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TOMUS XLIX.

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TESSELLATION-LIKE ROD-JOINT FRAMEWORKS

By

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(Received July 13, 2000)

1. Introduction

One of the simplest structures in statics are the tessellation-like rod-joint frameworks including grid frameworks.

Consider a rod-joint framework in the form of a rectangular grid. This framework is not rigid. How can one add diagonal braces to some of the squares in such a way as to make the framework rigid? This problem was solved generally by Maxwell [5], but in his result the time complexity of deciding the rigidity is $O(n^3)$ where n is the number of the joints. We are interested in an algorithm with time complexity $O(n)$ or less. Such a result for grid bracing problem was given by Bolker and Crapo [1, 2] and also by Gáspár, Radics, and Recski [4].

In this paper we give a new proof of Bolker–Crapo’s theorem and we consider the eight semiregular (Archimedean) tessellations, and the special hexagon bracing problem in the plane. The word “special” means that we assume the opposite edges of the regular polygon in the tessellation remain parallel during any motion of the vertices. These results are useful from an algorithmic point of view, for instance the number of the joints is $O(n^2)$ in case of the $n \times n$ grid and the size of the auxiliary graph of the framework, and hence the time complexity of the proposed algorithm is $O(n^2)$ while according to Maxwell the time complexity would be $O(n^6)$ if we use Gaussian elimination for deciding the rank of the rigidity matrix.

2. Tessellation-like rod-joint framework

2.1. c-graph

Define the **c**-graph (corresponding graph) of the framework F as follows: the vertices of the **c**-graph correspond to the joints of the framework F and there is an edge between two points of **c** if and only if there is a rod between the corresponding two joints of the framework. Consider a rod-joint framework in one dimension (on a line or on an arc).

LEMMA 2.1. *A framework is rigid in one dimension if and only if its **c**-graph is connected.*

PROOF. The connectivity of the **c**-graph means we can get from every point to every other point along edges of the graph, that means, the joints of the framework can move together to the same direction with the same velocity. If the **c**-graph is not connected then the frameworks corresponding to its components can move independently of each other. ■

2.2. Square tessellation

Consider the square tessellation on the plane with unit edges. Its squares are in rows and in columns.

A finite set of unit squares from the tessellation is semi convex if every line parallel with the row or the column intersects the set in an interval.

Let us correspond joints to the vertices of the semi convex tessellation and correspond rods to the sides of the semi convex tessellation. Hence we get to the square tessellation-like rod-joint framework.

Inserting braces in the diagonals of some squares we want to make the square grid rigid. We can see a square tessellation-like rod-joint framework with some braces in Fig. 1. The horizontal rods of the i -th column are parallel with each other during any motion of the joints so they can be denoted by a vector x_i . Similarly, the vertical rods of the j -th row are parallel with each other during any motion of the joints so they can be denoted by a vector y_j .

Thus we can describe the move of the square tessellation-like rod-joint framework with some vectors disregarding the translation of the framework. These vectors form an auxiliary gadget. The vectors of the auxiliary gadget can rotate independently of each other around the origin if there is no diagonal brace in the framework. We can see the auxiliary gadget of the framework of

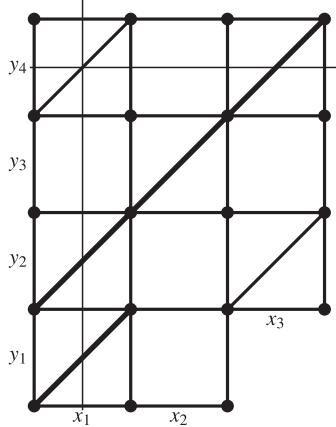


Fig. 1.

Fig. 1 on top of the right hand side in Fig. 2. We translate the vectors to top of the right direction because of the visibility.

The auxiliary graph of the braced square tessellation-like rod-joint framework is a bipartite graph. The point x_i in the first point class corresponds to vector x_i and the point y_j in the second point class corresponds to y_j , and an edge $x_i y_j$ exists if and only if there is a diagonal brace in the square determined by the i -th column and the j -th row. In this case the vector x_i is perpendicular to vector y_j in the auxiliary gadget.

If every square of the square tessellation-like rod-joint framework remains square during any motions of the joints then the square tessellation-like rod-joint framework is rigid in the plane. In this case every vector x_i is perpendicular to every vector y_j in the auxiliary gadget.

THEOREM 2.2. *The square tessellation-like rod-joint framework with some diagonal braces is rigid if and only if its auxiliary graph is connected.*

PROOF. The head of the vectors in the auxiliary gadget are on a unit circle and some of them are of distance $\frac{\pi}{2}$ from each other. These vectors are originally perpendicular in the gadget. Let us construct a framework on the circle. Its joints are the head of the vectors and its rods exist if there is a diagonal brace in the corresponding square, that means these two vectors must be perpendicular to each other. The square tessellation-like rod-joint framework is rigid in the plane if and only if the former framework is rigid on the circle. This framework lies on a one dimensional circular arc. Using Lemma 2.1 this framework is rigid if and only if its e-graph is connected.

But this **c**-graph is isomorphic to the auxiliary graph, because their points correspond to the columns and to the rows of the square tessellation-like framework, and their edges correspond to the diagonal braces. If the bracing graph of the square grid framework is not connected then the square grid framework is not rigid, see the graph in the bottom of the right hand side on Fig. 2, which is the bracing graph of the square grid framework on Fig. 1.

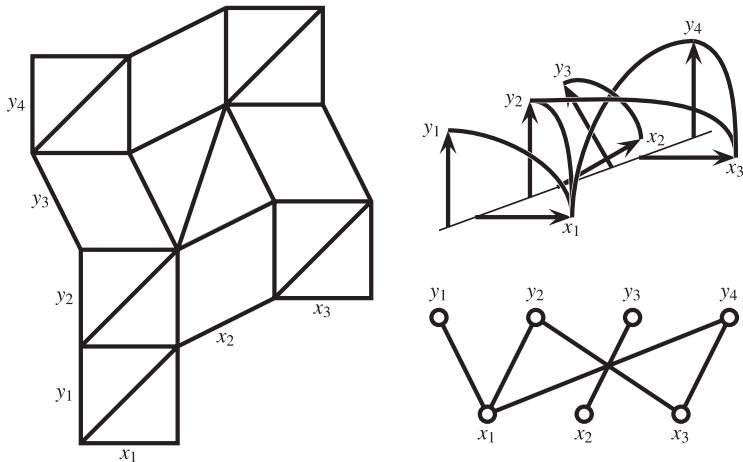


Fig. 2.

We can see a possible motion of the joints on Fig. 2. Since edge x_2y_3 is an independent component of the bracing graph, the corresponding motion of the gadget can be seen at the top of the right hand side of Fig. 2. The motion of the framework (the joints and the rods) is shown on the left hand side of Fig. 2. ■

This result can be generalized in several directions.

COROLLARY 2.1. *Theorem 2.1 is true for non degenerated parallelogram tessellations (this consists of parallelograms and its **c** graph is isomorphic to the **c** graph of the square tessellation-like rod-joint framework) instead of square tessellation, but not true for degenerated parallelogram tessellations.*

COROLLARY 2.2. *We regard the regular triangle tessellation as a non-degenerated parallelogram tessellation with each diagonal brace. Hence the regular triangle tessellation-like rod-joint framework is rigid.*

The result of Theorem 2.1 is useful from an algorithmic point of view.

COROLLARY 2.3. *The number of the joints is $O(mn)$ in case of the $m \times n$ square tessellation and the size of the auxiliary graph of the framework, and hence the time complexity of the proposed algorithm is $O(mn)$, because the time complexity of checking connectivity is $O(N)$, where N is the number of the vertices and edges in the graph, while according to Maxwell the time complexity would be $O((mn)^3)$.*

3. Hexagon tessellation and semiregular tessellations

While studying the rigidity of the hexagon tessellation Gábor Fejes Tóth suggested to consider the rigidity of all the semiregular (Archimedean) tessellations.

The semiregular (Archimedean) tessellations have incongruent regular polygons and equivalent surrounding of the vertices. Let us denote such a tessellation by a symbol giving the number of sides of the polygons surrounding a vertex in their proper cyclic order. There are eight semiregular (Archimedean) tessellation in the plane as follows $(3,12,12)$, $(4,8,8)$, $(4,6,12)$, $(3,4,6,4)$, $(3,3,3,4,4)$, $(3,3,4,3,4)$, $(3,6,3,6)$, $(3,3,3,3,6)$ [3].

3.1. The semiregular (Archimedean) tessellation-like rod-joint frameworks

The polygons of a semiregular (Archimedean) tessellation are in rows. In case of the square tessellation the columns will also be called rows. Fig. 1 shows four horizontal rows and three vertical rows. Among each set of parallel rows, one will be denoted by thin line in the figure of the tessellation.

A finite set of polygons from the tessellation is “row semi convex” if every line “parallel” with the rows intersects the set of the polygon in an interval.

Consider a “row semi convex” tessellation. Let us correspond joints to the vertices of the polygons and rods to the sides of the polygons. Hence we obtain a semiregular (Archimedean) tessellation-like rod-joint framework.

3.2. The $(3,3,3,3,6)$ -, $(3,3,3,4,4)$ - and the $(3,3,4,3,4)$ tessellation-like rod-joint frameworks

3.2.1. $(3,3,3,3,6)$ tessellation

Firstly we shall consider the $(3,3,3,3,6)$ tessellation with unit edges.

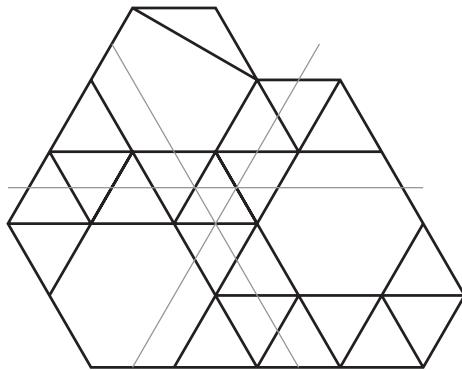


Fig. 3.

It is easy to see that the $(3,3,3,3,6)$ tessellation-like rod-joint framework is almost rigid, because there are many rigid triangles between the polygons of the tessellation (Fig. 3). Some hexagons on the boundary of the framework could be not rigid. These hexagons must be made rigid with short diagonal braces.

3.2.2. $(3,3,3,4,4)$ - and the $(3,3,4,3,4)$ tessellation

The rigidity of the $(3,3,3,4,4)$ - and the $(3,3,4,3,4)$ tessellation-like rod-joint frameworks is implied by Corollary 2.1. In Figures 4 and 5 we denote the rows by thin lines.

We can see the auxiliary gadget at the top of the right and the auxiliary graph at the bottom of the right. In the auxiliary graph some edges are originally included because the common edges of the neighboring triangles form diagonal braces in the parallelogram tessellation.

In the auxiliary graphs there will be new edges (denoted by thick line) if we put some diagonal braces into the square of the $(3,3,3,4,4)$ - and the $(3,3,4,3,4)$ tessellation-like rod-joint framework.

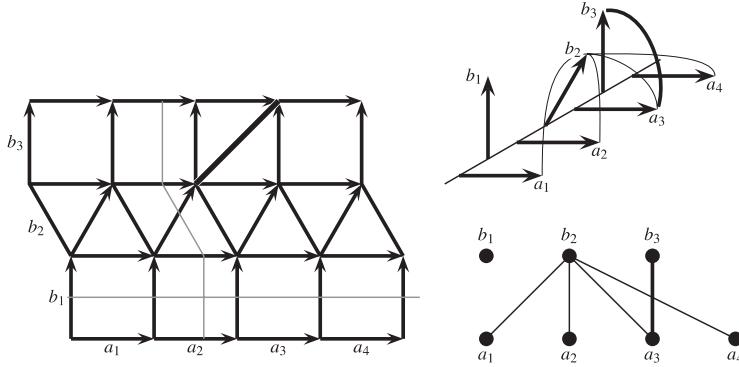


Fig. 4.

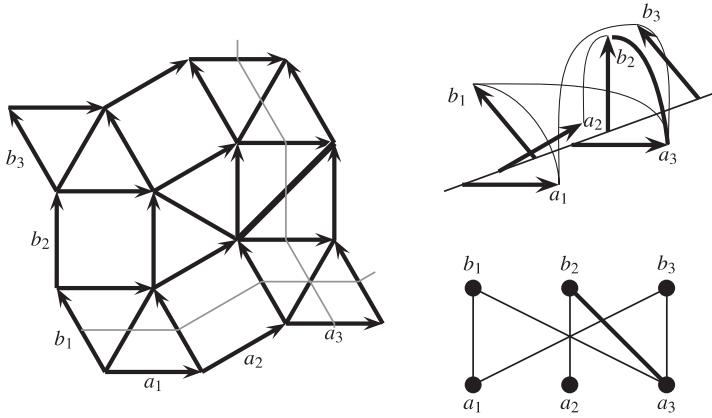


Fig. 5.

THEOREM 3.1. *The $(3,3,3,4,4)$ - and the $(3,3,4,3,4)$ tessellation-like rod-joint frameworks are rigid if and only if their auxiliary graphs are connected.*

PROOF. The statement of the theorem is implied by Corollary 2.1. ■

COROLLARY 3.1. *In case of the $(3,3,3,4,4)$ framework we have to insert into every square row one diagonal brace, this result is obvious. In case of the $(3,3,4,3,4)$ framework only one diagonal brace is enough to connect the auxiliary graph.*

3.3. The special tessellation frameworks

3.3.1. The special (4,6,12) framework

Let us consider the rest of the special semiregular (Archimedean) tessellation, and the special hexagon tessellation bracing problem in the plane. The word “special” means that we assume the opposite edges of the regular polygon in the tessellation remain parallel during any motion of the vertices. The special assumption is unnecessary in the former cases, but it is necessary for the rest of the special semiregular tessellations.

We can construct this kind of “special” framework using some new rods or joints. In case of the hexagon we add a new joint into the center of the hexagon, and three new rods from the center joint to every second one of the hexagon joints.

To make these tessellations rigid we use bracing elements along the shortest diagonals of the polygons because the two neighboring sides and the shortest diagonals form triangles. Hence the two neighboring sides cannot rotate around the common joint.

The rigidity problems of these tessellations are similar to each other, hence for simplicity we shall restrict our consideration to one of the former frameworks, for example the (4,6,12) framework. The drawing of the framework and auxiliary graph of the (4,6,12) tessellation framework are shown in Fig. 6.

Given a (4,6,12) tessellation with unit edges. Its polygons are in parallel rows that are in six kinds of different direction. The six directions are denoted by thin lines.

The special assumption implies the next statement. The rods of a row that are perpendicular to the direction of the row are parallel with each other during any motion of the joints so they can be denoted by a vector. Thus we can describe the move of the special (4,6,12) tessellation-like rod-joint framework with some vectors disregarding the translation of the framework only. These vectors form an auxiliary gadget. The vectors of the auxiliary gadget can rotate independently each other around the origin if there is no diagonal brace in the framework. The auxiliary graph of the braced special (4,6,12) tessellation-like rod-joint framework is a sixpartite graph. The point a_i in the first point class corresponds to vector a_i and the point b_j in the second point class corresponds to b_j , and an edge $a_i b_j$ exists if and only if there is a short diagonal brace between the a_i -th row and b_j -th row. In this case the head of vector a_i is braced to vector b_j in the auxiliary gadget. The

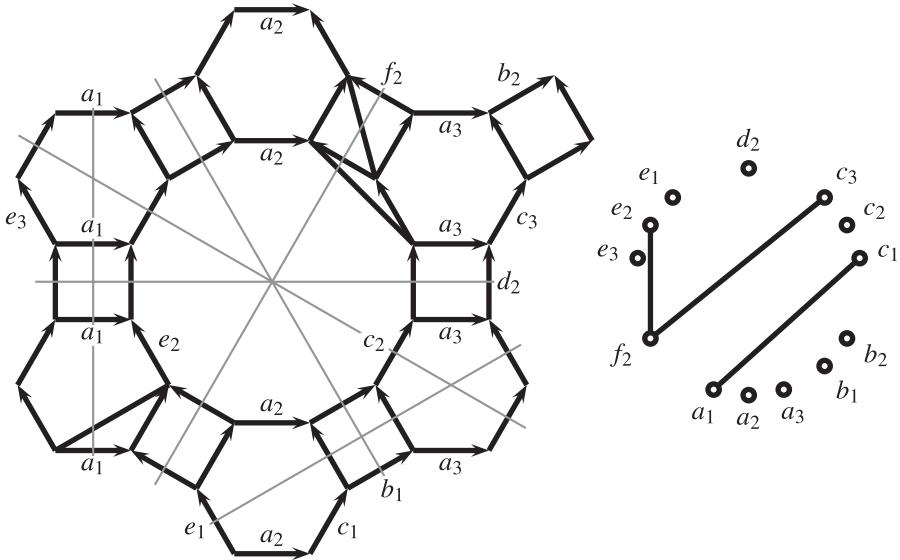


Fig. 6.

auxiliary gadget is rigid if and only if the special (4,6,12) tessellation-like rod-joint framework is rigid in the plane.

THEOREM 3.2. *The special regular hexagon-, the (4,8,8)- and the (4,6,12) tessellation-like rod-joint framework with some diagonal braces is rigid if and only if its auxiliary graph is connected.*

PROOF. This proof is similar to that of Theorem 1.1. The head of the vectors in the auxiliary gadget are on a unit circle. Let a framework be on this circle. Its joints are the heads of the vectors and its rods exist if there is a diagonal brace in the corresponding rows, that means these two vectors can move together. The (4,6,12) tessellation-like rod-joint framework is rigid in the plane if and only if the former framework is rigid on the circle. This framework lies on a one dimensional circular arc. Using Lemma 1.1 this framework is rigid if and only if its \mathbf{c} -graph is connected. But the \mathbf{c} -graph is isomorphic to the auxiliary graph of the framework, because their points correspond to the rows of the framework, and their edges correspond to the diagonal bracing. If the bracing graph of the square grid framework is not connected then the square grid framework is not rigid. ■

3.3.2. The (3,6,3,6), (3,12,12) and (3,4,6,4) special framework

Let us consider the (3,6,3,6), (3,12,12) and (3,4,6,4) special semiregular (Archimedean) tessellation bracing problem in the plane.

The rigidity problems of these tessellations are similar to each other, hence for simplicity we shall restrict our consideration to one of the former frameworks for instance the (3,4,6,4) framework.

The illustration of the framework, auxiliary gadget and auxiliary graph of tessellation (3,4,6,4) framework are shown on Fig. 7. The three different directions of the row are denoted by thin lines.

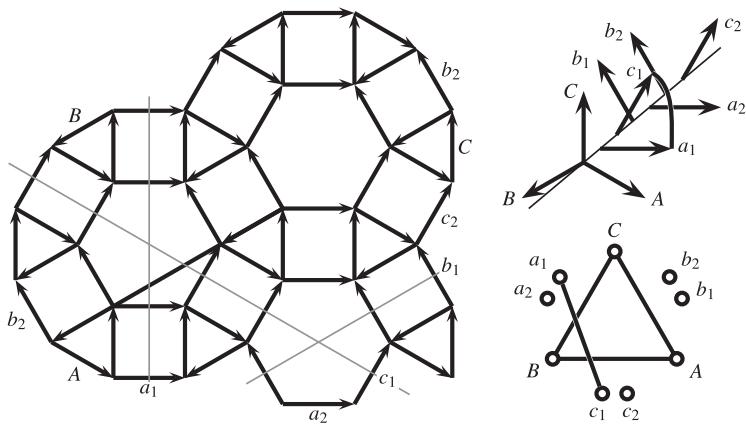


Fig. 7.

We can describe the move of the special (3,4,6,4) tessellation-like rod-joint framework with some vectors disregarding the translation of the framework only. These vectors form the auxiliary gadget of the framework. The vectors of the squares and the hexagons can rotate independently each other around the origin if there is no diagonal brace in the framework, but the vectors of the triangles denoted by capital letter can move together only, because they form a rigid triangle. The auxiliary graph is a sixpartite graph. The point a_i in the first point class corresponds to vector a_i and the points b_j , c_k in the second and the third point classes correspond to vector b_j and c_k respectively, and an edge $a_i - b_j$ exists if and only if there is a short diagonal brace between the a_i -th row and the b_j -th row. In this case the vector a_i is braced to vector b_j in the auxiliary gadget. The vectors of the triangles denoted by A , B and C are braced with each other, hence between the corresponding points of them there are edges in the auxiliary graph. The auxiliary gadget is rigid if

and only if the special (3,4,6,4) tessellation-like rod-joint framework is rigid in the plane.

