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BIFURCATIONS IN A SYSTEM MODELLING HANTAVIRUS EPIDEMICS I: TRANSCRITICAL BIFURCATION

By

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Abstract. In this paper, a simple mathematical model is studied that describes the dynamics of Hantavirus epidemics in a rodent population. Sensitivity analysis is performed on an epidemic threshold value and it is then used to show that at its critical value a transcritical bifurcation takes place. This bifurcation is forward: a super-threshold endemic equilibrium exists, the global asymptotic stability of which is also shown.

1. Introduction

There are situations in many disciplines which can be described, at least up to a crude first approximation, by a simple system of first order differential equations. Such are for example epidemic models where the population is divided into some groups such as susceptibles who can catch the disease and infectives who have the disease and can transmit it. Therefore, as it is usual in some of these models, we make the following assumptions (cf. [9], [8]):

- all newborns are susceptible and the birth rate is proportional to the total density of the population, because all individuals contribute equally to the procreation;
- the population is "well stirred", meaning that every individual has an equal chance to meet any other member of the population;
- the gain in the infective class is at a rate proportional to the member of infectives and susceptibles;
- the incubation period is short enough to be negligible; that is a susceptible who contracts the disease is infective right away;
- the infection is chronic, infectives do not die of it and they do not lose the infectiousness probably for their whole life.

Studies of the dynamical properties of such models usually consist of finding constant equilibrium solutions, and then carrying out a linearized analysis to determine their stability with respect to small disturbances.

In [1] the following simple mathematical model is proposed to study the Hantavirus epidemics of a mice population:

$$\left.\begin{aligned}\dot{M}_S &= F_1(M_S, M_I) := bM - cM_S - \frac{M_S M}{K} - aM_S M_I,\\ \dot{M}_I &= F_2(M_S, M_I) := -cM_I - \frac{M_I M}{K} + aM_S M_I.\end{aligned}\right\} \tag{1}$$

Here the dot means differentiation with respect to time t; $M_S(t) \geq 0$ and $M_I(t) \geq 0$ are the numbers or densities of susceptible and infected mice, $M(t) = M_S(t)+M_I(t)$ denotes the total population, respectively. $a > 0$, $b > 0$, $c > 0$ and $K > 0$ are the infection rate (the measure of the effectiveness of the infection between the two groups), the birth rate, the death rate and the carrying capacity of the environment, respectively.

In [3] a short analysis of the dynamical properties of system (1) was given (by using a Liapunov function) and two non-standard finite difference schemes were used for the simulation of (1).

The aim of the present paper is to give a detailed analysis of dynamical properties of (1) (by using other methods than it was used in [3]). We perform a sensitivity analysis on the so called basic reproduction number and show that transcritical bifurcation takes place.

2. The model

We shall present some results, including the positivity and boundedness of solutions, furthermore existence and stability of equilibria.

First of all, Picard–Lindelöf's Theorem guarantees that solutions of the initial value problem for system (1) exist locally and are unique.

We show now that interior of the positive quadrant of the phase space $[M_S, M_I]$ is an invariant region.

LEMMA 2.1. *All solutions of* (1) *with positive initial conditions* $M_S(0)>0$, $M_I(0) > 0$ *remain positive for all* $t \geq 0$ *in their domain of existence.*

PROOF. Let us assume contrary to the statement that there exists $t > 0$ at which $M_S(t)$ or $M_I(t)$ is equal to zero. Denote

$$t^* := \min\{t > 0 : \; M_S(t) \cdot M_I(t) = 0\},$$

then

- assuming that $M_I(t^*) = 0$, it follows that $M_S(t) \geq 0$ $(t \in [0, t^*])$. If we define
$$C := \min\left\{aM_S(t) - \frac{M_S(t) + M_I(t)}{K} - c : \ t \in [0, t^*]\right\}$$
then for $t \in [0, t^*]$, $\dot{M}_I(t) \geq CM_I(t)$. Therefore
$$M_I(t^*) > M_I(0)\exp(Ct^*) > 0,$$
which is a contradiction. Thus $M_I(t) > 0$ for all $t \geq 0$.
- assuming that $M_S(t^*) = 0$, it follows that
$$\dot{M}_S(t^*) = bM(t^*) - cM_S(t^*) - \frac{M_S(t^*)M(t^*)}{K} - aM_S(t^*)M_I(t^*) =$$
$$= bM_I(t^*) > 0.$$
Since $M_S(0) > 0$, for $M_S(t^*) = 0$ we must have $\dot{M}_S(t^*) \leq 0$, which is a contradiction. Thus $M_S(t) > 0$ for all $t \geq 0$. ∎

We shall consider system (1) restricted to $\mathbb{R}^2_+$ and prove that all solutions stay bounded in $t \in [0, +\infty)$ which implies the existence of solutions for every $t > 0$.

LEMMA 2.2. *System (1) is dissipative, i.e. all solutions are bounded.*

PROOF. We define the function $\sigma(M_S, M_I) :\equiv M_S + M_I$. The time derivative along a solution of (1) is

$$\dot{\sigma}(M_S, M_I) = \dot{M}_S + \dot{M}_I = M\left(b - c - \frac{M}{K}\right). \tag{2}$$

Thus, if $b \leq c$ or if $b > c$ but $M_S + M_I > K(b - c)$ then this derivative is negative. This means that the trajectories of the restricted system cross the line $\sigma(M_S, M_I) = L$ from outside to inside if $L > 0$ is sufficiently large. ∎

REMARK 2.1. Equation (2) has the same character as the logistic equation for M. Thus, for every positive initial $\varphi(0)$ the solution

$$\varphi(t) = \frac{K(b - c)\varphi(0)}{\varphi(0) + \left[K(b - c) - \varphi(0)\right] e^{-(b-c)t}} \quad (t \in [0, +\infty))$$

of (2) is positive and tends to $K(b - c)$ or to 0 if $b > c$ or $b < c$, respectively.

On the boundary of the positive quadrant the system (1) has two equilibrium points: the trivial equilibrium $(0,0)$ for all parameter values and the uninfected equilibrium $(K(b-c),0)$ provided that

$$b > c. \tag{3}$$

A standard stability analysis based on the Jacobian

$$J(M_S, M_I) :=$$
$$:= \frac{1}{K}\begin{pmatrix} K(b-c) - 2M_S - (1+aK)M_I & bK - (1+aK)M_S \\ (aK-1)M_I & (aK-1)M_S - cK - 2M_I \end{pmatrix}$$

shows that $(0,0)$ is stable iff the inequality (3) doesn't hold (i.e. when the birth rate is not higher than the death rate, the whole mice population may die) and (3) with $K \leq b/a(b-c)$ imply the stability of the uninfected equilibrium, respectively. In fact,

$$J(0,0) = \begin{pmatrix} b-c & b \\ 0 & -c \end{pmatrix}, \text{ resp. } J(K(b-c),0) = \begin{pmatrix} c-b & c+aK(c-b) \\ 0 & -b-aK(c-b) \end{pmatrix}$$

has the eigenvalues $b-c$, c resp. $c-b$, $-b-aK(c-b)$.

Based on the next generation approach (c.f. [5]) a value $\mathcal{R}_0 := aK(1-c/b)$ is in [3] introduced for the so called basic reproductive number which has for this model the following interpretation: "$\mathcal{R}_0$ is the number of infected mice resulting from each infected mouse during its infected lifetime". To examine the sensitivity of $\mathcal{R}_0$ to each of its parameters we calculate the normalized sensitivity indices (cf. [4])

$$\begin{aligned}
\Psi_a &= \frac{a}{\mathcal{R}_0}\cdot\frac{\partial\mathcal{R}_0}{\partial a} = \frac{a}{aK(1-c/b)}\cdot K(1-c/b) = 1,\\
\Psi_b &= \frac{b}{\mathcal{R}_0}\cdot\frac{\partial\mathcal{R}_0}{\partial b} = \frac{b}{aK(1-c/b)}\cdot\frac{aKc}{b^2} = \frac{c}{b-c},\\
\Psi_c &= \frac{c}{\mathcal{R}_0}\cdot\frac{\partial\mathcal{R}_0}{\partial c} = \frac{c}{aK(1-c/b)}\cdot\left(-\frac{aK}{b}\right) = \frac{c}{c-b},\\
\Psi_K &= \frac{K}{\mathcal{R}_0}\cdot\frac{\partial\mathcal{R}_0}{\partial K} = \frac{K}{aK(1-c/b)}\cdot a(1-c/b) = 1.
\end{aligned}$$

For the values of the parameters used in this model, the sensitivity indices Ψ_a resp. Ψ_K are unit (which means that an increase in a or K of 1% will result in an increase on $\mathcal{R}_0$ 1%), $\Psi_b = -\Psi_c$. Furthermore, since both of the indices Ψ_b, Ψ_c are functions of the parameters, these sensitivity indices will change as the parameter values change.

Clearly, the uninfected boundary equilibrium is locally asymptotically stable if $\mathcal{R}_0 < 1$ holds; whereas if $\mathcal{R}_0 > 1$ then it is unstable. Thus, $\mathcal{R}_0$ is a threshold parameter for this model. The following analysis of the local center manifold yields the existence and local stability of a super-threshold endemic equilibrium for $\mathcal{R}_0$ near one.

Let us denote by A the Jacobian of $\mathbf{F} := \mathrm{col}(F_1, F_2)$ evaluated at the critical value $K^* := b/(a(b-c))$ (i.e. when $\mathcal{R}_0 = 1$) and at the equilibrium point $(K^*(b-c), 0)$, i.e.

$$A := \begin{bmatrix} c-b & c-b \\ 0 & 0 \end{bmatrix}.$$

Clearly, the zero eigenvalue of A is simple, the other eigenvalue of A is negative and the vectors

$$\mathbf{q} := \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \quad \mathbf{p} := \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

are right and left nullvectors of A, i.e. $A\mathbf{q} = \mathbf{0}$, $\quad A^T\mathbf{p} = \mathbf{0}$ such that $\mathbf{p}^T\mathbf{q} \equiv \equiv \langle \mathbf{p}, \mathbf{q} \rangle = 1$. Let

$$\alpha := \frac{\langle \mathbf{p}, \mathbf{B}(\mathbf{q}, \mathbf{q}) \rangle}{2}, \quad \beta := \left\langle \mathbf{p}, \widetilde{\mathbf{B}}(\mathbf{q}) \right\rangle$$

where the functions $\mathbf{B} : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$, $\widetilde{\mathbf{B}} : \mathbb{R}^2 \to \mathbb{R}^2$ are given by

$$B_i(\mathbf{x}, \mathbf{y}) := \sum_{j,k=1}^{2} \left. \frac{\partial^2 F_i(\boldsymbol{\xi}, K^*)}{\partial \xi_j \, \partial \xi_k} \right|_{\boldsymbol{\xi}=(K^*(b-c),0)} x_j y_k \quad (i \in \{1, 2\}),$$

$$\widetilde{B}_i(\mathbf{x}) := \sum_{j=1}^{2} \left. \frac{\partial^2 F_i(\boldsymbol{\xi}, K^*)}{\partial \xi_j \, \partial K} \right|_{\boldsymbol{\xi}=(K^*(b-c),0)} x_j \quad (i \in \{1, 2\}).$$

All second derivatives of F_i are the following:

$$\left. \frac{\partial^2 F_1(\boldsymbol{\xi}, K^*)}{\partial M_S \partial M_I} \right|_{\boldsymbol{\xi}} = \frac{1}{K^*} - a = \left. \frac{\partial^2 F_1(\boldsymbol{\xi}, K^*)}{\partial M_I \partial M_S} \right|_{\boldsymbol{\xi}},$$

$$\left. \frac{\partial^2 F_1(\boldsymbol{\xi}, K^*)}{\partial M_S^2} \right|_{\boldsymbol{\xi}} = \frac{2}{K^*}, \qquad \left. \frac{\partial^2 F_1(\boldsymbol{\xi}, K^*)}{\partial M_I^2} \right|_{\boldsymbol{\xi}} = 0,$$

$$\left. \frac{\partial^2 F_2(\boldsymbol{\xi}, K^*)}{\partial M_S \partial M_I} \right|_{\boldsymbol{\xi}} = a - \frac{1}{K^*} = \left. \frac{\partial^2 F_2(\boldsymbol{\xi}, K^*)}{\partial M_I \partial M_S} \right|_{\boldsymbol{\xi}},$$

$$\left.\frac{\partial^2 F_2(\boldsymbol{\xi}, K^*)}{\partial M_S^2}\right|_{\boldsymbol{\xi}} = 0, \qquad \left.\frac{\partial^2 F_2(\boldsymbol{\xi}, K^*)}{\partial M_I^2}\right|_{\boldsymbol{\xi}} = -\frac{2}{K^*};$$

resp.

$$\left.\frac{\partial^2 F_1(\boldsymbol{\xi}, K^*)}{\partial M_S \partial K}\right|_{\boldsymbol{\xi}} = \frac{2(b-c)}{K^*}, \qquad \left.\frac{\partial^2 F_1(\boldsymbol{\xi}, K^*)}{\partial M_I \partial K}\right|_{\boldsymbol{\xi}} = \frac{b-c}{K^*},$$

$$\left.\frac{\partial^2 F_2(\boldsymbol{\xi}, K^*)}{\partial M_S \partial K}\right|_{\boldsymbol{\xi}} = 0, \qquad \left.\frac{\partial^2 F_2(\boldsymbol{\xi}, K^*)}{\partial M_I \partial K}\right|_{\boldsymbol{\xi}} = \frac{b-c}{K^*}.$$

Hence

$$\alpha = -a < 0 \quad \text{and} \quad \beta = \frac{b-c}{K^*} > 0.$$

As a consequence (cf. Therorem 4 in [6]) there exists a $\delta > 0$ such that for $1 < \mathcal{R}_0 < \delta$ system (1) has at least one locally asymptotically stable endemic equilibrium, i.e. a transcritical bifurcation takes place which is forward meaning that there is a transfer of stability from the infection-free steady state to the endemic equilibrium, and vice versa (cf. [2]). The interior equilibria are the intersections of the susceptibles null-cline

$$M_I = h_1(M_S) := \frac{K(b-c)M_S - M_S^2}{(aK+a)M_S - bK}$$

and the infecteds null-cline

$$M_I = h_2(M_S) := (aK-1)M_S - cK$$

(c.f. Fig. 1). The intersection $(\overline{M}_S, \overline{M}_I) := (b/a, K(b-c) - b/a)$ lies in the interior of the positive quadrant if and only if

$$b > c \quad \text{and} \quad K > \frac{b}{a(b-c)}. \tag{4}$$

The Jacobian

$J\left(\overline{M}_S, \overline{M}_I\right) =$

$$= \frac{1}{aK}\begin{pmatrix} a^2K^2(c-b) + b(aK-1) & -b \\ b(1-2aK) + acK - a^2K^2(c-b) & acK - b(aK-1) \end{pmatrix}$$

of $\mathbf{F}$ evaluated at $\left(\overline{M}_S, \overline{M}_I\right)$ has the eigenvalues $c-b$ and $b + aK(c-b)$ (cf. [3]) which proves again that $\left(\overline{M}_S, \overline{M}_I\right)$ is locally asymptotically stable if $\mathcal{R}_0 > 1$ i.e. (4) holds.

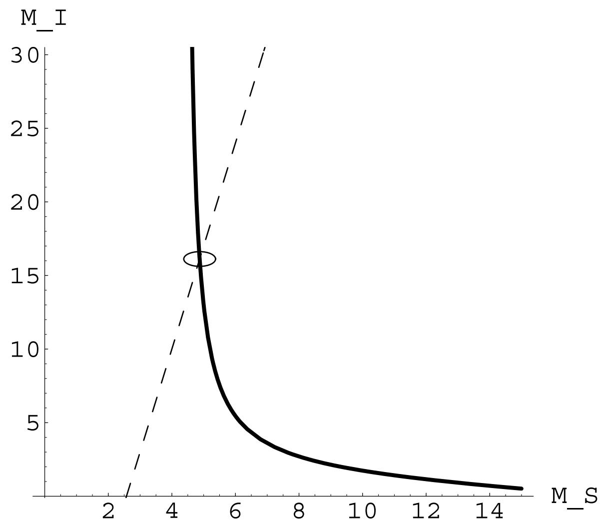

Fig. 1. The zero isoclines (dashed: h_2, bold: h_1) and the endemic equilibrium of system (1) (MATHEMATICA®)

Summarizing the results about the local stability of the two essential equilibria we can establish the following: If (3) holds then $(K(b-c), 0)$ is stable along the M_S-direction and stable or unstable along the M_I-direction according as $K \leq$ or $> b/(a(b-c))$. Thus, if $\left(\overline{M}_S, \overline{M}_I\right)$ does not exist, then $(K(b-c), 0)$ is an attractor or sink. But if $\left(\overline{M}_S, \overline{M}_I\right)$ exists, then $(K(b-c), 0)$ is a saddle point with ingoing trajectories on the M_S-axis. The phase portrait of system (1) for given values $a = 0.2$, $b = 0.9$, $c = 0.45$ and $K = 10$ resp. $K = 50$ is shown in Fig. 2.

By showing that system (1) admits no periodic orbits we extend the local stability result to a global one.

LEMMA 2.3. *The system* (1) *has no limit cycle in the positive quadrant of the phase plane.*

PROOF. Let us define the function $\mathfrak{h}$ by

$$\mathfrak{h}(M_S, M_I) := \frac{1}{M_S \cdot M_I} \quad (M_S > 0, M_I > 0).$$

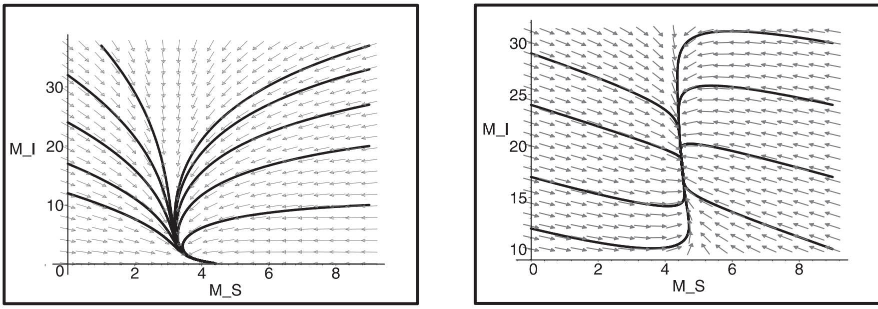

Fig. 2. Phase portraits of the system (1) for $K < b/(a(b-c))$ in the left-hand graph and $K > b/(a(b-c))$ in the right-hand graph (MAPLE®)

Then, we have

$$(\operatorname{div}(\mathfrak{h}\mathbf{F}))(M_S, M_I) = (\partial_1(\mathfrak{h}F_1))(M_S, M_I) + (\partial_2(\mathfrak{h}F_2))(M_S, M_I) =$$

$$= -\frac{b}{M_S} - \frac{1}{KM_I} - \frac{1}{KM_S} < 0.$$

Therefore by the Dulac's negative criterion (see e.g. [7]) system (1) has no limit cycle in the positive quadrant of the $[M_S, M_S]$ plane. ∎

Thus, we can summarize our results in the following:

THEOREM 2.1. *If*

1. $b \leq c$ *then system* (1) *has only the trivial equilibrium which is globally asymptotically stable;*
2. $b > c$ *and*
   (a) $K < b/(a(b-c))$ *then apart from the trivial (unstable) one there is only the uninfected equilibrium* $(K(b-c), 0)$, *which is globally asymptotically stable;*
   (b) $K \geq b/(a(b-c))$ *then a new (endemic) equilibrium* $(\overline{M}_S, \overline{M}_I)$ *bifurcates from* $(K(b-c), 0)$

$$((\overline{M}_S, \overline{M}_I) = (K(b-c), 0)$$

as $K = b/(a(b-c))$ *by a forward transcritical bifurcation and the new equilibrium becomes the globally asymptotically stable one as* $K > b/(a(b-c))$, *whereas* $(K(b-c), 0)$ *is a repeller as* $K > b/(a(b-c))$.

In a subsequent paper we are going to study what happens if delay is introduced into the system. It will be shown that at the critical value Poincaré–

Andronov–Hopf bifurcation takes place: a small amplitude periodic solution occurs.

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ABOUT THE NON-INTEGER PROPERTY OF HYPERHARMONIC NUMBERS

By

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Abstract. It was proven in 1915 by Leopold Theisinger that the H_n harmonic numbers are never integers. In 1996 Conway and Guy have defined the concept of hyperharmonic numbers. The question naturally arises: are there any integer hyperharmonic numbers? The author gives a partial answer to this question and conjectures that the answer is "no".

The n-th harmonic number is the n-th partial sum of the harmonic series:

$$H_n = \sum_{k=1}^{n} \frac{1}{k}.$$

Conway and Guy in [2] defined the harmonic numbers of higher orders, also known as the hyperharmonic numbers: $H_n^{(1)} := H_n$, and for all $r > 1$ let

$$H_n^{(r)} = \sum_{k=1}^{n} H_k^{(r-1)}$$

be the n-th harmonic number of order r. These numbers can be expressed by binomial coefficients and ordinary harmonic numbers:

$$H_n^{(r)} = \binom{n+r-1}{r-1}(H_{n+r-1} - H_{r-1}).$$

The prominent role of these numbers has been realized recently in combinatory. The $\left[{n \atop k} \right]_r$ r-Stirling number is the number of the permutations of

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the set $\{1,\dots,n\}$ having k disjoint, non-empty cycles, in which the elements 1 through r are restricted to appear in different cycles.

In [7] one can find the following interesting equality:

$$H_n^{(r)} = \frac{\left[{n+r \atop r+1} \right]_r}{n!}.$$

Let us turn our attention to the main question of this paper. It is known that any number of consecutive terms not necessarily beginning with 1 will never sum to an integer (see [4]). As a corollary, we get that the H_n harmonic numbers are never integers ($n > 1$). Theisinger proved this latter result directly in 1915 [1] . The question appears obviously: are there any integer hyperharmonic numbers?

Theisinger's main tool was the 2-adic norm. We give a short summary of his method. Every rational number $x \neq 0$ can be represented by $x = \frac{p^\alpha r}{s}$, where p is a fixed prime number, r and s are relative prime integers to p. α is a unique integer. We can define the p-adic norm of x by

$$|x|_p = p^{-\alpha}, \text{ and let } |0|_p = 0.$$

This norm fulfills the properties of the usual norms, namely

$$|x|_p = 0 \iff x = 0,$$

$$|xy|_p = |x|_p |y|_p \quad (x, y \in \mathbf{Q}),$$

$$|x + y|_p \leq |x|_p + |y|_p \quad (x, y \in \mathbf{Q}).$$

Furthermore, the so-called strong triangle inequality also holds:

$$|x + y|_p \leq \max\{|x|_p, |y|_p\} (\leq |x|_p + |y|_p).$$

We shall use the following property of integer numbers:

$$x \in \mathbf{Z} \Longrightarrow x = p^\alpha r \Longrightarrow |x|_p = \frac{1}{p^\alpha} \leq 1,$$

where r and the prime p are relative prime integers. This means that if the p-norm of a rational x is greater than 1 then x is necessarily non-integer.

Let us introduce the order of a natural number n: if $2^m \leq n < 2^{m+1}$, then $\mathrm{Ord}_2(n) := m$. It is obvious that $\mathrm{Ord}_2(n) = \lfloor \ln(n)/\ln(2) \rfloor$.

THEOREM 1.
$$|H_n|_2 = 2^{\mathrm{Ord}_2(n)} \quad (n \in \mathbf{N}),$$
that is – by our observation above – H_n is never integer.

PROOF. First, let n be even. Since $|x|_2 = |-x|_2$ for all $x \in \mathbf{Q}$, by the strong triangle inequality we get

$$\max\{|H_n|_2, |1|_2\} = \max\left\{|H_n|_2, |1|_2, \left|\frac{1}{3}\right|_2, \left|\frac{1}{5}\right|_2, \dots, \left|\frac{1}{n-1}\right|_2\right\} \geq$$
$$\geq \left|H_n - 1 - \frac{1}{3} - \frac{1}{5} - \dots - \frac{1}{n-1}\right|_2 =$$
$$= \left|\frac{1}{2} + \frac{1}{4} + \dots + \frac{1}{n-2} + \frac{1}{n}\right|_2 =$$
$$= \left|\frac{1}{2}\right|_2 \left|1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n/2}\right|_2 = 2\left|H_{n/2}\right|_2.$$

If n is odd, the situation is the same:
$$\max\{|H_n|_2, |1|_2\} \geq 2\left|H_{(n-1)/2}\right|_2.$$
The reader may verify it.

So we get that the 2-adic norm of the harmonic numbers is monotone increasing. Since $|H_2|_2 = \left|\frac{3}{2}\right|_2 = 2$, the 2-adic norm of all the harmonic numbers are greater than 1. As a corollary, this means that the harmonic numbers are not integers because of the property of the 2-adic norm mentioned above.

We can continue the calculations on $H_{n/2}$ (or on $H_{(n-1)/2}$) instead of H_n. For instance let us consider that $n/2$ is even. Then the method described above gives that
$$|H_{n/2}|_2 \geq \left|\frac{1}{2}\right|_2 |H_{n/4}|_2 = 2|H_{n/4}|_2.$$
This and the previous estimation implies that
$$|H_n|_2 \geq \left|\frac{1}{2}\right|_2 |H_{n/2}|_2 \geq \left|\frac{1}{2}\right|_2 \left|\frac{1}{2}\right|_2 |H_{n/4}|_2 = 4|H_{n/4}|_2.$$

And so on. If $n/2$ is odd then we choose $(n/2-1)/2$ instead of $n/4$. We can perform these steps exactly $\mathrm{Ord}_2(n)$ times.

After all, we shall have the following:

$$|H_n|_2 \geq \left|\frac{1}{2^{\mathrm{Ord}_2(n)}}\right|_2 |H_1|_2 = 2^{\mathrm{Ord}_2(n)}.$$

On the other hand,

$$|H_n|_2 \leq \max\left\{|1|_2, \left|\frac{1}{2}\right|_2, \dots, \left|\frac{1}{n}\right|_2\right\} = \left|\frac{1}{2^{\mathrm{Ord}_2(n)}}\right|_2 = 2^{\mathrm{Ord}_2(n)},$$

because the greatest 2-power occuring between 1 and n is $\mathrm{Ord}_2(n)$.

The inequalities detailed above give the statement. ∎

A different approach can be found in [3] , [5] and in their references. The next proof comes from these sources.

PROOF. Let us fix the order of n, i.e.: $\mathrm{Ord}_2(n) := m$. This implies that the denominator of $\frac{2^{m-1}}{n}$ is odd, unless $n = 2^m$. We get that the number

$$2^{m-1}H_n - \frac{1}{2}$$

can be represented by the sum of rationals with odd denominators. Write

$$2^{m-1}H_n - \frac{1}{2} = \frac{a_1}{b_1} + \dots + \frac{a_s}{b_s} = \frac{c}{\mathrm{lcm}(b_1, \dots, b_s)},$$

where b_i is odd for all $i = 1, \dots, s$. It means that $b := \mathrm{lcm}(b_1, \dots, b_s)$ is odd. The last formula gives the result

$$H_n = \frac{\frac{c}{b} + \frac{1}{2}}{2^{m-1}} = \frac{2c + b}{2^m b}. \qquad ∎$$

Let us turn our attention to hyperharmonic numbers. We need a lemma which can be found in [6] :

LEMMA 2.

$$|n!|_p = p^{(A_p(n)-n)/(p-1)},$$

where $A_p(n)$ is the sum of the digits of the p-adic expansion of n.

EXAMPLE 3. Let $p = 2$ and $n = 11$. Then $n = 1011_2$, that is, $A_2(n) = 3$.

$$|n!|_2 = |39916800|_2 = |256 \cdot 155925|_2 = |256|_2 |155925|_2 = |2^8|_2 \cdot 1 = 2^{-8}.$$

We can apply the lemma: $A_2(n) - n = 3 - 11 = -8$, whence $|n!|_2 = 2^{-8}$.

