_{ij} \right), \quad \rho_{ij}^* = \frac{1}{2n} (\kappa^* g_{ij} + \kappa F_{ij}).$$

Substituting this into (20) we find that  $(M, g, f)$  is of constant totally real sectional curvatures. So, we have to examine the case when (23) does not hold, i.e. the case  $a = b = 0$ . Substituting (20) into (7) we find

$$\begin{aligned} \mathcal{L}_1\Gamma + \mathcal{L}_2f \circ \Gamma &= \frac{1}{2(n-2)}M - \left( \frac{\kappa}{4(n-1)(n-2)} + \mathcal{L}_3 \right) G + \\ &\quad + \left( \frac{\overset{*}{\kappa}}{4(n-1)(n-2)} - \mathcal{L}_4 \right) f \circ G. \end{aligned}$$

Also we have

$$\begin{aligned} -\mathcal{L}_2\Gamma + \mathcal{L}_1f \circ \Gamma &= \frac{1}{2(n-2)}f \circ M - \left( \frac{\overset{*}{\kappa}}{4(n-1)(n-2)} - \mathcal{L}_4 \right) G - \\ &\quad - \left( \frac{\kappa}{4(n-1)(n-2)} + \mathcal{L}_3 \right) f \circ G, \end{aligned}$$

where  $f \circ G$ , resp.  $f \circ M$ , denotes the  $(0, 4)$ -tensor with the local components  $f_h^r G_{rjik}$ , resp.  $f_h^r M_{rjik}$ . The last two relations imply

$$\begin{aligned} \Gamma - \frac{\mathcal{L}_1}{2(n-2)(\mathcal{L}_1^2 + \mathcal{L}_2^2)}M + \frac{\mathcal{L}_2}{2(n-2)(\mathcal{L}_1^2 + \mathcal{L}_2^2)}f \circ M &= \\ &= \left( -\frac{\mathcal{L}_1}{\mathcal{L}_1^2 + \mathcal{L}_2^2} \left( \frac{\kappa}{4(n-1)(n-2)} + \mathcal{L}_3 \right) + \frac{\mathcal{L}_2}{\mathcal{L}_1^2 + \mathcal{L}_2^2} \left( \frac{\overset{*}{\kappa}}{4(n-1)(n-2)} - \mathcal{L}_4 \right) \right) G + \\ &\quad + \left( \frac{\mathcal{L}_1}{\mathcal{L}_1^2 + \mathcal{L}_2^2} \left( \frac{\overset{*}{\kappa}}{4(n-1)(n-2)} - \mathcal{L}_4 \right) + \frac{\mathcal{L}_2}{\mathcal{L}_1^2 + \mathcal{L}_2^2} \left( \frac{\kappa}{4(n-1)(n-2)} + \mathcal{L}_3 \right) \right) f \circ G. \end{aligned}$$

To made this clear we set

$$l_1 = \frac{\mathcal{L}_1}{2(n-2)(\mathcal{L}_1^2 + \mathcal{L}_2^2)}, l_2 = \frac{\mathcal{L}_2}{2(n-2)(\mathcal{L}_1^2 + \mathcal{L}_2^2)},$$

such that

$$\begin{aligned} \frac{1}{2(n-2)}(\Gamma - l_1M + l_2f \circ M) &= \\ &= \left( -l_1 \left( \frac{\kappa}{4(n-1)(n-2)} + \mathcal{L}_3 \right) + l_2 \left( \frac{\overset{*}{\kappa}}{4(n-1)(n-2)} - \mathcal{L}_4 \right) \right) G + \\ (24) \quad &+ \left( l_1 \left( \frac{\overset{*}{\kappa}}{4(n-1)(n-2)} - \mathcal{L}_4 \right) + l_2 \left( \frac{\kappa}{4(n-1)(n-2)} + \mathcal{L}_3 \right) \right) f \circ G. \end{aligned}$$

On the other hand, we have the following expression for  $\Gamma - l_1 M + l_2 f \circ M$

$$\begin{aligned}
& (\rho_{im} - l_1 g_{im} + l_2 F_{im})(\rho_{jl} - l_1 g_{jl} + l_2 F_{jl}) - \\
& - (\rho_{il} - l_1 g_{il} + l_2 F_{il})(\rho_{jm} - l_1 g_{jm} + l_2 F_{jm}) - \\
& - (\overset{*}{\rho}_{im} - l_2 g_{im} - l_1 F_{im})(\overset{*}{\rho}_{jl} - l_2 g_{jl} - l_1 F_{jl}) + \\
& + (\overset{*}{\rho}_{il} - l_2 g_{il} - l_1 F_{il})(\overset{*}{\rho}_{jm} - l_2 g_{jm} - l_1 F_{jm}) = \\
& = \Gamma_{ijlm} - l_1 M_{ijlm} + l_2 f_i^a M_{ajlm} + (l_1^2 - l_2^2) G_{ijlm} - 2l_1 l_2 f_i^a G_{ajlm}.
\end{aligned}$$

Substituting this into (24) we obtain

$$\begin{aligned}
& (\rho_{im} - l_1 g_{im} + l_2 F_{im})(\rho_{jl} - l_1 g_{jl} + l_2 F_{jl}) - \\
& - (\rho_{il} - l_1 g_{il} + l_2 F_{il})(\rho_{jm} - l_1 g_{jm} + l_2 F_{jm}) - \\
& - (\overset{*}{\rho}_{im} - l_2 g_{im} - l_1 F_{im})(\overset{*}{\rho}_{jl} - l_2 g_{jl} - l_1 F_{jl}) + \\
& + (\overset{*}{\rho}_{il} - l_2 g_{il} - l_1 F_{il})(\overset{*}{\rho}_{jm} - l_2 g_{jm} - l_1 F_{jm}) = \\
& = \left( l_1^2 - l_2^2 - \frac{1}{2(n-1)} (l_1 \kappa - l_2 \overset{*}{\kappa}) - 2(n-2)(l_1 \mathcal{L}_3 + l_2 \mathcal{L}_4) \right) G_{ijlm} + \\
& + \left( -2l_1 l_2 + \frac{1}{2(n-1)} (l_1 \overset{*}{\kappa} + l_2 \kappa) - 2(n-2)(l_1 \mathcal{L}_4 - l_2 \mathcal{L}_3) \right) f_i^a G_{ajlm}.
\end{aligned}$$

Applying Lemma 2.1 we find

$$\begin{aligned}
\rho_{im} - l_1 g_{im} + l_2 F_{im} &= \\
&= \overline{C} g_{im} + \overline{B} F_{im} + \tau (V_i V_m - \overline{V}_i \overline{V}_m) + \sigma (\overline{V}_i V_m + V_i \overline{V}_m),
\end{aligned}$$

i.e.

$$(25) \quad \rho_{im} = C g_{im} + B F_{im} + \tau V_{im} + \sigma \overline{V}_{im},$$

where  $B$ ,  $\overline{B}$ ,  $C$  and  $\overline{C}$  are some scalar functions. Substituting (25) into (7) we get

$$\begin{aligned}
R_{ijlm} &= (\mathcal{L}_1(C^2 - B^2) - 2\mathcal{L}_2 BC + \mathcal{L}_3) G_{ijlm} + \\
&+ (2\mathcal{L}_1 BC + \mathcal{L}_2(C^2 - B^2) + \mathcal{L}_4) f_i^a G_{ajlm} + \\
&+ (\mathcal{L}_1(\tau C - \sigma B) - \mathcal{L}_2(\sigma C + \tau B))(g_{im} V_{jl} + g_{jl} V_{im} - g_{il} V_{jm} - g_{jm} V_{il} - \\
&- F_{im} \overline{V}_{jl} - F_{jl} \overline{V}_{im} + F_{il} \overline{V}_{jm} + F_{jm} \overline{V}_{il}) + \\
&+ (\mathcal{L}_1(\sigma C + \tau B) + \mathcal{L}_2(\tau C - \sigma B))(g_{im} \overline{V}_{jl} + g_{jl} \overline{V}_{im} - g_{il} \overline{V}_{jm} - g_{jm} \overline{V}_{il} + \\
&+ F_{im} V_{jl} + F_{jl} V_{im} - F_{il} V_{jm} - F_{jm} V_{il}).
\end{aligned}$$

Finally, if we set

$$\begin{aligned}\alpha &= \mathcal{L}_1(\tau C - \sigma B) - \mathcal{L}_2(\sigma C + \tau B), \\ \beta &= \mathcal{L}_1(\sigma C + \tau B) + \mathcal{L}_2(\tau C - \sigma B), \\ \lambda &= \mathcal{L}_1(C^2 - B^2) - 2\mathcal{L}_2 BC + \mathcal{L}_3, \\ \mu &= 2\mathcal{L}_1 BC + \mathcal{L}_2(C^2 - B^2) + \mathcal{L}_4,\end{aligned}$$

then we get (6).

Evidently, if  $\tau = \sigma = 0$  then (25) reduces to (23) and therefore  $(M, g, f)$  is a manifold of constant totally real sectional curvatures.

We note that according (22) and because of  $a = b = 0$  we have

$$\begin{aligned}p &= \frac{\kappa}{2(n-1)} - 2(n-2)^2 \mathcal{L}_3, \\ q &= \frac{\overset{*}{\kappa}}{2(n-1)} + 2(n-2)^2 \mathcal{L}_4,\end{aligned}$$

so that by (17) we obtain

$$\begin{aligned}\frac{n-2}{n-1} \kappa &= \frac{\mathcal{L}_1}{\mathcal{L}_1^2 + \mathcal{L}_2^2} - 4(n-2)^2 \mathcal{L}_3, \\ (26) \quad \frac{n-2}{n-1} \overset{*}{\kappa} &= \frac{\mathcal{L}_2}{\mathcal{L}_1^2 + \mathcal{L}_2^2} + 4(n-2)^2 \mathcal{L}_4.\end{aligned}$$

Thus we can state

**THEOREM 3.1.** *The anti-Kähler manifold  $(M, g, f)$  of quasi-constant totally real curvatures is HC-flat and satisfies the equation of Roter type (15). Conversely, HC-flat anti-Kähler manifold satisfying the equation of Roter type (7) is of constant totally real sectional curvatures or of quasi-constant totally real sectional curvatures. In the last case the functions  $\mathcal{L}_1, \dots, \mathcal{L}_4$  satisfy (26).*

#### 4. Special case

The relation (15) reduces to

$$(27) \quad R_{ijlm} = \frac{1}{2(n-2)(L^2 + N^2)} (L\Gamma_{ijlm} - Nf_i^a \Gamma_{ajlm})$$

if and only if

$$\lambda = \frac{L}{2(n-2)}, \quad \mu = \frac{N}{2(n-2)}.$$

In view of (11) this means that

$$\lambda = -\frac{1}{2}(\alpha + \beta), \quad \mu = \frac{1}{2}(\alpha - \beta).$$

Thus in virtue of (10) we get

$$\lambda = \frac{\kappa}{4(n-1)(n-2)}, \quad \mu = -\frac{\overset{*}{\kappa}}{4(n-1)(n-2)},$$

and therefore

$$(28) \quad L = \frac{\kappa}{2(n-1)}, \quad N = -\frac{\overset{*}{\kappa}}{2(n-1)}.$$