THEOREM 3.3. *The special (3,6,3,6), (3,12,12) and (3,4,6,4) tessellation-like rod-joint framework with some diagonal braces is rigid if and only if its auxiliary graph is connected.*

PROOF. The proof is similar to that of Theorem 3.2. ■

COROLLARY 3.2. *Theorems 3.1, 3.2 and 3.3 are true for non degenerated special affine regular hexagon tessellation-like rod-joint frameworks and the affine semiregular tessellation-like rod-joint frameworks respectively (these consist of affine regular polygons and their \mathbf{c} graphs are isomorphic to the \mathbf{c} graphs of the special regular hexagon tessellation-like rod-joint frameworks and the semiregular tessellation-like rod-joint frameworks respectively).*

The result of Theorems 3.1, 3.2, 3.3 are useful from an algorithmic point of view:

COROLLARY 3.3. *Let n be the sum of the number of the rows in the different directions. Hence n equals the number of the points of the auxiliary graph of the special tessellation-like rod-joint framework. The time complexity of the rigidity algorithm of the former framework braced with diagonals is linear in the size of the graph, hence $O(n^2)$, because the time complexity of checking connectivity is $O(N)$, where N is the number of the vertices and edges in the graph.*

Acknowledgment

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IMPROVING SIZE-BOUNDS FOR SUBCASES OF SQUARE-SHAPED SWITCHBOX ROUTING*

By

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1. Introduction

Several different models are known for the detailed routing phase of VLSI-design [6], namely, for the case when the position of pins to be interconnected is given already, and the connections must be established using a minimal size cubic grid.

Regarding either the position of the pins, or the routing model of the layers, we can distinguish between different problems and models. Although these problems are under intensive research, the exact number of layers needed is still not known for most cases.

In this paper we focus on routing problems having square-shaped layers. We summarize the earlier results, upper and lower bounds given for the different subcases of the Switchbox Routing problem. Some results are improved to decrease the gap between the bounds.

2. Basic definitions

A *switchbox* is a rectangular grid G consisting of horizontal *tracks* (numbered from 0 to $w + 1$) and vertical *columns* (numbered from 0 to $h + 1$), where w is the *width* and h is the *height* of the switchbox. In this paper, we assume *square-shaped* layers, namely $w = h$. The boundary points of G are called *terminals*. Depending on which boundary of the switchbox they are

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situated on, the terminals are called *Northern*, *Southern*, *Eastern* or *Western*. The “corners” of the switchbox are not regarded as terminals, routings must not use them either.

A *net* is a collection of terminals. A *switchbox routing problem* is a set $\mathcal{N} = \{N_1, \dots, N_n\}$ of pairwise disjoint nets.

The *solution* of a routing problem is a set $\mathcal{H} = \{H_1, \dots, H_n\}$ of pairwise vertex disjoint subgraphs (also called *wires*, these are usually Steiner trees) of the k -layer rectangular grid such that H_i connects the terminals of N_i for $i = 1, \dots, n$. The wires can access the terminals on any layer. Edges of the wires that join adjacent vertices of two consecutive layers are called *vias*.

A specific routing problem can be solved either in the *unconstrained k -layer model*, or in the *Manhattan model*.

In the *unconstrained k -layer model* no further restrictions are applied to layers or to subgraphs H_i . Contrarily, in the multilayer *Manhattan model* consecutive layers must contain wire segments of different directions. Thus, in this model, layers with horizontal (East–West) and with vertical (North–South) wire segments alternate.

According to the position of terminals, several subcases of the switchbox routing problem are considered, as follows:

- Single Row Routing problem, if all terminals are Northern;
- Channel Routing problem, if all terminals are either Southern or Northern;
- Γ -Routing problem, if all terminals are either Northern or Western;
- C-Routing problem, if all terminals are either Southern or Western or Northern;
- General Switchbox problem, if terminals are situated on all four boundaries of the grid.

3. Square-shaped switchbox routing

In this paper, a survey is given on the results concerning the number of layers needed in both models. Table 1 shows the different routing problems and gives the most important references. Our new results on C-shape routing are printed in bold face letters.

Note that the figures in the table show the minimum number of layers that enables to solve *any* routing problem. Hence, some specific problems can be solved using fewer layers than the corresponding universal bound. On the

other hand, obviously, any upper bound in the Manhattan model is an upper bound in the unconstrained model, and any lower bound in the unconstrained model is a lower bound in the Manhattan model.

TABLE 1. Known bounds for the minimum number of layers in different square-shaped routing problems

	Manhattan model	Unconstrained model
	= 2 [2]	= 2 [2, 8, 12]
	= 3 [2, 5]	≥ 2 ≤ 3 [7, 8, 10]
	≥ 3 [1, 12] ≤ 4	≥ 2 ≤ 3 [12]
	≥ 4 ≤ 5 [11]	≥ 3 ≤ 5 [11]
	= 6 [3, 11]	≥ 4 [1] ≤ 6 [11]

Before giving the details of these results, we recall a theorem of Hambrusch [4]. There exists a switchbox routing problem of the $w \times h$ grid which cannot be solved using $k \leq m$ layers, where $m = \max(\frac{h}{w}, \frac{w}{h})$.

This theorem implies the restriction of any further analysis to the special case where m has a bounded value. This paper focuses on the case of $m = 1$, namely $h = w$.

A simple algorithm for Single Row Routing in the 2-layer Manhattan model is based on the fundamental studies of Gallai [2]. Further results on the effect of using more layers are given in [9], while finding the minimum wire-length solution turned out to be NP-hard [12].

Channel Routing problems cannot always be wired on two layers in the Manhattan model, as Fig. 1(a) shows an example. Note that this example can be extended to an arbitrary sized one. However, three layers are always enough in the Manhattan model [2, 5], and even two layers can be enough in the unconstrained model [7, 8, 10].

Wiring a Gamma Routing problem in the Manhattan model can require at least three layers [12], and can be trivially solved on four layers in the

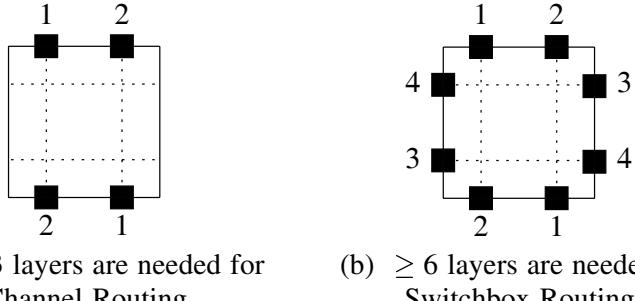


FIG. 1. Examples where lower bounds are sharp in the Manhattan model

Manhattan model. For the sake of completeness we mention here a short proof of this latter claim.

PROPOSITION 3.1. *Every square-shaped Gamma Routing problem can be wired on four layers in the Manhattan model.*

PROOF. Let us have four layers, the first and the third are used for vertical wire segments only, while the second and fourth are used for horizontal segments only.

For all terminals of the Western side introduce a horizontal wire segment alongside the whole height of the grid on the second layer. Similarly, for all terminals of the Northern side introduce a vertical wire segment alongside the whole width of the grid on the third layer.

If a net has at least one terminal on the Western side and at least one on the Northern side, connect their corresponding wire segments with a via hole between the second and third layers.

If a net has terminals only on the Western side, we can use the first layer to connect its corresponding wire segments on the second layer. As at most $\frac{w}{2}$ nets can have this property, the Gallai-algorithm can solve this using a single vertical layer.

Similarly, the fourth layer is sufficient to accomodate all horizontal segments for nets having terminals on the Northern side only. ■

Gamma Routing problems are always solvable on three layers in the unconstrained model. Furthermore, the following theorem can be stated as a straightforward consequence of the method in [12].

THEOREM 3.2. *Every Gamma Routing problem can be solved in the unconstrained model using three layers. Moreover, if there is no net having*

at least two Northern and at least two Western terminals then such a wiring exists that uses the third layer for a single net only.

Szeszlér [11] gave an efficient linear time algorithm to solve the general Switchbox problem in the Manhattan model using at most $2\lceil m \rceil + 4$ layers.

This yields a 6-layer solution for every square-shaped switchbox routing problem. Moreover, the construction includes a 5-layer solution for all C-shaped routing problem where $w = h$.

As the problem of Fig. 1(b) cannot be solved on five layers [3], this bound is sharp. Note that for this case, only a small size example is known, unlike in the case of Gamma Routing, where we have examples of arbitrary size. In the unconstrained model, no such example is known, but a lower bound of four layers is already proved.

As mentioned above, Sziszler's algorithm gives a trivial way to solve C-shaped routing problems on five layers. Our new results give a lower bound of four layers in the Manhattan model and a lower bound of three layers in the unconstrained model as stated in the following theorems:

THEOREM 3.3. *For all $w \geq 3$ there exists a C-shaped routing problem on the $w \times w$ grid which cannot be solved on three layers in the Manhattan model.*

PROOF. Suppose that the routing problem of Fig. 2 can be solved on 3 layers in the Manhattan model. Let l_h , and l_v denote the number of horizontal, and vertical layers, respectively.

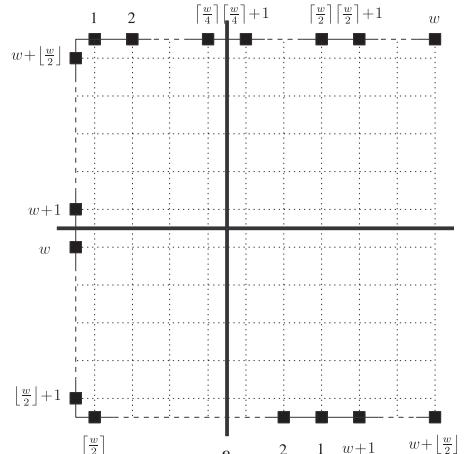


FIG. 2. ≥ 4 layers are needed in the Manhattan model

Consider a vertical cut e and a horizontal cut f , as shown in Fig. 2. Each of them crosses w grid edges on each layer. Thus, wires of at most $w l_h$ nets can cross e , and at most $w l_v$ nets can cross f .

However, in this example, all $\lfloor \frac{3w}{2} \rfloor$ nets must be wired through f , and through e , as well. Note that if $w = 4k + 1$ or $w = 4k + 2$, then wires of net $\lceil \frac{w}{4} \rceil$ may not cross e .

Since

$$w l_v \geq \left\lceil \frac{3w}{2} \right\rceil, \quad \text{and}$$

$$w l_h \geq \begin{cases} \left\lfloor \frac{3w}{2} \right\rfloor - 1, & \text{if } w > 3 \\ 4, & \text{if } w = 3 \end{cases}$$

we obtain $l_h \geq 2$ and $l_v \geq 2$, proving that at least two horizontal and at least two vertical layers are needed for wiring this problem. ■

In the following, we show that C-shaped routing problems cannot always be solved on 2 layers, even in the unconstrained model.

Consider a C-shaped routing problem of size $w \times w$ (Fig. 3), where w is an arbitrary non-negative even integer.

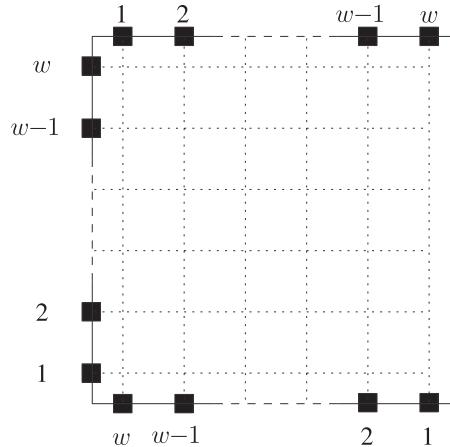


FIG. 3. ≥ 3 layers are needed in the unconstrained model

To prove the unsolvability of this problem on 2 layers we introduce the concepts of SW_k -corner and NW_k -corner.

DEFINITION 3.4. A South-Western corner of size $1 \leq k \leq \frac{w}{2}$ (SW_k -corner) is a set of grid-points containing a point (i, j) if and only if

- $i = 0$ AND $j \leq k$ (Southern part of Western terminals) OR
- $i \leq k$ AND $j = 0$ (Western part of Southern terminals) OR
- $1 \leq i, j \leq k$ AND $i + j \leq k + 1$ (internal points).

DEFINITION 3.5. A North-Western corner of size $1 \leq k \leq \frac{w}{2}$ (NW_k -corner) is a set of grid-points containing a point (i, j) if and only if

- $i = 0$ AND $j > w - k$ (Northern part of Western terminals) OR
- $i \leq k$ AND $j = w + 1$ (Western part of Northern terminals) OR
- $1 \leq i \leq k$ AND $w - k < j \leq w$ AND $j - i \geq w - k$ (internal points).

Note that each of these corners defines a cut in the grid. Fig. 4 shows an SW_4 -corner and a NW_3 -corner of a grid for $w = 8$.

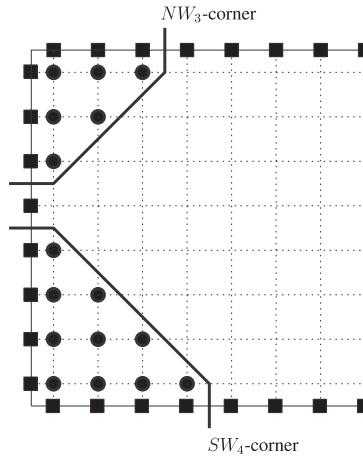


FIG. 4. Example of corners

DEFINITION 3.6. An edge is a *border edge* of the SW_k -corner (NW_k -corner), if it has exactly one end-point in the corner.

A point of the SW_k -corner (NW_k -corner) is a *border point* of the corner, if there is a border edge of the SW_k -corner (NW_k -corner) incident to it.

(Wiring of) a net *leaves* the SW_k -corner (NW_k -corner), if it has at least one terminal inside and at least one terminal outside the corner.

At first, some basic propositions are proved on the SW_k -corners (NW_k -corners) of any *feasible 2-layer solution* of our example. From this point we focus on the properties of such a solution.

PROPOSITION 3.7. *An SW_k -corner (NW_k -corner) is left by exactly $2k$ nets, and has exactly $2k$ border edges (on both layers). Furthermore, each border point of an SW_k -corner (NW_k -corner) is used by the wiring of exactly two nets in any feasible solution.*

PROOF. On one hand, the first part of the proposition is a trivial consequence of the construction of our example. On the other hand, the facts that $2k$ nets must leave the SW_k -corner (NW_k -corner), and that the number of border points is k imply the second part. ■

PROPOSITION 3.8. *Wires leaving terminals k and $w - k + 1$ in the SW_k -corner (NW_k -corner) must not enter the SW_{k-1} -corner (NW_{k-1} -corner).*

PROOF. The SW_{k-1} -corner (NW_{k-1} -corner) must be left by wires of $2(k - 1)$ other nets, which occupy all of its border points, and so wires realizing nets k and $w - k + 1$ cannot even enter the SW_{k-1} -corner (NW_{k-1} -corner). ■

PROPOSITION 3.9. *If a solution exists then on each border edge of the SW_k -corner (NW_k -corner) this solution uses exactly one layer.*

PROOF. At first, consider the edges $(1, k) - (1, k + 1)$ of the SW_k -corner.

As wires of nets k and $k + 1$ must both enter the grid, the only possibility to use both layers on this edge would be to route wires of nets k and $k + 1$ through it. However, as it was shown above, wires of net $k + 1$ cannot enter the SW_k -corner.

The same logic can be applied to the edge $(k, 1) - (k + 1, 1)$ of the SW_k -corner, and edges $(k, w) - (k + 1, w)$, and $(1, w - k) - (1, w - k + 1)$ of the NW_k -corner.

Let us consider the pairs of border edges of the SW_k -corner leaving it from (or to) a common end-point. On these pairs of edges one can route the wires of at most two nets, due to their common point. Hence, it is easy to see that if a solution uses two layers on any of them, then it should use two layers on one of the edges $(1, k) - (1, k + 1)$, or $(k, 1) - (k + 1, 1)$ to enable all nets to leave the SW_k -corner. As this is not allowed, the only way to let all nets leave the SW_k -corner is to use exactly one layer on each edge of it.

Note that the same logic can be applied for the NW_k -corner. ■

As a result, one can see that there is a bijection between the border edges of the $S W_{\frac{w}{2}}$ -corner ($N W_{\frac{w}{2}}$ -corner) and the wires of nets leaving it.

As there is a common border edge of the two corners, namely $(1, \frac{w}{2}) - (1, \frac{w}{2} + 1)$, a single net m must leave here both the $S W_{\frac{w}{2}}$ -corner, and the $N W_{\frac{w}{2}}$ -corner.

Let us define a cut C by taking the union of the points of these corners. All nets except m leave this cut twice, namely both from the $S W_{\frac{w}{2}}$ -corner, and the $N W_{\frac{w}{2}}$ -corner.

Net m should leave cut C in any feasible solution, but there is no room to do this, which contradicts the fact that a feasible 2-layer solution exists.

As the value of w can be arbitrary large in the above example, a C-shaped routing of any size can be unsolvable in the 2-layer unconstrained model.

Note that the same considerations can be taken when w has an odd value. In this case, $S W_{\lceil \frac{w}{2} \rceil}$ -corner ($N W_{\lceil \frac{w}{2} \rceil}$ -corner) is defined by adding point $(1 - \lceil \frac{w}{2} \rceil)$ to the $S W_{\lfloor \frac{w}{2} \rfloor}$ -corner ($N W_{\lfloor \frac{w}{2} \rfloor}$ -corner). The union of the $S W_{\lceil \frac{w}{2} \rceil}$ -corner and the $N W_{\lceil \frac{w}{2} \rceil}$ -corner should be left by the wires of all w nets but it has only $w - 1$ available border edges.

This implies the following theorem:

THEOREM 3.10. *For all $w \geq 3$ there exists a C-shaped routing problem on the $w \times w$ grid which cannot be solved on two layers in the unconstrained model.* ■

4. Summary

Our new results improve lower bounds for the number of layers needed to solve any C-shaped routing problem. Namely, we gave two problem instances that cannot be solved using less than 4 or 3 layers in the Manhattan or in the unconstrained model, respectively. Both can be extended to an arbitrary large size.

However, for 6 problems out of 10 mentioned here (see Table 1) the exact number of needed layers is still not known; this can be a subject of further research.

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SEMI-SEPARATED SETS IN GENERALIZED CLOSURE SPACES

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1. Introduction

In what follows, we denote by $\mathcal{P}(X)$ the power set of a non-empty set X . An arbitrary set-valued set function $u : \mathcal{P}(X) \rightarrow \mathcal{P}(X)$ is called *generalized closure operator* and the pair (X, u) is said to be a *generalized closure space* (briefly, u is a *GCO* and (X, u) is a *GCS*). The dual of the operator u is the *u -interior operator*, $u\text{-Int} : \mathcal{P}(X) \rightarrow \mathcal{P}(X)$ defined by $u\text{-Int}(A) = X \setminus u(X \setminus A)$.

A set $A \subset X$ is said to be *u -closed* or *closed in* (X, u) (respectively, *almost u -closed* or *almost closed in* (X, u)) if $u(A) = A$ (respectively, $u(A) \subset A$). A set $A \subset X$ is said to be *u -open* or *open in* (X, u) (respectively, *almost u -open* or *almost open in* (X, u)) if $A = u\text{-Int}(A)$ (respectively, $A \subset u\text{-Int}(A)$).

Due to its generality, the notion of generalized closure operator, viewed as a link between topology and algebra, has an increasing importance for applications [8].

In a topological space (X, τ) , with the usual closure operator Cl , two subsets A and B are said to be semi-separated if $\text{Cl}(A) \cap B = A \cap \text{Cl}(B) = \emptyset$. A subset of a topological space is connected if and only if it is not the union of two non-empty semi-separated sets.

In the study of various forms of connectedness, the notion of pair of semi-separated sets is an important prerequisite. The notion of pair of γ -separated sets [3], which generalizes and unifies several modifications of the

semi-separation property in topological spaces [10], [15], [13], is a particular case of the notion of pair of semi-separated sets in a GCS. The aim of the present paper is to prove the basic properties of semi-separated sets in generalized closure spaces, under minimal assumptions on the corresponding generalized closure operators. The fact that our assumptions can not be weakened is proven by examples, for almost every result. We give general sufficient conditions for semi-separation, and, in some special cases, necessary conditions for semi-separation. Characterizations in terms of semi-separation of complete graphs and connected graphs, as well as of continuous functions on graphs, are given. We study the invariance of semi-separation property to the relativization to a subspace. We prove that the preimages of two semi-separated sets, under a continuous function between GCS's, are semi-separated. A similar invariance result is proven for images under an almost open injective function between GCS's.

2. Preliminaries

2.1. Generalized closure operators

Let $u, v : \mathcal{P}(X) \rightarrow \mathcal{P}(X)$.

REMARK 2.1. (i) $v = u\text{-Int}$ if and only if $u = v\text{-Int}$.

(ii) A set is u -closed (respectively, almost u -closed) if and only if its complement is u -open (respectively, almost u -open).

(iii) The family of u -closed (respectively, u -open) sets is contained in the family of almost u -closed (respectively, almost u -open) sets. The corresponding families agree if u is expansive.

(iv) X is almost u -closed, the empty set is almost u -open.

DEFINITION 2.1. u is called *isotone* if $A \subset B \subset X$ implies $u(A) \subset u(B)$, *expansive* if $A \subset u(A)$ for every $A \subset X$, *idempotent* if $u(u(A)) = u(A)$ for every $A \subset X$, *sub-linear* if $u(A \cup B) \subset u(A) \cup u(B)$ whenever $A, B \subset X$.

Note that $u : \mathcal{P}(X) \rightarrow \mathcal{P}(X)$ is isotone (respectively, idempotent) if and only if $u\text{-Int}$ is isotone (respectively, idempotent), while u is expansive if and only if $u\text{-Int}(A) \subset A$ for every $A \subset X$.

The properties of isotony and expansivity are independent, as the following examples show.