THEOREM 4. *If* $\mathrm{Ord}_2(n+r-1) > \mathrm{Ord}_2(r-1)$ *then*

$$|H_n^{(r)}|_2 = 2^{A_2(n+r-1)-A_2(n)-A_2(r-1)+\mathrm{Ord}_2(n+r-1)},$$

else

$$|H_n^{(r)}|_2 = 2^{A_2(n+r-1)-A_2(n)-A_2(r-1)+\max\left\{\left|\frac{1}{r}\right|_2,\left|\frac{1}{r+1}\right|_2,\dots,\left|\frac{1}{n+r-1}\right|_2\right\}}.$$

PROOF. Let a, b, c, d be odd numbers. Then

$$\left|H_n^{(r)}\right|_2 = \left|\binom{n+r-1}{r-1}(H_{n+r-1}-H_{r-1})\right|_2 =$$

$$= \left|\binom{n+r-1}{r-1}\right|_2 \left|\frac{a}{2^{\mathrm{Ord}_2(n+r-1)}b} - \frac{c}{2^{\mathrm{Ord}_2(r-1)}d}\right|_2 =$$

$$= \left|\binom{n+r-1}{r-1}\right|_2 \left|\frac{2^{\mathrm{Ord}_2(r-1)}ad - 2^{\mathrm{Ord}_2(n+r-1)}bc}{2^{\mathrm{Ord}_2(n+r-1)+\mathrm{Ord}_2(r-1)}bd}\right|_2.$$

Because of the condition $\mathrm{Ord}_2(n+r-1) > \mathrm{Ord}_2(r-1)$ we get

$$\left|H_n^{(r)}\right|_2 = \left|\binom{n+r-1}{r-1}\right|_2 \left|\frac{ad - 2^{\mathrm{Ord}_2(n+r-1)-\mathrm{Ord}_2(r-1)}bc}{2^{\mathrm{Ord}_2(n+r-1)}bd}\right|_2.$$

Since the nominator is odd, we get the following:

$$\left|\frac{ad - 2^{\mathrm{Ord}_2(n+r-1)-\mathrm{Ord}_2(r-1)}bc}{2^{\mathrm{Ord}_2(n+r-1)}bd}\right|_2 = 2^{\mathrm{Ord}_2(n+r-1)}.$$

To compute the 2-adic norm of the binomial coefficient, we use the previous lemma.

$$\left|\binom{n+r-1}{r-1}\right|_2 = \left|\frac{(n+r-1)!}{(r-1)!n!}\right|_2 =$$

$$= \frac{2^{A_2(n+r-1)-n-r+1}}{2^{A_2(r-1)-r+1}2^{A_2(n)-n}} = 2^{A_2(n+r-1)-A_2(n)-A_2(r-1)}.$$

This, and the previous equality give the result with respect to the condition $\mathrm{Ord}_2(n+r-1) > \mathrm{Ord}_2(r-1)$.

Let us fix an arbitrary n for which $\mathrm{Ord}_2(n+r-1) = \mathrm{Ord}_2(r-1)$.

$$H_{n+r-1} - H_{r-1} = \frac{1}{r} + \frac{1}{r+1} + \dots + \frac{1}{n+r-1}.$$

Let us substract all of the fractions with odd denominators. Then we can take $\frac{1}{2}$ out of the remainder and continue the recursive method described in the first proof of Theorem 1. We can make such substraction steps

$$\max\left\{\left|\frac{1}{r}\right|_2, \left|\frac{1}{r+1}\right|_2, \dots, \left|\frac{1}{n+r-1}\right|_2\right\}$$

times. The result:

$$|H_{n+r-1} - H_{r-1}|_2 \geq \max\left\{\left|\frac{1}{r}\right|_2, \left|\frac{1}{r+1}\right|_2, \dots, \left|\frac{1}{n+r-1}\right|_2\right\}.$$

On the other hand, by the strong triangle inequality

$$|H_{n+r-1} - H_{r-1}|_2 \leq \max\left\{\left|\frac{1}{r}\right|_2, \left|\frac{1}{r+1}\right|_2, \dots, \left|\frac{1}{n+r-1}\right|_2\right\}. \qquad \blacksquare$$

COROLLARY 5. *The sum of the harmonic numbers cannot be integer:*

$$H_1 + H_2 + \cdots + H_n \notin \mathbf{N} \quad (n > 1).$$

PROOF. $H_1 + H_2 + \cdots + H_n = H_n^{(2)}$. The condition with respect to the order of n and r holds because $\mathrm{Ord}_2(n+2-1) > \mathrm{Ord}_2(2-1) = 0$ for all $n \geq 1$. Furthermore,

$$\left|H_n^{(2)}\right|_2 = 2^{A_2(n+1)-A_2(n)-A_2(1)+\mathrm{Ord}_2(n+1)}.$$

Let $m = \mathrm{Ord}_2(n+1)$. Our goal is to minimize the power of 2. $\mathrm{Ord}_2(n+1) = m$ implies that $n+1 < 2^{m+1}$, therefore $1 \leq A_2(n+1) \leq m+1$ and $1 \leq A_2(n) \leq m$. The minimum in the power is taken when $A_2(n) = m$ and $A_2(n+1) = 1$. It is possible if and only if $n = 2^m - 1$. In this case

$$A_2(n+1) - A_2(n) - A_2(1) + \mathrm{Ord}_2(n+1) = 1 - m - 1 + m = 0.$$

We get that if $n \neq 2^m - 1$ for some m, then $|H_n^{(2)}|_2 > 1$, that is, $H_n^{(2)} \notin \mathbf{N}$. On the other hand, let us assume that n has the form $2^m - 1$. This implies that

$$H_n^{(2)} = \binom{n+2-1}{2-1}(H_{n+2-1} - H_{2-1}) =$$

$$= (n+1)(H_{n+1} - 1) = 2^m\left(\frac{a}{2^{\mathrm{Ord}_2(n+1)}b} - 1\right) = \frac{a}{b} - 2^m \notin \mathbf{N}. \qquad \blacksquare$$

One can easily prove the following, using the method in the previous proof.

COROLLARY 6. *$H_n^{(3)} \notin \mathbf{N}$ for all $n > 1$.*

As we can see, the method to prove the non-integer property of harmonic numbers does not work for hyperharmonic numbers, because there are n and r integers for which $|H_n^{(r)}|_2 = 1$. In spite of this fact, we believe that Theisinger's theorem holds for all hyperharmonic numbers, too.

CONJECTURE 7. *None of the hyperharmonic numbers can be integers $(r, n \geq 2)$.*

EXAMPLE 8. We demonstrate that the theorem described above simplifies the calculation of the 2-norm of hyperharmonic numbers.

For instance,

$$H_{18}^{(8)} = \binom{18+8-1}{8-1}(H_{18+8-1} - H_{8-1}) =$$

$$= 480700\left(\frac{34052522467}{8923714800} - \frac{363}{140}\right) = \frac{10914604807}{18564}.$$

Since $|18564|_2 = 2^{-2}$, we get that $|H_{18}^{(8)}|_2 = 2^2 = 4$.

On the other hand, $A_2(18+8-1) = A_2(16+8+1) = 3$, $A_2(8-1) = A_2(4+ +2+1) = 3$, $A_2(18) = A_2(16+2) = 2$ and $\mathrm{Ord}_2(18+8-1) = \mathrm{Ord}_2(16+9) = 4$. By theorem 5,

$$|H_{18}^{(8)}|_2 = 2^{3-3-2+4} = 2^2 = 4.$$

Finally, we pose an interesting question:

PROBLEM 9. For which $n_1 \neq n_2$ and $r_1 \neq r_2$ does the equality

$$H_{n_1}^{(r_1)} = H_{n_2}^{(r_2)}$$

stand?

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ON SUBSETS DEFINED IN TERMS OF WEAK ENVELOPES AND ENVELOPES

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Abstract. We define and discuss the various characterizations and properties of some kind of sets in monotonic spaces, weak envelope spaces and envelope spaces which are similar to that of dense sets, nowheredense sets and δ-sets in topological spaces.

1. Introduction and preliminaries

Let X be a nonempty set and $\gamma\colon \wp(X) \to \wp(X)$. We say that $\gamma \in \Gamma(X)$ or simply $\gamma \in \Gamma$ if $\gamma(A) \subset \gamma(B)$ whenever $A \subset B$ where A and B are subsets of X. If $\gamma \in \Gamma$, we call the pair (X, γ), a *monotonic space*. A subset A of X is γ-open [1] if $A \subset \gamma(A)$. The complement of a γ-open set is said to be a γ-closed set. If $\gamma \in \Gamma$, then $\gamma^\bigstar\colon \wp(X) \to \wp(X)$ is defined by $\gamma^\bigstar(A) = X - \gamma(X - A)$ and $\gamma^\bigstar \in \Gamma$ [1]. A subset A of X is $\gamma^\bigstar$-closed if and only if $\gamma(A) \subset A$ [1, Proposition 1.8]. Let $\xi = \{A \subset X \mid A = \gamma(A)\}$. ξ is called the family of all γ-regularclosed (γ-regular [1]) sets. Therefore, a subset A of X is γ-regularclosed if and only if A is γ-open and $\gamma^\bigstar$-closed. The complement of a γ-regularclosed set is called a γ-regularopen set. Let μ be the family of all γ-regularopen sets. Then, $A \in \mu$ if and only if $A = \gamma^\bigstar(A)$ if and only if A is γ-closed and $\gamma^\bigstar$-open. We have the following subclasses of Γ.

$\Gamma_0 = \{\gamma \in \Gamma \mid \gamma(\emptyset) = \emptyset\}$,
$\Gamma_1 = \{\gamma \in \Gamma \mid \gamma(X) = X\}$,
$\Gamma_2 = \{\gamma \in \Gamma \mid \gamma(\gamma(A)) = \gamma(A)$ for every subset A of $X\}$,

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$\Gamma_- = \{\gamma \in \Gamma \mid \gamma(A) \subset A$ for every subset A of $X\}$ and
$\Gamma_+ = \{\gamma \in \Gamma \mid A \subset \gamma(A)$ for every subset A of $X\}$. If $I = \{0,1,2,+,-\}$ and $A \subset I$, then $\gamma \in \Gamma_A$ if and only if $\gamma \in \Gamma_i$ for every $i \in A$. If $\gamma \in \Gamma_+$, then γ is called a *weak envelope* [3]. The pair (X,γ) is called a *weak envelope* space. If $\gamma \in \Gamma_{2+}$, then γ is called an *envelope* [3]. The pair (X,γ) is called an *envelope* space. Weak envelope and envelope operations are further studied by Á. CSÁSZÁR, in [4].

A subset μ of $\wp(X)$ is called a *generalized topology* (briefly GT)[2] if $\emptyset \in \mu$ and arbitrary union of members of μ is again in μ. Elements of μ are called μ*-open* sets. Complements of μ-open sets are called μ*-closed* sets. In this paper, we define and discuss the various characterizations and properties of sets in monotonic spaces, weak envelope spaces and envelope spaces which are similar to that of dense sets, nowheredense sets and δ-sets in topological spaces.

2. rc-dense and rc-nwdense sets

Let (X,γ) be a monotonic space. A subset A of X is said to be *rc-dense* if $\gamma(A) = X$. It is clear that $\gamma(A) = X$ if and only if $\gamma^\star(X - A) = \emptyset$. Since $\gamma \in \Gamma$, it follows that every superset of an rc-dense set is rc-dense and so the existence of an rc-dense set implies that $\gamma(X) = X$ and so $\gamma \in \Gamma_1$ which says that, by Proposition 1.7(b) of [1], $\gamma^\star \in \Gamma_0$. Equivalently, if $\gamma^\star \notin \Gamma_0$, then no rc-dense sets exist. The following Theorem 2.1 gives a property of rc-dense sets. Example 2.2 shows that the converse of Theorem 2.1 is not true.

THEOREM 2.1. *Let (X,γ) be a monotonic space. If a subset A of X is rc-dense, then $A \cap V \neq \emptyset$ for every nonempty γ-regularopen set V.*

PROOF. Suppose $A \cap V = \emptyset$ for some nonempty γ-regularopen set V. $A \cap V = \emptyset$ implies that $V \subset X - A$ and so $V \subset \gamma^\star(X - A)$. Therefore, $X - \gamma^\star(X - A) \subset X - V$ and so $X = \gamma(A) \subset X - V$ which implies that $V = \emptyset$, a contradiction to the hypothesis. Therefore, $A \cap V \neq \emptyset$ for every nonempty γ-regularopen set V. ∎

EXAMPLE 2.2. Let $X = \{a,b,c\}$ and $\gamma : \wp(X) \to \wp(X)$ be defined by $\gamma(\emptyset) = \{b\}$, $\gamma(\{a\}) = \{a,b\}$, $\gamma(\{b\}) = \{b,c\}$, $\gamma(\{c\}) = \{b,c\}$, $\gamma(\{a,b\}) = X$, $\gamma(\{a,c\}) = X$, $\gamma(\{b,c\}) = \{b,c\}$, $\gamma(X) = X$. Then $\gamma \in \Gamma$, $\gamma \notin \Gamma_2$, $\xi = \{\{b,c\}, X\}$ and $\mu = \{\emptyset, \{a\}\}$. Now $V = \{a\}$ is the only nonempty γ-regularopen set such that $V \cap \{a\} \neq \emptyset$ but $\{a\}$ is not rc-dense. Note that $\{a,b\}$, $\{a,c\}$ and X are rc-dense sets.

THEOREM 2.3. *Let (X,γ) be a weak envelope space. If A is a subset of X such that $A \cap V \neq \emptyset$ for every nonempty γ-regularopen set V, then $\gamma(A) \cap V \neq \emptyset$ for every non-empty γ-regularopen set V.*

PROOF. The proof follows from the fact that $\gamma \in \Gamma_+$. ∎

The following Example 2.4 shows that the condition $\gamma \in \Gamma_+$ in Theorem 2.3 cannot be dropped.

EXAMPLE 2.4. Let $X = \{a,b,c\}$ and $\gamma : \wp(X) \to \wp(X)$ be defined by $\gamma(\emptyset) = \{a\}$, $\gamma(\{a\}) = \{a\}$, $\gamma(\{b\}) = \{a\}$, $\gamma(\{c\}) = X$, $\gamma(\{a,b\}) = \{a\}$, $\gamma(\{a,c\}) = X$, $\gamma(\{b,c\}) = X$, $\gamma(X) = X$. Then $\gamma \in \Gamma$, $\gamma \notin \Gamma_+$, $\xi = \{\{a\}, X\}$ and $\mu = \{\emptyset, \{b,c\}\}$. Let $A = \{a,b\}$. $V = \{b,c\}$ is the only nonempty γ-regularopen set such that $V \cap A = \{b\} \neq \emptyset$. But $\gamma(A) = \{a\}$ and so $\gamma(A) \cap V = \emptyset$.

THEOREM 2.5. *Let (X,γ) be a monotonic space and $A \subset X$. Then the following hold.*

(a) *If A is rc-dense, then $\gamma(A) \cap V \neq \emptyset$ for every nonempty γ-regularopen set V.*

(b) *If $\gamma \in \Gamma_2$ and $\gamma(A) \cap V \neq \emptyset$ for every nonempty γ-regularopen set V, then A is rc-dense.*

PROOF. (a) The proof is clear.

(b) Suppose $\gamma(A) \cap V \neq \emptyset$ for every nonempty γ-regularopen set V and A is not rc-dense. Then $X - \gamma(A) \neq \emptyset$. If $V = X - \gamma(A)$, then $\gamma^{\star}(V) = = \gamma^{\star}(X - \gamma(A)) = X - \gamma(X - (X - \gamma(A))) = X - \gamma(\gamma(A)) = X - \gamma(A) = V$ and so V is γ-regularopen. But $V \cap \gamma(A) = (X - \gamma(A)) \cap \gamma(A) = \emptyset$, a contradiction to the hypothesis. Therefore, A is rc-dense. ∎

The following Example 2.6 shows that the condition $\gamma \in \Gamma_2$ in the above Theorem 2.5(b) cannot be dropped. The proof of the Corollary 2.7 below follows from Theorems 2.1, 2.3 and 2.5.

EXAMPLE 2.6. Let (X,γ) be the monotonic space of Example 2.2. $\gamma \notin \Gamma_2$, $V = \{a\}$ is the only nonempty γ-regularopen set such that $V \cap \gamma(\{a\}) \neq \emptyset$ but $\{a\}$ is not rc-dense.

COROLLARY 2.7. *Let (X,γ) be an envelope space and $A \subset X$. Then the following are equivalent.*

(a) *A is rc-dense.*

(b) *$A \cap V \neq \emptyset$ for every nonempty γ-regularopen set V.*

(c) *$\gamma(A) \cap V \neq \emptyset$ for every nonempty γ-regularopen set V.*

Let (X,γ) be a monotonic space. We say that a subset A of X is said to be *rc-nowheredense* (in short, *rc-nwdense*) if $\gamma^{\bigstar}\gamma(A) = \emptyset$. It is clear that $\gamma^{\bigstar}\gamma(A) = \emptyset$ if and only if $\gamma\gamma^{\bigstar}(X - A) = X$. We will denote the family of all rc-nwdense sets in a monotonic space (X,γ) by $\mathcal{N}$. Since $\gamma \in \Gamma$, it follows that every subset of an rc-nwdense set is an rc-nwdense set and so the existence of an rc-nwdense set implies that $\emptyset$ is rc-nwdense and $\gamma^{\bigstar}\gamma \in \Gamma_0$. In other words, if $\gamma^{\bigstar}\gamma \notin \Gamma_0$, then rc-nwdense sets will not exist and so $\mathcal{N} = \emptyset$. The following Example 2.8 shows that in monotonic spaces, we can have either $\mathcal{N} = \emptyset$ or $\mathcal{N} = \{\emptyset\}$ or $\mathcal{N}$ has more than one element. Theorem 2.9 below gives a property of rc-nwdense sets.

EXAMPLE 2.8. (a) [1, Example 1.12] Let $X = \mathbf{R}$ be the set of all real numbers and $\gamma: \wp(X) \to \wp(X)$ be defined by $\gamma(A) = \{0\}$ if $0 \in A$ and $\emptyset$ otherwise. In this space, $\mathcal{N} = \emptyset$.

(b) Consider the monotonic space of Example 2.2. In this space, $\mathcal{N} = \{\emptyset\}$.

(c) Consider the monotonic space of Example 2.4. In this space,

$$\mathcal{N} = \{\emptyset, \{a\}, \{b\}, \{a,b\}\}.$$

THEOREM 2.9. *Let (X,γ) be a monotonic space and $A \subset X$ be rc-nwdense. If V is a nonempty γ-regularopen set, then V is not a subset of $\gamma(A)$.*

PROOF. Since A is rc-nwdense, $\gamma^{\bigstar}\gamma(A) = \emptyset$ and so $\gamma\gamma^{\bigstar}(X - A) = X$ which implies that $\gamma^{\bigstar}(X - A) = X - \gamma(A)$ is rc-dense. By Theorem 2.1, $V \cap (X - \gamma(A)) \neq \emptyset$ for every nonempty γ-regularopen set V. Therefore, $V - \gamma(A) \neq \emptyset$ which implies that $V \not\subset \gamma(A)$ for every nonempty γ-regularopen set V. This completes the proof. ∎

The following Example 2.10 shows that the converse of the above Theorem 2.9 is not true even if the space is a weak envelope space. Theorem 2.11 below shows that the converse is true if (X,γ) is an envelope space. Example 2.12 shows that either the condition $\gamma \in \Gamma_2$ or the condition $\gamma \in \Gamma_+$ cannot be dropped from Theorem 2.11.

EXAMPLE 2.10. Let (X,γ) be the monotonic space of Example 2.2. If $A = \{b,c\}$, then $V = \{a\}$ is the only nonempty γ-regularopen set such that $V \not\subset \gamma(A)$. But $\gamma^{\bigstar}\gamma(A) = \gamma^{\bigstar}(\{b,c\}) = \{c\} \neq \emptyset$ and so A is not rc-nwdense.

THEOREM 2.11. *Let (X,γ) be an envelope space and $A \subset X$. If for every nonempty γ-regularopen set V, V is not a subset of $\gamma(A)$, then A is rc-nwdense.*

PROOF. If V is a nonempty γ-regularopen set such that V is not a subset of $\gamma(A)$, then $V - \gamma(A) \neq \emptyset$ which implies that $V \cap (X - \gamma(A)) \neq \emptyset$. Since $\gamma \in \Gamma_+$, by Theorem 2.3, $V \cap \gamma(X - \gamma(A)) \neq \emptyset$. Since $\gamma \in \Gamma_2$, by Theorem 2.5, $X - \gamma(A)$ is rc-dense and so $\gamma(X - \gamma(A)) = X$ which implies that $X - \gamma^{\bigstar}\gamma(A) = X$. Therefore, $\gamma^{\bigstar}\gamma(A) = \emptyset$ and so A is rc-nwdense. ∎

EXAMPLE 2.12. (a) Example 2.10 shows that the condition Γ_2 cannot be dropped in Theorem 2.11.

(b) Consider the monotonic space of Example 2.8(a). Then $\mu = \{\mathbf{R}, \mathbf{R} - \{0\}\}$. Clearly, $\gamma \in \Gamma_2$. If $B \subset X$ such that $0 \in B$ and B has more than one point, then $\gamma(B) = \{0\} \not\supset B$ and so $\gamma \notin \Gamma_+$. If A is a nonempty subset of $\mathbf{R}$ not containing 0, then $\mathbf{R} \not\subset \gamma(A)$. But A is not rc-nwdense.

Let (X,γ) be a monotonic space. A subset A of X is said to be *weak rc-nwdense* (in short, *wrc-nwdense*) if for every nonempty $V \in \mu$, there exists a nonempty $W \in \mu$ with $W \subset V$ such that $W \cap A = \emptyset$. The following Examples 2.13 and 2.14 shows that rc-nwdenseness and wrc-nwdenseness are independent concepts.

EXAMPLE 2.13. Consider the monotonic space of Example 2.8(a). Then $\mu = \{\mathbf{R}, \mathbf{R} - \{0\}\}$ and $\{0\}$ is wrc-nwdense but not rc-nwdense. Therefore, a wrc-nwdense set need not be an rc-nwdense set.

EXAMPLE 2.14. Let $X = \{a,b,c\}$ and $\gamma : \wp(X) \to \wp(X)$ be defined by $\gamma(\emptyset) = \emptyset$, $\gamma(\{a\}) = \{a\}$, $\gamma(\{b\}) = \{a,b\}$, $\gamma(\{c\}) = X$, $\gamma(\{a,b\}) = X$, $\gamma(\{a,c\}) = X$, $\gamma(\{b,c\}) = X$, $\gamma(X) = X$. Then $\mu = \{\emptyset, X, \{b,c\}\}$. If $A = \{b\}$, then A is rc-nwdense but not wrc-nwdense.

Let (X,γ) be a monotonic space. We say that γ is *subadditive* if $\gamma(A \cup B) \subset \gamma(A) \cup \gamma(B)$ for every subsets A and B of X. Since γ is monotonic, if γ is subadditive, then γ is additive. That is, $\gamma(A \cup B) = \gamma(A) \cup \gamma(B)$ for every subsets A and B of X. The following Lemma 2.15 is essential to characterize rc-nwdense sets in Theorem 2.17 below.

LEMMA 2.15. *Let* (X,γ) *be a monotonic space and* $A \subset X$.
(a) $\gamma \in \Gamma_2$ *if and only if* $\gamma(A)$ *is* γ*-regularclosed for every subset* A *of* X.
(b) *If* γ *is subadditive, then the intersection of two* γ*-regularopen sets is a* γ*-regularopen set.*
(c) *If* $G \cap A = \emptyset$, *then* $G \cap \gamma(A) = \emptyset$ *for every nonempty* γ*-regularopen set* G. *The reverse direction is true, if* $\gamma \in \Gamma_+$.
(d) *If* $x \in \gamma(A)$, *then* $G \cap A \neq \emptyset$ *for every* γ*-regularopen set* G *containing* x.
(e) *If* $\gamma \in \Gamma_{2+}$ *and* $G \cap A \neq \emptyset$ *for every* γ*-regularopen set* G *containing* x, *then* $x \in \gamma(A)$.
(f) A *is rc-nwdense if and only if* $X - \gamma(A)$ *is rc-dense.*

PROOF. (a) The proof is clear.
(b) Let U and V be γ-regularopen. Now

$$\gamma^\star(U \cap V) = X - \gamma(X - (U \cap V)) = X - \gamma((X - U) \cup (X - V)) =$$
$$= X - (\gamma(X - U) \cup \gamma(X - V)) = (X - \gamma(X - U)) \cap (X - \gamma(X - V)) =$$
$$= \gamma^\star(U) \cap \gamma^\star(V) = U \cap V$$

and so $U \cap V$ is γ-regularopen.
(c) If $G \cap A = \emptyset$, then $A \subset X - G$ and so $\gamma(A) \subset \gamma(X - G) = X - \gamma^\star(G) = X - G$. Therefore, $G \cap \gamma(A) = \emptyset$. The proof of the converse is clear.
(d) If G is a γ-regularopen set containing x, then $\gamma(A) \cap G \neq \emptyset$. By (c), $G \cap A \neq \emptyset$.
(e) Suppose $x \notin \gamma(A)$. Since $\gamma \in \Gamma_2$, $G = X - \gamma(A)$ is a γ-regularopen set containing x by (a), such that $G \cap \gamma(A) = \emptyset$. Since $\gamma \in \Gamma_+$, by (c), $G \cap A = \emptyset$, a contradiction to the hypothesis which proves (e).
(f) A is rc-nwdense if and only if $\gamma^\star\gamma(A) = \emptyset$ if and only if $X - \gamma^\star\gamma(A) = X$ if and only if $X - (X - \gamma(X - \gamma(A))) = X$ if and only if $\gamma(X - \gamma(A)) = X$ if and only if $X - \gamma(A)$ is rc-dense. ∎

Example 2.16(a) shows that the condition $\gamma \in \Gamma_+$ cannot be dropped to prove the reverse direction in Lemma 2.15(c). Example 2.16(b) shows that the subadditivity cannot be dropped in the above Lemma 2.15(b). Also, it shows that *in an envelope space the intersection of two* γ*-regularopen sets need not be a* γ*-regularopen set.* Example 2.16(c) shows that the condition $\gamma \in \Gamma_2$ cannot be dropped in Lemma 2.15(e). That is, Lemma 2.15(e) is not true in a weak envelope space.

EXAMPLE 2.16. (a) Let $X = \{a,b,c\}$ and define $\gamma:\wp(X) \to \wp(X)$ by $\gamma(A) = \{a\}$ for every subset A of X. Then $\mu = \{\{b,c\}\}$ and $\gamma \notin \Gamma_+$. If $A = \{b\}$, then $G \cap \gamma(A) = \emptyset$ for every $G \in \mu$ but $G \cap A \neq \emptyset$.

(b) Consider $X = \{a,b,c\}$ and define $\gamma:\wp(X) \to \wp(X)$ by $\gamma(\emptyset) = \emptyset$, $\gamma(\{a\})=\{a\}$, $\gamma(\{b\}) = \{b\}$, $\gamma(\{c\}) = \{c\}$, $\gamma(\{a,b\}) = \{a,b\}$, $\gamma(\{a,c\}) = = \{a,c\}$, $\gamma(\{b,c\}) = \gamma(X) = X$. Then $\gamma \in \Gamma_+$ and $\gamma \in \Gamma_2$. If $A = \{b\}$ and $B = \{c\}$, then $\gamma(A) \cup \gamma(B) = \{b,c\}$. But $\gamma(A \cup B) = \gamma(\{b,c\}) = X \nsubseteq \{b,c\} = = \gamma(A) \cup \gamma(B)$. Therefore, γ is not subadditive. Here

$$\xi = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, X\}$$

and

$$\mu = \{\emptyset, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, X\}.$$

If $U = \{a,b\}$ and $V = \{a,c\}$, then U and V are γ-regularopen sets but $U \cap V = \{a\}$ is not a γ-regularopen set.

(c) Consider the monotonic space (X,γ) of Example 2.14. Then $\gamma \in \Gamma_+$ +, $\gamma \notin \Gamma_2$ and $\mu = \{\emptyset, X, \{b,c\}\}$. If $A = \{b\}$, then $G \cap A \neq \emptyset$ for every nonempty γ-regularopen set G containing c. Since $\gamma(A) = \gamma(\{b\}) = \{a,b\}$, $c \notin \gamma(A)$.

THEOREM 2.17. *Let (X,γ) be an envelope space and $A \subset X$. Then the following hold.*

(a) *If A is wrc-nwdense, then A is rc-nwdense.*

(b) *If γ is subadditive and A is rc-nwdense, then A is wrc-nwdense.*

PROOF. (a) Suppose A is not rc-nwdense. Then $G = \gamma^\bigstar\gamma(A) \neq \emptyset$. Since $\gamma \in \Gamma_2$, by Proposition 1.7(c) of [1], $\gamma^\bigstar \in \Gamma_2$ and so $\gamma^\bigstar(G) = \gamma^\bigstar(\gamma^\bigstar\gamma(A)) = = \gamma^\bigstar\gamma(A) = G$ and so G is a nonempty γ-regularopen set. Since $\gamma \in \Gamma_+$, by Proposition 1.7(d) of [1], $\gamma^\bigstar \in \Gamma_-$ and so $\gamma^\bigstar\gamma(A) \subset \gamma(A)$ which implies that $G \subset \gamma(A)$. Then for every nonempty $W \in \mu$ with $W \subset G$, $W \cap \gamma(A) \neq \emptyset$ and so by Lemma 2.15(c), $W \cap A \neq \emptyset$, a contradiction to the hypothesis.