Thus

$$\frac{L}{2(L^2 + N^2)} = \frac{(n-1)\kappa}{\kappa^2 + \overset{*}{\kappa}^2}, \quad -\frac{N}{2(L^2 + N^2)} = \frac{(n-1)\overset{*}{\kappa}}{\kappa^2 + \overset{*}{\kappa}^2},$$

and (27) becomes

$$(29) \quad R_{ijlm} = \frac{n-1}{n-2}(\kappa^2 + \overset{*}{\kappa}^2)^{-1}(\kappa \Gamma_{ijlm} + \overset{*}{\kappa} f_i^a \Gamma_{ajlm}),$$

while (8) takes the form

$$(30) \quad \begin{aligned} R_{ijlm} = & \frac{1}{4(n-1)(n-2)}(\kappa G_{ijlm} - \overset{*}{\kappa} f_i^a G_{ajlm}) + \\ & + g_{im}(\alpha V_{jl} + \beta \bar{V}_{jl}) + g_{jl}(\alpha V_{im} + \beta \bar{V}_{im}) - \\ & - g_{il}(\alpha V_{jm} + \beta \bar{V}_{jm}) - g_{jm}(\alpha V_{il} + \beta \bar{V}_{il}) - \\ & - F_{im}(-\beta V_{jl} + \alpha \bar{V}_{jl}) - F_{jl}(-\beta V_{im} + \alpha \bar{V}_{im}) + \\ & + F_{il}(-\beta V_{jm} + \alpha \bar{V}_{jm}) + F_{jm}(-\beta V_{il} + \alpha \bar{V}_{il}). \end{aligned}$$

As for (12), in view of (28), it is now

$$(31) \quad \rho_{jl} = \frac{\kappa}{2(n-1)}g_{jl} - \frac{\overset{*}{\kappa}}{2(n-1)}F_{jl} + 2(n-2)(\alpha V_{jl} + \beta \bar{V}_{jl}).$$

In [15] we considered HC-flat anti-Kähler manifolds satisfying the condition

$$(32) \quad R_{ijlm} = \mathcal{L}_1 \Gamma_{ijlm} + \mathcal{L}_2 f_i^a \Gamma_{ajlm},$$

corresponding, for the anti-Kähler manifolds, to the Roter equation (1). Proceeding in the similar way as in the section 3 of this paper we can prove that

$$(33) \quad \mathcal{L}_1 = \frac{(n-1)\kappa}{(n-2)(\kappa^2 + \kappa^{*2})} \mathcal{L}_2 = \frac{(n-1)\kappa^{*}}{(n-2)(\kappa^2 + \kappa^{*2})},$$

and the manifold is either of constant totally real sectional curvatures, or the curvature tensor and the Ricci tensor have the form (30) and (31), respectively. We mention that in [15] we have  $\frac{\alpha}{2(n-2)}$  and  $\frac{\beta}{2(n-2)}$  instead of  $\alpha$  and  $\beta$ .

Thus we can state

**THEOREM 4.1.** *The anti-Kähler manifold of quasi-constant totally real sectional curvatures (30) is HC-flat and satisfies the equation (29) of Roter type. Conversely, HC-flat anti-Kähler manifold satisfying the equation of Roter type (32) is either of constant totally real sectional curvatures, or it is of quasi-constant totally real sectional curvatures and (33) holds.*

**REMARK 4.1** In [15] it was written uncorrect

$$\rho(X, Y) =$$

$$= \frac{\kappa}{2(n-1)} g(X, Y) - \frac{\kappa^{*}}{2(n-1)} F(X, Y) + \alpha V(X)V(Y) + \beta \overline{V}(X)\overline{V}(Y)$$

instead

$$\rho(X, Y) = \frac{\kappa}{2(n-1)} g(X, Y) - \frac{\kappa^{*}}{2(n-1)} F(X, Y) + \alpha V(X, Y) + \beta \overline{V}(X, Y),$$

and similarly for  $R(X, Y, Z, W)$ .

## 5. H-pseudosymmetry

A semi-Riemannian manifold  $(M, g)$ ,  $\dim \geq 3$ , is said to be *semisymmetric* if  $R \cdot R = 0$  holds on  $M$  ([17]). As a proper generalization of *locally symmetric spaces* ( $\nabla R = 0$ ), semisymmetric manifolds were investigated by several authors. Some of those investigations gave rise to the next generalization, namely to *pseudosymmetric manifolds*, i.e. semi-Riemannian manifolds satisfying  $R \cdot R = \mathcal{L}_R Q(g, R)$  on the set  $U_R \subset M$  of all points at which its Riemann–Christoffel curvature tensor  $R$  is not proportional to the  $(0, 4)$ -tensor  $\gamma$  defined by (3) and  $\mathcal{L}_R$  is some function defined on  $U_R$  ([1], [2], [18]).

A semi-Riemannian manifold  $(M, g)$ , of dimension  $\geq 3$ , is said to be *Ricci-symmetric*, resp. *Ricci-pseudosymmetric*, if  $R \cdot \rho = 0$  holds on  $M$ , resp.  $R \cdot \rho = \mathcal{L}_\rho Q(g, \rho)$  holds on the set  $U_\rho \subset M$  of all points at which its Ricci tensor  $\rho$  is not proportional to the metric tensor  $g$  and  $\mathcal{L}_\rho$  is some function defined on  $U_\rho$  ([1], [2]). Recently, a geometric interpretation of pseudosymmetry, resp. Ricci-pseudosymmetry, was given in [9], resp. in [11] (see also [10]). In the local coordinates the condition of pseudosymmetry, resp. Ricci-pseudosymmetry, reads

$$(34) \quad \begin{aligned} (R \cdot R)_{ijklrs} &= -R_{ajkl} R_{irs}^a - R_{iakl} R_{jrs}^a - R_{ijal} R_{krs}^a - R_{ijka} R_{lrs}^a = \\ &= \mathcal{L}_R(g_{si} R_{rjkl} + g_{sj} R_{irkh} + g_{sk} R_{ijrl} + g_{sl} R_{ijkr} - \\ &\quad - g_{ri} R_{sjkl} - g_{rj} R_{iskl} - g_{rk} R_{ijsl} - g_{rl} R_{ijks}), \end{aligned}$$

resp.

$$(35) \quad \begin{aligned} (R \cdot S)_{ijrs} &= -\rho_{aj} R_{irs}^a - \rho_{ia} R_{jrs}^a = \\ &= \mathcal{L}_S(g_{si} \rho_{rj} + g_{sj} \rho_{ri} - g_{ri} \rho_{sj} - g_{rj} \rho_{si}). \end{aligned}$$

In section 3 it was proved that a manifold  $(M, g, f)$  satisfying the Roter type equation (7) also satisfies (19), which is like (35), but it is adopted to the complex structure  $f$  of the manifold. Therefore we introduce the following definition.

The anti-Kähler manifold  $(M, g, f)$ ,  $n \geq 2$ , is said to be *holomorphically pseudosymmetric*, in short *h-pseudosymmetric*, if

$$(36) \quad \begin{aligned} (R \cdot R)_{ijklrs} &= -R_{ajkl} R_{irs}^a - R_{iakl} R_{jrs}^a - R_{ijal} R_{krs}^a - R_{ijka} R_{lrs}^a = \\ &= \xi(g_{si} R_{rjkl} + g_{sj} R_{irkh} + g_{sk} R_{ijrl} + g_{sl} R_{ijkr} - \\ &\quad - g_{ri} R_{sjkl} - g_{rj} R_{iskl} - g_{rk} R_{ijsl} - g_{rl} R_{ijks} - \\ &\quad - f_r^a(F_{si} R_{ajkl} + F_{sj} R_{iakl} + F_{sk} R_{ijal} + F_{sl} R_{ijka}) + \\ &\quad + f_s^a(F_{ri} R_{ajkl} + F_{rj} R_{iakl} + F_{rk} R_{ijal} + F_{rl} R_{ijka}) + \\ &\quad + \psi(f_r^a(g_{si} R_{ajkl} + g_{sj} R_{iakl} + g_{sk} R_{ijal} + g_{sl} R_{ijka}) - \\ &\quad - f_s^a(g_{ri} R_{ajkl} + g_{rj} R_{iakl} + g_{rk} R_{ijal} + g_{rl} R_{ijka}) + \\ &\quad + F_{si} R_{rjkl} + F_{sj} R_{irkh} + F_{sk} R_{ijrl} + F_{sl} R_{ijkr} - \\ &\quad - F_{ri} R_{sjkl} - F_{rj} R_{iskl} - F_{rk} R_{ijsl} - F_{rl} R_{ijks}), \end{aligned}$$

on the appropriate set  $U \subset M$ , where  $\xi$  and  $\psi$  are some functions  $U$ .

The anti-Kähler manifold  $(M, g, f)$ ,  $n \geq 2$ , is said to be *Ricci-holomorphically pseudosymmetric*, in short *Ricci h-pseudosymmetric*, if

$$(R \cdot \rho)_{ijrs} = -\rho_{aj} R_{irs}^a - \rho_{ia} R_{jrs}^a =$$

$$\begin{aligned}
&= \xi(g_{si}\rho_{rj} + g_{sj}\rho_{ri} - g_{ri}\rho_{sj} - g_{rj}\rho_{si} - F_{si} \overset{*}{\rho}_{rj} - \\
&\quad - F_{sj} \overset{*}{\rho}_{ri} + F_{ri} \overset{*}{\rho}_{sj} + F_{rj} \overset{*}{\rho}_{si}) + \\
&\quad + \psi(F_{si}\rho_{rj} + F_{sj}\rho_{ri} - F_{ri}\rho_{sj} - F_{rj}\rho_{si} + g_{si} \overset{*}{\rho}_{rj} + g_{sj} \overset{*}{\rho}_{ri} - \\
(37) \quad &\quad - g_{ri} \overset{*}{\rho}_{sj} - g_{rj} \overset{*}{\rho}_{si}).
\end{aligned}$$

Thus we can state

**THEOREM 5.1.** *The anti-Kähler manifold satisfying the Roter type equation (7) is Ricci h-pseudosymmetric.*

But the class of Ricci h-pseudosymmetric manifolds contains the h-pseudosymmetric manifolds as a proper subset. It can be proved that an anti-Kähler manifold satisfying the Roter type equation (7) is h-pseudosymmetric. In fact if (7) holds then

$$\begin{aligned}
(R \cdot R)_{ijklrs} &= -R_{ajkl}R_{isr}^a - R_{iakl}R_{jsr}^a - R_{ijal}R_{ksr}^a - R_{ijka}R_{lsr}^a = \\
&= g_{si}(A\Gamma_{rjkl} + Bf_r^a\Gamma_{ajkl}) + g_{sj}(A\Gamma_{irkh} + Bf_r^a\Gamma_{iakl}) + \\
&\quad + g_{sk}(A\Gamma_{ijrl} + Bf_r^a\Gamma_{ijal}) + g_{sl}(A\Gamma_{ijkh} + Bf_r^a\Gamma_{ijka}) - \\
&\quad - F_{si}(Af_r^a\Gamma_{ajkl} - B\Gamma_{rjkl}) - F_{sj}(Af_r^a\Gamma_{iakl} - B\Gamma_{irkh}) - \\
&\quad - F_{sk}(Af_r^a\Gamma_{ijal} - B\Gamma_{ijrl}) - F_{sl}(Af_r^a\Gamma_{ijka} - B\Gamma_{ijkh}) - \\
&\quad - g_{ri}(A\Gamma_{sjkl} + Bf_s^a\Gamma_{ajkl}) - g_{rj}(A\Gamma_{iskl} + Bf_s^a\Gamma_{iakl}) - \\
&\quad - g_{rk}(A\Gamma_{ijsl} + Bf_s^a\Gamma_{ijal}) - g_{rl}(A\Gamma_{ijks} + Bf_s^a\Gamma_{ijka}) + \\
&\quad + F_{ri}(Af_s^a\Gamma_{ajkl} - B\Gamma_{sjkl}) + F_{rj}(Af_s^a\Gamma_{iakl} - B\Gamma_{iskl}) + \\
(38) \quad &\quad + F_{rk}(Af_s^a\Gamma_{ijal} - B\Gamma_{ijsl}) + F_{rl}(Af_s^a\Gamma_{ijka} - B\Gamma_{ijks})
\end{aligned}$$

where

$$\begin{aligned}
A &= (\mathcal{L}_1^2 - \mathcal{L}_2^2)c - 2\mathcal{L}_1\mathcal{L}_2d - (\mathcal{L}_1\mathcal{L}_3 - \mathcal{L}_2\mathcal{L}_4), \\
(39) \quad B &= (\mathcal{L}_1^2 - \mathcal{L}_2^2)d + 2\mathcal{L}_1\mathcal{L}_2c - (\mathcal{L}_1\mathcal{L}_4 + \mathcal{L}_2\mathcal{L}_3),
\end{aligned}$$

while  $c$  and  $d$  are given by (17).

