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BUTTERFLY POINTS' CURVE IN ISOTROPIC PLANE

by

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Abstract. There are many varieties of the Butterfly theorem in the Euclidean plane, [2], [5], [7], but also in the hyperbolic, [6], and the isotropic plane, [1]. In the paper [1] the Butterfly theorem is proved by using the analytical method on the affine model of the isotropic plane. In this paper, it is proved by using the synthetic method on the projective model.

It was shown earlier that with any quadrangle inscribed into a circle, an infinite number of butterfly points is associated which are located on a conic, [5], [6]. Here we prove that the analogues of those theorems also hold in the isotropic plane.

1. Introduction

The isotropic plane \mathcal{I}_2 is a real projective plane where the metric is induced by a real line f and a real point F , incidental with it, [4]. The ordered pair (f, F) is called the *absolute figure* of the isotropic plane.

In the affine model of the isotropic plane where the coordinates of the points are given by

$$x = \frac{x_1}{x_0}, \quad y = \frac{x_2}{x_0}$$

the absolute line f is determined by the equation $x_0 = 0$ and the absolute point F by the coordinates $(0, 0, 1)$.

All straight lines through the absolute point F are called *isotropic lines* and all points incidental with f are called *isotropic points*.

Two points $A(a_1, a_2)$ and $B(b_1, b_2)$ are called *parallel* if they are incidental with the same isotropic line. Their *span* is defined by $s(A, B) = b_2 - a_2$. For two non-parallel points, *distance* is defined by $d(A, B) = b_1 - a_1$.

The *midpoint* of the segment AB is point P_{AB} such that $(AB, P_{AB}X_{AB}) = -1$ holds, where X_{AB} is the isotropic point of the line AB . Its coordinates are $\left(\frac{a_1+b_1}{2}, \frac{a_2+b_2}{2}\right)$.

The pole of the absolute line with respect to the conic k is called the *center* of the conic, [4]. Lines through the center of the conic are its *diameters*.

A conic is a *circle* if it touches the absolute line at the absolute point. Therefore it is given by equation of the form $y = \lambda_2x^2 + \lambda_1x + \lambda_0$. It is possible to choose a coordinate system such that the equation becomes $y = \lambda x^2$.

2. Butterfly Theorem

The following theorem is proved in [1] by using the analytical method in the affine model of the isotropic plane. Here it is proved synthetically in the projective model. To make constructions simpler, the circles of the isotropic plane are represented in figures by the circles of the Euclidean plane.

THEOREM 1. *Let the complete quadrangle $ABCD$ be inscribed into the circle c of the isotropic plane, figure 1. Let l be a line conjugate to the diameter s of the circle c with respect to the circle and let P, P' , Q, Q' and R, R' be the intersections of the line l with the pairs of the opposite sides AB, CD , AC, BD and AD, BC of the quadrangle $ABCD$. If $L = s \cap l$ is the midpoint of one of the segments PP' , QQ' , RR' , then it is also the midpoint of the other two.*

PROOF. Let M and M' be intersection points of the line l and the circle c and S be the pole of the line s with respect to the circle. Since l and s are conjugated line equality $(MM', LS) = -1$ holds. From the assumption that L is the midpoint of the segment PP' it follows that $(PP', LS) = -1$. Thus, L and S are the fixed points of the involution determined on the line l by the conics of the pencil $ABCD$, [3]. The point L bisects every pair of intersection points of the line l and a conic of the pencil $ABCD$. Specially, it holds for the pairs Q, Q' and R, R' . ■

The point L , that on some line l bisects the segments formed by intersections of the pairs of opposite sides of the quadrangle $ABCD$ with the line l , is called *butterfly point* of the quadrangle $ABCD$ and the line l is called the *butterfly line*.

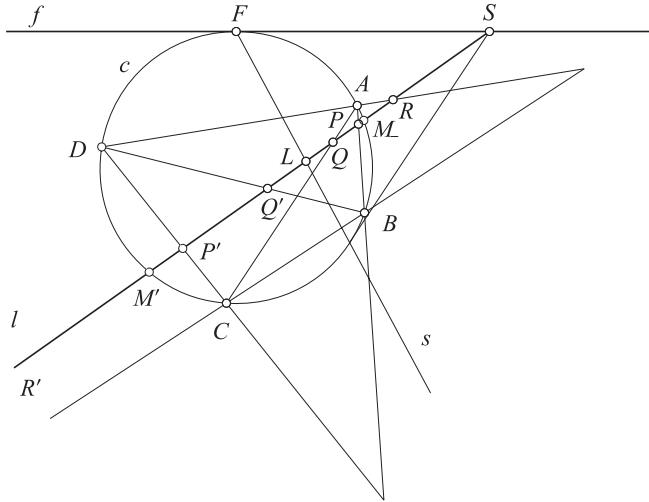


Figure 1.

It is obvious from the proof of Theorem 1 that the condition that the quadrangle $ABCD$ is inscribed into a circle is not necessary. It can be inscribed into any conic c . Therefore, the more general theorem is valid:

THEOREM 2. *Let the complete quadrangle $ABCD$ be inscribed into the conic c of the isotropic plane. Let l be a line conjugate to the diameter s of the conic c with respect to the conic and let P, P' , Q, Q' and R, R' be intersections of the line l with the pairs of the opposite sides AB, CD , AC, BD and AD, BC of the quadrangle $ABCD$. If $L = s \cap l$ bisects one of the segments PP' , QQ' , RR' , then it also bisects the other two.*

3. Butterfly points' curve

It is shown in papers [5] and [6] that in the Euclidean and the hyperbolic plane there is a butterfly point at every diameter of the circle c in which the quadrangle $ABCD$ is inscribed. The following theorem is analogous to those theorems in the isotropic plane.

THEOREM 3. *For a given complete quadrangle $ABCD$ inscribed into a circle c of the isotropic plane there is an infinite number of butterfly points, figure 2. All of these points are located on a special hyperbola.*

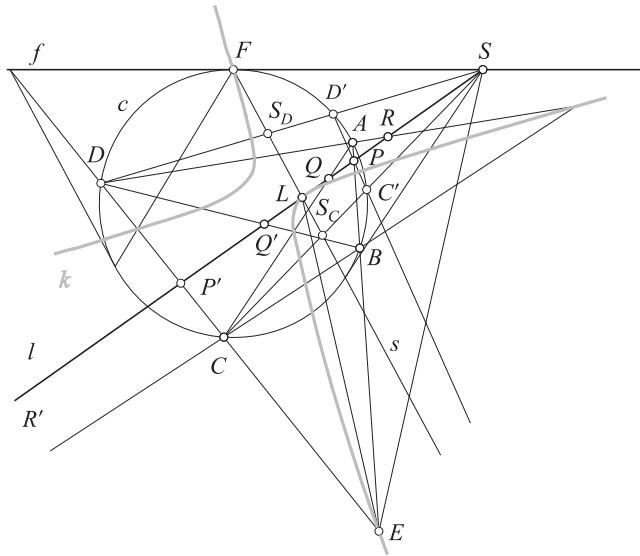


Figure 2.

PROOF. Let s be a diameter of the circle c and S be its pole. Let SC , SD intersect line s in the points S_C , S_D and the circle c in the points C' , D' . It holds $(CC', S_C S) = -1$ and $(DD', S_D S) = -1$. Lines CD , $C'D'$ and $s = S_C S_D$ are copunctal. Let $P = AB \cap C'D'$ and $l = SP$. Let L denote the intersection point of the line l and diameter s and let P' denote its intersection with CD . It is easy to see that $(PP', LS) = -1$. According to Theorem 1, L is a butterfly point of the quadrangle $ABCD$. Thus, it is possible to construct a butterfly point at each diameter of the circle c .

Let $E = AB \cap CD$ be a diagonal point of the quadrangle $ABCD$. After connecting E with P , P' , L and S we obtain a harmonic quadruple of lines. Therefore, for every diameter s the line EL is a harmonic conjugate to ES with respect to the sides AB and CD . The point L is the intersection of the lines s and EL . This reveals that all butterfly points lie on the second order curve k generated by a projectivity between the pencils (F) and (E) of lines. ■

The curve k from the Theorem 3 is called the *butterfly points' curve* or *butterfly curve* of the quadrangle $ABCD$.

REMARK. In the case when the quadrangle $ABCD$ cannot be inscribed into a circle, butterfly points associated with it form a conic, too. But that conic is not passing through the absolute point.

Theorem 3 can be proved analytically. More precisely, it holds:

THEOREM 4. Let the complete quadrangle $ABCD$ be inscribed into the circle c with equation $y = \lambda x^2$,

$$A(a, \lambda a^2), B(b, \lambda b^2), C(c, \lambda c^2), D(d, \lambda d^2).$$

Every point

$$L\left(s, \lambda \frac{2\alpha s^2 - 2\beta s + \gamma}{4s - \alpha}\right), \quad s \in \mathbb{R},$$

of the conic

$$(1) \quad k \dots 2\alpha\lambda x^2 - 4xy - 2\beta\lambda x + \alpha y + \lambda\gamma = 0,$$

where

$$\begin{aligned} \alpha &= a + b + c + d, \\ \beta &= ab + ac + ad + bc + bd + cd, \\ \gamma &= abc + abd + acd + bcd, \end{aligned}$$

is a butterfly point of the quadrangle $ABCD$. It has the property of a butterfly point on the straight line

$$l \dots y = 2\lambda sx - \lambda \frac{8s^3 - 4\alpha s^2 + 2\beta s - \gamma}{4s - \alpha}.$$

PROOF. The sides of the quadrangle $ABCD$ are lines

$$\begin{aligned} AB &\dots y = \lambda(a + b)x - \lambda ab, \\ AC &\dots y = \lambda(a + c)x - \lambda ac, \\ AD &\dots y = \lambda(a + d)x - \lambda ad, \\ BC &\dots y = \lambda(b + c)x - \lambda bc, \\ BD &\dots y = \lambda(b + d)x - \lambda bd, \\ CD &\dots y = \lambda(c + d)x - \lambda cd. \end{aligned}$$

The coordinates of the intersections

$P(x_P, y_P)$, $P'(x_{P'}, y_{P'})$, $Q(x_Q, y_Q)$, $Q'(x_{Q'}, y_{Q'})$ and $R(x_R, y_R)$, $R'(x_{R'}, y_{R'})$ of the line l and the pairs of the opposite sides AB, CD , AC, BD and AD, BC are given with

$$x_P = \frac{ab\alpha - \gamma + 2s(\beta - 2ab) - 4\alpha s^2 + 8s^3}{(\alpha - 4s)(a + b - 2s)},$$

$$y_P = \lambda \frac{-\gamma(a + b) + 2(ab\beta + (a + b)\beta)s - 4(\alpha(a + b) - 2ab)s^2 + 8(a + b)s^3}{(\alpha - 4s)(a + b - 2s)}$$

and analogously for the other points.

After short calculation we obtain

$$x_P + x_{P'} = 2s,$$

$$y_P + y_{P'} = 2\lambda \frac{2\alpha s^2 - 2\beta s + \gamma}{4s - \alpha}.$$

It follows that the point L is the midpoint of the segment PP' . The proof for RR' and QQ' is similar. ■

It is obvious that the conic k with the equation (1) is a special hyperbola. It intersects the absolute line in the absolute point $F(0, 0, 1)$ and the point $I\left(0, 1, \frac{\alpha\lambda}{2}\right)$. Those points are the contact points of the absolute line f and two conics of the pencil $ABCD$: the circle c and the parabola

$$\delta\lambda^2 + \frac{\alpha^2\lambda^2}{4}x^2 + y^2 - \gamma\lambda^2x + \frac{\lambda(2\beta - \varepsilon)}{4}y - \alpha\lambda xy = 0,$$

where

$$\delta = abcd,$$

$$\varepsilon = a^2 + b^2 + c^2 + d^2.$$

The diagonal points of the quadrangle

$$E\left(\frac{ab - cd}{a + b - c - d}, \lambda \frac{ab(c + d) - cd(a + b)}{a + b - c - d}\right),$$

$$G\left(\frac{ac - bd}{a - b + c - d}, \lambda \frac{ac(b + d) - bd(a + c)}{a - b + c - d}\right),$$

$$H\left(\frac{ad - bc}{a - b - c + d}, \lambda \frac{ad(b + c) - bc(a + d)}{a - b - c + d}\right)$$

satisfy the equation (1). It follows that they lie on the conic k .

The points E, F, G, H, I are the centers of five conics of the pencil $ABCD$: three degenerated conics, one circle and one parabola. It can be concluded that the butterfly conic k is in fact a curve of centers of the conics of the pencil $ABCD$.

Coordinates of the midpoints of the sides of the quadrangle $ABCD$

$$\begin{aligned} P_{AB} \left(\frac{a+b}{2}, \lambda \frac{a^2+b^2}{2} \right), & \quad P_{CD} \left(\frac{c+d}{2}, \lambda \frac{c^2+d^2}{2} \right), \\ P_{AC} \left(\frac{a+c}{2}, \lambda \frac{a^2+c^2}{2} \right), & \quad P_{BD} \left(\frac{b+d}{2}, \lambda \frac{b^2+d^2}{2} \right), \\ P_{AD} \left(\frac{a+d}{2}, \lambda \frac{a^2+d^2}{2} \right), & \quad P_{BC} \left(\frac{b+c}{2}, \lambda \frac{b^2+c^2}{2} \right) \end{aligned}$$

also satisfy the equation (1). In other words, they lie on the conic k , too.

Let us point out theorem proved above:

THEOREM 5. *For a given complete quadrangle $ABCD$ inscribed into a circle c of the isotropic plane the butterfly curve is identical to the curve of the centers of the conics of the pencil $ABCD$.*

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PROPER ISOMETRIC ACTIONS ON RIEMANNIAN MANIFOLDS

By

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The concept of a proper action was introduced by R. S. PALAIS in a paper where he managed also to extend some fundamental results of the theory of compact Lie group actions to actions of non-compact Lie groups [5]. A simple definition of proper action is given below for the case of locally compact groups.

DEFINITION. Let $\Phi: G \times M \rightarrow M$ be a continuous action of a locally compact topological group G on a Hausdorff space M and consider the subset

$$(A|B) = \{g \in G \mid \Phi(g, A) \cap B \neq \emptyset\}$$

of G . The action Φ is said to be *proper* if any two points $x, y \in M$ have neighbourhoods $U, V \subset M$ such that the closure

$$\overline{(U|V)} \subset G$$

is compact ([5]; [4], p. 2).

An important property of proper isometric actions was observed by S. T. YAU. Actually, considering an effective isometric action $\Phi: G \times M \rightarrow M$ of a Lie group G on a Riemannian manifold $(M, <, >)$, it has been stated by Yau that the action Φ is proper if G , as a subgroup of the full isometry group $\mathcal{I}(M, <, >)$ of the Riemannian manifold, is closed [6]. The question as to the validity of the converse of the above statement was raised by J. C. DIAZ-RAMOS. A positive answer to this question is presented below. The same positive answer was given in an elegant paper through entirely

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different argument by DIAZ-RAMOS himself, almost simultaneously with the present one [1].

In order to yield a self-contained presentation of the subject the following theorem is proved below:

THEOREM. *Let $(M, <, >)$ be a Riemannian manifold and $\Phi: G \times M \rightarrow M$ an effective isometric action of a Lie group. The action Φ is proper if and only if G , as a subgroup of the full isometry group $\mathcal{I}(M, <, >)$, is closed.*

First some basic properties of proper actions are given in the following lemma which will be applied subsequently.

LEMMA. (1) *Let $\Phi: G \times M \rightarrow M$ be an effective proper action of a locally compact topological group on a Hausdorff space. Then any stabilizer*

$$G_z \subset G, \quad z \in M$$

of the action is compact.

(2) *Let G be a Lie group, M a smooth manifold and $\Phi: G \times M \rightarrow M$ a smooth proper action. Then any orbit*

$$G(z) \subset M, \quad z \in M$$

is a closed smooth submanifold.

The first statement of the lemma is a simple consequence of the definition of proper actions ([4], pp. 3–4). The second statement is a basic property of smooth proper actions ([4], p. 18).

The proof of the following proposition is based on the method of Yau applied in a related situation [6].

PROPOSITION 1. *Let $(M, <, >)$ be an m -dimensional Riemannian manifold and*

$$\Phi: G \times M \rightarrow M$$

isometric action of such a Lie group G which, as a subgroup of the full isometry group $\mathcal{I}(M, <, >)$ of the Riemannian manifold, is closed. Then the action Φ is proper.

PROOF. The distance function $d: M \times M \rightarrow \mathbb{R}$ of the Riemannian manifold will be also applied below. Let $x, y \in M$ be arbitrary points and U, V their neighbourhoods which are open balls of center x, y respectively and of radius $\varepsilon > 0$ such that both the open balls of radius 3ε of x and of y are

normal neighbourhoods. It can be assumed that $(U|V) \neq \emptyset$ holds as well. In order to prove that $\overline{(U|V)}$ is compact it will be shown first that an arbitrary sequence

$$\{g_i \in G \mid i \in \mathbb{N}\} \subset (U|V)$$

has a convergent subsequence. In fact, it can be assumed without loss of generality that the point

$$x' = \lim_{i \rightarrow \infty} \Phi(g_i, x) \in \overline{W}$$

exists, where W is the open ball of center y and radius 3ε ; namely, this can be achieved by substituting the original sequence of isometries by a suitable subsequence, considering that $\Phi_{g_i}(x) \in W$, $i \in \mathbb{N}$ holds. Fix now a geodesic segment

$$\sigma: [0, \lambda] \rightarrow M$$

with $\sigma(0) = x$ such that its image $S = \sigma([0, \lambda])$ is included in the closed ball \overline{U} . Put $z = \sigma(\lambda)$, it can be assumed that the point

$$z' = \lim_{i \rightarrow \infty} \Phi_{g_i} z$$

exists as well; namely, the normal neighbourhood U is mapped to a normal neighbourhood U_i of $\Phi_{g_i}(x)$ with compact closure by the isometry Φ_{g_i} , $i \in \mathbb{N}$; and U_i is included in the normal neighbourhood W of y for $i \in \mathbb{N}$. Thus the original sequence of isometries can be replaced by a suitable one such that z' exists, if necessary. The sequence $\{\Phi_{g_i} \mid i \in \mathbb{N}\}$ is pointwise convergent on S . In fact, if $s \in S$, then

$$s' = \lim_{i \rightarrow \infty} \Phi_{g_i} s$$

exists and it is the metrically corresponding point of the geodesic segment S' with endpoints x', z' (see e. g. [3], vol. I, pp. 163–175). Fix a $0 < \delta < \varepsilon$ such that the normal neighbourhood

$$N_\delta^0 = \{q \in M \mid d(q, S) < \delta\}$$

of S exists and has a compact closure. The above normal neighbourhood of S is the union of a solid tube and of two half balls provided that δ is sufficiently small. There is no loss of generality by assuming that the sequence

$$\{\Phi_{g_i} \mid i \in \mathbb{N}\}$$

is pointwise convergent on N_δ^0 as well with limit points in $N_\delta'^0$, the normal neighbourhood of S' with radius δ ; in fact, this can be achieved again by

taking a suitable subsequence of the isometries, if necessary, considering the construction of the normal neighbourhoods of geodesic segments.

Consider now an arbitrary point $w \in M$ and a piecewise smooth curve from x to w . Then there is a geodesic polygon inscribed in the above curve consisting of geodesic segments $\sigma_1, \dots, \sigma_n$ such that there is a $\delta > 0$ for which the corresponding normal neighborhood

$$N_\delta^k$$

of the geodesic segment σ_k with radius δ , as constructed above, exists for $k = 1, \dots, n$. Then the sequence $\lim_{i \rightarrow \infty} \Phi_{g_i}$, which is pointwise convergent on the normal neighbourhood N_δ^0 is pointwise convergent on the set

$$N_\delta^1 \cup \dots \cup N_\delta^n$$

as well. Therefore the above sequence of isometries is convergent in the point w too. Thus the above sequence of isometries is pointwise convergent on M to an isometry $\Phi_g: M \rightarrow M$, where $g \in \mathcal{I}(M, <, >)$. Since an isometry of the Riemannian manifold is an isometry of the metric space (M, d) as well, the pointwise convergence of the above sequence is a uniform convergence on any compact set of (M, d) . Therefore the above sequence of isometries is convergent in the compact-open topology of $\mathcal{I}(M, <, >)$ (see e. g. [2], pp. 267–269). But the topology of $\mathcal{I}(M, <, >)$, which is given in the MYERS–STEENROD theorem, is the compact-open one (see e. g. [3], vol I, pp. 45–50); thus the sequence $\{g_i \mid i \in \mathbb{N}\}$ is convergent to the above given element g in the topology of $\mathcal{I}(M, <, >)$. But since G is closed in $\mathcal{I}(M, <, >)$,

$$g \in \overline{(U|V)} \subset G \subset \mathcal{I}(M, <, >).$$

is valid.

Consider now a sequence $\{g_i \in \overline{(U|V)} \mid i \in \mathbb{N}\}$. It will be shown that even this sequence has a convergent subsequence. In fact, fix a sequence

$$\{\mathcal{U}_j \subset \mathcal{I}(M, <, >) \mid j \in \mathbb{N}\}$$

of neighbourhoods of the identity e such that $\cap \{\mathcal{U}_j \mid j \in \mathbb{N}\} = \{e\}$ holds, and also a neighbourhood \mathcal{V}_j of e for each j such that

$$\mathcal{V}_j^2 \subset \mathcal{U}_j, \quad j \in \mathbb{N}$$

is valid. Take now for each j a sequence $\{g_{ij} \in (U|V) \mid i \in \mathbb{N}\}$ such that $g_i = \lim_{j \rightarrow \infty} g_{ij}$ and

$$g_{ii}^{-1} \in \mathcal{V}_i g_i^{-1}, \quad i \in \mathbb{N}$$

also holds. There is no loss of generality by assuming that

$$g = \lim_{i \rightarrow \infty} g_{ii}$$

exists; in fact, the sequence $\{g_{ii} \mid i \in \mathbb{N}\}$ can be substituted by a convergent subsequence if necessary. Then $g_{ii} \in g\mathcal{V}_i$ holds if i is sufficiently great. But then

$$g_i \in g_{ii}\mathcal{V}_i \subset g\mathcal{V}_i\mathcal{V}_i \subset g\mathcal{U}_i$$

is valid. Thus $g = \lim_{i \rightarrow \infty} g_i$ holds. This shows that any sequence of elements of $(\overline{U|V})$ has a subsequence which converges to an element of $(\overline{U|V})$. Consequently, the action Φ is proper. ■

PROPOSITION 2. *Let $\Phi: G \times M \rightarrow M$ be a proper effective isometric action of a connected Lie group G on a Riemannian manifold $(M, <, >)$. Then G , as a subgroup of the full isometry group of the Riemannian manifold, is closed.*

PROOF. Consider the canonical action $\Psi: \mathcal{I}(M, <, >) \times M \rightarrow M$ of the full isometry group and

$$\overline{\Phi}: \overline{G} \times M \rightarrow M$$

which is the restriction of Ψ to the action of the subgroup $\overline{G} \subset \mathcal{I}(M, <, >)$. It will be shown first that at any point $z \in M$ the stabilizer G_z of Φ is equal to the stabilizer \overline{G}_z of $\overline{\Phi}$. In fact, consider a $\overline{g} \in \overline{G}_z$ then there is a sequence $\{g_i \in G \mid i \in \mathbb{N}\}$ such that

$$\overline{g} = \lim_{i \rightarrow \infty} g_i$$

is valid. But then by the continuity of the action Ψ the equalities

$$z = \Psi(\overline{g}, z) = \overline{\Phi}(\overline{g}, z) = \overline{\Phi}(\lim_{i \rightarrow \infty} g_i, z) = \lim_{i \rightarrow \infty} \overline{\Phi}(g_i, z) = \lim_{i \rightarrow \infty} \Phi(g_i, z)$$

hold. Consider a ball B of center z in the metric space (M, d) , then there is no loss of generality by assuming that

$$\Phi(g_i, z) \in B, \quad i \in \mathbb{N}$$

is valid. Thus $g_i \in (B|B)$, $i \in \mathbb{N}$ holds. Therefore

$$\overline{g} \in (\overline{B|B}) \subset G$$

holds, since the action Φ is proper. Thus $\overline{G}_z \subset G_z$ follows. Since $G_z \subset \overline{G}_z$ is obviously valid, the equality of the stabilizers follows.

Fix now such a $z \in M$ that $G(z) \neq \{z\}$ holds. The orbit $G(z) \subset M$ is a closed submanifold according to the *Lemma* above, since the action Φ is proper; consequently, $G(z) = \overline{G(z)}$ holds. Therefore

$$\overline{G(z)} = G(z) \subset \overline{G(z)}$$

is valid. Since $\overline{G} \subset \mathcal{I}(M, <, >)$ is a closed subgroup, its action on M is proper by the preceding *Proposition 1*. Therefore the orbit $\overline{G}(z) \subset M$ is a closed submanifold as well by the *Lemma* above.

1st case: When $G(z) = \overline{G}(z)$ is valid. Consider now the sequence of diffeomorphisms

$$G/G_z \leftrightarrow G(z) \leftrightarrow \overline{G}(z) \leftrightarrow \overline{G}/\overline{G}_z,$$

where the first canonical diffeomorphism is G -equivariant, the second one is the identity and the last, the canonical one, is \overline{G} -equivariant. But since $G(z) = \overline{G}(z)$ and $G \subset \overline{G}$ hold, the last diffeomorphism is a G -equivariant one as well. Thus a G -equivariant diffeomorphism

$$G/G_z \leftrightarrow \overline{G}/\overline{G}_z$$

is obtained. If $G_z = \overline{G}_z = \{e\}$, then $G = \overline{G}$ follows. Otherwise consider the canonical principal bundles

$$G \rightarrow G/G_z, \quad \overline{G} \rightarrow \overline{G}/\overline{G}_z.$$

The fibre over $G_z \in G/G_z$ is $G_z \subset G$, and the fibre over $\overline{G}_z \in \overline{G}/\overline{G}_z$ is $\overline{G}_z \subset \overline{G}$. But $\overline{G}_z = G_z$ holds and this equality obviously extends to the fibres $gG_z \subset G$ and $g\overline{G}_z \subset \overline{G}$ over the points $gG_z \in G/G_z$, $g\overline{G}_z \in \overline{G}/\overline{G}_z$ by left translation. Thus $G = \overline{G}$ follows.

2nd case: When $G(z) \subsetneq \overline{G}(z)$ is valid. Consider now the following sequence of maps

$$G/G_z \leftrightarrow G(z) \hookrightarrow \overline{G}(z) \leftrightarrow \overline{G}/\overline{G}_z,$$

where the first map is the canonical G -equivariant diffeomorphism, the second one is the smooth embedding defined by $G(z) \subset \overline{G}(z)$ where both the orbits are closed in M , moreover, the third map is the canonical \overline{G} -equivariant diffeomorphism. Thus a smooth embedding

$$G/G_z \hookrightarrow \overline{G}/\overline{G}_z$$

is obtained. As in the first case the equality $G_z = \overline{G}_z$ will be applied. Considering again the canonical principal bundles

$$G \rightarrow G/G_z, \quad \overline{G} \rightarrow \overline{G}/\overline{G}_z,$$

the above embedding of the base manifold G/G_z into $\overline{G}/\overline{G}_z$ yields an embedding

$$G \hookrightarrow \overline{G}$$

of the bundle space G into \overline{G} , where $G \subset \overline{G}$ is a closed submanifold. Therefore

$$\overline{G} - G$$

is a non-empty open set in \overline{G} . But then \overline{G} cannot be the closure of G in $\mathcal{J}(M, <, >)$. Thus a contradiction is obtained. Therefore,

$$G(z) \subsetneq \overline{G}(z)$$

is a case which cannot happen. ■

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SEVERAL TYPES OF (ψ, ϕ) -OPEN FUNCTIONS ON GENERALIZED NEIGHBOURHOOD SYSTEMS

By

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Abstract. The notion of (ψ, ϕ) -open functions are introduced in [3]. We introduce the notions of weakly (ψ, ϕ) -open function, almost (ψ, ϕ) -open function and upper (ψ, ϕ) -open function, which are generalizations of (ψ, ϕ) -open functions. And we investigate characterizations for such functions and the relationships among (ψ, ϕ) -continuous functions and several types of (ψ, ϕ) -open functions.

1. Introduction

In [1], CSÁSZÁR introduced the notions of generalized neighbourhood systems and generalized topological spaces. He also introduced the notions of continuous functions and associated interior and closure operators on generalized neighbourhood systems and generalized topological spaces. In particular, he investigated characterizations for the generalized continuous functions ((ψ, ϕ) -continuous, (g, g') -continuous) in [1, 2]. In [3], the author introduced the notion of (ψ, ϕ) -open functions on generalized neighbourhood systems and investigated characterizations for such functions.

In this paper, we introduce the notions of weakly (ψ, ϕ) -open, almost (ψ, ϕ) -open and upper (ψ, ϕ) -open functions on generalized neighbourhood systems. We investigate characterizations for such functions and the relationships among several types of (ψ, ϕ) -open functions.

2. Preliminaries

We recall some notions and notations defined in [1]. Let X be a nonempty set. Let $\psi: X \rightarrow \exp(\exp(X))$ satisfy $x \in V$ for $V \in \psi(x)$. Then $V \in \psi(x)$ is called a *generalized neighbourhood* of $x \in X$ and ψ is called a *generalized neighbourhood system* (briefly GNS) on X . Denote the set of all generalized neighbourhood systems on X by $\Psi(X)$. And if ψ is a generalized neighbourhood system on X and $A \subseteq X$, the interior and closure of A with respect to ψ (denoted by $\iota_\psi(A)$, $\gamma_\psi(A)$, respectively) are defined as following:

$$\iota_\psi(A) = \{x \in A: \text{there exists } V \in \psi(x) \text{ such that } V \subseteq A\};$$

$$\gamma_\psi(A) = \{x \in X: V \cap A \neq \emptyset \text{ for all } V \in \psi(x)\}.$$

Then a function $f:(X,\psi) \rightarrow (Y,\psi')$ on GNS's is said to be (ψ,ψ') -continuous [1] if for $x \in X$ and $U \in \psi'(f(x))$, there is $V \in \psi(x)$ such that $f(V) \subset U$.

LEMMA 2.1. ([1]) *Let $\psi \in \Psi(X)$ and $A \subseteq X$. Then*

$$(1) \gamma_\psi(A) = X - \iota_\psi(X - A).$$

$$(2) \iota_\psi(A) = X - \gamma_\psi(X - A).$$

THEOREM 2.2. ([2, 3]) *Let $\psi \in \Psi(X)$, $\phi \in \Phi(Y)$ and $f:(X,\psi) \rightarrow (Y,\phi)$ a function. Then the following are equivalent:*

$$(1) f \text{ is } (\psi, \phi)\text{-continuous.}$$

$$(2) f^{-1}(\iota_\phi(B)) \subseteq \iota_\psi(f^{-1}(B)) \text{ for all } B \subseteq Y.$$

$$(3) \gamma_\psi(f^{-1}(B)) \subseteq f^{-1}(\gamma_\phi(B)) \text{ for all } B \subseteq Y.$$

$$(4) f(\gamma_\psi(A)) \subseteq \gamma_\phi(f(A)) \text{ for all } A \subseteq X.$$

DEFINITION 2.3. ([3]) Let (X,ψ) and (Y,ϕ) be two GNS's. Then $f:X \rightarrow Y$ is said to be (ψ,ϕ) -open if for each $x \in X$ and $U \in \psi(x)$, there exists an element $V \in \phi(f(x))$ such that $V \subseteq f(U)$.

THEOREM 2.4. ([3]) *Let (X,ψ) and (Y,ϕ) be two GNS's and let $f:X \rightarrow Y$ be a function. Then the following are equivalent:*

$$(a) f \text{ is } (\psi, \phi)\text{-open;}$$

$$(b) \iota_\psi(f^{-1}(A)) \subseteq f^{-1}(\iota_\phi(A)) \text{ for } A \subseteq Y;$$

$$(c) f^{-1}(\gamma_\phi(A)) \subseteq \gamma_\psi(f^{-1}(A)) \text{ for } A \subseteq Y;$$

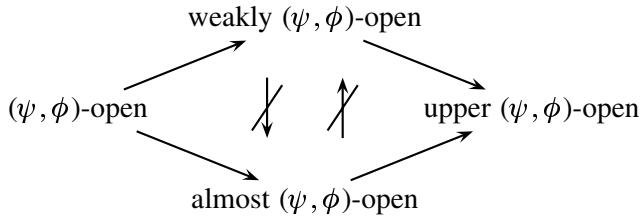
$$(d) f(\iota_\psi(B)) \subseteq \iota_\phi(f(B)) \text{ for } B \subseteq X.$$

3. Main Results

DEFINITION 3.1. Let (X, ψ) and (Y, ϕ) be two GNS's. Then $f: X \rightarrow Y$ is said to be

- (1) *weakly* (ψ, ϕ) -open if for each $x \in X$ and $U \in \psi(x)$, there exists an element $V \in \phi(f(x))$ such that $V \subseteq f(\gamma_\psi(U))$;
- (2) *almost* (ψ, ϕ) -open if for each $x \in X$ and $U \in \psi(x)$, there exists an element $V \in \phi(f(x))$ such that $V \subseteq \gamma_\phi(f(U))$;
- (3) *upper* (ψ, ϕ) -open if for each $x \in X$ and $U \in \psi(x)$, there exists an element $V \in \phi(f(x))$ such that $V \subseteq \gamma_\phi(f(\gamma_\psi(U)))$.

REMARK 3.2. We have the following diagram from the above definitions.



In the above diagram, the converses may not be true in general as shown in the next examples.

EXAMPLE 3.3. Let $X = \{a, b\}$. Consider a generalized neighbourhood system ψ defined as follows

$$\psi(a) = \{\{a\}\}, \psi(b) = \{X\}.$$

Let $f: (X, \psi) \rightarrow (X, \psi)$ be a function defined by $f(a) = b, f(b) = a$. Let $\phi = \psi$. Then f is both weakly (ψ, ϕ) -open and almost (ψ, ϕ) -open function but it is not a (ψ, ϕ) -open function.

EXAMPLE 3.4. (1) Let $X = \{a, b, c\}$. Consider two generalized neighbourhood systems ψ, ϕ defined as follows

$$\psi(a) = \{\{a\}\}, \psi(b) = \{X\}, \psi(c) = \{\{c\}\},$$

$$\phi(a) = \{\{a, b\}\}, \phi(b) = \{\{a, b\}\}, \phi(c) = \{X\}.$$

Let $f: (X, \psi) \rightarrow (X, \phi)$ be a function defined by $f(a) = b, f(b) = a, f(c) = c$. For $c \in X$, we have $\gamma_\psi(\{c\}) = \{b, c\}$ and $f(\gamma_\psi(\{c\})) = \{a, c\}$. But since $\phi(f(c))$ has the only element X , f is not weakly (ψ, ϕ) -open. However, f is upper (ψ, ϕ) -open.

(2) Let $X = \{a, b, c\}$. Consider two generalized neighbourhood systems ψ, ϕ defined as follows

$$\psi(a) = \{\{X\}\}, \psi(b) = \{\{X\}\}, \psi(c) = \{X\},$$

$$\phi(a) = \{\{a\}\}, \phi(b) = \{X\}, \phi(c) = \{\{c\}\}.$$

Note that:

$$\gamma_\phi(\{a\}) = \{a, b\}; \gamma_\phi(\{b\}) = \{b\}; \gamma_\phi(\{c\}) = \{b, c\}.$$

Consider a function $f: (X, \psi) \rightarrow (X, \phi)$ defined as follows $f(a) = b, f(b) = a, f(c) = c$. Then f is weakly (ψ, ϕ) -open but it is not almost (ψ, ϕ) -open.

EXAMPLE 3.5. (1) Let $X = \{a, b\}$. Consider two generalized neighbourhood systems ψ, ϕ defined as follows

$$\psi(a) = \{X\}, \psi(b) = \{X\}.$$

$$\phi(a) = \{\{a\}\}, \phi(b) = \{\{b\}\}.$$

Consider a function $f: (X, \psi) \rightarrow (X, \phi)$ defined by $f(a) = b, f(b) = a$. Then f is upper (ψ, ϕ) -open but it is not almost (ψ, ϕ) -open.

(2) Let $X = \{a, b\}$. Consider two generalized neighbourhood systems ψ, ϕ defined as follows

$$\psi(a) = \{\{a\}\}, \psi(b) = \{\{b\}\}.$$

$$\phi(a) = \{\{a\}\}, \phi(b) = \{X\}.$$

Consider a function $f: (X, \psi) \rightarrow (X, \phi)$ defined by $f(a) = b, f(b) = a$. Then f is upper (ψ, ϕ) -open but it is not weakly (ψ, ϕ) -open.

THEOREM 3.6. *Let (X, ψ) and (Y, ϕ) be two GNS's and let $f: X \rightarrow Y$ be a function. Then f is upper (ψ, ϕ) -open if and only if*

$$f(\iota_\psi(B)) \subseteq \iota_\phi(\gamma_\phi(f(\gamma_\psi(B))))$$

for $B \subseteq X$.

PROOF. Suppose f is upper (ψ, ϕ) -open. Then for each $x \in \iota_\psi(B)$, there exists $U \in \psi(x)$ such that $U \subseteq B$. From upper (ψ, ϕ) -openness, there exists an element $V \in \phi(f(x))$ such that $V \subseteq \gamma_\phi(f(\gamma_\psi(U)))$. This implies $f(x) \in \iota_\phi(\gamma_\phi(f(\gamma_\psi(B))))$ and hence $f(\iota_\psi(B)) \subseteq \iota_\phi(\gamma_\phi(f(\gamma_\psi(B))))$.

For the converse, let $U \in \psi(x)$ for $x \in X$. Since $x \in \iota_\psi(U)$, by hypothesis, we have $f(x) \in \iota_\phi(\gamma_\phi(f(\gamma_\psi(U))))$. Thus this implies that there

exists an element $V \in \phi(f(x))$ such that $V \subseteq \gamma_\phi(f(\gamma_\psi(U)))$. Hence f is upper (ψ, ϕ) -open. \blacksquare

THEOREM 3.7. *Let (X, ψ) and (Y, ϕ) be two GNS's and let $f: X \rightarrow Y$ be a function. Then f is weakly (ψ, ϕ) -open if and only if*

$$f(\iota_\psi(B)) \subseteq \iota_\phi(f(\gamma_\psi(B)))$$

for $B \subseteq X$.

PROOF. It is similar to the proof of Theorem 3.6. \blacksquare

THEOREM 3.8. *Let (X, ψ) and (Y, ϕ) be two GNS's and let $f: X \rightarrow Y$ be a function. Then the following are equivalent:*

- (a) f is almost (ψ, ϕ) -open;
- (b) $\iota_\psi(f^{-1}(A)) \subseteq f^{-1}(\iota_\phi(\gamma_\phi(A)))$ for $A \subseteq Y$;
- (c) $f^{-1}(\gamma_\phi(\iota_\phi(A))) \subseteq \gamma_\psi(f^{-1}(A))$ for $A \subseteq Y$;
- (d) $f(\iota_\psi(B)) \subseteq \iota_\phi(\gamma_\phi(f(B)))$ for $B \subseteq X$.

PROOF. (a) \Rightarrow (b) Let f be almost (ψ, ϕ) -open. For $x \in \iota_\psi(f^{-1}(A))$, there is $U \in \psi(x)$ such that $U \subseteq f^{-1}(A)$. From almost (ψ, ϕ) -openness of f , there is $V \in \phi(f(x))$ such that $V \subseteq \gamma_\phi(f(U)) \subseteq \gamma_\phi(A)$. This implies $f(x) \in \iota_\phi(\gamma_\phi A)$.

(b) \Rightarrow (a) Let $U \in \psi(x)$ for $x \in X$; then $x \in \iota_\psi(U) \subseteq \iota_\psi(f^{-1}(f(U)))$. By (b), $x \in f^{-1}(\iota_\phi(\gamma_\phi(f(U))))$ and so $f(x) \in \iota_\phi(\gamma_\phi(f(U)))$. From definition of the interior operator, there exists an element $V \in \phi(f(x))$ such that $V \subseteq \gamma_\phi(f(U))$. Hence f is almost (ψ, ϕ) -open.

(b) \Leftrightarrow (c) From Theorem 2.1, it is obvious.

(b) \Rightarrow (d) It is easily obtained from (b).

(d) \Rightarrow (a) Let $U \in \psi(x)$ for $x \in X$. Then by (d), we have $f(x) \in \iota_\phi(\gamma_\phi(f(U)))$. Thus there exists an element $V \in \phi(f(x))$ such that $V \subseteq \gamma_\phi(f(U))$. Hence f is almost (ψ, ϕ) -open. \blacksquare

DEFINITION 3.9. Let (X, ψ) and (Y, ϕ) be two GNS's. Then $f: X \rightarrow Y$ is said to be (ψ, ϕ) -closed if $\gamma_\phi(f(B)) \subseteq f(\gamma_\psi(B))$ for $B \subseteq X$.

THEOREM 3.10. Let (X, ψ) and (Y, ϕ) be two GNS's and let $f: X \rightarrow Y$ be a bijective function. Then the following are equivalent:

- (a) f is (ψ, ϕ) -closed;
- (b) $f^{-1}(\gamma_\phi(A)) \subseteq \gamma_\psi(f^{-1}(A))$ for $A \subseteq Y$;
- (c) $\iota_\psi(f^{-1}(A)) \subseteq f^{-1}(\iota_\phi(A))$ for $A \subseteq Y$.

PROOF. (a) \Rightarrow (b) Suppose f is (ψ, ϕ) -closed. Then for $A \subseteq Y$, it satisfies $\gamma_\phi(f(f^{-1}(A))) \subseteq f(\gamma_\psi(f^{-1}(A)))$. Since f is surjective, it is $\gamma_\phi(A) \subseteq f(\gamma_\psi(f^{-1}(A)))$. And from injectivity, it follows

$$f^{-1}(\gamma_\phi(A)) \subseteq f^{-1}(f(\gamma_\psi(f^{-1}(A)))) = \gamma_\psi(f^{-1}(A)).$$

Hence (b) is obtained.

- (b) \Rightarrow (a) For $B \subseteq X$, from (b) and injectivity, it follows

$$f^{-1}(\gamma_\phi(f(B))) \subseteq \gamma_\psi(f^{-1}(f(B))) = \gamma_\psi(B).$$

Finally from surjectivity, we have $\gamma_\phi(f(B)) \subseteq f(\gamma_\psi(B))$.

- (b) \Leftrightarrow (c) It follows from Theorem 2.1. ■

THEOREM 3.11. Let (X, ψ) and (Y, ϕ) be two GNS's and let $f: X \rightarrow Y$ be a function. Then if the closure operator γ_ψ satisfies $\gamma_\psi \gamma_\psi = \gamma_\psi$ and iff f is upper (ψ, ϕ) -open and (ψ, ϕ) -closed, then it is weakly (ψ, ϕ) -open.