EXAMPLE 2.1. Let X be a set with at least two elements and $x_0 \in X$. Define $u : \mathcal{P}(X) \rightarrow \mathcal{P}(X)$ by $u(A) = A \setminus \{x_0\}$ for all $A \subset X$. Then u is isotone and $A \subset u(A)$ if and only if $x_0 \notin A$, consequently u is not expansive.

EXAMPLE 2.2. Let $X = \{a, b, c\}$ and $u : \mathcal{P}(X) \rightarrow \mathcal{P}(X)$ be such that $u(A) = A$ for every $A \subset X$, with the following exceptions: $u(\{c\}) = \{a, c\}$ and $u(\{b\}) = \{b, c\}$. Then u is expansive. u is not isotone, since $u(\{c\})$ is not included in $u(\{b, c\})$.

PROPOSITION 2.1. *If $u : \mathcal{P}(X) \rightarrow \mathcal{P}(X)$ is isotone, the following properties are equivalent:*

- (1) u is expansive;
- (2) $u(A) \cap A \neq \emptyset$ for every non-empty subset A of X .

PROOF. (1) \Rightarrow (2): Obviously, even if u is not isotone.

(2) \Rightarrow (1): Assume that u is not expansive. Then there exists $A \subset X$ such that $B := A \setminus u(A) \neq \emptyset$. By isotony, $u(B) \subset u(A)$, hence $u(B) \cap B = \emptyset$, i.e., (2) is false. ■

REMARK 2.2. If u is isotone, then every union of almost u -open sets (every intersection of almost u -closed sets) is almost u -open (respectively, almost u -closed). If u is isotone and expansive, then every union of u -open sets (every intersection of u -closed sets) is u -open (respectively, u -closed). If u is isotone, expansive and idempotent, then the union of all u -open sets contained in an arbitrary set $A \subset X$ is u -Int(A) and the intersection of all u -closed sets containing A is $u(A)$.

EXAMPLE 2.3. If $\mathcal{C} \subset \mathcal{P}(X)$ is a closure system, i.e., a non-empty family closed to arbitrary intersections, one defines $c : \mathcal{P}(X) \rightarrow \mathcal{P}(X)$ by

$$c(A) = \bigcap \{B : B \in \mathcal{C}, A \subset B\}.$$

The set operator c , called the GCO generated by \mathcal{C} , is expansive, isotone and idempotent. The given closure system \mathcal{C} agrees with the family of all c -closed subsets of X . Conversely, if $u : \mathcal{P}(X) \rightarrow \mathcal{P}(X)$ is expansive and isotone, then the family of u -closed sets is a closure system, that generates the GCO u provided that u is, in addition, idempotent.

EXAMPLE 2.4. Let X be a vector space. For every $A \subset X$ let $u(A)$ be the linear hull (respectively, the convex hull) of A . Then the u -closed sets are

exactly the linear subspaces of X (respectively, the convex subsets of X) and, in both cases, u is expansive, isotone and idempotent, but it is not sub-linear.

Next we discuss the connections between some closure operators introduced in [9], [3], [13].

EXAMPLE 2.5. Let m_X be a minimal structure on X , i.e., $\{\emptyset, X\} \subset \subset m_X \subset \mathcal{P}(X)$ [7]. For every subset A of X , the m_X -closure of A and the m_X -interior of A are defined by $m_X\text{-Cl}(A) = \cap\{B : A \subset B, X \setminus B \in m_X\}$ and $m_X\text{-Int}(A) = \cup\{C : C \subset A, C \in m_X\}$, respectively. Then $m_X\text{-Cl}$ is expansive, isotone and idempotent, and its dual $m_X\text{-Int}$ is isotone and idempotent [6], [9]. If $u = m_X\text{-Cl}$, then the family of all open sets in the GCS (X, u) is the family of all unions of sets that belong to m_X [7].

REMARK 2.3. Consider, as in [2] and [3], an isotone operator $\gamma : \mathcal{P}(X) \rightarrow \mathcal{P}(X)$ and the sets $A \subset X$ with $A \subset \gamma(A)$, called γ -open. The family of all γ -open sets is a *generalized topology* [2], i.e., contains the empty set and is closed to arbitrary unions. The complements of all γ -open sets form a closure system, that generates a GCO denoted in [3] by c_γ , which is expansive, isotone and idempotent. A subset A of X is γ -open in the sense of [2], [3] if and only if A is open in the GCS (X, c_γ) . If $\gamma(X) = X$, then the family of all γ -open sets is a minimal structure m_X and $c_\gamma = m_X\text{-Cl}$.

REMARK 2.4. Assume that (X, τ) is a topological space, with the closure operator and the interior operator denoted by Cl and Int , respectively. For $\gamma = \text{Int}$ (respectively, $\gamma = \text{Cl} \circ \text{Int}$, $\gamma = \text{Int} \circ \text{Cl}$, $\gamma = \text{Int} \circ \text{Cl} \circ \text{Int}$, $\gamma = \text{Cl} \circ \text{Int} \circ \text{Cl}$) a set $A \subset X$ is γ -open if and only if A is open (respectively, semi-open [5], preopen [11], α -open [12], β -open [1]).

In [13], a set operator $\alpha : \mathcal{P}(X) \rightarrow \mathcal{P}(X)$ is said to be *associated with the topology* τ if $U \subset \alpha(U)$ for all $U \in \tau$, and a set $A \subset X$ is said to be α -semi open if there is an open set U such that $U \subset A \subset \alpha(U)$. Assume in the following that the set operator α , associated with the topology τ , is isotone. Then the family of all α -semi open sets is a generalized topology containing τ , in particular is a minimal structure, which will be denoted in the following by m_X . The union of all α -semi open sets contained in $A \subset X$ is an α -semi open set, denoted by $\alpha\text{-sInt}(A)$. Note that $\alpha\text{-sInt} = m_X\text{-Int}$. If in Remark 2.3 we take $\gamma = \alpha\text{-sInt}$, then the family of all γ -open sets is m_X .

2.2. Neighborhoods

DEFINITION 2.2 ([14]). Let $u : \mathcal{P}(X) \rightarrow \mathcal{P}(X)$. The *neighborhood function* $\mathcal{N}_u : X \rightarrow \mathcal{P}(\mathcal{P}(X))$ assigns to each $x \in X$ the family $\mathcal{N}_u(x) = \{N \in \mathcal{P}(X) : x \in u\text{-Int}(N)\}$ of its u -neighborhoods. $N \in \mathcal{P}(X)$ is said to be an u -neighborhood of a set A (write $N \in \mathcal{N}_u(A)$) if $N \in \mathcal{N}_u(x)$ for each $x \in A$.

N is a u -neighborhood of A if and only if $A \subset u\text{-Int}(N)$. If no confusion can occur, we will use the term neighborhood instead u -neighborhood.

REMARK 2.5. Each almost u -open set is a neighborhood of itself. If $u : \mathcal{P}(X) \rightarrow \mathcal{P}(X)$ is expansive, then every set which is a neighborhood of itself is u -open; the assumption “ u expansive” cannot be removed from this statement, as the following example shows.

EXAMPLE 2.6. Let $u : \mathcal{P}(X) \rightarrow \mathcal{P}(X)$ be as in Example 2.1 and assume that X has at least two elements. Then $u\text{-Int}(A) = A \cup \{x_0\}$ for every subset A of X , hence every subset is a neighborhood of itself. On the other hand, A is u -open if and only if $x_0 \in A$.

The following characterizations in terms of neighborhoods are known (see [14]). u is isotone if and only if for each $x \in X$ every superset of a neighborhood of x is a neighborhood of x . u is expansive if and only if each neighborhood of an arbitrary $x \in X$ contains x . u is idempotent if and only if, for each $x \in X$, $N \in \mathcal{N}_u(x)$ is equivalent to $u\text{-Int}(N) \in \mathcal{N}_u(x)$.

PROPOSITION 2.2. Let $u : \mathcal{P}(X) \rightarrow \mathcal{P}(X)$ be isotone and expansive. The following properties are equivalent:

- (1) u is idempotent;
- (2) For each $x \in X$, every neighborhood of x contains a u -open neighborhood of x .

PROOF. (1) \Rightarrow (2): Let $x \in X$ and $N \in \mathcal{N}_u(x)$. Since u is idempotent, $u\text{-Int}(N)$ is an u -open neighborhood of x and, since u is expansive, $u\text{-Int}(N) \subset N$.

(2) \Rightarrow (1): Assume that (2) holds and u is not idempotent. Then there exists $A \subset X$ such that $u\text{-Int}(A) \neq u\text{-Int}(u\text{-Int}(A))$. Since u is expansive, $u\text{-Int}(u\text{-Int}(A)) \subset u\text{-Int}(A)$. Let $x \in u\text{-Int}(A) \setminus u\text{-Int}(u\text{-Int}(A))$. We have $A \in \mathcal{N}_u(x)$. By (2), there exists an u -open set U such that $x \in U \subset A$. Since u is isotone, $U = u\text{-Int}(u\text{-Int}(U)) \subset u\text{-Int}(u\text{-Int}(A))$, hence $x \in u\text{-Int}(u\text{-Int}(A))$, a contradiction. ■

REMARK 2.6. To prove the implication $(1) \Rightarrow (2)$ it suffices to assume that u is expansive.

PROPOSITION 2.3. *Let $u : \mathcal{P}(X) \rightarrow \mathcal{P}(X)$, $A \subset X$ and $x \in X$. Consider the following properties:*

- (1) $x \in u(A)$;
- (2) $V \cap A \neq \emptyset$ for every u -neighborhood V of x .

Then $(2) \Rightarrow (1)$. The converse holds provided that u is isotone.

PROOF. Note that (1) is false if and only if $X \setminus A \in \mathcal{N}_u(x)$. If (1) is false, then taking $V := X \setminus A$ we see that (2) is false. Suppose that u is isotone. If we assume that $V \cap A = \emptyset$ for some u -neighborhood V of x , we get $x \in u\text{-Int}(V) \subset u\text{-Int}(X \setminus A)$, hence $X \setminus A \in \mathcal{N}_u(x)$. ■

If u is not isotone, implication $(1) \Rightarrow (2)$ from Proposition 2.3 does not hold in general, as the following example shows.

EXAMPLE 2.7. Let (X, u) be the GCS from Example 2.2. Let $x = c$, $A = \{b\}$ and $V = \{c\}$. Then $x \in \mathcal{N}_u(x)$, since $u\text{-Int}(V) = X \setminus u(X \setminus V) = \{c\}$, and $x \in u(A)$, but $V \cap A = \emptyset$.

We recall the comparison between two closure spaces on the same set.

DEFINITION 2.3 ([14]). Let u and v be GCO's on X . We say that v is *finer* than u , or u is *coarser* than v , if $v(A) \subset u(A)$ for all $A \subset X$ and denote this by $u \prec v$.

PROPOSITION 2.4 ([14]). *Let $u, v : \mathcal{P}(X) \rightarrow \mathcal{P}(X)$. The following statements are equivalent:*

- (1) $u \prec v$;
- (2) $u\text{-Int}(A) \subset v\text{-Int}(A)$ for all $A \subset X$;
- (3) $\mathcal{N}_u(x) \subset \mathcal{N}_v(x)$ for every $x \in X$.

COROLLARY 2.1. *If $u, v : \mathcal{P}(X) \rightarrow \mathcal{P}(X)$ then $u = v$ if and only if $\mathcal{N}_u(x) = \mathcal{N}_v(x)$ for every $x \in X$.*

2.3. Subspaces of generalized closure spaces

The relativization of an operator $u : \mathcal{P}(X) \rightarrow \mathcal{P}(X)$ to a non-empty subset Y of X is $u_Y : \mathcal{P}(Y) \rightarrow \mathcal{P}(Y)$, $u_Y(A) = u(A) \cap Y$. A GCS (Y, v) is said to be a subspace of (X, u) if $Y \subset X$ and v is the relativization of u to Y .

LEMMA 2.1. *Let (X, u) and (Y, v) be GCS's such that $Y \subset X$. The following are equivalent:*

- (1) (Y, v) is a subspace of (X, u) ;
- (2) $v\text{-Int}(A) = Y \cap u\text{-Int}(A \cup (X \setminus Y))$ for all $A \subset Y$.

LEMMA 2.2. *Let $u : \mathcal{P}(X) \rightarrow \mathcal{P}(X)$, $Y \subset X$ and $v = u_Y$. Then $\mathcal{N}_v(x) \subset \{Y \cap U : U \in \mathcal{N}_u(x)\}$ for every $x \in Y$.*

PROOF. Let $x \in Y$. If $\mathcal{N}_v(x) = \emptyset$ there is nothing to prove. Let $N \in \mathcal{N}_v(x)$. Then $x \in v\text{-Int}(N) = Y \cap u\text{-Int}(N \cup (X \setminus Y))$, by Lemma 2.8, hence $P := N \cup (X \setminus Y) \in \mathcal{N}_u(x)$ and $N = Y \cap P$. ■

THEOREM 2.1. *Let $u : \mathcal{P}(X) \rightarrow \mathcal{P}(X)$, Y a non-empty subset of X and $v : \mathcal{P}(Y) \rightarrow \mathcal{P}(Y)$. If u is isotone, the following properties are equivalent:*

- (1) v is the relativization of u to Y ;
- (2) $\mathcal{N}_v(x) = \{Y \cap N : N \in \mathcal{N}_u(x)\}$ for every $x \in Y$.

PROOF. (1) \Rightarrow (2): By Lemma 2.2, $\mathcal{N}_v(x) \subset \{Y \cap N : N \in \mathcal{N}_u(x)\}$ for every $x \in Y$. Let $x \in Y$ and $N \in \mathcal{N}_u(x)$. Then $x \in Y \cap u\text{-Int}(N) \subset Y \cap u\text{-Int}(N \cup (X \setminus Y)) = Y \cap u\text{-Int}((Y \cap N) \cup (X \setminus Y)) = v\text{-Int}(Y \cap N)$, by isotony and Lemma 2.1, hence $Y \cap N \in \mathcal{N}_v(x)$. It follows that $\mathcal{N}_v(x) \supset \{Y \cap N : N \in \mathcal{N}_u(x)\}$.

(2) \Rightarrow (1): Using the preceding implication we see that $\mathcal{N}_v(x) = \mathcal{N}_{u_Y}(x)$ for every $x \in Y$, hence $v = u_Y$ by Corollary 2.1. ■

The following proposition shows the importance of the isotony condition in Theorem 2.1.

PROPOSITION 2.5. *Let $u : \mathcal{P}(X) \rightarrow \mathcal{P}(X)$ be expansive. If $\mathcal{N}_{u_Y}(x) = \{Y \cap N : N \in \mathcal{N}_u(x)\}$ for every nonempty subset Y of X and every $x \in Y$, then u is isotone.*

PROOF. Assume by contrary that $\mathcal{N}_{u_Y}(x) = \{Y \cap N : N \in \mathcal{N}_u(x)\}$ for every nonempty subset Y of X and every $x \in Y$, but u is not isotone. Then there exists $A \subset B \subset X$ such that $u\text{-Int}(A) \setminus u\text{-Int}(B) \neq \emptyset$. Since u

is expansive, $u\text{-Int}(A) \subset A \neq \emptyset$. Consider $U := A$, $Y := A \cup (X \setminus B)$ and $x \in u\text{-Int}(A) \setminus u\text{-Int}(B)$. Then $x \in U \subset Y$, $U \in \mathcal{N}_u(x)$ and $x \notin u\text{-Int}(U \cup (X \setminus Y)) = u\text{-Int}(B)$, hence $x \notin Y \cap u\text{-Int}(U \cup (X \setminus Y)) = u_Y\text{-Int}(U)$. Then $U = Y \cap U \in \{Y \cap N : N \in \mathcal{N}_u(x)\} \setminus \mathcal{N}_{u_Y}(x)$, a contradiction. ■

3. Semi-separated sets

DEFINITION 3.1 ([14]). Let (X, u) be a GCS. Two subsets A and B of X are said to be *semi-separated* in (X, u) if there are neighborhoods U and V of A and B , respectively, such that $U \cap B = V \cap A = \emptyset$.

REMARK 3.1. (1) Semi-separation is hereditary: if A and B are semi-separated in (X, u) , $A_1 \subset A$ and $B_1 \subset B$, then A_1 and B_1 are semi-separated in (X, u) . The empty set is semi-separated from itself.

(2) Every two disjoint almost u -open subsets are semi-separated: if A and B are disjoint almost u -open subsets of X , then $U := A$ and $V := B$ satisfy all the conditions of Definition 3.1.

(3) Every two disjoint almost u -closed subsets are semi-separated. Indeed, if A and B are disjoint almost u -closed subsets of X , then $B \subset X \setminus A \subset u\text{-Int}(X \setminus A)$ and $A \subset X \setminus B \subset u\text{-Int}(X \setminus B)$, hence $U := X \setminus B$ and $V := X \setminus A$ satisfy all the conditions of Definition 3.1.

(4) If $u, v : \mathcal{P}(X) \rightarrow \mathcal{P}(X)$ satisfy $u \prec v$ then every two sets which are semi-separated in (X, u) are also semi-separated in (X, v) , by Proposition 2.4.

THEOREM 3.1. Let (X, u) be a GCS and A, B be subsets of X .

- (a) If $u(A) \cap B = A \cap u(B) = \emptyset$, then A and B are semi-separated in (X, u) .
- (b) The converse of the implication in (a) holds provided that u is isotone.

PROOF. (a) Assume that $u(A) \cap B = A \cap u(B) = \emptyset$, that is, $A \subset X \setminus u(B) = u\text{-Int}(X \setminus B)$ and $B \subset X \setminus u(A) = u\text{-Int}(X \setminus A)$. Then $U := X \setminus B \in \mathcal{N}_u(A)$, $V := X \setminus A \in \mathcal{N}_u(B)$ and $U \cap B = V \cap A = \emptyset$, hence A and B are semi-separated in (X, u) .

(b) Assume that u is isotone and A, B are semi-separated in (X, u) . There exist $U' \in \mathcal{N}_u(A)$, $V' \in \mathcal{N}_u(B)$ such that $U' \cap B = V' \cap A = \emptyset$. Then $A \subset u\text{-Int}(U') \subset u\text{-Int}(X \setminus B) = X \setminus u(B)$ and $B \subset u\text{-Int}(V') \subset u\text{-Int}(X \setminus A) = X \setminus u(A)$, hence $A \cap u(B) = \emptyset$ and $B \cap u(A) = \emptyset$. ■

COROLLARY 3.1 ([14]). *Let (X, u) be a GCS, with u isotone and let A, B be subsets of X . Then A and B are semi-separated in (X, u) if and only if $u(A) \cap B = A \cap u(B) = \emptyset$.*

COROLLARY 3.2. *Let (X, u) be a GCS with u isotone. If the subsets A_1 and A_2 of X are both almost u -open or almost u -closed, then the sets $A_1 \setminus A_2$ and $A_2 \setminus A_1$ are semi-separated.*

PROOF. Let $(i, j) \in \{(1, 2); (2, 1)\}$. Since u is isotone, $u(A_i \setminus A_j) \cap u(A_j \setminus A_i) \subset u(A_i) \cap u(X \setminus A_j) \cap A_j \cap (X \setminus A_i)$. If A_i is almost u -closed or A_j is almost u -open then $u(A_i) \cap (X \setminus A_i) = \emptyset$ or $u(X \setminus A_j) \cap A_j = \emptyset$, hence $u(A_i \setminus A_j) \cap (A_j \setminus A_i) = \emptyset$. ■

REMARK 3.2. The assumption “ u is isotone” cannot be removed from Theorem 3.1 (b), as the following example shows.

EXAMPLE 3.1. Let (X, u) be as in Example 2.2. For $M := \{a, b\}$ we have $u(u\text{-Int}(M)) \not\subseteq u(M)$, since $u\text{-Int}(M) = X \setminus u(\{c\}) = \{b\}$, $u(u\text{-Int}(M)) = \{b, c\}$ and $u(M) = \{a, b\}$. Set $A := X \setminus u(M)$, $B := u\text{-Int}(M)$, $U := X \setminus M$ and $V := M$. We have $A = u\text{-Int}(U)$, $B = u\text{-Int}(V)$ and, since u is expansive, $U \cap B = V \cap A = \emptyset$. Then A and B are semi-separated in (X, u) , by Definition 3.1. But $u(B) \cap A = u(u\text{-Int}(M)) \cap (X \setminus u(M)) = \{c\} \neq \emptyset$.

REMARK 3.3. Using the notations of Remark 2.3, we point out that two sets $U, V \subset X$ are γ -separated in the sense of [3] if and only if U, V are semi-separated in the GCS (X, c_γ) . If (X, τ) is a topological space and $\alpha : \mathcal{P}(X) \rightarrow \mathcal{P}(X)$ is an isotone operator associated to τ , two subsets A, B of X are said to be α -semi-separated if and only if A, B are γ -separated for $\gamma = \alpha\text{-sInt}$ [13] (see Remark 2.4). If (X, τ) is a topological space and $\gamma = \text{Cl} \circ \text{Int}$ (respectively, $\gamma = \text{Int} \circ \text{Cl}$), then two sets $A, B \subset X$ which are γ -separated are termed semiseparated [10] (respectively, preseparated [15]).

It is natural to think that every two semi-separated sets are disjoint, but this is not true in general, as the following proposition shows.