On the other hand, substituting (7) into the right hand side of (36), we obtain

$$\begin{aligned}
(R \cdot R)_{ijklrs} &= -R_{ajkl}R_{isr}^a - R_{iakl}R_{jsr}^a - R_{ijal}R_{ksr}^a - R_{ijka}R_{lsr}^a = \\
&= g_{si}((\xi\mathcal{L}_1 - \psi\mathcal{L}_2)\Gamma_{rjkl} + (\xi\mathcal{L}_2 + \psi\mathcal{L}_1)f_r^a\Gamma_{ajkl}) +
\end{aligned}$$

$$\begin{aligned}
& + g_{sj}((\xi\mathcal{L}_1 - \psi\mathcal{L}_2)\Gamma_{irk} + (\xi\mathcal{L}_2 + \psi\mathcal{L}_1)f_r^a\Gamma_{iakl}) + \\
& + g_{sk}((\xi\mathcal{L}_1 - \psi\mathcal{L}_2)\Gamma_{ijrl} + (\xi\mathcal{L}_2 + \psi\mathcal{L}_1)f_r^a\Gamma_{ijal}) + \\
& + g_{sl}((\xi\mathcal{L}_1 - \psi\mathcal{L}_2)\Gamma_{ijkr} + (\xi\mathcal{L}_2 + \psi\mathcal{L}_1)f_r^a\Gamma_{ijka}) - \\
& - g_{ri}((\xi\mathcal{L}_1 - \psi\mathcal{L}_2)\Gamma_{sjkl} + (\xi\mathcal{L}_2 + \psi\mathcal{L}_1)f_s^a\Gamma_{ajkl}) - \\
& - g_{rj}((\xi\mathcal{L}_1 - \psi\mathcal{L}_2)\Gamma_{iskl} + (\xi\mathcal{L}_2 + \psi\mathcal{L}_1)f_s^a\Gamma_{iakl}) - \\
& - g_{rk}((\xi\mathcal{L}_1 - \psi\mathcal{L}_2)\Gamma_{ijsl} + (\xi\mathcal{L}_2 + \psi\mathcal{L}_1)f_s^a\Gamma_{ijal}) - \\
& - g_{rl}((\xi\mathcal{L}_1 - \psi\mathcal{L}_2)\Gamma_{ijks} + (\xi\mathcal{L}_2 + \psi\mathcal{L}_1)f_s^a\Gamma_{ijka}) - \\
& - F_{si}((\xi\mathcal{L}_1 - \psi\mathcal{L}_2)f_r^a\Gamma_{ajkl} - (\xi\mathcal{L}_2 + \psi\mathcal{L}_1)\Gamma_{rjkl}) - \\
& - F_{sj}((\xi\mathcal{L}_1 - \psi\mathcal{L}_2)f_r^a\Gamma_{iakl} - (\xi\mathcal{L}_2 + \psi\mathcal{L}_1)\Gamma_{irk}) - \\
& - F_{sk}((\xi\mathcal{L}_1 - \psi\mathcal{L}_2)f_r^a\Gamma_{ijal} - (\xi\mathcal{L}_2 + \psi\mathcal{L}_1)\Gamma_{ijrl}) - \\
& - F_{sl}((\xi\mathcal{L}_1 - \psi\mathcal{L}_2)f_r^a\Gamma_{ijka} - (\xi\mathcal{L}_2 + \psi\mathcal{L}_1)\Gamma_{ijkr}) + \\
& + F_{ri}((\xi\mathcal{L}_1 - \psi\mathcal{L}_2)f_s^a\Gamma_{ajkl} - (\xi\mathcal{L}_2 + \psi\mathcal{L}_1)\Gamma_{sjkl}) + \\
& + F_{rj}((\xi\mathcal{L}_1 - \psi\mathcal{L}_2)f_s^a\Gamma_{iakl} - (\xi\mathcal{L}_2 + \psi\mathcal{L}_1)\Gamma_{iskl}) + \\
& + F_{rk}((\xi\mathcal{L}_1 - \psi\mathcal{L}_2)f_s^a\Gamma_{ijal} - (\xi\mathcal{L}_2 + \psi\mathcal{L}_1)\Gamma_{ijsl}) + \\
& + F_{rl}((\xi\mathcal{L}_1 - \psi\mathcal{L}_2)f_s^a\Gamma_{ijka} - (\xi\mathcal{L}_2 + \psi\mathcal{L}_1)\Gamma_{ijks}). \tag{40}
\end{aligned}$$

The relations (38) and (40) show that if

$$(41) \quad -A = \xi\mathcal{L}_1 - \psi\mathcal{L}_2, \quad -B = \xi\mathcal{L}_2 + \psi\mathcal{L}_1,$$

then  $(M, g, f)$  is h-pseudosymmetric. It follows from (41) that

$$\xi = -\frac{A\mathcal{L}_1 + B\mathcal{L}_2}{\mathcal{L}_1^2 + \mathcal{L}_2^2}, \quad \psi = \frac{A\mathcal{L}_2 - B\mathcal{L}_1}{\mathcal{L}_1^2 + \mathcal{L}_2^2},$$

so, in view of (39), we have

$$\xi = -c\mathcal{L}_1 + d\mathcal{L}_2 + \mathcal{L}_3, \quad \psi = -c\mathcal{L}_2 - d\mathcal{L}_1 + \mathcal{L}_4.$$

Finally, using (17) and (39), we get

$$(42) \quad \xi = -(n-2)\mathcal{L}_3, \quad \psi = -(n-2)\mathcal{L}_4.$$

If  $\mathcal{L}_3 = \mathcal{L}_4 = 0$ , then (36) reduces to the condition of semisymmetry, while (7) turns into (32). Thus we have

**THEOREM 5.2.** *The anti-Kähler manifold  $(M, g, f)$  satisfying the Roter type equation (7) is h-pseudosymmetric and (42) holds. If  $(M, g, f)$  satisfies the Roter type equation (32) then it is semisymmetric.*

We have seen that every manifold of quasi-constant totally real sectional curvatures  $(M, g, f)$ , i.e. a manifold satisfying (8), satisfies the Roter type equation (15), that is the Roter type equation (7), where

$$\begin{aligned}\mathcal{L}_1 &= \frac{L}{2(n-2)(L^2+N^2)}, & \mathcal{L}_2 &= -\frac{N}{2(n-2)(L^2+N^2)}, \\ \mathcal{L}_3 &= \lambda - \frac{L}{2(n-2)}, & \mathcal{L}_4 &= \mu - \frac{N}{2(n-2)}.\end{aligned}$$

Taking into account (11) we get

$$\mathcal{L}_3 = -\frac{1}{n-2} \left( \lambda + \frac{\alpha+\beta}{2} \right), \quad \mathcal{L}_4 = -\frac{1}{n-2} \left( \mu - \frac{\alpha-\beta}{2} \right),$$

or, in view of (41),

$$(43) \quad \xi = \lambda + \frac{\alpha+\beta}{2}, \quad \psi = \mu - \frac{\alpha-\beta}{2}.$$

Thus we have

**COROLLARY 5.1.** *The anti-Kähler manifold whose curvature tensor has the form (8) is h-pseudosymmetric such that (43) holds. In the special case (30) it is semisymmetric.*

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## ON $\theta$ -COMPACTNESS IN IDEAL TOPOLOGICAL SPACES

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**Abstract.** In this paper, we present and study  $\theta$ -*I*-compactness and countable  $\theta$ -*I*-compactness which are two kinds of compactness with respect to an ideal by utilizing the notion of  $\theta$ -open sets.

### 1. Introduction

It is a matter of fact that ideals played an important and decisive role in topology for several years. It was the works of Newcomb [19], Rancin [23], Samuels [25] and Hamlett and Janković ([9], [10], [11], [12]) which motivated the research in applying topological ideals to generalize the most fundamental properties in General Topology. During 90's several important topological properties were generalized by utilizing the topological ideals such as, compactness, connectedness, submaximality, resolvability and so on. In 1995, Dontchev [6] investigated Hausdorff spaces via topological ideals and quite recently Arenas et al. [1] studied some weak separation axioms between  $T_0$  and  $T_1$  via topological ideals.

In 1967, Newcomb introduced the notion of compactness modulo an ideal. Rancin [23], Hamlett and Janković [9] further investigated this notion and obtained some more properties of compactness modulo an ideal. The notion of countable compactness modulo an ideal was also introduced by Newcomb in the same paper. Hamlett et al. [13] further studied this notion which they called countably *I*-compact. It is the aim of this paper to present and study two kinds of compactness via ideals called  $\theta$ -*I*-compactness and countable  $\theta$ -*I*-compactness.

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In 1943, Fomin [8] (see, also [14]) introduced the notion of  $\theta$ -continuity. The notions of  $\theta$ -open subsets,  $\theta$ -closed subsets and  $\theta$ -closure were introduced by Veličko [26] for the purpose of studying the important class of  $H$ -closed spaces in terms of arbitrary filterbases. Dickman and Porter [4], [5], Joseph [17] continued the work of Veličko. Recently Noiri and Jafari [20] have also obtained several new and interesting results related to these sets.

In what follows  $(X, \tau)$  and  $(Y, \sigma)$  (or  $X$  and  $Y$ ) denote topological spaces. By  $(X, \tau, I)$ , we denote an ideal topological space. Let  $A$  be a subset of  $X$ . An ideal topological space is a topological space  $(X, \tau)$  with an ideal  $I$  on  $X$ , where the ideal is defined as a nonempty collection of subsets of  $X$  satisfying the following two conditions: (i) If  $A \in I$  and  $B \subset A$ , then  $B \in I$ ; (ii) If  $A \in I$  and  $B \in I$ , then  $A \cup B \in I$ . We denote the interior and the closure of a set  $A$  by  $Int(A)$  and  $Cl(A)$ , respectively. A point  $x \in X$  is called a  $\theta$ -cluster point of  $A$  if  $A \cap Cl(U) \neq \emptyset$  for every open set  $U$  of  $X$  containing  $x$ . The set of all  $\theta$ -cluster points of  $A$  is called the  $\theta$ -closure of  $A$ . A subset  $A$  is called  $\theta$ -closed if  $A$  and its  $\theta$ -closure coincide. The complement of a  $\theta$ -closed set is called  $\theta$ -open. We denote the collection of all  $\theta$ -open sets by  $\theta(X, \tau)$ . It is shown in [18] that the collection of  $\theta$ -open sets in a space  $X$  forms a topology denoted by  $\tau_\theta$ . A topological space  $(X, \tau)$  is called  $\theta$ -compact [15] if every cover of the space by  $\theta$ -open sets has a finite subcover.