PROOF. Suppose f is upper (ψ, ϕ) -open and (ψ, ϕ) -closed. For each $x \in X$ and $U \in \psi(x)$, from upper (ψ, ϕ) -openness of f , there exists an element $V \in \phi(f(x))$ such that $V \subseteq \gamma_\phi(f(\gamma_\psi(U)))$. From definition of (ψ, ϕ) -closed functions, it follows

$$V \subseteq \gamma_\phi(f(\gamma_\psi(U))) \subseteq f(\gamma_\psi(\gamma_\psi(U))) = f(\gamma_\psi(U)).$$

Consequently f is weakly (ψ, ϕ) -open. ■

THEOREM 3.12. Let (X, ψ) and (Y, ϕ) be two GNS's and let $f: X \rightarrow Y$ be a function. Then iff f is almost (ψ, ϕ) -open and (ψ, ϕ) -closed, then it is weakly (ψ, ϕ) -open.

PROOF. Suppose f is almost (ψ, ϕ) -open and (ψ, ϕ) -closed. For each $x \in X$ and $U \in \psi(x)$, from almost (ψ, ϕ) -openness of f , there exists an element $V \in \phi(f(x))$ such that $V \subseteq \gamma_\phi(f(U))$. From definition of (ψ, ϕ) -closed functions, it follows

$$V \subseteq \gamma_\phi(f(U)) \subseteq f(\gamma_\psi(U)).$$

Consequently f is weakly (ψ, ϕ) -open. ■

THEOREM 3.13. *Let (X, ψ) and (Y, ϕ) be two GNS's and let $f: X \rightarrow Y$ be a function. Then if the closure operator γ_ϕ satisfies $\gamma_\phi \gamma_\phi = \gamma_\phi$ and iff is upper (ψ, ϕ) -open and (ψ, ϕ) -continuous, then it is almost (ψ, ϕ) -open.*

PROOF. For each $x \in X$ and $U \in \psi(x)$, from upper (ψ, ϕ) -openness of f , there exists an element $V \in \phi(f(x))$ such that $V \subseteq \gamma_\phi(f(\gamma_\psi(U)))$. Since f is (ψ, ϕ) -continuous, from Theorem 2.2, it follows

$$V \subseteq \gamma_\phi(f(\gamma_\psi(U))) = \gamma_\phi(\gamma_\phi(f(U))) = \gamma_\phi(f(U)).$$

Hence f is almost (ψ, ϕ) -open. ■

THEOREM 3.14. *Let (X, ψ) and (Y, ϕ) be two GNS's and let $f: X \rightarrow Y$ be a function. Then iff is weakly (ψ, ϕ) -open and (ψ, ϕ) -continuous, then it is almost (ψ, ϕ) -open.*

PROOF. For each $x \in X$ and $U \in \psi(x)$, from weakly (ψ, ϕ) -openness of f , there exists an element $V \in \phi(f(x))$ such that $V \subseteq f(\gamma_\psi(U))$. Since f is (ψ, ϕ) -continuous, from Theorem 2.2, it follows

$$V \subseteq f(\gamma_\psi(U)) \subseteq \gamma_\phi(f(U)).$$

Hence f is almost (ψ, ϕ) -open. ■

THEOREM 3.15. *Let $f: X \rightarrow Y$ be a function on GNS's (X, ψ) and (Y, ϕ) . Then f is (ψ, ϕ) -continuous and (ψ, ϕ) -closed if and only if $\gamma_\psi(f(B)) = f(\gamma_\phi(B))$ for $B \subseteq X$.*

PROOF. It follows from Theorem 2.2 and Definition 3.9. ■

THEOREM 3.16. *Let $f: X \rightarrow Y$ be a function on GNS's (X, ψ) and (Y, ϕ) . Then the following are equivalent:*

- (a) f is (ψ, ϕ) -continuous and (ψ, ϕ) -open;
- (b) $f^{-1}(\gamma_\phi(A)) = \gamma_\psi(f^{-1}(A))$ for $A \subseteq Y$;
- (c) $\iota_\psi(f^{-1}(A)) = f^{-1}(\iota_\phi(A))$ for $A \subseteq Y$.

PROOF. It follows from Theorem 2.2 and Definition 2.3. ■

From Theorem 3.15 and Theorem 3.16, we have the following.

COROLLARY 3.17. *Let $f: X \rightarrow Y$ be a function on GNS's (X, ψ) and (Y, ϕ) . If f is bijective, then the following are equivalent:*

- (a) f is (ψ, ϕ) -continuous and (ψ, ϕ) -closed;
- (b) $\gamma_\psi(f(B)) = f(\gamma_\phi(B))$ for $B \subseteq X$;

- (c) $f^{-1}(\gamma_\phi(A)) = \gamma_\psi(f^{-1}(A))$ for $A \subseteq Y$;
- (d) $\iota_\psi(f^{-1}(A)) = f^{-1}(\iota_\phi(A))$ for $A \subseteq Y$.

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MANY DIFFERENCES, FEW SUMS

By

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*(Received January 15, 2009)***1. Introduction**

Let A be a set of integers, $|A| = n$. The minimal possible cardinalities of the sumset $A + A$ and the difference set $A - A$ are both $2n - 1$, and in both cases equality holds if and only if A is an arithmetic progression. If instead of integers we consider sets in an arbitrary commutative group, then the value of the possible minima decreases to n , with equality if A is a coset of a subgroup.

The maximal possible values differ, for the sumset it is $n(n + 1)/2$ and for the difference set $n^2 - n + 1$, but it is still true that these maximal values are attained at the same class of sets, which are generally called Sidon sets.

If we move away from the extremities, there is no more any strong connection between the number of sums and differences. Still, the equality of minima displays a remarkable stability: if $|A + A| = \alpha n$, then (see [1, 2, 4, 5])

$$(1.1) \quad \sqrt{\alpha}n \leq |A - A| \leq \alpha^2 n.$$

The maximum is a lot less stable: the assumption $|A - A| > 0.99n^2$ does not imply $|A + A| > 0.01n^2$, it can even happen that

$$(1.2) \quad |A - A| > n^2 - n^c, \quad |A + A| < n^c$$

with some $c < 2$, and a similar discrepancy is possible in the other direction, see [3].

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However, some weaker connection does exist. Already inequality (1.1) shows that the exponent c in (1.2) cannot be less than $3/2$. Our aim is to improve this easy estimate.

Our results are comfortably expressed in terms of the “deficits”, the distances from the extremal case.

DEFINITION 1.1. The *difference deficit* and *sum deficit* of a nonempty set A with $|A| = n$ are the quantities

$$\Delta_-(A) = n^2 - n + 1 - |A - A|,$$

$$\Delta_+(A) = \frac{n(n+1)}{2} - |A + A|.$$

For the empty set we put $\Delta_-(\emptyset) = \Delta_+(\emptyset) = 0$.

Our main result sounds as follows.

THEOREM 1.2. *Let A be a finite set in a commutative group, $|A| = n$. We have*

$$(1.3) \quad |A + A| \left(\Delta_-(A)^2 + \frac{n^3}{6} \right) \geq \frac{n^5}{20}.$$

In particular, if $|A + A| < n^2/10$, then

$$|A + A| \Delta_-(A)^2 \geq \frac{n^5}{30}.$$

This implies that the value of c in (1.2) is at least $5/3$. I have no conjecture about its proper value.

This theorem shows that for $\Delta_- \ll n^{3/2}$ the number of sums is comparable to n^2 . It does not, however, yield further improvements for very small values of Δ_- .

The following example shows that this assumption on Δ_- is insufficient to guarantee the stronger conclusion $|A + A| \sim n^2/2$. Write $n = m^2 + k$, and consider a set of the form $A = (B_1 + B_2) \cup C$, where $|B_i| = m$, $|C| = k$, and there is no other coincidence among the sums $a + a'$, $a, a' \in A$ than those that follow from commutativity. A simple calculation gives that for this example we have

$$\Delta_-(A) = 2m(m-1)^2, \quad \Delta_+(A) = \frac{m^2(m-1)^2}{4},$$

so $\Delta_+ \asymp \Delta_-^{4/3}$; when $m \sim \sqrt{n}$, we have $\Delta_- \ll n^{3/2}$ and $\Delta_+ \gg n^2$.

I think this is the correct maximal order of Δ_+ for small values of Δ_- .

CONJECTURE 1.3. *For all sets A we have*

$$\Delta_+(A) \ll \Delta_-(A)^{4/3},$$

in particular, $|A + A| \sim n^2/2$ if $\Delta_- = o(n^{3/2})$.

In this direction we show the following weaker result.

THEOREM 1.4 *Let A be a finite set in a commutative group. We have*

$$(1.4) \quad \Delta_+(A) \leq \frac{1}{2} \Delta_-(A)^{3/2} + \Delta_-(A).$$

2. Sums, differences, and the maximal number of representations

Let A be a finite set in a commutative group, $|A| = n$. We write $r(x)$ and $d(x)$ to denote the number of representations of any element x as a sum or difference from A , that is,

$$r(x) = |\{(a, a') : a, a' \in A, a + a' = x\}|,$$

$$d(x) = |\{(a, a') : a, a' \in A, a - a' = x\}|.$$

Clearly $d(0) = n$. For understanding the number of sums and differences the quantity

$$(2.1) \quad m = \max_{x \neq 0} d(x)$$

will be important.

LEMMA 2.1. *Let A be a finite set in a commutative group, $|A| = n$, $|A + A| = s$, $\Delta_-(A) = t$, and define m by (2.1) above. We have*

$$(2.2) \quad s(mt + 2n^2 - n) \geq n^4.$$

PROOF. Write $D = (A - A) \setminus \{0\}$. We have $\sum d(x) = n^2$, hence

$$(2.3) \quad \sum_{x \in D} d(x) = n^2 - n,$$

$$(2.4) \quad \sum_{x \in D} (d(x) - 1) = n^2 - n - |D| = n^2 - n - |A - A| + 1 = \Delta_-(A) = t.$$

(The first and last terms are also equal for the empty set.)

Since each summand is at most $m - 1$, we obtain

$$(2.5) \quad \sum_{x \in D} (d(x) - 1)^2 \leq (m - 1)t.$$

By adding the previous three equations we get

$$\sum_{x \in D} d(x)^2 \leq mt + n^2 - n.$$

Finally adding the contribution of $x = 0$ we conclude

$$(2.6) \quad \sum d(x)^2 \leq mt + 2n^2 - n.$$

The two representation functions are connected by the familiar equality

$$\sum d(x)^2 = \sum r(x)^2$$

(both count the number of solutions of $a_1 + a_2 = a_3 + a_4$), and we also have $\sum r(x) = n^2$. The inequality of arithmetic and quadratic mean gives

$$\sum d(x)^2 = \sum r(x)^2 \geq \frac{(\sum r(x))^2}{|A + A|} = \frac{n^4}{|A + A|}.$$

By substituting (2.6) we obtain (2.2). ■

LEMMA 2.2. *If $A = A_1 \cup A_2$, $A_1 \cap A_2 = \emptyset$, then*

$$\Delta_-(A) \geq \Delta_-(A_1) \cup \Delta_-(A_2).$$

Consequently if $B \subset A$, then $\Delta_-(B) \leq \Delta_-(A)$.

PROOF. This easily follows by writing identity (2.4) for A_1, A_2 and adding, or can be visualized as follows. Take a set A , and group the pairs (a, a') according the value of $a - a'$. Within each group color one pair black and the others red; then Δ_- is the number of red pairs. When we form the union, some black pairs may turn red and new pairs appear. ■

LEMMA 2.3. *Let A be a finite set in a commutative group, and define m by (2.1) above. Assume $m \geq 2$. There is a subset $B \subset A$ such that $m \leq |B| \leq 2m$ and*

$$(2.7) \quad \Delta_-(B) \geq m |B| / 4.$$

PROOF. Take an $x \neq 0$ with $d(x) = m$. The set B will contain all the elements $a, a + x$ whenever both belong to A . Clearly $m \leq |B| \leq 2m$.

Split B into a union of maximal arithmetical progressions with difference x :

$$B = P_1 \cup \dots \cup P_k, |P_1| \geq \dots \geq |P_k| \geq 2.$$

If x is of finite order, then some of the P_i 's may be cosets of the group generated by x .

Put $B_j = P_1 \cup \dots \cup P_j$. We prove by induction that

$$\Delta_-(B_j) \geq \frac{1}{4} |B_j|^2 + j - 1 - \delta,$$

where $\delta = 1$ if $|P_1| \leq 3$ and it is not a coset, and $\delta = 0$ otherwise.

Write $|P_i| = p_i$, $p_j = p$.

Consider first the case $j = 1$. If P_1 is a coset, we have

$$|P_1 - P_1| = p, \Delta_-(P_1) = (p - 1)^2 \geq \frac{p^2}{2},$$

with equality for $p = 2$ and strict inequality otherwise.

If P_1 is a proper progression, then

$$|P_1 - P_1| \leq 2p - 1$$

(inequality is possible if x is of finite order $< 2p - 1$), hence

$$\Delta_-(P_1) \geq (p - 1)(p - 2) \begin{cases} > p^2/4 & \text{if } p \geq 4, \\ \geq p^2/4 - 1 & \text{if } p = 2 \text{ or } 3. \end{cases}$$

For the inductive step, observe that

$$B_j - B_j = (B_{j-1} - B_{j-1}) \cup (B_{j-1} - P_j) \cup (P_j - B_{j-1}),$$

since $P_j - P_j \subset P_1 - P_1 \subset B_{j-1} - B_{j-1}$, consequently

$$|B_j - B_j| \leq |B_{j-1} - B_{j-1}| + 2 |B_{j-1} - P_j|.$$

To estimate the second summand observe that for two progressions with a common difference we have

$$|P_i - P_j| \leq p_i + p_j - 1 \leq \frac{3}{4} p_i p_j,$$

and adding this for $1 \leq i \leq j - 1$ we get

$$|B_{j-1} - P_j| \leq \frac{3}{4} |B_{j-1}| p.$$

Hence

$$|B_j - B_j| \leq |B_{j-1} - B_{j-1}| + \frac{3}{2}p |B_{j-1}|,$$

which can be rewritten as

$$\Delta_-(B_j) \geq \Delta_-(B_{j-1}) + \frac{1}{2}p |B_{j-1}| + p^2 - p.$$

By substituting the inductive assumption we obtain

$$\Delta_-(B_j) \geq \frac{1}{4} |B_{j-1}|^2 + j - 2 - \delta + \frac{3}{4}p^2 - p.$$

To finish the induction observe that $\frac{3}{4}p^2 - p \geq 1$, with equality for $p = 2$ and strict inequality otherwise.

In particular we have

$$\Delta_-(B) \geq \frac{1}{4} |B|^2 + k - 1 - \delta \geq \frac{1}{4} |B|^2 \geq \frac{1}{4}m |B|,$$

unless $k = 1$ and $\delta = 1$. This exception holds if B is a single arithmetic progression of length 2 or 3 and not a coset. Since a progression of length 2 has $m = 1$, then only remaining case is a three-term progression, where $m = 2$ and $\Delta_- = 2$ so the claim of the lemma still holds. ■

REMARK. Equality holds if (and only if) B is a coset of a two-element subgroup.

With some care the constant $1/4$ can be improved to $1/2 - \varepsilon$ for large m , and this could be used to improve the constants in the theorems for large values.

COROLLARY 2.4.

$$(2.8) \quad m \leq 2\sqrt{\Delta_-(A)}.$$

PROOF. Indeed, with the B of the previous lemma we have

$$\Delta_-(A) \geq \Delta_-(B) \geq m |B| / 4 \geq m^2 / 4. \blacksquare$$

This is sufficient to prove our estimate for very small Δ_- .

PROOF OF THEOREM 1.4. We retain the notations of Lemma 2.1.

By Lemma 2.1 we know that

$$s \geq \frac{n^4}{mt + 2n^2 - n} \geq \frac{n^2}{2} + \frac{n}{4} - \frac{mt}{4},$$

hence

$$\Delta_+(A) \leq \frac{n}{4} + \frac{mt}{4}.$$

Substituting inequality (2.8) we get

$$\Delta_+(A) \leq \frac{n}{4} + \frac{1}{2}t^{3/2}.$$

This settles the case $t \geq 4n$. If $t \leq 4n$, consider all pairs (a, a') with $a, a' \in A$, $d(a - a') \geq 2$. The number of such pairs is at most $2t$, hence at most $4t$ elements of A occur in these pairs. Let $A' \subset A$ be the set of those elements. Clearly $\Delta_-(A') = \Delta_-(A)$, $\Delta_+(A') = \Delta_+(A)$ and the already proved case yields

$$\Delta_+(A) = \Delta_+(A') \leq \frac{|A'|}{4} + \frac{1}{2}t^{3/2} \leq t + \frac{1}{2}t^{3/2}. \blacksquare$$

3. Proof of Theorem 1.2

Write $|A + A| = s$, $\Delta_-(A) = t$. Our aim is to prove

$$(3.1) \quad st^2 \geq \alpha n^5 - \beta s n^3,$$

with $\alpha = 1/20$ and $\beta = 1/6$.

The proof is an induction on the value of t . If $t = 0$, then A is a Sidon set, $s = n(n + 1)/2$ and (3.1) becomes

$$\alpha n^5 \leq \beta n^4(n + 1)/2,$$

which holds if $\alpha \leq \beta/2$.

Assume now that the statement is true for any smaller value of Δ_- . As in the previous section, let m denote the maximal multiplicity of a nonzero difference. An application of Lemma 2.3 gives us a set $B \subset A$ satisfying $m \leq |B| \leq 2m$, $\Delta_-(B) \geq m|B|/4$.

Write $|B| = z$. The set $A' = A \setminus B$ satisfies

$$|A'| = n' = n - z, \quad \Delta_-(A') = t' \leq \Delta_-(A) - \Delta_-(B) \leq t - \frac{mz}{4}.$$

Also obviously $|A' + A'| \leq s$. The induction hypothesis now yields

$$(3.2) \quad s(t - mz/4)^2 \geq \alpha(n - z)^5 - \beta s(n - z)^3.$$

To obtain (3.1) it is sufficient to prove

$$(3.3) \quad s(t^2 - (t - mz/4)^2) \geq \alpha(n^5 - (n - z)^5) - \beta s(n^3 - (n - z)^3).$$

The right hand side of (3.3) can be written as $f(n) - f(n - z)$, where $f(x) = \alpha x^5 - \beta s x^3$, so it is equal to $z f'(y)$ for some $y \in [n - z, n]$.

The function

$$f'(y) = 5\alpha y^4 - 3\beta s y^2$$

is negative for $y^2 < 3\beta s/(5\alpha)$. Since the left hand side of (3.3) is positive, (3.3) holds in this case and we now consider the case $y^2 \geq 3\beta s/(5\alpha)$.

The second derivative $f''(y) = 20\alpha y^3 - 6\beta s y$ is positive for $y^2 > 3\beta s/(10\alpha)$, so f' is increasing in our range and we conclude

$$f'(y) \leq f'(n) = 5\alpha n^5 - 3\beta s n^2.$$

We estimate the left hand side of (3.3) as

$$s(t^2 - (t - mz/4)^2) \geq smtz/4.$$

Hence to prove (3.3) it is sufficient to guarantee

$$(3.4) \quad smt \geq 20\alpha n^5 - 12\beta s n^2$$

Lemma 2.1 tells us that

$$s(mt + 2n^2 - n) \geq n^4,$$

hence

$$smt \geq n^4 - 2sn^2$$

which is exactly (3.4) under the choice $\alpha = 1/20$, $\beta = 1/6$.

4. Meditation on the reverse question

Our results tell that if the number of differences is near to n^2 , then the number of sums cannot be very small. It is very likely that similar results hold for the reverse problem, when Δ_+ is assumed to be small and estimates are sought for $|A - A|$. I expect that the following analogue of Theorem 1.2 will be valid.

CONJECTURE 4.1. *If $|A - A| < cn^2$, then*

$$|A - A| \Delta_+(A)^2 \geq cn^5$$

with a suitable positive constant c.

We will show the following weaker result.

STATEMENT 4.2 *If $|A - A| < n^2/2$, then*

$$|A - A| \Delta_+(A)^2 \geq n^4/9$$

for n sufficiently large.

For a lower estimate of $\min(|A - A|, \Delta_+(A))$ this yields only $n^{4/3}$, weaker than the $n^{3/2}$ which is an immediate consequence of (1.1).

For very small values of Δ_+ we have the following analogue of Theorem 1.4.

THEOREM 4.3. *Let A be a finite set in a commutative group. We have*

$$(4.1) \quad \Delta_-(A) \leq 2(\Delta_+(A)^2 + \Delta_+(A)).$$

For given positive integers n, k such that $2k+2 \leq n$ there is a set A satisfying $|A| = n$, $\Delta_+(A) = k$ and $\Delta_-(A) = 2(k^2 + k)$.

The proof will be analogous to the arguments in Section 2, the main difference being that there is no analogue of Lemma 2.3.

We retain the notations $r(x)$, $d(x)$ and D of Section 2, but now we define

$$m = \max r(x)$$

and $t = \Delta_+(A)$, to emphasize the analogous role of these quantities.

The analogue of Lemma 2.1 sounds as follows.

LEMMA 4.4.

$$(4.2) \quad |A - A| \geq \frac{(n^2 - n)^2}{n^2 - n + 2mt} + 1.$$

PROOF. Let

$$S = \{x : r(x) \geq 2\}.$$

Since $r(x) = 1$ is possible only if $x = 2a$ with $a \in A$, we can write

$$|S| = |A + A| - n', \quad n' \leq n.$$

We have

$$\sum r(x) = n^2,$$

hence

$$\sum_{x \in S} r(x) = n^2 - n',$$

$$(4.3) \quad \sum_{x \in S} (r(x) - 2) = n^2 - n' - 2|S|$$

$$(4.4) \quad = n^2 + n' - 2|A + A| = 2t + n' - n.$$

Since each term is $\leq m - 2$, we get

$$\sum_{x \in S} (r(x) - 2)^2 \leq (m - 2)(2t + n' - n).$$

To this we add

$$\sum_{x \in S} (4r(x) - 4) = 2n^2 - 2n + 4t$$

to conclude

$$\sum_{x \in S} r(x)^2 \leq 2mt + (m - 2)n' + 2n^2 - mn.$$

Finally by adding the contribution of the n' values with $r(x) = 1$ we obtain

$$\sum_{x \in S} r(x)^2 \leq 2mt + (m - 1)n' + 2n^2 - mn \leq 2mt + 2n^2 - n.$$

This is equal to $\sum d(x)^2$, from which we subtract the contribution of $d(0) = n$ to get

$$\sum_{x \in D} d(x)^2 \leq 2mt + n^2 - n.$$

Since

$$\sum_{x \in D} d(x) = n^2 - n,$$

the inequality of arithmetic and square mean yields

$$|D| \geq \frac{(n^2 - n)^2}{n^2 - n + 2mt},$$

to which we add 1 to obtain (4.2). ■

LEMMA 4.5.

$$\Delta_-(A) \leq m\Delta_+(A).$$

PROOF. The weaker estimate

$$\Delta_-(A) \leq 2m\Delta_+(A)$$

immediately follows from (4.2). To improve this by a factor of 2 we argue as follows. For every integer k we have $k^2 - 3k + 2 = (k - 1)(k - 2) \geq 0$. We apply this for $d(x)$, the motivation being that the typical values of $d(x)$ are 1 and 2:

$$\begin{aligned} 0 &\leq \sum_{x \in D} (d(x)^2 - 3d(x) + 2) \\ &\leq (2m\Delta_+ + n^2 - n) - 3(n^2 - n) + 2|D| \\ &= 2(m\Delta_+ - \Delta_-). \blacksquare \end{aligned}$$

PROOF OF THEOREM 4.3. We can estimate m by $2\Delta_+ + 2$, since on the left hand side of (4.3) there is at least one term equal to $m - 2$. Now to get the upper estimate we apply Lemma 4.5.

The example of equality is as follows. Let $n = 2k + 2 + l$, and consider a set of the form $A = B \cup (-B) \cup C$, where $|B| = k + 1$, $|C| = l$ and there is no nontrivial coincidence among sums and differences. An easy calculation gives

$$\Delta_+(A) = k, \quad \Delta_-(A) = 2k^2 + 2k. \blacksquare$$

PROOF OF STATEMENT 4.2. This follows immediately from Lemmas 4.4 and 4.5. ■

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**ON α -OPEN SETS, \mathcal{A}^* -SETS AND DECOMPOSITIONS OF
CONTINUITY AND SUPER-CONTINUITY**

By

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Abstract. In this paper, we introduce the notion of α -open sets and obtain properties of this class of sets. Also, we introduce the class of \mathcal{A}^* -sets via α -open sets. By using these sets, new decompositions of continuous functions are provided. Finally, we study some new weaker forms of δ -open sets called $\delta\mathcal{E}$ -sets, $\delta\mathcal{E}^*$ -sets, $\delta\mathcal{P}$ -sets, $\delta\mathcal{S}$ -sets, $\delta\mathcal{A}$ -sets, $\delta\delta$ -sets, $\delta\mathcal{R}$ -sets and investigate new decompositions of super-continuity by using these weaker forms.

1. Introduction

In recent two papers [4, 6], Ekici has introduced two new classes of sets called e^* -open sets and e -open sets. By using these sets Ekici has established some new decompositions of continuous functions. The main purpose of this paper is to obtain new decompositions of continuous functions and super-continuous functions. We have introduced and studied the notions of α -open sets and \mathcal{A}^* -sets. The relationships among α -open sets, \mathcal{A}^* -sets and the related sets are investigated. By using these notions we obtain new decompositions of continuous functions. Moreover, we introduce the concepts of $\delta\mathcal{E}$ -sets, $\delta\mathcal{E}^*$ -sets, $\delta\mathcal{P}$ -sets, $\delta\mathcal{S}$ -sets, $\delta\mathcal{A}$ -sets, $\delta\delta$ -sets, $\delta\mathcal{R}$ -sets and obtain decompositions of super-continuity by using $\delta\mathcal{E}$ -continuity, $\delta\mathcal{E}^*$ -continuity, $\delta\mathcal{P}$ -continuity, $\delta\mathcal{S}$ -continuity, $\delta\mathcal{A}$ -continuity, $\delta\delta$ -continuity and $\delta\mathcal{R}$ -continuity.

In this paper (X, τ) and (Y, σ) represent topological spaces. For a subset A of a space X , $cl(A)$ and $int(A)$ denote the closure of A and the interior of A , respectively. A subset A of a space (X, τ) is called α -open [10] (resp. preopen [7]) if $A \subset int(cl(int(A)))$ (resp. $A \subset int(cl(A))$). A subset A of a space (X, τ) is called regular open (resp. regular closed) [13] if $A = int(cl(A))$.

(resp. $A = cl(int(A))$). A subset A is said to be δ -open [15] if for each $x \in A$ there exists a regular open set V such that $x \in V \subset A$ and is said to be δ -closed if its complement is δ -open. A point $x \in X$ is called a δ -cluster points of A [15] if $A \cap int(cl(U)) \neq \emptyset$ for each open set U containing x . The set of all δ -cluster points of A is called the δ -closure of A and is denoted by $\delta-cl(A)$. The δ -interior of A is the union of all regular open sets contained in A and is denoted by $\delta-int(A)$. A subset A of a space (X, τ) is called δ -preopen [12] (resp. δ -semiopen [11]) if $A \subset int(\delta-cl(A))$ (resp. $A \subset cl(\delta-int(A))$). The complement of a δ -semiopen set (resp. a δ -preopen set) is called δ -semiclosed (resp. δ -preclosed).

A space X is called submaximal [2] if every dense subset of X is open.

DEFINITION 1. A subset K of a space (X, τ) is called

- (1) e -open [4] if $K \subset cl(\delta-int(K)) \cup int(\delta-cl(K))$ and e -closed [4] if $cl(\delta-int(K)) \cap int(\delta-cl(K)) \subset K$,
- (2) e^* -open [6] if $K \subset cl(int(\delta-cl(K)))$ and e^* -closed [6] if $int(cl(\delta-int(K))) \subset K$,
- (3) a \mathcal{D} -set [5] if $K = A \cap B$, where A is open and B is δ -closed,
- (4) a \mathcal{DS} -set [5] if $K = A \cap B$, where A is open and B is δ -semiclosed,
- (5) an LC -set [1] if $K \in LC(X) = \{A \cap B : A \in \tau, cl(B) = B\}$,
- (6) a \mathcal{B} -set [14] if $K \in \mathcal{B}(X) = \{A \cap B : A \in \tau, int(cl(B)) \subset B\}$.

2. a -open sets

DEFINITION 2. A subset A of a space X is called

- (1) a -open if $A \subset int(cl(\delta-int(A)))$,
- (2) a -closed if $cl(int(\delta-cl(A))) \subset A$.

The family of all a -open (respectively a -closed) sets of X is denoted by $aO(X)$ (resp. $aC(X)$).

DEFINITION 3. Let A be a subset of a space X .

- (1) The intersection of all a -closed sets containing A is called the a -closure of A and is denoted by $a-cl(A)$.
- (2) The a -interior of A is defined by the union of all a -open sets contained in A and is denoted by $a-int(A)$.

THEOREM 4. Let A be a subset of a space X . Then

- (1) $\alpha\text{-cl}(A) = A \cup cl(int(\delta\text{-cl}(A)))$.
- (2) $\alpha\text{-int}(A) = A \cap int(cl(\delta\text{-int}(A)))$.

PROOF. (1): Since $\alpha\text{-cl}(A)$ is α -closed,

$$cl(int(\delta\text{-cl}(A))) \subset cl(int(\delta\text{-cl}(\alpha\text{-cl}(A)))) \subset \alpha\text{-cl}(A).$$

Thus, $A \cup cl(int(\delta\text{-cl}(A))) \subset \alpha\text{-cl}(A)$.

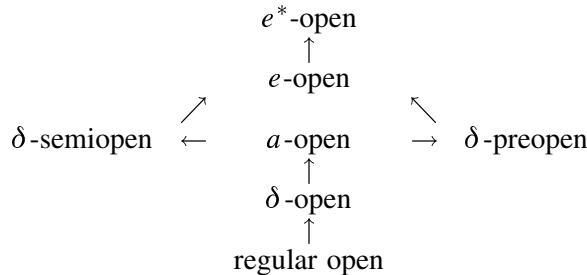
Conversely, we have

$$\begin{aligned} cl(int(\delta\text{-cl}[A \cup cl(int(\delta\text{-cl}(A)))])) &= cl(int(\delta\text{-cl}(A) \cup \delta\text{-cl}(cl(int(\delta\text{-cl}(A)))))) \\ &= cl(int(\delta\text{-cl}(A))) \\ &\subset A \cup cl(int(\delta\text{-cl}(A))). \end{aligned}$$

Since $A \cup cl(int(\delta\text{-cl}(A)))$ is α -closed containing A , then $\alpha\text{-cl}(A) \subset A \cup cl(int(\delta\text{-cl}(A)))$.

(2): It is obvious from (1). ■

REMARK 5. (1) The following diagram holds for a subset A of a space X :



(2) Every a -open set is α -open.

(3) None of these implications is reversible as shown in the following examples. The other examples are as shown in [4, 6].

EXAMPLE 6. Let $X = \{a, b, c, d\}$ and let $\tau = \{\emptyset, X, \{a\}, \{c\}, \{a, b\}, \{a, c\}, \{a, b, c\}, \{a, c, d\}\}$. Then the set $\{a, c, d\}$ is α -open but it is not a -open. The set $\{a, b, d\}$ is δ -semiopen but it is not a -open.

EXAMPLE 7. Let $X = \{a, b, c, d\}$ and let $\tau = \{\emptyset, X, \{a, d\}, \{c\}, \{a, c, d\}\}$. Then the set $\{b, c, d\}$ is δ -preopen but it is not a -open.

EXAMPLE 8. Let $X = \{a, b, c, d\}$ and let $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}, \{a, b, d\}\}$. Then the set $\{a, b, c\}$ is a -open but it is not δ -open.

3. \mathcal{A}^* -sets

DEFINITION 9. A subset K of a space X is said to be an \mathcal{A}^* -set if there exist an open set A and an a -closed set B such that $K = A \cap B$.

The family of all \mathcal{A}^* -sets of X is denoted by $\mathcal{A}^*(X)$.

REMARK 10. (1) The following diagram holds for a subset P of a space X :

$$\begin{array}{ccc} \mathcal{A}^*\text{-set} & \rightarrow & \mathcal{DS}\text{-set} \\ \uparrow & & \searrow \\ \mathcal{D}\text{-set} & \rightarrow & LC\text{-set} \end{array}$$

(2) In a topological space, every open set and every a -closed set is an \mathcal{A}^* -set.

(3) None of the above implications is reversible as shown in the following examples.

EXAMPLE 11. Let $X = \{a, b, c, d\}$ and let $\tau = \{\emptyset, X, \{a\}, \{c\}, \{a, b\}, \{a, c\}, \{a, b, c\}, \{a, c, d\}\}$. Then the set $\{b, c, d\}$ is an LC -set but it is not an \mathcal{A}^* -set.

EXAMPLE 12. Let $X = \{a, b, c, d\}$ and let $\tau = \{\emptyset, X, \{b\}, \{d\}, \{b, d\}\}$. Then the set $\{c\}$ is an \mathcal{A}^* -set but it is neither a \mathcal{D} -set nor an LC -set and also it is not open. The set $\{c, d\}$ is a \mathcal{DS} -set but it is not an \mathcal{A}^* -set. The set $\{b, d\}$ is an \mathcal{A}^* -set but it is not a -closed.

THEOREM 13. *Let N be a subset of a space X . Then*

(1) *If N is e^* -open, then $\delta\text{-cl}(N)$ is regular closed.*

(2) *$N \in \mathcal{A}^*(X)$ if and only if $N = A \cap a\text{-cl}(N)$ for some open set A .*

PROOF. (1) Since N is e^* -open, then $\delta\text{-cl}(N) \subset \delta\text{-cl}(cl(int(\delta\text{-cl}(N)))) = \delta\text{-cl}(\delta\text{-cl}(\delta\text{-int}(\delta\text{-cl}(N)))) = cl(\delta\text{-int}(\delta\text{-cl}(N)))$. On the other hand, we have $cl(\delta\text{-int}(\delta\text{-cl}(N))) \subset cl(\delta\text{-cl}(N)) = \delta\text{-cl}(N)$. Thus, $\delta\text{-cl}(N) = cl(\delta\text{-int}(\delta\text{-cl}(N))) = cl(int(\delta\text{-cl}(N)))$ and hence, $\delta\text{-cl}(N)$ is regular closed.

(2) (\Rightarrow): Since $N \in \mathcal{A}^*(X)$, $N = A \cap B$ where A is open and B is a -closed. Since $N \subset B$, $a\text{-cl}(N) \subset a\text{-cl}(B) = B$. Thus $A \cap a\text{-cl}(N) \subset A \cap B = N \subset A \cap a\text{-cl}(N)$ and hence $N = A \cap a\text{-cl}(N)$.

(\Leftarrow): Since $N = A \cap a\text{-cl}(N)$ for some open set A and $a\text{-cl}(N)$ is a -closed, then $N \in \mathcal{A}^*(X)$. ■

THEOREM 14. *Let N be a subset of a space X . The following are equivalent:*

- (1) N is open,
- (2) N is α -open and an \mathcal{A}^* -set,
- (3) N is preopen and an \mathcal{A}^* -set.

PROOF. (1) \Rightarrow (2): Since every open set is α -open and an \mathcal{A}^* -set, the proof is obvious.

(2) \Rightarrow (3): Obvious.

(3) \Rightarrow (1): Let N be preopen and an \mathcal{A}^* -set. Then N is a $\mathcal{D}\mathcal{S}$ -set. Since N is preopen and a $\mathcal{D}\mathcal{S}$ -set, by Theorem 13 [5], it is open. ■

THEOREM 15. *Let X be a space. Then X is submaximal if and only if every dense subset of X is an \mathcal{A}^* -set.*

PROOF. (\Rightarrow): Since X submaximal, every dense subset is open and so is an \mathcal{A}^* -set.

(\Leftarrow): It is known that every dense set is preopen. Also, by hypothesis, every dense set is \mathcal{A}^* -set. So, by Theorem 14, it is open. Thus, X is submaximal. ■

THEOREM 16. *Let X be a space. Then X is indiscrete if and only if the \mathcal{A}^* -sets in X are only the trivial ones.*

PROOF. (\Rightarrow): Let G be an \mathcal{A}^* -set in X . There exist an open set U and an α -closed set V such that $G = U \cap V$. If $G \neq \emptyset$, then $U \neq \emptyset$. We obtain $U = X$ and $G = V$. Hence, $X = \alpha\text{-cl}(G) \subset G$ and $G = X$.

(\Leftarrow): Every open set is an \mathcal{A}^* -set. So, open sets in X are only the trivial ones. Hence, X is indiscrete. ■

4. Weaker forms of δ -open sets

DEFINITION 17. A subset N of a space X is said to be a $\delta\mathcal{E}$ -set (resp. a $\delta\mathcal{E}^*$ -set, a $\delta\mathcal{P}$ -set, a $\delta\mathcal{S}$ -set, a $\delta\mathcal{A}$ -set, a $\delta\delta$ -set, a $\delta\mathcal{R}$ -set) if there exist a δ -open set A and an e -closed (resp. e^* -closed, δ -preclosed, δ -semiclosed, α -closed, δ -closed, regular closed) set B such that $N = A \cap B$.

REMARK 18. The following diagram holds for a subset P of a space X :

$$\begin{array}{ccccccc} & & & \delta\mathcal{S}\text{-set} & & & \\ & & & \downarrow & & & \\ \delta\mathcal{R}\text{-set} & \rightarrow & \delta\delta\text{-set} & \rightarrow & \delta\mathcal{A}\text{-set} & \nearrow & \delta\mathcal{E}\text{-set} \\ & & & & \searrow & & \uparrow \\ & & & & \delta\mathcal{P}\text{-set} & & \end{array}$$

None of these implications is reversible as shown in the following examples:

EXAMPLE 19. Let $X = \{a, b, c, d\}$ and let $\tau = \{\emptyset, \{a\}, \{c\}, \{a, b\}, \{a, c\}, \{a, b, c\}, \{a, c, d\}, X\}$. Then the set $\{b, c, d\}$ is a $\delta\mathcal{E}$ -set and $\delta\mathcal{P}$ -set but it is neither $\delta\mathcal{A}$ -set nor $\delta\mathcal{S}$ -set. The set $\{d\}$ is a $\delta\delta$ -set but it is not a $\delta\mathcal{R}$ -set.

EXAMPLE 20. Let $X = \{a, b, c, d\}$ and let $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}, \{a, b, d\}\}$. Then the set $\{b, d\}$ is a $\delta\mathcal{E}$ -set and a $\delta\mathcal{S}$ -set but it is neither a $\delta\mathcal{P}$ -set nor a $\delta\mathcal{A}$ -set. The set $\{d\}$ is a $\delta\mathcal{A}$ -set but it is not a $\delta\delta$ -set.

EXAMPLE 21. Let $X = \{a, b, c, d\}$ and let $\tau = \{\emptyset, X, \{b\}, \{d\}, \{b, d\}\}$. Then the set $\{a, b\}$ is a $\delta\mathcal{E}^*$ -set but it is not a $\delta\mathcal{E}$ -set.

THEOREM 22. *Let K be a subset of a space X .*

(1) *If K is an $\delta\mathcal{R}$ -set, then it is δ -semiopen.*

(2) *$K \in \delta\delta(X)$ if and only if $K = U \cap \delta\text{-cl}(K)$ for some δ -open set U .*

PROOF. (1): Let K be an $\delta\mathcal{R}$ -set. Then $K = A \cap B$, where A is δ -open and B is regular closed. On the other hand, $\delta\text{-int}(K) = \delta\text{-int}(A \cap B) = A \cap \delta\text{-int}(B)$. Suppose that $x \notin cl(\delta\text{-int}(K)) = \delta\text{-cl}(\delta\text{-int}(A \cap B))$. Then there exists a δ -open set V containing x such that $V \cap \delta\text{-int}(A \cap B) = V \cap A \cap \delta\text{-int}(B) = V \cap A \cap int(B) = \emptyset$. We obtain $V \cap A \cap B = V \cap K = V \cap A \cap cl(int(B)) = \emptyset$ and then $x \notin K$. Thus, $K \subset cl(\delta\text{-int}(K))$ and hence, K is δ -semiopen.

(2): It is similar to that of Theorem 13. ■

The following example shows that Theorem 22 (1) is not reversible.

EXAMPLE 23. Let $X = \{a, b, c, d\}$ and let $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}, \{a, b, d\}\}$. Then the set $\{a, b, d\}$ is δ -semiopen but it is not a $\delta\mathcal{R}$ -set.

LEMMA 24. Let X be a topological space and $A \subset X$. Then A is δ -semiopen if and only if $\delta\text{-cl}(A) = \delta\text{-cl}(\delta\text{-int}(A))$.

PROOF. Since A is δ -semiopen, $\delta\text{-cl}(A) \subset cl(\delta\text{-int}(A)) = \delta\text{-cl}(\delta\text{-int}(A)) \subset \delta\text{-cl}(A)$. Hence, $\delta\text{-cl}(A) = \delta\text{-cl}(\delta\text{-int}(A))$.

Conversely, let $\delta\text{-cl}(A) = \delta\text{-cl}(\delta\text{-int}(A))$. We have $A \subset \delta\text{-cl}(A) = \delta\text{-cl}(\delta\text{-int}(A))$. Hence, A is δ -semiopen. ■

THEOREM 25. Let K be a subset of a space X . Then the following are equivalent:

- (1) K is δ -open,
- (2) K is a -open and a $\delta\mathcal{R}$ -set,
- (3) K is a -open and a $\delta\mathcal{A}$ -set,
- (4) K is a -open and a $\delta\mathcal{P}$ -set,
- (5) K is a -open and a $\delta\mathcal{S}$ -set,
- (6) K is a -open and a $\delta\mathcal{E}$ -set,
- (7) K is a -open and a $\delta\mathcal{E}^*$ -set,
- (8) K is a -open and a $\delta\delta$ -set,
- (9) K is δ -preopen and a $\delta\mathcal{R}$ -set,
- (10) K is δ -preopen and a $\delta\delta$ -set,
- (11) K is δ -preopen and a $\delta\mathcal{A}$ -set,
- (12) K is δ -preopen and a $\delta\mathcal{S}$ -set.

PROOF. (1) \Rightarrow (2): It follows from the fact that every δ -open set is a -open and an $\delta\mathcal{R}$ -set.

(2) \Rightarrow (3): Obvious.

(3) \Rightarrow (4) \Rightarrow (6) \Rightarrow (7): It follows from Remark 18.

(3) \Rightarrow (5) \Rightarrow (6) \Rightarrow (7): It follows from Remark 18.

(7) \Rightarrow (1): Since K is a -open and a $\delta\mathcal{E}^*$ -set, $K = G \cap F$, where G is δ -open and F is e^* -closed and also $K = G \cap F \subset int(cl(\delta\text{-int}(G \cap F))) = \delta\text{-int}(\delta\text{-cl}(\delta\text{-int}(G) \cap \delta\text{-int}(F))) \subset \delta\text{-int}(\delta\text{-cl}(\delta\text{-int}(G)) \cap \delta\text{-cl}(\delta\text{-int}(F))) = \delta\text{-int}(\delta\text{-cl}(G) \cap \delta\text{-cl}(\delta\text{-int}(F))) = \delta\text{-int}(\delta\text{-cl}(G)) \cap \delta\text{-int}(\delta\text{-cl}(\delta\text{-int}(F)))) = \delta\text{-int}(\delta\text{-cl}(G)) \cap \delta\text{-int}(F)$ by Theorem 13. This implies that $K = G \cap F \subset \delta\text{-int}(\delta\text{-cl}(G)) \cap \delta\text{-int}(F) \cap G = G \cap \delta\text{-int}(F)$ and then $K = G \cap \delta\text{-int}(F)$. Hence, K is δ -open.