PROPOSITION 3.1. *Let $u : \mathcal{P}(X) \rightarrow \mathcal{P}(X)$. The following properties are equivalent:*

- (1) *u is not expansive;*
- (2) *The set $\{x\}$ is semi-separated from itself for some $x \in X$;*
- (3) *There is a non-empty subset of X which is semi-separated from itself;*
- (4) *There exist two non-empty semi-separated subsets of X , one of which*

containing the other;

(5) There exist two non-empty semi-separated subsets of X which are not disjoint.

PROOF. (1) \Rightarrow (2): Since u is not expansive, there is $M \subset X$ such $u\text{-Int}(M) \setminus M \neq \emptyset$. Let $A \subset u\text{-Int}(M) \setminus M$ be non-empty. Then A is semi-separated by itself, because M is a u -neighborhood of A and $M \cap A = \emptyset$. We can choose $A = \{x\}$, where $x \in u\text{-Int}(M) \setminus M$.

Implications (2) \Rightarrow (3), (3) \Rightarrow (4) and (4) \Rightarrow (5) are obvious.

(5) \Rightarrow (1): Assume that (5) is true and (1) is false. Let A, B be semi-separated in (X, u) which are not disjoint. There exist $U \in \mathcal{N}_u(A)$ and $V \in \mathcal{N}_u(B)$ such that $U \cap B = V \cap A = \emptyset$. Since u is expansive, we have $A \subset u\text{-Int}(U) \subset U \subset X \setminus B$, a contradiction with $A \cap B \neq \emptyset$. ■

REMARK 3.4. If u is isotone, a subset A of X is semi-separated from itself if and only if $u(A) \cap A = \emptyset$. See also Proposition 3.1 and Proposition 2.1.

The following proposition is useful in the study of connectedness in GCS's.

PROPOSITION 3.2. *Let (X, u) be a GCS. Assume that $X = A \cup B$. Consider the following statements:*

- (1) *The sets A and B are semi-separated in (X, u) ;*
- (2) *A and B are disjoint and almost u -open.*

Then:

- (a) (2) implies (1);
- (b) (1) implies (2) provided that u is expansive.

PROOF. (a) See Remark 3.1 (2).

(b) Suppose that A and B are semi-separated in (X, u) . Since u is expansive, A and B are disjoint. Then $B = X \setminus A$. For every generalized closure space (X, u) and any $A \subset X$, the sets A and $X \setminus A$ are semi-separated in (X, u) if and only if there exist some subsets U and V of X such that

$$U \subset A \subset u\text{-Int}(U) \quad \text{and} \quad V \subset X \setminus A \subset u\text{-Int}(V).$$

Since u is expansive, the inclusions above imply $A = u\text{-Int}(U) = U$ and $B = u\text{-Int}(V) = V$, hence A and B are u -open. ■

REMARK 3.5. Let (X, u) be a GCS and $A \subset X$. Then A and $X \setminus A$ are semi-separated in (X, u) if and only if there exist some (almost u -open)

subsets U and V of X such that $U \subset A \subset u\text{-Int}(U)$ and $V \subset X \setminus A \subset u\text{-Int}(V)$.

We study the relativization of the semi-separation property:

THEOREM 3.2. *Let (X, u) be a GCS and let u_Y be the relativization of u to $Y \subset X$. Then:*

- (a) *Every subsets A, B of Y which are semi-separated in (Y, u_Y) are semi-separated in (X, u) .*
- (b) *If u is isotone, every subsets A, B of Y which are semi-separated in (X, u) are semi-separated in (Y, u_Y) .*

PROOF. Denote u_Y by v . Let A, B be subsets of Y .

(a) Assume that A and B are semi-separated in (Y, v) . There exist some subsets U and V of Y such that $A \subset v\text{-Int}(U)$, $B \subset v\text{-Int}(V)$ and $U \cap B = V \cap A = \emptyset$. By Lemma 2.1, $A \subset u\text{-Int}(U \cup (X \setminus Y))$ and $B \subset u\text{-Int}(V \cup (X \setminus Y))$. Then $U' := U \cup (X \setminus Y) \in \mathcal{N}_u(A)$, $V' := V \cup (X \setminus Y) \in \mathcal{N}_u(B)$ and $U' \cap B = V' \cap A = \emptyset$, hence A and B are semi-separated in (X, u) .

(b) Assume that u is isotone and A and B are semi-separated in (X, u) . Then $u(A) \cap B = \emptyset = u(B) \cap A$, by Theorem 3.1 (b). Obviously $v(A) \cap B = \emptyset = v(B) \cap A$, hence A and B are semi-separated in (Y, v) by Theorem 3.1 (a). ■

The assumption “ u is isotone” cannot be removed from Theorem 3.2 (b), according to the following example.

EXAMPLE 3.2. Let X, u, A and B be as in Example 3.1. Then $A = \{c\}$ and $B = \{b\}$ are semiseparated in (X, u) , but they are not semi-separated in (Y, u_Y) , where $Y := \{b, c\}$ and u_Y is the relativization of u to Y . Indeed, if we suppose that A and B are semi-separated in (Y, u_Y) , then there exist M and N subsets of Y such that $A \subset u_Y\text{-Int}(Y \setminus M)$, $B \subset u_Y\text{-Int}(Y \setminus N)$, $A \subset N$ and $B \subset M$. Taking into account that u is expansive, this implies $\{c\} \subset N \subset u(N) \cap Y \subset \{c\}$ and $\{b\} \subset M \subset u(M) \cap Y \subset \{b\}$. But $\{b\} \subset M \subset Y$ implies $M \in \{\{b\}, \{b, c\}\}$, hence $u(M) \cap Y = \{b, c\} \not\subseteq \{b\}$. This contradiction shows that A and B are not semi-separated in (Y, u_Y) .

REMARK 3.6. Taking into account Remark 3.3 we can see that Remark 3.1 (2), Theorem 3.2 and Corollary 3.2 furnish Theorem 2, Theorem 4 and Theorem 5 in [10], as well as Theorem 2, Theorem 4 and Theorem 5 in [15].

EXAMPLE 3.3. Let $G = (V, E)$ be an undirected graph, where V is the vertex set and E is the set of edges. For every $A \subset V$, we define $u_1(A) \subset V$

(respectively, $u_2(A) \subset V$) such that a vertex $x \in V$ belongs to $u_1(A)$ (respectively, $u_2(A)$) if and only if, either $x \in A$, or x has a neighbor belonging to A (respectively, x can be joined by a path in G to a vertex belonging to A). Clearly, $u_2 \prec u_1$. Moreover, $u_2(A) = \cup\{u_1^m(A) : m = 1, 2, \dots\}$, for every $A \subset V$, where $u_1^m = \underbrace{u_1 \circ \dots \circ u_1}_m$. The set operators $u_k : \mathcal{P}(V) \rightarrow \mathcal{P}(V)$, $k = 1, 2$, are expansive and isotone, u_2 is idempotent, while u_1 is not idempotent in general. Note that $u_1\text{-Int}(A)$ (respectively, $u_2\text{-Int}(A)$) is the set of vertices belonging to A which have no neighbors in $X \setminus A$ (respectively, which cannot be joined by paths in G to vertices in $X \setminus A$). We see that $A \subset u_k\text{-Int}(B)$ if and only if $u_k(A) \subset B$, for $k = 1, 2$, whenever $A, B \subset V$.

The sets $A, B \subset V$ are semi-separated in (V, u_1) (respectively, in (V, u_2)) if and only if any two vertices $x \in A$ and $y \in B$ are not neighbors (respectively, cannot be joined by a path in G). Note the following symmetry: $u_k(A) \cap B = \emptyset$ implies $u_k(B) \cap A = \emptyset$, for $k = 1, 2$, whenever $A, B \subset V$.

In graph theory, a graph G is called complete if any two distinct vertices are neighbors, and connected if any two distinct vertices can be joined by a path in G . We get the following characterizations.

THEOREM 3.3. *Let $G = (V, E)$ be a graph, having more than one vertex.*

(a) G is not complete if and only if there exist two non-empty subsets of V , which are semi-separated in (V, u_1) .

(b) G is not connected if and only if there exist two non-empty subsets of V , which are semi-separated in (V, u_2) .

PROOF. If two non-empty sets $A, B \subset V$ are semi-separated in (V, u_1) (respectively, in (V, u_2)), taking $x \in A$ and $y \in B$ we see that vertices x and y are not neighbors (respectively, cannot be joined by a path in G), hence the graph G is not complete (respectively, G is not connected). Conversely, if the graph G is not complete (respectively, G is not connected), then there exist $x, y \in V$ such that x and y are not neighbors (respectively, cannot be joined by a path in G) and it follows that the sets $\{x\}$ and $\{y\}$ are semi-separated in (V, u_1) (respectively, in (V, u_2)). Moreover, if x and y cannot be joined by a path in G , then $u(\{x\}) \cap u(\{y\}) = \emptyset$. ■

Example 3.3 suggests the following result.

PROPOSITION 3.3. *Let $u : \mathcal{P}(X) \rightarrow \mathcal{P}(X)$ such that $u(B) \cap (X \setminus B) = \emptyset$ implies $u(X \setminus B) \cap B = \emptyset$, whenever $B \subset X$. Then a set $A \subset X$ is almost u -closed if and only if A is almost u -open.*

PROOF. For all $A \subset X$ the following relations are equivalent: $u(A) \subset A$, $u(A) \cap (X \setminus A) = \emptyset$, $u(X \setminus A) \cap A = \emptyset$, $A \subset X \setminus u(X \setminus A)$, $A \subset u\text{-Int}(A)$. ■

COROLLARY 3.3. *With the notations of Example 3.3, the following are equivalent for a set $A \subset V$:*

- (1) *A is closed in (V, u_2) ;*
- (2) *A is open in (V, u_2) ;*
- (3) *A and $V \setminus A$ are semi-separated in (V, u_2) .*

Moreover, if A is a non-empty proper subset of V , then each of the above statements is equivalent to the following:

- (4) *Any vertices $x \in A$ and $y \in V \setminus A$ cannot be joined by a path in G .*

4. Images and preimages of semi-separated sets

Recall the generalization of continuity to generalized closure spaces.

DEFINITION 4.1 ([14], [4]). Let (X, u) and (Y, v) be generalized closure spaces. A function $f : (X, u) \rightarrow (Y, v)$ is said to be *continuous at $x \in X$* if for all $V \in \mathcal{N}_v(f(x))$ implies $f^{-1}(V) \in \mathcal{N}_u(x)$. $f : (X, u) \rightarrow (Y, v)$ is said to be *continuous* if it is continuous at every point $x \in X$.

The following characterizations of continuity in GCS's are known.

LEMMA 4.1 ([14]). *For $f : (X, u) \rightarrow (Y, v)$ the following conditions are equivalent:*

- (1) *f is continuous;*
- (2) *$u(f^{-1}(B)) \subset f^{-1}(v(B))$ for every subset B of Y ;*
- (3) *$f^{-1}(v\text{-Int}(B)) \subset u\text{-Int}(f^{-1}(B))$ for every subset B of Y .*

PROPOSITION 4.1. *Let $f : (X, u) \rightarrow (Y, v)$. Consider the following statements:*

- (1) *f is continuous;*
- (2) *The preimage under f of each almost v -open subset of Y is an almost u -open subset of X .*

Then:

- (a) $(1) \Rightarrow (2)$;
- (b) $(2) \Rightarrow (1)$ provided that v is idempotent and expansive, and u is isotone.

PROOF. (a) Assume that f is continuous and let $B \subset Y$ be almost v -open, i.e., $B \subset v\text{-Int}(B)$. By Lemma 4.1,

$$f^{-1}(B) \subset f^{-1}(v\text{-Int}(B)) \subset u\text{-Int}(f^{-1}(B)),$$

hence $f^{-1}(B)$ is almost u -open.

(b) Let v be idempotent and expansive and let u be isotone. Assume that (2) is true. Let $V \in \mathcal{N}_v(f(x))$, i.e., $f(x) \in v\text{-Int}(V)$, which is equivalent to $x \in f^{-1}(v\text{-Int}(V))$. Since v is idempotent, the set $v\text{-Int}(V)$ is v -open, in particular almost v -open. According to our assumption,

$$f^{-1}(v\text{-Int}(V)) \subset u\text{-Int}(f^{-1}(v\text{-Int}(V))).$$

Since v is expansive and u is isotone, we have

$$u\text{-Int}(f^{-1}(v\text{-Int}(V))) \subset u\text{-Int}(f^{-1}(V)).$$

It follows that $x \in u\text{-Int}(f^{-1}(V))$, i.e., $f^{-1}(V) \in \mathcal{N}_u(x)$. ■

THEOREM 4.1. *Let $f : (X, u) \rightarrow (Y, v)$ be a continuous function. If the sets A and B are semi-separated in (Y, v) , then $f^{-1}(A)$ and $f^{-1}(B)$ are semi-separated in (X, u) .*

PROOF. Assume that A and B are semi-separated in (Y, v) . Let $U \in \mathcal{N}_v(A)$ and $V \in \mathcal{N}_v(B)$ such that $U \cap B = V \cap A = \emptyset$. Applying Lemma 4.1, we get $f^{-1}(A) \subset f^{-1}(v\text{-Int}(U)) \subset u\text{-Int}(f^{-1}(U))$ and $f^{-1}(B) \subset f^{-1}(v\text{-Int}(V)) \subset u\text{-Int}(f^{-1}(V))$. It follows that $f^{-1}(U) \in \mathcal{N}_u(f^{-1}(A))$, $f^{-1}(V) \in \mathcal{N}_u(f^{-1}(B))$ and $f^{-1}(U) \cap f^{-1}(B) = f^{-1}(V) \cap f^{-1}(A) = \emptyset$, hence $f^{-1}(A)$ and $f^{-1}(B)$ are semi-separated in (X, u) . ■

REMARK 4.1. In [14] Theorem 4.1 was proven under the assumption that u and v are isotone, expansive and preserve the empty set. There the proof makes use of the characterization of semi-separated sets in isotonic spaces, given by Corollary 3.1.

Let $G = (V, E)$ and $G' = (V', E')$ be two undirected graphs.

DEFINITION 4.2. We say that a function $f : V \rightarrow V'$ is *connectivity-preserving*, with respect to the graphs G and G' , if $f(x)$ and $f(y)$ can be joined by a path in G' whenever $x, y \in V$ can be joined by a path in G .

Note that, if the graph G has only isolated vertices, then every function $f : V \rightarrow V'$ is connectivity-preserving.

Denote by u_2 and u'_2 the set operators associated as in Example 3.3 to the graphs G and G' , respectively.

THEOREM 4.2. A function $f : V \rightarrow V'$ is connectivity-preserving, with respect to the graphs G and G' , if and only if $f : (V, u_2) \rightarrow (V', u'_2)$ is continuous.

PROOF. Using Definition 4.1 and the characterizations of neighborhoods given in Example 3.3, we see that the following are equivalent:

- (1) $f : (V, u_2) \rightarrow (V', u'_2)$ is continuous;
- (2) For every $x \in V$ and all $W' \subset V'$ such that $u'_2(\{f(x)\}) \subset W'$, we have $u_2(\{x\}) \subset f^{-1}(W')$.

Necessity. Assume that $f : V \rightarrow V'$ is connectivity-preserving, with respect to the graphs G and G' . Let $x \in V$ and $W' \subset V'$ such that $u'_2(\{f(x)\}) \subset W'$. If $y \in u_2(\{x\})$, then x and y can be joined by a path in G . Since f is connectivity-preserving, $f(y) \in u'_2(\{f(x)\}) \subset W'$, hence $y \in f^{-1}(W')$. We proved that (2) holds.

Sufficiency. Assume that (2) holds, but f is not connectivity-preserving. Then there exist $x, y \in V$ such that x and y can be joined by a path in G , but $f(x)$ and $f(y)$ cannot be joined by a path in G' . It follows that $u'_2(\{f(x)\}) \subset V' \setminus \{f(y)\}$. We deduce that $u_2(\{x\}) \subset f^{-1}(V' \setminus \{f(y)\}) \subset V \setminus \{y\}$, hence $y \notin u_2(\{x\})$, a contradiction. ■

COROLLARY 4.1. Let $f : V \rightarrow V'$ such that $f^{-1}(A)$ and $f^{-1}(B)$ are semi-separated in (V, u_2) whenever A and B are semi-separated in (V', u'_2) . Then $f : (V, u_2) \rightarrow (V', u'_2)$ is continuous.

PROOF. Assume, by contrary, that $f : (V, u_2) \rightarrow (V', u'_2)$ is not continuous. Then there exist $x, y \in V$ such that x and y can be joined by a path in G , but $f(x)$ and $f(y)$ cannot be joined by a path in G' . Then the sets $\{f(x)\}$ and $\{f(y)\}$ are semi-separated in (V, u_2) , but $f^{-1}(\{f(x)\})$ and $f^{-1}(\{f(y)\})$ are not semi-separated in (V, u_2) , which is a contradiction. ■

COROLLARY 4.2. $f : (V, u_2) \rightarrow (V', u'_2)$ is continuous if and only iff pulls back every pair of semi-separated sets into a pair of semi-separated sets.

PROOF. Apply Theorem 4.1 and Corollary 4.1. ■

DEFINITION 4.3. We say that $f : (X, u) \rightarrow (Y, v)$ is *almost open* if $f(u\text{-Int}(A)) \subset v\text{-Int}(f(A))$ for all $A \subset X$.

REMARK 4.2. Let $f : (X, u) \rightarrow (Y, v)$ be a bijection. By Lemma 4.1, f is almost open if and only if f^{-1} is continuous.

We prove a counterpart of Proposition 4.1 for almost open functions.

PROPOSITION 4.2. *Let $f : (X, u) \rightarrow (Y, v)$. Consider the following statements:*

- (1) f is almost open;
- (2) The image under f of each almost u -open subset of X is an almost v -open subset of Y .

Then:

- (a) (1) \Rightarrow (2);
- (b) (2) \Rightarrow (1) provided that u is idempotent and expansive, and v is isotone.

PROOF. (a) Assume that f is continuous and let $U \subset X$ be almost u -open, i.e., $U \subset u\text{-Int}(U)$. According to Definition 4.3,

$$f(U) \subset f(u\text{-Int}(U)) \subset v\text{-Int}(f(U)),$$

hence $f(U)$ is almost v -open.

(b) Let u be idempotent and expansive and let v be isotone. Assume that (2) holds.

Since u is idempotent, $u\text{-Int}$ is also idempotent, hence $u\text{-Int}(A)$ is (almost) u -open for all $A \subset X$. Let $A \subset X$. Then $f(u\text{-Int}(A))$ is almost v -open, i.e.,

$$f(u\text{-Int}(A)) \subset v\text{-Int}(f(u\text{-Int}(A))).$$

Since u is expansive and v is isotone, we have

$$v\text{-Int}(f(u\text{-Int}(A))) \subset v\text{-Int}(f(A)).$$

The latter two inclusions imply $f(u\text{-Int}(A)) \subset v\text{-Int}(f(A))$. ■

THEOREM 4.3. *If $f : (X, u) \rightarrow (Y, v)$ is almost open and injective, u and v are isotone and U, V are semi-separated in (X, u) , then $f(U), f(V)$ are semi-separated in (Y, v) .*

PROOF. Assume that $f : (X, u) \rightarrow (Y, v)$ is almost open and injective, u and v are isotone and U, V are semi-separated in (X, u) (i.e., $U \subset u\text{-Int}(X \setminus V)$ and $V \subset u\text{-Int}(X \setminus U)$). Then

$$\begin{aligned} f(U) &\subset v\text{-Int}(f(X \setminus V)) \subset v\text{-Int}(Y \setminus f(V)) \quad \text{and} \\ f(V) &\subset v\text{-Int}(f(X \setminus U)) \subset v\text{-Int}(Y \setminus f(U)), \end{aligned}$$

hence $f(U), f(V)$ are semi-separated in (Y, v) . ■

REMARK 4.3. Let $\gamma : \mathcal{P}(X) \rightarrow \mathcal{P}(X)$ and $\gamma' : \mathcal{P}(Y) \rightarrow \mathcal{P}(Y)$ be isotone operators. In [3] a function $f : X \rightarrow Y$ is said to be (γ, γ') -continuous iff $f^{-1}(V)$ is γ -open whenever $V \subset Y$ is γ' -open. In other words, $f : X \rightarrow Y$ is (γ, γ') -continuous if $f^{-1}(V)$ is open in the GCS (X, c_γ) whenever V is open in the GCS $(Y, c_{\gamma'})$. Similarly, $f : X \rightarrow Y$ is said to be (γ, γ') -open if $f(U)$ is γ' -open whenever $U \subset X$ is γ -open [3], i.e., $f(U)$ is open in the GCS $(Y, c_{\gamma'})$ whenever U is open in the GCS (X, c_γ) .

Since the operators c_γ and $c_{\gamma'}$ are isotone, expansive and idempotent, we may apply Proposition 4.1 (respectively, Proposition 4.2) to show that $f : X \rightarrow Y$ is (γ, γ') -continuous (respectively, (γ, γ') -open) if and only if $f : (X, c_\gamma) \rightarrow (Y, c_{\gamma'})$ is continuous in the sense of Definition 4.1 (respectively, almost open in the sense of Definition 4.3). It follows that Theorem 4.1 and Theorem 4.3 furnish Lemma 2.1 and Lemma 2.3 in [3], respectively.