## 2. $\theta$ -*I*-compact spaces

Throughout this paper, we write a space  $(X, \tau, I)$  instead of an ideal topological space when there is no chance for confusion. We begin with the following notion:

**DEFINITION 1.** A space  $(X, \tau, I)$  is called  $\theta$ -compact modulo an ideal or simply  $\theta$ -*I*-compact if for every  $\theta$ -open cover  $\{U_k \mid k \in K\}$  of  $X$ , there exists a finite subset  $K_0$  of  $K$  such that  $X \setminus \bigcup\{U_k \mid k \in K_0\} \in I$ .

**THEOREM 2.1.** *The following statements are equivalent for a space  $(X, \tau, I)$ :*

(1)  $(X, \tau, I)$  is  $\theta$ -*I*-compact;

(2) for each family  $\{P_k \mid k \in K\}$  of  $\theta$ -closed sets of  $X$  such that  $\bigcap\{P_k \mid k \in K\} = \emptyset$ , there exists a finite subset  $K_0$  of  $K$  such that  $\bigcap\{P_k \mid k \in K_0\} \in I$ .

PROOF. (1)  $\Rightarrow$  (2): Suppose that  $\{P_k \mid k \in K\}$  is a family of  $\theta$ -closed sets of  $X$  such that  $\bigcap\{P_k \mid k \in K\} = \emptyset$ . Thus  $\{X \setminus P_k \mid k \in K\}$  is a  $\theta$ -open cover of  $X$ . Since  $(X, \tau, I)$  is  $\theta$ - $I$ -compact, there exists a finite subset  $K_0$  of  $K$  such that  $X \setminus \bigcup\{X \setminus P_k \mid k \in K_0\} \in I$ . It follows that  $\bigcap\{P_k \mid k \in K_0\} \in I$ .

(2)  $\Rightarrow$  (1): Assume that  $\{U_k \mid k \in K\}$  is a  $\theta$ -open cover of  $X$ . Then  $\{X \setminus U_k \mid k \in K\}$  is a collection of  $\theta$ -closed sets and  $\bigcap\{X \setminus U_k \mid k \in K\} = \emptyset$ . Now there exists a finite subset  $K_0$  of  $K$  such that  $\bigcap\{X \setminus U_k \mid k \in K_0\} \in I$ . It follows that  $X \setminus \bigcup\{U_k \mid k \in K_0\} \in I$ . Therefore  $(X, \tau, I)$  is  $\theta$ - $I$ -compact.

**THEOREM 2.2.** *A space  $(X, \tau, I)$  is  $\theta$ - $I$ -compact if and only if  $(X, \tau_\theta, I)$  is  $I$ -compact.*

In a topological space  $(X, \tau)$ , we denote the ideal of finite (resp. countable) subsets of  $X$  by  $I_f$  (resp.  $I_c$ ).

**THEOREM 2.3.** *The following statements are equivalent for a space  $(X, \tau, I)$ :*

- (1) *a topological space  $(X, \tau)$  is  $\theta$ -compact;*
- (2) *the space  $(X, \tau, I_f)$  is  $\theta$ - $I_f$ -compact;*
- (3) *the space  $(X, \tau, \{\emptyset\})$  is  $\theta$ - $\{\emptyset\}$ -compact.*

PROOF. Straightforward.

**THEOREM 2.4.** *Let  $(X, \tau, I)$  be  $\theta$ - $I$ -compact. If  $I \subseteq J$ , where  $J$  is an ideal on  $X$ , then  $(X, \tau, J)$  is  $\theta$ - $J$ -compact*

PROOF. Straightforward.

Recall that a topological space  $(X, \tau)$  is said to be Lindelöf if every open cover has a countable subcover.

**DEFINITION 2.** A topological space  $(X, \tau)$  is said to be  $\theta$ -Lindelöf if every  $\theta$ -open cover has a countable subcover.

**THEOREM 2.5.** *If a space  $(X, \tau, I_c)$  is  $\theta$ - $I_c$ -compact, then  $(X, \tau)$  is  $\theta$ -Lindelöf.*

PROOF. Suppose that  $\{U_k \mid k \in K\}$  is a  $\theta$ -open cover of  $X$ . By hypothesis,  $(X, \tau, I_c)$  is  $\theta$ - $I_c$ -compact. This means that there exists a finite subcover  $K_0$  of  $K$  such that  $X \setminus \bigcup\{U_k \mid k \in K_0\} \in I_c$  and hence the proof.

COROLLARY 2.6. *If the space  $(X, \tau, I_c)$  is  $I_c$ -compact, then  $(X, \tau)$  is Lindelöf.*

LEMMA 2.7. (Newcomb [19]). *The following properties hold:*

- (1) *If  $f : (X, \tau, I) \rightarrow (Y, \sigma)$  is surjective, then  $f(I) = \{f(A) \mid A \in I\}$  is an ideal on  $Y$ .*
- (2) *If  $f : (X, \tau) \rightarrow (Y, \sigma, J)$  is injective, then  $f^{-1}(J) = \{f^{-1}(B) \mid B \in J\}$  is an ideal on  $X$ .*

DEFINITION 3. A function  $f : (X, \tau) \rightarrow (Y, \sigma)$  is said to be

- (1) quasi  $\theta$ -continuous [21] (see also [15]) if the inverse image of each  $\theta$ -open set of  $Y$  is a  $\theta$ -open set in  $X$ .
- (2)  $M$ - $\theta$ -open [3] if the image of each  $\theta$ -open set in  $X$  is a  $\theta$ -open set in  $Y$ .

THEOREM 2.8. *If  $f : (X, \tau, I) \rightarrow (Y, \sigma)$  is a surjective quasi  $\theta$ -continuous function and the space  $(X, \tau, I)$  is  $\theta$ - $I$ -compact, then  $(Y, \sigma, f(I))$  is  $\theta$ - $f(I)$ -compact.*

PROOF. Suppose that  $\{V_k \mid k \in K\}$  is a  $\theta$ -open cover of  $Y$ . Since  $f$  is a quasi  $\theta$ -continuous function,  $\{f^{-1}(V_k) \mid k \in K\}$  is a  $\theta$ -open cover of  $X$ . Since the space  $(X, \tau, I)$  is  $\theta$ - $I$ -compact, there exists a finite subset  $K_0$  of  $K$  such that  $X \setminus \bigcup\{f^{-1}(V_k) \mid k \in K_0\} \in I$ . By the surjectivity of  $f$  and Lemma 2.7, we have  $Y \setminus \bigcup\{V_k \mid k \in K_0\} = f(X \setminus \bigcup\{f^{-1}(V_k) \mid k \in K_0\}) \in f(I)$ . This shows that  $(Y, \sigma, f(I))$  is  $\theta$ - $f(I)$ -compact.

Recall that a function  $f : (X, \tau) \rightarrow (Y, \sigma)$  is called strongly  $\theta$ -continuous [22] if the inverse image of each open set in  $Y$  is  $\theta$ -open in  $X$ .

THEOREM 2.9. *If  $f : (X, \tau, I) \rightarrow (Y, \sigma)$  is a surjective strongly  $\theta$ -continuous function and the space  $(X, \tau, I)$  is  $\theta$ - $I$ -compact, then  $(Y, \sigma, f(I))$  is  $f(I)$ -compact.*

PROOF. Similar to the proof of Theorem 2.8.

THEOREM 2.10. *If  $f : (X, \tau) \rightarrow (Y, \sigma, J)$  is an  $M$ - $\theta$ -open bijection and  $(Y, \sigma, J)$  is  $\theta$ - $J$ -compact, then  $(X, \tau, f^{-1}(J))$  is  $\theta$ - $f^{-1}(J)$ -compact.*

PROOF. By hypothesis,  $f$  is  $M$ - $\theta$ -open and bijective and therefore  $f^{-1} : (Y, \sigma, J) \rightarrow (X, \tau)$  is a quasi- $\theta$ -continuous surjection. By the fact that  $(Y, \sigma, J)$

is  $\theta$ -J-compact, it follows from Theorem 2.8 that  $(X, \tau, f^{-1}(J))$  is  $\theta$ - $f^{-1}(J)$ -compact.

### 3. Countably $\theta$ -I-compact spaces

We say that a topological space  $(X, \tau)$  is countably  $\theta$ -compact if every countable  $\theta$ -open cover of  $(X, \tau)$  has a finite subcover.

**DEFINITION 4.** Let  $N$  be the set of natural numbers. A space  $(X, \tau, I)$  is said to be countably  $\theta$ -I-compact if for each countable  $\theta$ -open cover  $\{U_n \mid n \in N\}$  of  $X$ , there exists a finite subset  $N_0$  of  $N$  such that  $X \setminus \bigcup\{U_n \mid n \in N_0\} \in I$ .

**THEOREM 3.1.** *For a space  $(X, \tau, I)$  the following statements are equivalent:*

- (1)  $(X, \tau, I)$  is countably  $\theta$ -I-compact;
- (2) for any countable family  $\{P_n \mid n \in N\}$  of  $\theta$ -closed sets of  $X$  such that  $\bigcap\{P_n \mid n \in N\} = \emptyset$ , there exists a finite subset  $N_0$  of  $N$  such that  $\bigcap\{P_n \mid n \in N_0\} \in I$ .

**PROOF.** The proof is similar to the proof of Theorem 2.1.

**THEOREM 3.2.** *Let  $(X, \tau, I)$  be countably  $\theta$ -I-compact. If  $I \subset J$ , where  $J$  is an ideal on  $X$ , then  $(X, \tau, J)$  is countably  $\theta$ -J-compact.*

**THEOREM 3.3.** *The following statements are equivalent for a space  $(X, \tau, I)$ :*

- (1) a topological space  $(X, \tau)$  is countably  $\theta$ -compact;
- (2) the space  $(X, \tau, I_f)$  is countably  $\theta$ - $I_f$ -compact;
- (3) the space  $(X, \tau, \{\emptyset\})$  is countably  $\theta$ - $\{\emptyset\}$ -compact.

**THEOREM 3.4.** *If a space  $(X, \tau, I)$  is countably  $\theta$ -I-compact and  $(X, \tau)$  is  $\theta$ -Lindelöf, then  $(X, \tau, I)$  is  $\theta$ -I-compact.*

**PROOF.** Suppose that  $\{U_k \mid k \in K\}$  is a  $\theta$ -open cover of  $X$ . By  $\theta$ -Lindelöfness of  $(X, \tau)$ , there exists a countable subset  $K_1$  of  $K$  such that  $X = \bigcup\{U_k \mid k \in K_1\}$ . Since  $(X, \tau, I)$  is countably  $\theta$ -I-compact, there exists a finite subset  $K_2$  of  $K_1$  such that  $X \setminus \bigcup\{U_k \mid k \in K_2\} \in I$ . This shows that  $(X, \tau, I)$  is  $\theta$ -I-compact.