(2) \Rightarrow (8) \Rightarrow (10): Obvious.

(2) \Rightarrow (9) \Rightarrow (10): Obvious.

(10) \Rightarrow (1): Let K be δ -preopen and a $\delta\delta$ -set. Then $K \subset \text{int}(\delta\text{-cl}(K)) = \delta\text{-int}(\delta\text{-cl}(K))$ and $K = G \cap \delta\text{-cl}(K)$ where G is δ -open. We obtain $K \subset G \cap \delta\text{-cl}(K) \cap \delta\text{-int}(\delta\text{-cl}(K)) \subset G \cap \delta\text{-int}(\delta\text{-cl}(K)) = \delta\text{-int}(G \cap \delta\text{-cl}(K)) = \delta\text{-int}(K)$. Hence, K is δ -open.

(3) \Rightarrow (11) \Rightarrow (12): Obvious.

(12) \Rightarrow (1): Let K be δ -preopen and a $\delta\mathcal{S}$ -set. Then $K \subset \text{int}(\delta\text{-cl}(K)) = \delta\text{-int}(\delta\text{-cl}(K))$ and $K = G \cap A$ where G is δ -open and $\delta\text{-int}(\delta\text{-cl}(A)) = \delta\text{-int}(A)$ by Lemma 24. We have $K \subset \delta\text{-int}(\delta\text{-cl}(K)) = \delta\text{-int}(\delta\text{-cl}(G \cap A)) \subset \delta\text{-int}(\delta\text{-cl}(G)) \cap \delta\text{-int}(\delta\text{-cl}(A)) = \delta\text{-int}(\delta\text{-cl}(G)) \cap \delta\text{-int}(A)$. Thus, $K \subset \delta\text{-int}(\delta\text{-cl}(G)) \cap \delta\text{-int}(A) \cap G = G \cap \delta\text{-int}(A)$ and hence $K = G \cap \delta\text{-int}(A)$. Therefore, K is δ -open. \blacksquare

LEMMA 26. *Let N be a subset of a space X . Then $N \in \delta\mathcal{A}(X)$ if and only if $N = A \cap a\text{-cl}(N)$ for some δ -open set A .*

THEOREM 27. *Let N be a subset of a space X . The following are equivalent:*

- (1) N is an $\delta\mathcal{R}$ -set,
- (2) N is δ -semiopen and a $\delta\delta$ -set,
- (3) N is e -open and a $\delta\delta$ -set,
- (4) N is e^* -open and a $\delta\delta$ -set,
- (5) N is δ -semiopen and a $\delta\mathcal{A}$ -set,
- (6) N is e -open and a $\delta\mathcal{A}$ -set,
- (7) N is e^* -open and a $\delta\mathcal{A}$ -set.

PROOF. (1) \Rightarrow (2): It is obvious from Theorem 22 and Remark 18.

(2) \Rightarrow (3) \Rightarrow (4) \Rightarrow (7): By Remark 18, it is obvious.

(2) \Rightarrow (5) \Rightarrow (6) \Rightarrow (7): By Remark 18, it is obvious.

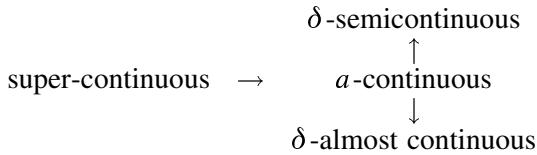
(7) \Rightarrow (1): Since N is a $\delta\mathcal{A}$ -set, then $N = G \cap a\text{-cl}(A)$, where G is δ -open. Since A is e^* -open, by Theorem 4, $a\text{-cl}(A) = cl(\text{int}(\delta\text{-cl}(A)))$. Hence, $a\text{-cl}(A)$ is regular closed and so N is an $\delta\mathcal{R}$ -set. \blacksquare

5. Decompositions of continuous functions and super-continuous functions

DEFINITION 28. A function $f:(X, \tau) \rightarrow (Y, \sigma)$ is said to be a -continuous if $f^{-1}(A)$ is a -open in X for every $A \in \sigma$.

DEFINITION 29. A function $f:(X, \tau) \rightarrow (Y, \sigma)$ is called super-continuous [9] (resp. δ -almost continuous [12], δ -semicontinuous [3]) if $f^{-1}(A)$ is δ -open (resp. δ -preopen, δ -semiopen) for each $A \in \sigma$.

REMARK 30. Let $f: X \rightarrow Y$ be a function. The following diagram holds:



These implications are not reversible as shown in the following examples:

EXAMPLE 31. Let $X = \{a, b, c, d\} = Y$ and let $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}, \{a, b, d\}\}$. Then

(1) the function $f:(X, \tau) \rightarrow (Y, \tau)$, defined as: $f(a) = a, f(b) = d, f(c) = c, f(d) = a$, is δ -semicontinuous but it is not a -continuous.

(2) the function $f:(X, \tau) \rightarrow (Y, \tau)$, defined as: $f(a) = a, f(b) = a, f(c) = d, f(d) = c$, is a -continuous but it is not super-continuous.

EXAMPLE 32. Let $X = \{a, b, c, d\} = Y$ and let $\tau = \{\emptyset, \{a\}, \{c\}, \{a, b\}, \{a, c\}, \{a, b, c\}, \{a, c, d\}, X\}$. Then the identity function $i:(X, \tau) \rightarrow (X, \tau)$ is δ -almost continuous but it is not a -continuous.

DEFINITION 33. A function $f:(X, \tau) \rightarrow (Y, \sigma)$ is said to be \mathcal{A}^* -continuous if $f^{-1}(A)$ is an \mathcal{A}^* -set in X for every $A \in \sigma$.

DEFINITION 34. A function $f:(X, \tau) \rightarrow (Y, \sigma)$ is called \mathcal{D} -continuous [5] (resp. \mathcal{DS} -continuous [5]) if $f^{-1}(A)$ is a \mathcal{D} -set (resp. a \mathcal{DS} -set) for each $A \in \sigma$.

REMARK 35. The following implications hold for a function $f: X \rightarrow Y$ but not conversely:

$$\mathcal{D}\text{-continuous} \rightarrow \mathcal{A}^*\text{-continuous} \rightarrow \mathcal{DS}\text{-continuous}$$

EXAMPLE 36. Let $X = \{a, b, c, d\} = Y$ and let $\tau = \{\emptyset, X, \{b\}, \{d\}, \{b, d\}\}$. Then

(1) the function $f:(X, \tau) \rightarrow (Y, \tau)$, defined as: $f(a) = a, f(b) = c, f(c) = b, f(d) = c$, is \mathcal{A}^* -continuous but it is not \mathcal{D} -continuous.

(2) the function $f:(X, \tau) \rightarrow (Y, \tau)$, defined as: $f(a) = a, f(b) = c, f(c) = b, f(d) = b$, is \mathcal{DS} -continuous but it is not \mathcal{A}^* -continuous.

DEFINITION 37. A function $f:(X, \tau) \rightarrow (Y, \sigma)$ is called α -continuous [8] (resp. precontinuous [7]) if $f^{-1}(A)$ is α -open (resp. preopen) for each $A \in \sigma$.

THEOREM 38. *The following are equivalent for a function $f: X \rightarrow Y$:*

- (1) f is continuous,
- (2) f is α -continuous and \mathcal{A}^* -continuous,
- (3) f is precontinuous and \mathcal{A}^* -continuous.

PROOF. It is an immediate consequence of Theorem 14. ■

DEFINITION 39. A function $f:(X, \tau) \rightarrow (Y, \sigma)$ is said to be $\delta\mathcal{E}^*$ -continuous (resp. $\delta\mathcal{E}$ -continuous, $\delta\mathcal{P}$ -continuous, $\delta\mathcal{S}$ -continuous, $\delta\mathcal{A}$ -continuous, $\delta\delta$ -continuous, $\delta\mathcal{R}$ -continuous) if $f^{-1}(A)$ is a $\delta\mathcal{E}^*$ -set (resp. a $\delta\mathcal{E}$ -set, a $\delta\mathcal{P}$ -set, a $\delta\mathcal{S}$ -set, a $\delta\mathcal{A}$ -set, a $\delta\delta$ -set, a $\delta\mathcal{R}$ -set) in X for every $A \in \sigma$.

REMARK 40. The following diagram holds for a function $f: X \rightarrow Y$:

$$\begin{array}{ccccccc} & & & & \delta\mathcal{S}\text{-cont.} & & \\ & & & & \downarrow & & \\ \delta\mathcal{R}\text{-cont.} & \rightarrow & \delta\delta\text{-cont.} & \rightarrow & \delta\mathcal{A}\text{-cont.} & \nearrow & \delta\mathcal{E}\text{-cont.} \rightarrow \delta\mathcal{E}^*\text{-cont.} \\ & & & & \searrow & & \\ & & & & \delta\mathcal{P}\text{-cont.} & & \end{array}$$

These implications are not reversible as shown in the following examples:

EXAMPLE 41. Let $X = \{a, b, c, d\} = Y$ and let $\tau = \{\emptyset, \{a\}, \{c\}, \{a, b\}, \{a, c\}, \{a, b, c\}, \{a, c, d\}, X\}$. Then

(1) the function $f:(X, \tau) \rightarrow (Y, \tau)$, defined as: $f(a) = d, f(b) = b, f(c) = b, f(d) = c$, is $\delta\mathcal{E}$ -continuous and $\delta\mathcal{P}$ -continuous but it is neither $\delta\mathcal{A}$ -continuous nor $\delta\mathcal{S}$ -continuous.

(2) the function $f:(X, \tau) \rightarrow (Y, \tau)$, defined as: $f(a) = d, f(b) = d, f(c) = d, f(d) = a$, is $\delta\delta$ -continuous but it is not $\delta\mathcal{R}$ -continuous.

EXAMPLE 42. Let $X = \{a, b, c, d\} = Y$ and let $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}, \{a, b, d\}\}$. Then

- (1) the function $f:(X, \tau) \rightarrow (Y, \tau)$, defined as: $f(a) = c, f(b) = d, f(c) = c, f(d) = b$, is $\delta\mathcal{S}$ -continuous and $\delta\mathcal{E}$ -continuous but it is neither $\delta\mathcal{P}$ -continuous nor $\delta\mathcal{A}$ -continuous.
- (2) the function $f:(X, \tau) \rightarrow (Y, \tau)$, defined as: $f(a) = d, f(b) = d, f(c) = d, f(d) = b$, is $\delta\mathcal{A}$ -continuous but it is not $\delta\delta$ -continuous.

EXAMPLE 43. Let $X = \{a, b, c, d\} = Y$ and let $\tau = \{\emptyset, X, \{b\}, \{d\}, \{b, d\}\}$. Then the function $f:(X, \tau) \rightarrow (Y, \tau)$, defined as: $f(a) = b, f(b) = b, f(c) = c, f(d) = a$, is $\delta\mathcal{E}^*$ -continuous but it is not $\delta\mathcal{E}$ -continuous.

THEOREM 44. *The following are equivalent for a function $f:X \rightarrow Y$:*

- (1) f is super-continuous,
- (2) f is α -continuous and $\delta\mathcal{R}$ -continuous,
- (3) f is α -continuous and $\delta\mathcal{A}$ -continuous,
- (4) f is α -continuous and $\delta\mathcal{P}$ -continuous,
- (5) f is α -continuous and $\delta\mathcal{S}$ -continuous,
- (6) f is α -continuous and $\delta\mathcal{E}$ -continuous,
- (7) f is α -continuous and $\delta\mathcal{E}^*$ -continuous,
- (8) f is α -continuous and $\delta\delta$ -continuous,
- (9) f is δ -almost continuous and $\delta\mathcal{R}$ -continuous,
- (10) f is δ -almost continuous and $\delta\delta$ -continuous,
- (11) f is δ -almost continuous and $\delta\mathcal{A}$ -continuous,
- (12) f is δ -almost continuous and $\delta\mathcal{S}$ -continuous

PROOF. It is an immediate consequence of Theorem 25. ■

DEFINITION 45. A function $f:(X, \tau) \rightarrow (Y, \sigma)$ is called e -continuous [4] (resp. e^* -continuous [6]) if $f^{-1}(A)$ is e -open (resp. e^* -open) for each $A \in \sigma$.

THEOREM 46. *The following are equivalent for a function $f:X \rightarrow Y$:*

- (1) f is $\delta\mathcal{R}$ -continuous,
- (2) f is δ -semicontinuous and $\delta\delta$ -continuous,
- (3) f is e -continuous and $\delta\delta$ -continuous,
- (4) f is e^* -continuous and $\delta\delta$ -continuous,
- (5) f is δ -semicontinuous and $\delta\mathcal{A}$ -continuous,

- (6) f is e -continuous and $\delta\mathcal{A}$ -continuous,
- (7) f is e^* -continuous and $\delta\mathcal{A}$ -continuous.

PROOF. It is an immediate consequence of Theorem 27. ■

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ON ALMOST PSEUDO SYMMETRIC MANIFOLDS

By

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Abstract. The object of the present paper is to study a type of semi-Riemannian manifold which is called almost pseudo symmetric manifold. Some properties of an almost pseudo symmetric manifold have been studied. Also we consider almost pseudo symmetric spacetime. Finally, the existence of an almost pseudo symmetric semi-Riemannian manifold is proved by non-trivial examples.

1. Introduction

In this paper the authors introduce a type of semi-Riemannian manifold (M^n, g) ($n \geq 2$), whose curvature tensor \tilde{R} of type $(0, 4)$ satisfies the condition

$$\begin{aligned} (\nabla_U \tilde{R})(X, Y, Z, W) = & [A(U) + B(U)]\tilde{R}(X, Y, Z, W) + \\ & + A(X)\tilde{R}(U, Y, Z, W) + A(Y)\tilde{R}(X, U, Z, W) + A(Z)\tilde{R}(X, Y, U, W) + \\ (1.1) \quad & + A(W)\tilde{R}(X, Y, Z, U), \end{aligned}$$

where A, B are two non-zero 1-forms defined by

$$(1.2) \quad g(X, P) = A(X), \quad g(X, Q) = B(X),$$

for all vector field X , ∇ denotes the operator of covariant differentiation with respect to the metric g , \tilde{R} is defined by $\tilde{R}(X, Y, Z, W) = g(R(X, Y)Z, W)$, where R is the curvature tensor of type $(1, 3)$. Such a manifold shall be called

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an almost pseudo symmetric manifold and the 1-forms A and B are called the associated 1-forms. The name almost pseudo symmetric is chosen because if $A = B$ in (1.1) then the manifold reduces to a pseudo symmetric manifold which is denoted by $(PS)_n$, introduced by M. C. Chaki [1]. In this connection we can mention the notion of weakly symmetric manifold introduced by Tamássy and Binh [9]. A semi-Riemannian manifold of dimension > 2 is said to be weakly symmetric [9] if there exist 1-forms A, B, C, D and E , not simultaneously zero, such that the curvature tensor satisfies the condition

$$\begin{aligned} (\nabla_U \tilde{R})(X, Y, Z, W) &= A(U)\tilde{R}(X, Y, Z, W) + B(X)\tilde{R}(U, Y, Z, W) + \\ &+ C(Y)\tilde{R}(X, U, Z, W) + D(Z)\tilde{R}(X, Y, U, W) \\ &+ E(W)\tilde{R}(X, Y, Z, U). \end{aligned}$$

It may be mentioned that almost pseudo symmetric manifold is not a particular case of a weakly symmetric manifold. Pseudo symmetric manifolds have been studied by many authors. An n -dimensional almost pseudo symmetric manifold will be denoted by $A(PS)_n$.

A semi-Riemannian manifold is said to have cyclic Ricci tensor if its Ricci tensor S of type $(0, 2)$ is non-zero and satisfies

$$(1.3) \quad (\nabla_X S)(Y, Z) + (\nabla_Y S)(Z, X) + (\nabla_Z S)(X, Y) = 0.$$

It is known that [4] Cartan hypersurfaces are manifolds with non-parallel Ricci tensor satisfying cyclic Ricci tensor.

WALKER'S LEMMA [10]: *If a_{ij}, b_i are numbers satisfying $a_{ij} = a_{ji}$, $a_{ij}b_k + a_{jk}b_i + a_{ki}b_j = 0$, for $i, j, k = 1, 2, \dots, n$, then either all $a_{ij} = 0$ or, all b_i are zero.*

The paper is organized as follows:

After preliminaries we have studied $A(PS)_n$ with non-zero constant scalar curvature. In section 4 of this paper it has been shown that in an $A(PS)_2$ one of the associated 1-forms is closed if and only if the other is also closed. In section 5, an $A(PS)_n$ with cyclic Ricci tensor is studied. The question whether an $A(PS)_n$ can be of constant curvature is answered and the Einstein $A(PS)_n$ is studied in section 6. Section 7 deals with conformally flat $A(PS)_n$ ($n > 3$). We prove that a conformally flat $A(PS)_n$ ($n > 3$) with non-zero constant scalar curvature is a quasi Einstein manifold introduced by Chaki and Maity [2]. In section 8 we prove that if an $A(PS)_n$ admits a parallel vector field which is not orthogonal to the associated vector field P then the manifold can not be conformally flat. In section 9 we deal with an $A(PS)_4$ perfect fluid spacetime with cyclic Ricci tensor. The last section contains two non-trivial examples of $A(PS)_n$ with non-zero non-constant scalar curvature.

2. Preliminaries

Let S and r denote the Ricci tensor of type $(0, 2)$ and the scalar curvature respectively and L denote the symmetric endomorphism of the tangent space at each point corresponding to the Ricci tensor, that is,

$$(2.1) \quad g(LX, Y) = S(X, Y),$$

for any vector field X, Y . If $\{e_i\}$, $i = 1, 2, \dots, n$, is an orthonormal basis of the tangent space at each point of the manifold so that $g(e_i, e_j) = \epsilon_i \delta_{ij}$, $\epsilon_i = \pm 1$, then

$$S(X, Y) = \sum_{i=1}^n \epsilon_i R(e_i, X, Y, e_i), \quad r = \sum_{i=1}^n \epsilon_i S(e_i, e_i).$$

Let \bar{A} and \bar{B} are two 1-forms defined by

$$(2.2) \quad A(LX) = \bar{A}(X), \quad B(LX) = \bar{B}(X).$$

Then \bar{A} and \bar{B} are called auxiliary 1-forms corresponding to the 1-forms A and B respectively.

Putting $X = W = e_i$ in (1.1) where $\{e_i\}$, $i = 1, 2, \dots, n$, is an orthonormal basis of the tangent space at each point of the manifold and taking summation over i , $1 \leq i \leq n$, we get

$$(2.3) \quad (\nabla_U S)(Y, Z) = [A(U) + B(U)]S(Y, Z) + A(Y)S(U, Z) + A(Z)S(Y, U) + \\ + \tilde{R}(U, Y, Z, P) + \tilde{R}(P, Y, Z, U).$$

Again putting $Y = Z = e_i$ in (2.3) yields

$$(2.4) \quad dr(U) = [A(U) + B(U)]r + 4\bar{A}(U).$$

3. $A(PS)_n$ ($n > 2$) of non-zero constant scalar curvature

We suppose that in an $A(PS)_n$ the scalar curvature r is a non-zero constant. Then from (2.4) we get

$$\bar{A}(X) = -\frac{r}{4}[A(X) + B(X)].$$

From this it follows that

$$(3.1) \quad S(X, P) = -\frac{r}{4}[A(X) + B(X)].$$

This shows that $S(X, P)$ can not be of the form $k A(X)$ where k is a scalar. Hence P can not be an eigen vector corresponding to any eigen value k of S . This leads to the following theorem:

THEOREM 3.1. *In an $A(PS)_n, n > 2$, of non-zero constant scalar curvature, P can not be an eigen vector corresponding to any eigen value of S .*

If in particular $A = B$ then from (3.1) we have $S(X, P) = -\frac{r}{2}A(X)$, that is, $\bar{A}(X) = -\frac{r}{2}A(X)$. From which we can state the following corollary:

COROLLARY 3.1. *In an $A(PS)_n$ of non-zero constant scalar curvature, in which $A = B$, we obtain $\bar{A}(X) = -\frac{r}{2}A(X)$.*

The above corollary has already been proved by M. C. Chaki [1] in his paper.

4. Two dimensional almost pseudo symmetric manifolds

In this section we consider an $A(PS)_2$. It is known [7] that every (M^2, g) is a recurrent manifold whose 1-form of recurrence Q is given by

$$(4.1) \quad Q(X) = X \cdot (\log r).$$

Since $dQ(X, Y) = XQ(Y) - YQ(X) - Q([X, Y])$, it follows from (4.1) that $dQ(X, Y) = 0$ which means that the 1-form Q is closed. Again since every (M^2, g) is an Einstein manifold we have

$$(4.2) \quad S(X, Y) = \frac{r}{2}g(X, Y).$$

So,

$$(4.3) \quad S(X, P) = \frac{r}{2}g(X, P) = \frac{r}{2}A(X).$$

Therefore from (2.4) and (4.3) we get

$$(4.4) \quad \begin{aligned} X.r &= [A(X) + B(X)]r + 4S(X, P) \\ &= [A(X) + B(X)]r + 2A(X) = [3A(X) + B(X)]r. \end{aligned}$$

Then by (4.1) the equation (4.4) reduces to

$$(4.5) \quad r[Q(X) - 3A(X) - B(X)] = 0.$$

Now in an (M^2, g) , $r = 0$ implies that the manifold is flat which is inadmissible by definition. Hence $r \neq 0$ and therefore from (4.5) we get

$$(4.6) \quad 3A(X) + B(X) = Q(X).$$

Since Q is closed it follows from (4.6) that A is closed if and only if B is closed. We can therefore state the following theorem:

THEOREM 4.1. *In an $A(PS)_2$ the associated 1-form A is closed if and only if the 1-form B is closed.*

It is to be noted that in an $A(PS)_2$ if the scalar curvature r is constant, then from (4.4) we get $3A(X) = -B(X)$ as $r \neq 0$ in M^2 . So we have the following corollary:

COROLLARY 4.1 *In an $A(PS)_2$ if the scalar curvature is constant, then the corresponding vector fields of the 1-forms A and B are in opposite direction.*

5. $A(PS)_n$ ($n > 2$) with cyclic Ricci tensor

In relation (2.3) if we replace Y , Z and U by X we get

$$(5.1) \quad (\nabla_X S)(X, X) = [3A(X) + B(X)]S(X, X),$$

since $\tilde{R}(X, X, X, Y) = 0$. Now if the Ricci tensor is non-zero then from (5.1) it follows that $(\nabla_X S)(X, X) = 0$ if and only if $3A(X) + B(X) = 0$. So we have the following theorem:

THEOREM 5.1. *In an $A(PS)_n$ if the Ricci tensor is non-zero, then the Ricci curvature $S(X, X)$ is covariantly constant in the direction of X if and only if $3A(X) + B(X) = 0$.*

Again from (2.3) we have

$$(5.2) \quad \begin{aligned} (\nabla_X S)(Y, Z) &= [A(X) + B(X)]S(Y, Z) + A(Y)S(X, Z) + A(Z)S(Y, X) + \\ &\quad + \tilde{R}(X, Y, Z, P) + \tilde{R}(P, Y, Z, X). \end{aligned}$$

Now interchanging X , Y , Z in (5.2) and then adding them and using the symmetric and skew-symmetric properties of curvature tensor we get

$$(5.3) \quad \begin{aligned} (\nabla_X S)(Y, Z) + (\nabla_Y S)(Z, X) + (\nabla_Z S)(X, Y) &= \\ &= T(X)S(Y, Z) + T(Y)S(X, Z) + T(Z)S(X, Y), \end{aligned}$$

where $T(X) = 3A(X) + B(X)$. Now if the Ricci tensor of the manifold be cyclic, then from (5.3) and (1.3) we have

$$(5.4) \quad T(X)S(Y, Z) + T(Y)S(X, Z) + T(Z)S(X, Y) = 0.$$

Then by Walker's lemma we can see that either $T = 0$ or, $S = 0$. But since $S \neq 0$, we have $T = 0$ which implies that

$$(5.5) \quad 3A(X) + B(X) = 0.$$

Conversely, if $3A(X) + B(X) = 0$ then $T = 0$ and from (5.3)

$$(\nabla_X S)(Y, Z) + (\nabla_Y S)(Z, X) + (\nabla_Z S)(X, Y) = 0$$

which implies that the Ricci tensor is cyclic. Thus we can state the following theorem:

THEOREM 5.2. *An $A(PS)_n$ admits a cyclic Ricci tensor if and only if the associated 1-forms A and B satisfy the relation (5.5).*

REMARK. For cyclic Ricci tensor $S \neq 0$. By THEOREM 5.2, an $A(PS)_n$ admits a cyclic Ricci tensor if and only if $3A(X) + B(X) = 0$. Again by THEOREM 5.1, in an $A(PS)_n$ if $S \neq 0$ then the Ricci curvature $S(X, X)$ is covariantly constant in the direction of X if and only if $3A(X) + B(X) = 0$. Hence we can state that “An $A(PS)_n$ admits a cyclic Ricci tensor if and only if the Ricci curvature $S(X, X)$ is covariantly constant in the direction of X .”

6. Einstein $A(PS)_n$ ($n > 2$)

In this section we suppose that an $A(PS)_n$ is an Einstein manifold. Then

$$(6.1) \quad S(X, Y) = \frac{r}{n}g(X, Y).$$

It is known [5] that in an Einstein manifold (M^n, g) ($n > 2$) r is constant. Hence in this case $dr(X) = 0$. So from (2.4) we get

$$(6.2) \quad [A(X) + B(X)]r + 4S(X, P) = 0.$$

Then from (6.1) and (6.2) we have

$$(6.3) \quad \left[\left(1 + \frac{4}{n} \right) A(X) + B(X) \right] r = 0,$$

which implies that either $r = 0$ or,

$$(6.4) \quad \left(1 + \frac{4}{n} \right) A(X) + B(X) = 0.$$

Hence we have the following theorem:

THEOREM 6.1. *In an Einstein $A(PS)_n$, $n > 2$ either the 1-forms A and B satisfy the condition (6.4) or, the scalar curvature vanishes.*

If in particular $A(X) = B(X)$ then an $A(PS)_n$ reduces to a $(PS)_n$ and the expression $\left(1 + \frac{4}{n} \right) A(X) + B(X)$ takes the form $\frac{2(n+2)}{n}A(X)$ which is not

zero because $A(X) \neq 0$. Thus it follows that an Einstein $(PS)_n$ with $n > 2$ is of zero scalar curvature—a result already proved by M. C. Chaki [1]. It is known that a manifold of constant curvature is an Einstein manifold, but the converse is not, in general, true. The question, therefore, arises whether an $A(PS)_n$ can be of constant curvature.

Suppose that an $A(PS)_n$ is of constant curvature. Then we can write

$$(6.5) \quad R(X, Y)Z = \kappa[g(Y, Z)X - g(X, Z)Y],$$

where κ is a constant. Being of constant curvature the $A(PS)_n$ under consideration is an Einstein manifold. Hence if (6.4) does not hold then from (6.3) it follows that $r = 0$ which implies, together with (6.5), that $\kappa = 0$ because from (6.5) we have $r = n(n - 1)\kappa$ by contraction. Consequently, from (6.5) it follows that $R(X, Y)Z = 0$, that is, the manifold is flat, which is not admissible by definition of an $A(PS)_n$. Therefore the answer to the question raised above we can state the following theorem:

THEOREM 6.2. *If an $A(PS)_n$ does not satisfy the condition (6.4), then it can not be of constant curvature.*

Since a 3-dimensional Einstein manifold is of constant curvature ([5] P. 293), we can state the following corollary of THEOREM 6.2:

COROLLARY 6.1. *An Einstein $A(PS)_3$ always satisfies the condition (6.4).*

7. Conformally flat $A(PS)_n$ ($n > 3$)

In this section we assume that the manifold $A(PS)_n$ is conformally flat. Then $\text{div } C = 0$, where C denotes the Weyl conformal curvature tensor and ‘div’ denotes the divergence. Hence we have [3]

$$(7.1) \quad (\nabla_X S)(Y, Z) - (\nabla_Z S)(X, Y) = \frac{1}{2(n - 1)}[g(Y, Z)dr(X) - g(X, Y)dr(Z)].$$

In virtue of (2.4) the equation (7.1) can be written as follows:

$$(7.2) \quad \begin{aligned} & 2(n - 1)[(\nabla_X S)(Y, Z) - (\nabla_Z S)(X, Y)] = \\ & = r[A(X)g(Y, Z) - A(Z)g(X, Y)] + r[B(X)g(Y, Z) - B(Z)g(X, Y)] + \\ & + 4[\bar{A}(X)g(Y, Z) - \bar{A}(Z)g(X, Y)]. \end{aligned}$$

Again by (2.3) we get

$$(7.3) \quad \begin{aligned} & (\nabla_X S)(Y, Z) - (\nabla_Z S)(X, Y) = [B(X)S(Y, Z) - B(Z)S(X, Y)] + \\ & + \tilde{R}(X, Y, Z, P) + 2\tilde{R}(P, Y, Z, X) - \tilde{R}(P, X, Y, Z). \end{aligned}$$

Now, in a conformally flat (M^n, g) , $(n > 3)$

$$\begin{aligned} \tilde{R}(X, Y, Z, W) = & \frac{1}{n-2}[g(Y, Z)S(X, W) + g(X, W)S(Y, Z) - \\ & - g(X, Z)S(Y, W) - g(Y, W)S(X, Z)] + \\ (7.4) \quad & + \frac{r}{(n-1)(n-2)}[g(X, Z)g(Y, W) - g(Y, Z)g(X, W)]. \end{aligned}$$

In virtue of (7.3) and (7.4) we get

$$\begin{aligned} 2(n-1)[(\nabla_X S)(Y, Z) - (\nabla_Z S)(X, Y)] = & \\ = 2(n-1)[B(X)S(Y, Z) - B(Z)S(X, Y)] + & \frac{6(n-1)}{(n-2)}[A(X)S(Y, Z) + \\ + \bar{A}(X)g(Y, Z) - A(Z)S(X, Y) - \bar{A}(Z)g(X, Y)] + & \\ (7.5) \quad & + \frac{6r}{(n-2)}[A(Z)g(X, Y) - A(X)g(Y, Z)]. \end{aligned}$$

From (7.2) and (7.5) we have

$$\begin{aligned} r[A(X)g(Y, Z) - A(Z)g(X, Y)] + r[B(X)g(Y, Z) - B(Z)g(X, Y)] + & \\ + 4[\bar{A}(X)g(Y, Z) - \bar{A}(Z)g(X, Y)] = & \\ = 2(n-1)[B(X)S(Y, Z) - B(Z)S(X, Y)] + & \frac{6(n-1)}{(n-2)}[A(X)S(Y, Z) + \\ + \bar{A}(X)g(Y, Z) - A(Z)S(X, Y) - \bar{A}(Z)g(X, Y)] + & \\ (7.6) \quad & + \frac{6r}{(n-2)}[A(Z)g(X, Y) - A(X)g(Y, Z)]. \end{aligned}$$

Now putting $Y = P$ in (7.6) we get

$$\begin{aligned} r[B(X)A(Z) - B(Z)A(X)] + 4[\bar{A}(X)A(Z) - \bar{A}(Z)A(X)] + & \\ (7.7) \quad & + 2(n-1)[\bar{A}(X)B(Z) - \bar{A}(Z)B(X)] = 0. \end{aligned}$$

Thus we have the following theorem:

THEOREM 7.1. *In a conformally flat $A(PS)_n$ the associated 1-forms satisfy the relation (7.7).*

If, in particular, $A = B$ then $A(PS)_n$ reduces to a $(PS)_n$ and the relation (7.7) reduces to $\bar{A}(X)A(Z) - \bar{A}(Z)A(X) = 0$, a result already proved by M. C. Chaki [1].

Now we suppose that the scalar curvature is a non-zero constant then $dr(X) = 0$ and from (2.4) we get

$$(7.8) \quad B(X) = - \left[A(X) + \frac{4}{r} \bar{A}(X) \right].$$

From (7.7) and (7.8) we have

$$\begin{aligned} & -r \left[\left(A(X) + \frac{4}{r} \bar{A}(X) \right) A(Z) - \left(A(Z) + \frac{4}{r} \bar{A}(Z) \right) A(X) \right] + \\ & \quad + 4[\bar{A}(X)A(Z) - \bar{A}(Z)A(X)] - \\ & - 2(n-1) \left[\left(A(Z) + \frac{4}{r} \bar{A}(Z) \right) \bar{A}(X) - \left(A(X) + \frac{4}{r} \bar{A}(X) \right) \bar{A}(Z) \right] = 0. \end{aligned}$$

or,

$$2(n-1)[\bar{A}(X)A(Z) - \bar{A}(Z)A(X)] = 0$$

or,

$$(7.9) \quad \bar{A}(Z) = t A(Z), \quad (\text{since } n > 3),$$

where $t = \frac{\bar{A}(X)}{A(X)}$ is a non-zero scalar. From (7.8) and (7.9) we get

$$(7.10) \quad B(X) = a A(X),$$

where

$$(7.11) \quad a = - \left(1 + \frac{4t}{r} \right),$$

is non-zero as $B \neq 0$. Now putting $Z = P$ in (7.6) and using (7.9), (7.10) and (1.2) we get

$$(7.12) \quad \alpha S(X, Y) = \beta g(X, Y) + \gamma A(X)A(Y),$$

where $\beta = \left[\frac{6r}{(n-2)} - \frac{6(n-1)}{(n-2)} + r(1+a) + 4t \right] A(P)$ is a scalar,

$$\gamma = \left[2(n-1)at + \frac{12(n-1)t}{(n-2)} - \frac{6r}{(n-2)} - r(1+a) - 4t \right]$$

is a scalar and $\alpha = (n-1) \left[2a + \frac{6}{(n-2)} \right] A(P)$ is a non-zero scalar. Because if $\alpha = 0$ then as $A \neq 0$ and $n > 3$ we have $2a + \frac{6}{(n-2)} = 0$. After some calculations by using (7.11) we then get $t = \frac{(5-n)r}{4(n-2)}$ which implies t is constant, that is, A is constant by the definition of t , which is not possible. Therefore, from (7.12) it follows that the manifold is a quasi-Einstein manifold [2]. Hence the following theorem holds.

THEOREM 7.2. *A conformally flat $A(PS)_n$ ($n > 3$) with non-zero constant scalar curvature is a quasi-Einstein manifold.*

8. $A(PS)_n$ admitting a parallel vector field

In this section we suppose that an $A(PS)_n$ admits a parallel vector field V ([5] P. 124, [8] P. 322) which is not orthogonal to the associated vector field P . Then

$$(8.1) \quad \nabla_X V = 0,$$

for all $X \in \chi(A(PS)_n)$. Applying Ricci identity to (8.1) we get

$$(8.2) \quad \tilde{R}(X, Y, Z, V) = 0.$$

Contracting Y and Z in (8.2) we get

$$(8.3) \quad S(X, V) = 0.$$

Now from (8.1) and (8.3) it follows that

$$(8.4) \quad (\nabla_X S)(Y, V) = 0.$$

From (2.3) we get by (8.2), (8.3) and (8.4)

$$(8.5) \quad A(V)S(X, Y) = 0.$$

Now since, by assumption, $A(V) \neq 0$, so from (8.5) we have $S(X, Y) = 0$ for all vector fields X, Y . Hence $C(X, Y, Z) = R(X, Y, Z)$, where C is the Weyl conformal curvature tensor. Then $C = 0$ implies $R = 0$, that is, the manifold is flat which is inadmissible by definition of $A(PS)_n$. Thus we have the following theorem:

THEOREM 8.1. *If an $A(PS)_n$ admits a parallel vector field which is not orthogonal to the associated vector field P then the manifold can not be conformally flat.*

9. $A(PS)_4$ perfect fluid spacetime with cyclic Ricci tensor

This section is concerned with certain investigations in general relativity by the coordinate free method of differential geometry. In this method of study the spacetime of general relativity is regarded as a connected four-dimensional semi-Riemannian manifold (M^4, g) with Lorentz metric g with signature $(-, +, +, +)$. The geometry of the Lorentz manifold begins with the study of the causal character of vectors of the manifold. It is due to this causality that the Lorentz manifold becomes a convenient choice for the study of general relativity.

Here we consider a special type of spacetime which is called almost pseudo symmetric spacetime.

If the Ricci tensor of an $A(PS)_n$, $n > 2$ is cyclic then from (5.4) we have

$$(9.1) \quad B(X) = -3A(X).$$

Now we consider an almost pseudo symmetric relativistic spacetime $A(PS)_4$ satisfying cyclic Ricci tensor as a perfect fluid spacetime with cosmological constant λ in which the associated vector field P is the velocity vector field of the fluid. It is known [6] that a general relativistic spacetime (M^4, g) is a 4-dimensional Lorentzian manifold having the matter content as a perfect fluid with unit time-like vector field. The Einstein equation can be written as

$$S - \frac{1}{2}rg + \lambda g = \kappa T, \quad \text{where} \quad T = (\sigma + p)A \otimes A + pg.$$

Now this can be written as

$$(9.2) \quad S(X, Y) - \frac{1}{2}rg(X, Y) + \lambda g(X, Y) = \kappa[(\sigma + p)A(X)A(Y) + pg(X, Y)],$$

where σ and p denote the density and pressure of the fluid respectively and A is given by $g(X, P) = A(X)$, for all vector field X , P is the flow vector field of the fluid such that $g(P, P) = -1$. Since $A(PS)_4$ spacetime satisfies cyclic Ricci tensor, by taking a frame field and contracting (1.3) we get $dr(X) = 0$. Hence from (2.4) we obtain $4\bar{A}(X) = -[A(X) + B(X)]r$, which implies

$$(9.3) \quad 4S(X, P) = -[A(X) + B(X)]r.$$

Using (9.1) in (9.3) we get

$$(9.4) \quad S(X, P) = \frac{r}{2}A(X) = \frac{r}{2}g(X, P).$$

Now putting $Y = P$ in (9.2) we get

$$(9.5) \quad S(X, P) - \frac{1}{2}rg(X, P) + \lambda g(X, P) = \kappa[(\sigma + p)A(X)A(P) + pg(X, P)].$$

In virtue of (9.4) and taking account of the fact that $A(P) = -1$ we can write (9.5) as $\lambda = -\kappa\sigma$, which implies that

$$(9.6) \quad \sigma = -\frac{\lambda}{\kappa}.$$

Again taking a frame field and contracting (9.2) we get by using (9.6) that

$$(9.7) \quad p = \frac{3\lambda - r}{3\kappa}.$$

Since $A(PS)_4$ satisfies cyclic Ricci tensor then the scalar curvature r becomes constant, so from (9.7) we see that p is constant. Also from (9.6) we have σ is constant. Since $\text{div } T = 0$, we get the energy and force equation as follows:

$$P\sigma = -(\sigma + p) \text{div } P \quad [\text{Energy equation}].$$

$$(\sigma + p)\nabla_P P = -\text{grad } p - (Pp)P \quad [\text{Force equation}].$$

Since in this case both P and σ are constants, it follows that $\text{div } P = 0$ and $\nabla_P P = 0$. But $\text{div } P$ represents the expansion scalar and $\nabla_P P$ represents the acceleration vector. Thus in this case both the expansion scalar and the acceleration vector are zero. Summing up we can state the following result:

THEOREM 9.1. *If an $A(PS)_4$ spacetime satisfies cyclic Ricci tensor, the matter content is perfect fluid whose velocity vector field is the associated vector field of the spacetime, then the acceleration vector of the fluid must be zero and the expansion scalar also be so.*

10. Examples of $A(PS)_n$

In this section we prove the existence of $A(PS)_n$ with non-zero non-constant scalar curvature by constructing two non-trivial examples.

EXAMPLE 10.1. Let us consider a Lorentzian metric g on \mathbb{R}^4 by

$$(10.1) \quad ds^2 = g_{ij}dx^i dx^j = (x^4)^{\frac{4}{3}}[(dx^1)^2 + (dx^2)^2 + (dx^3)^2] - (dx^4)^2,$$

($i, j = 1, 2, 3, 4$). Here the signature of g is $(+, +, +, -)$ which is Lorentzian. Then the only non-vanishing components of the Christoffel symbols and the curvature tensors are

$$\Gamma_{14}^1 = \Gamma_{24}^2 = \Gamma_{34}^3 = \frac{2}{3x^4}, \quad \Gamma_{11}^4 = \Gamma_{22}^4 = \Gamma_{33}^4 = \frac{2}{3}(x^4)^{\frac{1}{3}},$$

$$R_{1441} = R_{2442} = R_{3443} = -\frac{2}{9(x^4)^{2/3}}$$

and the components obtained by the symmetry properties. The non-vanishing components of the Ricci tensors and covariant derivatives of the components of curvature tensors are

$$(10.2) \quad R_{11} = R_{22} = R_{33} = \frac{2}{9(x^4)^{2/3}}, \quad R_{44} = -\frac{2}{3(x^4)^2},$$

$$R_{1441,4} = R_{2442,4} = R_{3443,4} = \frac{4}{9(x^4)^{5/3}}.$$

It can be easily shown that the scalar curvature r of the resulting space (\mathbb{R}^4, g) is $r = \frac{4}{3(x^4)^2}$, which is non-vanishing and non-constant. We shall now show that \mathbb{R}^4 is an $A(PS)_4$. Let us consider the associated 1-forms as follows:

$$(10.3) \quad A_i(x) = \begin{cases} -\frac{1}{3x^4}, & \text{for } i = 4 \\ 0, & \text{otherwise,} \end{cases}$$

$$(10.4) \quad B_i(x) = \begin{cases} -\frac{1}{x^4}, & \text{for } i = 4 \\ 0, & \text{otherwise,} \end{cases}$$

at any point $x \in \mathbb{R}^4$. To verify the relation (1.1), it is sufficient to check the followings:

$$(10.5) \quad R_{1441,4} = (3A_4 + B_4)R_{1441},$$

$$(10.6) \quad R_{2442,4} = (3A_4 + B_4)R_{2442},$$

$$(10.7) \quad R_{3443,4} = (3A_4 + B_4)R_{3443},$$

since for the other cases (1.1) holds trivially. By (10.3) and (10.4) we get

$$\begin{aligned} \text{R.H.S. of (10.5)} &= (3A_4 + B_4)R_{1441} \\ &= \left(-\frac{1}{x^4} - \frac{1}{x^4}\right) \left(-\frac{2}{9(x^4)^{2/3}}\right) \\ &= \frac{4}{9(x^4)^{5/3}} = R_{1441,4} \quad (\text{by (10.2)}) \\ &= \text{L.H.S. of (10.5)}. \end{aligned}$$

By similar argument it can be shown that (10.6) and (10.7) are also true. Hence \mathbb{R}^4 equipped with the metric g , given in (10.1), is an $A(PS)_4$.

It is to be noted that (1.1) can be satisfied by a number of 1-forms A , B namely by those which fulfil (10.5), (10.6), (10.7). Thus we can state the following:

THEOREM 10.1. *Let (\mathbb{R}^4, g) be a 4-dimensional Lorentzian space endowed with the Lorentzian metric g given by*

$$ds^2 = g_{ij}dx^i dx^j = (x^4)^{\frac{4}{3}}[(dx^1)^2 + (dx^2)^2 + (dx^3)^2] - (dx^4)^2,$$

($i, j = 1, 2, 3, 4$). Then (\mathbb{R}^4, g) is an $A(PS)_4$ with non-zero and non-constant scalar curvature.