(b) Suppose A is rc-nwdense. Then by Lemma 2.15(a), $X - \gamma(A)$ is a γ-regularopen set. By Lemma 2.15(f), $X - \gamma(A)$ is rc-dense. By Theorem 2.1, $(X - \gamma(A)) \cap V = W$ is nonempty and $W \subset V$ for every nonempty γ-regularopen set V. By Lemma 2.15(b), W is γ-regularopen. Since $\gamma \in \Gamma_+$, $A \cap W = A \cap ((X - \gamma(A)) \cap V) = \emptyset$. Therefore, A is wrc-nwdense. ∎

COROLLARY 2.18. *Let (X,γ) be an envelope space, $A \subset X$ and γ be subadditive. Then the following are equivalent.*

(a) *A is wrc-nwdense.*

(b) *A is rc-nwdense.*

(c) *If V is a nonempty γ-regularopen set, then V is not a subset of $\gamma(A)$.*

PROOF. The proof follows from Theorems 2.17, 2.9 and 2.11. ∎

The following Example 2.19 shows that in Theorem 2.17(a), the condition *envelope* cannot be replaced by *weak envelope*. Example 2.20 shows that in Theorem 2.17(b), the condition *subadditive* on γ cannot be dropped.

EXAMPLE 2.19. Let $X = \{a,b,c\}$ and $\gamma:\wp(X) \to \wp(X)$ be defined by $\gamma(\emptyset) = \{a\}$, $\gamma(\{a\}) = \{a,b\}$, $\gamma(\{b\}) = \{a,b\}$, $\gamma(\{c\}) = X$, $\gamma(\{a,b\}) = = \gamma(\{a,c\}) = \gamma(\{b,c\}) = \gamma(X) = X$. Then (X,γ) is a weak envelope space. Since $\mu = \{\emptyset\}$, every nonempty subset of X is wrc-nwdense and so $\{c\}$ is wrc-nwdense but it is not rc-nwdense.

EXAMPLE 2.20. Let (X,γ) be an envelope space where $X = \{a,b,c,d\}$ and $\gamma:\wp(X) \to \wp(X)$ be defined by

$$\gamma(\emptyset) = \emptyset,\ \gamma(\{a\}) = \{a\},\ \gamma(\{b\}) = \{b\},\ \gamma(\{c\}) = \{a,b,c\},\ \gamma(\{d\}) = \{a,d\},$$
$$\gamma(\{a,b\}) = \gamma(\{a,c\}) = \gamma(\{b,c\}) = \{a,b,c\},$$
$$\gamma(\{a,d\}) = \{a,d\},\quad \gamma(\{b,d\}) = \gamma(\{c,d\}) = \gamma(X) = X,$$
$$\gamma(\{a,b,c\}) = \{a,b,c\},\quad \gamma(\{a,b,d\}) = \gamma(\{a,c,d\}) = \gamma(\{b,c,d\}) = X.$$

Then $\mu = \{\emptyset, \{d\}, \{b,c\}, \{a,c,d\}, \{b,c,d\}, X\}$. We show that γ is not subadditive. If $A = \{a\}$ and $B = \{b\}$, then $\gamma(A \cup B) = \{a,b,c\}$ and $\gamma(A) \cup \gamma(B) = \{a,b\}$. Therefore, γ is not subadditive. If $A = \{b\}$, then $\gamma^{\bigstar}\gamma(A) = \gamma^{\bigstar}(\{b\}) = \emptyset$ and so A is rc-nwdense. If $V = \{b,c\}$, then $V \cap A \neq \emptyset$ and so A is not wrc-nwdense.

A nonempty collection $\mathcal{I}$ of subsets of X is said to be an *ideal* [6] if it satisfies the following: (i) If $A \in \mathcal{I}$ and $B \subset A$, then $B \in \mathcal{I}$ and (ii) $A \cup B \in \mathcal{I}$ whenever $A \in \mathcal{I}$ and $B \in \mathcal{I}$. In the rest of this section, we discuss some properties of rc-nwdense sets and analyze under what additional conditions on γ, $\mathcal{N}$ is an ideal on X.

THEOREM 2.21. *Let (X,γ) be a monotonic space and $A \subset X$. If A is the union of a γ-regularopen set and an rc-nwdense set, then $A \cap \gamma(X - A)$ is an rc-nwdense set.*

PROOF. Let $A = G \cup N$ where G is γ-regularopen and N is rc-nwdense. If $M = N - G$, then $\gamma^{\bigstar}\gamma(M) = \gamma^{\bigstar}\gamma(N - G) \subset \gamma^{\bigstar}(\gamma(N) \cap \gamma(X - G)) \subset \subset \gamma^{\bigstar}(\gamma(N)) \cap \gamma^{\bigstar}(\gamma(X - G)) = \emptyset \cap \gamma^{\bigstar}(\gamma(X - G)) = \emptyset$ and so M is rc-nwdense.

Again, $M \cup G = (N - G) \cup G = N \cup G = A$. Since G is γ-regularopen such that $G \subset A$, we have $G \subset \gamma^{\bigstar}(A)$. Now

$A \cap \gamma(X - A) = (G \cup N) \cap \gamma((X - G) \cap (X - N)) \subset (G \cup N) \cap \gamma(X - G) =$

$= (G \cup N) \cap (X - \gamma^{\bigstar}(G)) = (G \cup N) \cap (X - G) = N \cap (X - G) = N - G = M.$

Since $A \cap \gamma(X - A)$ is a subset of an rc-nwdense set M, $A \cap \gamma(X - A)$ is rc-nwdense. ∎

The following Theorem 2.22 shows that the converse of Theorem 2.21 is true if the space (X,γ) is an envelope space. Example 2.23 below shows that the condition envelope on the space cannot be dropped in Theorem 2.22. Theorem 2.24 gives a characterization of rc-nwdense sets in an envelope space.

THEOREM 2.22. *Let (X,γ) be an envelope space and $A \subset X$. If $A \cap \gamma(X - A)$ is an rc-nwdense set, then A is the union of a γ-regularopen set and an rc-nwdense set.*

PROOF. If $A \cap \gamma(X - A) = \emptyset$, then $A \subset X - \gamma(X - A) = \gamma^{\bigstar}(A)$ which implies that $A = \gamma^{\bigstar}(A)$, since by Proposition 1.7(d) of [1] $\gamma \in \Gamma_+$ if and only if $\gamma^{\bigstar} \in \Gamma_-$. Now $\gamma^{\bigstar}(A) \cup (A \cap \gamma(X - A)) = A \cup (A \cap (X - \gamma^{\bigstar}(A))) = A$. Suppose $A \cap \gamma(X - A) \neq \emptyset$. Then

$$\gamma^{\bigstar}(A) \cup (A \cap \gamma(X - A)) = (\gamma^{\bigstar}(A) \cup A) \cap (\gamma^{\bigstar}(A) \cup (X - \gamma^{\bigstar}(A))) =$$
$$= \gamma^{\bigstar}(A) \cup A.$$

Since $\gamma \in \Gamma_2, \gamma^{\bigstar}(A) \in \mu$ and since $\gamma \in \Gamma_+, \gamma^{\bigstar}(A) \subset A$. Therefore, $\gamma^{\bigstar}(A) \cup (A \cap \gamma(X - A)) = A$. This completes the proof. ∎

EXAMPLE 2.23. (a) Let $X = \mathbf{R}$ be the set of all real numbers and $\gamma : \wp(X) \to \wp(X)$ be defined by $\gamma(A) = A$ if $0 \in A$ and $\emptyset$ otherwise. Then (X,γ) is a monotonic space, $\gamma \in \Gamma_2, \mathcal{N} = \{\emptyset\} \cup \{A \mid 0 \notin A\}$ and $\mu = \{\emptyset, \mathbf{R}\} \cup \{A \mid 0 \notin A\}$. If B is a nonempty subset of $\mathbf{R}$ such that $0 \notin B$, then $\gamma(B) = \emptyset$ and so $\gamma \notin \Gamma_+$. If $A = [0,1)$, then $A \cap \gamma(X - A) = \emptyset$ and so $A \cap \gamma(X - A)$ is rc-nwdense but A is not the union of an rc-nwdense set and a γ-regularopen set.

(b) Let $X = \{a,b,c\}$ and $\gamma : \wp(X) \to \wp(X)$ be defined by $\gamma(\emptyset) = \emptyset$, $\gamma(\{a\}) = \{a\}$, $\gamma(\{b\}) = \{a,b\}$, $\gamma(\{c\}) = X$, $\gamma(\{a,b\}) = \gamma(\{a,c\}) = \gamma(\{b,c\}) = \gamma(X) = X$. Then $\gamma \in \Gamma_+$, $\mu = \{\emptyset, \{b,c\}, X\}$ and $\mathcal{N} = \{\emptyset, \{a\}, \{b\}\}$. Since $\gamma(\gamma(\{b\})) \neq \gamma(\{b\})$, $\gamma \notin \Gamma_2$. If $A = \{a,c\}$, then $A \cap \gamma(X - A) = \{a\}$, which is rc-nwdense but A cannot be written as the union of an rc-nwdense set and a γ-regularopen set.

THEOREM 2.24. *Let (X,γ) be an envelope space and $A \subset X$. Then A is rc-nwdense if and only if $A \subset \gamma(X - \gamma(A))$.*

PROOF. If A is *rc-nwdense*, then $\gamma^\bigstar\gamma(A) = \emptyset$. Now,

$$\gamma(X - \gamma(A)) = X - \gamma^\bigstar\gamma(A) = X \supset A.$$

Conversely, if $A \subset \gamma(X - \gamma(A))$, then $A \subset X - \gamma^\bigstar\gamma(A)$ and so

$$\gamma^\bigstar\gamma(A) \subset \gamma^\bigstar\gamma(X - \gamma^\bigstar\gamma(A)) = \gamma^\bigstar(X - \gamma^\bigstar\gamma^\bigstar\gamma(A)) =$$
$$= \gamma^\bigstar(X - \gamma^\bigstar\gamma(A)) \subset X - \gamma^\bigstar\gamma(A). \text{ (cf. [1], 1.7).}$$

Therefore, $\gamma^\bigstar\gamma(A) = \emptyset$ which implies that A is rc-nwdense. ∎

THEOREM 2.25. *Let (X,γ) be an envelope space and γ be subadditive. Then the union of two rc-nwdense sets is again an rc-nwdense set and so, if $\mathcal{N}$ is nonempty, then $\mathcal{N}$ is an ideal.*

PROOF. Let A and B be two rc-nwdense subsets of X. Then $\gamma^\bigstar\gamma(A) = \emptyset$ and $\gamma^\bigstar\gamma(B) = \emptyset$ and so $X - \gamma^\bigstar\gamma(A) = X$ and $X - \gamma^\bigstar\gamma(B) = X$. This implies that $\gamma(X - \gamma(A)) = X$ and $\gamma(X - \gamma(B)) = X$ and so $X - \gamma(A)$ and $X - \gamma(B)$ are rc-dense sets. Let $\emptyset \neq G \in \mu$. Since $X - \gamma(A)$ is rc-dense, by Theorem 2.1, $G \cap (X - \gamma(A)) \neq \emptyset$. Since $\gamma \in \Gamma_2$, $X - \gamma(A) \in \mu$ by Lemma 2.15(a). By Lemma 2.15(b), $G \cap (X - \gamma(A)) \in \mu$. Since $X - \gamma(B)$ is rc-dense, $(G \cap (X - \gamma(A))) \cap (X - \gamma(B)) \neq \emptyset$ which implies that $G \cap (X - (\gamma(A) \cup \gamma(B))) \neq \emptyset$ and so $G \cap (X - (\gamma(A \cup B))) \neq \emptyset$, since γ is additive. By Corollary 2.7, $X - (\gamma(A \cup B))$ is rc-dense and so $\gamma(X - (\gamma(A \cup B))) = X$ which implies that $X - \gamma(X - (\gamma(A \cup B))) = \emptyset$. Therefore, $\gamma^\bigstar\gamma(A \cup B) = \emptyset$ and so $A \cup B$ is rc-nwdense. This completes the proof. ∎

Example 2.23(a) above shows that if the finite union of rc-nwdense subsets of the space (X,γ) is rc-nwdense, then γ need not be additive. The following Example 2.26 shows that the condition subadditive on γ in Theorem 2.25 cannot be dropped.

EXAMPLE 2.26. Consider the monotonic space (X,γ) of Example 2.19. Then $\gamma \in \Gamma_+$ and $\gamma \notin \Gamma_2$. If $A = \{a\}$ and $B = \{b\}$, then $\gamma(A) \cup \gamma(B) = \{a,b\}$. But $\gamma(A \cup B) = \gamma(\{a,b\}) = X \not\subseteq \{a,b\} = \gamma(A) \cup \gamma(B)$. Therefore, γ is not subadditive. Here $\xi = \{X\}$ and $\mu = \{\emptyset\}$. Also, $\mathcal{N} = \{\emptyset, \{a\}, \{b\}\}$. If $C = \{a\}$

and $D = \{b\}$, then $C \cup D = \{a, b\}$ and $\gamma^{\star}\gamma(C \cup D) = \gamma^{\star}(X) = X - \gamma(\emptyset) = = X - \{a\} = \{b, c\} \neq \emptyset$ and so $C \cup D$ is not an rc-nwdense set.

THEOREM 2.27. *Let (X, γ) be an envelope space, $A \subset X$ and γ be subadditive. Then the following hold.*

(a) $\gamma(A) \cap V \subset \gamma(A \cap V)$ *for every γ-regularopen set V.*

(b) $\gamma(\gamma(A) \cap V) = \gamma(A \cap V)$ *for every γ-regularopen set V.*

(c) *If A is rc-dense, then $\gamma(V) = \gamma(A \cap V)$ for every γ-regularopen set V.*

PROOF. (a) Suppose $x \in \gamma(A) \cap V$. Then $x \in \gamma(A)$ and $x \in V$. If G is a γ-regularopen set containing x, by Lemma 2.15(b), $G \cap V$ is a γ-regularopen set containing x and so by Lemma 2.15(d), $(G \cap V) \cap A = G \cap (V \cap A) \neq \emptyset$. By Lemma 2.15(e), $x \in \gamma(V \cap A)$. Therefore, $\gamma(A) \cap V \subset \gamma(A \cap V)$.

(b) Since $\gamma \in \Gamma_+$, $A \cap V \subset \gamma(A) \cap V$ and so $\gamma(A \cap V) \subset \gamma(\gamma(A) \cap V) \subset \subset \gamma\gamma(A \cap V) = \gamma(A \cap V)$ and so $\gamma(\gamma(A) \cap V) = \gamma(A \cap V)$.

(c) By (b), if V is γ-regularopen, then

$$\gamma(A \cap V) = \gamma(\gamma(A) \cap V) = \gamma(X \cap V) = \gamma(V)$$

and so (c) follows. ∎

Let $\lambda \subset \wp(X)$. $\gamma \in \Gamma$ is said to be *λ-friendly* [5] if $\gamma(A) \cap V \subset \gamma(A \cap V)$ for every subset A of X and $V \in \lambda$. A generalized topology ψ is said to be a *quasi-topology* [5] if $M \cap M_1 \in \psi$ whenever $M \in \psi$ and $M_1 \in \psi$.

THEOREM 2.28. *Let (X, γ) be a monotonic space. Then the following hold.*

(a) *If γ is a weak envelope, then μ is a generalized topology. In addition, if γ is subadditive, then μ is a quasi-topology.*

(b) *If γ is an envelope, then γ is subadditive if and only if γ is μ-friendly.*

(c) *γ is subadditive if and only if $\gamma^{\star}(A \cap B) = \gamma^{\star}(A) \cap \gamma^{\star}(B)$ for every subsets A and B of X.*

(d) *If γ is an envelope, γ is subadditive, G is γ-regularopen and $A \subset X$, then $G \cap \gamma^{\star}(A) = \gamma^{\star}(G \cap A)$.*

(e) *If γ is an envelope, γ is subadditive, F is γ-regularclosed and $A \subset X$, then $\gamma^{\star}(A \cup F) \subset \gamma^{\star}(A) \cup F$.*

(f) *If γ is an envelope, γ is subadditive, F is γ-regularclosed and $A \subset X$, then $\gamma(A \cup F) = \gamma(A) \cup F$.*

PROOF. (a) By Lemmas 1.3 and 1.4 of [3], μ is a generalized topology. By Lemma 2.15(b), μ is a quasi-topology.

(b) If γ is subadditive, then by Theorem 2.27(a), γ is μ-friendly. Conversely, suppose γ is μ-friendly. For subsets A and B of X,

$$\gamma(A \cup B) - \gamma(B) = \gamma(A \cup B) \cap (X - \gamma(B)).$$

Since $X - \gamma(B) \in \mu$ by Lemma 2.15(a), by Theorem 2.27(a),

$$\gamma(A \cup B) - \gamma(B) \subset \gamma((A \cup B) \cap (X - \gamma(B))) = \gamma(A \cap (X - \gamma(B))) \subset \gamma(A)$$

and so $\gamma(A \cup B) \subset \gamma(A) \cup \gamma(B)$.

(c) The proof follows from the definition of $\gamma^\bigstar$.

(d) Let G be γ-regularopen and A be any subset of X. Then $G \cap \gamma^\bigstar(A)$ is a γ-regularopen set by Lemma 2.15(b), such that $G \cap \gamma^\bigstar(A) \subset G \cap A$, since $\gamma^\bigstar \in \Gamma_-$. Therefore, $G \cap \gamma^\bigstar(A) \subset \gamma^\bigstar(G \cap A) = \gamma^\bigstar(G) \cap \gamma^\bigstar(A) = G \cap \gamma^\bigstar(A)$, by (c). Therefore, $G \cap \gamma^\bigstar(A) = \gamma^\bigstar(G \cap A)$.

(e) Now

$$X - \gamma^\bigstar(A \cup F) = \gamma(X - (A \cup F)) = \gamma((X - A) \cap (X - F)) \supset \gamma(X - A) \cap (X - F),$$

by Theorem 2.27(a). Therefore,

$$X - \gamma^\bigstar(A \cup F) \supset (X - \gamma^\bigstar(A)) \cap (X - F) = X - (\gamma^\bigstar(A) \cup F)$$

and so $\gamma^\bigstar(A \cup F) \subset \gamma^\bigstar(A) \cup F$.

(f) Now

$$X - \gamma(A \cup F) = \gamma^\bigstar(X - (A \cup F)) = \gamma^\bigstar((X - A) \cap (X - F)) =$$
$$= \gamma^\bigstar(X - A) \cap (X - F) = (X - \gamma(A)) \cap (X - F) = X - (\gamma(A) \cup F)$$

and so $\gamma(A \cup F) = \gamma(A) \cup F$. ∎

The following Example 2.29 shows that in a weak envelope space (X, γ), if γ is μ-friendly, then γ need not be subadditive. Also, this example shows that the reverse direction of Lemma 2.15(b) is not true.

EXAMPLE 2.29. Let (X, γ) be the space of Example 2.19. Then $\mu = \{\emptyset\}$. Since $\gamma(A) \cap V \subset \gamma(A \cap V)$ for every γ-regularopen set V and $A \subset X$, γ is μ-friendly. If $A = \{a\}$ and $B = \{b\}$, then $\gamma(\{a\}) = \{a, b\}$ and $\gamma(\{b\}) = = \{a, b\}$ so that $\gamma(A) \cup \gamma(B) = \{a, b\}$. But $\gamma(A \cup B) = X$ and so γ is not subadditive.

3. $r\delta$-sets and $r\delta^\star$-sets

A subset A of a monotonic space (X,γ) is said to be an *$r\delta$-set* if $\gamma^\star\gamma(A) \subset \gamma\gamma^\star(A)$. Clearly, every rc-nwdense set is an $r\delta$-set. The following Theorem 3.1 shows that γ-regularopen sets are $r\delta$-sets, if $\gamma \in \Gamma_+$.

THEOREM 3.1. *Let (X,γ) be a monotonic space. If $\gamma \in \Gamma_+$, then every γ-regularopen set is an $r\delta$-set. In particular, $\emptyset$ is an $r\delta$-set.*

PROOF. Suppose A is a γ-regularopen set. Then $\gamma^\star(A) = A$. Now, since $\gamma \in \Gamma_+$, $\gamma^\star\gamma(A) \subset \gamma(A) = \gamma\gamma^\star(A)$. Therefore, A is an $r\delta$-set. ∎

THEOREM 3.2. *Let (X,γ) be a monotonic space and $\gamma \in \Gamma_2$. If $A \subset B \subset \subset \gamma(A)$ and A is an $r\delta$-set, then B, $\gamma(A)$ and $\gamma(B)$ are $r\delta$-sets.*

PROOF. Since A is an $r\delta$-set, $\gamma^\star\gamma(A) \subset \gamma\gamma^\star(A)$. Again, $A \subset B$ implies that $\gamma^\star(A) \subset \gamma^\star(B)$ and so $\gamma\gamma^\star(A) \subset \gamma\gamma^\star(B)$. Since $\gamma \in \Gamma_2$, $A \subset B \subset \subset \gamma(A)$ implies that $\gamma(A) = \gamma(B)$. Now $\gamma^\star\gamma(B) = \gamma^\star\gamma(A) \subset \gamma\gamma^\star(A) \subset \subset \gamma\gamma^\star(B)$ and so B is an $r\delta$-set. Clearly, $\gamma(A)$ and $\gamma(B)$ are $r\delta$-sets. ∎

COROLLARY 3.3. *Let (X,γ) be a monotonic space and $\gamma \in \Gamma_2$. If A is an $r\delta$-set which is also an rc-dense set, then every superset of A is an $r\delta$-set.*

The following Example 3.4 shows that the condition $\gamma \in \Gamma_2$ in Theorem 3.2 cannot be dropped.

EXAMPLE 3.4. Consider the monotonic space (X,γ) of Example 2.2. Then $\gamma \notin \Gamma_2$. If $A = \{a\}$, then $\gamma^\star\gamma(A) = \{a\}$ and $\gamma\gamma^\star(A) = \{a,b\}$. Therefore, A is an $r\delta$-set. If $B = \{a,b\}$, then $A \subset B \subset \gamma(A) = \{a,b\}$, $\gamma^\star\gamma(B) = \{a,c\}$ and $\gamma\gamma^\star(B) = \{a,b\}$. Therefore, B is not an $r\delta$-set.

THEOREM 3.5. *Let (X,γ) be a monotonic space and $A \subset X$. If A is an $r\delta$-set, then $X - A$ is also an $r\delta$-set.*

PROOF. If A is an $r\delta$-set, then $\gamma^\star\gamma(A) \subset \gamma\gamma^\star(A)$ which implies that $X - \gamma\gamma^\star(A) \subset X - \gamma^\star\gamma(A)$ and so $\gamma^\star(X - \gamma^\star(A)) \subset \gamma(X - \gamma(A))$. Therefore, $\gamma^\star\gamma(X - A) \subset \gamma\gamma^\star(X - A)$ and so $X - A$ is an $r\delta$-set. ∎

THEOREM 3.6. *Let (X,γ) be a monotonic space. If there exists a singleton set which is both γ-regularopen and rc-dense in X, then every singleton set is an $r\delta$-set.*

PROOF. Suppose $\{x\}$ is both γ-regularopen and rc-dense. Then $\gamma^{\bigstar}(\{x\}) = \{x\}$ and $\gamma(\{x\}) = X$. Let $y \in X$ be arbitrary. If $y = x$, then $\gamma\gamma^{\bigstar}(\{y\}) = \gamma\gamma^{\bigstar}(\{x\}) = \gamma(\{x\}) = X$ and $\gamma^{\bigstar}\gamma(\{y\}) = \gamma^{\bigstar}\gamma(\{x\}) = \gamma^{\bigstar}(X)$. Therefore, $\gamma^{\bigstar}\gamma(\{y\}) \subset \gamma\gamma^{\bigstar}(\{y\})$ and so $\{y\}$ is an $r\delta$-set. If $y \neq x$, then $y \in X - \{x\}$ and so $\gamma^{\bigstar}\gamma(\{y\}) \subset \gamma^{\bigstar}\gamma(X - \{x\}) = \gamma^{\bigstar}(X - \gamma^{\bigstar}(\{x\})) = \gamma^{\bigstar}(X - \{x\}) = X - \gamma(\{x\}) = \emptyset$. Therefore, $\gamma^{\bigstar}\gamma(\{y\}) \subset \gamma\gamma^{\bigstar}(\{y\})$ and so $\{y\}$ is an $r\delta$-set in this case also. ∎

THEOREM 3.7. *Let (X,γ) be a monotonic space and $\mu = r\delta(X)$ where $r\delta(X)$ is the family of all $r\delta$-sets in (X,γ). Then the following hold.*

(a) *If $A \in \mu$, then $A \in \xi$.*

(b) *If $\gamma \in \Gamma_+$, then $\gamma^{\bigstar}\gamma(A) \neq \emptyset$ for every nonempty subset A of X.*

PROOF. (a) Suppose $A \in \mu$ such that $A \notin \xi$. $A \notin \xi$ implies that $X - A \notin \mu$. By hypothesis, $X - A \notin r\delta(X)$ which implies that $A \notin r\delta(X)$, by Theorem 3.5, a contradiction to the hypothesis. This completes the proof.

(b) Suppose $\gamma^{\bigstar}\gamma(A) = \emptyset$ for some nonempty subset A of X. Then A is an $r\delta$-set and so by hypothesis, $A \in \mu$ which implies that $\gamma^{\bigstar}(A) = A$. Since $\gamma \in \Gamma_+$, $A \subset \gamma(A)$ which implies that $\gamma^{\bigstar}(A) \subset \gamma^{\bigstar}\gamma(A) = \emptyset$ and so $A = \emptyset$, a contradiction to the hypothesis. Therefore, $\gamma^{\bigstar}\gamma(A) \neq \emptyset$ for every nonempty subset A of X. ∎

The following Theorem 3.8 gives a characterization of $r\delta$-sets in envelope spaces. Theorem 3.9 below gives a necessary condition for a set to be an $r\delta$-set.

THEOREM 3.8. *Let (X,γ) be an envelope space. Then a subset A of X is an $r\delta$-set if and only if $\gamma^{\bigstar}\gamma(A) = \gamma^{\bigstar}\gamma\gamma^{\bigstar}(A)$.*

PROOF. Suppose A is an $r\delta$-set. Then, $\gamma^{\bigstar}\gamma(A) \subset \gamma\gamma^{\bigstar}(A)$. Since $\gamma \in \Gamma_2$, we have $\gamma^{\bigstar}\gamma(A) \subset \gamma^{\bigstar}\gamma\gamma^{\bigstar}(A)$. Since $\gamma \in \Gamma_+$, we have $\gamma^{\bigstar}(A) \subset A$ which implies that $\gamma^{\bigstar}\gamma\gamma^{\bigstar}(A) \subset \gamma^{\bigstar}\gamma(A)$. Therefore, $\gamma^{\bigstar}\gamma(A) = \gamma^{\bigstar}\gamma\gamma^{\bigstar}(A)$. Conversely, suppose $\gamma^{\bigstar}\gamma(A) = \gamma^{\bigstar}\gamma\gamma^{\bigstar}(A)$. Since $\gamma \in \Gamma_+$, $\gamma^{\bigstar}\gamma(A) \subset \gamma\gamma^{\bigstar}(A)$ and so A is an $r\delta$-set. ∎

THEOREM 3.9. *Let (X,γ) be an envelope space and $A \subset X$ be an $r\delta$-set. Then the following hold.*

(a) *$A = B \cup C$ where B is γ-regularopen, C is rc-nwdense and $B \cap C = \emptyset$.*

(b) *If γ is subadditive, then $\gamma^\bigstar\gamma(A \cap B) = \gamma^\bigstar\gamma(A) \cap \gamma^\bigstar\gamma(B)$ for every subset B of X.*

PROOF. (a) Suppose that A is an $r\delta$-set. Then $\gamma^\bigstar\gamma(A) \subset \gamma\gamma^\bigstar(A)$. If $B = \gamma^\bigstar(A)$, then $\gamma^\bigstar(B) = \gamma^\bigstar\gamma^\bigstar(A) = \gamma^\bigstar(A) = B$ and so B is γ-regularopen. If $C = A - \gamma^\bigstar(A)$, then $B \cup C = \gamma^\bigstar(A) \cup (A - \gamma^\bigstar(A)) =$ $= A \cup \gamma^\bigstar(A) = A$. Now $C \subset A$ implies that $\gamma(C) \subset \gamma(A)$ and so $\gamma^\bigstar\gamma(C) \subset$ $\subset \gamma^\bigstar\gamma(A) \subset \gamma\gamma^\bigstar(A)$. Clearly, $B \cap C = \emptyset$. By Lemma 2.15(c), $B \cap \gamma(C) = \emptyset$ which implies that $B \cap \gamma^\bigstar\gamma(C) = \emptyset$. Since $\gamma^\bigstar\gamma(C)$ is γ-regularopen, by Lemma 2.15(c), $\gamma(B) \cap \gamma^\bigstar\gamma(C) = \emptyset$ and so $\gamma\gamma^\bigstar(A) \cap \gamma^\bigstar\gamma(C) = \emptyset$ which implies that $\gamma^\bigstar\gamma(C) = \emptyset$. Therefore, C is rc-nwdense.