**THEOREM 3.5.** *If  $f : (X, \tau, I) \rightarrow (Y, \sigma)$  is a surjective quasi  $\theta$ -continuous function and the space  $(X, \tau, I)$  is countably  $\theta$ - $I$ -compact, then  $(Y, \sigma, f(I))$  is countably  $\theta$ - $f(I)$ -compact.*

**PROOF.** The proof is similar to the proof of Theorem 2.8.

We close with the following questions:

**QUESTION 3.6.** *Is there a nontrivial example of a  $\theta$ -compact (resp. countably  $\theta$ -compact) space which is not compact (resp. countably compact)?*

**QUESTION 3.7.** *Is there a nontrivial example of a  $\theta$ - $I$ -compact (resp. countably  $\theta$ - $I$ -compact) space which is not  $I$ -compact (resp. countably  $I$ -compact)?*

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**PARTITIONABILITY TO TWO TREES IS NP-COMPLETE**

By

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**Abstract.** We show that P2T — the problem of deciding whether the edge set of a simple graph can be partitioned into two trees or not — is NP-complete.

It is a well known that deciding whether the edge set of a graph can be partitioned into  $k$  spanning trees or not is in  $P$  [1]. Recently András Frank asked what we know about partitioning the edge set of a graph into  $k$  (not necessarily spanning) trees. One can easily see that whether a simple graph is a tree or not is in  $P$ . It was shown by Király that the problem of deciding if the edge set of a simple graph is the disjoint union of three trees is NP-complete by reducing the 3-colorability problem to it [3]. Now we prove that for two trees the problem is also NP-complete. First we define the problem precisely.

**DEFINITION.** The input of the decision problem P2T is a graph  $G = (V, E)$  and the goal is to decide whether there is an  $E = E_1 \dot{\cup} E_2$  partitioning of the edge set such that both  $E_1$  and  $E_2$  form a tree.

**THEOREM.** P2T is NP-complete.

It is obvious that P2T belongs to NP. To prove its completeness, we will show that the NAE-SAT (NOT-ALL-EQUAL SAT problem) is reducible to P2T.

The NAE-SAT problem is the following. We are given polynomially many clauses over the variables  $x_1, \dots, x_n$  and we have to decide whether there is an evaluation of the variables such that each clause contains both a true and a false literal. This is called a *good* evaluation. Eg., if the formula

contains at least one clause of size one (like  $x_1$ ), it does *not* have a *good* evaluation. This problem is well known to be NP-complete [2].

Now we will construct a graph  $G$  from a given clause set  $\mathcal{C}$ . We denote its variables by  $x_1, \dots, x_n$ . The graph  $G$  will consist of two main parts,  $L_i$  and  $C_j$  type subgraphs. A subgraph  $L_i$  corresponds to each variable, while a subgraph  $C_j$  corresponds to each clause from  $\mathcal{C}$ . Beside the  $L_i$ 's corresponding to the variables, we also have two extra subgraphs of this type,  $L_0$  and  $L_{n+1}$ . The vertex sets corresponding to the clauses and variables are all disjoint, except for  $V(L_i) \cap V(L_{i+1})$ , what is a single vertex denoted by  $t_i$ .

A subgraph  $L_i$  corresponding to the variable  $x_i$  consists of four vertices that form a cycle in the following order:  $t_{i-1}, v_i, t_i$  and  $\bar{v}_i$ . We would like to achieve that one of the trees contains the edges from  $t_{i-1}$  through  $v_i$  to  $t_i$ , while the other from  $t_{i-1}$  through  $\bar{v}_i$  to  $t_i$ . For simplicity, we denote  $t_{-1}$  by  $\alpha$  and  $t_{n+1}$  by  $\omega$ . Both trees will have to contain a path from  $t_0$  to  $t_n$ . The idea is that we want to force one of the trees to go through exactly those  $v_i$ 's for which  $x_i$  is true.

Before we start the construction of the subgraphs corresponding to the clauses, we introduce a notation. We say that two vertices  $u$  and  $w$  are linked with a *purple* edge if

- (1) There is no edge between  $u$  and  $w$ .
- (2) The smaller connectivity component of  $G \setminus \{u, w\}$  (called *purple subgraph*) consists of four vertices:  $v_1^{uw}, v_2^{uw}, v_3^{uw}$  and  $v_4^{uw}$ .
- (3) The  $v_i^{uw}$  vertices form a cycle in this order.
- (4) The  $v_i^{uw}$  vertices are not connected to any other vertices, except for  $v_1^{uw}$  that is connected to  $u$  and  $w$ . (See Figure 1.) This is a very useful structure because if  $E(G)$  is the union of two trees, then they both have to enter this purple subgraph since a tree cannot contain a cycle. So if the vertices are linked with a purple edge and  $E(G) = E(T) \dot{\cup} E(F)$  (where  $T$  and  $F$  denote the two trees), then it means that  $uv_1^{uw} \in E(T)$  and  $wv_1^{uw} \in E(F)$  or  $uv_1^{uw} \in E(F)$  and  $wv_1^{uw} \in E(T)$ .

A subgraph  $C_j$  corresponding to the  $j$ th clause consists of  $3k$  vertices where  $k$  is the size of the  $j$ th clause whose literals are denoted by  $l_1^j, \dots, l_k^j$ . A cycle of length  $2k$  is formed by the following vertices in this order:  $p_1^j, q_1^j, p_2^j, q_2^j, \dots, p_k^j, q_k^j$ . The other  $k$  vertices are denoted by  $r_1^j, \dots, r_k^j$ . The vertex  $r_i^j$  is always connected to  $p_i^j$  and it is also connected to  $v_m$  if  $l_i^j$  is  $x_m$  or to

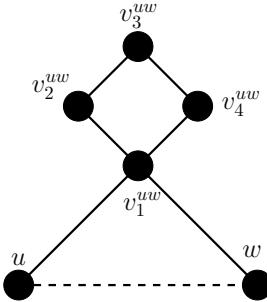
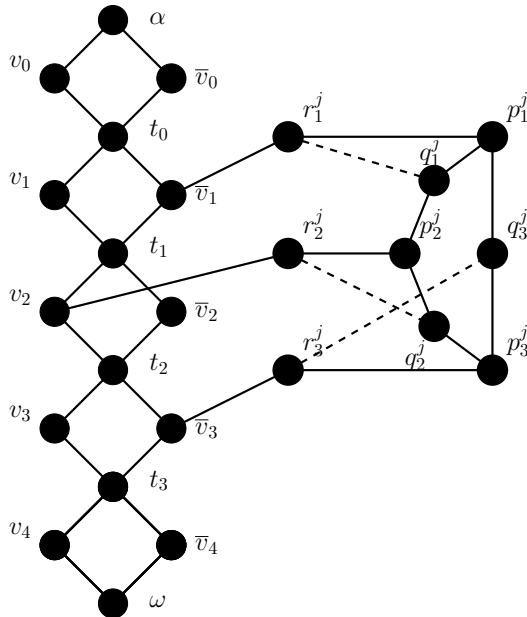


Figure 1. A purple edge

$\bar{v}_m$  if  $l_i^j$  is  $\bar{x}_m$ . Furthermore, there is a purple edge between  $r_i^j$  and  $q_i^j$ . This will ensure that a tree “entering” the clause subgraph through an  $r_i^j$ , cannot “leave” this subgraph. (See Figure 2. for a graph with one clause. The dashed edges mean purple edges.)

Figure 2. The graph for the single clause  $\bar{x}_1 \vee x_2 \vee \bar{x}_3$ 

The construction is finished, now we have to prove it's correctness. The easier part is to show that if our NAE-SAT problem has a good evaluation,

then we can partition the edges into two trees,  $T$  and  $F$ . First let us fix a good evaluation. Let the tree  $T$  contain the path from  $\alpha$  to  $\omega$  through the  $v_i$ 's for true  $x_i$ 's and through the  $\bar{v}_i$ 's for false  $x_i$ 's (consider the non-existing  $x_0$  and  $x_{n+1}$  true). Similarly  $F$  trails from  $\alpha$  to  $\omega$  through the  $\bar{v}_i$ 's for true  $x_i$ 's and through the  $v_i$ 's for false  $x_i$ 's. So each  $v_i$  and each  $\bar{v}_i$  belongs to exactly one of the trees. If a tree contains  $v_i$  (or  $\bar{v}_i$ ), let it also contain the  $v_i r_m^j$  (or  $\bar{v}_i r_m^j$ ) and  $r_m^j p_m^j$  edges if  $x_i$  (or  $\bar{x}_i$ ) is in the  $j$ th clause. This way both trees enter each clause subgraph since the evaluation satisfied our NAE-SAT problem. Let the edges  $p_i^j q_i^j$  and  $q_i^j p_{i+1}^j$  belong to the tree that does *not* contain  $r_i^j$ . This guarantees that we can enter the purple subgraph belonging to  $r_i^j$  and  $q_i^j$  by both trees. It can be also easily seen that the edges of  $T$  (and of  $F$ ) form a tree and every edge is assigned to one of them. So we are done with this part.

To prove the other part, let us suppose that  $E(G) = T \dot{\cup} F$  for two trees  $T$  and  $F$ . We know that  $t_0 \in V(T)$  and  $t_0 \in V(F)$ , because the  $L_0$  subgraph is a cycle. We can suppose that  $v_0 \in V(T)$ ,  $\bar{v}_0 \in V(F)$ ,  $\alpha \in V(T)$  and  $\alpha \in V(F)$ . We can similarly suppose  $\omega \in V(T)$  and  $\omega \in V(F)$ . Let us direct all the edges of the trees away from  $\alpha$ .

**PROPOSITION.** *There are no edges coming out of the purple subgraphs.*

**PROOF.** Both trees have to enter each purple subgraph since a tree cannot contain a cycle and since there are only two edges connecting a purple subgraph to the rest of the graph, both of them must be directed toward the purple subgraph. ■

We may conclude that the trees cannot “go through” purple edges.

**PROPOSITION.** *There are no edges coming out of the clause subgraphs.*

**PROOF.** Let us suppose that the edge from  $r_i^j$  going to some  $v_m$  (or  $\bar{v}_m$ ) is directed away from  $r_i^j$  and is in  $T$ . This implies  $p_i^j r_i^j \in T$  as well because  $T$  cannot enter  $r_i^j$  through the purple edge. But because  $r_i^j \notin V(F)$ , therefore  $q_i^j \in V(F)$  since they are linked with a purple edge. This shows  $q_i^j p_i^j \notin T$ , so  $T$  must have entered  $p_i^j$  from  $p_{i-1}^j$  through  $q_{i-1}^j$ . But then  $q_{i-1}^j \notin V(F)$ , so  $r_{i-1}^j \in V(F)$ . This means  $T$  entered  $p_{i-1}^j$  from  $p_{i-2}^j$ . And we can go on so until we get back to  $p_1^j$ , what gives a contradiction. ■

So now we know that the clauses are dead ends as well as the purple subgraphs. Since  $T$  and  $F$  trail from  $\alpha$  to  $\omega$ , each  $v_i$  (and  $\bar{v}_i$ ) must be contained in exactly one of them. So we can define  $x_i$  to be true if and only if  $v_i \in V(T)$ . Now the only property left to show is that the literals in the clauses are not equal. But if they were equal in the  $j$ th clause, then the  $C_j$  subgraph corresponding to this clause would be entered by only one of the trees and hence that tree would contain a cycle, contradiction. So we have shown that each tree partition yields a proper evaluation. This finishes the proof of the theorem. ■

Now we prove an upper bound on the maximum degree of the graph that we constructed. The degree of every vertex, except the  $v_i$ 's and  $\bar{v}_i$ 's, is at most four. A  $v_i$  (or  $\bar{v}_i$ ) has degree equal to two plus the number of occurrences of  $x_i$  (or  $\bar{x}_i$ ) in the clause set. But a NAE-SAT problem is easily reducible to a NAE-SAT-(2;2) problem (meaning that each literal can occur at most twice). If a literal  $l$  would occur in at least three clauses, then let us execute the following operation until we have at most two of each literal. Replace  $(C'_1 \vee l), (C'_2 \vee l), (C'_3 \vee l)$  with  $(l \vee \bar{z}), (C'_1 \vee l), (C'_2 \vee z), (C'_3 \vee z)$  where  $z$  is a new variable.