EXAMPLE 10.2. We define a Lorentzian metric g on \mathbb{R}^4 by

$$(10.8) \quad ds^2 = g_{ij} dx^i dx^j = (1 + 2q)[(dx^1)^2 + (dx^2)^2 + (dx^3)^2 - (dx^4)^2],$$

($i, j = 1, 2, 3, 4$), where $q = \frac{e^x}{k^2}$ and k is non-zero constant. Then the only non-vanishing components of the Christoffel symbols, the curvature tensors and the covariant derivatives of the components of curvature tensors are

$$(10.9) \quad \begin{aligned} \Gamma_{22}^1 &= \Gamma_{33}^1 = -\frac{q}{1+2q}, & \Gamma_{11}^1 &= \Gamma_{12}^2 = \Gamma_{13}^3 = \Gamma_{14}^4 = \Gamma_{44}^1 = \frac{q}{1+2q}, \\ R_{1221} &= R_{1331} = \frac{q}{1+2q}, & R_{1441} &= -\frac{q}{1+2q}, \\ R_{1221,1} &= R_{1331,1} = \frac{q(1-4q)}{(1+2q)^2}, & R_{1441,1} &= -\frac{q(1-4q)}{(1+2q)^2} \end{aligned}$$

and the components which can be obtained from these by symmetry properties. Using the above relations we find the non-vanishing components of the Ricci tensor are

$$R_{11} = \frac{3q}{(1+2q)^2}, \quad R_{22} = R_{33} = R_{44} = \frac{q}{(1+2q)^2}.$$

Also it can be easily shown that the scalar curvature of the resulting space (\mathbb{R}^4, g) is $r = \frac{4q}{(1+2q)^3}$, which is non-vanishing and non-constant. We shall now show that \mathbb{R}^4 is an $A(PS)_4$. Let us consider the associated 1-forms as follows:

$$(10.10) \quad A_i(x) = \begin{cases} \frac{1}{3(1+2q)}, & \text{for } i = 1 \\ 0, & \text{otherwise,} \end{cases}$$

$$(10.11) \quad B_i(x) = \begin{cases} -\frac{4q}{1+2q}, & \text{for } i = 1 \\ 0, & \text{otherwise,} \end{cases}$$

at any point $x \in \mathbb{R}^4$. To verify the relation (1.1), it is sufficient to check the followings:

$$(10.12) \quad R_{1221,1} = (3A_1 + B_1)R_{1221},$$

$$(10.13) \quad R_{1331,1} = (3A_1 + B_1)R_{1331},$$

$$(10.14) \quad R_{1441,1} = (3A_1 + B_1)R_{1441},$$

since for the other cases (1.1) holds trivially. By (10.10) and (10.11) we get

$$\begin{aligned}
 \text{R.H.S. of (10.12)} &= (3A_1 + B_1)R_{1221} \\
 &= \left(\frac{1}{1+2q} - \frac{4q}{1+2q}\right)\left(\frac{q}{1+2q}\right) \\
 &= \frac{q(1-4q)}{(1+2q)^2} = R_{1221,1} \quad (\text{by (10.9)}) \\
 &= \text{L.H.S. of (10.12).}
 \end{aligned}$$

By similar argument it can be shown that (10.13) and (10.14) are also true. Hence \mathbb{R}^4 equipped with the metric g , given in (10.8), is an $A(PS)_4$. Thus we can state the following:

THEOREM 10.2. *Let (\mathbb{R}^4, g) be a Lorentzian space endowed with the Lorentzian metric g given by*

$$ds^2 = g_{ij}dx^i dx^j = (1+2q)[(dx^1)^2 + (dx^2)^2 + (dx^3)^2 - (dx^4)^2],$$

($i, j = 1, 2, 3, 4$), where $q = \frac{e^x}{k^2}$ and k is non-zero constant. Then (\mathbb{R}^4, g) is an $A(PS)_4$ whose scalar curvature is non-zero and non-constant.

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STABLE MATCHINGS THROUGH FIXED POINTS AND GRAPHS

By

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Abstract. In this survey paper, we discuss some interconnections between fixed point theorems and the theory of stable matchings. Namely, we relate the bipartite matching problems to the Knaster–Tarski fixed point theorem and the nonbipartite ones to the Kakutani fixed point theorem. We study the natural lattice structure of stable matchings, and deduce some consequences of it, like linear characterizations of stable matching related polyhedra. Our approach to stable matching is “nonstandard” as it leans on graph-theoretic notions more than the “traditional” one.

1. Introduction

In the stable marriage problem of Gale and Shapley [31], there are n men and n women and each person ranks the members of the opposite gender by an arbitrary, strict preference order. A *marriage scheme* in this model is a set of marriages between different men and women. Such a scheme is *unstable* if there exist a man m and a woman w in such a way that m is either unmarried or m prefers w to his wife, and at the same time, w is either unmarried or prefers m to her partner. A marriage scheme is *stable* if it is not unstable, and a natural problem is finding a stable marriage scheme if it exists at all.

Nowadays, it is already folklore that for any preference rankings of the n men and n women, there exists a stable marriage scheme. This theorem was proved first by Gale and Shapley in [31]. They constructed a special stable marriage scheme with the help of a finite procedure, the so called

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deferred acceptance algorithm. It also turned out that for the existence of a stable scheme it is not necessary that the number of men is the same as the number of women or that for each person, all members of the opposite group are acceptable. Moreover, a natural modification of the deferred acceptance algorithm solves the more general stable admissions problem that allows polygamy on one side of the marriage market.

Although the paper of Gale and Shapley had a certain “recreational mathematics” flavour, later, the model turned out to be especially applicable both in theory and practice. Roth has observed in [49] that the two-sided market of medical students (intern candidates) and hospitals is fairly close to the stable admissions model of Gale and Shapley. Actually, in this particular segment of the economy, the “traditional” free competition approach failed to produce an equilibrium. The market has been stabilized only after a centralized program has been introduced that computes a stable admission scheme for volunteer participants. Roth lists several other practical applications of the framework on his homepage [48].

A most significant theoretical application of the stable marriage theorem is the solution of the Dinitz conjecture. Namely, Galvin proved that the list-chromatic index of any bipartite graph G equals the maximum degree Δ of G (see [33]). In other words, if for every edge of G there is a list of Δ possible colours then we can choose a colour for each edge from its list in such a way that adjacent edges receive different colours. The conjecture was open for fifteen years and the key idea in its ingenious elementary solution is the application of the Gale-Shapley theorem. Another theoretically interesting result is the observation that Pym’s linking theorem that proves that so called gammoids are matroids is an application of the stable marriage theorem (see [28] and Section 2.5 of this paper).

It seems that a vast amount of research on the theory of stable matchings was done by game theorists and economists. (For the state of art in the early 90’s, see the book of Roth and Sotomayor [54].) Thanks to this, we have a wide knowledge about the structure of stable matchings, much work was devoted to different generalizations, algorithmic and optimizational aspects. In this present work, we discuss a fairly novel approach to the theory of stable matchings. Our framework views stable matchings as fixed points of a certain set function and by this, we can generalize known facts about stable matchings and give simple proofs for different, seemingly unrelated theorems.

Probably, Feder [23] and Subramanian [61] were the first ones that pointed out a connection between stable matchings and fixed points. Both

of them were interested in the stable roommates problem (i.e. the nonbipartite generalization of the stable marriage problem). They reduced that to the so called network stability problem that is equivalent with the problem of deciding whether a certain set function on the edges has a fixed point. Subramanian has observed that in the bipartite (stable marriage) case there always exists a fixed point by the fixed point theorem of Tarski. Later, Adachi [5] reduced the stable marriage theorem and the lattice structure of stable matchings explicitly to Tarski's fixed point theorem. In this paper, we mainly follow the approach in [28] and describe a fixed point theorem based framework for stable matchings.

In Section 2, we define some basic notions and notations about stable matchings, recall Tarski's fixed point theorem and describe the basics of our framework in the bipartite case. We discuss questions that are related to multiple partner stable matchings in Section 2.1. Section 2.2 is devoted to the lattice structure of stable admissions and to point out a connection between path independent choice functions and monotone functions. We generalize the lattice operations on stable matchings in Section 2.3 and discuss some consequences. In particular, we prove a generalization of a theorem of Teo and Sethuraman (see Theorem 21) based on the lattice structure. Section 2.4 surveys linear descriptions of bipartite stable matching related polyhedra and Section 2.5 contains results on linking graph paths that are related to stable marriages. The nonbipartite version of the stable matching problem is studied in Section 3. In Section 3.1, we describe Irving's generalization of the Gale-Shapley algorithm to find a stable matching in the nonbipartite case. We introduce stable half-matchings in Section 3.2 and characterize by them the existence of a stable matching. We link this to Scarf's lemma and fixed points. Our last topic covers some extensions of the stable roommates problem in Section 3.3, some of them are tractable, others are not.

2. The bipartite case: stable marriages

Let G be a bipartite graph (parallel edges are allowed) with colour classes M and W and let E denote the set of edges of G . (It might be convenient to think that vertices of M and W represent men and women, respectively, and edges are along possible marriages.) For vertex v of G , let $E(v)$ denote the set of edges incident with v . (Note that $E(v)$ might not be finite, even if $V(G)$ is finite). We call (G, \emptyset) a *bipartite preference system* if graph G is as above and \emptyset is a set of orders $<_v$ for $v \in M \cup W$, such that $<_v$ is a well-order on $E(v)$. We say that edge e of G *M -dominates (W -dominates)* edge f of G

if e and f are incident with the same vertex m of M (w of W) and $e <_m f$ ($e <_{wf} f$). Edge e *dominates* edge f if e M -dominates or W -dominates f . Set F of edges M -dominates (W -dominates) edge set H if each edge h of H is M -dominated (W -dominated) by some edge f of F .

Let (G, \emptyset) be a bipartite preference system. Subset S of E is a *stable matching* if no edge of S is dominated by S , and S dominates $E \setminus S$. Note that any stable matching is necessarily a matching (i.e. a set of disjoint edges), as if two edges of a stable matching shares a vertex then one of them would dominate the other.

THEOREM 1. (Gale and Shapley 1962 [31]) *For any finite bipartite preference system (G, \emptyset) , there exists a stable matching.*

The original proof of Theorem 1 is a construction of a special stable matching with the so called deferred acceptance algorithm. This is an iterative procedure that alternatively repeat two steps. It starts with a proposal step in which each vertex of M selects its most preferred edge. In the next refusal step, certain edges that have been selected in the proposal step, get deleted. Namely, m 's edge e gets deleted, if there is another edge f selected by some other vertex with $f <_w e$ for some vertex w of W . If no edge is deleted in a refusal step then output the edges selected in the last proposal step. Otherwise start the procedure all over for the reduced bipartite preference system that we get after the deletions of the refusal step. Gale and Shapley have proved that this deferred acceptance algorithm constructs a stable matching of (G, \emptyset) which is “man-optimal”, that is, no edge that has been deleted in a refusal step can be in any stable matching, or equivalently, each M -vertex gets the best possible partner and each W -vertex receives the worst possible partner that he/she can have in a stable matching.

In the literature, the usual setting and the definition of a stable matching is somewhat different from ours. We describe that terminology as well, so as to have a “dictionary” between the two languages. If G is a graph, a *matching* N of G is a set of disjoint edges of G . Equivalently, we can regard a matching as an involution on the vertices of G , that is, a function $\mu: V(G) \rightarrow V(G)$ in such a way that $\mu(\mu(v)) = v$ for each vertex v of G . Indeed, if uv is an edge of matching N then $\mu(u) = v$ and $\mu(v) = u$, and if w is not covered by N then $\mu(w) = w$ for the corresponding involution. From involution μ , we can also reconstruct matching N as $uv \in N$ if and only if $\mu(u) = v$.

The conventional counterpart of a bipartite preference system is the following. We have given disjoint sets M and W . For vertex m of M (w of

(W) , let \prec_m (\prec_w) be a linear order on $W \cup \{m\}$ ($M \cup \{w\}$). In this model, matching μ is a stable matching if

- (1) $\mu(v) \leq_v v$ for each vertex v of $M \cup W$ and
- (2) for all $m \in M$ and $w \in W$, $w \prec_m \mu(m)$ implies $\mu(w) \prec_w m$.

That is, a matching is stable, if each person (vertex) is not worse off by his/her partner than by remaining single (or, in other words, matching μ is *individually rational*) and whenever man m (i.e. a vertex in the M side) prefers woman w to his eventual partner, then w must prefer $\mu(w)$ to the partnership with m . Note that traditionally, the preference orders are on the vertices, and not on the edges. (This makes the traditional model in some sense richer than ours, as it allows that a man m can accept woman w but w prefers to remain single to marry m). For simple graphs however, preference order \prec_v induces a linear order $<_v$ on $E(v)$, so a stable matching in our sense corresponds to a stable matching μ in the latter sense. Our terminology is more general in the sense that it readily allows graphs with parallel edges in a bipartite preference system. This feature is useful if we use a model where the vertices correspond to firms and workers, so parallel edges between a firm and a worker may mean different wages in case of an employment.

Let us now return to our terminology. If S is a stable matching then we can partition E into three disjoint parts as $E = S_M \cup S \cup S_W$, in such a way that S_M (and S_W) is M -dominated (W -dominated, respectively) by S . (Note that such a partition is unique only if no edge of $E \setminus S$ is both M - and W -dominated by S .) From here, it is easy to see that sets $S^M := S \cup S_M$ and $S^W := S \cup S_W$ have the following properties.

- (3) $S^M \cup S^W = E$
 - (4) no edge of $S^M \cap S^W$ dominates another edge of $S^M \cap S^W$
- (S^M) is M -dominated by $S^M \cap S^W$ and S^W is W -dominated by $S^M \cap S^W$

For convenience, we call (S^M, S^W) a *stable pair* if S^M and S^W have the above properties (3,4,5). Clearly, if (S^M, S^W) is a stable pair then $S := S^M \cap S^W$ is a stable matching. Equivalently, we can say that pair (S^M, S^W) is stable if besides (3) we have

$$(6) \quad \mathcal{C}_M(S^M) = S^M \cap S^W = \mathcal{C}_W(S^W),$$

where $\mathcal{C}_M(S^M)$ (and $\mathcal{C}_W(S^W)$) denotes those elements of S^M (and S^W) that are not M -dominated (and W -dominated) by other elements of S^M (S^W). This means that (S^M, S^W) is a stable pair if and only if

$$(7) \quad S^W = E \setminus (S^M \setminus \mathcal{C}_M(S^M)) \text{ and}$$

$$(8) \quad S^M = E \setminus (S^W \setminus \mathcal{C}_W(S^W))$$

holds. By substituting (7) into (8), we get that (S^M, S^W) is a stable pair if and only if (7) holds with

$$(9) \quad f(S^M) := E \setminus \left[\left(E \setminus [S^M \setminus \mathcal{C}_M(S^M)] \right) \setminus \mathcal{C}_W \left(E \setminus [S^M \setminus \mathcal{C}_M(S^M)] \right) \right] = S^M.$$

A key observation in our treatment that function f is *monotone*, that is, $f(A) \subseteq \subseteq f(B)$ whenever $A \subseteq B \subseteq E$. (To see this, it is useful to observe that function $A \mapsto A \setminus \mathcal{C}(A)$ is monotone for $\mathcal{C} = \mathcal{C}_M$ and $\mathcal{C} = \mathcal{C}_W$.) Thus we can invoke the Knaster–Tarski fixed point theorem.

THEOREM 2. (Knaster and Tarski 1928 [42]) *Iff: $2^E \rightarrow 2^E$ is a monotone function then f has a fixed point.*

As a consequence of Theorem 2, the function f that comes from the bipartite preference model according to (9) has a fixed point S^M . Using (7) as a definition for S^W , we can construct a stable pair (S^M, S^W) from the above fixed point, and a stable matching S from stable pair (S^M, S^W) . This proves the existence part of Theorem 1. What is more interesting, than this single reduction is that for finite ground sets, there is a most simple algorithmic proof of Theorem 2 that can be applied in our construction. (Note that in Theorem 2, E can be an arbitrarily large set.) Namely, $f(\emptyset) \subseteq f(f(\emptyset)) \subseteq \subseteq f(f(f(\emptyset))) \subseteq \dots$ by the monotonicity of f , so if E is finite then this increasing chain has to get stabilized at some fixed point A of f . This fixed point A is contained in any fixed point of f by the procedure. Similarly, chain $f(E) \supseteq f(f(E)) \supseteq f(f(f(E))) \supseteq \dots$ must get stabilized at the inclusionwise maximal fixed point of f . It turns out that the deferred algorithm itself is essentially the iteration of f starting from E . If we exchange the role of M and W then the deferred acceptance algorithm concludes with the woman-optimal stable matching that we can construct by iterating f starting from the \emptyset . This observation is enough to prove that the deferred acceptance algorithm outputs the man-optimal stable matching.

2.1. Multiple partner matchings

The stable marriage theorem of Gale and Shapley (Theorem 1) has several extensions and generalizations that nicely fit into our framework. We start with the stable admissions problem described in [31]. We have a bipartite preference system (G, \emptyset) and a function $b: M \cup W \rightarrow \mathbb{N}$ with $b(w) = 1$ for all $w \in W$. Set F of edges (M, b) -dominates ((W, b) -dominates) edge e of G if there is a vertex m of M (w of W) and different edges $f_1, f_2, \dots, f_{b(m)}$ ($f_1, f_2, \dots, f_{b(w)}$) of F so that $f_i <_m e$ for $i = 1, 2, \dots, b(m)$ ($f_i <_w e$ for $i = 1, 2, \dots, b(w)$). Set F of edges b -dominates edge set H if each edge h of H is (M, b) -dominated or (W, b) -dominated by F . Subset S of the edges of G is a *stable admission scheme* if no edge of S dominates another edge of S , and S dominates $E \setminus S$. By definition, each vertex in W is incident with at most one edge and vertex m of M is incident with at most $b(m)$ edges of a stable admission scheme. (The story is, that vertices of M represent colleges that offer admission, W stands for the set of students that look for admission in a college, and b is the quota of the colleges. We look for a stable admission scheme in which no college-student pair mutually prefer each other to their assignments.)

The same argument that we gave for the stable marriage problem goes through for the above admissions case, even in the more general case in which we do not require that $b(w) = 1$ for $w \in W$. (The generalization of a stable admission scheme to this case we call a *stable b-matching*.) The iteration of the corresponding monotone function describes the modified deferred acceptance algorithm that finds the optimal stable admissions. This implies the following theorem.

THEOREM 3. *For any bipartite preference system (G, \emptyset) and $b: V(G) \rightarrow \rightarrow \mathbb{N}$ there exists a stable b-matching. If M and W are the colour classes of G then there is an M -optimal stable b-matching S , in which each vertex m of M is incident with the most preferred $b(m)$ edges of $E(m)$ that can be present a stable b-matching. Simultaneously, each vertex w of W is incident with the least preferred $b(w)$ edges of $E(w)$ that can appear in a stable b-matching.*

We can generalize the notions of stable matchings and stable admissions. Let $\mathcal{C}_M, \mathcal{C}_W: 2^E \rightarrow 2^E$ be set functions. Pair (S^M, S^W) is a $\mathcal{C}_M\mathcal{C}_W$ -stable pair if (3,6) holds. Subset S of E is a $\mathcal{C}_M\mathcal{C}_W$ -stable set if S dominates exactly the elements of $E \setminus S$, that is, if

$$(10) \quad \mathcal{C}_M(S) = \mathcal{C}_W(S) = S \text{ and}$$

$$(11) \quad \mathcal{C}_M(S \cup \{e\}) = S \text{ or } \mathcal{C}_W(S \cup \{e\}) = S \text{ for any element } e \text{ of } E$$

What are the crucial properties of a dominance function \mathcal{C} that make our argument work? These are that

$$(12) \quad \mathcal{C}(A) \subseteq A \text{ for any } A$$

and that function

$$(13) \quad \overline{\mathcal{C}}(A) := A \setminus \mathcal{C}(A) \text{ is monotone.}$$

We call function \mathcal{C} *comonotone* if (12, 13) hold. These properties imply that function f defined in (9) is monotone and there is a stable pair. Still, it might happen that \mathcal{C}_M and \mathcal{C}_W are comonotone so there is a stable pair (S^M, S^W) , but no stable set exists. However, $S := S^M \cap S^W$ is a stable set if \mathcal{C}_M and \mathcal{C}_W have the additional property that

$$(14) \quad \mathcal{C}(A) = \mathcal{C}(B) \text{ whenever } \mathcal{C}(A) \subseteq B \subseteq A.$$

This is claimed in the following theorem.

THEOREM 4. (Fleiner 2000 [28]) *If $\mathcal{C}_M, \mathcal{C}_W: 2^E \rightarrow 2^E$ are comonotone set functions then there exists a $\mathcal{C}_M\mathcal{C}_W$ -stable pair (S^M, S^W) . If, moreover, both \mathcal{C}_M and \mathcal{C}_W have property (14) then there exists a $\mathcal{C}_M\mathcal{C}_W$ -stable set S .*

We give two more interesting examples of comonotone functions with property (14), hence extend the stable marriage theorem in two different directions. A *partial well-order* is a partial order so that each subset of the ground set has a minimal element in the induced order.

OBSERVATION 5. *If $<$ is a partial order on E and $\mathcal{C}(A)$ denotes the set of $<$ -minima of A for subset A of E then \mathcal{C} is comonotone. If $<$ is a partial well-order then \mathcal{C} has property (14).*

Theorem 4 and Observation 5 implies the following property of partial orders.

COROLLARY 6. (see Fleiner [28, 24, 26]) *If $<_1$ and $<_2$ are partial orders on E then there are subsets E_1, E_2 of E such that*

$$E_1 \cup E_2 = E \text{ and}$$

$E_1 \cap E_2$ is the set of $<_1$ -minima of E_1 and the set of $<_2$ -minima of E_2

If both $<_1$ and $<_2$ are partial well-orders then there is a common antichain S of $<_1$ and $<_2$ (i.e. no two elements of S are comparable in any of the orders) so that for any $e \in E$ there is an $s \in S$ with $s <_1 e$ or $s <_2 e$.

OBSERVATION 7. *Let $<$ be a linear order on finite set E , and \mathcal{M} be a matroid on E . Denote by $\mathcal{C}_{\mathcal{M}}^<(A)$ the output of the greedy algorithm on input*

A, where the algorithm goes through the elements of A in the order given by $<$. Then $\mathcal{C}_M^<$ is comonotone and has property (14).

Theorem 4 and Observation 7 extends the stable matching theorem to matroids.

COROLLARY 8. (see Fleiner [28, 24, 26]) *Let $<_1$ and $<_2$ be linear orders on finite set E, and M_1 and M_2 be matroids on E. Then there is a common independent set S of M_1 and M_2 so that for each element e of E we have*

$$e \in \text{span}_i \{s \in S : s \leq_i e\} \text{ for some } i \in \{1, 2\}.$$

There also is a common spanning set T of M_1 and M_2 such that for any element t of T, subset T is the lexicographically $<_i$ -minimal spanning subset of M_i that contains $T \setminus \{t\}$ for some $i \in \{1, 2\}$.

See [24, 28] for a reduction of the stable *b*-matching theorem to the first part of Corollary 8.

2.2. The lattice of stable sets and path independent choice functions

The marriage model of Gale and Shapley has attracted some interest in the theory of social choice. There are related results to the ones that we discussed so far. Again, the terminology is different from ours, so in this section we attempt to cover some basics and point out an interesting connection of this approach to the monotone function based framework.

Let (G, \emptyset) be a bipartite preference system and let F and W be the two colour classes of G . We will identify vertices of F with different firms and the vertices of W with workers. Edge fw of G represents that firm f considers w as a potential employee and worker w can accept f as an employer. Firms would like to get certain specific jobs to be done, and this is why they have a more complex preference function on workers than plain ranking. Namely, each firm f has a so called choice function \mathcal{C}_f that selects from any set W' of workers a subset $\mathcal{C}_f(W')$ of W' that firm f would hire if on the labour-market only firm f and workers of W' would be present. Set function $\mathcal{C}: 2^W \rightarrow 2^W$ is a *choice function* if there is a well-order $<$ on 2^W such that $\mathcal{C}(W')$ is the $<$ -minimal subset of W' , for any subset W' of W .

Continuing on a paper of Crawford and Knoer [20], Kelso and Crawford [40] extended the admissions model to a model where each firm has a choice function and each worker has an ordinary preference ranking on firms.

An assignment of workers to firms is called *stable* if it is not blocked by a worker-firm pair. Worker-firm pair (w, f) *blocks* an assignment if w prefers f to his/her employer and in the meanwhile firm f would take worker w if w would be available (that is $w \in \mathcal{C}_f(W_f \cup \{w\})$, where W_f is the set of workers assigned to firm f).

Note that in the above fairly general model there might be no stable assignment. However, if each choice function has the so-called substitutability property, then a stable assignment always exists. We say that set function $\mathcal{C}_f: 2^W \rightarrow 2^W$ of firm f has the *property of substitutability*, if

$$(15) \quad w \in \mathcal{C}_f(W') \text{ implies } w \in \mathcal{C}_f(W' \setminus \{w'\})$$

for any subset W' of the set W of workers and for any two different workers w, w' of W' . This means that if a firm would like to employ a certain worker, then it still would like to hire him/her if some other worker leaves the labour-market.

THEOREM 9. (Kelso-Crawford 1982 [40]) *If each firm's preference is a substitutable choice function in the worker-firm assignment model, then there is a stable assignment of workers to firms.*

The proof of Crawford and Kelso is via the accordingly modified deferred acceptance algorithm. They also observed that firm-proposing results in the firm-optimal assignment, and the worker-proposal based method leads to the worker-optimal situation. In [50, 51], Roth extended Theorem 9 to a many-to-many model. A *stable assignment* is a bipartite assignment graph A with colour classes F and W , such that for any $w \in W$ and $f \in F$ we have that $wf \in E(A)$ if and only if $f \in \mathcal{C}_w(\Gamma_A(w) \cup f)$ and $w \in \mathcal{C}_f(\Gamma_A(f) \cup w)$. (Notation $\Gamma_A(w)$ stands for the neighbours of w in A .)

THEOREM 10. (Roth 1984 [50, 51]) *Let F and W be disjoint finite sets, and for each $f \in F$ and $w \in W$ let $\mathcal{C}_w: 2^F \rightarrow 2^F$ and $\mathcal{C}_f: 2^W \rightarrow 2^W$ be choice functions with substitutability property (15). Then there is a stable assignment in the model.*

Clearly, the stable marriage theorem of Gale and Shapley is a special case of Theorem 10, where the choice functions simply select the highest ranked partner from the input. For the college model, the choice function of college c selects the best $b(c)$ choices.

A key observation in this section is that a choice function trivially has properties (12) and (14), and a little effort shows that for finite ground sets

substitutability property (15) implies property (13), so choice functions in Theorems 9 and 10 are comonotone. A fairly trivial construction shows that the above theorems are special cases of Theorem 4. Thus, our monotone framework is relevant for these results so we can prove that there always exists a stable set, that is, a stable assignment.

In [51], Roth studied three models: the one-to-one, the many-to-one and the many-to-many with substitutable preferences. He showed that for all three models there is a firm-optimal, “worker-pessimal” and a worker-optimal, “firm-pessimal” stable assignment. The name “polarization of interests” refers to this observation. Further on, Roth introduced the notion of the *consensus property*, that means the following. If each agent on one side of the market combines his/her most preferred assignment from a set of stable assignments, then this way another stable assignment is constructed. This is a generalization of the lattice property of stable schemes in the marriage model. (The observation that stable marriages in the marriage model have a natural lattice structure is attributed to John Conway.) Unfortunately, this property does not always hold in Theorem 10. In [51], Roth asked whether some lattice structure can still be defined on stable assignments. Blair answered this question positively in [14]. His idea was that instead of defining lattice operations, he introduced a more or less natural partial order on stable assignments. This partial order turned out to be a lattice order hence one can generalize the lattice property of bipartite stable matchings.

THEOREM 11. (Blair 1988 [14]) *Let F be a set of firms and W be a set of workers. Let, for each $f \in F$ ($w \in W$), set function $\mathcal{C}_f: 2^W \rightarrow 2^W$ ($\mathcal{C}_w: 2^F \rightarrow 2^F$) be given with properties (12, 14, 15). Let \mathcal{A} be the set of stable assignments of the model, and define for $A_1, A_2 \in \mathcal{A}$*

$$A_1 < A_2 \text{ if } \Gamma_{A_1}(f) = \mathcal{C}_f(\Gamma_{A_1}(f) \cup \Gamma_{A_2}(f))$$

holds for each firm f . (That is, each firm would choose A_1 if all choices provided by A_1 and A_2 would be offered.) Then \mathcal{A} is nonempty and $<$ is a lattice order, that is, any two stable assignments have a $<$ -maximal lower bound and a $<$ -minimal upper bound.

It seems that Theorem 11 is not general enough for the following example where choice functions do not have property 14.

In Hungary, university admissions are determined with the scores of the applicants. Applicants have strict rankings on their applications. Each application belongs to some applicant and describes a certain university and a subject. We call these university-subject pairs *colleges*. Each college assigns

to its applicants certain scores based on the applicants' previous grades and entrance exam results. Different applicants may have the same score at the same college and an applicant may have different scores at different colleges.

Each college has a strict quota on the number of its admitted students. After all information (that is rankings and scores) is known, each college has to declare a certain score limit. This score limit has to be *feasible*, that is if each applicant is admitted to the first college in her ranking where her score is above the limit then no college exceeds its quota. We say that a score limit is *stable* if it is feasible but decreasing the quota of any college results in a non-feasible score limit. The *SSL (stable score limit) problem* is described by the applicants, colleges, applicant rankings and scores and the task is the determination a stable score limit. For the details see Biró [12].

We can define choice functions for applicants and for colleges in the above model. Namely, the choice function C_a of applicant a selects from any set of colleges the first college in a 's ranking. If we fix a college c and a subset A of the applicants then choice function of college c selects subset $C_c(A)$ of A the following way. College c calculates the lowest score s such that the number of applicants in A with score above s is not exceeding the quota of c . Then the choice set $C_c(A)$ is the set of applicants of A with score more than s . It is not difficult to see that a stable assignment with these choice functions is exactly an admission scheme that is determined by a stable score limit.

Clearly, both the applicants' and colleges' choice functions have properties (12,15), and applicants choice function has the (14) property as well that does not hold for the colleges' function. A recent result of Jankó [38] shows that in spite of this, stable core limits form a lattice for the following partial order. We say that a score limit S is higher than score limit S' if for each college c , S assigns a greater or equal score limit to c than S' does.

THEOREM 12. (Jankó 2009 [38]) *If S_1 and S_2 are stable score limits in the SSL problem then there is unique lowest stable score limit $S_1 \vee S_2$ that is higher than S_1 and S_2 and there is a unique highest score limit $S_1 \wedge S_2$ that is lower than S_1 and S_2 .*

Theorem 12 implies that there is a unique highest stable score limit that describes the college-optimal admission scheme and a unique lowest stable score limit that is student-optimal and admits the maximum set of students.

We have mentioned earlier that (15) implies (13) for functions on finite ground sets, so the functions that describe the choice of the agents of the

market are comonotone with property (14). To prove the lattice property in our framework, we have to go back to the roots, that is to the fixed point theorem. Recall that we have cited the Knaster–Tarski fixed point theorem on monotone set functions. This theorem is often attributed to Tarski for the reason that 27 years after Knaster’s French paper in a Polish journal, he published a lattice theoretical generalization of the result in English in a much easier reachable journal [64]. (Actually, Tarski has also formulated a corollary of the fixed point theorem there in terms of Boolean algebras that is more general than Theorem 4.) Note that the proof of the set function theorem is just as difficult as the proof of the lattice generalization, but in the latter paper Tarski has explicitly stated the lattice property of fixed points that we need now.

If L is a lattice on E with partial order $<$ and lattice operations \wedge, \vee then function $f: E \rightarrow E$ is *monotone* if $a < b$ implies $f(a) < f(b)$. Lattice L' is a *sublattice* of L if its ground set E' is a subset of E and the lattice operations on L' are the original operations \wedge and \vee restricted to E' . Lattice L' is a *lattice subset* of L if its ground set E' is a subset of E and the lattice order on L' is the restriction of $<$ to E' . Lattice L is complete if any subset E' of its ground set has a meet (greatest lower bound) $\wedge E'$ and a join (least upper bound) $\vee E'$. (In particular, these lattices have a minimal and maximal element.) Note that any finite lattice is complete.

THEOREM 13. (Tarski 1955 [64]) *Let L be a complete lattice on ground set E and $f: E \rightarrow E$ be a monotone function. Then the fixed points of f form a nonempty complete lattice subset of L .*

It turns out that Theorem 13 is relevant in the setting of Theorem 11 and it implies that stable assignments exist and form a lattice as described. For the details, see [24]. In the next section, we discuss a property that implies that the lattice subset in Theorem 13 is a sublattice. If that is the case then stable pairs have a very rich structure that allows us to prove further results.

We finish this section with pointing out a connection between properties of set functions we have used so far. Set function $\mathcal{C}: 2^E \rightarrow 2^E$ is *path independent* if

$$(16) \quad \mathcal{C}(A \cup B) = \mathcal{C}(\mathcal{C}(A) \cup \mathcal{C}(B)) \text{ for any subsets } A, B \text{ of } E$$

The central notion of path independence has been introduced by Plott in 1973 [44] into the theory of social choice. Our observation is the following.

THEOREM 14. *Set function $\mathcal{C}: 2^E \rightarrow 2^E$ is path independent with property (12) if and only if it is comonotone and has property (14).*

LEMMA 15. (see [28, 24, 26]) *Set function $\mathcal{C}: 2^E \rightarrow 2^E$ is comonotone if and only if \mathcal{C} has property (12) and*

$$(17) \quad \mathcal{C}(B) \cap A \subseteq \mathcal{C}(A) \text{ whenever } A \subseteq B \subseteq E$$

PROOF. If (12) holds for \mathcal{C} then (17) is equivalent with the monotonicity of $\overline{\mathcal{C}}$. ■

PROOF. (Proof of Theorem 14) Assume that \mathcal{C} is path independent. Then

$$\mathcal{C}(A) = \mathcal{C}(A \cup A) = \mathcal{C}(\mathcal{C}(A) \cup \mathcal{C}(A)) = \mathcal{C}(\mathcal{C}(A)),$$

hence for $\mathcal{C}(A) \subseteq B \subseteq A$

$$\begin{aligned} \mathcal{C}(B) &= \mathcal{C}(\mathcal{C}(A) \cup B) = \mathcal{C}(\mathcal{C}(\mathcal{C}(A)) \cup \mathcal{C}(B)) = \\ &= \mathcal{C}(\mathcal{C}(A) \cup \mathcal{C}(B)) = \mathcal{C}(A \cup B) = \mathcal{C}(A), \end{aligned}$$

showing (14). For $A \subseteq B$ from (12) we get

$$\begin{aligned} \mathcal{C}(B) &= \mathcal{C}(A \cup (B \setminus A)) = \\ &= \mathcal{C}(\mathcal{C}(A) \cup \mathcal{C}(B \setminus A)) \subseteq \mathcal{C}(A) \cup \mathcal{C}(B \setminus A) \subseteq \mathcal{C}(A) \cup (B \setminus A), \end{aligned}$$

implying

$$\mathcal{C}(B) \cap A \subseteq (\mathcal{C}(A) \cup (B \setminus A)) \cap A = \mathcal{C}(A),$$

so \mathcal{C} is comonotone by Lemma 15.

Let now \mathcal{C} be comonotone with property (14) and A and B be subsets of E . From (17) and (12), we get that

$$\begin{aligned} (18) \quad \mathcal{C}(A \cup B) &= (\mathcal{C}(A \cup B) \cap A) \cup (\mathcal{C}(A \cup B) \cap B) \subseteq \\ &\subseteq \mathcal{C}(A) \cup \mathcal{C}(B) \subseteq A \cup B, \end{aligned}$$

and (14) and (18) implies (16). ■

We give an example indicating that comonotonicity and (14) is necessary in Theorem 14. That is, there is a function with properties (12) and (14) that is not path independent.

EXAMPLE 16. Let $|E| > k \geq 1$, fix element x of E and define $\mathcal{C}: 2^E \rightarrow 2^E$ by

$$\mathcal{C}(A) = \begin{cases} A & \text{if } |A| > k \\ A \setminus \{x\} & \text{if } |A| \leq k. \end{cases}$$

Then \mathcal{C} has properties (12,14) but it is neither path independent nor comonotone.

Example 16 and Theorem 14 together imply that there exists a comonotone function that is not path independent.

2.3. The stable matching lattice

In this section, we discuss the situation that the lattice subset of stable pairs in Theorem 4 and the lattice subset of fixed points in Theorem 13 are both sublattices. For a finite ground set E , we call function $f: 2^E \rightarrow 2^E$ *strongly monotone* if f is monotone with property

$$(19) \quad |f(A \cup \{e\})| \leq |f(A)| + 1 \text{ for any subset } A \text{ and element } e \text{ of } E.$$

Set function $\mathcal{C}: 2^E \rightarrow 2^E$ is *increasing**if

$$(20) \quad A \subseteq B \subseteq E \text{ implies } |\mathcal{C}(A)| \leq |\mathcal{C}(B)|.$$

Note that $|\mathcal{C}_M^<(A)| = \text{rank}(A)$ for comonotone function $\mathcal{C}_M^<$ described in Observation 7, hence $\mathcal{C}_M^<$ is increasing.

First we give a sufficient condition for a monotone function on subset lattices so that the lattice subset of its fixed points is a sublattice. The interested reader may find the details of an even more general treatment in [26].

THEOREM 17. (see [28, 24, 26]) *If $f: 2^E \rightarrow 2^E$ is a strongly monotone function on finite set E , then fixed points of f form a nonempty sublattice of $(2^E, \cap, \cup)$.*

The following Lemma is a link between strongly monotone and increasing comonotone functions.

LEMMA 18. (see [28, 24, 26]) *If function $\mathcal{C}: 2^E \rightarrow 2^E$ is increasing and comonotone then $\overline{\mathcal{C}}$ is strongly monotone.*

Based on Lemma 18, we can give a sufficient condition for the property that stable pairs in Theorem 4 form a sublattice. Note that independently from our work, Alkan [9] has also found Theorem 19 (see also [32]). He used the name *cardinal monotonicity* for our increasing notion.

* Independently of us, Hatfield and Milgrom also observed that this property is key to prove the sublattice property of stable sets [34]. They call the same notion “law of aggregate demand”.

THEOREM 19. (Alkan 2000 [9], Fleiner 2000 [28]) *If E is finite and $\mathcal{C}_M, \mathcal{C}_W: 2^E \rightarrow 2^E$ are increasing, comonotone functions then $\mathcal{C}_M\mathcal{C}_W$ -stable sets have the same cardinality and form a nonempty, complete lattice with lattice operations $S_1 \vee S_2 := \mathcal{C}_M(S_1 \cup S_2)$ and $S_1 \wedge S_2 := \mathcal{C}_W(S_1 \cup S_2)$. Moreover, $S_1 \cap S_2 = (S_1 \wedge S_2) \cap (S_1 \vee S_2)$ and $S_1 \cup S_2 = (S_1 \wedge S_2) \cup (S_1 \vee S_2)$, or equivalently,*

$$(21) \quad \chi(S_1) + \chi(S_2) = \chi(S_1 \wedge S_2) + \chi(S_1 \vee S_2)$$

holds for any two $\mathcal{C}_M\mathcal{C}_W$ -stable sets S_1, S_2

Theorem 19 can be regarded as a generalization of the “consensus property” observed by Roth in [51]. Namely, from Theorem 19, it follows that if \mathcal{S} is a set of $\mathcal{C}_M\mathcal{C}_W$ -stable sets then $\mathcal{C}_M(\bigcup \mathcal{S})$ (the *first choice of \mathcal{C}_M from \mathcal{S}*) is a $\mathcal{C}_M\mathcal{C}_W$ -stable set.

But more is true. Let us denote by \mathcal{S}_i^M the *i*th choice of \mathcal{C}_M from \mathcal{S} defined as the first choice of \mathcal{C}_M from the support of

$$\sum_{S \in \mathcal{S}} \chi(S) - \sum_{j=1}^{i-1} \chi(\mathcal{S}_j^M).$$

If \mathcal{S} is a chain of k $\mathcal{C}_M\mathcal{C}_W$ -stable sets then $\mathcal{S}_i^M \in \mathcal{S}$ for $i = 1, 2, \dots, k$. Otherwise, there are two incomparable stable sets of \mathcal{S} , say S_1 and S_2 of \mathcal{S} that we can exchange into $S_1 \wedge S_2$ and $S_1 \vee S_2$. By (21), this uncrossing operation does not change $\sum_{S \in \mathcal{S}} \chi(S)$. If we apply a sequence of uncrossing operations on \mathcal{S} then we can transform collection \mathcal{S} into a chain of $\mathcal{C}_M\mathcal{C}_W$ -stable sets in finite number of steps (see [28] for the details). As the uncrossing steps do not change $\sum_{S \in \mathcal{S}} \chi(S)$, this chain must be the chain of the *i*th choices of \mathcal{C}_M . The above argument also holds for \mathcal{C}_W and gives the following.

THEOREM 20. *If E is a finite ground set, $\mathcal{C}_M, \mathcal{C}_W: 2^E \rightarrow 2^E$ are increasing comonotone functions and \mathcal{S} is a set of n (not necessarily different) $\mathcal{C}_M\mathcal{C}_W$ -stable sets then the *i*th choice of \mathcal{C}_M from \mathcal{S} is a $\mathcal{C}_M\mathcal{C}_W$ -stable set and coincides with the $(n+1-i)$ th choice of \mathcal{C}_W from \mathcal{S} .*

Theorem 20 generalizes the following nice structural result of Teo and Sethuraman on stable matchings. The original proof used linear programming tools.

THEOREM 21. (Teo and Sethuraman 1998 [65]) *Let (G, \emptyset) be a bipartite preference system with colour classes M and W of G and let S_1, S_2, \dots, S_n be (not necessarily different) stable matchings. For each vertex v of $M \cup W$*

order edges of $\bigcup_{i=1}^n S_i \cap E(v)$ as $e_v^1 \leq_v e_v^2 \leq_v \dots$ in such a way that each edge is listed as many times as it appears in an S_i . (So the length of this chain is the number of S_i 's that cover v .) Then

$$S_M^k := \{e_m^k : m \in M\} \text{ and } S_W^k := \{e_w^k : w \in W\}$$

are stable matchings for $k \in \{1, 2, \dots, n\}$ and $S_M^k = S_W^{n+1-k}$.

We sketch a short direct proof of this result. We need the consequence of Theorem 19 that if each man chooses his best partner from a set of stable marriages then this induces a stable matching scheme in which each woman receives the worst husband from the given set.