(b) Suppose that A is an $r\delta$-set. Clearly, $\gamma^\bigstar\gamma(A \cap B) \subset \gamma^\bigstar\gamma(A) \cap \gamma^\bigstar\gamma(B)$. Since $\gamma \in \Gamma_2$ and γ is subadditive,

$$\begin{gathered}\gamma^\bigstar\gamma(A) \cap \gamma^\bigstar\gamma(B) = \\ = \gamma^\bigstar(\gamma^\bigstar\gamma(A) \cap \gamma^\bigstar\gamma(B)) \subset \\ \subset \gamma^\bigstar\gamma(\gamma^\bigstar\gamma(A) \cap \gamma^\bigstar\gamma(B)) \subset \\ \subset \gamma^\bigstar\gamma(\gamma\gamma^\bigstar(A) \cap \gamma^\bigstar\gamma(B)) = \\ = \gamma^\bigstar\gamma(\gamma^\bigstar(A) \cap \gamma^\bigstar\gamma(B)),\end{gathered}$$

by Theorem 2.27(b). Therefore, $\gamma^\bigstar\gamma(A) \cap \gamma^\bigstar\gamma(B) \subset \gamma^\bigstar\gamma(\gamma^\bigstar(A) \cap \gamma(B)) \subset$ $\subset \gamma^\bigstar\gamma\gamma(\gamma^\bigstar(A) \cap B) = \gamma^\bigstar\gamma(\gamma^\bigstar(A) \cap B) \subset \gamma^\bigstar\gamma(A \cap B)$. Therefore, $\gamma^\bigstar\gamma(A \cap B) = \gamma^\bigstar\gamma(A) \cap \gamma^\bigstar\gamma(B)$. ∎

The following Theorem 3.10 shows that in an envelope space (X, γ), if γ is subadditive, then the finite intersection of $r\delta$-sets is an $r\delta$-set and the finite union of $r\delta$-sets is again an $r\delta$-set.

THEOREM 3.10. *Let (X, γ) be an envelope space and γ be subadditive. If A and B are $r\delta$-sets of X, then the following hold.*

(a) *$A \cap B$ is an $r\delta$-set.*

(b) *$A \cup B$ is an $r\delta$-set.*

PROOF. (a) Suppose A and B are $r\delta$-sets. Now

$$\gamma^\bigstar\gamma(A \cap B) \subset \gamma^\bigstar\gamma(A) \cap \gamma^\bigstar\gamma(B) \subset \gamma\gamma^\bigstar(A) \cap \gamma^\bigstar\gamma(B) \subset \gamma(\gamma^\bigstar(A) \cap \gamma^\bigstar\gamma(B)),$$

by Theorem 2.27(a). Since B is an $r\delta$-set,

$$\gamma^\bigstar\gamma(A\cap B)\subset\gamma(\gamma^\bigstar(A)\cap\gamma\gamma^\bigstar(B))\subset\gamma\gamma(\gamma^\bigstar(A)\cap\gamma^\bigstar(B))=$$
$$=\gamma(\gamma^\bigstar(A)\cap\gamma^\bigstar(B))=\gamma\gamma^\bigstar(A\cap B),$$

by Theorem 2.28(c). Hence $A\cap B$ is an $r\delta$-set.

(b) Now $A\subset A\cup B$ implies that $\gamma^\bigstar(A)\subset\gamma^\bigstar(A\cup B)$ which in turn implies that $\gamma\gamma^\bigstar(A)\subset\gamma\gamma^\bigstar(A\cup B)$. Similarly, $\gamma\gamma^\bigstar(B)\subset\gamma\gamma^\bigstar(A\cup B)$ and so $\gamma\gamma^\bigstar(A)\cup\gamma\gamma^\bigstar(B)\subset\gamma\gamma^\bigstar(A\cup B)$. Since γ is subadditive and hence additive, $\gamma^\bigstar\gamma(A\cup B)=\gamma^\bigstar(\gamma(A)\cup\gamma(B))=\gamma^\bigstar\gamma^\bigstar(\gamma(A)\cup\gamma(B))\subset\gamma^\bigstar(\gamma(A)\cup\gamma^\bigstar\gamma(B))$, by Lemma 2.15(a) and Theorem 2.28(e). Since B is an $r\delta$-set, $\gamma^\bigstar\gamma(A\cup B)\subset$ $\subset\gamma^\bigstar(\gamma(A)\cup\gamma\gamma^\bigstar(B))\subset\gamma^\bigstar\gamma(A)\cup\gamma\gamma^\bigstar(B)\subset\gamma\gamma^\bigstar(A)\cup\gamma\gamma^\bigstar(B)$, since A is an $r\delta$-set. Therefore, $\gamma^\bigstar\gamma(A\cup B)\subset\gamma\gamma^\bigstar(A\cup B)$ and so $A\cup B$ is an $r\delta$-set. ∎

The following Example 3.11 shows that the condition subadditive on γ cannot be dropped in the above Theorem 3.10. Also, it shows that subsets of an $r\delta$-set need not be an $r\delta$-set.

EXAMPLE 3.11. Consider the monotonic space of Example 2.19. If $A=\{a\}$ and $B=\{b\}$, then

$$\gamma^\bigstar\gamma(A)=\gamma^\bigstar(\{a,b\})=X-\gamma(\{c\})=X-X=\emptyset\subset\gamma\gamma^\bigstar(A).$$

Also, $\gamma^\bigstar\gamma(B)=\gamma^\bigstar(\{a,b\})=\emptyset\subset\gamma\gamma^\bigstar(B)$. Therefore, A and B are $r\delta$-sets. But $A\cup B=\{a,b\}$ is not an $r\delta$-set. For,

$$\gamma^\bigstar\gamma(A\cup B)=\gamma^\bigstar(X)=X-\gamma(\emptyset)=X-\{a\}=\{b,c\}$$

and

$$\gamma\gamma^\bigstar(A\cup B)=\gamma(X-\gamma(\{c\}))=\gamma(X-X)=\gamma(\emptyset)=\{a\}\not\supset\gamma^\bigstar\gamma(A\cup B)$$

and so $A\cup B$ is not an $r\delta$-set.

If $C=\{a,c\}$ and $D=\{b,c\}$, then

$$\gamma^\bigstar\gamma(C)=\gamma^\bigstar(X)=X-\gamma(\emptyset)=X-\{a\}=\{b,c\}$$

and

$$\gamma\gamma^\bigstar(C)=\gamma(X-\gamma(\{b\}))=\gamma(X-\{a,b\})=\gamma(\{c\})=X\supset\gamma^\bigstar\gamma(C).$$

Also,

$$\gamma^\bigstar\gamma(D)=\gamma^\bigstar(X)=X-\gamma(\emptyset)=X-\{a\}=\{b,c\}$$

and

$$\gamma\gamma^{\bigstar}(D) = \gamma(X - \gamma(\{a\})) = \gamma(X - \{a,b\}) = \gamma(\{c\}) = X \supset \gamma^{\bigstar}\gamma(D)$$

and so C and D are $r\delta$-sets. But $C \cap D = \{c\}$ is not an $r\delta$-set. For,

$$\gamma^{\bigstar}\gamma(C \cap D) = \gamma^{\bigstar}(X) = X - \gamma(\emptyset) = X - \{a\} = \{b,c\}$$

and

$$\gamma\gamma^{\bigstar}(C \cap D) = \gamma(X - \gamma(\{a,b\})) = \gamma(X - X) = \gamma(\emptyset) = \{a\} \not\supset \gamma^{\bigstar}\gamma(C \cap D)$$

and so $C \cap D$ is not an $r\delta$-set.

COROLLARY 3.12. *Let (X,γ) be an envelope space and γ be subadditive. If $A = B \cup C$ where B is γ-regularopen and C is rc-nwdense, then A is an $r\delta - set$.*

PROOF. The proof follows from Theorem 3.1 and Theorem 3.10(b). ∎

A subset A of a monotonic space (X,γ) is said to be an $r\delta^{\bigstar}$*-set* if A is an $r\delta$*-set* and every subset of A is also an $r\delta$*-set*. Clearly, every rc-nwdense set is an $r\delta^{\bigstar}$*-set*. In fact, the following Theorem 3.13, the proof of which follows from Theorem 3.10(b), shows that the family of all $r\delta^{\bigstar}$*-sets* is an ideal in an envelope space (X,γ), if γ is subadditive.

THEOREM 3.13. *Let (X,γ) be an envelope space and γ be subadditive. Then the family of all $r\delta^{\bigstar}$-sets of X is an ideal.*

A subset A of a monotonic space (X,γ) is said to be *locally rc-closed* if $A = G \cap F$ where G is a γ-regularopen set and F is a γ-regularclosed set. If γ is a weak envelope, then X is γ-regularclosed. Therefore, every γ-regularopen set is a locally rc-closed set. Clearly, A is locally rc-closed if and only if $X - A$ is the union of a γ-regularopen set and a γ-regularclosed set. The following Theorem 3.14, gives characterizations of locally rc-closed sets.

THEOREM 3.14. *Let (X,γ) be an envelope space, γ be subadditive and $A \subset X$. Then the following are equivalent.*

(a) *A is locally rc-closed.*

(b) *$A = G \cap \gamma(A)$ for some γ-regularopen set G.*

(c) *$\gamma(A) - A$ is γ-regularclosed.*

(d) *$A \cup (X - \gamma(A))$ is γ-regularopen.*

PROOF. (a)⇒(b). Suppose $A = G \cap F$ where G is γ-regularopen and F is γ-regularclosed. $A = G \cap F$ implies that $A \subset F$ and so $\gamma(A) \subset \gamma(F) = F$.

By Theorem 2.27(a), $\gamma(A) = \gamma(G \cap F) \supset G \cap \gamma(F) = G \cap F = A$ and so $A \subset \gamma(A)$. Now, $A \subset \gamma(A)$ implies that

$$A = A \cap \gamma(A) = (G \cap F) \cap \gamma(A) = G \cap (F \cap \gamma(A)) = G \cap \gamma(A).$$

(b)⇒(c). Suppose that $A = G \cap \gamma(A)$ for some γ-regularopen set G. Now, $\gamma(A) - A = \gamma(A) \cap (X - A) = \gamma(A) \cap (X - (G \cap \gamma(A))) = \gamma(A) \cap ((X - G) \cup \cup (X - \gamma(A))) = \gamma(A) \cap (X - G)$. By Lemma 2.15(a) and Theorem 2.28(a), $\gamma(A) \cap (X - G)$ is γ-regularclosed. Hence $\gamma(A) - A$ is γ-regularclosed.

(c)⇒(d). $\gamma(A) - A$ is γ-regularclosed implies that $X - (\gamma(A) - A)$ is γ-regularopen which in turn implies that $X - (\gamma(A) \cap (X - A))$ is γ-regularopen and so $(X - \gamma(A)) \cup (X - (X - A))$ is γ-regularopen. Therefore, $(X - \gamma(A)) \cup A$ is γ-regularopen.

(d)⇒(a). $(A \cup (X - \gamma(A))) \cap \gamma(A) = (A \cap \gamma(A)) \cup ((X - \gamma(A)) \cap \gamma(A)) = = A \cap \gamma(A) = A$, since $A \subset \gamma(A)$. Hence A is locally rc-closed. ∎

The following Example 3.15 shows that γ-regularopen sets and locally rc-closed sets are independent concepts. Theorem 3.16 below shows that for rc-dense sets, γ-regularopen sets and locally rc-closed sets coincide in an envelope space (X,γ) where γ is subadditive. Theorem 3.17 below gives a property of locally rc-closed sets.

EXAMPLE 3.15. Let $X = \{a,b,c\}$ and $\gamma : \wp(X) \to \wp(X)$ be defined by

$$\gamma(\emptyset) = \gamma(\{a\}) = \gamma(\{b\}) = \gamma(\{c\}) = \gamma(\{a,c\}) = \gamma(\{b,c\}) = \{a\}$$

and

$$\gamma(\{a,b\}) = \gamma(X) = \{a,b\}.$$

Then $\gamma \in \Gamma$, $\xi = \{\{a\},\{a,b\}\}$ and $\mu = \{\{c\},\{b,c\}\}$. If $A = \{b\}$, then $A = \{a,b\} \cap \{b,c\}$ and so A is locally rc-closed but not γ-regularopen. If $B = \{b,c\}$, then B is γ-regularopen but cannot be written as the intersection of a γ-regularopen set and a γ-regularclosed set.

THEOREM 3.16. *Let (X,γ) be a weak envelope space and $A \subset X$. Then the following hold.*

(a) *Every γ-regularopen set is locally rc-closed.*

(b) *If $\gamma \in \Gamma_2$, γ is subadditive and A is an rc-dense subset of X which is also locally rc-closed, then A is γ-regularopen.*

PROOF. (a) The proof follows from the fact that if $\gamma \in \Gamma_+$, then X is γ-regularclosed.

(b) Suppose that A is both rc-dense and locally rc-closed. Then $A = G \cap \gamma(A)$ for some γ-regularopen set G (cf. 3.14(b)). Now, $A = G \cap \gamma(A)$ implies that $A = G \cap X = G$ and so A is a γ-regularopen set. ∎

THEOREM 3.17. *Let (X,γ) be a monotonic space and γ be subadditive. If G is γ-regularopen and A is locally rc-closed, then $A \cap G$ is also locally rc-closed.*

PROOF. A is locally rc-closed implies that $A = U \cap F$ where U is γ-regularopen and F is γ-regularclosed. $A \cap G = (U \cap F) \cap G = (U \cap G) \cap F$. Since γ is subadditive, $U \cap G$ is γ-regularopen, by Lemma 2.15(b). Hence $A \cap G$ is locally rc-closed. ∎

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A GENERAL DECOMPOSITION OF CONTINUITY

By

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Abstract. Quite recently, Sheik and Sundaram [22] have obtained the following theorem: a function $f:(X,\tau) \to (Y,\sigma)$ is continuous if and only if f is ω-continuous and slc^*-continuous. In this paper, by using the notion of mg^*-closed sets, we obtain the unified theory for the above decomposition of continuity in topological spaces.

1. Introduction

In 1970, Levine [10] introduced the notion of generalized closed (g-closed) sets in topological spaces. As modifications of g-closed sets, Murugalingam [13] introduced the notions of sg^*-closed (resp. αg^*-closed, pg^*-closed βg^*-closed) sets by using semi-open (resp. α-open, preopen, β-open) sets and studied their basic properties and characterizations. In [16], these notions are unified by the notion of mg^*-closed sets. The notion of sg^*-closed sets is also called ω-closed [23], semi-star-closed [19], or $\hat{g}$-closed [8]. Recently, by using the notion of ω-closed sets, Sheik and Sundaram [22] obtained the following theorem: a function $f:(X,\tau) \to (Y,\sigma)$ is continuous if and only if f is ω-continuous and slc^*-continuous.

The present authors [17], [18] introduced and investigated the notions of m-structures, m-spaces and m-continuity. In this paper, we introduce the notion of mlc-sets as a general form of locally closed sets. By using mg^*-closed sets and mlc-sets, we introduce the notions of mg^*-continuity and mlc-continuity, respectively, and obtain a general decomposition of continuity for the above theorem. Furthermore, we provide a sufficient condition for an mg^*-continuous function to be continuous. In the last section, we consider new forms of decomposition of continuity.

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2. Preliminaries

Let (X,τ) be a topological space and A a subset of X. The closure of A and the interior of A are denoted by $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$, respectively.

DEFINITION 2.1. A subset A of a topological space (X,τ) is said to be *semi-open* [9] (resp. *preopen* [12], *α-open* [14], *b-open* [3], *β-open* [1]) if $A \subset \mathrm{Cl}(\mathrm{Int}(A))$ (resp. $A \subset \mathrm{Int}(\mathrm{Cl}(A))$, $A \subset \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(A)))$, $A \subset \mathrm{Cl}(\mathrm{Int}(A)) \cup$ $\cup\, \mathrm{Int}(\mathrm{Cl}(A))$, $A \subset \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(A)))$).

For the subsets defined in Definition 2.1, the following relations are well-known:

DIAGRAM I

$$\begin{array}{ccccccc}
\text{open} & \Rightarrow & \alpha\text{-open} & \Rightarrow & \text{preopen} & & \\
 & & \Downarrow & & \Downarrow & & \\
 & & \text{semi-open} & \Rightarrow & b\text{-open} & \Rightarrow & \beta\text{-open}
\end{array}$$

The family of all semi-open (resp. preopen, α-open, b-open, β-open) sets in (X,τ) is denoted by $\mathrm{SO}(X)$ (resp. $\mathrm{PO}(X)$, $\alpha(X)$, $\mathrm{BO}(X)$, $\beta(X)$).

DEFINITION 2.2. Let (X,τ) be a topological space. A subset A of X is said to be *g-closed* [10] (resp. *sg^*-closed*, *pg^*-closed*, *αg^*-closed*, *bg^*-closed*, *βg^*-closed* [13]) if $\mathrm{Cl}(A) \subset U$ whenever $A \subset U$ and U is open (resp. semi-open, preopen, α-open, b-open, β-open) in (X,τ).

REMARK 2.1.

(1) An sg^*-closed set is also called ω-closed [23], semi-star-closed [19], or $\hat{g}$-closed [8].

(2) By the definitions, we obtain the following diagram:

DIAGRAM II

$$\begin{array}{ccccccc}
g\text{-closed} & \Leftarrow & \alpha g^*\text{-closed} & \Leftarrow & pg^*\text{-closed} & & \\
 & & \Uparrow & & \Uparrow & & \\
 & & sg^*\text{-closed} & \Leftarrow & bg^*\text{-closed} & \Leftarrow & \beta g^*\text{-closed}
\end{array}$$

Throughout the present paper, (X,τ) and (Y,σ) always denote topological spaces and $f:(X,\tau) \to (Y,\sigma)$ presents a function.

DEFINITION 2.3. A function $f:(X,\tau) \to (Y,\sigma)$ is said to be *g-continuous* [4] (resp. *ω-continuous* [21] or *$\hat{g}$-continuous* [8]) if $f^{-1}(F)$ is g-closed (resp. ω-closed) in (X,τ) for each closed set F of (Y,σ).

REMARK 2.2. It is known in [22] that the following implications hold and the converses are not necessarily true: continuity $\Rightarrow$ ω-continuity $\Rightarrow$ g-continuity.

DEFINITION 2.4. A subset A of a topological space (X,τ) is called an $slc*$-set [22] or an slc-set [5] if $A = U \cap F$, where $U \in \mathrm{SO}(X)$ and F is closed in (X,τ).

REMARK 2.3. It is known in [22] that every closed set is an $slc*$-set but not conversely and that an ω-closed set and an $slc*$-set are independent.

DEFINITION 2.5. A function $f:(X,\tau) \to (Y,\sigma)$ is said to be *$slc*$-continuous* [22] if $f^{-1}(F)$ is an $slc*$-set of (X,τ) for each closed set F of (Y,σ).

REMARK 2.4. It is known in [22] that every continuous function is $slc*$-continuous but not conversely and that ω-continuity and $slc*$-continuity are independent.

THEOREM 2.1. (Sheik and Sundaram [22]). *A function $f:(X,\tau) \to (Y,\sigma)$ is continuous if and only if f is ω-continuous and slc^*-continuous.*

3. m-Structures and mg^*-closed sets

DEFINITION 3.1. A subfamily m_X of the power set $\mathcal{P}(X)$ of a nonempty set X is called a *minimal structure* (briefly *m-structure*) [17], [18] on X if $\emptyset \in m_X$ and $X \in m_X$.

By (X, m_X), we denote a nonempty set X with a minimal structure m_X on X and call it an *m-space*. Each member of m_X is said to be *m_X-open* and the complement of an m_X-open set is said to be *m_X-closed.*

DEFINITION 3.2. A minimal structure m_X on a nonempty set X is said to have *property $\mathcal{B}$* [11] if the union of any family of subsets belonging to m_X belongs to m_X.

REMARK 3.1. Let (X,τ) be a topological space. Then the families $\mathrm{SO}(X)$, $\mathrm{PO}(X)$, $\alpha(X)$, $\mathrm{BO}(X)$ and $\beta(X)$ are all m-structures with property $\mathcal{B}$.

DEFINITION 3.3. Let (X,τ) be a topological space and m_X an m-structure on X. A subset A is said to be *mg^*-closed* [16] if $\mathrm{Cl}(A) \subset U$ whenever $A \subset U$ and $U \in m_X$.

REMARK 3.2. Let (X,τ) be a topological space and A a subset of X. If $m_X = \tau$ (resp. SO(X), PO(X), $\alpha(X)$, BO(X), $\beta(X)$) and A is mg^*-closed, then A is g-closed (resp. sg^*-closed, pg^*-closed, αg^*-closed, bg^*-closed, βg^*-closed).

LEMMA 3.1. (Noiri and Popa [16]). *Let (X,τ) be a topological space and m_X an m-structure on X such that $\tau \subset m_X$. Then the following implications hold:*

$$\textit{closed} \Rightarrow mg^*\textit{-closed} \Rightarrow g\textit{-closed}$$

LEMMA 3.2. (Noiri and Popa [16]). *If A is mg^*-closed and m_X-open, then A is closed.*

DEFINITION 3.4. Let (X,τ) be a topological space and m_X an m-structure on X. A subset A is called an *mlc-set* if $A = U \cap F$, where $U \in m_X$ and F is closed in (X,τ).

REMARK 3.3. Let (X,τ) be a topological space and A a subset of X.

(1) if $m_X = \tau$ (resp. SO(X), PO(X), $\alpha(X)$, BO(X), $\beta(X)$) and A is an mlc-set, then A is called a locally closed set [6] (briefly lc-set) (resp. an slc^*-set [22] or an slc-set [5], a plc-set [5], an αlc-set [2], a blc-set, a βlc-set [5]),

(2) every closed set is an mlc-set but not conversely by Remark 2.3,

(3) an mg^*-closed set and an mlc-set are independent by Remark 2.3,

(4) by the definitions, we obtain the following diagram:

DIAGRAM III

$$\begin{array}{ccccccccc}
lc\text{-sets} & \Rightarrow & \alpha lc\text{-sets} & \Rightarrow & plc\text{-sets} & & \\
& & \Downarrow & & \Downarrow & & \\
& & slc\text{-sets} & \Rightarrow & blc\text{-sets} & \Rightarrow & \beta lc\text{-sets}
\end{array}$$

4. Decompositions of continuity

THEOREM 4.1. *Let (X, τ) be a topological space and m_X a minimal structure on X such that $\tau \subset m_X$. Then a subset A of X is closed if and only if it is mg^*-closed and an mlc-set.*

PROOF. *Necessity.* Suppose that A is closed in (X, τ). Then, by Lemma 3.1, A is mg^*-closed. Furthermore, $A = X \cap A$, where $X \in m_X$ and A is closed and hence A is an mlc-set.

Sufficiency. Suppose that A is mg^*-closed and an mlc-set. Since A is an mlc-set, $A = U \cap F$, where $U \in m_X$ and F is closed in (X, τ). Therefore, we have $A \subset U$ and $A \subset F$. By the hypothesis, we obtain $\mathrm{Cl}(A) \subset U$ and $\mathrm{Cl}(A) \subset F$ and hence $\mathrm{Cl}(A) \subset U \cap F = A$. Thus, $\mathrm{Cl}(A) = A$ and A is closed.

COROLLARY 4.1. *Let (X, τ) be a topological space. Then, for a subset A of X, the following properties are equivalent:*

(1) A is closed;

(2) A is g-closed and a locally closed set;

(3) A is αg^*-closed and an αlc-set;

(4) A is pg^*-closed and a plc-set;

(5) A is sg^*-closed (resp. ω-closed) and slc-closed (resp. sl^*c-set);

(6) A is bg^*-closed and a blc-set;

(7) A is βg^*-closed and a βlc-set.

PROOF. This is an immediate consequence of Theorem 4.1.

DEFINITION 4.1. Let (X, τ) be a topological space and m_X a minimal structure on X. A function $f : (X, \tau) \to (Y, \sigma)$ is said to be *mg^*-continuous* (resp. *mlc-continuous*) if $f^{-1}(F)$ is mg^*-closed (resp. an mlc-set) in (X, τ) for each closed set F of (Y, σ).

REMARK 4.1. Let (X, τ) be a topological space and m_X an m-structure on X such that $\tau \subset m_X$. Then,

(1) by Lemma 3.1, the following implications hold: continuity $\Rightarrow$ mg^*-continuity $\Rightarrow$ g-continuity,

(2) every continuous function is mlc-continuous but not conversely by Remark 2.4,

(3) mg^*-continuity and mlc-continuity are independent by Remark 2.4.

THEOREM 4.2. *Let (X,τ) be a topological space and m_X a minimal structure on X such that $\tau \subset m_X$. Then a function $f:(X,\tau) \to (Y,\sigma)$ is continuous if and only if it is mg^*-continuous and mlc-continuous.*

PROOF. This is an immediate consequence of Theorem 4.1

DEFINITION 4.2. A function $f:(X,\tau) \to (Y,\sigma)$ is said to be *g-continuous* [4] (resp. *αg^*-continuous*, *pg^*-continuous*, *sg^*-continuous* or *ω-continuous* [21], *bg^*-continuous*, *βg^*-continuous*) if $f^{-1}(F)$ is g-closed (resp. αg^*-closed, pg^*-closed, sg^*-closed, bg^*-closed, βg^*-closed) in (X,τ) for each closed set F of (Y,σ).

DEFINITION 4.3. A function $f:(X,\tau) \to (Y,\sigma)$ is said to be *LC-continuous* [6] (resp. *αlc-continuous*, *plc-continuous*, *slc-continuous* or *slc^*-continuous* [22], *blc-continuous*, *βlc-continuous*) if $f^{-1}(F)$ is a locally closed set (resp. αlc-set, plc-set, slc-set, blc-set, βlc-set) of (X,τ) for each closed set F of (Y,σ).

COROLLARY 4.2. *For a function $f:(X,\tau) \to (Y,\sigma)$, the following properties are equivalent:*

(1) f is continuous;

(2) f is g-continuous and LC-continuous;

(3) f is αg^*-continuous and αlc-continuous;

(4) f is pg^*-continuous and plc-continuous;

(5) f is ω-continuous and slc^*-continuous;

(6) f is bg^*-continuous and blc-continuous;

(7) f is βg^*-continuous and βlc-continuous.

Proof. This is an immediate consequence of Corollary 4.1.

DEFINITION 4.4. Let (X,τ) be a topological space and m_X a minimal structure on X. A function $f:(X,\tau) \to (Y,\sigma)$ is said to be *contra-m-continuous* [15] at $x \in X$ if for each closed set F of Y containing $f(x)$, there exists $U \in m_X$ containing x such that $f(U) \subset F$. f is said to be *contra-m-continuous* if it has this property at each point $x \in X$.

LEMMA 4.1. (Noiri and Popa [15]). *Let (X,m_X) be an m-space such that m_X has property $\mathscr{B}$. Then a function $f:(X,m_X) \to (Y,\sigma)$ is contra-m-continuous if and only if $f^{-1}(F)$ is m_X-open for every closed set F of (Y,σ).*

THEOREM 4.3. *Let (X,τ) be a topological space and m_X a minimal structure on X such that $\tau \subset m_X$ and m_X has property $\mathscr{B}$. If a function $f:(X,\tau) \to (Y,\sigma)$ is mg^*-continuous and contra-m-continuous, then f is continuous.*

PROOF. Let F be any closed set of (Y,σ). Since f is contra-m-continuous and m_X has property $\mathscr{B}$, by Lemma 4.1 $f^{-1}(F)$ is m_X-open. Since f is mg^*-continuous, $f^{-1}(F)$ is mg^*-closed and hence, by Lemma 3.2, $f^{-1}(F)$ is closed. Therefore, f is continuous.