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THE FOURIER INTEGRALS OF FUNCTIONS OF BOUNDED VARIATION

By

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1. Introduction

Various summability properties of number series as well as their applications to Fourier series are well studied. In his papers [11, 12], G. Goes developed the theory of C^α -complementary spaces of Fourier coefficients, that is with respect to Cesàro methods of summability. In [13] he used this as a bridge between known theorems on summability factors for the Cesàro summability method C^α and effective sufficient conditions for the multipliers of Fourier series. These results were extended to general Toeplitz methods by M. Tynnov [17, 18]. A comprehensive exposition may be found in [1, § 28]. This concept allows to study summability and multiplier problems for Fourier series by means of summability methods for numerical series.

Similar results for integrals are less investigated – some references may be found in [2] where an attempt to build a general theory was made. Of course, this is impossible without understanding, generally speaking, the Fourier transform in the distributional sense. The main object of this topic are various spaces \hat{X} of the Fourier integrals

$$f^\circ(t) \sim \int_{\mathbf{R}} \varphi(x) e^{ixt} dx$$

such that φ is the Fourier transform \hat{f} of $f \in X$

$$\varphi(x) = \hat{f}(x) = (2\pi)^{-1} \int_{\mathbf{R}} f(t) e^{-ixt} dt.$$

Given X a normed space, we define $\|f^\circ\|_{X^\wedge} = \|f\|_X$.

In a sense, here we attempt to do for Fourier integrals what we did in [3] for double Fourier series. In particular, our scope is such that we avoid the distributional approach.

The paper is organized as follows. After the present introduction we give in the next section the formulations of our main results as well as examples and corollaries. We prove the results in the last section.

2. Problems and results

We first observe that the Fourier transform of a function of bounded variation vanishing at infinity, written $F \in V_0$, exists as an improper integral everywhere except maybe the origin (see, e.g., [4, Ch. 1, §2]).

For functions of bounded variation the notion of the Fourier–Stieltjes transform proved to be more natural in many respects as the usual Fourier transform:

$$\widehat{dF}(x) = (2\pi)^{-1} \int_{\mathbf{R}} e^{-ixt} dF(t).$$

We study conditions under which f° belongs to the space $d^\alpha V^\wedge$, $\alpha \geq 0$, that is, when

$$(1) \quad (ix)^{1-\alpha} \varphi(x) = \widehat{dF}(x),$$

where F is a function of bounded variation. For this we need a notion of fractional derivative. The one corresponding to our scope is naturally defined via the Fourier transform: $g^{(\alpha)}$, the α th derivative of g , is the function for which

$$\widehat{g^{(\alpha)}}(x) = (ix)^\alpha \widehat{g}(x).$$

We shall study these spaces in connection with summability. Let the summability method be defined by a single function λ , a multiplier, as follows

$$(2) \quad (\Lambda_N f)(x) = \int_{\mathbf{R}} \lambda\left(\frac{t}{N}\right) \varphi(t) e^{ixt} dt.$$

It is clear that λ should be defined at each point, so let λ be continuous. The following representation is useful in many cases (when $f, \lambda \in L(\mathbf{R})$) it is

merely equivalent to (2), see, e.g., [16, Ch. I, Th. 1.16]; moreover, this is true under any assumptions which provide the validity of the Parseval identity)

$$(3) \quad (\Lambda_N f)(x) = \int_{\mathbf{R}} N \widehat{\lambda}(N(t-x)) f(t) dt.$$

In what follows we assume that

$$(4) \quad \text{both } \lambda \text{ and } \widehat{\lambda} \text{ are integrable on } \mathbf{R}$$

and

$$(5) \quad \lambda(0) = 1.$$

We shall make use of either (2) or (3) as a definition of the linear means Λ_N just according to which of the two formulas is valid.

When a function is already represented by a Fourier integral, to take its α th derivative means to multiply the integrand by $(it)^\alpha$. Therefore, by $(\Lambda_N f)^{(1-\alpha)}$ we understand

$$(\Lambda_N f)^{(1-\alpha)}(x) = \int_{\mathbf{R}} \lambda\left(\frac{t}{N}\right) \varphi(t) (it)^{(1-\alpha)} e^{ixt} dt.$$

Our first result is the following:

THEOREM 1. *Let φ and λ be such that $\lambda(t/N)\varphi(t)(it)^{(1-\alpha)}$ are integrable for all N . In order that f° belong to $d^\alpha V^\wedge$, it is necessary and sufficient that*

$$(6) \quad \|(\Lambda_N f)^{(1-\alpha)}\|_{L(\mathbf{R})} = O(1).$$

REMARK. Clearly, $\alpha = 0$ and $\alpha = 1$ are two main cases. Slightly less general versions of these cases are known from [10] and [9], respectively. For more information, see [7] or [15].

Here and for the sequel we recall that a function s is called a *multiplier of class (X, Y)* if for each $f^\circ \in X^\wedge$ we have $sf^\circ \in Y^\wedge$.

Thus this theorem is a criterion for $(ix)^{1-\alpha}\varphi$ to be a multiplier of class (L, L) (see, e.g., [7, Th. 6.5.6]).

Let $\varphi(t) = Si(|t|) = \int_0^{|t|} u^{-1} \sin u du$, which is bounded and non-negative, and $\lambda(t) = (1 - |t|)_+$. We wish to check that $(\Lambda_N f)$ are uniformly integrable for this choice. The value $\int_0^{1/N} |(\Lambda_N f)(x)| dx$ is uniformly bounded,

as for any bounded φ . Given $x > 1/N$, we obtain for $y = Nx > 1$ by changing the order of integration

$$(\Lambda_N f)(y) =$$

$$= N \int_0^1 u^{-1} \sin Nu \left[-y^{-2} \cos y - y^{-1}(1-u) \sin(yu) + y^{-2} \cos(yu) \right] du.$$

The main term for $\int_{\mathbf{R}} |(\Lambda_N f)(x)| dx = N^{-1} \int_{\mathbf{R}} |(\Lambda_N f)(y)| dy$ is

$$\int_1^\infty y^{-1} \left| \int_0^1 u^{-1} \sin(Nu) \sin(yu) du \right| dy,$$

the uniform integrability of the rest is rather obvious. For $y > 2N$ and $1 < y < N/2$, we obtain the needed bounds by estimating the inner integral as

$$\frac{1}{2} \left| \int_{|y-N|}^{y+N} u^{-1} \cos u du \right| \leq \frac{1}{2} \log \left| \frac{y+N}{y-N} \right|$$

and straightforward calculations, e.g. for $y > 2N$ use that $\log |(y+N)/(y-N)| \leq 2N/(N-y)$. For the integrals from $N/2$ to $N-1$ and $N+1$ to $2N$ use that $\int_1^\infty u^{-1} \cos u du$ exists and $\int_{N/2}^{2N} y^{-1} dy = \log 4$. The estimate over $N-1 \leq y \leq N+1$ is simple. Hence $Si(|t|)$ is a multiplier of class (L, L) .

Taking the same λ and $\varphi(t) = 1$ for $|t| \leq 1$ and $\varphi(t) = 0$ otherwise, the partial Fourier integrals, we easily derive that this function cannot be a multiplier of class (L, L) . Indeed, we immediately arrive at

$$\int_0^1 \left(1 - \frac{t}{N} \right) \cos xt dt = \left(1 - \frac{1}{N} \right) x^{-1} \sin x + N^{-1} x^{-2} (1 - \cos x),$$

which is non-integrable for any $N > 1$.

The other negative example is delivered, again with the same λ , by $\varphi(t) = 1$ for $t > 0$ and $\varphi(t) = -1$ for $t < 0$. The calculations are simple:

$$\int_0^N \left(1 - \frac{t}{N} \right) \sin xt dt = x^{-1} - N^{-1} x^{-2} \sin Nx,$$

and the right-hand side is non-integrable for any N . This gives a different proof that the Hilbert transform is not a bounded operator on L .

Further, we introduce the following (cf. [2])

DEFINITION. We define the complementary space to $d^\alpha V^\wedge$, written V_α^* , as the space of all Fourier integrals

$$g^\circ(t) \sim \int_{\mathbf{R}} \hat{g}(t) e^{ixt} dt$$

for which the integral

$$\int_{\mathbf{R}} \hat{f}(x) \hat{g}(x) dx$$

is Λ -summable for all $f^\circ \in d^\alpha V^\wedge$.

THEOREM 2. *In order that $g^\circ \in V_\alpha^*$ it is necessary and sufficient that the sequence $(\Lambda_N g)^{(\alpha-1)}$ is boundedly convergent.*

For $f, g \in X$ we define the inner product by the integral

$$\langle f, g \rangle := \int_{\mathbf{R}} f(t) g(t) dt.$$

Further, define the function e by the formula $e(t) := 1$ for all $t \in \mathbf{R}$.

A function $s \in X$ is called a *summability factor of type (Λ, Λ)* if for every Λ -summable functional with the values $\langle \hat{h}, e \rangle$ for $\hat{h} \in X^\wedge$ the functional with the values $\langle s \hat{h}, e \rangle = \langle \hat{h}, s \rangle$ is Λ -summable.

THEOREM 3. *If a function $s \in X$ is a summability factor of type (Λ, Λ) then s is a multiplier of class (V_α^*, V_α^*) .*

Let $s^{(\alpha)}$ mean the well-known fractional derivative in the Riemann–Liouville sense.

COROLLARY 4. *Let $\alpha \geq 0$. If s is even, $s^{(\alpha)}$ is absolutely continuous, s has a finite limit at infinity, and*

$$\int_{\mathbf{R}} |t^\alpha s^{(\alpha+1)}(t)| dt < +\infty,$$

then s is a multiplier of the class (V_α^, V_α^*) .*

3. Proofs

PROOF OF THEOREM 1. *Necessity.* Let $f^\circ \in d^\alpha V^\wedge$. Then

$$\begin{aligned} (\Lambda_N f)^{(1-\alpha)}(x) &= \int_{\mathbf{R}} \lambda\left(\frac{t}{N}\right) (it)(it)^{-\alpha} \varphi(t) e^{ixt} dt \\ &= \int_{\mathbf{R}} \lambda\left(\frac{t}{N}\right) \widehat{dF}(t) e^{ixt} dt. \end{aligned}$$

By the Stieltjes version of (3) (see, e.g., [7, Th. 6.5.6]) we have

$$(\Lambda_N f)^{(1-\alpha)}(x) = \int_{\mathbf{R}} N \hat{\lambda}(N(t-x)) dF(t).$$

Hence

$$\|(\Lambda_N f)^{(1-\alpha)}\|_{L(\mathbf{R})} \leq \int_{\mathbf{R}} \int_{\mathbf{R}} N |\hat{\lambda}(N(t-x))| d|F(t)| dx.$$

Since $\hat{\lambda}$ is integrable on \mathbf{R} , and F is of bounded variation we may use here and in what follows the Fubini theorem freely. This yields

$$\begin{aligned} \|(\Lambda_N f)^{(1-\alpha)}\|_{L(\mathbf{R})} &\leq \int_{\mathbf{R}} d|F(t)| \int_{\mathbf{R}} N |\hat{\lambda}(N(t-x))| dx \\ &= \|F\|_{BV} \|\hat{\lambda}\|_{L(\mathbf{R})}, \end{aligned}$$

and we are done.

Sufficiency. Denote

$$\Phi_N(x) = \int_{-\infty}^x (\Lambda_N f)^{(1-\alpha)}(t) dt.$$

This sequence possesses the following properties. First, (6) provides that both the family $\{\Phi_N\}$ and the family of variations of these functions are uniformly bounded (by the same constant $\|(\Lambda_N f)^{(1-\alpha)}\|_{L(\mathbf{R})}$). In virtue of the boundedness of $\{\Phi_N\}$ along with their variations, the first Helly's theorem (see, e.g., [14, Th. 9.1.1]) ensures the existence of a subsequence $\{N_k\}$, $N_k \rightarrow \infty$ as $k \rightarrow \infty$, such that

$$\lim_{k \rightarrow \infty} \Phi_{N_k} = \Phi(x)$$

at any point x on the whole \mathbf{R} , where Φ is a function of bounded variation (bounded by $\|\Lambda_N f\|_{L(\mathbf{R})}$ along with its total variation). Our next (and final) step is to show that

$$(7) \quad d\widehat{\Phi}(x) = (2\pi)^{-1} \int_{\mathbf{R}} e^{-ixt} d\Phi(t) = \lim_{k \rightarrow \infty} (2\pi)^{-1} \int_{\mathbf{R}} e^{-ixt} d\Phi_{N_k}(t).$$

Generally speaking, the second Helly's theorem is true only under additional assumptions (see, e.g., [15, Th. 9.1.3]). For example, it holds true if the last limit is uniform on every finite interval (see [5, Lemma 1]). As in [9], we have

$$\lambda \left(\frac{t}{N_k} \right) (it)^{1-\alpha} \varphi(t) = (2\pi)^{-1} \int_{\mathbf{R}} e^{-ixt} d\Phi_{N_k}(x).$$

The right-hand side is continuous as well as $\lambda(t/N_k)$, hence $(it)^{1-\alpha} \varphi(t)$ almost everywhere coincides with a continuous function $\psi(t)$. By (4) and (5), $\lambda(t/N_k) \psi(t)$ converges, as $N \rightarrow \infty$, to $\psi(t)$ uniformly on every finite interval. Thus, (7) is true, which completes the proof. ■

PROOF OF THEOREM 2. We are checking the conditions for convergence of

$$(8) \quad \int_{\mathbf{R}} \lambda \left(\frac{x}{N} \right) \hat{g}(x) \hat{f}(x) dx.$$

Since $f^\circ \in d^\alpha V^\wedge$, we have

$$\int_{\mathbf{R}} \lambda \left(\frac{x}{N} \right) \hat{g}(x) (ix)^{\alpha-1} (ix)^{1-\alpha} \hat{f}(x) dx = \int_{\mathbf{R}} \lambda \left(\frac{x}{N} \right) \hat{g}(x) (ix)^{\alpha-1} d\widehat{F}(x) dx$$

for some F of bounded variation. By Parseval's identity this is

$$\int_{\mathbf{R}} (\Lambda_N g)^{\alpha-1}(t) dF(t).$$

Taking

$$F(t) = \begin{cases} 1, & -\infty < t < x \\ 0, & \text{otherwise} \end{cases}$$

yields

$$\int_{\mathbf{R}} \lambda \left(\frac{x}{N} \right) \hat{g}(x) \hat{f}(x) dx = (\Lambda_N g)^{(\alpha-1)}(x),$$

and the right-hand side is boundedly convergent provided $g^\circ \in V_\alpha^*$.

Conversely, let $(\Lambda_N g)^{(\alpha-1)}$ be boundedly convergent. Then the operation similar to that above and Parseval's identity yield

$$\begin{aligned} \int_{\mathbf{R}} \lambda \left(\frac{x}{N} \right) \hat{g}(x) \hat{f}(x) dx &= \int_{\mathbf{R}} \lambda \left(\frac{x}{N} \right) \hat{g}(x) (ix)^{\alpha-1} (ix)^{1-\alpha} \hat{f}(x) dx \\ &= \int_{\mathbf{R}} (\Lambda_N g)^{(\alpha-1)}(t) dF(t). \end{aligned}$$

Since $(\Lambda_N g)^{(\alpha-1)}(x)$ is boundedly convergent and F is of bounded variation, Lebesgue's dominated convergence theorem shows that the integral (8) converges as $N \rightarrow \infty$, which completes the proof. ■

PROOF OF THEOREM 3. Let $f^\circ \in V_\alpha^*$. It means that

$$\langle \Lambda_N \hat{f}, \check{g} \rangle = \langle \Lambda_N \hat{f} \check{g}, e \rangle$$

converge for every $g^\circ \in V_\alpha^*$. This yields, since s is a summability factor of type (Λ, Λ) , that

$$\langle \Lambda_N s \hat{f}, \check{g} \rangle = \langle \Lambda_N s \hat{f}, \check{g} \rangle$$

converge for every $g^\circ \in V_\alpha^*$. This just means that s acts as a multiplier taking the space V_α^* into itself. ■

PROOF OF COROLLARY 4. We recall, that C^α -summability with $\alpha \geq 0$ is the case when $\lambda(t) = (1 - |t|)_+^\alpha$. We apply a result of J. Cossar [8] on the Cesàro C^α -summability of integrals (see [6] for the complete version of this result; the partial case of integer α is due to G. H. Hardy [14]) due to which if λ and α satisfy the conditions of Corollary 4, then λ is a summability factor of type (C^α, C^α) for integrals. In view of Theorem 3 the function s is a multiplier of the indicated class. ■

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INTERSECTIONS OF OPEN SETS IN GENERALIZED TOPOLOGICAL SPACES

By

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0. Introduction

Let X be a (non-empty) set and τ a topology on X . In the literature, a lot of papers use the following construction: for a subset $A \subset X$, let kA denote the intersection of all τ -open sets containing A (usually called *kernel* of A) and then they consider the class of all sets A such that $A = kA$ (see e.g. [1], [12] or [17]). Another often used construction is the following: one considers all sets $A \subset X$ such that, if $A \subset G \in \tau$, the closure of A is contained in G . In these constructions, the role of open sets is often given to sets taken from another class (see e.g. [9], [14], [15], [16]).

The purpose of the present paper is to formulate a rather general form of the above constructions and, in particular, to look for conditions under which the result is a *generalized topology* (briefly GT), i.e., a subset μ of the power set $\exp X$ of X such that $\emptyset \in \mu$ and every union of elements of μ belongs to μ (see [6]).

1. Preliminaries

For a set $X \neq \emptyset$, let us consider a subset $\lambda \subset \exp X$ and $A \subset X$. Let us call λ -*open* the elements of λ and λ -*closed* their complements; let λ^* be the family of all λ -closed sets. According to [9], $i_\lambda A$ denotes the union of all λ -open sets contained in A , or let $i_\lambda A = \emptyset$ if there is no such set in λ .

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Similarly, let $c_\lambda A$ denote the intersection of all elements of λ^* containing A and let $c_\lambda A = X$ if there is no such set in λ^* . By [9],

$$(1.1) \quad i_\lambda i_\lambda A = i_\lambda A \subset A \subset c_\lambda A = c_\lambda c_\lambda A$$

and

$$(1.2) \quad A \subset B \subset X \text{ implies } i_\lambda A \subset i_\lambda B \text{ and } c_\lambda A \subset c_\lambda B$$

so that, by [5], the sets A such that $A = i_\lambda A$ (that is $A \subset i_\lambda A$) constitute a GT ν . Moreover

$$(1.3) \quad i_\lambda(X - A) = X - c_\lambda A \text{ and } c_\lambda(X - A) = X - i_\lambda A.$$

Again by [9], if λ^* is a GT then

$$(1.4) \quad c_\lambda \left(\bigcup_{k \in K \neq \emptyset} A_k \right) = \bigcup_{k \in K} c_\lambda A_k$$

so that ν is an *ultratopology* (i.e., a topology such that every intersection of a non-empty family of ν -open sets is ν -open). Therefore ν^* is an ultratopology as well.

A family $\beta \subset \exp X$ is said to be a *base* for a GT μ if μ is composed of \emptyset and all unions of subfamilies of β (see [10] or [11]).

2. The case of a GT λ^*

Consider a family $\lambda \subset \exp X$ and the GT ν defined as above.

PROPOSITION 2.1. *The family λ is a base for the GT ν .*

PROOF. If $N \in \nu$ then $N = i_\lambda N$ is either empty or a union of some elements of λ . Conversely, if $N = \emptyset$ then $N \in \nu$ as ν is a GT. If $\emptyset \neq N \in \lambda$ then clearly $N = i_\lambda N$ and $N \in \nu$. Hence if N is the union of some elements of λ then N belongs to the GT ν . ■

Cf. [9], 2.8.

COROLLARY 2.2. *If λ^* is a GT then the family λ of the λ^* -closed sets is a base for the ultratopology ν .* ■

3. Generalized closed sets

Let us now consider $\lambda, \lambda' \subset \exp X$. Let us denote by $\phi(\lambda, \lambda')$ the family of the sets $A \subset X$ having the following property: if $A \subset L' \in \lambda'$ then $c_\lambda A \subset L'$. The elements of this family are called (λ, λ') -closed.

The terminology is justified by

LEMMA 3.1. *Every λ -closed set belongs to $\phi(\lambda, \lambda')$.*

PROOF. $F \in \lambda^*$ implies $X - F = i_\lambda(X - F)$ so that, by (1.3), $F = c_\lambda F$. ■

With the help of the operations introduced above, it is easy to characterize the family $\phi(\lambda, \lambda')$:

PROPOSITION 3.2. *The following statements are equivalent:*

- a) $A \in \phi(\lambda, \lambda')$,
- b) $c_\lambda A \subset c_{\lambda'^*} A$.