PROOF. For vertex v of G and stable matchings S and S' let $S \preceq_v S'$ denote that v prefers S to S' . List S_1, S_2, \dots, S_n as $S_v^1 \preceq_v S_v^2 \preceq_v \dots \preceq_v S_v^n$ for each vertex v . Observe that $S_M^k = \bigwedge_{m \in M} \bigvee_{i=1}^k S_m^i$ and $S_W^k = \bigwedge_{w \in W} \bigvee_{i=1}^k S_w^i$, Chains $S_M^1, S_M^2, \dots, S_M^n$ and $S_W^1, S_W^2, \dots, S_W^n$ are opposite, hence $S_M^k = S_W^{n+1-k}$. ■

With the help of Theorem 22, it is straightforward to generalize the above direct proof of Theorem 21 to stable b -matchings. Independently of us, after our result, this was done by Klaus and Klijn in [41], for the many-to-one case, i.e. when $b \equiv 1$ on one colour class of the underlying graph. Klaus and Klijn have a little different proof: they simply join the meets of all k -subsets of the given set of stable matchings.

In what follows, we point out the so called splitting property of stable b -matchings. To this end, we use the generalization of the Comparability Theorem of Roth and Sotomayor [53] to the many-to-many model by Baïou and Balinski. The Comparability Theorem states that in a fixed bipartite preference system, if two stable b -matchings are different for some agent a , then a strictly prefers one b -matching to the other (that is, a would choose one of the b -matchings if all options of the two b -matchings were offered). For a short and direct proof see [27].

THEOREM 22. (Baïou and Balinski 2000 [10]) *Let S and S' be two stable b -matchings for bipartite preference system (G, \emptyset) , let v be a vertex of graph G and $S_v := S \cap E(v)$ and $S'_v := S' \cap E(v)$. If $S_v \neq S'_v$ then $|S_v| = |S'_v| = b(v)$ and the $b(v) <_v$ -best edges of $S_v \cup S'_v$ are either S_v or S'_v .*

A consequence of Theorem 22 that is interesting in itself is that in the polygamous stable marriage problem each participating person p can partition

the members of the other gender into as many groups as p 's quota in such a way that in any polygamous stable marriage scheme p receives at most one partner from each group. This result turns out to be useful for the linear characterization of the stable b -matching polytope. See Section 2.4 for the details.

COROLLARY 23. (Fleiner 2002 [27]) *Let (G, \emptyset) be a bipartite preference system and $b: V(G) \rightarrow \mathbb{N}$ be a quota function. Then for each vertex v of G , there is a partition of $E(v)$ into $b(v)$ parts $E_1(v), E_2(v), \dots, E_{b(v)}(v)$ so that $|S \cap E_i(v)| \leq 1$ for any stable b -matching S and any integer i with $1 \leq i \leq b(v)$.*

Note that Corollary 23 has also been observed by Sethuraman et al. in [59] and turned out to be crucial in giving an alternative proof for Theorem 28, the characterization of the stable admissions polytope.

An interesting corollary of Theorem 22 and Theorem 20 is the following “middle choice” property of stable b -matchings.

COROLLARY 24. *If (G, \emptyset) is a finite bipartite preference system, $b: V(G) \rightarrow \mathbb{N}$ is a quota function and $M_1, M_2, \dots, M_{2k-1}$ are stable b -matchings then there is a stable b -matching M of (G, \emptyset) that assigns each vertex v of G with the edges of the the k th best assignment of v out of $M_1, M_2, \dots, M_{2k-1}$.*

PROOF. Theorem 22 implies that each vertex v has a linear preference order on the possible assignments that v can get in a stable b -matching. This follows that the k th choice of v from $\mathcal{S} = \{M_1, M_2, \dots, M_{2k-1}\}$ is the k th best assignment of v out of $M_1, M_2, \dots, M_{2k-1}$. As $k = n + 1 - k$ for $n = 2k - 1$, Corollary 24 follows directly from Theorem 20. ■

The last result in this section generalizes a well-known fact in the stable admissions model (that is also valid in the many-to-many case). In that model, those colleges that cannot fill up their quota in some stable admission scheme receive the very same set of students in any stable assignment. A special case of this property is that in any bipartite preference system always the same persons get married in each stable marriage scheme. Theorem 25 is a direct corollary of Theorem 19.

THEOREM 25. (Fleiner 2000 [28]) *If $\mathcal{M}_1, \mathcal{M}_2$ are matroids on a common ground set and S_1, S_2 have the property of S in Corollary 8 for linear orders $<_1, <_2$, then $\text{span}_{\mathcal{M}_i}(S_1) = \text{span}_{\mathcal{M}_i}(S_2)$ for $i \in \{1, 2\}$.*

2.4. Stable matching polyhedra

Recently, linear programming become a popular tool to study bipartite and nonbipartite stable matchings, see Abeledo, Blum, Roth, Rothblum, Sethuraman, Teo, Vande Vate and others in [3, 4, 1, 2, 50, 63, 57]. In this section, we survey linear descriptions of bipartite stable matching polyhedra. The earliest such result is that of Vande Vate.

We denote by $P^b(G, \emptyset)$ the convex hull of characteristic vectors in \mathbb{R}^E of stable b -matchings of bipartite preference system (G, \emptyset) . (So $P^1(G, \emptyset)$ is the polytope of ordinary stable matchings.) As usual in linear programming, we define $x(S) := \sum\{x(e) : e \in S\}$ for a vector $x \in \mathbb{R}^E$ and subset S of E .

THEOREM 26. (Vande Vate 1989 [66]) *Let (G, \emptyset) be a bipartite preference system with colour classes M and W , $|M| = |W|$ and $E = M \times W$. Then*

$$P^1(G, \emptyset) =$$

$$\{x \in \mathbb{R}^E : x \geq \mathbf{0}, x(E(v)) = 1 \forall v \in M \cup W, x(\psi(e)) \leq 1 \forall e \in E\}$$

where $\psi(mw) := \{f \in E : f \geq_m mw \text{ or } f \geq_w mw\}$.

Rothblum gave a shorter proof of a modified description for a more general problem in [55], and his proof was further simplified by Roth *et al.* in [52].

THEOREM 27. (Rothblum 1992 [55]) *Let (G, \emptyset) be a bipartite preference system with colour classes M and W . Then*

$$P^1(G, \emptyset) =$$

$$= \{x \in \mathbb{R}^E : x \geq \mathbf{0}, x(E(v)) \leq 1 \forall v \in M \cup W, x(\phi(e)) \geq 1 \forall e \in E\}$$

where $\phi(mw) := \{f \in E : f \leq_m mw \text{ or } f \leq_w mw\}$.

Based on these results, standard tools of linear programming allow us to find a maximum weight stable matching in polynomial time.

But these results handle only the stable matching problem and do not say much about stable b -matchings. The following theorem of Baïou and Balinski [11] gives a linear description of the stable admissions polytope and generalizes Theorem 27. Note that Sethuraman et al. gave an alternative proof for the following Theorem 28 based on Corollary 23.

THEOREM 28. (Baïou and Balinski 1999 [11], see also Sethuraman et al. [59]) *Let (G, \emptyset) be a bipartite preference system and $b: M \cup W \rightarrow \mathbb{N}$ be a quota function so that $b(w) = 1$ for all nodes w of W . Then*

$$\begin{aligned} P^b(G, \emptyset) = \{x \in \mathbb{R}^E : & x \geq \mathbf{0}, \\ & x(E(w)) \leq 1 \quad \forall w \in W, \quad x(E(m)) \leq b(m) \quad \forall m \in M, \\ & x(C(m, w_1, w_2, \dots, w_{b(m)})) \geq b(m) \\ & \text{for all combs } C(m, w_1, w_2, \dots, w_{b(m)})\}, \end{aligned}$$

where a comb is defined for $m \in M$ and $m w_1 >_m m w_2 >_m \dots >_m m w_{b(m)}$ as

$$\begin{aligned} C(m, w_1, w_2, \dots, w_{b(m)}) = \{mw \in E : mw \leq_m mw_1\} \cup \\ \cup \{mw'_i \in E : m'w_i \leq_{w_i} mw_i \text{ for some } i = 1, 2, \dots, b(m)\} \end{aligned}$$

Because of the comb constraints, the above characterization can consist of $\Omega(n^B)$ linear inequalities, where n is the number of “colleges” and B is the maximum of all quotas. But in spite of the exponential number of constraints, it is still possible to find an optimum weight stable admission by the ellipsoid method, using the separation algorithm of Baïou and Balinski.

In [28, 26], with the help of the theory of blocking polyhedra and lattice polyhedra, Fleiner gave a linear description of certain polyhedra that are related to $\mathcal{C}_M \mathcal{C}_W$ -stable sets. Fix functions $\mathcal{C}_M \mathcal{C}_W: 2^E \rightarrow 2^E$ and let us denote by

$$\begin{aligned} \mathcal{S} &:= \{S \subseteq W : S \text{ is an } \mathcal{C}_M \mathcal{C}_W\text{-stable set}\} \\ \mathcal{B} &:= \{B \subseteq E : B \cap S \neq \emptyset \text{ for any } S \in \mathcal{S}\} \\ \mathcal{A} &:= \{A \subseteq E : |A \cap S| \leq 1 \text{ for any member } S \text{ of } \mathcal{S}\} \\ K &:= E \setminus \bigcup \mathcal{S} \end{aligned}$$

family of $\mathcal{C}_M \mathcal{C}_W$ -stable sets, the *blocker*, the *antiblocker* of \mathcal{S} , and the set of nonstable elements, respectively. Define further the $\mathcal{C}_M \mathcal{C}_W$ -stable set polytope, its dominant and submissive polyhedra by

$$(22) \quad P(\mathcal{C}_M, \mathcal{C}_W) := \text{conv}\{\chi^S : S \in \mathcal{S}\}$$

$$(23) \quad P(\mathcal{C}_M, \mathcal{C}_W)^{\uparrow} := P(\mathcal{C}_M, \mathcal{C}_W) + \mathbb{R}_+^E = \{x + y : x \in P(\mathcal{C}_M, \mathcal{C}_W), y \geq 0\}$$

$$(24) \quad P(\mathcal{C}_M, \mathcal{C}_W)^{\downarrow} := \{x - y : x \in P(\mathcal{C}_M, \mathcal{C}_W), y \geq 0\} \cap \mathbb{R}_+^E.$$

THEOREM 29. (Fleiner 2000 [28, 26]) *If $\mathcal{C}_M, \mathcal{C}_W: 2^E \rightarrow 2^E$ are increasing comonotone functions then*

$$(25) \quad P(\mathcal{C}_M, \mathcal{C}_W)^\uparrow = \{x \in \mathbb{R}^E : x \geq \mathbf{0}, x(B) \geq 1 \text{ for } B \in \mathcal{B}\},$$

$$P(\mathcal{C}_M, \mathcal{C}_W)^\downarrow = \{x \in \mathbb{R}^E : x \geq \mathbf{0}, x(K) = 0 \text{ and}$$

$$(26) \quad x(A) \leq 1 \text{ for any } A \in \mathcal{A}\},$$

$$P(\mathcal{C}_M, \mathcal{C}_W) = \{x \in \mathbb{R}^E : x \geq \mathbf{0}, x(B) \geq 1 \text{ for } B \in \mathcal{B} \text{ and}$$

$$(27) \quad x(A) \leq 1 \text{ for } A \in \mathcal{A}\}.$$

If Theorem 29 is applied to the bipartite stable b -matching problem then it gives the following linear description of the stable b -matching polytope.

THEOREM 30. (Fleiner 2000 [28, 25, 26]) *Let (G, \emptyset) be a bipartite preference system and $b: V(G) \rightarrow \mathbb{N}$ be a quota function. Then*

$$P^b(G, \emptyset) = \{x \in \mathbb{R}^E : x \geq \mathbf{0}, x(A) \leq 1 \forall A \in \mathcal{A}, x(B) \geq 1 \forall B \in \mathcal{B}\}$$

where

$$\mathcal{A} := \{A \subseteq E : |A \cap S| \leq 1 \text{ for any stable } b\text{-matching } S\} \text{ and}$$

$$\mathcal{B} := \{B \subseteq E : B \cap S \neq \emptyset \text{ for any stable } b\text{-matching } S\}.$$

Note that the constraints in Theorem 27 are special cases of the ones in Theorem 30. However, there are two important differences between Theorem 30 and the above earlier results. A shortage of Theorem 30 is that it uses implicit constraints, hence if it is specialized to the stable marriage problem, it might require more constraints than Rothblum's explicit description. (This is why Theorem 30 is rather an extension than a generalization of Theorem 27.) A positive feature of Theorem 30 is that unlike Theorem 28, both the matrix and the right hand side vector in the description contains only 0 and 1 entries.

The following result is a strengthening of Theorem 30 for the stable b -matching polytope and it is a genuine generalization of Theorem 27.

THEOREM 31. (Fleiner 2002 [27]) *Let (G, \emptyset) be a bipartite preference system and $b: M \cup W \rightarrow \mathbb{N}$ be a quota function. Then the star-partitions $E_1(v), E_2(v), \dots, E_{b(v)}(v)$ of $E(v)$ described in Corollary 23 satisfy*

$$P^b(G, \emptyset) = \{x \in \mathbb{R}^E : x \geq \mathbf{0},$$

$$x(E_i(v)) \leq 1 \forall v \in M \cup W, 1 \leq i \leq b(v),$$

$$x(\phi_{i,j}(mw)) \geq 1 \forall mw \in E, 1 \leq i \leq b(m), 1 \leq j \leq b(w)\},$$

where $\phi_{i,j}(mw) := \{mw\} \cup [\phi(mw) \cap (E_i(m) \cup E_j(w))]$

Note that star-partitions in Corollary 23 can be found with m deferred acceptance algorithms, where m is the number of edges of G (see [27]).

It is interesting to compare the linear descriptions of the stable matching polytopes, that is Rothblum's Theorem 27, Theorem 28 of Baïou and Balinski and Theorem 31 by Fleiner. Rothblum's result gives a linear description of the stable matching polytope for the one-to-one case with $O(n + m)$ constraints, the one of Baïou and Balinski does it for many-to-one markets with $O(n + mD^B)$ constraints and Fleiner characterized the many-to-many polytope by $O(n + mB^2)$ linear inequalities. (Here n denotes the number of agents, m is the number of possible relationships, B is the maximum quota, and maximum degree D is the maximum number of possible relationships that an agent can have.) An advantage of the first two characterizations is that the linear constraints are explicit, while in Theorem 31 we need some preprocessing to write down the inequalities. An advantage of Theorem 31 is that it handles the most general problem.

However both Theorem 27 and 28 can handle more general situations. Due to Observation 42, we can optimize over the stable admissions polytope by applying Theorem 27 on b -splitting (G^b, \emptyset^b) . If we apply the b -splitting only on one side of the market, then Theorem 28 allows us to optimize over the stable b -matching polytope. The first construction gives us a linear programming problem in $O(mB)$ dimensions with $O((n + m)B)$ constraints, while the second creates an LP problem in $O(mB)$ dimensions with $O(nB + Bm(BD)^B)$ constraints. But this is not the end of the story. If we apply Observation 42 and 43 as in the proof of Theorem 41, then we can readily use Rothblum's characterization (Theorem 27) to optimize over the stable b -matching polytope, and for this we solve an LP problem with $O((n + m)B)$ constraints in $O(mB)$ dimensions. Hence in selecting the algorithm for optimizing stable matchings, we have some tradeoff between the number of constraints, the dimension of the problem and implicitness of the linear inequalities.

It is a natural question whether there is a similar polyhedral description of the stable matching polytope for the nonbipartite (one-sided) case. Feder proved in [23] that there is no hope for such a characterization: finding a minimum weight stable matching in the stable roommates problem is NP-complete.

2.5. Stable matchings and graph paths

In this section, we discuss two consequences of the stable marriage theorem on paths of graphs. Recall Corollary 6 that roughly said that if we have two partial orders on a ground set then there is a common antichain that dominates every element of the ground set. If we consider the Hasse diagrams of the posets (that is, the directed graph in which arcs go along covering elements) then we can formulate Corollary 6 as follows. For any two acyclic directed graphs on a common vertex set there is a subset S of the ground set such that there is no directed path between two vertices of S in any of the graphs but from any vertex outside S there is a directed path into S in one or both of the graphs.

Actually, one can drop the above acyclicity condition to get the following well-known result of Sands *et al.*. See [26] for an alternative proof with comonotone functions.

THEOREM 32. (Sands *et al.* 1982 [56]) *If A_1 and A_2 are arc-sets on vertex-set V , such that there is no infinite path in any of the A_i 's that starts at some vertex then there is a subset S of V such that*

(28) *for each element $v \in V$ there is a simple path in A_1 or in A_2 from v to S , and*

(29) *there is neither a simple A_1 -, nor a simple A_2 -path between different vertices of S .*

The following result of Pym is an unexpected application of the stable marriage theorem. One proof of Pym actually implies the stable marriage theorem, but it seems that the author was unaware of this application.

THEOREM 33. (Pym 1969 [45, 46]) *Let $D = (V, A)$ be a directed graph and X, Y be subsets of V . Let moreover \mathcal{P} and \mathcal{Q} be families of vertex-disjoint simple XY -paths. Then there exists a family \mathcal{R} of vertex-disjoint simple XY -paths, such that*

(30) *any path of \mathcal{R} consists of a (possibly empty) initial segment of a path of \mathcal{P} and of a (possibly empty) end segment of a path of \mathcal{Q} , moreover*

$$(31) \quad In(\mathcal{P}) \subseteq In(\mathcal{R}) \subseteq In(\mathcal{P} \cup \mathcal{Q})$$

$$(32) \quad End(\mathcal{Q}) \subseteq End(\mathcal{R}) \subseteq End(\mathcal{P} \cup \mathcal{Q}).$$

PROOF. (Sketch of the proof.) Define a bipartite preference system in such a way that men and women correspond to paths of \mathcal{P} and \mathcal{Q} , respectively.

Each common vertex v of a path P of \mathcal{P} and Q of \mathcal{Q} yields an edge e_v between P and Q . Let path P of \mathcal{P} prefer edge e_v to e_u if v is closer to the initial vertex of P on P than u . Similarly, path W of \mathcal{Q} prefers edge e_v to e_u if v is closer to the terminal vertex of Q on Q . Let S be a stable matching in the bipartite preference system, that corresponds to vertex set R in D . Merge initial segments of \mathcal{P} -paths to end segments of \mathcal{Q} -paths along the vertices of R . This results in a collection \mathcal{R} of paths with the desired property. ■

Note that Theorem 33 proves that gammoids are matroids. Independently of us, essentially the above proof of Theorem 33 has been found by Diestel and Thomassen [22]. A special case of Theorem 33 is the following infinite generalization of the Mendelsohn-Dulmage theorem.

THEOREM 34. (see Mendelsohn-Dulmage [43]) *Assume that G is a bipartite graph with colour classes A and B . Then for any matchings M_1 and M_2 of G there is a matching M of G such that M covers each vertex of A that is covered by M_1 and M also covers each vertex of B that is covered by M_2 .*

PROOF. Let us orient each edge of G from colour class A to colour class B . Then both M_1 and M_2 become families of vertex-disjoint simple AB paths. According to Theorem 33, there is a family M of vertex-disjoint AB path that covers each vertex of A that is covered by M_1 and each vertex of B that is covered by M_2 . By the construction, each path of M has exactly one edge, hence M is a matching. ■

Note that a consequence of Theorem 34 is the well-known set theoretical Cantor-Bernstein Theorem claiming that if $f: A \rightarrow B$ and $g: B \rightarrow A$ are injections then there exists a bijection between A and B . To see this, it is enough to observe that injection f and g correspond to matchings between A and B .

Brualdi and Pym proved a modified version of the original linking theorem of Pym (Theorem 33 without (30)) where they require condition (34) but allow generalized directed paths [16]. A *generalized directed path* is either a finite directed path or a (finite) circuit or an infinite path. A *circuit path* is a sequence $v_0, a_1, v_1, \dots, v_{t-1}, a_t, v_t$, where a_i is a $v_{i-1}v_i$ arc, $v_t = v_0$, otherwise all other vertices are different in the sequence. An *infinite path* is an infinite sequence $v_0, a_1, v_1, a_2, \dots$ or $\dots, a_{-1}, v_{-1}, a_0, v_0$ or $\dots, a_{-1}, v_{-1}, a_0, v_0, a_1, v_1, a_2, \dots$, in such a way that a_i is a $v_{i-1}v_i$ arc and all vertices v_i are different in the sequence. The initial and terminal vertex of the above circuit is v_0 , the first type of infinite path has initial vertex v_0 , and has no terminal vertex, the second infinite path has no initial vertex, but v_0 is

its terminal vertex, and the third infinite path has neither initial nor terminal vertex.

THEOREM 35. (Brualdi-Pym [16]) *In digraph $D = (V, A)$, let \mathcal{P} and \mathcal{Q} be families of vertex-disjoint generalized paths. There exists a family \mathcal{R} of vertex-disjoint general paths of D such that*

$$(33) \quad \begin{aligned} In(\mathcal{P}) &\subseteq In(\mathcal{R}) \subseteq In(\mathcal{P} \cup \mathcal{Q}) \\ End(\mathcal{Q}) &\subseteq End(\mathcal{R}) \subseteq End(\mathcal{P} \cup \mathcal{Q}) \\ V(\mathcal{P}) \cap V(\mathcal{Q}) &\subseteq V(\mathcal{R}) \subseteq V(\mathcal{P} \cup \mathcal{Q}) \\ A(\mathcal{P}) \cap A(\mathcal{Q}) &\subseteq A(\mathcal{R}) \subseteq A(\mathcal{P} \cup \mathcal{Q}). \end{aligned}$$

Note that although this theorem sounds similar to Theorem 33, it seems to be substantially different. To be able to prove condition (34), we *must* drop condition (30), as even if both \mathcal{P} and \mathcal{Q} consist of finite simple paths, it might be necessary to use both circular and infinite paths in \mathcal{R} (see [16]).

Ingleton and Piff [35] found the following simple proof of Theorem 35.

PROOF. (Sketch of the proof of Theorem 35) Substitute each common vertex v of a path of \mathcal{P} and of \mathcal{Q} with two vertices v_{in} and v_{out} such that all arc arriving at v will arrive at v_{in} and all arcs leaving v will leave v_{out} . This node-splitting cuts the paths both of \mathcal{P} and of \mathcal{Q} into new sets of vertex disjoint paths \mathcal{P}' and \mathcal{Q}' such that a path of \mathcal{P}' and a path of \mathcal{Q}' may share their initial or end vertex but no inner vertices can be common. Paths of $\mathcal{P}' \cup \mathcal{Q}'$ induce a directed bipartite graph G' such that one colour class consists of all vertices v_{in} and the other class is formed by vertices v_{out} . (For this we have to introduce some “artificial” initial and end vertices for the infinite paths of $\mathcal{P}' \cup \mathcal{Q}'$.) In this graph G' , each families \mathcal{P}' and \mathcal{Q}' correspond to a matching.

According to Theorem 34, there is a matching R' in G' that covers each vertex v_{out} that is a initial vertex of a path of \mathcal{P}' and each vertex v_{in} that is an end vertex of some path of \mathcal{Q}' . If we substitute each arc of R' with the corresponding path of \mathcal{P}' or of \mathcal{Q}' and we merge each pair v_{in}, v_{out} of vertices into vertex v then we get a vertex disjoint family \mathcal{R} of path that has properties (33) and (34). ■

The following corollary is also observed by others (see e.g. [19]). It provides an interesting application of Theorem 33 on families of edge-disjoint (rather than vertex-disjoint) paths. In [19], by Conforti *et al.*, Corollary 36 is deduced directly from the stable matching theorem on bipartite multigraphs, using the framework we described after the proof of Theorem 33.

COROLLARY 36. *Let $G = (V, E)$ be an undirected graph and x, y, z be different vertices of V . Let \mathcal{P} be a set of k edge-disjoint xy -paths and \mathcal{Q} be a set of k edge-disjoint yz -paths. Then there exist a set \mathcal{R} of k edge-disjoint xz -paths such that each path of \mathcal{R} is the union of a (possibly empty) initial segment of a path of \mathcal{P} and of a (possibly empty) end segment of a path of \mathcal{Q} .*

To prove the above result, we apply Theorem 33 on the line-graphs of paths of \mathcal{P} and \mathcal{Q} . (A line-graph of a path is a path again.) There still remain some tiny details to take care of. This is done in the following.

PROOF. Let vertex-disjoint path-families $\mathcal{P}', \mathcal{Q}'$ be the collection of the line-graphs of the paths in \mathcal{P} and in \mathcal{Q} , respectively. By applying Theorem 33 on \mathcal{P}' and \mathcal{Q}' we get a vertex-disjoint path collection \mathcal{R}' . Family \mathcal{R}' is the set of line-graphs of a set \mathcal{R}^* of edge-disjoint walks. (These walks are not necessarily paths). Clearly, $|\mathcal{R}^* \cap \mathcal{P}| = |\mathcal{R}^* \cap \mathcal{Q}|$, so we can pair those paths and merge them via y . By this operation, \mathcal{R}^* becomes a collection of edge-disjoint xz -walks. To obtain \mathcal{R} as described in the corollary, we have to shortcut the possible circles on each element of \mathcal{R}^* . When no more shortcut is possible, we get edge-disjoint xz -paths switching exactly once, as stated. ■

Using Corollary 36 in [19], Conforti *et al.* described a Gomory-Hu based maxflow-representing structure. For each edge uv of a Gomory-Hu tree of a graph G , they store a list of $\lambda_G(u, v)$ edge disjoint uv paths. They also do it for some other $|V(G)|$ pairs uv of vertices of G . Then, by applying the stable marriage algorithm $O(\alpha(n))$ times as in Corollary 36, they can construct a collection of $\lambda_G(x, y)$ edge-disjoint xy -paths of G for any two vertices x and y of G (where $\alpha(n)$ is the inverse Ackerman-function of n that is regarded almost as good as a constant function).

3. The nonbipartite case: the stable roommates problem

We have seen that the bipartite nature of the stable marriage theorem was crucial for the application of the Knaster–Tarski fixed point theorem. There is however a natural nonbipartite model in which similar questions can be asked. We discuss certain nonbipartite versions in this section. The interested reader is referred to [6] for further details.

A *graphic preference system* is a pair (G, \emptyset) where G is a graph and $\emptyset = \{<_v : v \in V(G)\}$ so that $<_v$ is a linear order on $E(v)$. Let $b: V(G) \rightarrow \mathbb{N}$. Set F of edges of G *b-dominates* edge e of G if there is a vertex v of e and different edges $f_1, f_2, \dots, f_{b(v)}$ of F so that $f_i <_v e$ for

$i = 1, 2, \dots, b(v)$. Set F of edges b -dominates edge set H if each edge h of H is b -dominated by F . A *stable b -matching* is a subset S of the edges of G such that no edge of S is dominated by S , and S dominates $E \setminus S$. By definition, each vertex v of G is incident with at most $b(v)$ edges of a stable b -matching. A *stable matching* is the short name of a stable 1-matching.

An important difference from the bipartite model is that in nonbipartite graphic preference systems there might exist no stable matching whatsoever. An example is a 3-cycle where preferences are also cyclic. The first efficient algorithm to decide the existence of a stable matching for this case is due to Irving [36]. Although, after Irving's result, several different algorithms were designed for the same (or for a more general) problem (see [23, 61, 63, 37]), Irving's algorithm still plays an important role in studying stable matchings. Our nonstandard description in the next section is based on the graph terminology.

3.1. Irving's algorithm

To find a stable matching in a nonbipartite graphic preference model, Irving's algorithm uses a so called “rotation elimination” step besides the proposal and rejection steps of the deferred acceptance algorithm of Gale and Shapley. To describe the algorithm, let us call edge e of graphic preference system (G, \emptyset) a *first choice* of agent v if e is the most preferred edge by v .

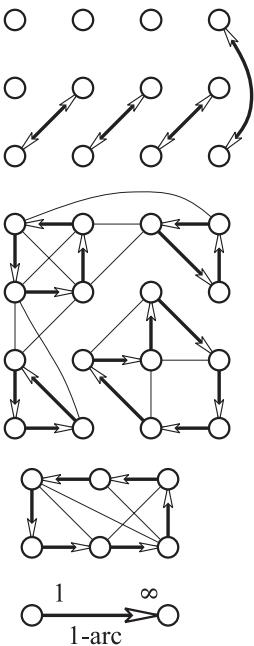
Proposal step. If (G, \emptyset) is a graphic preference system and edge $e = vw$ is a 1st choice of v in graphic preference system (G, \emptyset) then orient e from v to w and call the hence created arc vw a *1-arc*.

The proposal step roughly means that an agent points to his/her best possible partner. As we do not change the underlying preference system, the set of stable matchings is unchanged after a proposal step. So we can safely make all possible proposal steps. If no more proposal steps are possible then we might make a rejection step.

Rejection step. If 1-arc e points to v and v prefers e to some other edge f then we delete f from preference system (G, \emptyset) .

That is, if a proposal e arrives to agent v then v can forget about any partnership that is inferior to e . In particular, if v receives more than one proposals then he/she keeps only the best one and turns down the others. It is not difficult to prove that a rejection step does not change the set of stable matchings. Clearly, if 1-arc $f = uv$ is deleted in a rejection step then agent u can propose again. So in the so called *1st phase* of Irving's algorithm we

make all possible proposal steps, then all possible rejection steps, and then we propose again, reject, and so on. In other words, we keep on deleting edges of the underlying graph until it is not possible any more. As soon as we arrive to this situation the 1st phase ends and we proceed to the 2nd phase.

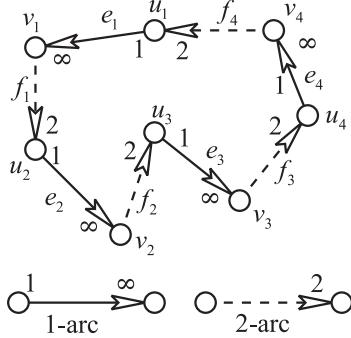


Observe that if no rejection takes place then each agent receives at most one proposal and these proposals represent the agents worse partnerships. As each agent sends one proposal, each of them have to receive exactly one, and our preference system has the 2nd phase property (or the so called *first-last property*): everybody's first choice is the last choice of the corresponding partner. In this situation, we need different kind of reduction steps. Note that if the first-last property holds then the components of the underlying graph belong to the following three categories. A component can be a singleton, corresponding to an agent that does not have a partner in any stable matching. It can be an edge (a bioriented 1-arc) that is disjoint from other edges of the graph. These edge components represent partnerships that have to be present in each stable matching. The third type components represents the essential problem. Observe that each vertex of these third type components have degree at least two, i.e. these agents have a different first and last choice.

Assume that our preference system (G, \emptyset) has the first-last property. We call an edge $e = uv$ the *2nd choice* of u if e is the second partnership in u 's preference list, succeeding the 1-arc that leaves u . (It is possible that 2nd choice $e = uv$ of u is also a 1st choice of v , hence vu is a 1-arc.) If $e = uv$ is a 2nd choice of u then (somewhat unnaturally) we orient e from v to u and call arc vu a *2-arc*. We have observed that our preference system has three type of components and we only have to work in those components where each agent have different first and last choices, i.e. with agents that have a well-defined second choice.

Let's start from any agent u_1 and follow a directed path that alternatingly uses 1-arcs and 2-arcs. As we work on a finite graph, the sequence of arcs will be periodic from some point on. The period is called a *rotation*. So a rotation is a sequence $(e_1, f_1, e_2, f_2, \dots, e_k, f_k)$ of joining arcs such that $e_i = u_i v_i$ is a

1-arc and $f_i = v_i u_{i+1}$ is a 2-arc, and addition is modulo k . The figure shows a “nice” rotation that we can further work with.



Not all rotations are “nice”: in fact, there are two types of them we have to deal with.

THEOREM 37. *If $(e_1, f_1, e_2, f_2, \dots, e_k, f_k)$ is a rotation then we have two alternatives.*

1. *Either $\{e_1, e_2, \dots, e_k\} = \{f_1, f_2, \dots, f_k\}$ and there exists no stable matching in our preference system.*
2. *Or $\{e_1, e_2, \dots, e_k\}$ and $\{f_1, f_2, \dots, f_k\}$ are disjoint and the following holds. If a stable matching S contains some edge e_i of the rotation then $e_1, e_2, \dots, e_k \in S$ and $S' := S \setminus \{e_1, e_2, \dots, e_k\} \cup \{f_1, f_2, \dots, f_k\}$ is also a stable matching. Moreover, each stable matching of $G \setminus \{e_1, e_2, \dots, e_k\}$ is a stable matching of G .*

Theorem 37 justifies that if neither a proposal nor a rejection step can be made then we can execute the following step.

Rotation elimination step. If $(e_1, f_1, e_2, f_2, \dots, e_k, f_k)$ is a rotation ($e_i = u_i v_i$, $f_i = v_i u_{i+1}$) and $\{e_1, e_2, \dots, e_k\}$ and $\{f_1, f_2, \dots, f_k\}$ are disjoint then for all i delete all edges that f_i dominates at v_i .

If, in the 2nd phase of the algorithm, we find a rotation with the same 1-arc and 2-arc sets then we can conclude according to Theorem 37 that no stable matching exists. Otherwise, after deleting all 1-arcs of a rotation we might kill some stable matchings but we do not lose all of them. So no matter what kind of rotation we find, we either conclude or reduce our problem to a graphic preference system with less edges, that possibly lacks the first-last property. So we can return to the 1st phase and continue to reduce the problem. This is enough to design an efficient algorithm to find a stable matching in a graphic preference system. However, Irving could speed

it up with the following observation that was made also by Cechlárová and Borbel'ová in some more general settings [15].

Observe that after deleting all 1-arcs of a rotation, its 2-arcs become 1-arcs with opposite orientation and all other 1-arcs remain 1-arcs. So we need no extra time for the proposal steps. However, the new 1-arcs might involve some rejection steps. Luckily, all rejections occur at vertices v_i and no 1-arc is deleted in the meanwhile. Hence after at most k rejection steps we restore the first-last property. This means that once we started the 2nd phase, we never have to go back to the 1st phase if we execute the following steps.

Clearly, as long as we have an agent in our model who has different first and last choices then we can delete an edge either in a rejection or in a rotation elimination step. This follows that if we cannot make any further step then our graph is a matching that has to be stable in the original preference system, as well.

3.2. Tan's characterization

Based on Irving's proof, Tan in [62] gave a compact characterization of those models that contain a stable matching. In this section, we study Tan's result.

Recall that a function w assigning non-negative weights to edges in G is called a *fractional matching* if $\sum_{v \in e} w(e) \leq 1$ for every vertex v . A fractional matching w is called *stable* if every edge e contains a vertex v such that $\sum_{v \in f, f \leq_v e} w(f) = 1$.

THEOREM 38. (Tan, 1991 [62]) *In any graphic preference system, there exists a half-integral fractional stable matching. In other words, there exists a set S of edges whose connected components are single edges and cycles, such that every edge e of the graph contains a vertex v of $V(S)$ such that $e \leq_v s$ for each $s \in S$ containing v .*

Tan's original proof is based on Irving's algorithm [36]. Tan observed that if we do not stop the algorithm when we find a rotation with identical 1-arc and 2-arc sets, but instead we conclude that the edges of these rotations will have weight half, then we have a polynomial time algorithm for finding a stable fractional matching in a graphic preference systems. (Note that Tan's terminology was pretty different from the above one that comes from Aharoni [7].)

Observe that if the support of a half-integral fractional stable matching contains only even cycles then there obviously exists a stable matching: we only have to throw away each second edge of the cycle components of the support. Tan also proved the following curious fact. (For a short and direct proof that is independent from Irving's algorithm, see [7].)

THEOREM 39. (Tan 1991 [62]) *Let (G, \emptyset) be a graphic preference system. If an odd cycle appears in the support of some fractional stable matching of (G, \emptyset) , then this very cycle appears in the support of any fractional stable matching of (G, \emptyset) .*

As the characteristic vector of a stable matching is a half-integral fractional stable matching, the presence of an odd cycle in the support of a half-integral fractional stable matching is equivalent to the non-existence of a stable matching.

Theorem 38 follows directly from a well-known game theoretical lemma of Scarf. An advantage of this reduction is that Scarf's lemma is very flexible and it allows us to deduce a generalization of Theorem 38 to stable b -matchings in nonbipartite graphs. We call function $w: E(G) \rightarrow \mathbb{N}$ a *fractional b -matching* if $\sum_{v \in e} w(e) \leq b(v)$ for every vertex v of G . A fractional b -matching w is called *stable* if for every edge e either $w(e) = 1$ or e contains a vertex v such that $\sum_{v \in f, f \subseteq_e} w(f) = b(v)$.

THEOREM 40. *If (G, \emptyset) is a graphic preference system and $b: V(G) \rightarrow \mathbb{N}$ then there exists a half-integral fractional stable b -matching. In other words, there exist disjoint subsets S and S^{half} of edges such that*

- the components of S^{half} are cycles,
- for any vertex v of G ,

$$(35) \quad |E(v) \cap S| + \frac{1}{2}|E(v) \cap S^{half}| \leq b(v)$$

- if S^{half} covers some vertex v then (35) holds with equality and the $<_v$ -maximal edge of $E(v) \cap (S \cup S^{half})$ belongs to S^{half} , and at last
- each edge e of $E \setminus S$ has a vertex v such that (35) holds with equality and $s <_v e$ for any $s \in E(v) \cap (S \cup S^{half})$.

Obviously, if all components of S^{half} are even in Theorem 40 then S together with each second edge of S^{half} is a stable b -matching. Just like in case of nonbipartite stable matchings, a half-integral fractional stable b -matching characterizes the existence of a stable b -matching.

THEOREM 41. *Let (G, \emptyset) be a graphic preference system, $b: V(G) \rightarrow \mathbb{N}$ and C be an odd cycle of G . If w is a half-integral stable b -matching for (G, \emptyset) and $w(e) = \frac{1}{2}$ for each edge e of C then the edges of C receive weight $\frac{1}{2}$ in any half-integral fractional stable b -matching.*

That is, if S^{half} contains an odd cycle then this very odd cycle is contained in the half-support of any half-integral stable b -matching, hence no (integral) stable b -matching can exist. We prove Theorem 41 with the help of two constructions that reduce the stable b -matching problem to the stable matching problem. Let graphic preference system (G, \emptyset) and quota function $b: V(G) \rightarrow \mathbb{N}$ be given. Applying a b -splitting to preference system (G, \emptyset) results in another preference system (G^b, \emptyset^b) such that

$$\begin{aligned} V(G^b) &:= \{v(i) : v \in V(G) \text{ and } i = 1, 2, \dots, b(v)\} \\ E(G^b) &:= \{u(i)v(j) : uv \in E(G) \text{ and } u(i), v(j) \in V(G^b)\} \\ \emptyset^b &:= \{<_{v(i)} : v(i) \in V(G^b)\} \\ u(i)v(j) <_{u(i)} u(i)w(k) &\iff \begin{cases} uv <_u uw \text{ or} \\ v = w \text{ and } j < k \end{cases} \end{aligned}$$

That is, we substitute each vertex v of G by $b(v)$ different copies, and two vertices of G^b are joined by an edge if the corresponding vertices of G are different and adjacent. Preferences are inherited from (G, \emptyset) we only have to take extra care of that if the two edges come from the same edge.

OBSERVATION 42. *Let (G, \emptyset) be a graphic preference system and $b: V(G) \rightarrow \mathbb{N}$ in such a way that $b(u) = 1$ or $b(v) = 1$ for any edge uv of G . Then there is a one-to-one correspondence of half-integral fractional stable b -matchings of (G, \emptyset) to half-integral fractional stable matchings of (G^b, \emptyset^b) in such a way that an odd cycle of weight half of the fractional stable b -matching corresponds to an odd cycle of weight half in the fractional stable matching and vice versa.*

That is, there exists a stable b -matching of (G, \emptyset) if and only if there is a stable matching of (G^b, \emptyset^b) . The problem is that the condition on b in Observation 42 does not hold in general. The next construction solves this difficulty. The subdivided preference system of (G, \emptyset) is preference system (G^s, \emptyset^s) such that

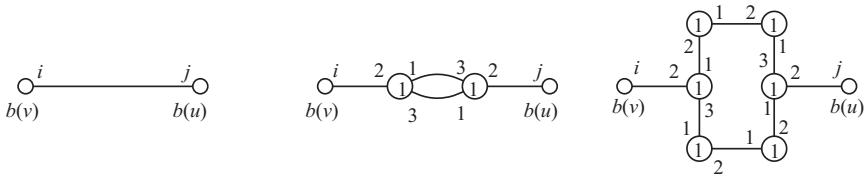
$$\begin{aligned} V(G^s) &:= V(G) \cup \{v(e) : v \in V(G) \text{ and } e \in E(v)\} \\ E(G^s) &:= \{vv(e) : v \in V(G) \text{ and } e \in E(v)\} \cup \{e_{uv} : uv = e \in E(G)\}, \end{aligned}$$

where e_{uv} joins $u(e)$ to $v(e)$

$\emptyset^s := \{\prec_v : v \in V(G)\} \cup \{<_{v(e)} : v(e) \in V(G^s)\}$, such that

$uu(e) \prec_u uu(f)$ if $e <_u f$, and $e_{uv} <_{v(e)} vv(e) <_{v(e)} e_{vu}$ for $uv = e \in E(G)$.

That is, we subdivide each edge of G by two vertices and introduce a new edge between the subdividing vertices. (In other words, we substitute each edge with a path on four vertices so that the middle edge has a parallel copy.) Note that in [17] a similar construction was applied that used a 6-cycle to avoid parallel edges.



The preference order of the old vertices come from the original preference order and the preference order of a subdividing vertex is such that each of the two parallel edges is the best at one of its ends and the worst at the other end. Define $b^s: V(G^s) \rightarrow \mathbb{N}$ by $b^s(v) := b(v)$ if $v \in V(G)$ and $b^s(v(e)) := 1$ for $v \in V, e \in E_G(v)$.

OBSERVATION 43. Let (G, \emptyset) be a graphic preference system and $b: V(G) \rightarrow \mathbb{N}$. Preference system (G^s, \emptyset^s) and quota function b^s has the property needed in Observation 42, that is, $b^s(x) = 1$ or $b^s(y) = 1$ for any edge xy of G^s .

Moreover, any half-integral fractional stable b -matching of (G, \emptyset) induces a half-integral fractional stable b^s -matching of (G^s, \emptyset^s) and vice versa. In both constructions, an odd cycle of weight half induces another odd cycle of weight half.

In particular, there is a stable b -matching of (G, \emptyset) if and only if there is a stable matching of (G^s, \emptyset^s) . Moreover, G^s is 3-chromatic, hence any stable (b -)matching problem can be reduced to one on a 3-chromatic graph. If G is bipartite then G^s is also bipartite, and we have already used this fact in Section 2.4.

PROOF. (Sketch of the proof of Theorem 41.) Let w and C be as in Theorem 41. By Observations 42 and 43, there is a fractional stable matching w' of $((G^s)^b, (\emptyset^s)^b)$ that corresponds to w , hence C induces an odd cycle C' of $(G^s)^b$ with w' -weight $\frac{1}{2}$. By Theorem 39, any half-integral fractional stable

matching of $((G^s)^b, (\emptyset^s)^b)$ assigns weight $\frac{1}{2}$ to each edge of C' . This means that any half-integral fractional stable b -matching of (G, \emptyset) must induce a half-integral stable matching of $((G^s)^b, (\emptyset^s)^b)$ that assigns weight $\frac{1}{2}$ to C' , hence any half-integral stable b -matching of (G, \emptyset) must assign weight $\frac{1}{2}$ to each edge of C . \blacksquare

There is another interesting consequence of Theorem 40 on approximate stable b -matchings.