REMARK 4.2. Let (X,τ) be a topological space and $m_X = \tau$ (resp. SO(X), PO(X), $\alpha(X)$, BO(X), $\beta(X)$). Then by Theorem 4.3, we can obtain several sufficient conditions for a function to be continuous. For example, in case $m_X = \tau$ we have the following.

COROLLARY 4.3. *If a function $f:(X,\tau) \to (Y,\sigma)$ is g-continuous and contra-continuous, then f is continuous.*

5. New forms of decomposition of continuity

First, we recall the θ-closure and the δ-closure of a subset in a topological space. Let (X,τ) be a topological space and A a subset of X. A point $x \in X$ is called a θ-cluster (resp. δ-cluster) point of A if $\mathrm{Cl}(V) \cap A \neq \emptyset$ (resp. $\mathrm{Int}(\mathrm{Cl}(V)) \cap A \neq \emptyset$) for every open set V containing x. The set of all θ-cluster (resp. δ-cluster) points of A is called the *θ-closure* (resp. *δ-closure*) of A and is denoted by $\mathrm{Cl}_\theta(A)$ (resp. $\mathrm{Cl}_\delta(A)$) [24].

DEFINITION 5.1. A subset A of a topological space (X,τ) is said to be

(1) *δ-preopen* [20] (resp. *θ-preopen* [16]) if $A \subset \mathrm{Int}(\mathrm{Cl}_\delta(A))$ (resp. $A \subset \mathrm{Int}(\mathrm{Cl}_\theta(A))$),

(2) *δ-β-open* [7] (resp. *θ-β-open* [16]) if $A \subset \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}_\delta(A)))$ (resp. $A \subset \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}_\theta(A)))$).

By δ PO(X) (resp. $\delta\beta(X)$, θ PO(X), $\theta\beta(X)$), we denote the collection of all δ-preopen (resp. δ-β-open, θ-preopen, θ-β-open) sets of a topological space (X,τ). These four collections are m-structures with property $\mathscr{B}$. In [16], the following diagram is known:

DIAGRAM IV

$$\begin{array}{ccccccc}
\alpha\text{-open} & \Rightarrow & \text{preopen} & \Rightarrow & \delta\text{-preopen} & \Rightarrow & \theta\text{-preopen} \\
\Downarrow & & \Downarrow & & \Downarrow & & \Downarrow \\
\text{semi-open} & \Rightarrow & \beta\text{-open} & \Rightarrow & \delta\text{-}\beta\text{-open} & \Rightarrow & \theta\text{-}\beta\text{-open}
\end{array}$$

For subsets of a topological space (X,τ), we can define many new variations of g-closed sets. For example, in case $m_X = \delta$ PO(X), $\delta\beta(X)$, θ PO(X), $\theta\beta(X)$, we can define new types of g-closed sets as follows:

DEFINITION 5.2. A subset A of a topological space (X,τ) is said to be *δpg^*-closed* (resp. *θpg^*-closed*, *$\delta\beta g^*$-closed*, *$\theta\beta g^*$-closed*) if $\mathrm{Cl}(A) \subset U$ whenever $A \subset U$ and U is δ-preopen (resp. θ-preopen, δ-β-open, θ-β-open) in (X,τ).

By DIAGRAM IV and Definitions 5.2, we have the following diagram:

DIAGRAM V

$$\begin{array}{ccccccccccc}
g\text{-closed} & \Leftarrow & \alpha pg^*\text{-closed} & \Leftarrow & pg^*\text{-closed} & \Leftarrow & \delta pg^*\text{-closed} & \Leftarrow & \theta pg^*\text{-closed} & & \\
 & & \Uparrow & & \Uparrow & & \Uparrow & & \Uparrow & & \\
 & & sg^*\text{-closed} & \Leftarrow & \beta g^*\text{-closed} & \Leftarrow & \delta\beta g^*\text{-closed} & \Leftarrow & \theta\beta g^*\text{-closed} & \Leftarrow & \text{closed}
\end{array}$$

DEFINITION 5.3. A subset A of a topological space (X,τ) is called a *δplc-set* (resp. *θplc-set*, *$\delta\beta lc$-set*, *$\theta\beta lc$-set*) if $A = U \cap F$, where U is δ-preopen (resp. θ-preopen, δ-β-open, θ-β-open) in (X,τ) and F is closed in (X,τ).

COROLLARY 5.1. *For a subset A of a topological space (X,τ), the following properties are equivalent:*

(1) A is closed;

(2) A is δpg^*-closed and a δplc-set;

(3) A is θpg^*-closed and a θplc-set;

(4) A is $\delta\beta g^*$-closed and a $\delta\beta lc$-set;

(5) A is $\theta\beta g^*$-closed and a $\theta\beta lc$-set.

PROOF. Let $m_X = \delta$ PO(X), θ PO(X), $\delta\beta(X)$ and $\theta\beta(X)$. Then this is an immediate consequence of Theorem 4.1.

By defining functions similarly to Definitions 4.2 and 4.3, we obtain the following decompositions of continuity:

COROLLARY 5.2. *For a function $f:(X,\tau) \to (Y,\sigma)$, the following properties are equivalent:*

(1) f is continuous;

(2) f is δpg^*-continuous and δplc-continuous;

(3) f is θpg^*-continuous and θplc-continuous;

(4) f is $\delta\beta g^*$-continuous and $\delta\beta lc$-continuous;

(5) f is $\theta\beta g^*$-continuous and $\theta\beta lc$-continuous.

PROOF. This is an immediate consequence of Theorem 4.2.

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ITERATED DIFFERENCE SETS IN σ-FINITE GROUPS

By

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Abstract. We improve on a previous result on iterated difference sets in arbitrary $\sigma-$finite groups.

1. Introduction

We investigate here the concept of iterated difference sets in the following way: for a given subset X of an arbitrary additively written group G, we define $D(X) = X - X = \{x - x' : x, x' \in X\}$ called difference set of X. We put $D_1 = D$, and for $k \geq 2$, $D_k(X) = D(D_{k-1}(X))$ for any $X \subseteq G$. In the case where G is the set of integers, Stewart and Tijdeman in [5] investigated the so-called iterated positive difference operation: for an infinite set A of positive integers, let $D^+(A)$ be the positive difference set defined by $D^+(A) = \{a - a' \mid a \geq a', a, a' \in A\}$. The k-fold iterated positive difference sequence $\{D^+{}_k(A);\ k \geq 0\}$ of A is defined by $D^+{}_0(A) = A$ and $D^+{}_k(A) = D^+(D^+{}_{k-1}(A))$ for $k \geq 1$. Stewart and Tijdeman observed that if a sequence A has positive upper density i.e.

$$\overline{d}(A) := \limsup_{n\to\infty} \frac{|A \cap \{1, 2, \dots, n\}|}{n} > 0,$$

then the sequence $\{D^+{}_k(A);\ k \geq 0\}$ is stable i.e. there exists a k_0 such that, $D^+{}_{k+1}(A) = D^+{}_k(A)$ for every $k \geq k_0$.

We define the time of stability of A by $T(A) = \min\{k \mid D^+{}_{k+1}(A) = D^+{}_k(A)\}$. For instance, if $\overline{d}(A) > 1/2$, it is readily seen that $D^+(A)$ is

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the whole set of nonnegative integers, hence $T(A) \leq 1$. In [5] Stewart and Tijdeman gave an upper bound for $T(A)$ if the upper density of A is positive. They proved that if $0 < \overline{d}(A) \leq 1/2$ then $T(A) \leq 2\log_2(\overline{d}(A)^{-1})$, where $\log_2$ denotes the logarithmic function in base 2. This result was improved by Ruzsa in [4] where it is shown that under the same assumption on $\overline{d}(A)$, we have $T(A) \leq 2 + \log_2(\overline{d}(A)^{-1} - 1)$ and this bound is sharp.

At this point we note that a seemingly similar question is to consider the sequence $\{D_k(A);\ k \geq 1\}$ of the iterated difference sets without any restriction. The advantage of this question is that it can be handled in more general groups. Let G be a countable torsion group and let $H_1 \subseteq H_2 \subseteq \cdots \subseteq$ $\subseteq H_n \subseteq \cdots$ be a sequence of finite subgroups of G. Then G is said to be σ-finite with respect to $\{H_n\}$ if $G = \bigcup_{n=1}^{\infty} H_n$.

We assume that G is a such group. Let $A \subseteq G$. The asymptotic upper density of A is defined by

$$\overline{d}(A) = \limsup_{n \to \infty} \frac{|A \cap H_n|}{|H_n|}. \tag{1}$$

We introduce the time of stability in groups as well.

Assume the sequence $\{D_k(A);\ k \geq 0\}$ is stable (i.e. for some k, $D_{k+1}(A) = D_k(A)$). Let $T(A, G)$ be the time of stability defined by

$$T(A, G) = \min\{k \mid D_{k+1}(A) = D_k(A)\}.$$

In [1] the first named author extended the results of Stewart, Tijdeman and Ruzsa to σ-finite Abelian groups. He proved

Theorem A. *Let G be a σ-finite abelian group with respect to $\{H_n\}$ and let A be a non empty subset of G. Let $\overline{d}(A)$ be the upper density of A defined by (1). If $\overline{d}(A) > 0$, then*

$$T(A, G) \leq \log_2(\overline{d}(A)^{-1}) + 2.$$

It is worth mentioning that generalization to arbitrary linear operations (i.e. instead of $D(X)$, we consider operation $\Gamma(X) = aX - bX$) of this kind of problem is investigated in [2].

In the next section, we present some basic multiplicative results which are used in the rest of the section in order to show that Theorem A holds with an optimal bound in some sense without assuming G to be abelian.

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2. Iterated difference sets in finite group and σ-finite group

In this section, groups are not necessarily abelian and are written multiplicatively with identity element denoted by 1. Let G be any group and $A, A_1, A_2, \dots, A_k$ be subsets of G. We denote by $A_1 A_2 \cdots A_k$ the subset of G of all products $x_1 x_2 \dots x_k$ with $x_j \in A_j$, $j = 1 \dots, k$. We also define for $k \geq 1$, the k-fold product set $A^k = A \cdots A$ (k times), $A^{-1} = \{x^{-1} \mid x \in A\}$, and we put $D(A) = AA^{-1}$. We denote by $|A|$ the cardinality of A. Finally let $D_0(A) = A$, $D_1(A) = D(A)$ and $D_k(A) = D(D_{k-1}(A))$ for $k \geq 2$.

2.1. Preliminary results

LEMMA 2.1. *Let A and B be subsets of a finite group G such that $|A| + |B| \geq |G| + 1$. Then $AB = G$.*

PROOF. Indeed if there is a $g \in G$ which is not in AB, then $A^{-1}g \cap B = \emptyset$ and so $|A^{-1}g| + |B| = |A| + |B| \leq |G|$, a contradiction. ∎

LEMMA 2.2. *Let A be a generating subset of a finite group G such that $1 \in A$. For any non-empty subset X of G*

$$|XA| \geq \min\{|G|, |X| + |A|/2\}.$$

PROOF. It is Theorem 1 in [3]. ∎

A straightforward consequence, which is obtained by an iterated application of this lemma, is that for any generating subsets $A_1, \dots, A_j$ of a finite group G such that $1 \in A_i$, $i = 1, \dots, j$, one has

$$|A_1 A_2 \dots A_j| \geq \min\left(|G|, |A_1| + \frac{|A_2| + \cdots + |A_j|}{2}\right).$$

2.2. Results for finite and σ-finite groups

We extend Theorem A to arbitrary finite groups and σ-finite groups. We first consider the case of arbitrary groups.

THEOREM 2.3. *Let A be a generating subset of a finite group G such that $1 \in A$. Let k_0 defined by*

$$k_0 = \begin{cases} 1 & \text{if } |G|/2 < |A| \leq |G|, \\ \left\lfloor \log_2 \left(\frac{|G|}{|A|} - 1 \right) \right\rfloor + 2 & \text{if } |A| \leq |G|/2 \end{cases}$$

where $\lfloor u \rfloor$ denotes the greatest integer less than or equal to the real number u.

Then, for any integer $k \geq k_0$

$$D_k(A) = G. \tag{2}$$

PROOF. If $|A| > |G|/2$ then by Lemma 2.1. with $B = A^{-1}$, we get $D_1(A) = G$. In the remaining of the proof, we assume that $|A| \leq |G|/2$. To see that (2) holds if $k \geq k_0$, we shall use the remark following Lemma 2.2. For any $k \geq 1$, $D_{k-1}(A)$ is a product of 2^{k-1} subsets, the factors being alternatively A or A^{-1}, which are both generating subsets of G and containing 1. It follows that $|D_{k-1}(A)| > |G|/2$ whenever $k > \log_2(|G|/|A| - 1) + 1$. By Lemma 2.1., we conclude that $D_k(A) = G$ under the same assumption on k. This gives our theorem. ∎

These bounds allow us to improve that on [1, Proposition1]. We obtain for k_0 defined in Theorem 2.3. that

$$T(A, G) \leq k_0 \tag{3}$$

for any subset A of an arbitrary finite group G.

We now show that Theorem A holds for any non abelian σ-finite group as well. In the case where A is a subset of a σ-finite group G with upper density $\overline{d}(A)$ larger than $1/2$, it is readily seen that $A - A = G$, hence $T(A, G) \leq 1$. We then formulate the remaining case:

THEOREM 2.4. *Let G be a σ-finite group with respect to $\{H_n\}$ and let A be a non empty subset of G. Assume that A has a positive upper density such that $\alpha := \overline{d}(A)^{-1} \geq 2$. Then*

$$T(A, G) \leq \lfloor \log_2(\alpha - 1) \rfloor + 2. \tag{4}$$

PROOF. Since the function $\lfloor\ \rfloor$ is right-continuous, there exists a real number $0 < \eta < 1$ such that the right-hand side of (4) is equal to

$$k := \lfloor \log_2(\alpha - \eta) \rfloor + 2.$$

Let

$$\varepsilon := \min(-\log_2(\eta), 1 - \{\log_2(\alpha - \eta)\}) \tag{5}$$

where $\{u\} = u - \lfloor u \rfloor$ denotes the fractional part of u. Note that $\varepsilon > 0$ hence $2^{\varepsilon} > 1$, hence there exists an increasing sequence of integers $\{n_1 < n_2 < < \cdots < n_i < \cdots\}$ such that

$$\overline{d}(A) < 2^{\varepsilon} \frac{|A \cap H_{n_i}|}{|H_{n_i}|}, \quad i \geq 1. \tag{6}$$

We claim that $T(A, G) \leq k$. Suppose that it is not the case. Thus

$$D_k(A) \neq D_{k+1}(A). \tag{7}$$

Let $A_n = A \cap H_n$. By (7) we infer that there exists an integer $n \in \{n_i \,;\, i \geq 1\}$ such that (6) holds and

$$D_k(A_n) \neq D_{k+1}(A_n). \tag{8}$$

Then by (5) and (6), we get

$$\alpha - \eta > 2^{-\varepsilon} \frac{|H_n|}{|A_n|} - \eta \geq 2^{-\varepsilon} \left(\frac{|H_n|}{|A_n|} - 1 \right),$$

hence, by (5) again,

$$k \geq \log_2 (\alpha - \eta) + 1 + \varepsilon > \log_2 \left(\frac{|H_n|}{|A_n|} - 1 \right) + 1 + \varepsilon - \log_2(2^{\varepsilon}) =$$
$$= \log_2 \left(\frac{|H_n|}{|A_n|} - 1 \right) + 1.$$

By Theorem 2.3. and (3), we get $k \geq T(A_n, H_n)$, i.e. A_n is stable after k steps, a contradiction to (8). This ends the proof. ∎

3. Concluding remarks

In order to show that bounds (3) for $T(A, G)$ deduced from Theorem 2.3. (and thus the bounds in Theorem 2.4.) are sharp, we provide the following example:

Let m be a positive integer and put $n = 2^{m+1} + 1$. We denote by U_n the abelian multiplicative group formed by the complex n-th roots of unity and let $A = \{1, \omega := \exp(2i\pi/n)\}$. It is clear that for any $k \geq 1$

$$D_k(A) = \{\omega^j,\ -2^{k-1} \leq j \leq 2^{k-1}\},$$

hence $T(A, U_n) = m+1$, which coincides with the corresponding upper bound given in Theorem 2.3.

We may extend this example as follows. Let G be a finite group and H be a normal subgroup of G such that the factor group G/H is cyclic generated by gH for some $g \in G$. We let $A = H \cup gH$. Then clearly $T(A, G) = T(\overline{A}, G/H)$ where $\overline{A}$ is the image of A by the canonical morphism from G onto G/H. If we assume further that G/H has order $n = 2^{m+1} + 1$ for some $m \geq 1$, we get

$$T(A, G) = T(\overline{A}, G/H) = T(\{1, \omega\}, U_n) = m + 1 = \left\lfloor \log_2 \left(\frac{|G|}{|A|} - 1 \right) \right\rfloor + 2.$$

To conclude, we stress the fact that upper bounds in Theorems 2.3. and 2.4. can be sligthly improved if we consider particular groups G and subsets A of G. For instance, we easily deduce from Cauchy-Davenport theorem that $T(A, \mathbb{Z}/p\mathbb{Z}) \leq \log_2 \left(\frac{p-1}{k-1}\right)$ where p is a prime number and A any subset of $\mathbb{Z}/p\mathbb{Z}$ with cardinality $k \geq 2$. Another way to derive better bounds is to observe that in fact we have $|XA| \geq \min(|G|, |X| + \lceil |A|/2 \rceil)$ in Lemma 2.2. where $\lceil u \rceil$ denotes the smallest integer larger than or equal to the real number u.

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A GENERALIZATION OF NEARLY CONTINUOUS MULTIFUNCTIONS

By

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Abstract. Recently Ekici [9] has introduced the notion of upper/lower nearly continuous multifunctions as a generalization of continuous multifunctions and N-continuous functions [16]. In this paper, we obtain the unified form of several generalizations of upper/lower nearly continuous multifunctions.

1. Introduction

The notion of N-closed sets in a topological space is introduced in [6] and studied in [20], [21] and other papers. Ekici [9] introduced and studied upper/lower nearly continuous multifunctions as a generalization of upper/lower semi-continuous multifunctions and N-continuous functions. In [26], the present authors introduced the notion of upper/lower m-continuous multifunctions.

In this paper we introduce and study the notion of upper/lower nearly m-continuous multifunctions as multifunctions from a set satisfying some minimal conditions into a topological space. The multifunction is a generalization of upper/lower m-continuous multifunctions and upper/lower nearly continuous multifunctions. We obtain several characterizations and properties of such multifunctions by generalizing the results established in [9] and other results. In the last section, we recall some types of modifications of open sets and point out the possibility for new forms of nearly continuous multifunctions.

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2. Preliminaries

Let (X,τ) be a topological space and A a subset of X. The closure of A and the interior of A are denoted by Cl(A) and Int(A), respectively. A subset A is said to be regular open (resp. regular closed) if Int(Cl(A)) = A (resp. Cl(Int(A)) = A).

DEFINITION 2.1. A subset A of a topological space (X,τ) is said to be *N-closed relative to X* (briefly *N-closed*) [6] if every cover of A by regular open sets of X has a finite subcover.

DEFINITION 2.2. Let (X,τ) be a topological space. A subset A of X is said to be *α-open* [19] (resp. *semi-open* [14], *preopen* [17], *β-open* [1] or *semi-preopen* [3], *b-open* [4]) if $A \subset$ Int(Cl(Int(A))) (resp. $A \subset$ Cl(Int(A)), $A \subset$ Int(Cl(A)), $A \subset$ Cl(Int(Cl(A))), $A \subset$ Int(Cl(A)) $\cup$ Cl(Int(A))).

The family of all semi-open (resp. preopen, α-open, β-open, semi-preopen, b-open) sets in X is denoted by SO(X) (resp. PO(X), $\alpha(X)$, $\beta(X)$, $SPO(X)$, BO(X)).

DEFINITION 2.3. The complement of a semi-open (resp. preopen, α-open, β-open, semi-preopen, b-open) set is said to be *semi-closed* [8] (resp. *preclosed* [11], *α-closed* [18], *β-closed* [1], *semi-preclosed* [3], *b-closed* [4]).

DEFINITION 2.4. The intersection of all semi-closed (resp. preclosed, α-closed, β-closed, semi-preclosed, b-closed) sets of X containing A is called the *semi-closure* [8] (resp. *preclosure* [11], *α-closure* [18], *β-closure* [2], *semi-preclosure* [3], *b-closure* [4]) of A and is denoted by sCl(A) (resp. pCl(A), α Cl(A), $_\beta$ Cl(A), spCl(A), bCl(A)).

DEFINITION 2.5. The union of all semi-open (resp. preopen, α-open, β-open, semi-preopen, b-open) sets of X contained in A is called the *semi-interior* (resp. *preinterior*, *α-interior*, *β-interior*, *semi-preinterior*, *b-interior*) of A and is denoted by sInt(A) (resp. pInt(A), α Int(A), $_\beta$ Int(A), spInt(A), bInt(A)).

DEFINITION 2.6. A function $f:(X,\tau) \to (Y,\sigma)$ is said to be *N-continuous* at $x \in X$ [16] if for each open set V of Y containing $f(x)$ and having N-closed complement, there exists an open set U containing x such that $f(U) \subset V$. The function is said to be *N-continuous* if it has this property at each point of X.

Throughout the present paper, (X,τ) and (Y,σ) (briefly X and Y) always denote topological spaces and $F\colon X \to Y$ (resp. $f\colon X \to Y$) presents a multivalued (resp. single valued) function. For a multifunction $F\colon X \to Y$, we shall denote the upper and lower inverse of a subset B of a space Y by $F^+(B)$ and $F^-(B)$, respectively, that is

$F^+(B) = \{x \in X : F(x) \subset B\}$ and $F^-(B) = \{x \in X : F(x) \cap B \neq \emptyset\}$.

DEFINITION 2.7. A multifunction $F\colon (X,\tau) \to (Y,\sigma)$ is said to be

(1) *upper nearly continuous* (briefly *u.n.c.*) at a point $x \in X$ [9] if for each open set V containing $F(x)$ and having N-closed complement, there exists an open set U of X containing x such that $F(U) \subset V$,

(2) *lower nearly continuous* (briefly *l.n.c.*) at a point $x \in X$ [9] if for each open set V meeting $F(x)$ and having N-closed complement, there exists an open set U of X containing x such that $F(u) \cap V \neq \emptyset$ for each $u \in U$,

(3) *upper/lower nearly continuous* on X if it has this property at each point of X.

3. Nearly m-continuous multifunctions

DEFINITION 3.1. A subfamily m_X of the power set $\mathcal{P}(X)$ of a nonempty set X is called a *minimal structure* (briefly *m-structure*) [24], [25] on X if $\emptyset \in m_X$ and $X \in m_X$.

By (X, m_X) (briefly (X, m)), we denote a nonempty set X with a minimal structure m_X on X and call it an *m-space*. Each member of m_X is said to be *m_X-open* (briefly *m-open*) and the complement of an m_X-open set is said to be *m_X-closed* (briefly *m-closed*).

REMARK 3.1. Let (X,τ) be a topological space. Then the families τ, SO(X), PO(X), $\alpha(X)$, BO(X) and SPO(X) are all m-structures on X.

DEFINITION 3.2. Let (X, m_X) be an m-space. For a subset A of X, the *m_X-closure* of A and the *m_X-interior* of A are defined in [15] as follows:

(1) $\mathrm{mCl}(A) = \cap\{F\colon A \subset F, X - F \in m_X\}$,

(2) $\mathrm{mInt}(A) = \cup\{U : U \subset A, U \in m_X\}$.

REMARK 3.2. Let (X,τ) be a topological space and A be a subset of X. If $m_X = \tau$ (resp. SO(X), PO(X), $\alpha(X)$, BO(X), SPO(X)), then we have

(a) $\mathrm{mCl}(A) = \mathrm{Cl}(A)$ (resp. $\mathrm{sCl}(A)$, $\mathrm{pCl}(A)$, $\alpha\,\mathrm{Cl}(A)$, $\mathrm{bCl}(A)$, $\mathrm{spCl}(A)$),

(b) $\mathrm{mInt}(A) = \mathrm{Int}(A)$ (resp. $\mathrm{sInt}(A)$, $\mathrm{pInt}(A)$, $\alpha\,\mathrm{Int}(A)$, $\mathrm{bInt}(A)$, $\mathrm{spInt}(A)$).

LEMMA 3.1. (Maki et al. [15]). *Let (X, m_X) be an m-space. For subsets A and B of X, the following properties hold:*

(1) $\mathrm{mCl}(X - A) = X - \mathrm{mInt}(A)$ *and* $\mathrm{mInt}(X - A) = X - \mathrm{mCl}(A)$,

(2) If $(X - A) \in m_X$, *then* $\mathrm{mCl}(A) = A$ *and if* $A \in m_X$, *then* $\mathrm{mInt}(A) = A$,

(3) $\mathrm{mCl}(\emptyset) = \emptyset$, $\mathrm{mCl}(X) = X$, $\mathrm{mInt}(\emptyset) = \emptyset$ *and* $\mathrm{mInt}(X) = X$,

(4) If $A \subset B$, *then* $\mathrm{mCl}(A) \subset \mathrm{mCl}(B)$ *and* $\mathrm{mInt}(A) \subset \mathrm{mInt}(B)$,

(5) $A \subset \mathrm{mCl}(A)$ *and* $\mathrm{mInt}(A) \subset A$,

(6) $\mathrm{mCl}(\mathrm{mCl}(A)) = \mathrm{mCl}(A)$ *and* $\mathrm{mInt}(\mathrm{mInt}(A)) = \mathrm{mInt}(A)$.

LEMMA 3.2. (Popa and Noiri [25]). *Let (X, m_X) be an m-space and A a subset of X. Then $x \in \mathrm{mCl}(A)$ if and only if $U \cap A \neq \emptyset$ for every $U \in m_X$ containing x.*

DEFINITION 3.3. A minimal structure m_X on a nonempty set X is said to have *property* $\mathscr{B}$ [15] if the union of any family of subsets belonging to m_X belongs to m_X.

REMARK 3.3. Let (X, τ) be a topological space. Then the families τ, $\mathrm{SO}(X)$, $\mathrm{PO}(X)$, $\alpha(X)$, $\mathrm{BO}(X)$ and $\mathrm{SPO}(X)$ have property $\mathscr{B}$.

LEMMA 3.3. (Popa and Noiri [27]). *For an m-structure m_X on a nonempty set X, the following properties are equivalent:*

(1) m_X has property $\mathscr{B}$;

(2) If $\mathrm{mInt}(A) = A$, *then* $A \in m_X$;

(3) If $\mathrm{mCl}(A) = A$, *then A is m_X-closed.*

DEFINITION 3.4. *Let (X, m_X) be an m-space and (Y, σ) a topological space.* A multifunction $F\colon (X, m_X) \to (Y, \sigma)$ is said to be

(1) *upper m-continuous* (briefly *u.m.c.*) [26] at a point $x \in X$ if for each open set V containing $F(x)$, there exists an m_X-open set U containing x such that $F(U) \subset V$,

(2) *lower m-continuous* (briefly *l.m.c.*) [26] at a point $x \in X$ if for each open set V meeting $F(x)$, there exists an m_X-open set U containing x such that $F(u) \cap V \neq \emptyset$ for each $u \in U$,

(3) *upper/lower m-continuous* on X if it has this property at every point of X.

DEFINITION 3.5. *Let* (X, m_X) *be an* m*-space and* (Y, σ) *a topological space.* A multifunction $F: (X, m_X) \to (Y, \sigma)$ is said to be

(1) *upper nearly* m*-continuous* (briefly *u.n.m.c.*) at a point $x \in X$ if for each open set V containing $F(x)$ and having N-closed complement, there exists an m_X-open set U containing x such that $F(U) \subset V$,

(2) *lower nearly* m*-continuous* (briefly *l.n.m.c.*) at a point $x \in X$ if for each open set V meeting $F(x)$ and having N-closed complement, there exists an m_X-open set U containing x such that $F(u) \cap V \neq \emptyset$ for each $u \in U$,

(3) *upper/lower nearly* m*-continuous* on X if it has this property at every point of X.

REMARK 3.4. Every upper/lower m-continuous multifunction is upper/lower nearly m-continuous. The converse is not true by Example 4 of [9].