**COROLLARY.** *The decision of whether the edge set of a simple graph is the disjoint union of two trees or not, is NP-complete even for graphs with maximum degree four.*

**Acknowledgment.** I would like to thank Zoltán Király for discussions and the anonymous referee for his useful comments.

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# ON QUASI RICCI RIEMANNIAN MANIFOLDS

By

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**Abstract.** A new type of Riemannian manifold has been defined and studied. This manifold has been established by giving an existence theorem. The perfect fluid space-time  $(M^4, g)$  of general relativity is also studied.

## Introduction

In a recent paper [7] the present author introduced and studied a type of Riemannian manifold called Ricci Riemannian manifold. According to him a non-flat Riemannian manifold  $(M^n, g)$ ,  $n > 2$ , is called Ricci Riemannian manifold if its Ricci tensor  $L$  of type  $(1, 1)$  is not identically zero and satisfies the condition

$$(1) \quad L^2 X = \frac{r}{n-1} LX,$$

for every vector field  $X$ , where  $r$  is the scalar curvature of the manifold. Such an  $n$ -dimensional manifold was denoted by the symbol  $(RRM)_n$ .

The object of this present paper is to study a type of non-flat Riemannian manifold  $(M^n, g)$ ,  $n > 2$ , whose Ricci tensor  $L$  of type  $(1, 1)$  is not identically zero and satisfies the condition

$$(2) \quad L^2 X = \frac{r}{n-1} LX + aX,$$

for every vector field  $X$ , where  $a$  is a non-zero scalar called associated scalar and  $r$  is the scalar curvature of the manifold. Such a manifold shall be called Quasi Ricci Riemannian manifold and an  $n$ -dimensional manifold of this kind

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shall be denoted by the symbol  $(\text{QRRM})_n$ . If  $a = 0$ , then the manifold defined by (2) reduces to Ricci Riemannian manifold defined by (1). This justifies the name Quasi Ricci Riemannian manifold and the use of the symbol  $(\text{QRRM})_n$ .

After preliminaries, in Section 2, it is shown that, in particular, if  $U$  is a vector field orthogonal to  $LU$  in a  $(\text{QRRM})_n$  then the Ricci curvature in the direction of  $LU$  is  $\frac{r}{n-1}$  and if  $U$  is a unit vector field orthogonal to  $LU$ , then the square of the length of the vector field  $LU$  is equal to the associated scalar  $a$ . In the next section it is shown that a conformally flat  $(M^n, g)$ ,  $n > 2$ , is a  $(\text{QRRM})_n$ , if and only if its curvature transformation and Ricci transformation commute.

In section 4, it is shown that in an Einstein  $(\text{QRRM})_n$  the value of the associated scalar is  $-\frac{r^2}{n^2(n-1)}$ . In section 5 it is shown that in a Ricci recurrent  $(\text{QRRM})_n$ ,  $(n > 3)$ ,  $\frac{2a(n-1)}{(n-3)r}$  ( $r \neq 0$ ) is an eigenvalue of the Ricci tensor  $L$  of type  $(1, 1)$  corresponding to the vector of recurrence. Finally, considering semi Riemannian  $(\text{QRRM})_4$  perfect fluid space-time of general relativity whose energy momentum tensor obeys time like convergence condition and Einstein equation without cosmological constant holds it is shown that such a  $(\text{QRRM})_4$  space-time contains pure matter if its associated scalar is positive.

## 1. Preliminaries

Let  $L$  be the symmetric endomorphism corresponding to the Ricci tensor  $S$  of type  $(0, 2)$  at each point of the tangent space and is defined by

$$(1.1) \quad g(LX, Y) = S(X, Y).$$

In virtue of (1.1), (2) can be expressed as

$$(1.2) \quad S(LX, Y) = \frac{r}{n-1}S(X, Y) + ag(X, Y).$$

Putting  $X = e_i$  and  $Y = e_i$  where  $\{e_i\}_{i=1,2,3,\dots,n}$  is an orthonormal basis of the tangent space at each point and  $i$  is summed for  $1 \leq i \leq n$ , we get

$$(1.3) \quad S(Le_i, e_i) = \frac{r^2}{n-1} + an.$$

These formulas will be used in sequel.

## 2. Scalar Curvature of $(\text{QRRM})_n$ and its associated scalar

Let  $|S|$  be the length of the Ricci tensor  $S$ , and is defined by  $|S|^2 = S(Le_i, e_i)$ . Hence we get from (1.3)

$$(2.1) \quad r^2 = (n - 1)[|S|^2 - an].$$

Hence we can state the following theorem:

**THEOREM 1.** *In a  $(\text{QRRM})_n$ ,  $n > 2$ , the scalar curvature of the manifold is given by (2.1).*

Putting  $LY$  for  $Y$  in (1.2) we get

$$(2.2) \quad S(LX, LY) = \frac{r}{n-1}S(X, LY) + ag(X, LY).$$

Putting  $X = Y = U$  in (2.2), where  $U$  is a vector field and  $U$  is not collinear with  $LU$  we get

$$(2.3) \quad S(LU, LU) = \frac{r}{n-1}g(LU, LU) + ag(U, LU).$$

Here  $U$  is not in general orthogonal to  $LU$ , but, in particular, if  $U$  is orthogonal to  $LU$ , then

$$(2.4) \quad g(U, LU) = 0.$$

Hence from (2.3) we get

$$(2.5) \quad \frac{S(LU, LU)}{g(LU, LU)} = \frac{r}{n-1},$$

i.e. the Ricci curvature [8] in the direction  $LU$  is  $\frac{r}{n-1}$ .

If  $U$  is a unit vector field, then

$$(2.6) \quad g(U, U) = 1.$$

Putting  $X = Y = U$  in (1.2) and using (2.6) and (2.4) we get

$$(2.7) \quad g(LU, LU) = a,$$

i.e. the length of the vector field  $LU$  is  $\sqrt{a}$ .

Hence we can state the following theorem:

**THEOREM 2.** *If, in particular,  $U$  is a vector field orthogonal to  $LU$  in a  $(\text{QRRM})_n$ , then the Ricci curvature in the direction of  $LU$  is  $\frac{r}{n-1}$ . Again if  $U$  is a unit vector field orthogonal to  $LU$ , then the length of the vector field  $LU$  is  $\sqrt{a}$ , where  $a$  is the associated scalar of  $(\text{QRRM})_n$*

### 3. Existence theorem of Q(RRM)<sub>n</sub>

In this section we prove the following:

**THEOREM 3.** *If in a conformally flat  $(M^n, g)$ ,  $n \geq 3$ , the curvature transformation and the Ricci transformation commute, then it is a (QRRM)<sub>n</sub>.*

**PROOF.** Let  $R$  and  $C$  be the curvature tensor and conformal curvature tensor of  $(M^n, g)$ ,  $n \geq 3$  of type  $(1, 3)$  respectively then  $C$  [2] is defined by

$$(3.1) \quad \begin{aligned} -\frac{1}{n-2}C(X, Y)Z &= R(X, Y)Z - \\ &- [g(Y, Z)LX - g(X, Z)LY + S(Y, Z)X - S(X, Z)Y] + \\ &+ \frac{r}{(n-1)(n-2)}[g(Y, Z)X - g(X, Z)Y] \end{aligned}$$

where  $L$  and  $r$  are the Ricci tensor of type  $(1, 1)$  and scalar curvature of the manifold respectively.

Let  $(M^n, g)$  be conformally flat, then

$$(3.2) \quad C(X, Y)Z = 0,$$

From (3.1) and (3.2) we get

$$(3.3) \quad \begin{aligned} R(X, Y)Z &= \frac{1}{n-2}[g(Y, Z)LX - g(X, Z)LY + S(Y, Z)X - S(X, Z)Y] - \\ &- \frac{r}{(n-1)(n-2)}[g(Y, Z)X - g(X, Z)Y]. \end{aligned}$$

Again it is known that  $R(X, Y)$  is a skew symmetric endomorphism of the tangent space at each point and  $R(X, Y)$  is also called curvature transformation. We suppose that the curvature transformation  $R(X, Y)$  and the Ricci transformation  $L$  of the manifold commute i.e.,  $R(X, Y) \circ L = L \circ R(X, Y)$  which implies in virtue of (1.1)

$$(3.4) \quad S(R(X, Y)Z, W) + S(Z, R(X, Y)W) = 0.$$

Hence from (3.4) and (3.3) we get

$$(3.5) \quad \begin{aligned} &g(Y, Z)S(LX, W) - g(X, Z)S(LY, W) + \\ &+ g(Y, W)S(LX, Z) - g(X, W)S(LY, Z) - \\ &- \frac{r}{n-1}[g(Y, Z)S(X, W) - g(X, Z)S(Y, W) + \\ &+ g(Y, W)S(X, Z) - g(X, W)S(Y, Z)] = 0. \end{aligned}$$

Putting  $Y = Z = e_i$ , where  $\{e_i\}_{i=1,2,\dots,n}$  is an orthonormal basis of the tangent space at each point and summing for  $1 \leq i \leq n$ , we get

$$(3.6) \quad S(LX, W) = \frac{r}{n-1} S(X, W) + \frac{1}{n} \left[ S(Le_i, e_i) - \frac{r^2}{n-1} \right] g(X, W)$$

which is of the form

$$(3.7) \quad S(LX, W) = \frac{r}{n-1} S(X, W) + a g(X, W)$$

where  $a = \frac{1}{n} \left[ S(Le_i, e_i) - \frac{r^2}{n-1} \right]$  which gives in virtue of (1.1),

$$(3.8) \quad L^2 X = \frac{r}{n-1} LX + a X$$

i. e. the manifold is  $(\text{QRRM})_n$ . This completes the proof.

Again let a conformally flat  $(M^n, g)$  be a  $(\text{QRRM})_n$  given by (3.7). From (3.3) and (3.7) we get

$$S(R(X, Y)Z, W) + S(Z, R(X, Y)W) = 0$$

i. e.  $R(X, Y) \circ L = L \circ R(X, Y)$ .

Hence in virtue of the theorem 3, we can state the following theorem:

**THEOREM 4.** *A conformally flat  $(M^n, g)$ ,  $n \geq 3$ , is a  $(\text{QRRM})_n$  if and only if its curvature transformation and Ricci transformation commute.*

#### 4. Einstein $(\text{QRRM})_n$

It is known that a Riemannian manifold  $(M^n, g)$  is called Einstein manifold if its Ricci tensor  $S$  satisfies the condition

$$(4.1) \quad S(X, Y) = \frac{r}{n} g(X, Y) \quad \text{for all vector fields } X, Y.$$

Putting  $LX$  for  $X$  in (4.1) and using (1.1) we get

$$(4.2) \quad S(LX, Y) = \frac{r}{n} S(X, Y).$$

From (4.1), (4.2) and (1.2) we get

$$(4.3) \quad \left[ \frac{r^2}{n^2(n-1)} + a \right] g(X, Y) = 0.$$

Az  $g(X, Y) \neq 0$ , we have from (4.3)

$$(4.4) \quad a = -\frac{r^2}{n^2(n-1)}$$

This leads to the following theorem.