THEOREM 44. *If (G, \emptyset) is a graphic preference system and $b: V(G) \rightarrow \mathbb{N}$ then there is a subset U of $V(G)$ with $|U| \leq \frac{1}{3}|V(G)|$ such that for any $b': V(G) \rightarrow \mathbb{N}$ of the form*

$$b'(v) := \begin{cases} b(v) & \text{if } v \notin U \\ b(v) \pm 1 & \text{if } v \in U \end{cases}$$

there is a stable b' -matching of (G, \emptyset) .

PROOF. Let S, S^{half} be as in Theorem 40 and construct U by choosing one vertex from each odd cycle of S^{half} . As each odd cycle is of length at least 3, the size of U is at most the third of $|V(G)|$. Construct subset T of S^{half} by throwing away each second edge of each even cycle of S^{half} and by selecting each second edge of each odd component with the exception of points of U , where we select both or none of the edges depending on whether $b'(u) = b(u) + 1$ or $b'(u) = b(u) - 1$. Clearly, $S \cup T$ is a stable b' -matching of (G, \emptyset) . \blacksquare

Our next observation is that Corollary 23 has a direct generalization to nonbipartite models.

THEOREM 45. *Let (G, \emptyset) be a preference system and $b: V(G) \rightarrow \mathbb{N}$ be a quota function. Then for each vertex v of G , there is a partition of $E(v)$ into $b(v)$ parts $E_1(v), E_2(v), \dots, E_{b(v)}(v)$ so that $|S \cap E_i(v)| \leq 1$ for any stable b -matching S and any integer i with $1 \leq i \leq b(v)$.*

The proof is the reduction to Corollary 23 via a third construction. The *duplicated preference system* of (G, \emptyset) is (G^d, \emptyset^d) , where

$$V(G^d) := \{\bar{v} : v \in V(G)\} \cup \{\underline{v} : v \in V(G)\}$$

$$E(G^d) := \{\bar{u}\underline{v}, \underline{u}\bar{v} : uv \in E(G)\}$$

$$\emptyset^d := \{<_{\bar{v}}, <_{\underline{v}} : v \in V(G)\}, \text{ where}$$

$$\underline{u}\bar{v} < \underline{u}\underline{u}\bar{w} \iff uv < _u u w \iff \bar{u}\underline{v} < \bar{u}\bar{u}\underline{w} \quad \text{for any } uv \in E(G).$$

That is, we take two disjoint copies of $V(G)$, and the edges go along the original edges between the two copies. Preference orders are induced naturally by the original preference orders. Note that (G^d, \emptyset^d) is a bipartite preference system. Define $b^d: V(G^d) \rightarrow \mathbb{N}$ by $b^d(\bar{v}) := b^d(\underline{v}) := b(v)$ for any vertex v of G .

It is easy to see that stable (b -)matchings of (G, \emptyset) bijectively correspond to symmetric stable (b -)matchings of (G^d, \emptyset^d) , where we call an edge set F of G^d *symmetric* if $u\bar{v} \in F \iff \underline{u}\bar{v} \in F$ holds.

PROOF. (Proof of Theorem 45.) Apply Theorem 23 to quota function b^d and to bipartite preference system (G^d, \emptyset^d) . For each vertex v of G , we get a partition of $E_{G^d}(\bar{v})$ into $b^d(\bar{v}) = b(v)$ parts in such a way that any stable b^d -matching of (G^d, \emptyset^d) contains at most one edge from each part. This partition induces a partition on $E_G(v)$ that satisfies the property of Theorem 45. This is true because $S^d := \{\bar{u}\underline{v}, \underline{u}\bar{v} : uv \in S\}$ is a stable b -matching of (G^d, \emptyset^d) for any stable b -matching S of (G, \emptyset) . ■

Note that the duplication construction is useful to generalize other properties of bipartite stable matchings to nonbipartite ones. For example, it follows from Corollary 24 that if we have $2k - 1$ stable (b -)matchings in a bipartite preference system (G, \emptyset) and everybody selects his or her k th choice out of the $2k - 1$ possible assignments then a new stable (b -)matching is created. To see the same property of any nonbipartite preference systems (G, \emptyset) , we only have to observe that if we have $2k - 1$ symmetric stable (b -)matchings of bipartite preference system (G^d, \emptyset^d) and everybody selects his or her k th choice then this results in a stable (b -)matching by Corollary 24. It is easy to check that this very (b -)matching will be a *symmetric* one, so the above “middle choice” property holds for nonbipartite preference systems as well.

The proof of Theorem 40 is an application of the following lemma of Scarf to vector b , the extended incidence matrix and the extended domination matrix of the preference system. Notation $[n]$ stands for the set of the first n positive integers.

THEOREM 46. (Scarf 1967 [57]) *Let $n < m$ be positive integers, b be a vector in \mathbb{R}_+^n and $B = (b_{i,j})$, $C = (c_{i,j})$ be matrices of dimensions $n \times m$, satisfying the following three properties: the first n columns of B form an $n \times n$ identity matrix (i.e. $b_{i,j} = \delta_{i,j}$ for $i, j \in [n]$), the set $\{x \in \mathbb{R}_+^n : Bx = b\}$*

is bounded, and $c_{i,i} \leq c_{i,k} \leq c_{i,j}$ for any $i \in [n]$, $i \neq j \in [n]$ and $k \in [m] \setminus [n]$.

Then there is a nonnegative vector x of \mathbb{R}_+^m such that $Bx = b$ and the columns of C that correspond to $\text{supp}(x)$ form a dominating set, that is, for any column $i \in [m]$ there is a row $k \in [n]$ of C such that $c_{k,i} \geq c_{k,j}$ for any $j \in \text{supp}(x)$.

Theorem 46 can be interpreted such that in any weighted hypergraphic preference system there always exists a fractional stable b -matching. It turned out that Theorem 46 is a close relative of the topological fixed point theorem of Kakutani (see also [8, 7]). A *simplicial complex* is a non-empty family \mathcal{C} of subsets of a finite ground set such that $A \subset B \in \mathcal{C}$ implies $A \in \mathcal{C}$. Members of \mathcal{C} are called *simplices* or *faces*. Let us call simplicial complex \mathcal{C} *manifold-like* if, denoting its rank by n (that is, the maximum cardinality of a simplex in it is $n + 1$), every face of cardinality n of \mathcal{C} is contained in two faces of cardinality $n + 1$. The *dual* \mathcal{C}^* of a complex \mathcal{C} is the set of complements of its simplices. Just like in the case of complexes, members of a dual complex are also called *faces*.

LEMMA 47. (Aharoni 2001 [7]) *If \mathcal{C} and \mathcal{C}' are two manifold-like complexes on the same ground set, then the number of maximum cardinality faces of \mathcal{C} that are also minimum cardinality faces of \mathcal{C}'^* is even.*

What examples are there of manifold-like complexes? Of course, a triangulation of a closed manifold is of this sort. (We call this complex a *manifold-complex*.) Another well known example of a dual manifold-like complex is the *cone complex*: let X be a set of vectors in \mathbb{R}^n , and b a vector not lying in the positive cone spanned by any $n - 1$ elements of X . A third example of a manifold-like complex is the *domination complex*. Let C be a matrix as in Theorem 46 with the additional property that in each row of C all entries are different. Then the family of dominating column sets together with the extra member $[n]$ is a manifold-like complex. (For the details, see [8].)

If we plug the cone complex and the domination complex in Lemma 47 and apply a general position argument then we can deduce Theorem 46. The application of Lemma 47 to the cone complex and the manifold complex yields the following discrete version of Kakutani's fixed point theorem.

THEOREM 48. (see [6]) *Let us label the vertices of an n -dimensional simplex S by the unit vectors of \mathbb{R}^{n+1} , and label the other vertices of a*

triangulation T of S by vectors of \mathbb{R}^{n+1} in such a way that the label of any vertex v of T is in the positive hull of the labels of those vertices of S that lie on the minimal face of S that contain v . For any vector $b \in \mathbb{R}_+^{n+1}$, there is a elementary simplex in the triangulation of S whose vertex labels contain b in their cone.

Theorem 48 and a continuity argument implies Kakutani's fixed point theorem.

THEOREM 49. (Kakutani 1941 [39]) *Let $S \subseteq \mathbb{R}^n$ be an n -dimensional simplex and $K(S)$ be the family of closed convex subsets of S . Let $\Phi: S \rightarrow \rightarrow K(S)$ be upper semicontinuous, that is, whenever $y_i \rightarrow y$, $x_i \rightarrow x$ and $y_i \in \Phi(x_i)$ then $y \in \Phi(x)$. Then there is a point x of S so that $x \in \Phi(x)$.*

Brouwer's fixed point theorem is the special case of Theorem 49 where $\Phi(x)$ is one point for all x . The discrete version of Brouwer's fixed point theorem is Sperner's lemma and Theorem 48, the discrete version of Kakutani's theorem is a genuine generalization of Sperner's lemma. Note that a result of Shapley in [60] that generalize Sperner's lemma is formally a special case of Theorem 48. Still, Shapley's method in [60] proves Theorem 48.

It is interesting to see the close relationship of the bipartite stable matching theorem to the lattice theoretical fixed point theorem of Tarski and of the nonbipartite version to the topological fixed point theorem of Kakutani. One might ask himself whether there is some fixed-point theorem that implies a generalization of Tan's results on nonbipartite preference systems.

3.3. The stable roommates problem with free and forbidden edges

In this section, we study three generalizations of the one sided stable matching problem. We consider the problem with special edges, look at preference orders that are not necessarily strict, and we look for stable partnerships, where instead of linear orders, agents' preferences are described by some personal choice functions.

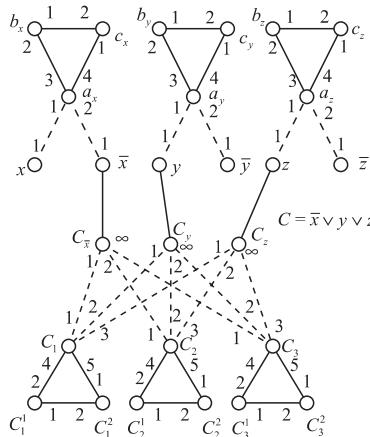
In the stable roommates problem, ordinary edges between two agents have two different features: on one hand, an edge can be present in a matching and dominate other edges, and on the other hand it may block a matching. A possible way to generalize the stable roommates problem is to allow edges that have only one of those properties. That is, we call an edge *forbidden* if it may block a matching, but it cannot be present in the stable matching that we

look for. So the presence of a forbidden edge makes it more difficult to find a stable matching. A *free* edge is the opposite: we may use it in our matching, but it never blocks. The stable roommates problem with forbidden and free edges is given by preference system (G, \emptyset) , disjoint subsets $E_{\text{forbidden}}$ and E_{free} of E and we ask if there exists a matching S of $E \setminus E_{\text{forbidden}}$ that is not blocked by an edge of $E \setminus E_{\text{free}}$.

By the definition, if we declare an ordinary edge free then all stable matchings remain stable and some new may emerge, hence it becomes easier to find one. Forbidding an ordinary edge may kill some stable matchings but never creates a new one, so it makes it more difficult to find one. We shall show that for the decision problem the opposite holds: the problem with forbidden edges is tractable and the presence of free edges makes it hard.

THEOREM 50. (Cechlárová and Fleiner 2008 [18]) *For a preference system (G, \emptyset) and subset $F \subseteq E(G)$ of free edges it is NP-complete to decide the existence of a stable matching. The problem is NP-complete already if F consists of disjoint edges.*

PROOF. (Sketch of the proof.) We show a polynomial reduction of the 3-SAT problem to the the stable roommates problem with free edges. For this reason we have to construct in polynomial time for each 3-CNF boolean formula Φ a preference system (G, \emptyset) and set of free edges such that Φ is satisfiable if and only if there is a matching in (G, \emptyset) that can be blocked only by free edges.



For each variable x of Φ , let us define vertices a_x, b_x, c_x, x and \bar{x} , edges $a_x b_x, b_x c_x, c_x a_x$ free edges $a_x x$ and $a_x \bar{x}$. These latter edges are first choices

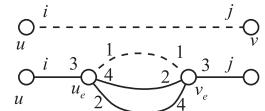
of x and \bar{x} respectively, the preference order of a_x is $a_x x, a_x \bar{x}, a_x b_x, a_x c_x$, b_x prefers $b_x c_x$ to $b_x a_x$ and c_x prefers $c_x a_x$ to $c_x b_x$. For each clause C we have vertices C_i and C_i^j for $i = 1, 2, 3, j = 1, 2$ and C_l , for each literal l in C . Construct free edges $C_l C_i$ that are the i th choices of C_l for $i = 1, 2$ and $C_l C_3$ is the very last choice of C_l . For each C_i , the edges $C_i C_l$ are the first three choices in an arbitrary order.

For $i = 1, 2, 3$, add all edges between C_i, C_i^1 and C_i^2 such that these edges are ordinary and preferences along these triangles are cyclic. To finish the construction of G , for each literal l in some clause C , connect vertex l with C_l . All nonspecified preferences are arbitrary. The figure shows the part of G corresponding to clause $C = \bar{x} \vee y \vee z$ and variables x, y and z . It is easy to see that if there is a truth assignment to Φ then there is a matching in G that is not blocked by ordinary edges. For this, we choose edges xv_x for each true variable and edges $\bar{x}v_x$ for each false one, all edges $b_x c_x$ and $C_i^1 C_i^2$. It is straightforward to complete this matching it to one we need.

If there is a matching S that is not blocked by an ordinary edge then for each variable x , exactly one of $a_x x$ and $a_x \bar{x}$ is present in S , as otherwise S would contain a stable matching of triangle $a_x b_x c_x$ that does not exist. Similarly, each vertex C_i is covered by a free edge of S as otherwise S would contain a stable matching of triangle $C_i C_i^1 C_i^2$, which is impossible. This means that S contains no edge $l C_l$.

If $xv_x \in S$ then declare x true, otherwise let x be false. We prove that this is a truth assignment of Φ , that is, each clause has a true literal. If C is a clause then there is an edge $C_3 C_l$ of S . As $l C_l$ cannot block this means that l is covered by a free edge, hence l is a true literal in C . This follows that Φ has a truth assignment.

To show the second part of the Theorem, we construct for each stable roommates problem with free edges an equivalent problem one where free edges are disjoint.



To do this it is enough to substitute each free edge with a little graph similar to the construction we had in the subdivided preference system. The difference is that for an edge $e = uv$, we add a free edge $u_e v_e$ that is first choice for both u_e and v_e . (See the figure.) After this change all free edges are disjoint and there is a matching that is not blocked by ordinary edges in (G, \emptyset) if and only if there is one after the construction. ■

Péter Biró [13] asked the complexity of the stable roommates problem with free edges in the special case where vertices of the underlying graph G are partitioned into disjoint sets and free edges are exactly those ones that connect different partition sets. Note that the above construction for the reduction of 3-SAT has this property, so this special case is NP-complete. Rob Irving asked the same question for the case where free edges are the ones that connect two vertices of the same partition set. As a graph where free edges are disjoint has this property, our second construction in the above proof shows the NP-completeness of this problem as well.

Now we turn our attention to forbidden edges. The first result on this is probably the following.

THEOREM 51. (Dias et al. [21]) *If (G, \emptyset) is a bipartite preference system and subset F of edges is forbidden then there is a linear time algorithm to find a stable matching that does not contain an edge of F , if such exists.*

Fleiner et al. in [29] extended Theorem 51 to nonbipartite preference systems, where one may allow indifferences in the preferences. Our goal here is to describe an extension of Irving's algorithm to solve an even more general problem. To define that problem, we focus on choice functions that come from not necessarily linear preference orders.

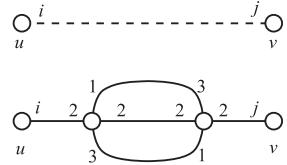
In both the stable marriage and the stable roommates problems, strict (linear) preferences of the participating agents play a crucial role. However, in many practical situations, one has to deal with indifferences in the preference orders. A natural model for this is that preference orders are partial (rather than linear) orders. One can extend the notion of a stable matching to this model in at least three different ways. One possibility is that a matching is *weakly stable* if no pair of agents a, b exists such that they mutually strictly prefer one another to their eventual partner. Ronn proved that deciding the existence of a weakly stable matching is NP-complete [47]. Based on Theorem 50, there is an alternative proof.

THEOREM 52. (see Ronn [47]) *It is NP-complete to decide the existence of a weakly stable matching in the stable roommates problem where each vertex has a weak linear preference order on the incident edges with at most two edges in tie.*

PROOF. (Sketch of the proof.) Construct graph G' the following way. Substitute each free edge of G with the gadget we used for the subdivided preference system. That is, instead of each free edge create a 3-path with two parallel edges in the middle. Further, add a third parallel edge to each gadget

such that this third edge is tied with the nonparallel edges at both endvertices. (See the figure.)

It is easy to check that there is a matching of G not blocked by any ordinary edge if and only if there is a weakly stable matching in (G', \emptyset') , where preferences of \emptyset' are the same as of \emptyset at vertices of G and given by the figure at the new vertices.



Hence for each instance of the stable roommates problem with free edges, one can construct in polynomial time an equivalent instance of the problem described in Theorem 52, that is we have a polynomial reduction of the former problem to the latter. As the former problem is NP-complete by Theorem 50, the latter one has this property as well. ■

A more restrictive notion than weak stability is the following. A matching is *strongly stable* if there are no agents a and b such that a strictly prefers b to his eventual partner and b does not prefer his eventual partner to a . Scott gave an algorithm that finds a strongly stable matching or reports if none exists in $O(m^2)$ time [58]. The most restrictive notion is that of super-stability. A matching is *super-stable* if there exist no two agents a and b such that neither of them prefers his eventual situation to being a partner of the other. In other words, a matching is super-stable, if it is stable for any linear extensions of the preference orders of the agents.

For the case where indifference is transitive (preferences are weak linear orders), Irving and Manlove gave an $O(m)$ algorithm to find a super-stable matching, if exists [37]. Interestingly, the algorithm has two phases, just like Irving's [36], but its second phase is completely different. The authors remark in [37] that the algorithm works without modification for the more general case when preference orders are partial orders.

Through indifferences of the agents in a preference system with weak linear orders, we may create special type of edges. Namely, it is possible that there are two parallel edges (say e and e') between two vertices and both vertices are indifferent between the parallel copies. This means that both e and e' may block a matching but nor e , neither e' can be present in a super-stable matching, as its parallel copy would block. It is easy to see that the super-stable roommates problem is equivalent with the one where we delete e' and forbid e .

The stable matching problem with forbidden edges is given by a preference system (G, \emptyset) and a subset F of E , the set of *forbidden edges*. The problem is to find a stable matching S of (G, \emptyset) that does not contain any

forbidden edge of F . Clearly, this problem is a special case of the super-stable matching problem. Fleiner *et al.* exhibited a reduction of the super-stable matching problem with forbidden edges this problem to 2-SAT [29] and in [30], the same authors extended Irving's algorithm to this case. Here we describe this latter result, and hence solve the stable roommates problem with forbidden pairs and the super-stable roommates problem, as well.

We have seen that a forbidden edge in the stable roommates problem can be regarded as special edge in the super-stable roommates problem. Interestingly, in the extension of Irving's algorithm we explicitly use forbidden edges: while Irving's original algorithm had only edge deletions, in the extension we eventually need to forbid an edge. Recall that if Irving's algorithm deletes an edge then no new stable matching is created by this and if the deletion kills some stable matching then some other has to survive. Our algorithm deletes an edge in two steps. First it is forbidding edge e if either there exists no super-stable matching in the current instance or if there is a super-stable matching avoiding e . Hence if we find a super-stable matching after forbidding e then it is super-stable before that, and if we conclude that no super-stable matching exists then the conclusion is valid before that step. Our algorithm eventually deletes a forbidden edge e if no super-stable matching exists that is blocked only by e , that is, if no new super-stable matching is created by the deletion.

There is a hierarchy of the different type steps of the following algorithm. Every time we try to execute the first possible of them. To describe these step types, we say that edge $e = E(v)$ of G (forbidden or not) is a *first choice edge of v* , if there is no edge $f \in E(v) \setminus F$ with $f <_v e$ (i.e., if no free edge can dominate e at vertex v). Note that there can be more than one 1st choices of v present.

Proposal step. If $e = vw$ is a 1st choice of v then orient e from v to w . Just like in Irving's algorithm, arcs created in a proposal step are called *1-arcs*. Note that it is possible that a vertex sends more than one 1-arc and a 1-arc can also be bioriented. After we found all 1-arcs, the algorithm looks for a

Mild rejection step. If 1-arc e of G_i points to v and $E(v) \ni f \not\prec_v e$ (that is, f is not better than e according to v in G) then forbid f .

After all such steps are done we move on to find a

Firm rejection step. If some free 1-arc e of G_i points to v and $e <_v f \in E(v)$ (e is better than f according to v) then we delete f .

Note that the above f is already forbidden by a mild rejection step.

As soon as no more proposal and (mild or firm) rejection step can be executed, the following generalization of the first-last property holds: each vertex v that is incident with a free edge sends and receives exactly one 1-arc, and these represents the unique best and worse choices of v , respectively. If some vertex v is not incident with a free edge then v still may be incident with forbidden edges. If this happens then no super-stable matching exist. Otherwise each vertex is either a singleton of G or it sends and receives exactly one 1-arc. The algorithm works further with these latter vertices.

Assume that in G no proposal or rejection step can be executed. An edge $e \in E(v)$ is a *second choice of v* if $e >_v f \notin F$ implies that f is the unique 1st choice of v . In other words, e is a second choice, if the only free edge that dominates e at v is the unique 1-arc leaving v . Every vertex v of G with degree at least two is incident with at least one free second choice edge: if not other, then the unique 1-arc pointing to v . Now the algorithm can make a

2nd choice step. If $e = vw$ is a second choice of v then (counterintuitively) orient e from w to v . Arcs created at this step are called *2-arcs*.

What is the meaning of a 2-arc? It can be interpreted as an implication: if wv is a 2-arc, a is the 1-arc entering w and a' is the 1-arc leaving v then for any super-stable matching S , $a \in S$ implies $a' \in S$. This observation allows us to build an implication structure on the set of 1-arcs. In this implication structure 1-arc e *sm-implies* 1-arc f if there is a directed path starting with e ending with f and using 1-arcs and 2-arcs in an alternating manner. Clearly, if e sm-implies f and S is a super-stable matching then $f \in S$ whenever $e \in S$. We say that 1-arcs e and f are called *sm-equivalent*, if e sm-implies f and f sm-implies e , or, in other words if there is a directed cycle D formed by 1-arcs and 2-arcs in an alternating manner such that D contains both e and f . (Note that D may use the same vertex more than once.) Sm-equivalence is clearly an equivalence relation and if C is an sm-class and S is a super-stable matching then either C is disjoint from S or C is contained in S .

Beyond determining sm-equivalence classes, 2-arcs yield further implications between sm-classes: if uu' is a 1-arc of sm-class C and vv' is a 1-arc of sm-class C' and $u'v$ is a 2-arc, then sm-class C “implies” sm-class C' in such a way that if C is not disjoint from super-stable matching S then S contains both classes C and C' . Assume that sm-class C is on the top of this implication structure, i.e. C is not implied by any other sm-class (but C may imply certain other classes). We can summarize this in the formula that C has the property that

$$(36) \quad \begin{aligned} &\text{if } vv' \text{ is a 1-arc of } C \text{ and } w'v \text{ is a 2-arc} \\ &\text{then (the unique) 1-arc } ww' \text{ is sm-equivalent to } vv'. \end{aligned}$$

To find a top sm-class C , introduce an auxiliary digraph on the vertices of G , such that if uu' is a 1-arc and $u'v$ is a 2-arc, then we introduce an arc uv of the auxiliary graph. We can find a source strong component of the auxiliary graph in linear time by depth first search. If it contains vertices u_1, u_2, \dots, u_k then it determines a top sm-class $C = \{u_1u'_1, u_2u'_2, \dots, u_ku'_k\}$ formed by 1-arcs. Note that it is possible here that $u_i = u'_j$ for different i and j .

2nd choice elimination step. If for 1-arcs $u_iu'_i, u_ju'_j \in C$ there are 2-arcs vu_i and vu_j with $v u_i \not\prec v v u_j$ then forbid $v u_i$.

Note that a 2nd choice elimination step might create some new 2-arcs so we might have to take further 2nd choice steps. After these steps even the top sm-class C may change. Sooner or later we arrive to a situation where none of the above steps are possible. At that moment, if uu' is a 1-arc of C then there is unique 2-arc leaving u' and there is a unique 2-arc pointing to u , that is, u has a unique 2nd choice. This is exactly the same situation that we have in the ordinary stable roommates problem in case of a rotation. That is, if $C \subseteq S$ for a super-stable matching S then matching $S \setminus C \cup C'$ is also super-stable, where C' denotes the set of 2-arcs within C . So if none of the above steps can be executed then we try our last option.

Rotation elimination step. Forbid all edges of C in G .

After a rotation elimination step, the 2-arcs within C become 1-arcs in the opposite direction. This may involve certain mild and firm rejection steps (e.g. we shall delete the forbidden 1-arcs of C), 2nd choice steps, 2nd choice elimination steps and further rotation eliminations. None of these steps kill all super-stable matchings and none of them creates a super-stable matching. Furthermore, if none of the above steps are possible then we are left with a matching on the vertices that are adjacent with ordinary (non-forbidden) edges. So if no vertex is adjacent to a forbidden edge in this situation then what we left with is a super-stable matching. If there are some forbidden edges as well then all these edges must be bidirected 1-arcs and no super-stable matching exists in the particular problem.

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COMPLEMENTARY SUBALGEBRAS. PROBLEMS TO SOLVE

By

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Quantum complementarity is a notion originated by one of the fathers of quantum mechanics, NIELS BOHR. According to WOLFGANG PAULI, the new quantum theory could have been called the theory of complementarity [7]. In traditional quantum theory complementarity is the special relations of the self-adjoint operators P and Q . Our setting can be connected to finite level quantum systems [1, 10].

In abstract mathematical approach we shall consider subalgebras of the algebra $M_n(\mathbb{C})$ of $n \times n$ complex matrices. It is assumed that the subalgebras contain the unit matrix I and closed under taking the adjoint (unital *-subalgebras). The algebra $M_n(\mathbb{C})$ becomes a Hilbert space with the Hilbert–Schmidt inner product:

$$\langle A, B \rangle := \text{Tr } A^* B.$$

Let $\mathcal{A}_1, \mathcal{A}_2$ be subalgebras. Their complementarity is actually a kind of orthogonality relation:

$$\{A \in \mathcal{A}_1 : \text{Tr } A = 0\} \perp \{A \in \mathcal{A}_2 : \text{Tr } A = 0\}$$

The subalgebras \mathcal{A}_1 and \mathcal{A}_2 cannot be orthogonal, since $I \in \mathcal{A}_1, \mathcal{A}_2$, but the above relation is the same as

$$(\mathcal{A}_1 \ominus \mathbb{C}I) \perp (\mathcal{A}_2 \ominus \mathbb{C}I).$$

Sometimes instead of complementarity quasi-orthogonality is used. The content of these notes is an overview of some related unsolved problems.

First we consider commutative subalgebras. Maximal Abelian subalgebras (called MASA's) are one-to-one correspondence with the orthonormal bases of the Hilbert space. The matrices diagonal in a given basis form a

MASA. Assume that \mathcal{A}_1 and \mathcal{A}_2 are MASA's and the corresponding bases are $\xi_1, \xi_2, \dots, \xi_n$ and $\eta_1, \eta_2, \dots, \eta_n$. The subalgebras \mathcal{A}_1 and \mathcal{A}_2 are complementary if and only if

$$|\langle \xi_i, \eta_j \rangle|^2 = \frac{1}{n} \quad (1 \leq i, j \leq n).$$

It is easy to give an example. Fix the basis $\xi_1, \xi_2, \dots, \xi_n$ and let U be the unitary making a cyclic permutation. If the vectors η_j are eigenvectors of U , then ξ_i and η_j satisfy the above property. They are called mutually unbiased bases, MUB's. The maximal number of MUB's is $n + 1$. If n is a power of a prime number, then this upper bound can be reached [16, 13]. It is little known about the case $n = 6$ [5, 6].

A MASA is n -dimensional, a two dimensional commutative algebra has the form $\mathcal{A} = \{\lambda P_1 + \mu P_2 : \lambda, \mu \in \mathbb{C}\}$, where P_1 and P_2 are orthogonal projections. Assume that P_1 is of rank one, then it has the form $|\xi\rangle\langle\xi|$, where $|\xi\rangle$ is a unit vectors. Such subalgebra is determined by the vector $|\xi\rangle$. These kind of subalgebras $\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_k$ are complementary if

$$|\langle \xi_i, \xi_j \rangle|^2 = \frac{1}{n} \quad (1 \leq i < j \leq k).$$

Denote by $q(n)$ the maximum number of such vectors. The construction of such vectors is in relation with another problem. In the n -dimensional space one tries to get n^2 vectors $\eta_1, \eta_2, \dots, \eta_{n^2}$ such that

$$(1) \quad |\langle \eta_i, \eta_j \rangle|^2 = c \quad (1 \leq i < j \leq n^2) \quad \text{and} \quad \sum_{i=1}^{n^2} |\eta_i\rangle\langle\eta_i| = dI$$

for some constants $c, d \in \mathbb{R}$. These conditions imply $c = 1/(n + 1)$. The existence of vectors is known for $n = 2, 3, 4, 5$ and some numerical arguments are available up to $n = 45$ [5, 20]. If there are n^2 vectors with the property of (1) in n dimension, then

$$q(n + 1) \geq n^2.$$

This is a lower estimate. $q(3) \geq 5$ can be shown by construction [2]. Nothing else is known about $q(n)$ for $n > 3$.

Now we move to noncommutative subalgebras. The simplest situation is find complementary subalgebras of $M_4(\mathbb{C}) \simeq M_2(\mathbb{C}) \otimes M_2(\mathbb{C})$ which are isomorphic to $M_2(\mathbb{C})$. Since the dimension of $M_4(\mathbb{C})$ is 16 and the dimension of $M_4(\mathbb{C})$ is 4, the maximum number is $(16 - 1)/(4 - 1) = 5$. There are

no 5 such subalgebras [12] and even more is true. If $\mathcal{A}_1, \dots, \mathcal{A}_4$ are such subalgebras, then their orthogonal complement together with I is a MASA [9]. It seems that the prime number 2 is particular. Assume that the large algebra is a k fold product $M_n(\mathbb{C}) \otimes M_n(\mathbb{C}) \otimes \dots \otimes M_n(\mathbb{C})$ and we want to find complementary subalgebras isomorphic to $M_n(\mathbb{C})$. If n is a power of a prime number larger than 2, then the maximum number

$$\frac{n^{2k} - 1}{n^2 - 1}$$

of subalgebras $M_n(\mathbb{C})$ is possible [8]. If $n = 2$, there is a construction for

$$\frac{2^{2k} - 1}{2^2 - 1} - 1$$

complementary subalgebras isomorphic to $M_2(\mathbb{C})$, but it is not known if this is the maximal number [9]. If it is, then one can ask about the orthogonal complement. Is it a subalgebra if the I is added?

The complementary commutative subalgebras give the smallest over-heating information about a quantum state. This makes some optimality in state estimation [18, 3] and has application in quantum cryptography [4]. Complementary MASA's are studied also in the setting of von Neumann algebras [14]. The physical background of noncommutative complementary algebras has not been studied. From the point of view of information, the essential property must be similar to the case of commutative subalgebras.

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A CHANGE OF VARIABLES THEOREM FOR THE MULTIDIMENSIONAL RIEMANN INTEGRAL

By

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Abstract. The most general change of variables theorem for the Riemann integral of functions of a single variable has been published in 1961 by H. Kestelman. In this theorem, the substitution is made by an ‘indefinite integral’, that is, by a function of the form $t \mapsto c + \int_a^t g =: G(t)$ where g is Riemann integrable on $[a, b]$ and c is any constant. We prove a multidimensional generalization of this theorem for the case where G is injective – using the fact that the Riemann primitives are the same as those Lipschitz functions which are almost everywhere strongly differentiable in (a, b) . We prove a generalization of Sard’s lemma for Lipschitz functions of several variables that are almost everywhere strongly differentiable, which enables us to keep all our proofs within the framework of the Riemannian theory which was our aim.

1. Introduction

As far as we know, the following theorem appeared first in [5].

THEOREM 1. *If $g : [a, b] \rightarrow \mathbb{R}$ is Riemann integrable, $c \in \mathbb{R}$, $\forall t \in [a, b]$ $G(t) := c + \int_a^t g$, and the function f is Riemann integrable on the range of G , then $(f \circ G) \cdot g$ is Riemann integrable, and*

$$\int_{G(a)}^{G(b)} f = \int_a^b (f \circ G) \cdot g.$$

Notice that the first statement of Theorem 1 is somewhat surprising because the composition $f \circ G$ need not be Riemann integrable even if G is C^∞ (see [3, Example 34 in Chapter 8.]). Some years later, D. Preiss and J. Uher

in [12] proved the converse: boundedness of f and integrability of $(f \circ G)g$ implies integrability of f .

The aim of the present paper is to formulate and prove a multidimensional version of these two theorems for the case where G is injective on the interior of its domain — with a proof that remains within the framework of the Riemann theory. In textbooks, the usual assumptions on G are that it is injective and it has a continuously differentiable extension to an open set that covers the closure of the original – Jordan measurable – domain. Observe that the latter assumption implies Lipschitz continuity (see Theorem 22). Moreover, in almost every textbook, the theorem is proved under the additional assumption that $G'(x)$ is everywhere invertible. Probably M. Spivak was the first who proved that this assumption can be omitted (see [14]). An interesting version of the theorem was proved by P. Lax in [6]. In his theorem, both the absolute value in the formula and the injectivity assumption are omitted, but there are other assumptions instead.

The starting point to the generalization mentioned above is the fact (which seems not to be well-known) that a function $G: [a, b] \rightarrow \mathbb{R}$ is a Riemann primitive if and only if it is Lipschitz and almost everywhere strongly differentiable (see Definition 9 and Theorem 29).

In the next section, after introducing some notation and terminology, we summarize some well-known facts about Riemann integrability, and give a basic theorem, Theorem 8, about the change of variables with easy proof and 'hard-to-check' assumptions. In section 3, we investigate the notion of 'strong differentiability' and other auxiliary tools, then in section 4 we prove that for injective functions G that are Lipschitz and almost everywhere strongly differentiable, and for properly chosen g , conditions a), b) and c) of Theorem 8 are fulfilled (injectivity will be assumed only on the difference of $D(G)$ and a set of Lebesgue measure zero).

2. Terminology and some basic facts about Riemann integrability

For any $H \subset \mathbb{R}^m$, the set of Jordan measurable subsets of H will be denoted by \mathcal{J}_H , in the case $H := \mathbb{R}^m$ the subscript will be omitted. The volume or Jordan content of a Jordan measurable set $X \subset \mathbb{R}^m$ will be denoted by $V(X)$ and the outer Jordan content of a bounded set Y by $V^*(Y)$.

By a Jordan partition of $X \in \mathcal{J}$ we mean a finite collection of pairwise non-overlapping sets in \mathcal{J}_X the union of which is X . The set of all Jordan partitions of $X \in \mathcal{J}$ will be denoted by $\Pi(X)$. By the norm of a Jordan

partition $\Phi \in \Pi(X)$ we mean the number $|\Phi| := \max\{\text{diam}(H) : H \in \Phi\}$. The lower sum, upper sum and sum of oscillations of a bounded function $f: X \rightarrow \mathbb{R}$ corresponding to the partition $\Phi \in \Pi(X)$ is defined by $s_f(\Phi) := \sum_{H \in \Phi} \inf f|_H V(H)$, $S_f(\Phi) := \sum_{H \in \Phi} \sup f|_H V(H)$,

$$\mathcal{O}_f(\Phi) := S_f(\Phi) - s_f(\Phi) = \sum \text{osc}_f(H) V(H)$$

respectively, where

$$\text{osc}_f(H) := \sup f|_H - \inf f|_H = \sup\{|f(y) - f(x)| : x \in H, y \in H\}.$$

The lower and upper Darboux integral of f is

$$\underline{\int}_X f := \sup s_f, \quad \text{and} \quad \overline{\int}_X f := \inf S_f,$$

respectively. The bounded function $f: X \rightarrow \mathbb{R}$ is integrable (with integral $\alpha \in \mathbb{R}$) if its lower and upper Darboux integrals agree (and are equal to α).

By a dotted Jordan partition of $X \in \mathcal{J}$ we mean a finite set of ordered pairs

$$\eta := \{(H_1, y_1), \dots, (H_n, y_n)\}$$

such that $D(\eta) := \{H_1, \dots, H_n\} \in \Pi(X)$, and $y_i \in H_i$ for $i = 1, \dots, n$. The Riemann sum of the function $f: X \rightarrow \mathbb{R}$ corresponding to the dotted Jordan partition η is $\sigma_f(\eta) := \sum_{i=1}^n f(y_i) V(H_i)$.

We will make use of the following well-known statements:

THEOREM 2. (Generalized Darboux Theorem) *For each $X \in \mathcal{J}$ and for each bounded $f: X \rightarrow \mathbb{R}$,*

$$\lim_{|\Phi| \rightarrow 0} s_f(\Phi) = \underline{\int}_X f, \quad \lim_{|\Phi| \rightarrow 0} S_f(\Phi) = \overline{\int}_X f.$$

THEOREM 3. (modified Riemann's condition) *For each $X \in \mathcal{J}$ and for each bounded $f: X \rightarrow \mathbb{R}$, integrability of f is equivalent to the condition $\lim_{|\Phi| \rightarrow 0} \mathcal{O}(\Phi) = 0$.*

THEOREM 4. *For each $X \in \mathcal{J}$, $\alpha \in \mathbb{R}$ and $f: X \rightarrow \mathbb{R}$, the following two statements are equivalent: 1. f is integrable with integral α , 2. $\lim_{|D(\eta)| \rightarrow 0} \sigma_f(\eta) = \alpha$.*

The definition of the integral based on Riemann sums can be used in the matrix-valued case, too. In $\mathbb{R}^{m \times n}$, any metric induced by a norm can

be used. In particular, for each $X \in \mathcal{J}$ and integer $m > 1$, a matrix-valued function $h: X \rightarrow \mathbb{R}^{m \times m}$ is integrable if and only if all the entries $h_{ik}: X \rightarrow \mathbb{R}$ ($i, k = 1, \dots, m$) are integrable. This fact will be used in order to simplify the formulation of our last theorem.

THEOREM 5. *For each $X \in \mathcal{J}$, $H \in \mathcal{J}_X$ and integrable $f: X \rightarrow \mathbb{R}$, the restriction $f|_H$ is integrable, for each $\Phi \in \Pi(X)$, $\int_X f = \sum_{H \in \Phi} \int_H f$.*

THEOREM 6. *For each $X \in \mathcal{J}$ and integrable $f: X \rightarrow \mathbb{R}$, the function $|f|$ is integrable, and the inequality $|\int_X f| \leq \int_X |f|$ holds.*

DEFINITION 7. If $X \subset \mathbb{R}^m$, $\mathcal{D} \subset \mathcal{J}_X$, $\cup \mathcal{D} = X$ and $\Psi: \mathcal{D} \rightarrow \mathbb{R}$, then by a density function of Ψ we mean a function $g: X \rightarrow \mathbb{R}$ for which integrability of $g|_H$ and $\Psi(H) = \int_H g$ hold for each $H \in \mathcal{D}$.

THEOREM 8. *Let $X \in \mathcal{J}$, $G: X \rightarrow \mathbb{R}^m$ a Lipschitz function, $Y := G(X)$, $f: Y \rightarrow \mathbb{R}$ bounded and $g: X \rightarrow \mathbb{R}$ integrable. Suppose that*

- a) *for each $H \in \mathcal{J}_X$, $G(H) \in \mathcal{J}$,*
- b) *for each pair of non-overlapping sets $A \in \mathcal{J}_X$, $B \in \mathcal{J}_X$, $G(A)$ and $G(B)$ are non-overlapping,*
- c) *g is a density function of the function $\mathcal{J}_X \ni H \mapsto V(G(H))$,*
- d) *f or $(f \circ G)g$ is integrable.*

Then both f and $(f \circ G)g$ are integrable and $\int_X (f \circ G)g = \int_Y f$.

PROOF. Let $L > 0$ be a Lipschitz constant for G , $K > 0$ such that for every $x \in Y$, $|f(x)| \leq K$, and use the notation $\psi := (f \circ G)g$.