THEOREM 3.1. *For a multifunction* $F: (X, m_X) \to (Y, \sigma)$*, the following properties are equivalent:*

(1) F *is u.n.m.c.;*

(2) $F^+(V) = \mathrm{mInt}(F^+(V))$ *for each open set* V *of* Y *having* N*-closed complement;*

(3) $F^-(K) = \mathrm{mCl}(F^-(K))$ *for every* N*-closed and closed set* K *of* Y*;*

(4) $\mathrm{mCl}(F^-(B)) \subset F^-(\mathrm{Cl}(B))$ *for every subset* B *of* Y *having the* N*-closed closure;*

(5) $F^+(\mathrm{Int}(B)) \subset \mathrm{mInt}(F^+(B))$ *for every subset* B *of* Y *such that* $Y - \mathrm{Int}(B)$ *is* N*-closed.*

PROOF. (1) $\Rightarrow$ (2): Let V be any open set of Y having N-closed complement and $x \in F^+(V)$. Then $F(x) \subset V$ and there exists $U \in m_X$ containing x such that $F(U) \subset V$. Therefore, $x \in U \subset F^+(V)$ and hence $x \in \mathrm{mInt}(F^+(V))$. This shows that $F^+(V) \subset \mathrm{mInt}(F^+(V))$. Therefore, by Lemma 3.1 we obtain $F^+(V) = \mathrm{mInt}(F^+(V))$.

(2) $\Rightarrow$ (3): Let K be any N-closed and closed set of Y. Then, by Lemma 3.1 we have $X - F^-(K) = F^+(Y - K) = \mathrm{mInt}(F^+(Y - K)) = \mathrm{mInt}(X - F^-(K)) = X - \mathrm{mCl}(F^-(K))$. Therefore, we obtain $F^-(K) = \mathrm{mCl}(F^-(K))$.

(3) $\Rightarrow$ (4): Let B be any subset of Y having the N-closed closure. By Lemma 3.1, we have $F^-(B) \subset F^-(\mathrm{Cl}(B)) = \mathrm{mCl}(F^-(\mathrm{Cl}(B)))$. Hence $\mathrm{mCl}(F^-(B)) \subset \mathrm{mCl}(F^-(\mathrm{Cl}(B))) = F^-(\mathrm{Cl}(B))$.

(4) $\Rightarrow$ (5): Let B be a subset of Y such that $Y - \mathrm{Int}(B)$ is N-closed. Then by Lemma 3.1 we have

$$\begin{aligned} X - \mathrm{mInt}(F^+(B)) = \mathrm{mCl}(X - F^+(B)) = \mathrm{mCl}(F^-(Y - B)) \subset \\ \subset F^-(\mathrm{Cl}(Y - B)) \subset F^-(Y - \mathrm{Int}(B)) = X - F^+(\mathrm{Int}(B)). \end{aligned}$$

Therefore, we obtain $F^+(\mathrm{Int}(B)) \subset \mathrm{mInt}(F^+(B))$.

(5) $\Rightarrow$ (1): Let $x \in X$ and V be any open set of Y containing $F(x)$ and having N-closed complement. Then $x \in F^+(V) = F^+(\mathrm{Int}(V)) \subset \mathrm{mInt}(F^+(V))$. There exists $U \in m_X$ containing x such that $U \subset F^+(V)$; lsoedhence $F(U) \subset V$. This shows that F is *u.n.m.c.*

THEOREM 3.2. *For a multifunction $F\colon (X, m_X) \to (Y, \sigma)$, the following properties are equivalent:*

(1) F is l.n.m.c.;

(2) $F^-(V) = \mathrm{mInt}(F^- V))$ for each open set V of Y having N-closed complement;

(3) $F^+(K) = \mathrm{mCl}(F^+(K))$ is for every N-closed and closed set K of Y;

(4) $\mathrm{mCl}(F^+(B)) \subset F^+(\mathrm{Cl}(B))$ for every subset B of Y having the N-closed closure;

(5) $F^-(\mathrm{Int}(B)) \subset \mathrm{mInt}(F^-(B))$ for every subset B of Y such that $Y - \mathrm{Int}(B)$ is N-closed.

PROOF. The proof is similar to that of Theorem 3.1.

COROLLARY 3.1. *Let (X, m_X) be an m-space and m_X have property $\mathscr{B}$. For a multifunction $F\colon (X, m_X) \to (Y, \sigma)$, the following properties are equivalent:*

(1) F is u.n.m.c. (resp. l.n.m.c.);

(2) $F^+(V)$ (resp. $F^-(V)$) is m_X-open for each open set V of Y having N-closed complement;

(3) $F^-(K)$ (resp. $F^+(K)$) is m_X-closed for every N-closed and closed set K of Y.

PROOF. This is an immediate consequence of Theorems 3.1 and 3.2 and Lemma 3.3.

REMARK 3.5. Let (X, τ) and (Y, σ) be topological spaces. If $m_X = \tau$ and $F\colon (X, m_X) \to (Y, \sigma)$ is upper/lower nearly m-continuous, then by Theorems 3.1 and 3.2 we obtain the results established in Theorem 2 of [9].

COROLLARY 3.2. *A multifunction* $F:(X,m_X)\to(Y,\sigma)$ *is u.n.m.c. (resp. l.n.m.c.) if* $F^-(K)=\mathrm{mCl}(F^-(K))$ *(resp.* $F^+(K)=\mathrm{mCl}(F^+(K))$*) for every* N*-closed set* K *of* Y*.*

PROOF. Let G be any open set of Y having N-closed complement. Then $Y-G$ is N-closed and closed. By the hypothesis, $X-F^+(G)=F^-(Y-G)=\mathrm{mCl}(F^-(Y-G))=\mathrm{mCl}(X-F^+(G))=X-\mathrm{mInt}(F^+(G))$ and hence, $F^+(G)=\mathrm{mInt}(F^+(G))$. It follows from Theorem 3.1 that F is *u.n.m.c.* The proof of lower near m-continuity is entirely similar.

DEFINITION 3.6. A function $f:(X,m_X)\to(Y,\sigma)$ is said to be *nearly* m*-continuous* if for each point $x\in X$ and each open set V containing $f(x)$ and having N-closed complement, there exists an m_X-open set U containing x such that $f(U)\subset V$.

COROLLARY 3.3. *For a function* $f:(X,m_X)\to(Y,\sigma)$*, the following properties are equivalent:*

(1) f *is nearly* m*-continuous;*

(2) $f^{-1}(V)=\mathrm{mInt}(f^{-1}(V))$ *for each open set* V *of* Y *having* N*-closed complement;*

(3) $f^{-1}(K)=\mathrm{mCl}(f^{-1}(K))$ *for every* N*-closed and closed set* K *of* Y*;*

(4) $\mathrm{mCl}(f^{-1}(B))\subset f^{-1}(\mathrm{Cl}(B))$ *for every subset* B *of* Y *having the* N*-closed closure;*

(5) $f^{-1}(\mathrm{Int}(B))\subset\mathrm{mInt}(f^{-1}(B))$ *for every subset* B *of* Y *such that* $Y-\mathrm{Int}(B)$ *is* N*-closed.*

COROLLARY 3.4. *For a function* $f:(X,m_X)\to(Y,\sigma)$*, where* m_X *has property* $\mathscr{B}$*, the following properties are equivalent:*

(1) f *is nearly* m*-continuous;*

(2) $f^{-1}(V)$ *is* m_X*-open for each open set* V *of* Y *having N-closed complement;*

(3) $f^{-1}(K)$ *is* m_X*-closed for every N-closed and closed set* K *of* Y*.*

REMARK 3.6. Let (X,τ) and (Y,σ) be topological spaces. If $m_X=\tau$ and $f:(X,m_X)\to(Y,\sigma)$ is nearly m-continuous, then by Corollary 3.4 we obtain the results established in Theorem 1 of [16].

DEFINITION 3.7. A subset A of a topological space (X,τ) is said to be

(1) *α-paracompact* [30] if every cover of A by open sets of X is refined by a cover of A which consists of open sets of X and is locally finite in X,

(2) *α-regular* [13] if for each $a \in A$ and each open set U of X containing a, there exists an open set G of X such that $a \in G \subset \mathrm{Cl}(G) \subset U$.

For a multifunction $F: X \to (Y,\sigma)$, a multifunction $\mathrm{Cl}\, F: X \to (Y,\sigma)$ is defined in [5] as follows: $(\mathrm{Cl}\, F)(x) = \mathrm{Cl}(F(x))$ for each point $x \in X$. Similarly, we can define $\alpha\, \mathrm{Cl}\, F$, $\mathrm{sCl}\, F$, $\mathrm{pCl}\, F$, $\mathrm{spCl}\, F$, and $\mathrm{bCl}\, F$.

LEMMA 3.4. (Popa and Noiri [26]). *If $F: (X, m_X) \to (Y,\sigma)$ is a multifunction such that $F(x)$ is α-paracompact and α-regular for each $x \in X$, then for each open set V of Y $F^+(V) = G^+(V)$, where G denotes* $\mathrm{Cl}\, F$, $\alpha\, \mathrm{Cl}\, F$, $\mathrm{sCl}\, F$, $\mathrm{pCl}\, F$, $\mathrm{bCl}\, F$ *or* $\mathrm{spCl}\, F$.

PROOF. The proof is similar to that of Lemma 3.3 of [24].

THEOREM 3.3. *Let $F: (X, m_X) \to (Y,\sigma)$ be a multifunction such that $F(x)$ is α-regular and α-paracompact for each $x \in X$. Then F is u.n.m.c. if and only if $G: (X, m_X) \to (Y,\sigma)$ is u.n.m.c., where G denotes* $\mathrm{Cl}\, F$, $\alpha\, \mathrm{Cl}\, F$, $\mathrm{sCl}\, F$, $\mathrm{pCl}\, F$, $\mathrm{bCl}\, F$ *or* $\mathrm{spCl}\, F$.

PROOF. *Necessity*. Suppose that F is *u.n.m.c.* Let V be any open set of Y having N-closed complement. By Lemma 3.4 and Theorem 3.1, we have $G^+(V) = F^+(V) = \mathrm{mInt}(F^+(V)) = \mathrm{mInt}(G^+(V))$. This shows that G is *u.n.m.c.*

Sufficiency. Suppose that G is *u.n.m.c.* Let V be any open set of Y having N-closed complement. By Lemma 3.4 and Theorem 3.1, $F^+(V) = G^+(V) = \mathrm{mInt}(G^+(V)) = \mathrm{mInt}(F^+(V))$. By Theorem 3.1, F is *u.n.m.c.*

LEMMA 3.5. (Popa and Noiri [26]). *If $F: (X, m_X) \to (Y,\sigma)$ is a multifunction, then for each open set V of Y $F^-(V) = G^-(V)$, where G denotes* $\mathrm{Cl}\, F$, $\alpha\, \mathrm{Cl}\, F$, $\mathrm{sCl}\, F$, $\mathrm{pCl}\, F$, $\mathrm{bCl}\, F$ *or* $\mathrm{spCl}\, F$.

THEOREM 3.4. *A multifunction $F: (X, m_X) \to (Y,\sigma)$ is l.n.m.c. if and only if $G: (X, m_X) \to (Y,\sigma)$ is l.n.m.c., where G denotes* $\mathrm{Cl}\, F$, $\alpha\, \mathrm{Cl}\, F$, $\mathrm{sCl}\, F$, $\mathrm{pCl}\, F$, $\mathrm{bCl}\, F$ *or* $\mathrm{spCl}\, F$.

PROOF. By using Lemma 3.5, this is shown similarly as in Theorem 3.3.

4. Some properties

LEMMA 4.1. (Popa and Noiri [26]). *A multifunction* $F: (X, m_X) \to (Y, \sigma)$ *is u.m.c. (resp. l.m.c.) if and only if* $F^+(V) = \mathrm{mInt}(F^+(V))$ *(resp.* $F^-(V) = \mathrm{mInt}(F^-(V))$*) for every open set* V *of* Y.

THEOREM 4.1. *Let* $F: (X, m_X) \to (Y, \sigma)$ *be a multifunction such that* (Y, σ) *has a base of open sets having* N*-closed complements and* m_X *has property* $\mathscr{B}$. *If F is l.n.m.c., then F is l.m.c.*

PROOF. Let V be any open set of Y. By the hypothesis, $V = \cup_{i \in I} V_i$, where V_i is an open set having N-closed complement for each $i \in I$. Since m_X has property $\mathscr{B}$, by Corollary 3.1 $F^-(V_i) \in m_X$ for each $i \in I$. Moreover, $F^-(V) = F^-(\cup\{V_i : i \in I\}) = \cup\{F^-(V_i) : i \in I\}$. Therefore, we have $F^-(V) \in m_X$. Then by Lemma 4.1 and Lemma 3.3 F is *l.m.c.*

REMARK 4.1. Let (X, τ) and (Y, σ) be topological spaces and

$$F: (X, m_X) \to (Y, \sigma)$$

be *l.n.m.c.* If $m_X = \tau$, then by Theorem 4.1 we obtain the result established in Theorem 5 of [9].

THEOREM 4.2. *Let* $F: (X, m_X) \to (Y, \sigma)$ *and* $G: (Y, \sigma) \to (Z, \theta)$ *be multifunctions. If F is u.m.c. (resp. l.m.c.) and G is u.n.c. (resp. l.n.c.), then* $G \circ F: (X, m_X) \to (Z, \theta)$ *is u.n.m.c. (resp. l.n.m.c.).*

PROOF. Let V be any open set of V having N-closed complement. Since G is *u.n.c.* (resp. *l.n.c.*), by Theorem 2 of [9] $F^+(V)$ (resp. $F^-(V)$) is an open set of Y. Since F is *u.m.c.* (resp. *l.m.c.*), by Lemma 4.1 $(G \circ F)^+(V) = F^+(G^+(V)) = \mathrm{mInt}(F^+(G^+(V))) = \mathrm{mInt}((G \circ F)^+(V))$ (resp. $(G \circ F)^-(V) = F^-(G^-(V)) = \mathrm{mInt}(F^-(G^-(V))) = \mathrm{mInt}((G \circ F)^-(V))$). By Theorem 3.1 (resp. Theorem 3.2) F is *u.n.m.c.* (resp. *l.n.m.c.*).

REMARK 4.2. If $F: (X, \tau) \to (Y, \sigma)$ is a multifunction and $m_X = \tau$, then by Theorem 4.2 we obtain the result established in Theorem 6 of [9].

DEFINITION 4.1. A topological space (Y, σ) is said to be N*-normal* [9] if for each disjoint closed sets K and H of Y, there exist open sets U and V having N-closed complement such that $K \subset U$, $H \subset V$ and $U \cap V = \emptyset$.

DEFINITION 4.2. An m-space (X, m_X) is said to be m-T_2 [24] if for each distinct points $x, y \in X$, there exist U, $V \in m_X$ such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$.

THEOREM 4.3. *If $F: (X, m_X) \to (Y, \sigma)$ is an u.n.m.c. multifunction satisfying the following conditions:*

(1) $F(x)$ is closed in Y for each $x \in X$,

(2) $F(x) \cap F(y) = \emptyset$ for each distinct points $x, y \in X$,

(3) m_X has property $\mathscr{B}$, and

(4) (Y, σ) is an N-normal space,

then (X, m_X) is m-T_2.

PROOF. Let x and y be distinct points of X. Then, we have $F(x) \cap F(y) = \emptyset$. Since $F(x)$ and $F(y)$ are closed and Y is N-normal, there exist disjoint open sets U and V having N-closed complement such that $F(x) \subset U$ and $F(y) \subset V$. By Corollary 3.1, we obtain $x \in F^+(U) \in m_X$, $y \in F^+(V) \in m_X$ and $F^+(U) \cap F^+(V) = \emptyset$. This shows that X is m-T_2.

REMARK 4.3. If $F: (X, \tau) \to (Y, \sigma)$ is a multifunction and $m_X = \tau$, then by Theorem 4.3 we obtain the result established in Theorem 19 of [9].

THEOREM 4.4. *Let (X, m_X) be an m-space. If for each pair of distinct points x_1 and x_2 in X, there exists a multifunction F from (X, m_X) into an N-normal space (Y, σ) satisfying the following conditions:*

(1) $F(x_1)$ and $F(x_2)$ are closed in Y,

(2) F is u.n.m.c. at x_1 and x_2, and

(3) $F(x_1) \cap F(x_2) = \emptyset$,

then (X, m_X) is m-T_2.

PROOF. Let x_1 and x_2 be distinct points of X. Then, we have $F(x_1) \cap F(x_2) = \emptyset$. Since $F(x_1)$ and $F(_2)$ are closed and Y is N-normal, there exist disjoint open sets V_1 and V_2 having N-closed complement such that $F(x_1) \subset V_1$ and $F(x_2) \subset V_2$. Since F is *u.n.m.c.* at x_1 and x_2, there exist $U_1 \in m_X$ and $U_2 \in m_X$ containing x_1 and x_2, respectively, such that $F(U_1) \subset V_1$ and $F(U_2) \subset V_2$. This implies that $U_1 \cap U_2 = \emptyset$. Hence (X, m_X) is an m-T_2-space.

DEFINITION 4.3. A subset A of an m-space (X, m_X) is said to be *m-dense* on X if $\mathrm{mCl}(A) = X$.

THEOREM 4.5. *Let X be a nonempty set with two minimal structures m_1 and m_2 such that $U \cap V \in m_2$ whenever $U \in m_1$ and $V \in m_2$ and (Y, σ) be an N-normal space. If the following conditions are satisfied:*

(1) $F: (X, m_1) \to (Y, \sigma)$ is u.n.m.c.,

(2) $G: (X, m_2) \to (Y, \sigma)$ is u.n.m.c.,

(3) $F(x)$ and $G(x)$ are closed in Y for each $x \in X$, and

(4) $A = \{x \in X : F(x) \cap G(x) \neq \emptyset\}$,

then $A = \mathrm{m}_2\,\mathrm{Cl}(A)$. If $F(x) \cap G(x) \neq \emptyset$ for each point in an m-dense set D of (X, m_2), then $F(x) \cap G(x) \neq \emptyset$ for each point $x \in X$.

PROOF. Suppose that $x \in X - A$. Then $F(x) \cap G(x) = \emptyset$. Since $F(x)$ and $G(x)$ are closed sets and Y is N-normal, there exist disjoint open sets V and W in Y having N-closed complement such that $F(x) \subset V$ and $G(x) \subset W$. Since F is *u.n.m.c.* at x, there exists $U_1 \in m_1$ containing x such that $F(U_1) \subset V$. Since G is *u.n.m.c.* at x, there exists $U_2 \in m_2$ containing x such that $G(U_2) \subset W$. Now set $U = U_1 \cap U_2$, then $U \in m_2$ and $U \cap A = \emptyset$. Therefore, by Lemma 3.2 we have $x \in X - \mathrm{m}_2\,\mathrm{Cl}(A)$ and hence $A = \mathrm{m}_2\,\mathrm{Cl}(A)$. On the other hand, if $F(x) \cap G(x) \neq \emptyset$ on an m-dense set D of an m-space (X, m_2), then we have $X = \mathrm{m}_2\,\mathrm{Cl}(D) \subset \mathrm{m}_2\,\mathrm{Cl}(A) = A$. Therefore, $F(x) \cap G(x) \neq \emptyset$ for each $x \in X$.

COROLLARY 4.1. (Ekici [9]). *Let F and G be upper nearly continuous and point closed multifunctions from a topological space (X, τ) into an N-normal space (Y, σ). Then the set $\{x : F(x) \cap G(x) \neq \emptyset\}$ is closed in X.*

DEFINITION 4.4. A topological space (X, τ) is said to be *N-connected* [10] if X cannot be written as the union of two disjoint nonempty open sets having N-closed complements.

DEFINITION 4.5. An m-space (X, m_X) is said to be *m-connected* [22] if X cannot be written as the union of two disjoint nonempty m_X-open sets.

THEOREM 4.6. *Let (X, m_X) be an m-space, where m_X has property $\mathscr{B}$. If $F: (X, m_X) \to (Y, \sigma)$ is an u.n.m.c. or l.n.m.c. surjective multifunction such that $F(x)$ is connected for each $x \in X$ and (X, m_X) is m-connected, then (Y, σ) is N-connected.*

PROOF. Suppose that (Y, σ) is not N-connected. There exist nonempty open sets U and V of Y having N-closed complement such that $U \cap V = \emptyset$ and $U \cup V = Y$. Since $F(x)$ is connected for each $x \in X$, either $F(x) \subset U$ or $F(x) \subset V$. If $x \in F^+(U \cup V)$, then $F(x) \subset U \cup V$ and hence $x \in$ $\in F^+(U) \cup F^+(V)$. Moreover, since F is surjective, there exist x and y such that $F(x) \subset U$ and $F(y) \subset V$; hence $x \in F^+(U)$ and $y \in F^+(V)$. Therefore, we obtain the following:

(1) $F^+(U) \cup F^+(V) = F^+(U \cup V) = X$,

(2) $F^+(U) \cap F^+(V) = \emptyset$,

(3) $F^+(U) \neq \emptyset$ and $F^+(V) \neq \emptyset$.

Next, we show that $F^+(U)$ and $F^+(V)$ are m_X-open sets. (i) In case F is *u.n.m.c.* by Corollary 3.1 $F^+(U)$ and $F^+(V)$ are m_X-open sets. (ii) In case F is *l.n.m.c.* by Corollary 3.1 $F^+(V)$ is m_X-closed because U is clopen in (Y, σ), therefore, $F^+(V)$ is m_X-open. Similarly $F^+(U)$ is m_X-open. Therefore, (X, m_X) is not m-connected.

DEFINITION 4.6. Let (X, m_X) be an m-space and A a subset of X. The *m-frontier* of A [25], denoted by mFr(A), is defined as follows:

$$\mathrm{mFr}(A) = \mathrm{mCl}(A) \cap \mathrm{mCl}(X - A) = \mathrm{mCl}(A) - \mathrm{mInt}(A).$$

THEOREM 4.7. *The set of all points $x \in X$ at which a multifunction $F\colon (X, m_X) \to (Y, \sigma)$ is not u.n.m.c. (resp. l.n.m.c.) is identical with the union of the m-frontiers of the upper (resp. lower) inverse images of open sets containing (resp. meeting) $F(x)$ and having N-closed complement.*

PROOF. Let x be a point of X at which F is not *u.n.m.c.* Then, there exists an open set V of Y containing $F(x)$ and having N-closed complement such that $U \cap (X - F^+(V)) \neq \emptyset$ for every m_X-open set U containing x. Hence, by Lemma 3.2 we have $x \in \mathrm{mCl}(X - F^+(V))$. On the other hand, we have $x \in F^+(V) \subset \mathrm{mCl}(F^+(V))$ and hence $x \in \mathrm{mFr}(F^+(V))$.

Conversely, suppose that F is *u.n.m.c.* Then for each open set V having N-closed complement and containing $F(x)$, we have $x \in \mathrm{mInt}(F^+(V))$. This is a contradiction. In case F is *l.n.m.c.*, the proof is similar.

5. New forms of nearly m-continuous multifunctions

For modifications of open sets defined in Definition 2.1, the following relationships are known:

$$\begin{array}{ccccccc} \text{open} & \Rightarrow & \alpha\text{-open} & \Rightarrow & \text{preopen} & & \\ & & \Downarrow & & \Downarrow & & \\ & & \text{semi-open} & \Rightarrow & b\text{-open} & \Rightarrow & \text{semi-preopen} \end{array}$$

First, we can define the following modifications of upper/lower nearly continuous multifunctions.

DEFINITION 5.1. A multifunction $F\colon (X, \tau) \to (Y, \sigma)$ is said to be

(1) *upper nearly α-continuous* (resp. *upper nearly precontinuous*, *upper nearly semi-continuous*, *upper nearly b-continuous*, *upper nearly sp-continuous*) at a point $x \in X$ if for each open set V containing $F(x)$ and

having N-closed complement, there exists an α-open (resp. preopen, semi-open, b-open, semi-preopen) set U containing x such that $F(U) \subset V$,

(2) *lower nearly α-continuous* (resp. *lower nearly precontinuous*, *lower nearly semi-continuous*, *lower nearly b-continuous*, *lower nearly sp-continuous*) at a point $x \in X$ if for each open set V meeting $F(x)$ and having N-closed complement, there exists an α-open (resp. preopen, semi-open, b-open, semi-preopen) set U containing x such that $F(u) \cap V \neq \emptyset$ for each $u \in U$,

(3) *upper/lower nearly α-continuous* (resp. *upper/lower nearly precontinuous*, *upper/lower nearly semi-continuous*, *upper/lower nearly b-continuous*, *upper/lower nearly sp-continuous*) on X if it has this property at each $x \in X$.

For multifunctions defined in Definition 5.1, the following relationships hold:

$$
\begin{array}{ccccccc}
\text{upper } n\text{-con.} & \Rightarrow & \text{upper } n\text{-}\alpha\text{-con.} & \Rightarrow & \text{upper } n\text{-precon.} & & \\
 & & \Downarrow & & \Downarrow & & \\
 & & \text{upper } n\text{-semi-con.} & \Rightarrow & \text{upper } n\text{-}b\text{-con.} & \Rightarrow & \text{upper } n\text{-}sp\text{-con.}
\end{array}
$$

REMARK 5.1. In the diagram above, "n" and "con." means near and continuity, respectively. And also the analogous diagram holds for the case "lower".

Let define the further modifications of upper/lower nearly continuous multifunctions. For the purpose, we recall the definitions of the θ-closure and the δ-closure due to Veličko [29]. Let (X, τ) be a topological space and A a subset of X. A point $x \in X$ is called a θ-cluster (resp. δ-cluster) point of A if $\mathrm{Cl}(V) \cap A \neq \emptyset$ (resp. $\mathrm{Int}(\mathrm{Cl}(V)) \cap A \neq \emptyset$) for every open set V containing x. The set of all θ-cluster (resp. δ-cluster) points of A is called the *θ-closure* (resp. *δ-closure*) of A and is denoted by $\mathrm{Cl}_\theta(A)$ (resp. $\mathrm{Cl}_\delta(A)$) [29]. A subset A is said to be *θ-closed* (resp. *δ-closed*) if $\mathrm{Cl}_\theta(A) = A$ (resp. $\mathrm{Cl}_\delta(A) = A$). The complement of a θ-closed (resp. δ-closed) set is said to be *θ-open* (resp. *δ-open*). The union of all θ-open (resp. δ-open) sets contained in the subset A is called the θ-interior (resp. δ-interior) of A and is denoted by $\mathrm{Int}_\theta(A)$ (resp. $\mathrm{Int}_\delta(A)$).

DEFINITION 5.2. A subset A of a topological space (X, τ) is said to be

(1) *δ-semiopen* [23] (resp. *θ-semiopen* [7]) if $A \subset \mathrm{Cl}(\mathrm{Int}_\delta(A))$ (resp. $A \subset \mathrm{Cl}(\mathrm{Int}_\theta(A))$),

(2) *δ-preopen* [28] (resp. *θ-preopen* [22]) if $A \subset \mathrm{Int}(\mathrm{Cl}_\delta(A))$ (resp. $A \subset \mathrm{Int}(\mathrm{Cl}_\theta(A))$),

(3) δ-*sp-open* [12] (resp. θ-*sp-open* [22]) if $A \subset \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}_\delta(A)))$ (resp. $A \subset \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}_\theta(A))))$.

By δ SO(X) (resp. δ PO(X), δ SPO(X), θ SO(X), θ PO(X), θ SPO(X)), we denote the collection of all δ-semiopen (resp. δ-preopen, δ-sp-open, θ-semiopen, θ-preopen, θ-sp-open) sets of a topological space (X,τ). These six collections are all m-structures with property $\mathscr{B}$. It is known that the families of all θ-open sets and δ-open sets of (X,τ) are topologies for X, respectively. In [22] and [7], the following relationships are known:

$$\begin{array}{ccccccccccc}
\theta\text{-open} & \Rightarrow & \delta\text{-open} & \Rightarrow & \text{open} & \Rightarrow & \text{preopen} & \Rightarrow & \delta\text{-preopen} & \Rightarrow & \theta\text{-preopen} \\
\Downarrow & & \Downarrow & & \Downarrow & & \Downarrow & & \Downarrow & & \Downarrow \\
\theta\text{-semiopen} & \Rightarrow & \delta\text{-semiopen} & \Rightarrow & \text{semi-open} & \Rightarrow & sp\text{-open} & \Rightarrow & \delta\text{-}sp\text{-open} & \Rightarrow & \theta\text{-}sp\text{-open}
\end{array}$$

Definition 5.3. A multifunction $F\colon (X,\tau) \to (Y,\sigma)$ is said to be

(1) *upper nearly θ-continuous* (resp. *upper nearly θ-precontinuous*, *upper nearly θ-semi-continuous*, *upper nearly θ-sp-continuous*) at a point $x \in X$ if for each open set V containing $F(x)$ and having N-closed complement, there exists a θ-open (resp. θ-preopen, θ-semiopen, θ-sp-open) set U containing x such that $F(U) \subset V$,

(2) *lower nearly θ-continuous* (resp. *lower nearly θ-precontinuous*, *lower nearly θ-semi-continuous*, *lower nearly θ-sp-continuous*) at a point $x \in X$ if for each open set V meeting $F(x)$ and having N-closed complement, there exists a θ-open (resp. θ-preopen, θ-semiopen, θ-sp-open) set U containing x such that $F(u) \cap V \neq \emptyset$ for each $u \in U$,

(3) *upper/lower nearly θ-continuous* (resp. *upper/lower nearly θ-precontinuous*, *upper/lower nearly θ-semi-continuous*, *upper/lower nearly θ-sp-continuous*) on X if it has this property at each $x \in X$.