**THEOREM 5.** *In an Einstein (QRRM)<sub>n</sub>, the associated scalar a is  $-\frac{r^2}{n^2(n-1)}$ .*

## 5. Ricci recurrent (QRRM)<sub>n</sub>

A Riemannian manifold  $(M^n, g)$ ,  $n > 2$  is said to be Ricci recurrent [5] if its Ricci tensor  $S$  of type  $(0, 2)$  is not proportional to the metric tensor  $g$  and satisfies the condition

$$(5.1) \quad (\nabla_X S)(Y, Z) = A(X)S(Y, Z)$$

for all vector field  $X, Y, Z$  where  $A$  is a non-zero 1-form, is defined by  $g(X, \rho) = A(X)$  and  $\nabla$  denotes the operator of covariant differentiation.

Contracting (5.1) we get

$$(5.2) \quad (\text{div } L)(X) = A(LX)$$

and

$$(5.3) \quad X.r = A(X)r.$$

Now, since  $(\text{div } L)(X) = \frac{1}{2}X.r$  we get from (5.3),

$$(5.4) \quad (\text{div } L)(X) = \frac{1}{2}A(X)r.$$

Putting  $LX$  for  $X$  in (5.2) we get

$$\begin{aligned} (5.5) \quad (\text{div } L)(LX) &= A(L^2 X) \\ &= A\left(\frac{r}{n-1}LX + aX\right) \quad \text{by [2]} \\ &= \frac{r}{n-1}A(LX) + aA(X). \end{aligned}$$

Putting  $LX$  for  $X$  in (5.4) we get

$$(5.6) \quad (\text{div } L)(LX) = \frac{1}{2}A(LX)r.$$

From (5.5) and (5.6) we get

$$(5.7) \quad A(LX) = \frac{2a(n-1)}{(n-3)r}A(X), \quad n > 3, r \neq 0.$$

From (1.1), (5.7) and using  $g(X, \rho) = A(X)$ , we get

$$L\rho = \frac{2a(n-1)}{(n-3)r}, \quad \text{for } n > 3$$

which shows that  $\frac{2a(n-1)}{(n-3)r}$ ,  $r \neq 0$ , is an eigenvalue of the Ricci tensor  $L$  of type  $(1, 1)$  corresponding to the eigenvector  $\rho$  which is the vector of recurrence. This leads to the following theorem:

**THEOREM 6.** *In a Ricci recurrent (QRRM)<sub>n</sub>,  $\frac{2a(n-1)}{(n-3)r}$ ,  $r \neq 0$ , is an eigenvalue of the Ricci tensor  $L$  of type  $(1, 1)$  corresponding to the vector of recurrence  $\rho$ .*

## 6. Semi Riemannian (QRRM)<sub>4</sub>

Let a semi Riemannian (QRRM)<sub>4</sub> be a general relativistic space-time  $(M^4, g)$  where  $g$  is a Lorentz metric with signature  $(+, +, +, -)$ .

We know [6] that if the Ricci tensor  $S$  of type  $(0, 2)$  of the space-time satisfies the condition

$$(6.1) \quad S(X, X) > 0,$$

for every timelike vector field  $X$ , then (6.1) is called the time like convergence condition .We consider the general relativistic perfect fluid space-time  $(M^4, g)$  with unit time like velocity vector field  $U$ , then we have

$$(6.2) \quad g(U, U) = -1.$$

From (1.2) we get in (QRRM)<sub>4</sub>,

$$(6.3) \quad S(LX, Y) = \frac{r}{3}S(X, Y) + agX, Y).$$

Let  $\{e_i\}$ ,  $i = 1, 2, 3, 4$  be an orthonormal basis of a frame field at a point of the space-time and contracting over  $X$  and  $Y$ , we obtain

$$(6.4) \quad S(Le_i, e_i) = \frac{r^2}{3} + 4a.$$

The sources of any gravitational field (matter and energy) are represented in relativity by a type of  $(0, 2)$  symmetric tensor  $T$  called the energy momentum tensor [4] and is given by

$$(6.5) \quad T(X, Y) = (\sigma + p)A(X)A(Y) + pg(X, Y),$$

where  $X, Y$  are any two vector fields and  $\sigma$  and  $p$  denote the energy density and the isotropic pressure of the perfect fluid respectively and  $A$  is defined by

$$(6.6) \quad g(X, U) = A(X),$$

for all  $X$  and we suppose that  $T$  obeys time like convergence condition.

The Einstein equation without cosmological constant [4], [1] can be written as

$$(6.7) \quad S(X, Y) - \frac{r}{2}g(X, Y) = KT(X, Y),$$

where  $K$  is the gravitational constant.

From (6.5) and (6.7) we get

$$(6.8) \quad S(X, Y) - \frac{r}{2}g(X, Y) = K[(\sigma + p)A(X)A(Y) + pg(X, Y)].$$

Let  $\{e_i\}$ ,  $i = 1, 2, 3, 4$  be an orthonormal basis of the frame field at a point of the space-time and contracting over  $X$  and  $Y$ , we obtain

$$(6.9) \quad r = K(\sigma - 3p).$$

Putting  $X = Y = U$  in (6.8) and using (6.2) and (6.9) we get

$$(6.10) \quad S(U, U) = \frac{\kappa}{2}(\sigma + 3p)$$

Putting  $LX$  for  $X$  in (6.8) and again taking orthonormal basis  $\{e_i\}$ ,  $i = 1, 2, 3, 4$  of a frame field and contracting over  $X$  and  $Y$  and using (6.2), (6.4), (6.9) and (6.10) we get

$$(6.12) \quad a = \frac{1}{6}K^2\sigma(\sigma + 3p).$$

Since  $K > 0$ ,  $\sigma + 3p > 0$  [by (6.1) and (6.10)],  $\sigma > 0$  if  $a > 0$ .

It is known that if pure matter exists then  $\sigma > 0$ .

Hence this  $(QRRM)_4$  perfect fluid space-time contains pure matter if  $a > 0$ . This leads to the following theorem:

**THEOREM 7.** *If in a  $(\text{QRRM})_4$  perfect fluid space-time in which Einstein equation without cosmological constant holds, the energy momentum tensor obeys the time like convergence condition, then such a space-time contains pure matter if the associated scalar of  $Q(\text{RRM})_4$  is positive.*

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**CONFERENCES**

**Károly Böröczky is 70 years old.**

**February 27, 2009**

**I. Bárány: It is only triangles that do not fix**

**B. Csikós: On spherical polytopes of critical volume with given inradius**

**A. Florian: An extremum problem for convex polyhedra**

**A. Heppes: Helly type transversal problems in the plane**

**E. Makai: Some characterizations of the ball**

**H. Martini: Elementary geometry from a modern point of view**

**Topology and real analysis.**  
**Ákos Császár is 85 years old.**  
**March 26, 2009.**

**Z. Buczolich: 1-Hausdorff dimensional rotation sets of functions not in  $L^1$**

Suppose that  $f : \mathbb{R} \rightarrow \mathbb{R}$  is a given measurable function, periodic by 1. For an  $\alpha \in \mathbb{R}$  put  $M_n^\alpha f(x) = \frac{1}{n+1} \sum_{k=0}^n f(x + k\alpha)$ . Let  $\Gamma_f$  denote the set of those  $\alpha$ 's in  $(0, 1)$  for which  $M_n^\alpha f(x)$  converges for almost every  $x \in \mathbb{R}$ . We call  $\Gamma_f$  the rotation set of  $f$ . We proved earlier that if  $\Gamma_f$  is of positive Lebesgue measure then  $f$  is integrable on  $[0, 1]$ , and hence, by Birkhoff's Ergodic Theorem all  $\alpha \in [0, 1]$  belongs to  $\Gamma_f$ . However,  $\Gamma_f \setminus \mathbb{Q}$  can be dense (even  $c$ -dense) for non- $L^1$  functions as well. In this talk we discussed the construction which shows that there are non- $L^1$  functions for which  $\Gamma_f$  is of Hausdorff dimension one.

There are some interesting recent papers with respect to ergodic averages of non- $L^1$  functions and rotations by Ya. Sinai and C. Ulcigrai. Their results were not used explicitly in our construction, but they supplied inspiration for our result.

**M. Elekes: A characterization of continuous rigid functions**

A function  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  is called *rigid* if the graph of  $cf$  is isometric to the graph of  $f$  for every  $c > 0$ . In this talk we sketch the proof of Janković's conjecture; a continuous univariate function is rigid iff it is of the form  $a + bx$  or  $a + be^{dx}$ . We also describe the analogous characterisation for continuous functions of two variables. The case of at least three variables remains open. In the proofs we apply methods from surprisingly many areas; linear algebra, topology, group theory, and functional equations.

**I. Juhász: Convergence and character spectra of compact spaces**

An infinite set  $A$  in a space  $X$  converges to a point  $p$  (denoted by  $A \rightarrow p$ ) if for every neighbourhood  $U$  of  $p$  we have  $|A \setminus U| < |A|$ . We call  $cS(p, X) = \{|A| : A \subset X \text{ and } A \rightarrow p\}$  the convergence spectrum of  $p$  in  $X$  and  $cS(X) = \cup\{cS(x, X) : x \in X\}$  the convergence spectrum of  $X$ . The character

spectrum of a point  $p \in X$  is  $\chi S(p, X) = \{\chi(p, Y) : p \text{ is non-isolated in } Y \subset X\}$  and  $\chi S(X) = \cup\{\chi S(x, X) : x \in X\}$  is the character spectrum of  $X$ . If  $\kappa \in \chi S(p, X)$  for a compactum  $X$  then  $\{\kappa, cf(\kappa)\} \subset c S(p, X)$ .

A selection of our results ( $X$  is always a compactum):

1. If  $\chi(p, X) > \lambda = \lambda^{<\hat{i}(X)}$  then  $\lambda \in \chi S(p, X)$ ; in particular, if  $X$  is countably tight and  $\chi(p, X) > \lambda = \lambda^\omega$  then  $\lambda \in \chi S(p, X)$ .
2. If  $\chi(X) > 2^\omega$  then  $\omega_1 \in \chi S(X)$  or  $\{2^\omega, (2^\omega)^+\} \subset \chi S(X)$ .
3. If  $\chi(X) > \omega$  then  $\chi S(X) \cap [\omega_1, 2^\omega] \neq \emptyset$ .
4. If  $\chi(X) > 2^\kappa$  then  $\kappa^+ \in c S(X)$ , in fact there is a converging discrete set of size  $\kappa^+$  in  $X$ .
5. If we add  $\lambda$  Cohen reals to a model of GCH then in the extension for every  $\kappa \leq \lambda$  there is  $X$  with  $\chi S(X) = \{\omega, \kappa\}$ . In particular, it is consistent to have  $X$  with  $\chi S(X) = \{\omega, \aleph_\omega\}$ .
6. If all members of  $\chi S(X)$  are limit cardinals then

$$|X| \leq (\sup\{|\overline{S}| : S \in [X]^\omega\})^\omega.$$

7. It is consistent that  $2^\omega$  is as big as you wish and there are arbitrarily large  $X$  with  $\chi S(X) \cap (\omega, 2^\omega) = \emptyset$ .

It remains an open question if, for all  $X$ ,  $\min c S(X) \leq \omega_1$  (or even  $\min \chi S(X) \leq \omega_1$ ) is provable in ZFC.