First, suppose that f is integrable and let ε be a positive number. We will show that for some $\delta > 0$, and for each dotted partition η of X with $|D(\eta)| < \delta$, we have $|\sigma_\psi(\eta) - \int_Y f| < \varepsilon$ (see Theorem 4). Choose positive numbers δ_f and δ_g such that $\mathcal{O}_f(\Psi) < \varepsilon/2K$ whenever $\Psi \in \Pi(Y)$ and $|\Psi| < \delta_f$, resp. $\mathcal{O}_g(\Phi) < \varepsilon/2K$ whenever $\Phi \in \Pi(X)$ and $|\Phi| < \delta_g$ (see Theorem 3). Let $\{(H_k, y_k) : k = 1, \dots, n\}$ a dotted partition of X such that the norm of $\Phi := \{H_1, \dots, H_n\}$ is less than $\min\{\delta_g, \delta_f/L\} =: \delta$. Conditions a) and b) imply that $\Psi := \{G(H_1), \dots, G(H_n)\}$ is a Jordan partition of Y , Lipschitz condition and the definition of L imply that the norm of this latter partition is smaller than δ_f .

$$\left| \sum_{k=1}^n f(G(y_k))g(y_k) V(H_k) - \int_Y f \right| \stackrel{5}{=} \left| \sum_{k=1}^n \left(f(G(y_k))g(y_k) V(H_k) - \int_{G(H_k)} f \right) \right| =$$

$$\begin{aligned}
&= \left| \sum_{k=1}^n \left[f(G(y_k)) [g(y_k) V(H_k) - V(G(H_k))] + f(G(y_k)) V(G(H_k)) - \int_{G(H_k)} f \right] \right| \leq \\
&\stackrel{\text{c)}{\leq} \sum_{k=1}^n |f(G(y_k))| \left| g(y_k) V(H_k) - \int_{H_k} g + \sum_{k=1}^n \left| f(G(y_k)) V(G(H_k)) - \int_{G(H_k)} f \right| \right| = \\
&= \sum_{k=1}^n |f(G(y_k))| \left| \int_{H_k} [g(y_k) - g(x)] dx + \sum_{k=1}^n \left| \int_{G(H_k)} [f(G(y_k)) - f(y)] dy \right| \right| \leq \\
&\leq K \sum_{k=1}^n \int_{H_k} |g(y_k) - g(x)| dx + \sum_{k=1}^n \int_{G(H_k)} |f(G(y_k)) - f(y)| dy \leq \\
&\leq K \sum_{k=1}^n \text{osc}_g(H_k) V(H_k) + \sum_{k=1}^n \text{osc}_f(G(H_k)) V(G(H_k)) = \\
&= K \mathcal{O}_g(\Phi) + \mathcal{O}_f(\Psi) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2}.
\end{aligned}$$

Second, suppose that ψ is integrable, we prove that $\int_X \psi$ is equal to the upper Darboux integral of f . The proof of the fact that $\int_X \psi$ is equal to the lower integral of f is completely similar, therefore it will be omitted. Let ε be a positive number, we show that $\int_X \psi$ is in the ε -neighborhood of the upper integral of f . According to Theorem 4, one can choose a $\delta_\psi > 0$ such that $|\sigma_\psi(\eta) - \int_X \psi| < \varepsilon/4$ holds for every dotted partition η of X with $|D(\eta)| < \delta_\psi$, according to Theorem 3, one can choose a $\delta_g > 0$ such that $\mathcal{O}_g(\Phi) < \varepsilon/4K$ holds whenever the norm of $\Phi \in \mathcal{J}_X$ is less than δ_g and according to Theorem 2, one can choose a $\delta_f > 0$ such that $S_f(\Psi)$ lies in the $\varepsilon/4$ -neighborhood of the upper integral of f whenever the norm of $\Psi \in \mathcal{J}_Y$ is less than δ_f . Fix a Jordan partition $\Phi = \{H_1, \dots, H_n\} \in \mathcal{J}_X$ with

$$|\Phi| < \min\{\delta_\psi, \delta_g, \delta_f/L\} =: \delta,$$

and for each $k = 1, \dots, n$ an element $y_k \in H_k$ such that

$$f(G(y_k)) > \sup f|_{G(H_k)} - \frac{\varepsilon}{4(V(Y)+1)}.$$

Denoting the collection of sets $G(H_k)$ by Ψ and the set of pairs (H_k, y_k) by η ($k = 1, \dots, n$), we have

$$\begin{aligned} \int_X \psi - \bar{\int}_Y f &= \left[\int_X \psi - \sigma_\psi(\eta) \right] + \sum_{k=1}^n f(G(y_k)) [g(y_k) V(H_k) - V(G(H_k))] + \\ &\quad + \sum_{k=1}^n [f(G(y_k)) - \sup f|_{G(H_k)}] V(G(H_k)) + \left[S_f(\Psi) - \bar{\int}_Y f \right]. \end{aligned}$$

$\delta \leq \delta_\psi$, $\delta \leq \delta_g$, the choice of the points y_k and $\delta \leq \delta_f$ imply that the absolute value of the first, second, third, respectively the fourth term on the right hand side is less than $\varepsilon/4$. As for the second term, this is seen from the following estimate:

$$\begin{aligned} &\left| \sum_{k=1}^n f(G(y_k)) [g(y_k) V(H_k) - V(G(H_k))] \right| = \\ &\stackrel{\text{c)}}{=} \left| \sum_{k=1}^n f(G(y_k)) \int_{H_k} [g(y_k) - g(x)] dx \right| \leq \\ &\leq \sum_{k=1}^n |f(G(y_k))| \int_{H_k} |g(y_k) - g(x)| dx \leq K \sum_{k=1}^n \int_{H_k} \text{osc}_g(H_k) = K \mathcal{O}_g(\Phi). \blacksquare \end{aligned}$$

3. Auxiliary tools

3.1. Strong differentiability

DEFINITION 9. Let m and n be positive integers, $U \subset \mathbb{R}^m$, and u an interior point of U . The function $f: U \rightarrow \mathbb{R}^n$ is *strongly differentiable* at the point u , if there exists a linear map $A: \mathbb{R}^m \rightarrow \mathbb{R}^n$ such that

$$\lim_{(x,y) \rightarrow (u,u)} \frac{1}{\|x - y\|} [f(x) - f(y) - A(x - y)] = 0,$$

where on $\mathbb{R}^m \times \mathbb{R}^m$ one can use any metric induced by a norm, e.g.

$$d((x,y), (z,w)) := \max\{\|x - z\|, \|y - w\|\}.$$

We make some remarks about this notion.

Some authors use the term ‘strict differentiability’ instead of ‘strong differentiability.’

In the definition the spaces \mathbb{R}^m and \mathbb{R}^n could be replaced by any normed spaces, but in this case (if the first space is infinite dimensional) one says ‘continuous linear’ instead of ‘linear’.

If f is strongly differentiable at u then it is differentiable there and $f'(u) = A$ must hold.

Strong differentiability of f at u implies the existence of a neighborhood of u on which f is a Lipschitz function.

If one replaces the assumption on continuous differentiability of f at the point u by strong differentiability at u in the local inverse function theorem (see [4]), then one can state existence of a neighborhood U of u such that the restriction $f|_U$ is *injective*, its range is a neighborhood of $f(u)$, the local inverse (the inverse of this restriction) is strongly differentiable at the point $f(u)$, and the derivative of the local inverse at $f(u)$ is equal to the inverse of $f'(u)$, (see for example [7] or [8]). The most difficult part of the proof is essentially contained in the proof of the next theorem (the proof of the fact that $f(u)$ is an interior point of the range of the injective restriction).

THEOREM 10. *Let u be an element of the open set $\Omega \subset \mathbb{R}^m$, suppose that the function $G: \Omega \rightarrow \mathbb{R}^m$ is strongly differentiable at u and $M := G'(u)$ is regular; for each $x \in \Omega$, let $\ell_x: \mathbb{R}^m \rightarrow \mathbb{R}^m$ be the affine function $z \mapsto G(x) + M(z - x)$. Then for each $\varepsilon \in (0, 1)$, there exists a $\delta > 0$ such that $\forall (x, r) \in \Omega \times (0, +\infty)$,*

$\overline{B}(x, r) \subset \overline{B}(u, \delta) \implies \ell_x \overline{B}(x, (1 - \varepsilon)r) \subset G(\overline{B}(x, r)) \subset \ell_x \overline{B}(x, (1 + \varepsilon)r)$,
in particular, $G(u) \in \text{int } R(G)$.

PROOF. Define the function $\varrho: \Omega \times \Omega \rightarrow \mathbb{R}^m$ as follows: if $x, z \in \Omega$ and $x = z$ then $\varrho(x, z) := 0$, otherwise

$$\varrho(x, z) := \frac{1}{\|z - x\|}[G(z) - G(x) - M(z - x)].$$

The strong differentiability condition implies that for each $\varepsilon \in (0, 1)$, one can find a $\delta > 0$ such that

$$\|\varrho(x, z)\| < \frac{\varepsilon}{\|M^{-1}\|}, \quad \text{whenever } x, z \in \overline{B}(u, \delta).$$

Fix a pair (x, r) satisfying the condition $\overline{B}(x, r) \subset \overline{B}(u, \delta)$, and, in order to prove the first inclusion, fix an element $y = \ell_x(v)$ with $\|v - x\| \leq (1 - \varepsilon)r$

as well. One can apply Banach's fixed point theorem on the metric subspace $X := \overline{B}(x, r)$ of \mathbb{R}^m to the function

$$f: X \rightarrow \mathbb{R}^m, \quad z \mapsto z - M^{-1}[G(z) - y] = M^{-1}[y - G(z) + Mz]$$

(of course, the fixed point is a point $z \in \overline{B}(x, r)$ with $G(z) = y$). Indeed, f maps X into X , because for each $z \in X$ we have

$$\begin{aligned} \|f(z) - x\| &= \|M^{-1}[y - G(x) + G(x) - G(z) - M(x - z)]\| \leq \\ \|M^{-1}[\ell_x(v) - G(x)]\| &+ \|M^{-1}\| \|G(x) - G(z) - M(x - z)\| \leq \\ \|v - x\| + \varepsilon \|x - z\| &\leq (1 - \varepsilon)r + \varepsilon r = r, \end{aligned}$$

and it is a contraction with Lipschitz constant ε , because for each pair $(z, w) \in X \times X$ we have

$$\|f(z) - f(w)\| = \|M^{-1}[G(w) - G(z) - M(w - z)]\| \leq \|M^{-1}\| \frac{\varepsilon}{\|M^{-1}\|} \|w - z\|.$$

To prove the second inclusion, fix an element $v \in \overline{B}(x, r)$ and set

$$w := M^{-1}[\|v - x\| \varrho(x, v)].$$

Then $\|w\| \leq \|M^{-1}\| \cdot \|\varrho(x, v)\| \cdot \|v - x\|$, hence $\|v + w - x\| \leq r + \varepsilon r$, and $G(v) = G(x) + M(v - x) + \|v - x\| \varrho(x, v) = G(x) + M(v + w - x) = \ell_x(v + w)$.

To show that $G(u)$ is an interior point of the range, apply the first inclusion with $\varepsilon := 1/2$, $x := u$, $r := \delta$:

$$G(\overline{B}(u, \delta)) \supset G(u) + MB(0, \delta/2) \supset G(u) + B(0, \delta/(2\|M^{-1}\|)). \blacksquare$$

REMARK 11. Let m, n, U and u be the same as in Definition 9. The function $f: U \rightarrow \mathbb{R}^n$ is strongly differentiable at u if and only if each component of f has this property.

REMARK 12. Let m, n, U and u be the same as in Definition 9. If the function $f: U \rightarrow \mathbb{R}^n$ is strongly differentiable at u , then, without any assumption on the domain of f' , the function $x \mapsto f'(x)$ is continuous at u . For a proof, see [10].

REMARK 13. Let m, n, U and u be the same as in Definition 9. If the function $f: U \rightarrow \mathbb{R}^n$ is differentiable in a neighborhood of u and f' is continuous at u then f is strongly differentiable at u . As for the proof: apply the mean value inequality to the function $z \mapsto f(z) - f'(u)z$ on the line segment $[x, y]$.

Before our last remark we introduce a definition which is a slight modification of Nijenhuis' definition (see [10]).

DEFINITION 14. Let m and j be positive integers, $j \leq m$, $U \subset \mathbb{R}^m$, and u an interior point of U . The function $f: U \rightarrow \mathbb{R}$ is *strongly partially differentiable* with respect to the j -th variable at the point u , if there exists a real number $D_j^s f(u)$ such that for each $\varepsilon > 0$ there is a $\delta > 0$ with the following property: if $x, y \in B(u, \delta)$, $y_j \neq x_j$, but for all $i \neq j$ $y_i = x_i$, then

$$\left| \frac{f(x) - f(y)}{x_j - y_j} - D_j^s f(u) \right| < \varepsilon.$$

REMARK 15. Let U be a subset of \mathbb{R}^m , and u an interior point of U . The function $f: U \rightarrow \mathbb{R}$ is strongly differentiable at u if and only if f is strongly partially differentiable at u with respect to all its variables. (The proof is an easy exercise, for the case $m = 2$ see [10].)

3.2. Cubes, set-functions

A (closed) cube in \mathbb{R}^m is the product of m many closed intervals of equal length, by a cube-partition of a cube Q we mean a finite set of pairwise non-overlapping cubes, the union of which is Q . Analogously, by a dotted cube-partition of a cube Q we mean a finite set

$$\{(Q_1, y^1), \dots, (Q_n, y^n)\}$$

of ordered pairs, where $\{Q_1, \dots, Q_n\}$ forms a cube-partition of Q and $y^i \in Q_i$ for $i = 1, \dots, n$. Equivalently, dotted cube-partitions of a cube Q can be viewed as functions: a function $\eta: \mathcal{A} \rightarrow Q$ is a dotted cube-partition of the cube Q , if \mathcal{A} is a cube-partition of Q and for each cube $I \in \mathcal{A}$, $\eta_I := \eta(I) \in I$. In the following, (dotted) partition of a cube will always mean a (dotted) cube-partition.

In the space \mathbb{R}^m we use the norm $x \mapsto \max |x_i| =: \|x\|$, therefore the closed balls $\overline{B}(y, r)$ are cubes and the open balls $B(y, r)$ are open cubes.

Let δ be a positive valued function defined on a cube Q . A dotted partition η of Q is said to be δ -fine, if for each $I \in D(\eta)$, $I \subset B(\eta_I, \delta(\eta_I))$. The following statement will be called 'Cousin's lemma':

LEMMA 16. (Cousin's lemma) *For each cube Q and each function $\delta: Q \rightarrow (0, +\infty)$, Q has a δ -fine dotted partition.*

For a proof of this assertion, see for example the proof of [11, Lemma 7.3.2].

DEFINITION 17. A real valued function Φ defined on a set $\mathcal{C} \subset \mathcal{J}$ will be called

1. *additive*, if for each $H \in \mathcal{C}$ and each Jordan partition $\mathcal{A} \subset \mathcal{C}$ of H ,

$$\Phi(H) = \sum_{J \in \mathcal{A}} \Phi(J),$$

2. *Lipschitz*, if there exists a nonnegative number L such that for each $H \in \mathcal{C}$, $\Phi(H) \leq L \cdot V(H)$,

DEFINITION 18. Suppose that for some $r > 0$ and $u \in \mathbb{R}^m$, each closed subcube of $B(u, r)$ belongs to the domain of the real valued set-function Φ . Then Φ is called *differentiable* (resp. *strongly differentiable*) at u , if for some real number $\Phi'(u)$ and for each $\omega > 0$ there exists a $\delta > 0$ for which $u \in I \in D(\Phi)$ (resp. $I \in D(\Phi)$) and $I \subset B(u, \delta)$ imply

$$\left| \frac{\Phi(I)}{V(I)} - \Phi'(u) \right| < \omega.$$

Of course, if a cube-function Φ is differentiable at u then the number $\Phi'(u)$ in the definition of differentiability is unique. Note that if $m = 1$, $f: [a, b] \rightarrow \mathbb{R}$ and Φ is defined on the set of closed subintervals of $[a, b]$ by $[\alpha, \beta] \mapsto f(\beta) - f(\alpha)$, then Φ is additive, Φ is Lipschitz if and only if f is Lipschitz, Φ is (strongly) differentiable at $u \in (a, b)$ if and only if f is (strongly) differentiable there.

REMARK 19. Another important and well-known example: if $X \in \mathcal{J}$ and $g: X \rightarrow \mathbb{R}$ is integrable, then the set-function $\mathcal{J}_X \ni H \mapsto \int_H g =: \Psi(H)$ is additive, Lipschitz, and strongly differentiable at the continuity points u of g with $\Psi'(u) = g(u)$.

3.3. Lipschitz functions and Lipschitz set-functions

LEMMA 20. If (M, d) is a metric space, $\emptyset \neq X \subset M$ and $G: X \rightarrow \mathbb{R}^m$ is a Lipschitz function with Lipschitz constant L , then G has a Lipschitz extension $F: M \rightarrow \mathbb{R}^m$ with the same Lipschitz constant L .

PROOF. Because of our choice of the norm in \mathbb{R}^m , the lemma follows from the special case where $m = 1$, which can be applied to the component functions. But this special case is a well-known theorem, for a proof see for example [11, 6.6.5 and 6.6.6].

THEOREM 21. *Let $X \in \mathcal{J}$ and $G: X \rightarrow \mathbb{R}^m$ be a Lipschitz function, then the set-function $\mathcal{J}_X \ni H \mapsto V^*(G(H)) =: \Psi(H)$ is again Lipschitz.*

PROOF. Let $F: \mathbb{R}^m \rightarrow \mathbb{R}^m$ be an extension of G satisfying the Lipschitz condition with Lipschitz constant L (see Lemma 20), so for each cube $I = \overline{B}(u, r) \subset \mathbb{R}^m$, Lipschitz condition yields $F(I) \subset \overline{B}(F(u), Lr)$, consequently $V^*(F(I)) \leq L^m V(I)$. Let $H \in \mathcal{J}_X$, $\varepsilon > 0$ and $\{I_1, \dots, I_n\}$ be a finite set of cubes such that

$$H \subset \bigcup_{k=1}^n I_k \quad \text{and} \quad \sum_{k=1}^n V(I_k) < V(H) + \varepsilon.$$

V^* is monotonic and subadditive, therefore

$$\begin{aligned} V^*(G(H)) &\leq V^*\left[\bigcup_{k=1}^n F(I_k)\right] \leq \\ &\leq \sum_{k=1}^n V^*(F(I_k)) \leq L^m \sum_{k=1}^n V(I_k) < L^m(V(H) + \varepsilon) \end{aligned}$$

and this gives the inequality $\Psi \leq L^m V|_{\mathcal{J}_X}$. ■

THEOREM 22. *If (X_1, d_1) is a compact metric space, (X_2, d_2) a metric space and $f: X_1 \rightarrow X_2$ is locally Lipschitz at each point of X_1 then f is Lipschitz.*

PROOF. Using on $X_1 \times X_1$ – for example – the metric

$$(x, y), (u, v) \mapsto \max\{d_1(x, u), d_1(y, v)\},$$

$X_1 \times X_1$ is compact, therefore (being a closed subset of this compact space) the diagonal $\Delta := \{(x, x) : x \in X_1\}$ is also compact. This fact and the local Lipschitz condition gives a positive integer n , elements $z_1, \dots, z_n \in X_1$ and positive numbers $r_1, \dots, r_n, L_1, \dots, L_n$ such that for each $k = 1, \dots, n$, $f|_{B(z_k, r_k)}$ is Lipschitz with Lipschitz constant L_k , and

$$\Delta \subset \bigcup_{k=1}^n B(z_k, r_k) \times B(z_k, r_k) =: \Gamma.$$

$(X_1 \times X_1) \setminus \Gamma$ is again compact, the metrics and G are continuous, thus the restriction to $(X_1 \times X_1) \setminus \Gamma$ of the function

$$(X_1 \times X_1) \setminus \Delta \ni (x, y) \mapsto \frac{d_2(f(x), f(y))}{d_1(x, y)} =: h(x, y)$$

has an upper bound L_0 . This implies that $\max\{L_0, L_1, \dots, L_n\}$ is an upper bound of h . ■

3.4. Some consequences of Cousin's lemma

The following lemma is known from several proofs of Sard's lemma (see for example [14, proof of Theorem 3.14.]).

LEMMA 23. *Suppose that A is a subset of \mathbb{R}^m , $G: A \rightarrow \mathbb{R}^m$ is strongly differentiable at the interior point u of A and let $G'(u)$ be singular. Then for each $\varepsilon > 0$ there exists a $\delta > 0$ such that $B(u, \delta) \subset A$, and the inequality $V^*(G(I)) \leq \varepsilon V(I)$ holds for all cubes I covered by $B(u, \delta)$.*

Now we prove an interesting version of the so-called Sard's lemma. Observe that as A. Sard himself writes in [13], the real valued C^1 -case is due to A. P. Morse (see [9]). If G is differentiable at an interior point x of its domain then the Jacobian matrix of G at x will be denoted by $J_G(x)$.

THEOREM 24. *Suppose that Ω is an open subset of \mathbb{R}^m , $K \subset \Omega$ is a set of Lebesgue measure 0 and $G: \Omega \rightarrow \mathbb{R}^m$ is a Lipschitz function which is strongly differentiable at each point of $\Omega \setminus K$. Then the image under G of the set*

$$\{x \in \Omega \setminus K : J_G(x) \text{ is singular}\}$$

is of Lebesgue measure 0.

PROOF. Ω is a countable union of cubes, so it is enough to prove that for any cube $Q \subset \Omega$, the image under G of the set

$$S := \{x \in Q \setminus K : J_G(x) \text{ is singular}\}$$

has Jordan content 0. Set

$$R := \{x \in Q \setminus K : J_G(x) \text{ is regular}\}, \quad \text{and} \quad T := Q \cap K.$$

First, observe that $R \subset \text{ext } S$. Indeed, J_G is continuous at every point of R (see Remark 12) and so is the function $\det: \mathbb{R}^{m \times m} \rightarrow \mathbb{R}$, thus a neighborhood of a point of R in which for every x we have $\det J_G(x) \neq 0$ cannot intersect S . Second, observe that if L is a Lipschitz constant for G then for any cube $I := \overline{B}(u, r) \subset \Omega$ we have $G(I) \subset \overline{B}(G(u), Lr)$, so $V^*(G(I)) \leq L^m V(I)$. In order to apply Cousin's lemma, we define a positive valued function δ on Q . Let ε be a positive number, fix a countable set \mathcal{J} of open intervals with the sum of volumes being less than $\varepsilon / 2L^m$, the union of which covers the set T . If $u \in R$ then let $\delta(u) > 0$ be such that $B(u, \delta(u)) \cap S = \emptyset$. For each $u \in S$, using the previous lemma, select a $\delta(u) > 0$ such that $V^*(G(I)) \leq \varepsilon V(I)/2V(Q)$ holds for every cube I satisfying conditions $u \in I \subset B(u, \delta(u))$. For each $u \in T$, select first a $J_u \in \mathcal{J}$ that contains u and then a $\delta(u) > 0$ such that

$B(u, \delta(u)) \subset J_u$. Fix a δ -fine dotted partition η of Q and write its domain in the form $\mathcal{A} \cup \mathcal{B} \cup \mathcal{C}$ where for a cube $I \in D(\eta)$, $I \in \mathcal{A}$ means $\eta_I \in R$, $I \in \mathcal{B}$ means $\eta_I \in S$ and $I \in \mathcal{C}$ means $\eta_I \in T$. As η is δ -fine, each $I \in \mathcal{A}$ is disjoint from S , hence

$$G(S) \subset \bigcup_{I \in \mathcal{B} \cup \mathcal{C}} G(I).$$

Finally, using subadditivity of V^* and the definition of δ , we have

$$\begin{aligned} V^* \left(\bigcup_{I \in \mathcal{B} \cup \mathcal{C}} G(I) \right) &\leq \sum_{I \in \mathcal{B} \cup \mathcal{C}} V^*(G(I)) = \sum_{I \in \mathcal{B}} V^*(G(I)) + \sum_{I \in \mathcal{C}} V^*(G(I)) \leq \\ &\leq \sum_{I \in \mathcal{B}} \frac{\varepsilon}{2V(Q)} V(I) + L^m \sum_{I \in \mathcal{C}} V(I) \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \blacksquare \end{aligned}$$

LEMMA 25. 1. Let Q be a cube, denote by \mathcal{C} the set of subcubes of Q and let $\Phi: \mathcal{C} \rightarrow \mathbb{R}$ be an additive Lipschitz function such that $\Phi'(u) = 0$ holds for almost all interior points u of Q . Then Φ is the constant 0 function.

2. If $X \in \mathcal{J}$ and $\Psi: \mathcal{J}_X \rightarrow \mathbb{R}$ is an additive Lipschitz function such that $\Psi'(u) = 0$ holds for almost all interior points of X , then Ψ is the constant 0 function.

PROOF. 1. Suppose the contrary, then there exists a subcube K such that $\varepsilon := |\Phi(K)| > 0$. From the assumptions we have a positive L such that for each $I \in \mathcal{C}$, $|\Phi(I)| \leq L \cdot V(I)$, and we have a subset H of K with Lebesgue measure 0, which contains all the boundary points of K , such that for all points $u \in K \setminus H$, $\Phi'(u) = 0$. Consequently, we have a countable set \mathcal{I} of open intervals with $\sum_{J \in \mathcal{I}} V(J) < \varepsilon/2L$ the union of which covers H . To apply Cousin's lemma, define a positive valued function δ on K . Assign to each $u \in H$ a $J_u \in \mathcal{I}$ that contains the point u and then a positive $\delta(u)$ such that $B(u, \delta(u)) \subset J_u$, while to each $u \in K \setminus H$, a $\delta(u) > 0$ for which the following implication holds: if a cube $I \in \mathcal{C}$ satisfies the condition $u \in I \subset B(u, \delta(u))$ then

$$\left| \frac{\Phi(I)}{V(I)} \right| < \frac{\varepsilon}{2V(K)}.$$

Fix a δ -fine dotted partition η of K . The domain of η can be written as $\mathcal{A} \cup \mathcal{B}$ where for $I \in \mathcal{A}$ and for $I \in \mathcal{B}$ we have $\eta_I \in H$ and $\eta_I \in K \setminus H$, respectively. We get a contradiction in the form $\varepsilon < \varepsilon$:

$$\varepsilon = |\Phi(K)| \leq \sum_{I \in \mathcal{A}} |\Phi(I)| + \sum_{I \in \mathcal{B}} |\Phi(I)| \leq$$

$$(1) \quad \leq \sum_{I \in \mathcal{A}} L \cdot V(I) + \sum_{I \in \mathcal{B}} \frac{\varepsilon \cdot V(I)}{2V(K)} \leq L \cdot \sum_{I \in \mathcal{A}} V(I) + \frac{\varepsilon}{2} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Inequality (*) can be proved as follows. Using δ -fineness of η , each $I \in \mathcal{A}$ is a subset of J_{η_I} , so the sum of volumes $V(I)$ for cubes I belonging to the same J_u can be majorized by the volume of this common J_u , consequently, for some finite subset \mathcal{J}_0 of \mathcal{J} we have

$$\sum_{I \in \mathcal{A}} V(I) \leq \sum_{J \in \mathcal{J}_0} V(J) \leq \sum_{J \in \mathcal{J}} V(J) < \frac{\varepsilon}{2L}.$$

2. If $Y \subset X$ is Jordan measurable, ε is a positive number and $L > 0$ is a Lipschitz constant for Ψ , then there exists a set $H \subset \text{int } Y$ which is a finite union of cubes with $V(Y \setminus H) < \varepsilon/L$, therefore part 1. of the theorem yields

$$|\Psi(Y)| = |\Psi(Y \setminus H) + \Psi(H)| = |\Psi(Y \setminus H)| \leq \varepsilon. \blacksquare$$

REMARK 26. Repeating a part of the proof of assertion 1. we can get an elementary proof (which is essentially the same that one can find in [2, Theorem 4] for the case $m = 1$) of the fact that a bounded function f defined on a cube K which is continuous at almost all interior points of K , is integrable (consequently the same is true for a bounded function defined on a Jordan measurable set). Indeed, let ε be a positive number, define $L := \text{osc}_f(K)$, $H := \partial K \cup \text{dis}f$ where $\text{dis}f$ denotes the set of discontinuity points of f . For $u \in H$, let the definition of $\delta(u)$ be the same as in the previous proof, while for $u \in K \setminus H$, let $\delta(u)$ be any positive number satisfying the condition $\text{osc}_f(\overline{B}(u, \delta(u))) < \varepsilon/2V(K)$, and let the definitions of η , \mathcal{A} and \mathcal{B} be the same as before. Then a proof of the inequality

$$\sum_{I \in \mathcal{A}} \text{osc}_f(I) V(I) + \sum_{I \in \mathcal{B}} \text{osc}_f(I) V(I) < \varepsilon,$$

which implies integrability of f , can be formulated as follows. Write the left hand side of this inequality followed by a “ \leq ” sign and then switch to line (1) and copy the previous proof.

As three important corollaries, we give a characterization of the ‘indefinite integral’ of a given integrable function, a characterization of the density functions of the constant zero set-functions and a characterization of the set-functions $\Psi: \mathcal{J}_X \rightarrow \mathbb{R}$ possessing a density function.

THEOREM 27. *If X is a Jordan measurable set, $g: X \rightarrow \mathbb{R}$ is an integrable function and $\Psi: \mathcal{J}_X \rightarrow \mathbb{R}$ then the following two statements are equivalent:*

1. Ψ is an additive Lipschitz function such that $\Psi'(u) = g(u)$ holds for almost all interior points u of X ,
2. $\Psi(H) = \int_H g$ for each $H \in \mathcal{J}_X$ (in other words: g is a density function of Ψ).

PROOF. 1. \Rightarrow 2. Assertion 2. of Lemma 25 can be applied to the set-function $\mathcal{J}_X \ni H \mapsto \Psi(H) - \int_H g$.

2. \Rightarrow 1. Additivity and Lipschitz condition are well-known, $\Psi'(u) = g(u)$ holds in each continuity points $u \in \text{int } X$ of g . ■

THEOREM 28. *If X is a Jordan measurable set, $g: X \rightarrow \mathbb{R}$ is an integrable function and $\Psi: \mathcal{J}_X \rightarrow \mathbb{R}$ is the constant zero set-function then the following three statements are equivalent:*

1. $g(u) = 0$ holds for almost all points $u \in \text{int } X$,
2. g is a density function of Ψ ,
3. $g(u) = 0$ holds for all continuity points $u \in \text{int } X$ of g .

PROOF. 1. \Rightarrow 2. See assertion 1. \Rightarrow 2. of Theorem 27. 2. \Rightarrow 3. See Remark 19. 3. \Rightarrow 1. Integrability implies continuity at almost all interior points. ■

THEOREM 29. *Given a Jordan measurable set $X \subset \mathbb{R}^m$ and a set-function $\Psi: \mathcal{J}_X \rightarrow \mathbb{R}$, the following two assertions are equivalent:*

1. Ψ is an additive Lipschitz function that is strongly differentiable at almost all interior points of X ,
2. Ψ has a density function.

PROOF. 1. \Rightarrow 2. We prove that the function $g: X \rightarrow \mathbb{R}$ defined by

$$g(x) := \begin{cases} \inf \left\{ \sup \left\{ \frac{\Psi(I)}{V(I)} : \begin{array}{c} I \text{ is a subcube} \\ \text{of } B(x,r) \end{array} \right\} : r > 0 \right\}, & \text{if } x \in \text{int } X, \\ 0, & \text{if } x \in \partial X \cap X \end{cases}$$

is a density function of Ψ . First we show that g is integrable, that is bounded, and continuous in almost all points of $\text{int } X$. If L is a Lipschitz constant for Ψ , then the range of g is contained in the interval $[-L, L]$, thus it suffices to show that if Ψ is strongly differentiable at an interior point u of X , then g is continuous at u . Let $u \in \text{int } X$, from the definition of strong differentiability we have $\Psi'(u) = g(u)$. Let ε be a positive number and $\delta > 0$ such that $B(u, \delta) \subset \text{int } X$ and for each subcube I of $B(u, \delta)$ $|\Psi(I)/V(I) - g(u)| < \varepsilon$ holds. This implies that if $\|x - u\| < \delta$ and $r < \delta - \|x - u\|$, then – being each subcube I of $B(x, r)$ a subcube of $B(u, \delta)$ –

$$\sup \left\{ \frac{\Psi(I)}{V(I)} : I \text{ is a subcube of } B(x, r) \right\} \in [g(u) - \varepsilon, g(u) + \varepsilon],$$

therefore $g(x) \in [g(u)-\varepsilon, g(u)+\varepsilon]$, whenever $x \in B(u, \delta)$. Now, Theorem 27 implies that g is a density function of Ψ .

2. \Rightarrow 1. It is well-known that Ψ is additive, Lipschitz, and strongly differentiable at the continuity points $u \in \text{int } X$ of g . ■

4. Back to the change of variables

THEOREM 30. *If $A \subset \mathbb{R}^m$ is Jordan measurable, $G: A \rightarrow \mathbb{R}^m$ is a Lipschitz map and G is strongly differentiable at almost all interior points of A , then $G(A)$ is Jordan measurable.*

PROOF. Let L be a Lipschitz constant for G , T the set of those interior points of A where G is not strongly differentiable, R and S the set of those points $x \in (\text{int } A) \setminus T$, for which $J_G(x)$ is regular or singular, respectively. Finally, let F be the unique continuous extension of G defined on \overline{A} (which is again a Lipschitz function with Lipschitz constant L). $G(A)$ is bounded, because it is a subset of the compact set $F(\overline{A})$. Continuity of F implies $\overline{G(A)} = \overline{F(A)} = F(\overline{A})$, therefore

$$\begin{aligned}\partial G(A) &= \overline{G(A)} \setminus \text{int } G(A) = F(\overline{A}) \setminus \text{int } G(A) = \\ &= [G(R) \cup G(S) \cup G(T) \cup F(\partial A)] \setminus \text{int } G(A) \subset \\ &\subset [G(R) \setminus \text{int } G(A)] \cup G(S) \cup G(T) \cup F(\partial A) = G(S) \cup G(T) \cup F(\partial A).\end{aligned}$$

The last equality follows from the inclusion $G(R) \subset \text{int } G(A)$ which is a consequence of Theorem 10. Theorem 24 can be applied to the function $G|_{\text{int } A}$, from this we get that $G(S)$ is a Lebesgue-0-set. As both T and ∂A are Lebesgue-0-sets, to finish the proof it is enough to observe that the image under a Lipschitz map of a Lebesgue-0-set is a Lebesgue-0-set. ■

THEOREM 31. *If $X \in \mathcal{J}$, $K \subset X$ is a set of Lebesgue measure 0 and $G: X \rightarrow \mathbb{R}$ is a Lipschitz function which is injective on $\text{int } X \setminus K$, then for any two non-overlapping $A \in \mathcal{J}_X$, $B \in \mathcal{J}_X$, their images under G are also non-overlapping.*

PROOF. The inclusion

$$G(A) \cap G(B) \subset G(K) \cup G(\partial A \cap A) \cup G(\partial B \cap B)$$

follows from the fact that if $y = G(a) = G(b)$, $a \in A \setminus K$ and $b \in B \setminus K$, then the relations $a \in \text{int } A$, $b \in \text{int } B$ cannot hold at the same time: in the case $a = b$ this would contradict to the fact that A and B are non-overlapping,

in the case $a \neq b$ – to the injectivity assumption. This inclusion implies that $G(A) \cap G(B)$ is of Lebesgue measure 0, therefore

$$[\text{int } G(A)] \cap [\text{int } G(B)] = \text{int}[G(A) \cap G(B)] = \emptyset. \blacksquare$$

All the existing proofs of the change of variables formula use the following lemma, that we will also do.

LEMMA 32. *If the affine map $\ell: \mathbb{R}^m \rightarrow \mathbb{R}^m$ is defined by $\ell(x) := Ax + b$ where $A \in \mathbb{R}^{m \times m}$ and $b \in \mathbb{R}^m$, then for each cube (in fact for each Jordan measurable set) Q we have $V(\ell(Q)) = |\det A| \cdot V(Q)$.*

THEOREM 33. *Let Q be a cube in \mathbb{R}^m , $G: Q \rightarrow \mathbb{R}^m$ a Lipschitz function which is strongly differentiable at $u \in \text{int } Q$ and Φ the cube-function defined on the set of subcubes of Q by $\Phi(I) := V(G(I))$. Then Φ is strongly differentiable at u and $\Phi'(u) = |\det J_G(u)|$.*

PROOF. Set $D := |\det J_G(u)|$. Suppose that $D \neq 0$ as the other case has already been settled in Lemma 23. Let ω be a positive number and $\varepsilon \in (0, 1)$ such that

$$D - \omega < (1 - \varepsilon)^m D \leq (1 + \varepsilon)^m D < D + \omega.$$

Theorem 10 yields a δ for this ε ; we may and do suppose that $\overline{B}(u, \delta) \subset Q$. For each subcube $I = \overline{B}(x, r)$ of $\overline{B}(u, \delta)$ we use the notation

$$I_- := \overline{B}(x, (1 - \varepsilon)r), \quad I_+ := \overline{B}(x, (1 + \varepsilon)r)$$

and apply Theorem 10:

$$\begin{aligned} D - \omega &< (1 - \varepsilon)^m D = (1 - \varepsilon)^m \frac{V(\ell_x(I))}{V(I)} = \frac{V(\ell_x(I_-))}{V(I)} \leq \frac{V(G(I))}{V(I)} \leq \\ &\leq \frac{V(\ell_x(I_+))}{V(I)} \leq (1 + \varepsilon)^m \frac{V(\ell_x(I))}{V(I)} \leq (1 + \varepsilon)^m D < D + \omega. \blacksquare \end{aligned}$$

THEOREM 34. *If $X \subset \mathbb{R}^m$ is a Jordan measurable set, $K \subset X$ a Lebesgue-0-set, $G: X \rightarrow \mathbb{R}^m$ a Lipschitz map which is strongly differentiable at almost all points of $\text{int } X$, injective on $X \setminus K$ and the set-function $\mathcal{J}_X \ni H \mapsto V(G(H))$ is denoted by Ψ , then a density function g of Ψ can be constructed in this way: using the notation*

$$\begin{aligned} B^j(x, r) &:= \{(y, z) \in B(x, r) \times B(x, r) : y_j \neq z_j, y_i = z_i \text{ if } i \neq j\} \\ &(x \in \mathbb{R}^m, r > 0), \end{aligned}$$

let $g: X \rightarrow \mathbb{R}$ be the function $x \mapsto |\det(\bar{g}_{ij}(x))|$, where for all pairs of integers $1 \leq i, j \leq m$, the definition of the functions $\bar{g}_{ij}: X \rightarrow \mathbb{R}$ is

$$\bar{g}_{ij}(x) := \begin{cases} \inf \left\{ \sup \left\{ \frac{G_i(z) - G_i(y)}{z_j - y_j} : \substack{(y,z) \in \\ \in B^j(x,r)} \right\} : r > 0 \right\}, & \text{if } x \in \text{int } X, \\ 0, & \text{if } x \in \partial X \cap X \end{cases}$$

PROOF. First we show that for each pair (i, j) , \bar{g}_{ij} is integrable, that is bounded and almost everywhere in $\text{int } X$ continuous. If L is a Lipschitz constant for G , then the range of \bar{g}_{ij} is contained in the interval $[-L, L]$, thus it suffices to show that if G is strongly differentiable at an interior point u of X , then \bar{g}_{ij} is continuous at u . Let $u \in \text{int } X$ such that G is strongly differentiable at u and let ε be a positive number. From Remarks 11 and 15 we know that G_i is strongly partially differentiable at u with respect to the j -th variable and from Definition 14 it is clear that $\bar{g}_{ij}(u) = D_j^s G_i(u)$. Select a $\delta > 0$ such that

$$\frac{G_i(z) - G_i(y)}{z_j - y_j} \in [\bar{g}_{ij}(u) - \varepsilon, \bar{g}_{ij}(u) + \varepsilon], \quad \text{whenever } (y, z) \in B^j(x, \delta).$$

If $\|x - u\| < \delta$ and $r < \delta - \|x - u\|$, then – being $B^j(x, r) \subset B^j(u, \delta)$ –

$$\sup \left\{ \frac{G_i(z) - G_i(y)}{z_j - y_j} : (y, z) \in B^j(x, r) \right\} \in [\bar{g}_{ij}(u) - \varepsilon, \bar{g}_{ij}(u) + \varepsilon],$$

therefore $\bar{g}_{ij}(x) \in [\bar{g}_{ij}(u) - \varepsilon, \bar{g}_{ij}(u) + \varepsilon]$, whenever $x \in B(u, \delta)$. Now, using continuity of the function $\det: \mathbb{R}^{m \times m} \rightarrow \mathbb{R}$ and compactness of $[-L, L]^m$ in \mathbb{R}^m we get that g is bounded, and continuous at the strong differentiability points of G , thus g is integrable. Moreover, at the strong differentiability points $u \in \text{int } X$ of G , we have $\Psi'(u) = g(u)$ (see Theorem 33) and according to Theorems 31, 21, Ψ is an additive Lipschitz function. These facts and Theorem 27 imply that g is a density function of Ψ . ■

THEOREM 35. *If $X \subset \mathbb{R}^m$ is a Jordan measurable set, $K \subset X$ a Lebesgue-0-set, $G: X \rightarrow \mathbb{R}^m$ a Lipschitz map which is strongly differentiable in almost all points of $\text{int } X$, injective on $X \setminus K$, and $f: G(X) \rightarrow \mathbb{R}$ is any bounded function, then*

1. there exists an integrable function $\bar{g}: X \rightarrow \mathbb{R}^{m \times m}$ such that for almost all $x \in \text{int } X$, $\bar{g}(x) = J_G(x)$,

2. for each integrable function $\bar{h}: X \rightarrow \mathbb{R}^{m \times m}$ with this property, the function

$$\psi: X \rightarrow \mathbb{R}, \quad \psi(x) := f(G(x)) \cdot |\det \bar{h}(x)|$$

is integrable if and only if f is integrable, and

$$(2) \quad \int_{G(X)} f = \int_X \psi$$

holds whenever one of f and ψ is (that is both of f and ψ are) integrable.

PROOF. 1. From Theorem 34 we already know that the function \bar{g} defined there is integrable, and at the strong differentiability points x of G , $\bar{g}(x) = J_G(x)$ holds.

2. Theorem 8 can be applied with $X \ni x \mapsto |\det \bar{h}(x)| =: g(x)$. Indeed, Theorem 30 implies that condition a) of Theorem 8 is satisfied, while b) is implied by Theorem 31. As for condition c), g differs from the function $X \ni x \mapsto |\det \bar{g}(x)|$ on a set of Lebesgue measure 0; according to Theorem 34, the latter is a density function of $\mathcal{J}_X \ni H \mapsto V(G(H))$, so the same is true for g (see Theorem 28). ■

REMARK 36. In Theorem 35, the injectivity assumption cannot be omitted. In particular, Theorem 24.26 in [1] is false. This “theorem” asserts (using our notations) the following: Suppose that φ is a continuously differentiable function on an open set $\Omega \subset \mathbb{R}^m$ with values in \mathbb{R}^m such that $\det(J_\varphi(x)) \neq 0$ for each $x \in \Omega$. If X is a compact Jordan measurable subset of Ω and $f: \varphi(X) \rightarrow \mathbb{R}$ is continuous, then $\varphi(X)$ is Jordan measurable and $\int_{\varphi(X)} f = \int_X (f \circ \varphi) |\det J_\varphi|$.

Counterexample: $m := 2$, $\Omega := (0, +\infty) \times \mathbb{R}$, $\varphi(x, y) := (x \cos y, x \sin y)$, $X := [1, 2] \times [0, 4\pi]$ ($G := \varphi|_X$), $f(x, y) := 1$; the left hand side of the formula gives 3π , while the right hand side is equal to 6π .

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DESCRIPTION OF THE CLUBS

By

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Abstract. We give a complete algebraic and geometric description of two dimensional $\text{GF}(q)$ -linear pointsets of size $q^2 + 1$ of the projective line $\text{PG}(1, q^h)$.

Let $\text{GF}(q)$ denote the finite (Galois) field of order q , and $\text{PG}(n, q)$ the projective space of dimension n over $\text{GF}(q)$, where q is a prime power.

In [1] the same authors investigated regular plane-spreads of the projective space $\text{PG}(8, q)$ and they faced the problem how a plane can meet this spread. This problem led to the question of characterizing the (two dimensional) q -linear pointsets of size $q^2 + 1$ in the line $\text{PG}(1, q^3)$. In [1] we called such pointsets as ‘clubs’ and we sketched the proof of the fact that a club and a subline (of order q) can intersect only in 0, 1, 2, 3 or $q + 1$ points. Here we give the detailed proof and hence the complete description for a more general case, i.e. for clubs in $\text{PG}(1, q^h)$.

Let $L = \text{PG}(1, q^h)$ be a line of the projective plane $\text{PG}(2, q^h)$, $h \geq 2$. Let \mathcal{C} denote a set of $q^2 + 1$ points in the line L .

DEFINITION. We say that \mathcal{C} is a *club* of L if there exists a subplane $\Delta = \text{PG}(2, q)$ of order q in $\text{PG}(2, q^h)$, and there exists a point C on an extended line of Δ but not in Δ , such that \mathcal{C} is the projected image of Δ onto L from the center C .

By this projection, \mathcal{C} has a special (‘multiple’) point H called the *head* of the club, which is the projected image of the line of Δ whose extension contains the center of projection.

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REMARK 1. Projecting another subplane of order q onto L from the same center C can yield the same club, and even changing the center results in a not necessarily different club.

Different constructions of the same club might yield different *heads*, thus, if we say that the point H is the *head*, we always mean that it is the head *determined by a particular construction*.*

It is more natural to consider the *club* \mathcal{C} as a subset of points of the factor-geometry $\text{PG}(2, q^h)/C$. This factor-geometry is a projective line $\text{PG}(1, q^h)$ with additional structure: their points actually are the lines of $\text{PG}(2, q^h)$ through the point $C \in \text{PG}(2, q^h)$, in other words, it is a pencil. The *club* \mathcal{C} , as a subset of this pencil, is the set of those lines through C which meet Δ . The *head* of the club (determined by this construction) is the unique q -secant line of Δ that contains C .