PROOF. a) \Rightarrow b): Define $C = c_{\lambda'^*} A$. If $C = X$ then clearly $c_\lambda A \subset C$. If $C \neq X$ then C is the intersection of all sets L' such that $A \subset L' \in (\lambda'^*)^* = \lambda'$. For every such set L' , $c_\lambda A \subset L'$, consequently $c_\lambda A \subset C = c_{\lambda'^*} A$.

b) \Rightarrow a): If $A \subset L' \in \lambda'$ then $c_{\lambda'^*} A \subset L'$ so that $c_\lambda A \subset L'$ by b). Hence $A \in \phi(\lambda, \lambda')$. ■

THEOREM 3.3. *If λ^* is a GT then $\phi(\lambda, \lambda')$ is a GT as well.*

PROOF. As $\emptyset \in \lambda^*$, we have $c_\lambda \emptyset = \emptyset$ and $\emptyset \in \phi(\lambda, \lambda')$. If $A_k \in \phi(\lambda, \lambda')$ for $k \in K \neq \emptyset$ and $A = \bigcup_{k \in K} A_k$, choose $A \subset L' \in \lambda'$. Then $A_k \subset L'$ for each k , hence $c_\lambda A_k \subset L'$, so that (1.4) implies $c_\lambda A \subset L'$. ■

If μ is a GT on X , we can choose $\lambda = \mu^*$ and then the families ν, ν^* and $\phi(\lambda, \lambda)$ are all GT's. The following simple example shows that they can be distinct.

EXAMPLE 3.4. Let $X = \{a, b, c\}$, $\mu = \{\emptyset, \{a, b\}, \{a, c\}, X\}$. Then for $\lambda = \mu^*$, we have $\lambda = \{\emptyset, \{b\}, \{c\}, X\}$. Now $i_\lambda A$ is equal to \emptyset for $A = \emptyset$ or $\{a\}$, to $\{b\}$ for $A = \{b\}$ or $\{a, b\}$, to $\{c\}$ for $A = \{c\}$ or $\{a, c\}$, to $\{b, c\}$ for $A = \{b, c\}$ and to X for $A = X$. Therefore $A = i_\lambda A$ holds for $A = \emptyset, \{b\}, \{c\}, \{b, c\}, X$ so that $\nu = \{\emptyset, \{b\}, \{c\}, \{b, c\}, X\}$. Further

$v^* = \{\emptyset, \{a\}, \{a, b\}, \{a, c\}, X\}$. Finally, by $\lambda^* = \mu$, c_λ and c_{λ^*} are defined as follows:

$$\begin{aligned} A &= \emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, X; \\ c_\lambda A &= \emptyset, \{a\}, \{a, b\}, \{a, c\}, \{a, b\}, \{a, c\}, X, X; \\ c_{\lambda^*} A &= \emptyset, X, \{b\}, \{c\}, X, X, X. \end{aligned}$$

Therefore $\phi(\lambda, \lambda)$ is composed of the sets A with $c_\lambda A \subset c_{\lambda^*} A$, i.e., equals $\{\emptyset, \{a\}, \{a, b\}, \{a, c\}, \{b, c\}, X\}$.

4. The case $\lambda = \mu, \lambda' = m$

In the paper [12], the following particular case of $\phi(\lambda, \lambda')$ is considered: one chooses a GT μ on X and a *minimality* m (i.e., a subset $m \subset \exp X$ satisfying $\emptyset \in m$). Then $\phi(\mu, m)$ is composed of all sets $A \subset X$ such that $A \subset M \in m$ implies $c_\mu A \subset M$; these sets A are called μmg -closed in [12]. Different particular cases are studied in [4] and [18] so that the results in [12] produce generalizations of the results in [4] or [18].

Here we formulate a slight sharpening of [12], 4.2:

THEOREM 4.1. *If A is μmg -closed then $c_\mu A - A$ does not contain any m -closed set $F \neq \emptyset$. Conversely if m is a GT such that $m \supset \mu$ and the above condition is fulfilled then A is μmg -closed.*

PROOF. Suppose that A is μmg -closed and assume that $F \subset c_\mu A - A$ is m -closed. Then $M = X - F \in m$ and $F \subset X - A$ implies $A \subset M$. By hypothesis, $c_\mu A \subset M = X - F$ so that $c_\mu A \cap F = \emptyset$ in contradiction to $F \subset c_\mu A$, except if $F = \emptyset$.

Conversely (cf. the proof of [12], 4.2), if $m \supset \mu$ is a GT and $c_\mu A - A$ does not contain any m -closed set $F \neq \emptyset$, let $A \subset M \in m$. Then $F = c_\mu A \cap (X - M)$ is m -closed and $X - M \subset X - A$ implies $F \subset c_\mu A - A$. Hence $F = \emptyset$ and $c_\mu A \subset M$: A is μmg -closed. ■

The above theorem generalizes [4], Theorems 2.6 and 2.9.

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λ -OPEN SETS IN GENERALIZED TOPOLOGICAL SPACES

By

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The concept of a λ -open set is generalized in the way that the given topology is replaced by a generalized topology.

0. INTRODUCTION

Let (X, τ) be a topological space. According to [1], a set $L \subset X$ is said to be a Λ -set iff L is the intersection of all open supersets of L . A set $A \subset X$ is said to be λ -closed iff $A = L \cap D$ where L is a Λ -set and D is closed. A set $A \subset X$ is said to be λ -open iff $X - A$ is λ -closed.

The purpose of the present paper is to show that these constructions work in the more general case when the topology τ is replaced by a generalized topology in the sense of [3] and to show some simple applications.

1. PRELIMINARIES

Let X be a (non-empty) set. According to [3], a *generalized topology* (briefly GT) on X is a subset μ of the power set $\exp X$ of X such that $\emptyset \in \mu$ and every union of elements of μ belongs to μ . The elements of μ are called μ -*open*, their complements μ -*closed*. According to e.g. [5], the union of all

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μ -open subsets of $A \subset X$ defines $i_\mu A$ and the intersection of all μ -closed supersets of A defines $c_\mu A$. These operations fulfil, for $A, B \subset X$,

$$(1.1) \quad i_\mu i_\mu A = i_\mu A \subset A,$$

$$(1.2) \quad c_\mu c_\mu A = c_\mu A \supseteq A,$$

$$(1.3) \quad i_\mu A \subset i_\mu B \quad \text{and} \quad c_\mu A \subset c_\mu B \quad \text{if} \quad A \subset B$$

$$(1.4) \quad c_\mu A = X - i_\mu(X - A) \quad \text{and} \quad i_\mu A = X - c_\mu(X - A).$$

$i_\mu A$ is always μ -open and $c_\mu A$ is μ -closed.

If μ is a GT on X , then, in general, $X \in \mu$ need not hold. If, all the same, $X \in \mu$ then μ is said to be a strong GT (see [4]). In the general case $i_\mu X = M_\mu = \bigcup\{M : M \in \mu\}$, hence $c_\mu \emptyset = X - M_\mu$.

A family $\mathcal{B} \subset \mu$ is said to be a *base* for a GT μ (see [6] or [7]) iff every element of μ is union of elements of \mathcal{B} .

If μ is a GT, we define, according to [5], the GT α (α, π, ξ, β) to be composed of all sets $A \subset X$ satisfying $A \subset i_\mu c_\mu i_\mu A$ ($A \subset c_\mu i_\mu A$, $A \subset \subset i_\mu c_\mu A$, $A \subset c_\mu i_\mu A \cup i_\mu c_\mu A$, $A \subset c_\mu i_\mu c_\mu A$). For these GT's, the following inclusions always hold:

$$(1.5) \quad \mu \subset \alpha \subset \sigma \subset \xi \subset \beta$$

and

$$(1.6) \quad \alpha \subset \pi \subset \xi.$$

As a consequence of (1.4) and [2], 1.8, $A \subset X$ is σ -closed iff $A \supset i_\mu c_\mu A$, A is π -closed iff $A \supset c_\mu i_\mu A$, A is α -closed iff $A \supset c_\mu i_\mu c_\mu A$, A is β -closed iff $A \supset i_\mu c_\mu i_\mu A$, finally A is ξ -closed iff $A \supset i_\mu c_\mu A \cap c_\mu i_\mu A$.

2. λ -OPEN SETS IN A GT

Consider a GT μ on X . Let Λ be composed of all sets $L \subset X$ equal to the intersection of all μ -open supersets of L .

LEMMA 2.1. $L \in \Lambda$ iff L is the intersection of a family of μ -open sets. ■

Let us say that a set A is λ -closed iff $A = L \cap D$ where $L \in \Lambda$ or $L = X$ and D is μ -closed.

LEMMA 2.2. A set is λ -closed iff it is the intersection of a family of sets each of them being μ -open or μ -closed.

PROOF. If $A = L \cap D$, $L \in \Lambda$, D is μ -closed, then L is the intersection of a family of μ -open sets by 2.1, hence A is the intersection of these μ -open

sets and the μ -closed set D . In the case $L = X$, A is the intersection of the μ -closed sets X and D .

Conversely, if A is the intersection of a family of partly μ -open and partly μ -closed sets, then $A = L \cap D$ where L is the intersection of the μ -open factors, hence $L \in A$ by 2.1, and D is the intersection of the μ -closed factors, hence μ -closed as well. If all factors are μ -open, then we can put $D = X$, and if all factors are μ -closed then we put $L = X$. ■

Let us say that $A \subset X$ is λ -open iff $X - A$ is λ -closed. Let us denote by λ the family of all λ -open sets. We deduce from 2.2:

LEMMA 2.3. *λ is a GT and the family \mathcal{B} of all μ -open and all μ -closed sets is a base for the GT λ .* ■

More precisely:

THEOREM 2.4. *λ is a strong GT.*

PROOF. X is λ -open since $\emptyset = \emptyset \cap X$ is λ -closed. ■

If it is necessary, we write $\lambda(\mu)$ instead of λ .

LEMMA 2.5. *If μ and μ' are GT's on X such that $\mu \subset \mu'$ then $\lambda(\mu) \subset \lambda(\mu')$.*

PROOF. 2.3. ■

3. SOME APPLICATIONS

In the following, we examine the GT's $\lambda(\alpha)$, $\lambda(\sigma)$, $\lambda(\pi)$, $\lambda(\zeta)$, $\lambda(\beta)$ for a given GT μ and the GT's α , σ , π , ζ , β defined with the help of the given GT μ .

THEOREM 3.1. *If μ is an arbitrary GT on X , then $\lambda(\zeta)$ is the discrete topology on X .*

PROOF. We show that, for $x \in X$, the set $\{x\}$ is either ζ -open or ζ -closed, hence $\lambda(\zeta)$ -open. In fact, if $\{x\} \in \mu$ then $\{x\} \in \zeta$ by (1.5). On the other hand, if $\{x\} \notin \mu$ then $i_\mu \{x\} = \emptyset$, hence $c_\mu i_\mu \{x\} = X - M_\mu$, while $i_\mu c_\mu \{x\} \in \mu$ is contained in M_μ so that $c_\mu i_\mu \{x\} \cap i_\mu c_\mu \{x\} = \emptyset \subset \{x\}$ so that $\{x\}$ is ζ -closed. ■

COROLLARY 3.2. If μ is an arbitrary GT on X then $\lambda(\beta)$ is the discrete topology on X .

PROOF. (1.5) and 2.5. ■

For $\lambda(\pi)$, we have a weaker statement:

THEOREM 3.3. If μ is a strong GT on X then $\lambda(\pi)$ is the discrete topology on X .

PROOF. If $x \in X$ and $\{x\} \in \mu$ then $\{x\} \in \pi$ by (1.5). If $\{x\} \notin \mu$ then $i_\mu \{x\} = \emptyset$, hence $c_\mu i_\mu \{x\} = \emptyset$ as \emptyset is μ -closed by $X \in \mu$, consequently $\{x\}$ is π -closed. ■

The statement of 3.3 need not hold if the GT μ is not strong:

EXAMPLE 3.4. Let $X = \{a, b, c\}$ and $\mu = \{\emptyset, \{a\}\}$. Now we have $c_\mu \{b\} = \{b, c\}$, $i_\mu \{b, c\} = \emptyset$ does not contain $\{b\}$ so that $\{b\} \notin \pi$. On the other hand, $i_\mu \{b\} = \emptyset$, $c_\mu \emptyset = \{b, c\}$ is not contained in $\{b\}$ so that $\{b\}$ is not π -closed and $\{b\} \notin \lambda(\pi)$. Therefore $\lambda(\pi)$ cannot be equal to the discrete topology. ■

It can happen that μ is a topology on X and neither $\lambda(\alpha)$ nor $\lambda(\sigma)$ equals to the discrete topology:

EXAMPLE 3.5. Let $X = \{a, b\}$, $\mu = \{\emptyset, X\}$. Now $i_\mu \{a\} = \emptyset$, $c_\mu \emptyset = \emptyset$ so that $\{a\} \notin \sigma$, while $c_\mu \{a\} = X$, $i_\mu X = X$ and $\{a\}$ is not σ -closed. Therefore $\lambda(\sigma)$ does not be equal to the discrete topology. By (1.5) and 2.5 the same is true for $\lambda(\alpha)$. ■

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**ON \mathcal{E} -SETS AND \mathcal{F} -SETS AND DECOMPOSITIONS OF
 α -CONTINUITY, QUASI-CONTINUITY AND \mathcal{A} -CONTINUITY**

By

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1. Introduction

In general topology, mathematicians have introduced in several papers different and interesting new decompositions of continuous functions as well as of some other weak forms of continuous functions. In two recent papers, Al-Nashef [2] has introduced a decomposition of α -continuity and quasi-continuity via αLC -sets and $\alpha \mathcal{B}$ -sets in 2002. Aslim and Ayhan [4] have introduced an other decomposition of α -continuity via αA -sets and αLC -sets in 2004. In this paper, we introduce the notions of \mathcal{E} -sets, \mathcal{F} -sets, \mathcal{E} -continuity and \mathcal{F} -continuity. Also, the relationships between these classes of sets and other related sets are investigated. Together with some other forms of continuities, we obtain some new decompositions of α -continuity, quasi-continuity and \mathcal{A} -continuity.

In this paper (X, τ) and (Y, σ) represent topological spaces. For a subset K of a space X , $\text{cl}(K)$ and $\text{int}(K)$ denote the closure of K and the interior of K , respectively. A subset K of a space (X, τ) is called regular open (resp. regular closed) [16] if $K = \text{int}(\text{cl}(K))$ (resp. $K = \text{cl}(\text{int}(K))$). A subset K is said to be δ -open [18] if for each $x \in K$, there exists a regular open set G such that $x \in G \subset K$. A point $x \in X$ is called a δ -cluster point of K [18] if $K \cap \text{int}(\text{cl}(U)) \neq \emptyset$ for each open set U containing x . The set of all δ -cluster points of K is called the δ -closure of K and is denoted by $\delta\text{-cl}(K)$. If $\delta\text{-cl}(K) = K$, then K is said to be δ -closed. The set $\{x \in X : x \in U \subset K$ for some regular open set U of $X\}$ is called the δ -interior of K and is denoted by $\delta\text{-int}(K)$. A subset K of a space (X, τ) is called semiopen [10]

(resp. α -open [13], β -open [1], b -open [3] or γ -open [8] or sp-open [6], preopen [11], δ -preopen [15], δ -semiopen [14]) if $K \subset \text{cl}(\text{int}(K))$ (resp. $K \subset \text{int}(\text{cl}(\text{int}(K)))$, $K \subset \text{cl}(\text{int}(\text{cl}(K)))$, $K \subset \text{int}(\text{cl}(K)) \cup \text{cl}(\text{int}(K))$, $K \subset \text{int}(\text{cl}(K))$, $K \subset \text{int}(\delta\text{-cl}(K))$, $K \subset \text{cl}(\delta\text{-int}(K))$). The complement of a δ -semiopen set is called a δ -semiclosed set. The intersection of all δ -semiclosed sets, each containing a set K in a topological space X is called the δ -semiclosure of K and it is denoted by $\delta\text{-scl}(K)$ [14].

DEFINITION 1. A subset K of a space (X, τ) is called

- (1) a \mathcal{D} -set [7] if $K \in \mathcal{D}(X) = \{V \cap T : V \in \tau \text{ and } T \text{ is } \delta\text{-closed}\}$;
- (2) a \mathcal{DS} -set [7] if $K = V \cap T$, where V is open and T is δ -semiclosed;
- (3) an \mathcal{A} -set [17] if $K \in \mathcal{A}(X) = \{V \cap T : V \in \tau, T = \text{cl}(\text{int}(T))\}$;
- (4) an αLC -set [2] if $K \in \alpha LC(X) = \{V \cap T : V \text{ is } \alpha\text{-open and } T \text{ is closed}\}$;
- (5) an $\alpha\mathcal{B}$ -set [2] if $K \in \alpha\mathcal{B}(X) = \{V \cap T : V \text{ is } \alpha\text{-open and } T \text{ is semiclosed}\}$;
- (6) an αA -set [4] if $K \in \alpha A(X) = \{V \cap T : V \text{ is } \alpha\text{-open and } \text{cl}(\text{int}(T)) = T\}$;
- (7) a B_2 -set [4] if $K \in B_2(X) = \{V \cap T : V \text{ is } \alpha\text{-open and } \text{cl}(T) = X\}$;
- (8) an LC -set [5] if $K \in LC(X) = \{V \cap T : V \in \tau, \text{cl}(T) = T\}$.

DEFINITION 2. A function $f : (X, \tau) \rightarrow (Y, \sigma)$ is called β -continuous [1] (resp. α -continuous [12], γ -continuous [8], quasi-continuous [9], precontinuous [11], δ -almost continuous [15]) if $f^{-1}(V)$ is β -open (resp. α -open, γ -open, semiopen, preopen, δ -preopen) for each $V \in \sigma$.

2. \mathcal{E} -sets and \mathcal{F} -sets

DEFINITION 3. A subset K of a topological space X is called an \mathcal{E} -set if $K = V \cap T$, where V is α -open and T is δ -closed.

The family of all \mathcal{E} -sets of a space X will be denoted by $\mathcal{E}(X)$.

LEMMA 4 ([13]). *Let X be a topological space and $A \subset X$. Then A is α -open if and only if $A = V \cap T$ where V is open and $\text{int}(T)$ is dense.*

THEOREM 5. *The following are equivalent for a subset K of a space X :*

- (1) $K \in \mathcal{E}(X)$,
- (2) $K = V \cap T$ where V is a \mathcal{D} -set and $\text{int}(T)$ is dense.

PROOF. (\Rightarrow): Let $K \in \mathcal{E}(X)$. This implies $K = U \cap F$ where U is α -open and F is δ -closed. By Lemma 4, we have $U = W \cap T$ where W is open and

$\text{int}(T)$ is dense. Moreover, we have $K = U \cap F = W \cap T \cap F = (W \cap F) \cap T$ such that $W \cap F$ is a \mathcal{D} -set and $\text{int}(T)$ is dense.

(\Leftarrow): Let $K = V \cap T$ where V is a \mathcal{D} -set and $\text{int}(T)$ is dense. Since V is a \mathcal{D} -set, there exist an open set M and a δ -closed set R such that $V = M \cap R$. We have $K = V \cap T = M \cap R \cap T = (M \cap T) \cap R$ where $M \cap T$ is an α -open by Lemma 4. Thus, $K \in \mathcal{E}(X)$. ■

DEFINITION 6. A subset K of a topological space X is called an \mathcal{F} -set if $K = V \cap T$, where V is α -open and T is δ -semiclosed.

The family of all \mathcal{F} -sets of a space X will be denoted by $\mathcal{F}(X)$.

THEOREM 7. *The following are equivalent for a subset K of a space X :*

- (1) $K \in \mathcal{F}(X)$,
- (2) $K = V \cap T$ where V is a \mathcal{DS} -set and $\text{int}(T)$ is dense.

PROOF. It is similar to that of Theorem 5. ■

REMARK 8. (1) The following diagram holds for a subset K of a space X :

$$\begin{array}{ccc} \alpha LC\text{-set} & \Rightarrow & \alpha \mathcal{B}\text{-set} \\ \uparrow & & \uparrow \\ \mathcal{E}\text{-set} & \Rightarrow & \mathcal{F}\text{-set} \\ \uparrow & & \uparrow \\ \alpha A\text{-set} & & \end{array}$$

(2) Every α -open and every δ -closed set is an \mathcal{E} -set and every α -open and every δ -semiclosed set is an \mathcal{F} -set.

None of these implications is reversible as shown in the following examples.

EXAMPLE 9. Let $X = \{x, y, w, z\}$ and let $\tau = \{\emptyset, X, \{x\}, \{w\}, \{x, y\}, \{x, w\}, \{x, y, w\}, \{x, w, z\}\}$. Then

- (1) the set $\{w, z\}$ is an \mathcal{E} -set and so an \mathcal{F} -set but it is not α -open;
- (2) the set $\{x, z\}$ is an \mathcal{E} -set and so an \mathcal{F} -set but it is neither δ -closed nor δ -semiclosed;
- (3) the set $\{z\}$ is an \mathcal{E} -set but it is not an $\alpha \mathcal{A}$ -set;
- (4) the set $\{y, w, z\}$ is an αLC -set and an $\alpha \mathcal{B}$ -set but it is neither an \mathcal{E} -set nor an \mathcal{F} -set.

EXAMPLE 10. Let $X = \{a, b, c, d\}$ and let $\tau = \{\emptyset, X, \{b\}, \{d\}, \{b, d\}\}$. The set $\{c, d\}$ is an \mathcal{F} -set and it is neither an αLC -set nor an \mathcal{E} -set.

THEOREM 11. *The following are equivalent for a subset K of a space X :*

- (1) $K \in \mathcal{E}(X)$,
- (2) $K = V \cap \delta\text{-cl}(K)$ for some α -open set V .