## M. Laczkovich: The order of elementary functions, 2

### A. Máthé: Nice restrictions of measurable sets

In 1985 P. Humke and M. Laczkovich proved that a typical continuous function  $f : [0, 1] \rightarrow \mathbb{R}$  (in the Baire category sense) intersects every monotone function in a bilaterally strongly  $\phi$ -porous set, for every fixed porosity premeasure  $\phi$ . In particular, a typical continuous function cannot be restricted to any set of positive Hausdorff dimension so that the restriction is monotone.

It was an open question what happens when we look for restrictions of bounded variation. In 2004 M. Elekes proved that a typical continuous function is not of bounded variation on any set of Hausdorff dimension larger than  $1/2$ . He also raised the question what happens in the case of Hölder restrictions. He showed that for every  $0 < \alpha < 1$ , a typical continuous function is not Hölder continuous with exponent  $\alpha$  on any set of dimension larger than  $1 - \alpha$ .

I showed that both of these results of M. Elekes are sharp. That is, for every measurable function  $f : [0, 1] \rightarrow \mathbb{R}$  there exists a compact set  $K \subset [0, 1]$  of Hausdorff dimension  $1/2$  such that  $f$  restricted to  $K$  is of bounded variation. And for every  $0 < \alpha < 1$ , for every measurable function  $f : [0, 1] \rightarrow \mathbb{R}$  there exists a compact set  $K_\alpha \subset [0, 1]$  of Hausdorff dimension  $1 - \alpha$  such that  $f$  restricted to  $K_\alpha$  is Hölder continuous with exponent  $\alpha$ .

## L. Soukup: Cardinal sequences of scattered spaces

A topological space is *scattered* iff every non-empty subspaces has an isolated point. We let, for a scattered space  $X$  and an ordinal  $\alpha$ ,  $I_\alpha(X)$  denote the  $\alpha^{\text{th}}$  Cantor–Bendixson level of  $X$ . The *reduced height* of  $X$ ,  $ht^-(X)$ , is the minimal ordinal  $\beta$  with  $|I_\beta(X)| < \omega$ . The sequence

$$\langle |I_\alpha(X)| : \alpha < ht^-(X) \rangle$$

is called the *cardinal sequence* of  $X$  and is denoted by  $SEQ(X)$ .

Let  $\mathcal{C}(\alpha)$  denote the class of all cardinal sequences of length  $\alpha$  associated with compact scattered spaces. Also put

$$\mathcal{C}_\lambda(\alpha) = \{s \in \mathcal{C}(\alpha) : s(0) = \lambda = \min[s(\beta) : \beta < \alpha]\}.$$

First we show that it suffices to characterize  $\mathcal{C}_\lambda\alpha$  for all  $\lambda$  to get a characterization of  $\mathcal{C}(\alpha)$ .

Then we characterize, under GCH,  $\mathcal{C}_\lambda(\alpha)$  for  $\alpha < \omega_2$ .

For  $\alpha \geq \omega_2$  we do not have such a characterization from GCH, however we can show that for each uncountable regular cardinal  $\lambda$  and ordinal  $\alpha < \lambda^{++}$  it is consistent with GCH that  $\mathcal{C}_\lambda(\alpha)$  is “as large as possible”. In this way we can give a consistent characterization of  $\mathcal{C}_\lambda(\alpha)$  for regular cardinals  $\lambda$ .

The results are from joint works with I. Juhász, J. C. Martinez, and W. Weiss.

## Z. Szentmiklóssy: On decompositions of topological spaces

### István Juhász, Lajos Soukup and Zoltán Szentmiklóssy: Resolvability of topological spaces

It is well known, that there are  $\mathfrak{c} = 2^\omega$ -many disjoint dense subset of the real line.

Easy to see, that if we refine the usual topology of the rational numbers taken the finest 0-dimensional crowded topology on it, the space has no two disjoint dense subset.

Several questions may ask about the resolvability of topological spaces into disjoint dense subset. This questions were first studied by E. Hewitt, in 1943.

Given a cardinal  $\kappa > 1$ , a topological space  $X$  is called  $\kappa$ -resolvable iff it contains  $\kappa$  disjoint dense subsets.  $X$  is  $\kappa$ -irresolvable iff not  $\kappa$ -resolvable.  $X$  is resolvable iff it is 2-resolvable and irresolvable otherwise.

If  $X$  is  $\kappa$ -resolvable and  $G \subset X$  is any non-empty open set in  $X$  then clearly  $\kappa \leq |G|$ . Hence if  $X$  is  $\kappa$ -resolvable then we have  $\kappa \leq \Delta(X)$  where

$$\Delta(X) = \min\{|G| : G \text{ is a nonempty open set}\}.$$

This observation explains the following terminology of J. Ceder: a space  $X$  is called *maximally resolvable* iff it is  $\Delta(X)$ -resolvable.

We give a method ( $\mathcal{D}$ -forced spaces) which help us to construct some topological spaces (especially dense subsets of Cantor-cubes), which will answer some natural questions of resolvability. So we get strongly irresolvable and also resolvable but not maximally resolvable spaces.

We studies how can a topological space be maximally, or large enough resolvable. We got some interesting results if a space has only small discrete or discrete and closed subsets.

In the final section, we studies a special class of spaces, namely the *monotonically normal* spaces. We show that every crowded monotonically normal (in short: MN) space is  $\omega$ -resolvable and almost  $\mu$ -resolvable, where  $\mu = \min\{2^\omega, \omega_2\}$ . On the other hand, if  $\kappa$  is a measurable cardinal then there is a MN space  $X$  with  $\Delta(X) = \kappa$  such that no subspace of  $X$  is  $\omega_1$ -resolvable.

**Tamás Varga Conference on Mathematics Education**  
**November 6–7, 2009**

**Paul Andrews: Where do we look to learn how to teach mathematics more effectively?**

**István Lénárt: Shakespeare and geometry**

**Peter Appelbaum: Against Sense & Representation**

**Norbert Hegyvári: A nonstandard solution to some combinatorial problems**

**Paul Andrews: Some interesting proofs in Euclidean number theory**

**István Molnár: Application of the Erdős–Mordell theorem to the special points of the triangle**

**Workshop on the occasion of the Sixtieth Birthday of András Frank,  
June 5–6, 2009**

**T. Fleiner (Budapest University of Technology and Economics):  
A generalization of stable matchings**

We describe a nonstandard generalization of stable matchings that may turn out to be useful in practical applications.

**E. Győri (Rényi Institute): On 2-factors in graphs**

We survey sufficient (and mostly sharp) degree or degree sum conditions for the existence of 2-factors of exactly  $k$  cycles. Sometimes we have extra conditions on the cycles (say, prescribed edges) sometimes we have extra conditions (say, hamiltonicity).

**B. Jackson (University of London): Bounded direction-length frameworks**

A  $d$ -dimensional *direction-length framework* is a pair  $(G, p)$  where  $G = (V; D, L)$  is a ‘mixed’ graph whose edges are labeled as ‘direction’ or ‘length’ edges, and  $p$  is a map from  $V$  to  $R^d$ . The label of an edge  $uv$  represents a direction or length constraint between  $p(u)$  and  $p(v)$ . The framework  $(G, p)$  is *bounded* if there exists a  $K \geq 0$  such that every equivalent realisation  $(G, q)$  has  $|q(u) - q(v)| \leq K$  for all  $u, v$  in  $V$ . I will describe a characterisation of bounded frameworks and outline its implications for the open problem of characterising globally rigid 2-dimensional generic frameworks. This is joint work with Peter Keevash.

**T. Jordán (ELTE): When highly  $k$ -tree connected graphs are what we need**

Inductive constructions for certain families of graphs may be very useful in inductive proofs. Sometimes a nice construction already exists but such an application is yet to be found. This will be illustrated by the story of highly  $k$ -tree connected graphs.

**T. Király (ELTE/EGRES): Hyperedges: potatoes, stars, or cephalopods?**

Hyperedges may appear in several forms in connectivity augmentation problems: they can be members or bases of a matroid, nodes of a bipartite graph, or bi-sets of heads and tails. I will demonstrate the usefulness of these different viewpoints through some examples.

**Z. Király (ELTE): On partition connectivity**

In 1997 Frank and Szigeti asked the following question. When does a graph have a  $T$ -odd strongly connected orientation? I solved a special form of it: for which graphs we have a  $T$ -odd strongly connected orientation for each  $T$  with appropriate parity. In 1998 together with András we extend this result to  $k$ -arc connected orientations, introduced the notion of  $(k, l)$ -partition-connectedness, and asked several questions related to this notion. Since then most of these questions has been answered by different members of Egres group. But the basic question of Frank and Szigeti is still open.

**G. Maróti (Erasmus University, Rotterdam): Developing decision support tools—and actually using them**

The development of OR based decision support tools is a tremendous task which requires both a solid theoretical foundation and a deep practical insight. Today I am going to speak about another challenge: how to gain the practitioners' acceptance so that they actually end up using the tools. The talk will be illustrated by some lessons learnt from the rolling stock scheduling problem of Netherlands Railways.

**G. Pap (Cornell): Is there an alternating path algorithm for the path matching problem?**

The known combinatorial algorithm for the path matching problem augments a non-maximal path matching using alternating paths and cycles. It is open whether it is enough to use alternating paths—this is the question to be discussed in my talk.

**L. Schrijver (CWI, Amsterdam): Jacobi and the Hungarian Method****A. Sebő (Grenoble): From Seymour Graphs to the Odd Jungle**

Starting from a recent joint work with Ageev and Szigeti, a current leaf of my Frank-Arborescence, I follow a path back to the root: a question on Odd Forests, a first problem András Frank asked me. There are several recent and old stops on the way, and I will not hide the other branches of the arborescence either.

**J. Szabó (SZTAKI/ELTE): Matroid parity and jump systems**

András Recski conjectured that the matroid parity problem for linearly represented matroids remain polynomial time solvable if we change the parity requirement to any prescription without two consecutive gaps. In this lecture we outline a proof to this conjecture. The proof is based on Lovász's result

on the polynomial solvability of the matroid parity problem for linearly represented matroids and on a technique about jump systems, proved by Sebő.

**L. Szegő (Corvinus): A distorted point of view: about a stunning joint result with András**

When recalling a joint result of mine with András my heart still aches with pleasure. At the beginning of the millenium András and I gave the constructive characterization of graphs being the edge-disjoint union of  $k$  spanning trees after adding any new edge. The elementary case of  $k = 2$  must have been extended quite a lot. I would like to show an important part of the proof which evolved much after being published in the proceedings of IPCO 2001: the characterization of a node at which the inverse of the building operation can be applied. The question where other approaches can be found sounds interesting and promising.

**Z. Szigeti (Grenoble): Hypergraph Edge Connectivity Augmentation**

We present some old and new results on edge-connectivity augmentation in hypergraphs.

**É. Tardos (Cornell): Ad Auction Nash Equilibria with Conservative Bidder**

Generalized Second Price Auction and its variants has been the main mechanism used by search companies to auction positions for sponsored search links. In this paper we study the social welfare of the Nash equilibria of this game. It is known that socially optimal Nash equilibria exist, but it's not hard to see that in the general case there are also very bad equilibria: the gap between a Nash equilibrium and the socially optimal can be arbitrarily large. In this paper, we consider the case when the bidders are conservative, in the sense that they do not bid above their own valuations. We prove that for conservative bidders the worse Nash equilibrium and the socially optimal are within a factor of 1.618. Joint work with Renato Paes Leme.

**L. Végh (EGRES): Augmenting undirected connectivity by one**

We give a min-max formula for augmenting the node-connectivity of an undirected graph by one, proving the conjecture of Tibor Jordán.

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