REMARK 2. An arbitrary line L of $\text{PG}(2, q^h)$ that does *not* meet C can be naturally identified with the factor-geometry $\text{PG}(2, q^h)/C$ by identifying the point $P \in L$ with the line through P and C . This natural isomorphism connects the two concepts of clubs above. ■

From now on, by a subline (without any attribute) we always mean a $\text{PG}(1, q)$ contained in L .

DEFINITION. A subline completely contained by the club \mathcal{C} will be called *regular* (*according to this particular construction*) if it is the image of a line of Δ . The sublines completely contained by the club \mathcal{C} that cannot be got as an image of any subline of Δ will be called *irregular* or *deviant* (*according to this particular construction*).

REMARK 3. Note that a subline being deviant according to a construction can be regular according to another construction.

Obviously, the lines of Δ (except one, whose external point is C) are projected to different regular sublines, so the number of regular sublines is exactly $q^2 + q$, by each construction. Each regular subline will contain the head since its preimage in Δ intersects the line which is the preimage of H . ■

Let P , Q and H be three *distinct* points of the club $\mathcal{C} \subseteq L$ such that the point H is the *head* determined by the construction of \mathcal{C} that projects the subplane Δ onto L from the center C .

* Later we will show that either each point of the club can be the head (using a suitable construction) or the head is unique, independently of the construction.

LEMMA 4. *We can coordinatize the line $L = \text{PG}(1, q^h)$ by $\text{GF}(q^h) \cup \{\infty\}$ such that*

- $P = 0, Q = 1$ and $H = \infty$; and
- $\mathcal{C} \setminus \{H\} = \{a + b\omega \mid a, b \in \text{GF}(q)\}$, where $\omega \in \text{GF}(q^h) \setminus \text{GF}(q)$ is a suitable fixed element;
- the subline $\text{GF}(q) \cup \{\infty\}$ is regular according to the construction above.

PROOF. Let \bar{P} and \bar{Q} denote the unique inverse images of P and Q (along the projection of Δ from C onto L), respectively. Let \bar{L} denote the line $\bar{P}\bar{Q}$. Since P and Q are distinct, the line \bar{L} does *not* contain C , and thus, the projection from the center C is an isomorphism between L and \bar{L} (that fixes the point $L \cap \bar{L}$).

Using homogeneous coordinates, we can coordinatize the plane $\text{PG}(2, q^h)$ such that $\Delta = \{(x_1, x_2, x_3) \mid x_1, x_2, x_3 \in \text{GF}(q)\} \setminus \{(0, 0, 0)\}$, $\bar{P} = (0, 0, 1)$, $\bar{Q} = (1, 0, 1)$ (and so $\bar{L} = [0, 1, 0]$), and the $(q+1)$ -secant line of Δ (that contains the center of projection C as an external point) is in the line $[0, 0, 1]$. Since C is *not* in Δ (which is now the canonical subgeometry), $C = (-\omega, 1, 0)$, for some $\omega \in \text{GF}(q^h) \setminus \text{GF}(q)$.

By the definition, the projection of Δ onto \bar{L} from C is a *club*, let it be denoted by $\bar{\mathcal{C}}$. (Let us remark that this club is an isomorphic image of the club $\mathcal{C} \subseteq L$.) The projected image of any point $(a, b, 1)$ is $(a + b\omega, 0, 1)$, and the head is $\bar{H} = (1, 0, 0)$.

So the club $\bar{\mathcal{C}}$ consists of the head $\bar{H} = (1, 0, 0)$ and a two dimensional vectorspace over $\text{GF}(q)$ contained in $\text{AG}(1, q^h)$. Let us identify $\bar{L} \setminus \bar{H} = \text{AG}(1, q^h)$ and $\text{GF}(q^h)$ with $(x, 0, 1) \mapsto x$. Then this two dimensional vectorspace contains $\text{GF}(q)$ and it is generated over $\text{GF}(q)$ by the ‘vectors’ 1 and ω .

Finally, let A' denote the point $CA \cap \bar{L}$ for each $A \in L$ (i. e. A' is the preimage of A by the isomorphism above of L and \bar{L} that projects the club $\bar{\mathcal{C}}$ onto \mathcal{C}). Let the coordinate of A be the coordinate of A' defined above. Then the coordinates of P, Q and H are $\bar{P} = 0, \bar{Q} = 1$ and $\bar{H} = \infty$, respectively. The coordinates of the points of \mathcal{C} are the coordinates of the points of $\bar{\mathcal{C}}$. The subline $\text{GF}(q) \cup \{\infty\}$ of L is the projected image of $\Delta \cap \bar{L}$ from C onto L , so it is a regular subline. ■

PROPOSITION 5. *Let ℓ be a subline of order q (and let an arbitrary construction of the club \mathcal{C} be fixed). If the intersection $\ell \cap \mathcal{C}$ contains the head*

of the club \mathcal{C} and two other points of \mathcal{C} , then ℓ must be a regular subline of the club.

If $\ell \cap \mathcal{C}$ contains four (non-head) points, then ℓ must be completely contained in the club \mathcal{C} .

If the club \mathcal{C} is not equal to a subline $\text{PG}(1, q^2)$ of order q^2 (that automatically holds when h is odd), then each subline completely contained by the club \mathcal{C} is regular according to each possible construction of the club.

PROOF. Let a particular (but arbitrary) construction of the club \mathcal{C} be fixed, and let H denote the head determined by this fixed construction.

If $\ell \cap \mathcal{C}$ contains the head H and two other points $P, Q \in \mathcal{C}$, then let \bar{P} and \bar{Q} be the (unique) preimages of P and Q before the projection, and let \bar{H} be chosen among the preimages of H so that \bar{H} is collinear with \bar{P} and \bar{Q} . (It can be done because the preimages of H constitute a whole projective line of $\text{PG}(2, q^h)$.) There is a regular subline in the club (which is the projected image of the line $\bar{P}\bar{Q}$ of Δ) containing P, Q and H , and the subline through 3 points of L is unique, this regular subline and ℓ must coincide.

If A, B, P and Q are four distinct common points of ℓ and the club \mathcal{C} such that neither of these four points is the head H , then let us coordinatize L by the method of Lemma 4. We get $H = \infty$, $P = 0$, $Q = 1$, and this lemma says that the coordinates of A and B are $a = a_1 + a_2\omega$ and $b = b_1 + b_2\omega$, respectively, where $a_1, a_2, b_1, b_2 \in \text{GF}(q)$ and $\omega \in \text{GF}(q^h) \setminus \text{GF}(q)$. So 0, 1, a and b are contained in a subline, that means that their cross-ratio, $y = \frac{0-a}{0-b} \frac{1-b}{1-a}$ is in $\text{GF}(q)$. It gives $0 = (1-y)ab + yb - a$, substituting $a = a_1 + a_2\omega, b = b_1 + b_2\omega$ we get a quadratic equation for ω , with coefficients from $\text{GF}(q)$. The coefficient of ω^2 is $(1-y)a_2b_2$. Here $y = 1$ would mean $a = b$.

If $a_2b_2 = 0$ then one of a_2 and b_2 is 0, but $0 = (1-y)ab + yb - a$ implies that if one of a and b is in $\text{GF}(q)$ then the other is as well. So $\text{GF}(q) \cup \{\infty\}$ is the unique subline of order q that contains A, B, P and Q , and Lemma 4 says that it is a regular subline of the club.

If $a_2b_2 \neq 0$ then ω gives a quadratic extension of $\text{GF}(q)$ and in fact our two dimensional vectorspace $\{x + y\omega \mid x, y \in \text{GF}(q)\}$ is the unique $\text{GF}(q^2)$ itself. (It can happen only for $h = \text{even}$.) Now consider the points of the subline through $0, 1, a, b$. This consists of the points c for which the cross-ratio y of $0, 1, a$ and c is in $\text{GF}(q) \cup \{\infty\}$. We have $y = \frac{0-a}{0-c} \frac{1-c}{1-a}$,

and the values $y = 0, 1, \infty$ give $c = 1, a, 0$. Note that $c = \infty = H$ is impossible as it would imply $y = \frac{a}{a-1}$ which is not in $\text{GF}(q)$ unless $a \in \text{GF}(q)$ and the subline is $\text{GF}(q) \cup \{\infty\}$.

So for the subline we have $\left\{ c = \frac{a}{(1-y)a+y} \mid y \in \text{GF}(q) \cup \{\infty\} \right\}$. But when calculating $c = \frac{a}{(1-y)a+y}$, we work within $\text{GF}(q^2)$, so within the club, hence the subline through $0, 1, a, b$ will be contained in the club completely. ■

COROLLARY 6. *If the club \mathcal{C} is not equal to a subline $\text{PG}(1, q^2)$ of order q^2 then the head determined by one of the constructions is the head determined by each possible construction. In other words, in this case the head does not depend on the construction.*

PROOF. Suppose to the contrary that $H_1 \neq H_2$ are the heads determined by the constructions projecting Δ_1 from C_1 and Δ_2 from C_2 , respectively. Proposition 5 says that each sublines of \mathcal{C} is regular according to both constructions so they contains both H_1 and H_2 . But the preimage of H_2 is unique in the first construction, it cannot be contained by the preimages of all the sublines, a contradiction. ■

PROPOSITION 7. *If the club \mathcal{C} equals to a subline $\text{PG}(1, q^2)$ of order q^2 then for each subline ℓ completely contained in the club, there exist such constructions that ℓ is regular according to them and also such ones that ℓ is deviant according to them. In this case, for each construction there exist $q^3 - q^2$ deviant sublines.*

In this case the club, with the (regular and deviant) sublines it contains, forms a Moebius plane. I.e. for any 3 points of \mathcal{C} there is a unique subline containing them, and this subline is contained in \mathcal{C} as well.

Also in this case every point of the club is equivalent geometrically, any point can play the role of the head.

Note that the preimages of a deviant subline is a set of $q + 1$ points, neither three of them being collinear, so each deviant subline is the projected image of a proper conic of Δ .

Also note that even if h is even and at least 4, there are clubs which *do not* contain deviant sublines, this depends on wheter the center C of projection is in the quadratic extension of Δ or not.

PROOF. If the club \mathcal{C} equals to a subline $\text{PG}(1, q^2)$ of order q^2 then it is the projection of a Baer subplane of a subplane $\text{PG}(2, q^2)$ that contains the club and the center C of projection (that is not contained in the Baer subplane). For each subline of the club there exists a Baer subplane that contains this subline and does not contain C . Let H be an arbitrary point of $\text{PG}(1, q^2)$ then there exists such a Baer subplane that the line HC is a $(q + 1)$ -secant, and so H is the head.

Let ℓ be an arbitrary subline of the club $\text{PG}(2, q^2)$, and let H be a point of this club not contained by ℓ . Since we can construct this club such that H plays the role of the head and each regular subline must contain the head, we get ℓ as a deviant subline.

Since $\text{PG}(1, q^2)$ contains $\frac{\binom{q^2+1}{3}}{\binom{q+1}{3}} = (q^2 + 1)q$ sublines of order q (Baer-sublines) and each construction makes only $q^2 + q$ regular sublines, there must be $q^3 - q^2$ deviant sublines. ■

COROLLARY 8. *A club and a subline intersect in 0, 1, 2, 3 or $q + 1$ points. In the Moebius case the intersection size 3 does not occur.* ■

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SHELAH'S PROOF OF DIAMOND

By

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Abstract. Shelah's proof is given for \diamondsuit_{λ^+} when $\lambda > \aleph_0$ and $2^\lambda = \lambda^+$.

The aim of this note is to give an elaboration of Shelah's recent proof of diamond.

Notation and definitions

We use the standard axiomatic set theory notation. If V is a set, μ is a cardinal, $[V]^\mu = \{X \subseteq V : |X| = \mu\}$, $[V]^{\leq\mu} = \{X \subseteq V : |X| \leq \mu\}$. If $\kappa < \lambda$ are regular cardinal, then $S_\kappa^\lambda = \{\alpha < \lambda : \text{cf}(\alpha) = \kappa\}$, $S_{<\kappa}^\lambda = \{\alpha < \lambda : \text{cf}(\alpha) < \kappa\}$, $S_{\neq\kappa}^\lambda = \{\alpha < \lambda : \text{cf}(\alpha) \neq \kappa\}$.

If $X \subseteq \lambda \times \lambda^+$, $i < \lambda$ then $(X)_i = \{\alpha < \lambda^+ : \langle i, \alpha \rangle \in X\}$.

THEOREM 1. (Shelah) *If $2^\lambda = \lambda^+$, $\kappa = \text{cf}(\lambda)$, $S \subseteq S_{\neq\kappa}^{\lambda^+}$ is stationary, then \diamondsuit_S holds.*

PROOF. We start with two entirely trivial remarks.

REMARK 1. If π is a bijection from λ^+ onto $\lambda \times \lambda^+$ then for a club set of $\delta < \lambda^+$ (the restriction of) π is a bijection from δ onto $\lambda \times \delta$.

REMARK 2. If $\{X_\alpha : \alpha < \lambda^+\}$ is an enumeration of $[\lambda^+]^{\leq\lambda}$, $Z \subseteq \lambda^+$, then for a club set of $\delta < \lambda^+$, the following holds. There are arbitrarily large $\alpha < \delta$ such that for some $\beta < \delta$, $Z \cap \alpha = X_\beta$ holds.

Fix, for every $\delta < \lambda^+$ an increasing decomposition $\delta = \bigcup\{A_i^\delta : i < \kappa\}$ with $|A_i^\delta| < \lambda$ ($i < \kappa$). Fix an enumeration $[\lambda \times \lambda^+]^{\leq \lambda} = \{X_\alpha : \alpha < \lambda^+\}$.

LEMMA 1. *For some $i < \kappa$ the following holds. For every $Z \subseteq \lambda^+$ there are stationary many $\delta \in S$ such that there are arbitrary large $\alpha < \delta$, and some $\beta < \delta$ with $\alpha, \beta \in A_i^\delta$ such that $Z \cap \alpha = (X_\beta)_i$ holds.*

PROOF. Otherwise, for every $i < \kappa$ there are a $Z_i \subseteq \lambda^+$ and a closed, unbounded $E_i \subseteq \lambda^+$ such that if $\delta \in E_i \cap S$ then there are **not** arbitrarily large $\alpha < \delta$ and some $\beta < \delta$ such that $\alpha, \beta \in A_i^\delta$, and $Z_i \cap \alpha = (X_\beta)_i$. However, by the Remarks, there is some $\delta \in \bigcap\{E_i : i < \kappa\} \cap S$ such that there exist arbitrarily large $\alpha < \delta$ and some $\beta < \delta$ such that $Z_i \cap \alpha = (X_\beta)_i$ holds for every $i < \kappa$. As $\text{cf}(\delta) \neq \kappa$, for some $i < \kappa$ we have that there are arbitrarily large $\alpha < \delta$ with some $\beta < \delta$ as above, and $\alpha, \beta \in A_i^\delta$. But this contradicts the choice of Z_i, E_i . ■

If we now rewrite $(X_\beta)_i$ as X_β then we obtain the following form of Lemma 1.

LEMMA 2. *For some $i < \kappa$ there is a sequence $\langle X_\beta : \beta < \lambda^+ \rangle$ such that the following holds. For every $Z \subseteq \lambda^+$ there are stationary many $\delta \in S$ such that there are arbitrary large $\alpha < \delta$, and some $\beta < \delta$ with $\alpha, \beta \in A_i^\delta$ such that $Z \cap \alpha = X_\beta$ holds.*

If we apply a bijection between λ^+ and $\lambda \times \lambda^+$ and use Remark 1, we get the following statement.

LEMMA 3. *For some $i < \kappa$ there is a sequence $\langle X_\beta : \beta < \lambda^+ \rangle$ of subsets of $\lambda \times \lambda^+$ such that the following holds. For every $Z \subseteq \lambda \times \lambda^+$ there are stationary many $\delta \in S$ such that there are arbitrary large $\alpha < \delta$, and some $\beta < \delta$ with $\alpha, \beta \in A_i^\delta$ such that $Z \cap (\lambda \times \alpha) = X_\beta$ holds.*

For the rest of the proof, we fix i and $\langle X_\beta : \beta < \lambda^+ \rangle$ as in Lemma 3. Next, we try to define, by transfinite recursion, for every $\xi < \lambda$, a set $Y_\xi \subseteq \lambda^+$ and a closed, unbounded set $E_\xi \subseteq \lambda^+$ such that for every $\delta \in E_\xi \cap S$, if we define

$$V_\xi^\delta = \left\{ \langle \alpha, \beta \rangle \in A_i^\delta \times A_i^\delta : \eta < \xi \longrightarrow Y_\eta \cap \alpha = (X_\beta)_\eta \right\}$$

then either $\sup \left\{ \alpha : \langle \alpha, \beta \rangle \in V_\xi^\delta \right\} < \delta$ or else $V_\xi^\delta \neq V_{\xi+1}^\delta$. Notice that $V_\eta^\delta \supseteq V_\xi$ always holds ($\eta < \xi$).

CLAIM. *This process terminates at some stage $\xi < \lambda$.*

PROOF. Assume indirectly that we can define Y_ξ, E_ξ for every $\xi < \lambda$. Set $E = \bigcap \{E_\xi : \xi < \lambda\}$, a closed, unbounded set in λ^+ . By Lemma 3, there is a $\delta \in E \cap S$ such that there are arbitrarily large $\alpha < \delta$ such that for some $\beta < \delta$ we have $\alpha, \beta \in A_i^\delta$ and $Y_\eta = (X_\beta)_\eta$ ($\eta < \lambda$). Then the sequence $\{V_\eta^\delta : \eta < \lambda\}$ is strictly decreasing, which is impossible as these sets are subsets of $A_i^\delta \times A_i^\delta$, a set of cardinality less than λ . ■

By the Claim, there is a $\xi < \lambda$ such that the construction goes through up to step ξ but Y_ξ, E_ξ cannot be defined.

Define $S_\delta = \bigcup \left\{ (X_\beta)_\xi : \langle \alpha, \beta \rangle \in V_\xi^\delta \right\}$ for $\delta \in E_\xi \cap S$. We claim that this is a diamond-sequence for S .

Indeed, assume that for some $Y \subseteq \lambda^+$, the set $\{\delta \in E \cap S : S_\delta = Y \cap \delta\}$ is nonstationary, where $E = \bigcap \{E_\eta : \eta < \xi\}$. Then set $Y_\xi = Y$ and let E_ξ be a closed, unbounded set such that $S_\delta \neq Y_\xi \cap \delta$ holds for $\delta \in E \cap E_\xi \cap S$. Then, if $\delta \in E \cap E_\xi \cap S$, and $\sup \{\alpha : \langle \alpha, \beta \rangle \in V_\xi^\delta\} = \delta$ then $V_\xi^\delta \neq V_{\xi+1}^\delta$ holds as otherwise $Y_\xi \cap \alpha = (X_\beta)_\xi$ holds for every $\langle \alpha, \beta \rangle \in V_\xi^\delta$ and so

$$Y_\xi \cap \delta = \bigcup \left\{ Y_\xi \cap \alpha : \langle \alpha, \beta \rangle \in V_\xi^\delta \right\} = \bigcup \left\{ (X_\beta)_\xi : \langle \alpha, \beta \rangle \in V_\xi^\delta \right\} = S_\delta. \blacksquare$$

THEOREM 2. (Shelah) *If $2^\lambda = \lambda^+$, $\omega < \kappa = \text{cf}(\lambda) < \lambda$, $S \subseteq S_\kappa^{\lambda^+}$ is stationary and \Diamond_S fails, then there is a function $h : S_{<\kappa}^{\lambda^+} \rightarrow \kappa$ such that for almost every $\delta \in S$ there is a club set $C \subseteq \delta$ on which h is strictly increasing.*

PROOF. Let $\lambda = \sup \{\lambda_i : i < \kappa\}$ for some sequence of cardinals $\{\lambda_i : i < \kappa\}$. Assume that for $\alpha < \lambda^+$ α is decomposed into an increasing union as $\alpha = \bigcup \{A_i^\alpha : i < \kappa\}$ where $|A_i^\alpha| \leq \lambda_i$ and $\beta \in A_i^\alpha$ implies $A_i^\beta \subseteq A_i^\alpha$.

Considering the proof of Theorem 1. we get that Lemma 3. must be false, that is, for every $i < \kappa$ there are a set $Z_i \subseteq \lambda^+$ and a closed, unbounded

set $E_i \subseteq \lambda^+$ such that for $\delta \in E_i \cap S$ there are no arbitrarily large $\alpha < \delta$ and some $\beta < \delta$ such that $\alpha, \beta \in A_i^\delta$ and $Z_i \cap \alpha = (X_\beta)_i$.

There is a closed, unbounded set $E \subseteq \bigcap\{E_i : i < \kappa\}$ such that if $\delta \in E$ then there arbitrarily large $\alpha < \delta$ and some $\beta < \delta$ such that $Z_i \cap \alpha = (X_\beta)_i$ holds for every $i < \kappa$. We now define the function $h : S_{<\kappa}^{\lambda^+} \cap E \rightarrow \kappa$ as follows.

For $\xi \in S_{<\kappa}^{\lambda^+} \cap E$ let $h(\xi)$ be the least $j < \kappa$ such that there are arbitrarily large $\alpha < \xi$ and $\beta < \xi$ such that $\alpha, \beta \in A_j^\xi$ and $Z_i \cap \alpha = (X_\beta)_i$ holds for every $i < \kappa$. As $\text{cf}(\xi) < \kappa$, such a j clearly exists.

If $\delta \in E' \cap S$ then h cannot get the constant i value on a stationary subset of δ as that would mean that there are arbitrarily large $\alpha < \delta$ and $\beta < \delta$ with $\alpha, \beta \in A_i^\delta$ such that $Z_i \cap \alpha = (X_\beta)_i$ and we excluded this possibility.

That is, for every $i < \kappa$ the function h is $\geq i$ on a closed, unbounded subset of δ and we are done. ■

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PH. D. THESIS
COMPLEMENTARY USE OF MATRIX-ANALYTIC METHODS
AND LARGE-DEVIATION THEORY FOR A GENERAL
TELECOMMUNICATIONS QUEUEING SYSTEM

By

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Queueing theory is capable of supporting the task of provisioning Quality of Service (QoS) in a communications network in multiple ways. We model a communications queue with an infinite-buffer single-server queueing system and we use the overflow probability of the steady-state queue-length distribution to predict QoS parameters such as delay (by both Matrix-Analytic Methods and Large Deviations, as they complement each other: they are appropriate for “small” queues and “large” queues, respectively). We also inspect the resource-requirement view of the QoS behaviour.

Matrix-Analytic Methods: For the Markov-chain X_n and for the queue length $L_{n+1} = (L_n + X_n - C)^+$, an infinite Markov chain can be examined with diagonally repeating submatrices: \mathcal{Q}_i matrices are in the i -th diagonal¹. The steady-state queue-length distribution, and so the overflow probability $\Pr\{L(X, C) > B\}$, can be calculated from the minimal non-negative solution of the matrix equations $\mathcal{Q}_i^* = \mathcal{Q}_i + \sum_{j=1}^{\infty} \mathcal{Q}_j^* (I - \mathcal{Q}_0^*)^{-1} \mathcal{Q}_{i-j}^*$ ($i \leq 0$) and $\mathcal{Q}_i^* = \mathcal{Q}_i + \sum_{j=1}^{\infty} \mathcal{Q}_{i+j}^* (I - \mathcal{Q}_0^*)^{-1} \mathcal{Q}_{-j}^*$ ($i \geq 0$), where the probabilistic interpretation of \mathcal{Q}_i^* is: probability of going from level n to level $n+i$ ($n-i$ to level n) making arbitrary steps above level n for $i \leq 0$ (for $i \geq 0$). The straightforward iteration method for \mathcal{Q}_i^* , is called the Censoring-State-Reduction (CSR) method. By using $\mathcal{Q}_i^{<}[k]$ as the corresponding probabilities of going from level $n-i$ to level n making at most k steps above level n for $i \geq 0$ (and similarly for $i \leq 0$) we have been able to propose the

¹ For the Markov chain $X_n \in \{0, \dots, N\}$, it is a $GI/G/1$ -type (g, h) -banded process with $g = C$, $h = N - C$ and diagonals $-C \leq i \leq N - C$; $g = h = 1$ is a QBD process.

new, Finite–Sojourn (FS) method by controlling the steps above level n . The advantage of the FS method is that it does not use matrix inversion, however, its matrix addition/multiplication requirement is linearly increasing in the step size. Here follow the formal results.

DEFINITION 2.1. Define $\mathbf{Q}_i^{(j)}[k]$ as the matrix of the probabilities of going from the corresponding sub-levels of level n to the corresponding sub-levels of level $n + i$ for $i \leq 0$ making exactly $k \geq 0$ steps in the sojourn in states above level $n + i + j$ for $j \geq 0$.

THEOREM 2.2. *The convergence $\mathbf{Q}_i^{\leq}[k] \xrightarrow{k \rightarrow \infty} \mathbf{Q}_i^*$ is monotonically increasing, where the “main recursion²” is*

$$\begin{aligned}\mathbf{Q}_i^{\leq}[k+1] &= \mathbf{Q}_i^{\leq}[k] + \sum_{m=1}^{\min\{h,(k+1)g+i\}} \mathbf{Q}_m \mathbf{Q}_{i-m}^{(-i)}[k], \quad i \leq 0, \\ \mathbf{Q}_i^{\leq}[k+1] &= \mathbf{Q}_i^{\leq}[k] + \sum_{m=1}^{\min\{h-i,(k+1)g\}} \mathbf{Q}_{i+m} \mathbf{Q}_{-m}^{(0)}[k], \quad i \geq 0,\end{aligned}$$

and the “auxiliary recursion³” is

$$\mathbf{Q}_i^{(j)}[k] = \sum_{m=\max\{i+j+1,-g\}}^{\min\{h,kg+i\}} \mathbf{Q}_m \mathbf{Q}_{i-m}^{(j)}[k-1], \quad -(k+1)g \leq i \leq 0.$$

DEFINITION 2.3. Define $\mathbf{Q}_i^{(j)}[k]$ as the probabilities of going from level $n - i$ to level n for $i \geq 0$ making exactly $k \geq 0$ steps in the sojourn in states above level $n - i - j$ for $j \leq 0$.

THEOREM 2.4. *The convergence*

$$\mathbf{Q}_i^{\leq}[k] \xrightarrow{k \rightarrow \infty} \mathbf{Q}_i^*$$

is monotonically increasing, where the “main recursion²” is

$$\mathbf{Q}_i^{\leq}[k+1] = \mathbf{Q}_i^{\leq}[k] + \sum_{m=1}^{\min\{(k+1)h,g+i\}} \mathbf{Q}_m^{(0)}[k] \mathbf{Q}_{i-m}[k], \quad i \leq 0$$

² $-g \leq i \leq h, k \geq 0, \mathbf{Q}_i^{\leq}[0] = \mathbf{Q}_i$

³ $0 \leq j < g$ and $-h < j \leq 0, k > 0, \mathbf{Q}_i^{(j)}[0] = \mathbf{Q}_i$ for $-g \leq i \leq 0$ and $0 \leq i \leq h$

$$\mathcal{Q}_i^{\leq}[k+1] = \mathcal{Q}_i^{\leq}[k] + \sum_{m=1}^{\min\{(k+1)h-i,g\}} \mathcal{Q}_{i+m}^{(-i)}[k] \mathcal{Q}_{-m}, \quad i \geq 0$$

and the “auxiliary recursion³” is

$$\mathcal{Q}_i^{(j)}[k] = \sum_{m=\max\{-i,-h\}}^{\min\{kh-i,g\}} \mathcal{Q}_{i+m}^{(j)}[k-1] \mathcal{Q}_{-m}, \quad 0 \leq i \leq (k+1)h.$$

PROPOSITION 2.5. *The number of matrix operations of step k for the CSR, CSR_{QBD} and FS methods for $g \leq h$ can be found in the next Table:*

Method	Matrix sizes	Addition	Multiplication	Inverse
CSR	N	gh	$2gh$	1
CSR_{QBD}	Nh	1	2	1
FS	N	$M^{g,h} + A_+^{g,h}(k)$	$M^{g,h} + A_{\times}^{g,h}(k)$	0
$A_+^{g,h}(k) \leq g(g+h)((k+1)g+1)$				$M^{g,h} \leq gh + \frac{h(h-1)}{2}$
$A_{\times}^{g,h}(k) \leq g(g+h+1)((k+1)g+1)$				

COROLLARY 2.6. *For QBD-s, a K -step FS method needs $K+1$ auxiliary matrices, $K+1$ multiplications and $K+1$ additions for the main recursion, while the auxiliary recursion’s complexity can be approximately halved.*

Many-Sources Asymptotics: The Many-Sources Asymptotics (MSA) of the Large Deviations is true for a sequence of “scaled” systems: the “ N -scaled” discrete-time queueing model is a queue of infinite buffer capacity fed by the arrivals $X_n^{(N)} \stackrel{\text{def}}{=} \sum_{i=1}^N X_{i,n}$ for $X_{i,n}$ i.i.d.⁴ and server capacity $C \stackrel{\text{def}}{=} cN$. The queue length is $L_{n+1}(X_{n+1}^{(N)}, cN) = (L_n(X_n^{(N)}, cN) + X_n^{(N)} - cN)^+$ and $(L_n(X_n^{(N)}, cN), X_{1,n}, \dots, X_{N,n})$ is an infinite Markov chain. MSA states that under the stability condition $E(X^{(N)}) < C$ and for non-decomposable Markov chains: $\lim_{N \rightarrow \infty} \frac{1}{N} \log \Pr\{L(X^{(N)}, cN) > bN\} \stackrel{\text{def}}{=} -I_{(X,b,c)}$, where $-I_{(X,b,c)} = \sup_{n>0} \inf_{s>0} \{sn\alpha_X(s, n) - s(b+cn)\}$

⁴ $X_{i,n}$ are independent and identically distributed (i.i.d.) with X_n .

with the effective-bandwidth function, $\alpha_X(s, n) \stackrel{\text{def}}{=} \frac{1}{sn} \log E(e^{sX[0,n]})$. We use the asymptotic-constant form and the approximation forms of the overflow probability:

$$\begin{aligned} \Pr\{L(X^{(N)}, cN) > bN\} &\stackrel{\text{def}}{=} \gamma_N^{b,c} e^{-NI_{(X,b,c)}} \approx \\ &\approx \text{for large } N e^{-NI_{(X,b,c)}} \stackrel{\text{def}}{=} e^{-I_{(X^{(N)}, B, C)}}. \end{aligned}$$

Our small-scale (SS) improvement approximates the asymptotic constant $\gamma_N^{b,c}$ of the overflow probability using the values of smaller-scale queues:

$$\bar{\gamma}_N^{b,c} \stackrel{\text{def}}{=} \begin{cases} \gamma_N^{b,c} = e^{NI_{(X,b,c)}} \Pr\{L(X^{(N)}, cN) > bN\} & \text{for } N \leq N' \\ \gamma_{N'}^{b,c} = e^{N'I_{(X,b,c)}} \Pr\{L(X^{(N')}, cN') > bN'\} & \text{for } N > N'. \end{cases}$$

We set up a sufficient condition in order to obtain conservative (upper-bound) MSA and conservative SS improvement:

(A1) $\exists N' < \infty \forall N \geq N' : q(N) \leq q(N+1) \leq q(N+2) \leq \dots$, where $q(N) \stackrel{\text{def}}{=} \frac{\Pr\{L(X^{(N+1)}, c(N+1)) > b(N+1)\}}{\Pr\{L(X^{(N)}, cN) > bN\}}$, and

(A2) $\exists q_{(X,b,c)} \stackrel{\text{def}}{=} \lim_{N \rightarrow \infty} q(N) \leq 1$.

PROPOSITION 3.1. *Assume (A1) and (A2) hold. Then MSA exists and*

$$\lim_{N \rightarrow \infty} \frac{1}{N} \log \Pr\{L(X^{(N)}, cN) > bN\} = \log q_{(X,b,c)}$$

and furthermore

$$\Pr\{L(X^{(N)}, cN) > bN\} \leq \bar{\gamma}_N^{b,c} e^{-NI_{(X,b,c)}} \leq e^{-NI_{(X,b,c)}}.$$

MSA/Resource-Requirement Aspects: Consider the accumulated arrival process $X[0, t]$ having a predetermined QoS requirement towards the service-queue of capacity C , $\Pr\{L(X, C) > B\} \leq \varepsilon$, where $L(X, C)$ is the corresponding steady-state queue length in the system and where ε is generally a low target. The equivalent capacity or bandwidth requirement of the arrivals can be defined as the minimal server capacity for which the QoS target is satisfied: $C_{\text{equ}} \stackrel{\text{def}}{=} \inf \{C : \Pr\{L(X, C) > B\} \leq \varepsilon\}$, and similarly, the buffer requirement is defined through:

$$B_{\text{req}} \stackrel{\text{def}}{=} \inf \{B : \Pr\{L(X, C) > B\} \leq \varepsilon\},$$

which means triple optimisation for the corresponding resource requirements based on the MSA approximation of the overflow probability, $C_{\text{equ}}^{B,\varepsilon}$ and $B_{\text{req}}^{C,\varepsilon}$, as compared to the double optimisation of the overflow probability. We have developed the double-optimisation-form resource requirements: Introducing the rate-type, server-capacity-type and buffer-size-type functions $\mathcal{J}_{B,C}(s,t) \stackrel{\text{def}}{=} st\alpha(s,t) - s(B + Ct)$, $\mathcal{C}_{B,\varepsilon}(s,t) \stackrel{\text{def}}{=} \alpha(s,t) + \frac{-\log \varepsilon}{st} + \frac{B}{t}$ and $\mathcal{B}_{C,\varepsilon}(s,t) \stackrel{\text{def}}{=} t\alpha(s,t) + \frac{-\log \varepsilon}{s} - Ct$, MSA transforms to $-I_{(X,B,C)} = \sup_{t>0} \inf_{s>0} \mathcal{J}_{B,C}(s,t)$.

PROPOSITION 3.2.

$$\sup_{t>0} \inf_{s>0} \mathcal{J}_{B,C}(s,t) < \log \varepsilon \implies \begin{cases} \sup_{t>0} \inf_{s>0} \mathcal{C}_{B,\varepsilon}(s,t) \leq C \\ \sup_{t>0} \inf_{s>0} \mathcal{B}_{C,\varepsilon}(s,t) \leq B, \end{cases}$$

and similarly for $\mathcal{C}_{B,\varepsilon}(s,t)$ and $\mathcal{B}_{C,\varepsilon}(s,t)$.

(A_{MSA}) Assume that the double optimisation of the involved rate functions are attained for some parameters s and t , e.g.,

$$\sup_{t>0} \inf_{s>0} \mathcal{J}_{B,C}(s,t) = \max_{t>0} \min_{s>0} \mathcal{J}_{B,C}(s,t).$$

THEOREM 3.3. Assume (A_{MSA}). Then

$$\begin{aligned} \mathcal{J}_{B,C}(s_{B,C}, t_{B,C}) < \log \varepsilon &\Leftrightarrow \mathcal{C}_{B,\varepsilon}(s_{B,\varepsilon}, t_{B,\varepsilon}) < C \Leftrightarrow \mathcal{B}_{C,\varepsilon}(s_{C,\varepsilon}, t_{C,\varepsilon}) < B, \\ \mathcal{J}_{B,C}(s_{B,C}, t_{B,C}) = \log \varepsilon &\Leftrightarrow \mathcal{C}_{B,\varepsilon}(s_{B,\varepsilon}, t_{B,\varepsilon}) = C \Leftrightarrow \mathcal{B}_{C,\varepsilon}(s_{C,\varepsilon}, t_{C,\varepsilon}) = B, \\ \mathcal{J}_{B,C}(s_{B,C}, t_{B,C}) > \log \varepsilon &\Leftrightarrow \mathcal{C}_{B,\varepsilon}(s_{B,\varepsilon}, t_{B,\varepsilon}) > C \Leftrightarrow \mathcal{B}_{C,\varepsilon}(s_{C,\varepsilon}, t_{C,\varepsilon}) > B. \end{aligned}$$

LEMMA 3.4. For $t > 0$ such that inf is min:

$$\mathcal{J}_{B,C}(s_{B,C}(t), t) < \log \varepsilon \Leftrightarrow \mathcal{C}_{B,\varepsilon}(s_{B,\varepsilon}(t), t) < C \Leftrightarrow \mathcal{B}_{C,\varepsilon}(s_{C,\varepsilon}(t), t) < B.$$

LEMMA 3.5. For $t > 0$ such that inf is min:

$$\mathcal{J}_{B,C}(s_{B,C}(t), t) = \log \varepsilon \Leftrightarrow \mathcal{C}_{B,\varepsilon}(s_{B,\varepsilon}(t), t) = C \Leftrightarrow \mathcal{B}_{C,\varepsilon}(s_{C,\varepsilon}(t), t) = B.$$

LEMMA 3.6. For $t > 0$ such that inf is min:

$$\mathcal{J}_{B,C}(s_{B,C}(t), t) > \log \varepsilon \Leftrightarrow \mathcal{C}_{B,\varepsilon}(s_{B,\varepsilon}(t), t) > C \Leftrightarrow \mathcal{B}_{C,\varepsilon}(s_{C,\varepsilon}(t), t) > B.$$

MSA/On-Off-Type Approximation We define for, $X[0, t) = \sum_{i=1}^N X_i[0, t)$ independent but not necessarily i.i.d. sum, the on-off-type approximation of the effective-bandwidth function by

$$\overline{\alpha}(s, t) \stackrel{\text{def}}{=} \frac{1}{st} \log \left(1 + \frac{e^{stp} - 1}{tp} E(X[0, t)) \right)$$

for the aggregate and

$$\widehat{\alpha}(s, t) \stackrel{\text{def}}{=} \sum_{i=1}^N \frac{1}{st} \log \left(1 + \frac{e^{stp_i} - 1}{tp_i} E(X_i[0, t)) \right)$$

for the per-source approximation⁵.

For the bufferless multiplexing (Chernoff bound), briefly,

$$\Pr\{L(X, C) > B\} \leq \Pr\{X > C\} \leq e^{\inf_{s>0}\{s\alpha_X(s,0)-sC\}} \stackrel{\text{def}}{=} -I_{(X,0,C)}$$

with

$$\alpha_X(s, 0) \stackrel{\text{def}}{=} \frac{1}{s} \log E(e^{sX}),$$

$$C_{\text{equ}}^0 \stackrel{\text{def}}{=} \inf \left\{ C : e^{-I_{(X,0,C)}} \leq \varepsilon \right\} = \inf_{s>0} \left\{ \alpha(s, 0) + \frac{-\log \varepsilon}{s} \right\},$$

$$B_{\text{req}}^0 \stackrel{\text{def}}{=} \inf \left\{ B > 0 : e^{-I_{(X,0,C)}} \leq \varepsilon \right\} = 0,$$

and the on-off-type versions are

$$\overline{\alpha}(s, 0) \stackrel{\text{def}}{=} \frac{1}{s} \log \left(1 + \frac{e^{sp} - 1}{p} E(X) \right)$$

and

$$\widehat{\alpha}(s, 0) \stackrel{\text{def}}{=} \sum_{i=1}^N \frac{1}{s} \log \left(1 + \frac{e^{sp_i} - 1}{p_i} E(X_i) \right),$$

respectively.

The corresponding notation of the respective aggregate on-off-type approximations are $-\overline{I}_{(X,0,C)}$, $-\overline{I}_{(X,B,C)}$, $\overline{C}_{\text{equ}}^0$, $\overline{C}_{\text{equ}}^{B,\varepsilon}$ and $\overline{B}_{\text{req}}^{C,\varepsilon}$, and the per-source on-off-type approximations are $-\widehat{I}_{(X,0,C)}$, $-\widehat{I}_{(X,B,C)}$, $\widehat{C}_{\text{equ}}^0$, $\widehat{C}_{\text{equ}}^{B,\varepsilon}$ and

⁵ $e^x \leq 1 + \frac{e^p - 1}{p} x$ if $0 \leq x \leq p \stackrel{\text{def}}{=} \sum_{i=1}^N p_i$.

$\widehat{B}_{\text{req}}^{C,\varepsilon}$. We show that the on-off-type approximation reduces MSA to the bufferless multiplexing and that the aggregate on-off approximation often delivers the trivial (peak-rate) approximation:

THEOREM 3.7. $e^{-\overline{I}_{(X,B,C)}} = e^{-\overline{I}_{(X,0,C)}}, e^{-\widehat{I}_{(X,B,C)}} = e^{-\widehat{I}_{(X,0,C)}}, \overline{C}_{\text{equ}}^{B,\varepsilon} = \overline{C}_{\text{equ}}^0, \widehat{C}_{\text{equ}}^{B,\varepsilon} = \widehat{C}_{\text{equ}}^0,$

$$\overline{B}_{\text{req}}^{C,\varepsilon} = \begin{cases} 0 & \overline{C}_{\text{equ}}^0 \leq C \\ \infty & \overline{C}_{\text{equ}}^0 > C \end{cases} \quad \text{and} \quad \widehat{B}_{\text{req}}^{C,\varepsilon} = \begin{cases} 0 & \widehat{C}_{\text{equ}}^0 \leq C \\ \infty & \widehat{C}_{\text{equ}}^0 > C \end{cases}.$$

PROPOSITION 3.8.

$$\frac{E(X)}{\sum_{k=1}^N p_k} \geq \varepsilon \iff \overline{C}_{\text{equ}}^{B,\varepsilon} = \overline{C}_{\text{equ}}^0 = \sum_{k=1}^N p_k = p.$$

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I N D E X

DE, U. C., GAZI, A. K.: On almost pseudo symmetric manifolds	53
EKİCI, E.: On α -open sets, \mathcal{A}^* -sets and decompositions of continuity and super-continuity	39
FANCSALI, SZ. L., SZIKLAI, P.: Description of the clubs	141
FLEINER, T.: Stable matchings through fixed points and graphs	69
GAZI, A. K., DE, U. C.: On almost pseudo symmetric manifolds	53
JURKIN, E., SLIEPČEVIĆ, A.: Butterfly points' curve in isotropic plane	3
KOMJÁTH, P.: Shelah's proof of diamond	147
MIN, W. K.: Several types of (ψ, ϕ) -open functions on generalized neighbourhood systems	19
MOLNÁR, Z., NAGY, I., SZILÁGYI, T.: A change of variables theorem for the multidimensional Riemann integral	121
NAGY, I., SZILÁGYI, T., MOLNÁR, Z.: A change of variables theorem for the multidimensional Riemann integral	121
PETZ, D.: Complementary subalgebras. Problems to solve	117
RUZSA, I. Z.: Many differences, few sums	27
SLIEPČEVIĆ, A., JURKIN, E.: Butterfly points' curve in isotropic plane	3
SZENTHE, J.: Proper isometric actions on riemannian manifolds	11
SZIKLAI, P., FANCSALI, SZ. L.: Description of the clubs	141
SZILÁGYI, T., MOLNÁR, Z., NAGY, I.: A change of variables theorem for the multidimensional Riemann integral	121
SZLÁVIK, Á., ADVISOR: SZEIDL, L.: Complementary Use of matrix-analytic methods and large-deviation theory for a general telecommunications queueing system (Ph. D. Thesis)	151

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