PROOF. (\Rightarrow): Let $K \in \mathcal{E}(X)$. We have $K = V \cap F$ where V is α -open and F is δ -closed. Since $K \subset F$, $\delta\text{-cl}(K) \subset \delta\text{-cl}(F) = F$. This implies $V \cap \delta\text{-cl}(K) \subset V \cap F = K \subset V \cap \delta\text{-cl}(K)$ and hence $K = V \cap \delta\text{-cl}(K)$.

(\Leftarrow): Let $K = V \cap \delta\text{-cl}(K)$ for some α -open set V . Since $\delta\text{-cl}(K)$ is δ -closed, $K \in \mathcal{E}(X)$. ■

THEOREM 12. *The following are equivalent for a subset K of a space X :*

- (1) $K \in \mathcal{F}(X)$,
- (2) $K = V \cap \delta\text{-scl}(K)$ for some α -open set V .

PROOF. Similar to that of Theorem 11. ■

THEOREM 13. *The following are equivalent for a subset K of a space X :*

- (1) K is α -open,
- (2) K is preopen and an \mathcal{E} -set,
- (3) K is preopen and an \mathcal{F} -set,
- (4) K is δ -preopen and an \mathcal{E} -set,
- (5) K is δ -preopen and an \mathcal{F} -set,
- (6) K is a B_2 -set and an \mathcal{E} -set,
- (7) K is a B_2 -set and an \mathcal{F} -set,
- (8) K is preopen and an αA -set,
- (9) K is δ -preopen and an αA -set,
- (10) K is a B_2 -set and an αLC -set.

PROOF. (1) \Rightarrow (2): Since every α -open set is both preopen and \mathcal{E} -set, the proof is obvious.

(2) \Rightarrow (3) \Rightarrow (5): Obvious.

(2) \Rightarrow (4) \Rightarrow (5): Obvious.

(5) \Rightarrow (1): Let K be δ -preopen and an \mathcal{F} -set. Since K is δ -preopen, then $K \subset \text{int}(\delta\text{-cl}(K))$. From $K \in \mathcal{F}(X)$ we have $K = U \cap \delta\text{-scl}(K)$ for some α -open set U . On the other hand $\delta\text{-scl}(K) = K \cup \text{int}(\delta\text{-cl}(K)) = \text{int}(\delta\text{-cl}(K))$ since $K \subset \text{int}(\delta\text{-cl}(K))$. Hence, $K = U \cap \delta\text{-scl}(K) = U \cap \text{int}(\delta\text{-cl}(K))$. Thus, K is α -open.

(1) \Rightarrow (6): It follows from the fact that every α -open set is both

a B_2 -set and an \mathcal{E} -set.

(6) \Rightarrow (7): Obvious.

(7) \Rightarrow (1): Let K be a B_2 -set and an \mathcal{F} -set. Since K is a B_2 -set, by Proposition 3.11 [4] then it is preopen. Hence, K is preopen and an \mathcal{F} -set. Thus, by (3) K is α -open.

(1) \Rightarrow (8) \Rightarrow (9): Obvious.

(9) \Rightarrow (4): It follows from Remark 8.

(6) \Rightarrow (10): Obvious.

(10) \Rightarrow (1): Let K be a B_2 -set and an αLC -set. Since K is a B_2 -set, by Proposition 3.11 [4] then it is preopen. Hence, K is preopen and an αLC -set. Thus, by Theorem 3.13 [4] K is α -open. ■

THEOREM 14. *The following are equivalent for a subset K of a space X :*

- (1) K is an \mathcal{A} -set,
- (2) K is an αA -set and a \mathcal{D} -set.

PROOF. (1) \Rightarrow (2): It follows from the fact that every \mathcal{A} -set is an αA -set and an \mathcal{D} -set.

(2) \Rightarrow (1): Let K be an αA -set and a \mathcal{D} -set. Then K is an LC -set. By Theorem 3.9 [4], K is an \mathcal{A} -set. ■

THEOREM 15. *The following are equivalent for a subset K of a space X :*

- (1) K is semiopen,
- (2) K is γ -open and an \mathcal{E} -set,
- (3) K is β -open and an \mathcal{E} -set.

PROOF. (1) \Rightarrow (2): Let K be semiopen. This implies that K is γ -open. Also, by Proposition 3.2 [2], there exists a regular closed set A such that $K = A \cap B$ where $\text{int}(B)$ is dense. Since A is a \mathcal{D} -set, by Theorem 5 K is an \mathcal{E} -set.

(2) \Rightarrow (3): Obvious.

(3) \Rightarrow (1): Let K be β -open and an \mathcal{E} -set. By Theorem 5, there exists a \mathcal{D} -set A such that $K = A \cap B$ where $\text{int}(B)$ is dense. Since every \mathcal{D} -set is an LC -set, by Proposition 3.5 [2] K is an αLC -set. Thus, by Theorem 3.7 [2], K is semiopen. ■

THEOREM 16. *The following are equivalent for a subset K of a space X :*

- (1) K is open,
- (2) K is a B_2 and αA -set and a \mathcal{D} -set,
- (3) K is a preopen αA -set and a \mathcal{D} -set,
- (4) K is a δ -preopen αA -set and a \mathcal{D} -set.

PROOF. (1) \Rightarrow (2): Obvious.

(2) \Rightarrow (3): Since every B_2 -set is preopen by Proposition 3.11 [4], the proof is completed.

(3) \Rightarrow (4): Obvious.

(4) \Rightarrow (1): Let K be a δ -preopen αA -set and a \mathcal{D} -set. Since K is δ -preopen and a \mathcal{D} -set, it is δ -preopen and an \mathcal{E} -set. So, by Theorem 13 K is α -open. Since K is an αA -set and a \mathcal{D} -set, by Theorem 14 it is an \mathcal{A} -set. Thus, by Theorem 3.2 [17], K is open. ■

3. Decompositions of α -continuity, quasi-continuity and \mathcal{A} -continuity

DEFINITION 17. A function $f : (X, \tau) \rightarrow (Y, \sigma)$ is called \mathcal{E} -continuous (resp. \mathcal{F} -continuous) if $f^{-1}(V) \in \mathcal{E}(X)$ (resp. $f^{-1}(V) \in \mathcal{F}(X)$) for each $V \in \sigma$.

DEFINITION 18. A function $f : (X, \tau) \rightarrow (Y, \sigma)$ is called $\alpha\mathcal{B}$ -continuous [2] (resp. αA -continuous [4], αLC -continuous [2], B_2 -continuous [4], \mathcal{D} -continuous [7]) if $f^{-1}(V) \in \alpha\mathcal{B}(X)$ (resp. $f^{-1}(V) \in \alpha A(X)$, $f^{-1}(V) \in \alpha LC(X)$, $f^{-1}(V) \in B_2(X)$, $f^{-1}(V) \in \mathcal{D}(X)$) for each $V \in \sigma$.

REMARK 19. The following diagram holds for a function $f : X \rightarrow Y$:

$$\begin{array}{ccc} \alpha LC\text{-continuous} & \Rightarrow & \alpha\mathcal{B}\text{-continuous} \\ \uparrow & & \uparrow \\ \alpha A\text{-continuous} & \Rightarrow & \mathcal{E}\text{-continuous} \Rightarrow \mathcal{F}\text{-continuous} \end{array}$$

None of these implications is reversible as shown in the following examples:

EXAMPLE 20. Let $X = \{x, y, w, z\}$ and $\tau = \{\emptyset, \{x\}, \{w\}, \{x, y\}, \{x, w\}, \{x, y, w\}, \{x, w, z\}, X\}$. Then

- (1) the function $f : (X, \tau) \rightarrow (X, \tau)$, defined as $f(x) = z, f(y) = z, f(w) = x, f(z) = w$, is \mathcal{E} -continuous but it is not $\alpha\mathcal{A}$ -continuous;
- (2) the function $h : (X, \tau) \rightarrow (X, \tau)$, defined as $h(x) = z, h(y) = w, h(w) = y, h(z) = x$, is αLC -continuous and $\alpha\mathcal{B}$ -continuous but it is not \mathcal{E} -continuous and \mathcal{F} -continuous.

EXAMPLE 21. Let $X = \{a, b, c, d\}$ and $\tau = \{\emptyset, X, \{b\}, \{d\}, \{b, d\}\}$. Define a function $f : (X, \tau) \rightarrow (X, \tau)$ as follows: $f(a) = c, f(b) = a, f(c) = b, f(d) = b$. Then f is \mathcal{F} -continuous but it is neither \mathcal{E} -continuous nor αLC -continuous.

THEOREM 22. *For a function $f : X \rightarrow Y$, the following are equivalent:*

- (1) f is α -continuous,
- (2) f is precontinuous and \mathcal{E} -continuous,
- (3) f is precontinuous and \mathcal{F} -continuous,
- (4) f is δ -almost continuous and \mathcal{E} -continuous,
- (5) f is δ -almost continuous and \mathcal{F} -continuous,
- (6) f is B_2 -continuous and \mathcal{E} -continuous,
- (7) f is B_2 -continuous and \mathcal{F} -continuous,
- (8) f is precontinuous and αA -continuous,
- (9) f is δ -almost continuous and αA -continuous,
- (10) f is B_2 -continuous and αLC -continuous.

PROOF. It follows from Theorem 13. ■

THEOREM 23. *For a function $f : X \rightarrow Y$, the following are equivalent:*

- (1) f is \mathcal{A} -continuous,
- (2) f is αA -continuous and \mathcal{D} -continuous.

PROOF. The proof is an immediate consequence of Theorem 14. ■

THEOREM 24. *For a function $f : X \rightarrow Y$, the following are equivalent:*

- (1) f is quasi-continuous,
- (2) f is γ -continuous and \mathcal{E} -continuous,
- (3) f is β -continuous and \mathcal{E} -continuous.

PROOF. It follows from Theorem 15. ■

THEOREM 25. *For a function $f : X \rightarrow Y$, the following are equivalent:*

- (1) f is continuous;
- (2) f is B_2 -continuous, αA -continuous and \mathcal{D} -continuous;
- (3) f is precontinuous, αA -continuous and \mathcal{D} -continuous;
- (4) f is δ -almost continuous, αA -continuous and \mathcal{D} -continuous.

PROOF. The proof is an immediate consequence of Theorem 16. ■

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ON THE CLASS OF CONTRA λ -CONTINUOUS FUNCTIONS

By

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*(Received October 18, 2006)***1. Introduction**

Maki [18] in 1986 introduced the notion of Λ -sets in topological spaces. A Λ -set is a set A if it is equal to its kernel (= saturated set), i.e., to the intersection of all open supersets of A . Arenas et al. [1] introduced and investigated the notion of λ -closed sets by involving Λ -sets and closed sets. This enabled them to obtain some nice results. Quite recently, Caldas et al. [3] introduced the notion of λ -closure of a set by utilizing the notion of λ -open sets defined in [1]. Recently, Jafari and Noiri introduced and investigated the notions of contra-precontinuity [14], contra- α -continuity [12] and contra-super-continuity [13] as a continuation of research done by Dontchev [8] and Dontchev and Noiri [9] on the interesting notions of contra-continuity and contra-semi-continuity, respectively. Caldas and Jafari [6] introduced and investigated the notion of contra- β -continuous functions in topological spaces. The present paper has as its purpose to investigate some properties of contra λ -continuous functions, by using λ -open sets.

Throughout this paper, by (X, τ) and (Y, σ) (or X and Y) we always mean topological spaces. Let A be a subset of X . We denote the interior, the closure and the complement of a set A by $\text{Int}(A)$, $\text{Cl}(A)$ and $X \setminus A$ or A^c , respectively. A subset A of X is said to be regular open (resp. regular closed) if $A = \text{Int}(\text{Cl}(A))$ (resp. $A = \text{Cl}(\text{Int}(A))$). A subset A of a space X is called pre-open [17] (resp. semi-open [16], α -open [21], β -open [2]) if $A \subset \text{Int}(\text{Cl}(A))$ (resp. $A \subset \text{Cl}(\text{Int}(A))$, $A \subset \text{Int}(\text{Cl}(\text{Int}(A)))$, $A \subset \text{Cl}(\text{Int}(\text{Cl}(A)))$). The complement of a preopen (resp. semi-open, α -open, β -open) set is said to be

preclosed (resp. semi-closed, α -closed, β -closed). The collection of all closed (resp. clopen) subsets of X will be denoted by $C(X)$ (resp. $CO(X)$). We set $C(X, x) = \{V \in C(X) : x \in V\}$ for $x \in X$. We define similarly $CO(X, x)$.

A subset B of a topological space X is called g -closed in X [15] if $Cl(B) \subset G$ whenever $B \subset G$ and G is open in X . A subset A of (X, τ) is called λ -closed [1] if $A = L \cap D$, where L is a Λ -set and D is a closed set. The complement of a λ -closed set is called λ -open. We denote the collection of all λ -open sets (resp. λ -closed sets) by $\lambda O(X, \tau)$ (resp. $\lambda C(X, \tau)$). We set $\lambda O(X, x) = \{U : x \in U \in \lambda O(X, \tau)\}$ and $\lambda C(X, x) = \{U : x \in U \in \lambda C(X, \tau)\}$. A point x in a topological space (X, τ) is called a λ -cluster point of A [3] if every λ -open set U of X containing x fulfils $A \cap U \neq \emptyset$. The set of all λ -cluster points is called the λ -closure of A and is denoted by $Cl_\lambda(A)$ ([1, 3]). A subset B_x of a topological space X is said to be λ -neighborhood of a point [3] $x \in X$ if there exists a λ -open set U such that $x \in U \subset B_x$.

LEMMA 1.1 ([1, 3]). *Let A , B and A_i ($i \in I$) be subsets of a topological space (X, τ) . The following properties hold:*

- (1) *If A_i is λ -closed for each $i \in I$, then $\cap_{i \in I} A_i$ is λ -closed.*
- (2) *If A_i is λ -open for each $i \in I$, then $\cup_{i \in I} A_i$ is λ -open.*
- (3) *A is λ -closed if and only if $A = Cl_\lambda(A)$.*
- (4) $Cl_\lambda(A) = \cap\{F \in \lambda C(X, \tau) : A \subset F\}$.
- (5) $A \subset Cl_\lambda(A)$.
- (6) *If $A \subset B$, then $Cl_\lambda(A) \subset Cl_\lambda(B)$.*
- (7) *$Cl_\lambda(A)$ is λ -closed.*

Recall that a topological space (X, τ) is said to be:

- (i) $\lambda T_{1/2}$ ([3, 4]) if every singleton is λ -open or λ -closed.
- (ii) λT_2 ([3, 4]) if for any pair of distinct points x and y in X , there exist $U \in \lambda O(X, x)$ and $V \in \lambda O(X, y)$ such that $U \cap V = \emptyset$.
- (iii) Ultra Hausdorff [22] if for each pair of distinct points x and y in X there exist $U \in CO(X, x)$ and $V \in CO(X, y)$ such that $U \cap V = \emptyset$.

A space X is called locally indiscrete [20] if every open set is closed.

DEFINITION 1. function $f : X \rightarrow Y$ is said to be:

- (i) λ -continuous [1] if the inverse image of every closed set in Y is λ -closed in X , equivalently if the inverse image of every open set in Y is λ -open in X .
- (ii) LC -continuous [11] if the inverse image of every open set in Y is locally closed in X .

2. Contra λ -continuous functions

DEFINITION 2. A function $f : X \rightarrow Y$ is said to be contra λ -continuous if $f^{-1}(V)$ is λ -closed in X for each open set V of Y .

DEFINITION 3. Let A be a subset of a space (X, τ) . The set $\bigcap\{U \in \tau : A \subset U\}$ is called the kernel of A [19] and is denoted by $\ker(A)$.

LEMMA 2.1 (JAFARI AND NOIRI [13]). *The following properties hold for the subsets A, B of a space X :*

- (1) $x \in \ker(A)$ if and only if $A \cap F \neq \emptyset$ for any $F \in C(X, x)$.
- (2) $A \subset \ker(A)$ and $A = \ker(A)$ if A is open in X .
- (3) If $A \subset B$, then $\ker(A) \subset \ker(B)$.

THEOREM 2.2. *Let $f : X \rightarrow Y$ be a function from a topological space X into a topological space Y . The following statements are equivalent.*

- (1) f is contra λ -continuous;
- (2) The inverse image of each closed set in Y is λ -open in X ;
- (3) For each point x in X and each closed set V in Y with $f(x) \in V$, there exists a λ -open set U in X such that $x \in U, f(U) \subset V$;
- (4) For every subset A of X , $f(\text{Cl}_\lambda(A)) \subset \ker(f(A))$;
- (5) For each subset B of Y , $\text{Cl}_\lambda(f^{-1}(B)) \subset f^{-1}(\ker(B))$.

PROOF. (1) \leftrightarrow (2): see Definition 2.

(2) \rightarrow (3): Let $x \in X$ and V be a closed set containing $f(x)$. By (2), $U = f^{-1}(V)$ is a λ -open set containing x such that $f(U) \subset V$.

(3) \rightarrow (4): Let A be any subset of X . Suppose that $y \notin \ker(f(A))$. Then, by Lemma 2.1 there exists $V \in C(Y, y)$ such that $f(A) \cap V = \emptyset$. For any $x \in f^{-1}(V)$, by (3) there exists $U_x \in \lambda O(X, x)$ such that $f(U_x) \subset V$. Hence $f(A \cap U_x) \subset f(A) \cap f(U_x) \subset f(A) \cap V = \emptyset$ and $A \cap U_x = \emptyset$. This shows that $x \notin \text{Cl}_\lambda(A)$ for any $x \in f^{-1}(V)$. Therefore, $f^{-1}(V) \cap \text{Cl}_\lambda(A) = \emptyset$ and hence $V \cap f(\text{Cl}_\lambda(A)) = \emptyset$. Thus, $y \notin f(\text{Cl}_\lambda(A))$. Consequently, we obtain $f(\text{Cl}_\lambda(A)) \subset \ker(f(A))$.

(4) \rightarrow (5): Let B be any subset of Y . By (4) and Lemma 2.1, we have $f(\text{Cl}_\lambda(f^{-1}(B))) \subset \ker(f(f^{-1}(B))) \subset \ker(B)$ and $\text{Cl}_\lambda(f^{-1}(B)) \subset f^{-1}(\ker(B))$.

(5) \rightarrow (1): Let V be any open set of Y . Then, by Lemma 2.1(2) we have $\text{Cl}_\lambda(f^{-1}(V)) \subset f^{-1}(\ker(V)) = f^{-1}(V)$ and $\text{Cl}_\lambda(f^{-1}(V)) = f^{-1}(V)$. This shows that $f^{-1}(V)$ is λ -closed in X . This completes the proof. ■

DEFINITION 4. A function $f : X \rightarrow Y$ is said to be contra-continuous [8] (resp. contra- α -continuous [12], contra-pre-continuous [14], contra-semi-continuous [9], contra- β -continuous [6] and contra- g -continuous [5]) if for each open set V of Y , $f^{-1}(V)$ is closed (resp. α -closed, preclosed, semi-closed, β -closed and g -closed) in X .

For the functions defined above, we have the following implications:

$$\begin{array}{ccc} \text{contra-continuity} & \rightarrow & \text{contra-}\alpha\text{-continuity} & \rightarrow & \text{contra-pre-continuity} \\ \downarrow & & \downarrow & & \downarrow \\ \text{contra-}\lambda\text{-continuity} & & \text{contra-semi-continuity} & \rightarrow & \text{contra-}\beta\text{-continuity} \end{array}$$

REMARK 2.3. (1) The following Examples 2.4 and 2.5 show that λ -continuous and contra- λ -continuous are independent concepts.
(2) The following Examples 2.5 and 2.9 show that contra- λ -continuous and contra- g -continuous are independent concepts.

EXAMPLE 2.4. The identity function on the real line (with the usual topology) is continuous and hence λ -continuous but not contra λ -continuous, since the preimage of each singleton fails to be λ -open.

EXAMPLE 2.5. Let (X, τ) be a topological space such that, $X = \{a, b, c\}$ and $\tau = \{\emptyset, \{a\}, \{a, b\}, X\}$. Clearly $\lambda O(X, \tau) = \{\emptyset, \{a\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, X\}$. Let $f : X \rightarrow X$ be a function defined by $f(b) = a$, $f(c) = b$ and $f(a) = c$. Then f is contra λ -continuous, but f is not λ -continuous and also it is not contra- g -continuous.

REMARK 2.6. It should be mentioned that every contra-continuous function is contra- g -continuous and none of the implications in the above diagram is reversible as shown by the following examples.

EXAMPLE 2.7. Let $X = Y = \{a, b, c\}$, $\tau = \{\emptyset, \{a\}, \{c\}, \{a, c\}, X\}$ and $\sigma = \{\emptyset, \{a\}, Y\}$. Then the identity function $f : (X, \tau) \rightarrow (X, \sigma)$ is contra- λ -continuous but not contra-continuous.

EXAMPLE 2.8 ([12]). Let $X = \{a, b, c\}$, $\tau = \{\emptyset, \{a\}, X\}$ and $\sigma = \{\emptyset, \{b\}, \{c\}, \{b, c\}, X\}$. Then the identity function $f : (X, \tau) \rightarrow (X, \sigma)$ is contra- α -continuous but not contra-continuous.