Definition 5.4. A multifunction $F\colon (X,\tau) \to (Y,\sigma)$ is said to be

(1) *upper nearly δ-continuous* (resp. *upper nearly δ-precontinuous*, *upper nearly δ-semi-continuous*, *upper nearly δ-sp-continuous*) at a point $x \in X$ if for each open set V containing $F(x)$ and having N-closed complement, there exists a δ-open (resp. δ-preopen, δ-semiopen, δ-sp-open) set U containing x such that $F(U) \subset V$,

(2) *lower nearly δ-continuous* (resp. *lower nearly δ-precontinuous*, *lower nearly δ-semi-continuous*, *lower nearly δ-sp-continuous*) at a point $x \in X$ if for each open set V meeting $F(x)$ and having N-closed complement, there exists a δ-open (resp. δ-preopen, δ-semiopen, δ-sp-open) set U containing x such that $F(u) \cap V \neq \emptyset$ for each $u \in U$,

(3) *upper/lower nearly δ-continuous* (resp. *upper/lower nearly δ-precontinuous*, *upper/lower nearly δ-semi-continuous*, *upper/lower nearly δ-sp-continuous*) on X if it has this property at each $x \in X$.

For the multifunctions defined above, the following diagram hold, where u., n. and c. mean upper, near and continuity, respectively. And also the analogous diagram holds for the case “lower”.

$$\begin{array}{ccccccccccc}
\text{u.n.}\theta\text{-c.} & \Rightarrow & \text{u.n.}\delta\text{-c.} & \Rightarrow & \text{u.n.c.} & \Rightarrow & \text{u.n.p.c.} & \Rightarrow & \text{u.n.}\delta\text{-p.c.} & \Rightarrow & \text{u.n.}\theta\text{-p.c.} \\
\Downarrow & & \Downarrow & & \Downarrow & & \Downarrow & & \Downarrow & & \Downarrow \\
\text{u.n.}\theta\text{-s.c.} & \Rightarrow & \text{u.n.}\delta\text{-s.c.} & \Rightarrow & \text{u.n.s.c.} & \Rightarrow & \text{u.n.sp.c.} & \Rightarrow & \text{u.n.}\delta\text{-sp.c.} & \Rightarrow & \text{u.n.}\theta\text{-sp.c.}
\end{array}$$

CONCLUSION. We can apply the results established in Sections 3 and 4 to all multifunctions defined in Definitions 5.1, 5.2 and 5.3.

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PH. D. THESIS
A CONSTRUCTIVE APPROACH TO MATCHING AND ITS GENERALIZATIONS

By

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This paper is a summary of results presented in the author's Ph.D. thesis "A constructive approach to matching and its generalizations", Eötvös University, Budapest, 2007, written under the supervision of András Frank.

1. Matching in graphs

In the first chapter of the thesis we discuss some matching problems in graphs, including non-bipartite matching, square-free 2-factors, path-matching, and even factors. Our approach is slightly different from Edmonds' [5] well-known method of alternating forests and blossoms. The point is that our approach is easier to generalize to those more general problems, thus we obtain simpler proofs and algorithms than known before, and we also obtain some new results.

1.1. Restricted b-matching in bipartite graphs

Consider a simple bipartite graph $G = (A, B; E)$ and let $b \in \mathbb{N}^{A\cup B}$. A subset $M \subseteq E$ of edges is called a *b-matching* if it satisfies $\delta_M(v) \leq b(v)$ for all nodes $v \in A \cup B$. (Here $\delta_M(v)$ denotes the number of edges in M incident with a node v.) Furthermore, we are given a family $\mathcal{K}$ of some complete bipartite subgraphs of G, which will be considered as "forbidden". A b-matching is called $\mathcal{K}$*-free* if it does not contain every edge of any member of $\mathcal{K}$. Suppose that for every $K \in \mathcal{K}$ we have $|A\cap V(K)| \geq 2$, $|B \cap V(K)| \geq$

≥ 2, $b(v) = |B \cap V(K)|$ for all $v \in A \cap V(K)$, and $b(v) = |A \cap V(K)|$ for all $v \in B \cap V(K)$.

THEOREM 1.1. (Pap, [26], [27]) *If $G, b, \mathcal{K}$ satisfies the above assumptions, then the maximum cardinality of a $\mathcal{K}$-free b-matching is equal to*

$$\min_{Z \subseteq A \cup B} b(A \cup B - Z) + |E[Z]| - c_{\mathcal{K}}(G[Z]), \tag{1}$$

where $c_{\mathcal{K}}(G[Z])$ denotes the number of those components of induced subgraph $G[Z]$ which are members of $\mathcal{K}$. Moreover, a maximum $\mathcal{K}$-free b-matching can be constructed in polynomial time.

The formula generalizes a result of Frank [6] on K_{tt}-free t-matching, and results of Hartvigsen [10] and Z. Király [13] on square-free 2-factor. The algorithm generalizes Hartvigsen's algorithm, and is conceptually simpler than that.

1.2. Even factors

Consider a directed graph $D = (V, A)$. A *path/cycle* is the arcset of a unclosed/closed walk without repetition of nodes. An arc of a digraph is called *symmetric* if its reverse is in the arcset of the digraph, too. A cycle is called symmetric if all of its arcs are symmetric. D is called *odd-cycle-symmetric* if all of its odd cycles are symmetric. An *even factor* is the node-disjoint union of paths and even cycles. A *source-component* of a digraph is a strongly connected component of in-degree zero.

THEOREM 1.2. (Pap, Szegő, [19]) *If $D = (V, A)$ is odd-cycle-symmetric, then the maximum cardinality of an even factor is equal to*

$$\min_{Z \subseteq V} |V| + |\Gamma_D^+(Z)| - \sigma_{odd}(D[Z]), \tag{2}$$

where $\sigma_{odd}(D[Z])$ denotes the number of source-components of $D[Z]$ on an odd number of nodes.

Our original proof is non-constructive. In the dissertation (and in [21], [25]), we provide a simpler proof, which also implies a polynomial time algorithm that is conceptually simpler than that of Cunningham, Geelen [4]. Our result is sloghtly more general, since Cunningham and Geelen only claimed the result for so-called weakly-symmetric digraphs, a subclass of odd-cycle-symmetric digraphs. The even factor algorithm also implies a new algorithm for path-matching. The method of proof can be extended to hypo-matching in digraphs [20], [22], a common generalization of even factors and hypo-matching (see Cornuéjols, Hartvigsen [3]).

2. Matroid matching

In the second chapter we discuss matroid matching, in particular we focus on those special cases admitting a good characterization and a polynomial time algorithm. First we propose a matroid intersection algorithm, which is obtained from a generalization of the bipartite matching algorithm in the first chapter. Then we discuss Lovász' [14] linear matroid matching formula, and its algorithmic proof given by Orlin and Vande Vate [29]. Then we propose a new, algorithmic proof of a recent result of Makai and Szabó [17] on polymatroid matching, which implies a polynomial time algorithm for a theorem of Frank, Jordán, Szigeti [7]. Finally we discuss the lesser-known matroid fractional matching problem, results of Vande Vate [31], and Gijswijt [9]. We also provide a new application of matroid fractional matching, namely fractional packing of $\mathcal{A}$-paths.

2.1. Linear matroid matching

Let $E = \{l_1, \dots, l_n\}$ be a set of *lines*, i.e. a set of 2-dimensional subspaces of a given vectorspace V. A line is assumed given by a pair of spanning vectors. A subset $M \subseteq E$ of lines is called a **matching** if $r(M) = 2|M|$. Let $\nu(E)$ denote the maximum cardinality of a matching. A pair K, π is called a *cover* if K is a subspace, and $\pi = \{A_1, \dots, A_k\}$ is a partition of E. The *value* of a cover is defined by $val(K,\pi) := r(K) + \sum_{A_i \in \pi} \lfloor \frac{1}{2} r_{V/K}(A_i) \rfloor$.

THEOREM 2.1. (Lovász, [14]) $\nu(E) = \min val(K,\pi)$, *where the minimum is taken over covers* K, π.

In addition to this min-max formula, Lovász also provided a polynomial time algorithm, on the assumption that the lines are given by an explicit representation over a specific field, and arithmetic operations over this field take constant time.

In this section of the thesis, we discuss another matroid matching algorithm provided by Orlin and Vande Vate [29], which in many regards is analogous to those graph matching algorithms in the first chapter. It is based on the observation that, in Lovász' formula, the optimum cover is attained by a special kind of cover, a so-called *very strong cover*. The algorithm maintains a very strong cover of a growing subset of lines, and concludes by exhibiting a maximum matching and a minimum cover of the whole set of lines.

Note that, if the matroid is given by a linear representation over the rationals, then the original statement of Lovász' min-max formula is NOT a good characterization, since some subspace K may only be represented by a basis using very large numbers. Luckily, we show that its minimum attains with a very strong cover, which, using the following theorem, implies that Lovász' formula is a good characterization, and that the algorithm runs in polynomial time.

THEOREM 2.2. *If $K, \pi = \{A_1, \dots, A_k\}$ is a very strong cover, then there are maximum matchings $M_1, \dots, M_k$ such that*

$$K = \bigvee_{i=1}^{k} (sp(A_i) \wedge sp(M_i - A_i)).$$

2.2. Matching in ntcdc-free polymatroids

Consider a polymatroid function b on a finite groundset S, and let $\mathcal{P}(b)$ denote its induced polymatroid. A vector is called *even* if all of its entries are even. *Matchings* are the even vectors in $\mathcal{P}(b)$, the size of which is defined by half the sum of its entries. Let $\nu(b)$ denote the maximum size of a matching. A non-trivial compatible double-circuit (*ntcdc*, for short) is an integer vector $w \geq 0$ such that $w \notin \mathcal{P}(b)$, $w - \chi_s \notin \mathcal{P}(b)$ for all $s \in S$, and $supp(w)$ has a partition into at least three components such that $w - \chi_s - \chi_t \in \mathcal{P}(b)$ iff s, t is in distinct components of the partition. A polymatroid-function is called *ntcdc-free* if there is no ntcdc.

THEOREM 2.3. (Makai, Szabó, [17], Makai, Pap, Szabó, [18]) *If b is an ntcdc-free polymatroid-function, then $\nu(b) = \min_\pi \sum_{S_i \in \pi} \lfloor \frac{1}{2} b(S_i) \rfloor$, where the minimum is taken over partitions $\pi = \{S_1, \dots, S_t\}$ of S.*

Makai and Szabó originally provided a non-constructive proof. In the dissertation, we provide a different, constructive proof. Special cases of this result are: A result of T. Király, Szabó [12] on parity constrained orientations of a graph covering a given non-negative submodular function. This generalizes to a great extent the result of Frank, Jordán, Szigeti [7], and also Nebeský's [28] characterization of maximum genus graph embedding.

2.3. Matroid fractional matching

Consider an arbitrary matroid $\mathcal{M} = (S,\mathcal{I})$. Let E be a family of some two-element subsets of S, which are called pairs. For a set $K \subseteq S$, let $d_K \in \{0,1,2\}^E$ denote the vector such that $d_K(e) := r(K \cap sp(e))$ for all $e \in E$. A *fractional matching* is a vector $x \in \mathbb{R}^E$ satisfying $x \geq 0$ and $d_K \cdot x \leq r(K)$ for all $K \subseteq S$. The size of a fractional matching is defined by the sum of its entries. Let $\nu^*(E)$ denote the maximum size of a fractional matching.

THEOREM 2.4 (Vande Vate, [31]) $\nu^*(E) = \min\limits_{K\subseteq S}$

$$r(K) + \frac{1}{2} r_{\mathcal{M}/K}(\{e : r_{V/K}(e) = 2\}).$$

THEOREM 2.5. (Gijswijt, Pap, [9]) *The above description of the matroid fractional matching polytope is totally dual half-integer.*

Vande Vate provided two special cases of matroid fractional matching: graph fractional matching, and matroid intersection. In the dissertation we show that fractional packing of $\mathcal{A}$-paths is also a special case.

3. Packing A-paths

In the third chapter, we consider Mader's [16] path-packing min-max formulae, its generalizations, and polynomial time algorithms. Consider an undirected graph $G = (V, E)$ and a set $A \subseteq V$. A path is called an *A-path* if its two distinct endpoints are in A. Let $\widehat{\nu}(G, A)$ denote the maximum number of pairwise fully node-disjoint A-paths. Gallai [8] proved that determining $\widehat{\nu}(G, A)$ reduces to maximum matching in an auxiliary graph. If, moreover, we are given a partition $\mathcal{A} = \{A_1, A_2, \ldots, A_k\}$ of A, then an A-path is called an *$\mathcal{A}$-path* if it joins two distinct sets A_i. Let $\nu(G,\mathcal{A})$ denote the maximum number of pairwise fully node-disjoint $\mathcal{A}$-paths. Mader provided a min-max formula for $\nu(G,\mathcal{A})$. A polynomial time algorithm follows from Lovász' [15] linear matroid matching algorithm, and Schrijver's [30] linear representation.

3.1. Fractional packing of $\mathcal{A}$-paths

We assume $G, \mathcal{A}$ is given as above, and we are given a vector $b \in \mathbb{N}^V$ of node-capacities. A *fractional b-packing* of $\mathcal{A}$-paths is a vector x : "all $\mathcal{A}$-paths" $\to \mathbb{R}$ satisfying the inequalities (4)–(5). The linear program (3)–(5) is called the maximum fractional b-packing problem.

$$\max \mathbf{1}x \tag{1}$$

$$x(P) \geq 0 \qquad \text{for all } \mathcal{A}\text{-paths } P, \tag{2}$$

$$x(\{P : v \in V(P)\}) \leq b(v) \qquad \text{for all } v \in V. \tag{3}$$

THEOREM 3.1. *Both (3)–(5), and its dual admit a half-integral optimum solution.*

According to the following result, in case of $b \equiv \mathbf{1}$, the fractional b-packing problem not only admits a half-integral optimum, but the half-integral optimum attains at a special kind of half-integral solution. A half-integral packing is called an odd-A-cycle, if it arises in the following way. Suppose C is a cycle and $P_1, \dots, P_{2m+1}$ are pairwise node-disjoint paths such that one of its endpoints s_i is in A, and the other endpoint is in $V(C)$, in this cyclic order. A half-integral packing is given by assigning $\frac{1}{2}$ with those unique s_i–s_{i+1} paths. Half-integral packings of this kind are called *odd-A-cycles*.

THEOREM 3.2. *There is a maximum half-integral $\mathbf{1}$-packing x which decomposes into the fully node-disjoint union of odd-A-cycles and some paths P with $x(P) = 1$.*

This theorem is attractive by itself, but it also proved useful in the proof of the following result, claiming that fractional $\mathbf{1}$-packing reduces to a special case of matroid fractional matching. Let $\mathcal{E} = \mathcal{E}(G, \mathcal{A})$ denote the set of those lines of a euclidian vectorspace constructed by Schrijver [30], which implies $\nu(G, \mathcal{A}) = \nu(\mathcal{E}) - |V - A|$. The following theorem claims that applying matroid fractional matching for the same set of lines implies a solution of the problem of fractional packing.

THEOREM 3.3 *The maximum size of a fractional $\mathbf{1}$-packing is equal to $\nu^*(\mathcal{E}) - |V - A|$.*

3.2. Mader matroids are gammoids

The problem of packing $\mathscr{A}$-paths induces a matroid on the set A of terminals, which is called the Mader matroid. A subset of A is defined independent if there is a maximum packing covering each of those nodes in the subset. It is quite easy to show that this gives a matroid. Schrijver [30] published as an open question whether all Mader matroids are gammoids. (Gammoids are the minors of transversal matroids.) In the dissertation we show that the answer is positive.

THEOREM 3.4. *Every Mader matroid is a gammoid.*

3.3. Packing non-returning A-paths

We propose a model of packing A-paths which generalizes the problem of packing $\mathscr{A}$-paths. The problem is to pack a maximum number of non-returning A-paths in a permutation-labeled graph, for which an extension of Mader's formula is proven, and a polynomial time algorithm is constructed.

Consider an undirected graph $G = (V, E)$ and a given subset $A \subseteq V$ of nodes. Let Ω be an arbitrary set disjoint from the graph. We assign an element $\omega(v) \in \Omega$ with every node $v \in A$. We assign with every edge $ab = e \in E$ a reference-orientation, and a permutation $\pi(e)$ of S. We define $\pi(e,a) := = \pi(e)$ and $\pi(e,b) := \pi(e)^{-1}$, where we take the reference-orientation into consideration. An A-path $(v_1, \ldots, v_m)$ is called *non-returning* if $\pi(v_1v_2, v_1) \circ \cdots \pi(v_{m-1}v_m, v_{m-1})(\omega(v_1)) \neq \omega(v_m)$ holds.

The model of $\mathscr{A}$-paths easily reduces to this setting, and thus we obtain a generalization of Mader's formula below. This theorem is, in fact, a generalization of Chudnovsky et al.'s [2] result on packing non-zero A-paths in a group-labeled graph.

THEOREM 3.5. ([23], [24]) *The maximum number of pairwise node-disjoint non-returning A-paths is equal to* $\min \widehat{\nu}(G-X, A \cup V(F) - X)$*, where the minimum is taken over pairs $X \subseteq V, F \subseteq E$ such that $G - X - F$ does not contain any non-returning A-paths. The maximum may be determined in running time polynomial in $|V| + |E| + |\Omega|$.*

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CONFERENCES

Student Research Circle conference
Eötvös Loránd University, November 30, 2007

Kristóf Bérczi: Packings and covering in directed graphs

Zalán Gyenis: On finite substructures of certain stable strutures

Dávid Kunszenti-Kovács: Flows in networks—a probabilistic approach

Marcella Takáts: Vandermonde sets and super Vandermonde sets

Ramón Horváth: On lifting up an eversion

Tamás Horváth: The number of solutions of a boundary value problem containing a singular nonlinearity

János Geleji: Summing count matroids

Gabriella Pluhár: The free spectrum of the variety of bands

M60. Miklós Laczkovich is Sixty
A miniconference in Real Analysis. 22–23 February, 2008

http://www.cs.elte.hu/buczo/M60/M60.html

L. Lovász: Opening remarks

Á. Császár: Some results of Miklós Laczkovich

There are some common results of M. Laczkovich and the author from the years between 1975 and 1980. Let (f_n) be a sequence of real valued functions defined on a set X. We say that a function f is the *discrete* limit of this sequence iff, for each $x \in X$, there is an index n_0 such that $f_n(x) = f(x)$ whenever $n > n_0$. f is the *equal* limit of the sequence iff there is a sequence of positive numbers ϵ_n such that, for each $x \in X$, there exists an index n_0 such that $|f_n(x) - f(x)| < \epsilon_n$ whenever $n > n_0$. The common papers discuss results on these sorts of convergence, in particular, they consider Baire classes where the usual convergence is replaced with discrete or equal convergence.

P. Humke: A little bit of this and a little bit of that

Z. Daróczy: On a family of functional equations

Let $I \subset \mathbb{R}$ be a non-empty open interval. We consider the following family of functional equations

$$f(px + (1-p)y)\left[r(1-q)g(y) - (1-r)qg(x)\right] = \\ = \mu\,[p(1-q)f(x)g(y) - (1-p)qf(y)g(x)],$$

where $(p, q, r) \in (0, 1)^3$ and $\mu \neq 0, 1$ are constants, $f, g : I \to \mathbb{R}_+$ are unknown functions and the equation holds for all $x, y \in I$. We give a review on some special cases of this equation depending on the four parameters which are already solved.

Sz. Révész: Integral concentration of idempotent trigonometric polynomials on small sets

L. Székelyhidi: Spectral synthesis on hypergroups

Spectral analysis and spectral synthesis deal with the description of translation invariant function spaces over locally compact Abelian groups. As translation has a natural meaning on commutative hypergroups, too, it seems reasonable to consider similar problems on these structures. In this talk we present recent results on spectral analysis and spectral synthesis over discrete Abelian groups. The first problem is to define the basic building blocks of spectral analysis and synthesis: the exponential monomials. The reasonable definition is not straightforward. In the talk we show how to manage this in accordance with the group case on polynomial hypergroups and indicate possible extensions for Sturm–Liouville hypergroups, as well. The next problem is to study spectral analysis and synthesis. We deal with these problems again on polynomial hypergroups and solve them in the positive.

M. Balcerzak: On the Laczkovich–Komjáth property concerning sequences of analytic sets

The talk is based on the paper "On the Laczkovich-Komjáth property of sigma-ideals" (joint with Szymon G⁄ląb) accepted for publication in the Topology and Its Applications. In 1977 Laczkovich proved that, for each sequence (A_n) of Borel subsets of a Polish apace X, if $\limsup_{n\in H} A_n$ is uncountable for every $H \in [\mathbb{N}]^\omega$ then $\bigcap_{n\in G} A_n$ is uncountable for some $G \in [\mathbb{N}]^\omega$. In 1984 this result was generalized by Komjáth to the case when the sets A_n are analytic. His theorem, by our definition, means that the σ-ideal $[X]^{\leq\omega}$ has the Laczkovich-Komjáth property (in short (LK)). We prove that every σ-ideal generated by X/E has property (LK), for an equivalence relation $E \subset X^2$ of type F_σ with uncountably many equivalence classes. We also show the parametric version of this result and we study the invariance of property (LK) with respect to various operations.

Zs. Páles: The extension of Clarke's generalized derivative of real-valued locally Lipschitz functions to the Radon–Nikodym space-valued setting

Locally Lipschitz functions acting between infinite dimensional normed spaces are considered. Clarke's generalized Jacobian is extended to the setting when the range is a dual space and satisfies the Radon–Nikodým property. Characterization and fundamental properties of this extended generalized Jacobian are established including nonemptiness, compactness and upper semicontinuity with respect to a relevant topology, and a mean-value theorem. Connection to known notions of differentiation and chain rules are provided. The generalized Jacobian introduced is shown to enjoy all the properties required of a derivative like-set.

P. Holický: Decompositions of Borel bimeasurable mappings

We stated the nonseparable analogue of the classical results by Luzin, Novikov and Purves on the decomposition of Borel measurable mappings which map Borel sets to Borel sets. We demonstrated the method of proof on the classical result on countable decompositions of Borel bimeasurable mappings between Polish spaces, which reads as follows:

Let X, Y be Polish spaces, $f : X \to Y$ be a Borel measurable mapping. Then the following statements are equivalent:

(a) $f(Borel(X)) \subset Borel(Y)$;

(b) $f(G_\delta(X)) \subset Borel(Y)$;

(c) there are Borel subsets $X_0, X_1, \ldots$ of X such that $\bigcup_{i=0}^{\infty} X_i = X$, $f(X_0)$ is countable, and $f \restriction X_n$, $n = 1, 2, \ldots$, are injective.

J. Lindenstrauss, D. Preiss, J. Tišer: Fréchet differentiability of Lipschitz functions via a variational principle

In the talk (presented by the third author) we indicated how a new variational principle which in particular does not assume the completeness of the domain, can give a new, more natural, proof of the fact that a real valued Lipschitz function on an Asplund space has points of Fréchet differentiability. In more details, as an illustration, we showed that everywhere Gâteaux differentiable Lipschitz function on a a space with separable dual is somewhere Fréchet differentiable.

M. Csörnyei: Lipschitz image of sets of positive measure

We present two constructions to Laczkovich's problem about Lipschitz mappings of planar sets of positive measures onto balls. The first one uses discrete techniques, the second one (due to Khrushchev) is based on complex analytic methods and the Hahn-Banach theorem. The higher dimensional problem remains open.

B. Kirchheim: Convexity notions in the Calculus of Variations

42. Jahrestagung der Gesellschaft für Didaktik der Mathematik
13.–18. März 2008 BUDAPEST

Kurzfassungen
A) HAUPTVORTRÄGE

COHORS-FRESENBORG, Elmar: Mechanismen von Metakognition und Diskursivität im Mathematikunterricht

Es soll die Bedeutung von metakognitiven und diskursiven Aktivitäten von Lehrenden und Lernenden für die Qualität von Mathematikunterricht herausgearbeitet werden. Dazu werden die Konstrukte "Metakognition" und "Diskursivität" im Hinblick auf ihre Bedeutung für das Lehren und Lernen von Mathematik dekomponiert und ein Kategoriensystem zur Unterrichtsanalyse vorgestellt.
Im Einzelnen wird über den Effekt von metakognitiven Aktivitäten für den Lernerfolg und die Problemlösekompetenz berichtet und herausgearbeitet, inwieweit Diskursivität als Instrument geeignet ist, inhaltliche Klarheit (bzw. Unklarheit) aufzudecken. Anhand von videographierten Unterrichtsszenen wird exemplarisch dargelegt, wie Wirkmechanismen von Metakognition und Diskursivität funktionieren. Schließlich wird dargelegt, welche Rolle die Kategorisierung von Unterrichtstranskripten nach metakognitiven und diskursiven Aktivitäten bei der Lehreraus- und -weiterbildung sowie der Qualitätsanalyse von Mathematikunterricht spielen kann.

HERBER, Hans-Jörg: Psychologische Hintergrundsparadigmen von Innerer Differenzierung und Individualisierung

Wissenschaftlich fundierter Schulunterricht muss sich - wie die aktuelle Bildungsdiskussion zeigt - zunehmend mehr der individuellen Lernvoraussetzungen der Schüler in lern-, motivations- und entwicklungspsychologischer Hinsicht annehmen. Unter der Annahme der Abhängigkeit des schulischen Lernverhaltens von solchen Bedingungen, kann die Optimierung der Lehrer-Schüler-Interaktion durch rationale Analyse der relevanten Bedingungszusammenhänge und deren praktische Berücksichtigung verbessert werden: In unserem Begriffsverständnis von schülergerechtem Unterricht soll durch solcherart fundierte Maßnahmen der Inneren Differenzierung und Individualisierung dem heranwachsenden Menschen gemäß seiner je individuellen kognitiven und emotional-motivationalen Entwicklungsvoraussetzungen eine pädagogische Hilfestellung angeboten werden, durch die er seine Kompetenzen (intellektuelle Fähigkeiten, sachbezogne und soziale Motivationen, etc.) optimal entfalten kann. Kurz gesagt: Schulischer Unterricht soll

nach Möglichkeit die Selbstbildungsprozesse des Individuums behutsam unterstützen und vor allem nicht behindern. Dies erfordert einen strukturierten Lernraum, in dem wechselseitiges Vertrauen herrscht und selbständiges sowie kooperatives Lernen möglich ist. Ziel ist die Bildung von selbstverantwortlichen, lernmotivierten, autonomen Menschen mit hoher sozialer Kompetenz. Das über jahrzehntelange Forschung entwickelte Grundmodell der Inneren Differenzierung und Individualisierung (z.B. Herber & Vásárhelyi 2002) stützt sich auf die wichtigsten Theoreme zeitgemäßer psychologischer Hintergrundsparadigmen individueller und sozialer Lernprozesse und entsprechende Feldforschung im Zusammenhang schulischen Lernens.
Im aktuellen Vortrag werden durch prototypische Schlaglichter die wichtigsten Argumente für Innere Differenzierung und Individualisierung – theorienbezogen und empirisch gestützt – zusammengefasst und kritisch diskutiert.

Literatur: Herber, H.-J. & Vásárhelyi, É. (2002). Das Unterrichtsmodell ”Innere Differenzierung einschließlich Analogiebildung” - Aspekte einer empirisch veranlassten Modellentwicklung. Salzburger Beiträge zur Erziehungswissenschaft 6, Heft 2, 5-19

LOVÁSZ, László: Trends in Mathematics, and how they Change Education

Mathematical activity has changed a lot in the last 50 years. Some of these changes, like the use of computers, are very visible and are being implemented in mathematical education quite extensively. There are other, more subtle trends that may not be so obvious. We discuss some of these trends and how they could, or should, influence the future of mathematical education.

MEVARECH, Zemira R.: Why teaching facts is just not enough? The effects of meta-cognitive instruction on mathematics achievement

No child left behind is one of the most challenging issues of the 21st century. The fact that all children attend schools and the rate of dropout is quite low, raises the question of how to provide effective education to ALL: lower and higher achievers, LD as well as gifted children, and of course, “ordinary” children.
The challenge of ”no child left behind” is particularly applicable to mathematics education because on one hand a large proportion of school time is devoted to the studying of mathematics, and on the other hand it is considered to be one of the most difficult subjects taught in school.
Along the developments in the theoretical and empirical studies of cognition

and meta-cognition, major changes have been suggested also in intervention programs attempting to enhance mathematics reasoning via metacognitive guidance. The first intervention programs were based on meta-memory and the explicit teaching of facts, strategies, and algorithms. Although these methods have many advantages, mainly with regard to the easiness of its implementation in classes with a large number of students, recent studies have started to question its effectiveness. These findings raise three basic research questions: first, how to transform recent meta-cognitive theories into effective instructional methods? Second, who benefits from this kind of innovative instructional methods? And finally, at what age this kind of teaching methods are needed? The present presentation focuses on these issues with regard to mathematics education.

The presentation includes four parts:

(a) Metacognitive Framework - an overview and rationale;

(b) IMPROVE - an effective metacognitive teaching method in which no child left behind;

(c) Results of experimental and quasi-experimental studies showing the impact of IMPROVE on various measurements of mathematics reasoning and meta-cognitive skills of students at different age groups, and

(d) Metacognitive instructional methods - restructuring mathematics education.

The theoretical and practical implications of these studies will be discussed at the conference.

PLÉH, Csaba: Two traditions and two strategies of cognitive science

The talk shall outline a formal and a more content oriented strategy of cognitive science. During the late 19th century these two strategies were first outlined by Wilhelm Wundt and Gottlob Frege as the sensualistic and the propositional theory of thought processes. Frege in this regard treated his propositions as Platonic entities, thus denying their reality in individual minds.

The second half of the twentieth century can be seen as a renewal of Frege where propositions are treated as actual characterizations of human thought process. This lead to the victorious computational theories of modern cognitive science illustrated by names like Noam Chomsky and David Marr.

Not only computers but humans were to be subjected to the Turing test. The last decades of twentieth century however realized that propositions allocated to individual minds must have an origin in themselves too. This has lead to different levels of the Turing test and to present day neural network and evolution anchored theories of cognition.

ZIMMERMANN, Bernd: György Pólya, 1887-1985 – Zur Biographie, zum Lebenswerk und zu seiner Wirkung auf die Mathematikdidaktik

György Pólya gehört zweifellos zu den bedeutendsten Persönlichkeiten, die bis heute einen sehr starken Einfluss auf die internationale Diskussion über Mathematikunterricht haben. Pólyas Weg zur Mathematik war keineswegs gradlinig und zeugt von einem vielseitigem Talent und Engagement. Schon mit Beginn seiner beruflichen Tätigkeit in Mathematik befasste er sich auch mit Fragen des Unterrichtens von Mathematik. Seine Arbeiten in der Mathematik reichen von der Analysis über die Zahlentheorie und Geometrie bis zur Wahrscheinlichkeitsrechnung und Kombinatorik. Auch hierin erkennt man z. T. schon sein Interesse an Methoden des Entdeckens und des Lösens von Problemen. Hierauf konzentrieren sich seine mathematikdidaktischen Arbeiten, insbesondere seine unübertroffenen Werke zum mathematischen Problemlösen. Schließlich werden ein Ausblick auf die heutige Situation des Mathematikunterrichts insbesondere in Deutschland gegeben sowie mögliche oder wünschenswerte Wirkungen der Ideen von Pólya präsentiert.

First ELTE–NUS Science Forum between the Faculty of Science of Eötvös Loránd University and National University of Singapore Budapest, 21–22 May 2008 Mathematics Session

Chengbo Zhu:
Local theta correspondence: introduction and recent results

We give a brief introduction to local theta correspondence, which links representations of certain classical groups. We then explain some of its recent development as well as applications to the theory of invariant distributions.

István Ágoston:
On homological properties of quasi-hereditary algebras

Quasi-hereditary algebras were introduced in the late 1980's to deal with certain problems arising in the representation theory of complex semisimple Lie algebras and algebraic groups. A number of 'universality' results shows their importance also within the class of associative algebras. For example, every finite dimensional associative algebra is the endomorphism algebra of a projective module over a suitable quasi-hereditary algebra. The recursive construction of quasi-hereditary and more generally, of standardly stratified algebras makes it possible to give an explicit bound on the finitistic dimension of standardly stratified algebras. Results about the quasi-heredity of the Koszul dual of a quasi-hereditary algebra were also presented.

Hung Yean Loke: The smallest representation of non-linear covers of odd orthogonal groups

In this talk, I will first explain and motivate the definition of small representations of real reductive Lie groups. Then I will describe the construction of the smallest representation of the indefinite orthogonal groups. The latter is a joint work with Gordan Savin.

László Verhóczki: Cohomogenenity one isometric actions on compact symmetric spaces of type E_6/K

As is well-known, the exceptional compact Lie group E_6 has four symmetric subgroups up to isomorphisms. In this talk we discuss the four Riemannian symmetric spaces of type E_6/K and show that three of them admit a cohomogeneity one isometric action with a totally geodesic singular orbit. This implies that these symmetric spaces can be thought of as compact tubes. We describe the shape operators and the volumes of the principal orbits of the considered isometric actions. Hence, we obtain a simple method to compute the volumes of these exceptional symmetric spaces.

Zoltán Buczolich:
Pointwise convergence and divergence of ergodic averages

We discuss almost everywhere convergence results concerning the non-conventional ergodic averages

(*) $\frac{1}{N}\sum_{k=1}^{N} f(T^{n_k}x)$ as $N \to \infty$. Motivated by questions of A. Bellow and H. Furstenberg, J. Bourgain showed that for the sequence $n_k = k^2$ for $f \in L^p$, $p > 1$ the averages (*) converge μ almost everywhere and he raised the question of almost everywhere convergence of (*) for $f \in L^1$. In a joint paper with D. Mauldin we showed that there are $f \in L^1$ for which the averages (*) along the squares do not converge μ almost everywhere. Answering another related well-known problem I have managed to construct a sequence n_k with gaps $n_{k+1} - n_k \to \infty$ for which for any ergodic dynamical system (X, Σ, μ, T) and $f \in L^1(\mu)$ the averages (*) converge μ almost everywhere to the integral of f and this result disproves a conjecture of J. Rosenblatt and M. Wierdl.

Péter Komjáth: Paradoxical decompositions of Euclidean spaces

Ferenc Izsák: Error estimations in the numerical solutions of the Maxwell equations

An a posteriori error estimation technique was presented for the (finite element) numerical solution of the time harmonic Maxwell equations. One can prove that the estimate is a lower bound of the exact error and we exhibited a strong correlation with this as shown in some numerical experiments.

Kwok Pui Choi:
Asymptotics of the average of functions of order stateistics

László Márkus:
On the extremes of nonlinear time series models describing river flows

Jialiang Li, Xiao-Hua Andrew Zhou: Nonparametric and semiparametric estimations of the three way receiver operating characteristic surface

Tamás Király: An introduction to iterative relaxation

Iterative relaxation is a new method for obtaining approximate solutions to combinatorial optimization problems with degree constraints, when the corresponding feasibility problem is already NP-complete. The method allows a slight violation of the degree constraints, and finds a solution of this relaxation that has small cost. A prime example of this approach is the Minimum Bounded Degree Spanning Tree problem, where we have upper (and possibly lower) bounds on the degree of the spanning tree at each node. Singh and Lau showed that if the value of the optimal solution is OPT, then an iterative relaxation algorithm can find a spanning tree of cost at most OPT that violates the degree bounds by at most 1. In this talk we show how this technique can be extended to problems involving arbitrary matroids.

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SOME ASPECTS OF GEOMETRY THROUGH THE SCHOOL YEARS

By

RICHARD ASKEY

(Received November 30, 2008)

Abstract. We will focus on some aspects of triangles and some quadrilaterals which are both of interest for their own sake and because of the ideas which are involved. We start with early primary school and get to some results which in the United States have disappeared from school geometry and are not known by most high school teachers.

After students learn what a triangle is, a very important property should be introduced: a triangle is rigid. This can be illustrated with fingers, and contrasted with the fact that a quadrilateral is not rigid. By this I mean that when the sides are given, a triangle is determined, but a quadrilateral is not. Children like to show this with their fingers.

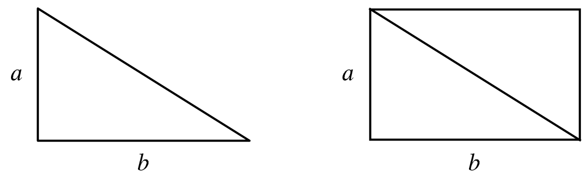
After students learn that a triangle is determined by knowing its sides, by knowing two sides and the included angle, and by knowing two angles and the included side, it is time to show that the sum of the angles of a triangle is two right angles, or 180°. First, students should learn how to form a right angle by folding paper, and then continue folding to make a rectangle so they develop a feel for right angles and learn that a rectangle has four right angles, and its angles add to 360°.

One way to study general triangles is to start with right triangles.

For a right triangle,

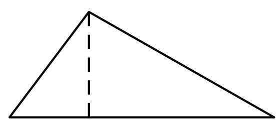
one can use it to form a rectangle by drawing lines perpendicular to the shorter sides of the triangle.

Plenary talk of Tamás Varga Conference on Mathematics Education held at ELTE, Budapest, November 7–8, 2008.



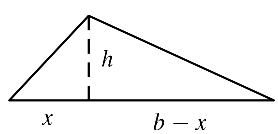
The two triangles are congruent since they have sides of the same length, so we know that they have the same area and the sum of the angles is the same. Since the area of the rectangle is ab, and the sum of the angles is 360°, the original right triangle has area ab/2 and the sum of its angles is 180°.

Take a general triangle, first with the longest side used as the base, and decompose it into two right triangles by dropping a perpendicular to the longest side from the opposite vertex.



The previous result for right triangles gives $180^\circ + 180^\circ$ for the two right triangles. Subtract 180° for the two right angles to get 180° for the sum of the angles in the triangle.

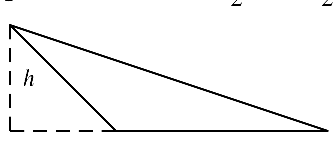
For the area, a different argument completes the derivation of a formula for the area of a triangle.



The area of the triangle is $\frac{xh}{2} + \frac{(b-x)h}{2} = \frac{bh}{2}$.

There is another case where a similar argument works.

The area of the triangle comes from $\frac{(x+b)h}{2} - \frac{xh}{2} = \frac{bh}{2}$.



There are many ways to prove the Pythagorean theorem. Here is the URL to a beautiful way:

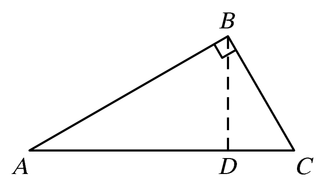
http://math.berkeley.edu/~giventh/papers/eu.pdf

This article was written for mathematicians, but if you skip the parts which use words you do not know, the essence is still there.

Here is a sketch.

The Pythagorean theorem says that the area of the square drawn on the hypotenuse is equal to the sum of the areas of squares drawn on the other sides of a right triangle. Squares are similar, and a little thought along with the knowledge that areas of similar figures scale as the square of the corresponding sides will be used.

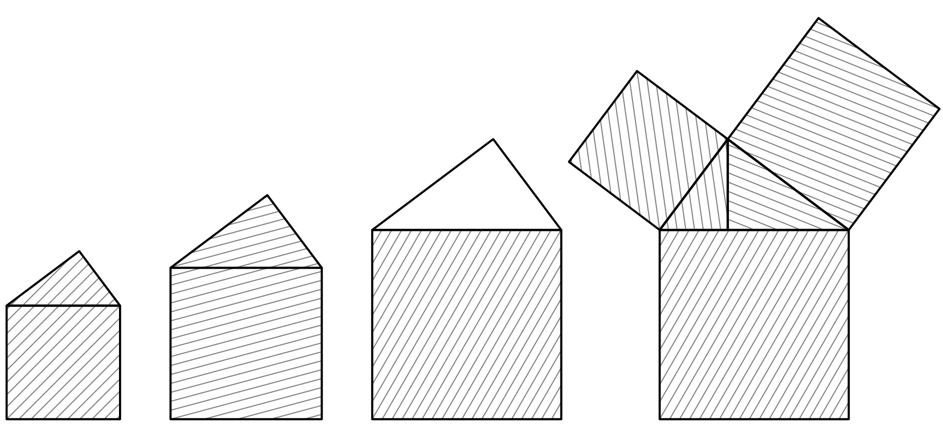
This is true in general when side is interpreted appropriately, but we just use this for triangles and squares so side means what you think it does). Thus it suffices to find similar figures placed on the sides of a right triangle so that their areas add as they should. Here is the picture.



Triangle ABC is a right triangle and it is similar to triangles ADB and BDC. Clearly the area of triangle ADB plus the area of triangle BDC is the area of triangle ABC.

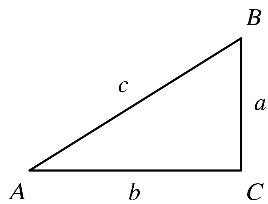
The area of triangle ABC is a constant k times the area of the square on AC, and this is true with the same constant k for the other two triangles and the corresponding squares of AB and BC.

Thus $k|AC|^2 = k|AB|^2 + k|BC|^2$ and this is the Pythagorean theorem.



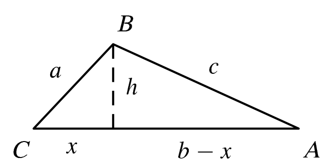
With some knowledge of similarity one can do much more, including introducing trigonometric functions.

$$\sin(A) = \frac{a}{c} \qquad\qquad \cos(A) = \frac{b}{c}$$



There is an extension of the Pythagorean theorem due to Euclid. It holds for a general triangle, but following Euclid we only give a proof in one of the two cases and leave the other one to you.

Drop a perpendicular from B to AC and use the



Pythagorean theorem twice.

$$a^2 = h^2 + x^2, \qquad c^2 = h^2 + (b - x)^2$$

Subtract and do a little algebra to get Euclid's version:

$$c^2 = a^2 - x^2 + b^2 + x^2 - 2bx, \qquad c^2 = a^2 + b^2 - 2bx$$

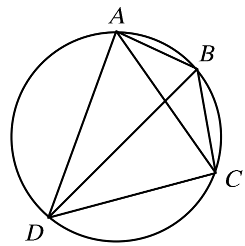
We now rewrite this as $c^2 = a^2 + b^2 - 2ab\cos(C)$.

This is usually called the law of cosines or the cosine rule.

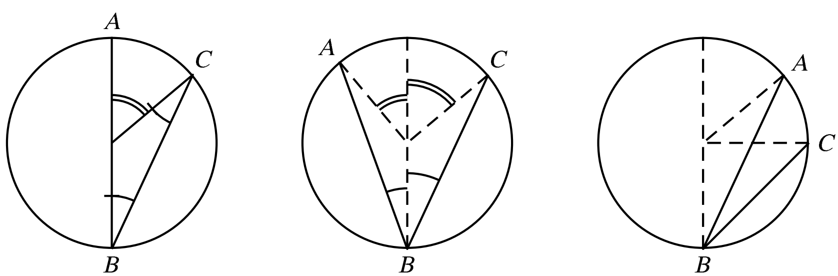
For the Greeks, trigonometry did not deal with right triangles, but with chords in circles. This was because of their serious interest in astronomy. When this was developed enough, they had to construct tables of lengths of chords for different angles. About 150 AD (or CE depending on your preference and possibly age), Ptolemy found a beautiful theorem which allowed him to construct such a table. Rather than give Ptolemy's proof, I will give one which was likely the proof found by the Indian mathematician

Brahmagupta [1] before 630. However, he did not give a proof, he just stated the result.

Given a circle, mark four points A, B, C, D on it and connect them with line segments.



AB, BC, CD, DA are the sides of this quadrilateral and AC and BD are diagonals. Quadrilaterals whose vertices lie on a circle are called cyclic quadrilaterals. Like triangles, they are rigid.



A condition which tells if a quadrilateral is cyclic was given in Euclid, the angles DAC and BCD add to 180°, and since a quadrilateral has 360° in its angles, the other angles ABC and CDA also add to 180°. Here is a proof without words that the inscribed angle ABC is half of the central angle AOC.

Given a cyclic quadrilateral as above, one should be able to find the length of the diagonal AC in terms of the lengths of the sides of the quadrilateral. Here is one way to do this. Use the cosine rule twice. To simplify the typing and reading, we will use $|AB| = a$, $|BC| = b$, $|CD| = c$, $|DA| = d$, $|AC| = x$, $|BD| = y$.

$$\begin{aligned} x^2 &= a^2 + b^2 - 2ab\cos(B) = \\ &= c^2 + d^2 - 2cd\cos(D) = \\ &= c^2 + d^2 + 2cd\cos(B) = \end{aligned}$$

since $B + D = 180°$ and $\cos(180° - B) = -\cos(B)$.

Multiply the first equation by cd, the third by ab and add them together. The result is

$$(ab+cd)x^2 = cd(a^2+b^2) + ab(c^2+d^2).$$

The left side is nice, but the right hand side is not. The mixture of squares and first powers makes this side less attractive, so let us fix it by taking each squared term, split it into two linear factors and put one with each of the other two factors. Thus $cd(a^2)$ becomes $(ac)(ad)$ which looks nicer.

When this is done with each term, the right hand side becomes

$$(ac)(ad) + (bc)(bd) + (ac)(bc) + (ad)(bd)$$

Put the first and third terms together by factoring out $(ac)$and factor (bd) from the other two terms. This gives

$$(ac)[(ad)+(bc)] + (bd)[(ad)+(bc)] = [(ac)+(bd)][(ad)+(bc)]$$

so

$$x^2 = \frac{[ac+bd][ad+bc]}{ab+cd}$$

By symmetry,

$$y^2 = \frac{[ac+bd][ab+cd]}{ad+bc}$$

Multiply these together and take a square root to get

$$xy = ac + bd,$$

which is Ptolemy's theorem. Notice the importance of factoring. Without it, the cancellation which was immediate would have been hard to see, and might have been missed.

Similarly, divide the expressions for x^2 and y^2 to get

$$\frac{x}{y} = \frac{ad+bc}{ab+cd}.$$

There are many things which can be done after getting Ptolemy's theorem. One which is little known is to find other parts of a cyclic quadrilateral. What parts, you ask? Brahmagupta tells us what else can be found.

If you call the intersection of the diagonals P, then the lengths of AP, BP, CP, and DP can be found. He wrote that these can be found by proportion, and they can. He also mentioned the needles. These are formed by extending the sides of the quadrilateral until they meet. He does not seem to mention the third diagonal, the line segment connecting the terminal points of the two needles. That can also be found.

There are two books on trigonometry published in England, one by Hobson [5] and one by Durell and Robson [2], which contain the formula for the length of the third diagonal, but both proofs are more complicated than necessary.

Here is a trigonometric reformulation of Ptolemy's theorem. Take a circle of diameter 1 to simplify formulas. Mark the angles as follows

$$CAB = t, \quad ABD = u, \quad CBD = v, \quad ACB = w.$$

From the inscribed angle result mentioned above, angle CAB is the same size as an angle cutting off the chord BC with the diameter as one side so $\sin t = |BC|$. Doing this for not only the angles above, but all of the angles between diagonals and sides, and sides and sides, Ptolemy's theorem can be given as

$$\sin(v + t)\sin(u + v) = \sin(t)\sin(u) + \sin(v)\sin(w).$$

There seem to be four variables, but there are really only three since $t + u + v + w = 180°$. Using this and $\sin(180° - x) = \sin(x)$, Ptolemy's theorem is equivalent to

$$\sin(v + t)\sin(u + v) = \sin(t)\sin(u) + \sin(v)\sin(t + u + v).$$

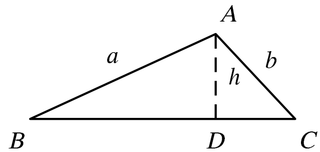
When $u + v = 90°$, which is the same as having a diagonal be a diameter of the circle, this reduces to the well known addition formula

$$\sin(x + y) = \sin(x)\cos(y) + \cos(x)\sin(y).$$

Other choices will give other versions of addition formulas. Thus it is not surprising that Ptolemy could use his theorem to compute the lengths of chords in a circle when an inscribed or central angle is given.

Ptolemy's theorem contains the addition formulas, so it is natural to ask if the addition formulas can be used to prove Ptolemy's theorem. When I taught a course which was used primarily to help prospective high school teachers learn the mathematics they would teach, I would give this problem after deriving the addition formulas. Here is the way I would derive the addition formula for $\sin(x + y)$.

Draw the same picture we have used so often, this time for a general triangle.



Twice the area of triangle BAC is $ab\sin(BAC)$ and is also

$$ah\sin(BAD) + bh\sin(CAD).$$

Replace h by $b\cos(CAD)$ and by $a\cos(BAD)$.

$$ab\sin(BAC)ab\sin(BAD)\cos(CAD) + ab\cos(BAD)\sin(CAD)$$

The result is the addition formula for $\sin(x + y)$ which was given above. It is amazing how many important results can be obtained from this simple picture.

When students tried to use the addition formulas to prove the trigonometric identity given above as equivalent to Ptolemy's theorem, they would all start by expanding $\sin(t + u + v)$ and a few of them were able to get the result after about three pages of calculations. However, there is a much easier way.

$$\cos(x - y) = \cos(x)\cos(y) + \sin(x)\sin(y)$$
$$\cos(x + y) = \cos(x)\cos(y) - \sin(x)\sin(y)$$

Subtract the second from the first to get

$$2\sin(x)\sin(y) = \cos(x - y) - \cos(x + y).$$

Use this in the three terms in the identity you want to prove and the next line completes the proof.

I told students that there are some things in mathematics which are of primary importance, and these need to be known backwards and forwards and in disguised forms. The addition formulas for trigonometric functions are examples. Then there are results of secondary importance. These you do not have to have in your working memory, but you have to know that something like this is true and be able to derive it easily from the results of primary importance. The linearization result above is such an example. Then there are results of lesser importance. These you might find easier to look up than to derive.

I recommend reading and working through Ptolemy's proof. You can find it on the web.

Here is one URL: http://www.cut-the-knot.org/proofs/ptolemy.shtml

This proof is also included in a great geometry text with a very interesting publication history. The most recent version is Kiselev's Geometry which was translated and edited by Alexander Givental [4]. He set up his own publishing firm in California with the most interesting name: Sumizdat. I highly recommend the book.

Here is an outline of one more proof of Ptolemy's theorem.

Take a circle with radius 1 and center at $(0, 0)$. Then take a line with slope t which passes through the point $(-1, 0)$. Find the point where these two curves intersect. The coordinates of this point are $\left(\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2}\right)$. The other point of intersection is $(-1, 0)$, which students find surprising until they think a little and see why this has to be true. We are interested in the first of these two points.

Do the same thing with a line of slope s. The distance between these two points is

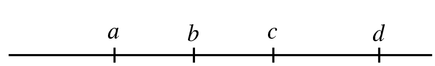
$$\frac{2|t-s|}{\left(1+t^2\right)^{1/2} \cdot \left(1+s^2\right)^{1/2}}.$$

Now compute the distances between each of the four points. There are six of them, since the lines connecting each pair of points is either a side or a diagonal of a cyclic quadrilateral.

Does this give a proof of Ptolemy's theorem? Of course, since I would not be writing about it if that did not happen. The question is what is needed to show that this gives such a proof. The answer is surprising.

Consider a very large circle and draw a tangent line to it. Place four points on the circle so that they are relatively close to the point of tangency. Then let the radius increase to infinity and have the points approach the tangent line. Here is what Ptolemy's theorem becomes on the tangent line.

$$(c-a)(d-b) = (d-c)(b-a) + (d-a)(c-b)$$



This identity is easy to prove. The right hand side can be expanded to give

$$db - da - cb + ca + dc - db - ac + ab$$

and some terms cancel to give $dc - da - cb + ab$ which factors as $(c-a)(d-b)$.

The distance between two points on the circle given above when used for all six distances reduces to the identity just proven since the same square roots occur for each product and the absolute values are removed by the order of the slopes.

There is another use of the representation of the interesting point of intersection of the line and the circle. Since this point is on the unit circle, it satisfies $x^2 + y^2 = 1$ where (x, y) is the point $\left(\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2}\right)$ in question. Take the values of x and y given above in terms of t and let $t = p/q$. A little algebra gives

$$(q^2 - p^2)^2 + (2pq)^2 = (q^2 + p^2)^2.$$

When $q = 2$, $p = 1$, this gives $3^2 + 4^2 = 5^2$; $q = 3$, $p = 2$ gives $5^2 + 12^2 = 13^2$, etc.

Of course there is a lot more. There is an extension of Ptolemy's theorem to a general quadrilateral. Most of the references to extensions are weak extensions; the equality becomes an inequality when the quadrilateral is not cyclic. However, there is an identity with a new term.

One nice way to derive it is to copy Ptolemy's proof as far as you can, and then complete the proof with the cosine rule as used in the first proof above. See Hobson [5] for details.

Here is a nice problem which is easy with Ptolemy's theorem. Inscribe an equilateral triangle in a circle. Take another point on the circle and connect it to the three vertices of the triangle. Show that the sum of the two shorter line segments is equal to the length of the longest segment.

There are other comments on Ptolemy's theorem [2].

ACKNOWLEDGMENTS AND THANKS. I thank Katalin Fried for help in preparing both the material for the talk on which this paper is based and for help with the manuscript, both the technical aspects and suggestions. Also, I thank the organizers of this meeting for the invitation. It is always a pleasure to visit Budapest, to return to the place where some beautiful mathematics was done which had a major influence on my mathematical work.

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Péter Maga: Generalized number theoretic functions

We define the number theoretic functions over an arbitrary unique factorization domain and study some important functions. We show that the number theoretic functions form a ring that is isomorphic to a certain power series ring. Finally, we prove that unique factorization holds in the ring of number theoretic functions.

Roland Paulin: Stability constants

Dániel Soukup: Planar topologies using the idea of the Sorgenfrey line

My primary goal was to generalize the convergence idea of the Sorgenfrey-line to the plane and investigate these new topologies. For every euclidean closed $S \subseteq S^1$ we define an $\mathbb{R}^2_S$ topology. In our spaces a sequence (x_n) converges to x iff (x_n) converges to x from directions which are in S. The paper deals with the characterization of the properties of $\mathbb{R}^2_S$ with the defining $S \subseteq S^1$ subsets. We examine such properties as countability, separability, compact subspaces and connectedness. Several interesting questions remained unanswered, such as: how many different topologies we get this way?

Péter Maga: Covers and dimension in infinite profinite groups

Answering a question of Miklós Abért we prove that an infinite profinite group cannot be the union of less than continuum many translates of a compact subset of box dimension less than 1. Furthermore, we show that in any infinite profinite group there exists a compact subset C of Hausdorff dimension 0 such that it is consistent with the axioms of set theory that less than continuum many translates of C cover the group.

Dániel Dobos: Sylow p-subgroups never intersect in a chainlike way

Call a finite group G *p-chainlike* if $\left|Syl_p(G)\right| \geq 3$ and the following two conditions hold for its Sylow p-subgroups $P_1, \ldots, P_\ell$:

$$\left|P_i \cap P_j\right| > 1 \Leftrightarrow i \equiv j \pm 1 \pmod{\ell} \quad \text{and} \quad O_p(G) = \bigcap_{i=1}^{\ell} P_i = e$$

(a suitable labelling is chosen).
We prove that actually p-chainlike groups *do not exist*.

György Hermann: Small lattice simplices in dimension d

Tamás Hubai: Competitive rectangle filling

Two players take alternating turns filling an $n \times m$ rectangular board with unit squares. Each square has to be aligned parallel to the board edges, but may otherwise be arbitrary. In particular, they are not forced to have integer coordinates. Squares may not overlap and the game ends when there is no space for the next one.

The result of the game is the area filled, or equivalently, the number of turns in the game. The *constructor* aims to maximize this quantity while the *destructor* wants to minimize it. We would like to determine this value, at least asymptotically, provided that both players use their optimal strategy.

For the case $n = 1$, which corresponds to the one dimensional variant of the problem, we show that about $\frac{3}{4}$ of the interval can be filled, which is exact if it evaluates to an integer. With a different and more complicated approach we are also able to determine the exact value for the case $n = 2$ (and large m), where we obtain that $\frac{9}{16}$ of the area gets covered. This result coincides with our conjecture for the general case when n and m are even or tend to infinity.

We also prove tight bounds for arbitrary n and m by specifying actual strategies for both players. Finally we look at some multi-dimensional and discrete variants, making some observations that lead to a conjecture for the product of such filling games.

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Fried Katalin PhD

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