EXAMPLE 2.9 ([5]). Let $X = \{a, b, c\}$, $\tau = \{\emptyset, \{a\}, X\}$ and $\sigma = \{\emptyset, \{b\}, \{c\}, \{b, c\}, X\}$. Then the identity function $f : (X, \tau) \rightarrow (X, \sigma)$ is contra- g -continuous but not contra-continuous and also it is not contra- λ -continuous.

EXAMPLE 2.10 ([9]). A contra-semi-continuous function need not be contra-pre-continuous. Let $f : R \rightarrow R$ be the function $f(x) = [x]$, where $[x]$ is the Gaussian symbol. If V is a closed subset of the real line, its preimage $U = f^{-1}(V)$ is the union of intervals of the form $[n, n+1]$, $n \in Z$; hence U is semi-open being union of semi-open sets. But f is not contra-pre-continuous, since $f^{-1}(0.5, 1.5) = [1, 2]$ is not preclosed in R .

EXAMPLE 2.11 ([9]). A contra-pre-continuous function need not be contra-semi-continuous. Let $X = \{a, b\}$, $\tau = \{\emptyset, X\}$ and $\sigma = \{\emptyset, \{a\}, X\}$. The identity function $f : (X, \tau) \rightarrow (Y, \sigma)$ is contra-pre-continuous as only the trivial subsets of X are open in (X, τ) . However, $f^{-1}(\{a\}) = \{a\}$ is not semi-closed in (X, τ) ; hence f is not contra-semi-continuous.

Recall that a topological space (X, τ) is called a $T_{1/2}$ -space [15] if every generalized closed subset of X is closed or equivalently if every singleton is open or closed [10].

LEMMA 2.12. *Let (X, τ) be a $T_{1/2}$ -space and $f : X \rightarrow Y$ be a function. If f is contra- β -continuous (resp. contra-semi-continuous, contra-pre-continuous, contra- α -continuous, contra- g -continuous), then f is contra- λ -continuous.*

PROOF. It follows directly from Theorem 2.6 of [1]. ■

Recall that a function $f : X \rightarrow Y$ is said to be RC -continuous [9] if for each open set V of Y , $f^{-1}(V)$ is regular-closed in X .

THEOREM 2.13. (1) *The following statements are equivalent for a function $f : X \rightarrow Y$.*

- (i) f is RC -continuous.
- (ii) f is contra-pre-continuous and semi-continuous.
- (iii) f is contra- α -continuous and β -continuous.
- (iv) f is contra-continuous and β -continuous.

(2) *If $f : X \rightarrow Y$ is RC -continuous, then f is contra λ -continuous.*

PROOF. (1) It follows directly from Theorem 3.9 of [14] and Theorem 3.11 of [9].

(2) Every RC -continuous function is contra-continuous and hence contra λ -continuous. ■

We present a new decomposition of contra-continuity.

THEOREM 2.14. *For a function $f : X \rightarrow Y$ the following conditions are equivalent:*

- (1) *f is contra-continuous.*
- (2) *f is contra- g -continuous and LC-continuous.*
- (3) *f is contra- g -continuous and contra λ -continuous.*

PROOF. It follows directly from [1, Theorem 2.4 and Lemma 2.2(i)]. ■

THEOREM 2.15. *If a function $f : X \rightarrow Y$ is contra λ -continuous and Y is regular, then f is λ -continuous.*

PROOF. Let x be an arbitrary point of X and V an open set of Y containing $f(x)$. Since Y is regular, there exists an open set W in Y containing $f(x)$ such that $\text{Cl}(W) \subset V$. Since f is contra λ -continuous, so by Theorem 2.2 there exists $U \in \lambda O(X, x)$ such that $f(U) \subset \text{Cl}(W)$. Then $f(U) \subset \text{Cl}(W) \subset V$. Hence, f is λ -continuous. ■

REMARK 2.16. By Example 2.4, a λ -continuous function $f : X \rightarrow Y$ is not always contra λ -continuous even if Y is regular.

Recall that Caldas et al. [3] define the λ -frontier of A denoted by $\text{Fr}_\lambda(A)$, as $\text{Fr}_\lambda(A) = \text{Cl}_\lambda(A) \setminus \text{Int}_\lambda(A)$, equivalently $\text{Fr}_\lambda(A) = \text{Cl}_\lambda(A) \cap \text{Cl}_\lambda(X \setminus A)$.

THEOREM 2.17. *The set of points $x \in X$ such that $f : (X, \tau) \rightarrow (Y, \sigma)$ is not contra λ -continuous at x is identical with the union of the λ -frontiers of the inverse images of closed sets of Y containing $f(x)$.*

PROOF. Necessity. Suppose that f is not contra λ -continuous at a point x of X . Then there exists a closed set $F \subset Y$ containing $f(x)$ such that $f(U)$ is not a subset of F for every $U \in \lambda O(X, x)$. Hence we have $U \cap (X \setminus f^{-1}(F)) \neq \emptyset$ for every $U \in \lambda O(X, x)$. It follows that $x \in \text{Cl}_\lambda(X \setminus f^{-1}(F))$. We also have $x \in f^{-1}(F) \subset \text{Cl}_\lambda(f^{-1}(F))$. This means that $x \in \text{Fr}_\lambda(f^{-1}(F))$.

Sufficiency. Suppose that $x \in \text{Fr}_\lambda(f^{-1}(F))$ for some $F \in C(Y, f(x))$. Now, we assume that f is contra λ -continuous at $x \in X$. Then there exists $U \in \lambda O(X, x)$ such that $f(U) \subset F$. Therefore, we have $x \in U \subset f^{-1}(F)$ and hence $x \in \text{Int}_\lambda(f^{-1}(F)) \subset X \setminus \text{Fr}_\lambda(f^{-1}(F))$. This is a contradiction. This means that f is not contra λ -continuous at x . ■

DEFINITION 5. A space (X, τ) is said to be λS -space if every λ -open subset of X is semi-open in X .

LEMMA 2.18. *A space X is locally indiscrete if and only if every λ -open set of X is open in X .*

THEOREM 2.19. *If a function $f : X \rightarrow Y$ is contra λ -continuous and X is a λS -space (resp. locally indiscrete), then f is contra-semi-continuous (resp. contra-continuous, continuous).*

PROOF. It follows immediately from definitions. ■

THEOREM 2.20. *If X is a topological space and for each pair of distinct points x_1 and x_2 in X , there exists a function f of X into a Urysohn topological space Y such that $f(x_1) \neq f(x_2)$ and f is contra λ -continuous at x_1 and x_2 , then X is λ - T_2 .*

PROOF. Let x_1 and x_2 be any distinct points in X . Then by hypothesis, there is a Urysohn space Y and a function $f : X \rightarrow Y$, which satisfies the conditions of the theorem. Let $y_i = f(x_i)$ for $i = 1, 2$. Then $y_1 \neq y_2$. Since Y is Urysohn, there exist open neighborhoods U_{y_1} and U_{y_2} of y_1 and y_2 , respectively, in Y such that $\text{Cl}(U_{y_1}) \cap \text{Cl}(U_{y_2}) = \emptyset$. Since f is contra λ -continuous at x_i , there exists a λ -open neighborhood W_{x_i} of x_i in X such that $f(W_{x_i}) \subset \text{Cl}(U_{y_i})$ for $i = 1, 2$. Hence we get $W_{x_1} \cap W_{x_2} = \emptyset$ since $\text{Cl}(U_{y_1}) \cap \text{Cl}(U_{y_2}) = \emptyset$. Hence X is λ - T_2 . ■

COROLLARY 2.21. *If f is a contra λ -continuous injection of a topological space X into a Urysohn space Y , then X is λ - T_2 .*

PROOF. For each pair of distinct points x_1 and x_2 in X , f is a contra λ -continuous function of X into a Urysohn space Y such that $f(x_1) \neq f(x_2)$ since f is injective. Hence by Theorem 2.20, X is λ - T_2 . ■

COROLLARY 2.22. *If f is a contra λ -continuous injection of a topological space X into an Ultra Hausdorff space Y , then X is λ - T_2 .*

PROOF. Let x_1 and x_2 be any distinct points in X . Since f is injective and Y is Ultra Hausdorff, $f(x_1) \neq f(x_2)$, and there exist $V_1, V_2 \in \text{CO}(Y)$ such that $f(x_1) \in V_1, f(x_2) \in V_2$ and $V_1 \cap V_2 = \emptyset$. Then $x_i \in f^{-1}(V_i) \in \lambda O(X)$ for $i = 1, 2$ and $f^{-1}(V_1) \cap f^{-1}(V_2) = \emptyset$. Thus X is λ - T_2 . ■

We say that the product space $X = X_1 \times \cdots \times X_n$ has property P_λ if, whenever A_i is a λ -open set in a topological space X_i , for $i = 1, 2, \dots, n$, then $A_1 \times \cdots \times A_n$ is also λ -open in the product space $X = X_1 \times \cdots \times X_n$.

THEOREM 2.23. *Let $f_1 : X_1 \rightarrow Y$ and $f_2 : X_2 \rightarrow Y$ be two functions, where*

- (1) $X = X_1 \times X_2$ has the property P_λ ,
- (2) Y is a Urysohn space,
- (3) f_1 and f_2 are contra λ -continuous.

Then $\{(x_1, x_2) : f_1(x_1) = f_2(x_2)\}$ is λ -closed in the product space $X = X_1 \times X_2$.

PROOF. Let A denote the set $\{(x_1, x_2) : f_1(x_1) = f_2(x_2)\}$. In order to show that A is λ -closed, we show that $(X_1 \times X_2) - A$ is λ -open. Let $(x_1, x_2) \notin A$. Then $f_1(x_1) \neq f_2(x_2)$. Since Y is Urysohn, there exist open neighborhoods V_1 and V_2 of $f_1(x_1)$ and $f_2(x_2)$ such that $\text{Cl}(V_1) \cap \text{Cl}(V_2) = \emptyset$. Since f_i ($i = 1, 2$) is contra λ -continuous, $f_i^{-1}(\text{Cl}(V_i))$ is a λ -open set containing x_i in X_i ($i = 1, 2$). Hence by (1), $f_1^{-1}(\text{Cl}(V_1)) \times f_2^{-1}(\text{Cl}(V_2))$ is λ -open. Furthermore $(x_1, x_2) \in f_1^{-1}(\text{Cl}(V_1)) \times f_2^{-1}(\text{Cl}(V_2)) \subset X_1 \times X_2 - A$. It follows that $X_1 \times X_2 - A$ is λ -open. Thus A is λ -closed in the product space $X = X_1 \times X_2$. ■

COROLLARY 2.24. *Assume that the product space $X \times X$ has the property P_λ . If $f : X \rightarrow Y$ is contra λ -continuous and Y is a Urysohn space, then $A = \{(x_1, x_2) : f(x_1) = f(x_2)\}$ is λ -closed in the product space $X \times X$.*

THEOREM 2.25. *Let $f : X \rightarrow Y$ be a function and $g : X \rightarrow X \times Y$ the graph function, given by $g(x) = (x, f(x))$ for every $x \in X$. Then f is contra λ -continuous if and only if g is contra λ -continuous.*

PROOF. Let $x \in X$ and let W be a closed subset of $X \times Y$ containing $g(x)$. Then $W \cap (\{x\} \times Y)$ is a closed set in $\{x\} \times Y$ containing $g(x)$. Also $\{x\} \times Y$ is homeomorphic to Y . Hence $\{y \in Y : (x, y) \in W\}$ is a closed subset of Y . Since f is contra λ -continuous, $\cup\{f^{-1}(y) : (x, y) \in W\}$ is a λ -open subset of X . Further $x \in \cup\{f^{-1}(y) : (x, y) \in W\} \subset g^{-1}(W)$. Hence $g^{-1}(W)$ is λ -open. Then g is contra λ -continuous.

Conversely, let F be a closed subset of Y . Then $X \times F$ is a closed subset of $X \times Y$. Since g is contra λ -continuous, $g^{-1}(X \times F)$ is a λ -open subset of X . Also $g^{-1}(X \times F) = f^{-1}(F)$. Hence f is contra λ -continuous. ■

THEOREM 2.26. *Let X and Z be any topological spaces and Y be locally indiscrete. Then the composition $g \circ f : X \rightarrow Z$ of a contra λ -continuous function $f : X \rightarrow Y$ and a λ -continuous function $g : Y \rightarrow Z$ is contra λ -continuous.*

PROOF. It follows from the definitions. ■

THEOREM 2.27. *Let $\{X_\lambda : \lambda \in \Omega\}$ be any family of topological spaces. If $f : X \rightarrow \prod X_\lambda$ is a contra λ -continuous function, then $\text{Pr}_\lambda \circ f : X \rightarrow X_\lambda$ is contra λ -continuous for each $\lambda \in \Omega$, where Pr_λ is the projection of $\prod X_\lambda$ onto X_λ .*

PROOF. We shall consider a fixed $\lambda \in \Omega$. Suppose U_λ is an arbitrary open set in X_λ . Then $\text{Pr}_\lambda^{-1}(U_\lambda)$ is open in $\prod X_\lambda$. Since f is contra λ -continuous, we have by definition $f^{-1}(\text{Pr}_\lambda^{-1}(U_\lambda)) = (\text{Pr}_\lambda \circ f)^{-1}(U_\lambda)$ is λ -closed in X . Therefore $\text{Pr}_\lambda \circ f$ is contra λ -continuous. ■

THEOREM 2.28. *If $f : X \rightarrow Y$ is a contra λ -continuous function and $g : Y \rightarrow Z$ is a continuous function, then $g \circ f : X \rightarrow Z$ is contra λ -continuous.*

DEFINITION 6. A topological space X is said to be:

- (1) λ -compact [3] if every λ -open cover of X has a finite subcover (resp. $A \subset X$ is λ -compact relative to X if every cover of X by λ -open sets of X has a finite subcover);
- (2) strongly S -closed [8] if every closed cover of X has a finite subcover (resp. $A \subset X$ is strongly S -closed if the subspace A is strongly S -closed [8]);
- (3) mildly λ -compact if every λ -clopen cover of X has a finite subcover.

Recall that for a function $f : X \rightarrow Y$, the subset $\{(x, f(x)) : x \in X\} \subset X \times Y$ is called the graph of f and is denoted by $G(f)$.

DEFINITION 7. A graph $G(f)$ of a function $f : X \rightarrow Y$ is said to be contra λ -closed if for each $(x, y) \in (X \times Y) \setminus G(f)$, there exist $U \in \lambda O(X)$ containing x and $V \in C(Y)$ containing y such that $(U \times V) \cap G(f) = \emptyset$.

LEMMA 2.29. *$G(f)$ is contra λ -closed in $X \times Y$ if and only if for each $(x, y) \in (X \times Y) \setminus G(f)$, there exist $U \in \lambda O(X)$ containing x and $V \in C(Y)$ containing y such that $f(U) \cap V = \emptyset$.*

THEOREM 2.30. *If $f : X \rightarrow Y$ is contra λ -continuous and Y is Urysohn, then $G(f)$ is contra λ -closed in $X \times Y$.*

PROOF. Let $(x, y) \in (X \times Y) \setminus G(f)$, then $f(x) \neq y$ and there exist open sets V, W such that $f(x) \in V, y \in W$ and $\text{Cl}(V) \cap \text{Cl}(W) = \emptyset$. Since f is contra λ -continuous, there exists $U \in \lambda O(X, x)$ such that $f(U) \subset \text{Cl}(V)$. Therefore, we obtain $f(U) \cap Cl(W) = \emptyset$. This shows that $G(f)$ is contra λ -closed in $X \times Y$. ■

THEOREM 2.31. *Let X be locally indiscrete. If $f : X \rightarrow Y$ has a contra λ -closed graph, then the inverse image of a strongly S -closed set K of Y is closed in X .*

PROOF. Assume that K is a strongly S -closed set of Y and $x \notin f^{-1}(K)$. For each $k \in K$, $(x, k) \notin G(f)$. By Lemma 2.29, there exist $U_k \in \lambda O(X, x)$ and $V_k \in C(Y, k)$ such that $f(U_k) \cap V_k = \emptyset$. Since $\{K \cap V_k : k \in K\}$ is a closed cover of the subspace K , there exists a finite subset $K_0 \subset K$ such that $K \subset \bigcup\{V_k : k \in K_0\}$. Set $U = \bigcap\{U_k : k \in K_0\}$, then U is open since X is locally indiscrete. Therefore $f(U) \cap K = \emptyset$ and $U \cap f^{-1}(K) = \emptyset$. This shows that $f^{-1}(K)$ is closed in X . ■

THEOREM 2.32. *If $f : X \rightarrow Y$ is contra λ -continuous and K is λ -compact relative to X , then $f(K)$ is strongly S -closed in Y .*

PROOF. Let $\{H_\alpha : \alpha \in I\}$ be any cover of $f(K)$ by closed sets of the subspace $f(K)$. For each $\alpha \in I$, there exists a closed set K_α of Y such that $H_\alpha = K_\alpha \cap f(K)$. For each $x \in K$, there exist $\alpha(x) \in I$ such that $f(x) \in K_{\alpha(x)}$. By Theorem 2.2, there exists $U_x \in \lambda O(X, x)$ such that $f(U_x) \subset K_{\alpha(x)}$. Since the family $\{U_x : x \in K\}$ is a λ -open cover of K , there exists a finite subset K_0 of K such that $K \subset \bigcup\{U_x : x \in K_0\}$. Therefore, we obtain $f(K) \subset \bigcup\{f(U_x) : x \in K_0\}$ which is a subset of $\bigcup\{K_{\alpha(x)} : x \in K_0\}$. Thus $f(K) = \bigcup\{H_{\alpha(x)} : x \in K_0\}$ and hence $f(K)$ is strongly S -closed in Y . ■

THEOREM 2.33. *If $f : X \rightarrow Y$ is a contra λ -continuous and λ -continuous surjection and X is mildly λ -compact, then Y is compact.*

PROOF. Let $\{V_\alpha : \alpha \in I\}$ be an open cover of Y . Since f is contra λ -continuous and λ -continuous, we have that $\{f^{-1}(V_\alpha) : \alpha \in I\}$ is a λ -clopen cover of X . Since X is mildly λ -compact, there exists a finite subset I_0 of I such that $X = \bigcup\{f^{-1}(V_\alpha) : \alpha \in I_0\}$. Since f is surjective, $Y = \bigcup\{V_\alpha : \alpha \in I_0\}$ and therefore Y is compact. ■

Recall that a function is called preclosed [7] if the image of every closed subset of X is preclosed in Y .

THEOREM 2.34. *Let $f : X \rightarrow Y$ be a surjective preclosed contra λ -continuous function. If X is locally indiscrete, then Y is locally indiscrete.*

PROOF. Let V be any open set of Y . By hypothesis, f is contra λ -continuous and therefore $f^{-1}(V) = U$ is λ -closed in X . Since X is locally

indiscrete, the set U is closed in X . Since f is preclosed, then V is also preclosed in Y . Now we have $\text{Cl}(V) = \text{Cl}(\text{Int}(V)) \subset V$. This means that V is closed and hence Y is locally indiscrete. ■

THEOREM 2.35. *Let $f : X \rightarrow Y$ be a contra λ -continuous function. If X is locally indiscrete, then f is contra-continuous.*

PROOF. Obvious. ■

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**A NOTE TO THE PAPER
“ON A FAST VERSION OF A PSEUDORANDOM GENERATOR”**

By

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1. Introduction

C. Mauduit and A. Sárközy [5, pp. 367–370] introduced the following measures of pseudorandomness of binary sequences:

For a finite binary sequence $E_N = \{e_1, e_2, \dots, e_N\} \in \{-1, +1\}^N$ write

$$U(E_N, t, a, b) = \sum_{j=0}^{t-1} e_{a+jb}$$

and, for $D = (d_1, \dots, d_k)$ with non-negative integers $d_1 < \dots < d_k$,

$$V(E_N, M, D) = \sum_{n=1}^M e_{n+d_1} e_{n+d_2}, \dots, e_{n+d_k}.$$

Then the *well-distribution measure* of E_N is defined as

$$W(E_N) = \max_{a,b,t} |U(E_N(t, a, b))| = \max_{a,b,t} \left| \sum_{j=0}^{t-1} e_{a+jb} \right|,$$

where the maximum is taken over all a, b, t such that $a, b, t \in \mathbb{N}$ and $1 \leq a \leq a + (t - 1)b \leq N$. The *correlation measure of order k* of E_N is defined

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as

$$C_k(E_N) = \max_{M,D} |V(E_N, M, D)| = \max_{M,D} \left| \sum_{n=1}^M e_{n+d_1} e_{n+d_2} \dots e_{n+d_k} \right|,$$

where the maximum is taken over all $D = (d_1, d_2, \dots, d_k)$ and M such that $d_1 < d_2 < \dots < d_k < M + d_k \leq N$.

A sequence E_N is considered as a “good” pseudorandom sequence if each of these measures $W(E_N)$, $C_k(E_N)$ (at least for small k) is “small” in terms of N (in particular, all are $o(N)$ as $N \rightarrow \infty$). Indeed, it was proved in [2, Theorem 1, 2] that for a truly random sequence $E_N \subseteq \{-1, +1\}^N$ each of these measures is $\ll \sqrt{N \log N}$ and $\gg \sqrt{N}$. Later Alon, Kohayakawa, Mauduit, Moreira and Rödl [1] improved on these bounds.

Numerous binary sequences have been tested for pseudorandomness by J. Cassaigne, S. Ferenczi, C. Mauduit, J. Rivat and A. Sárközy, and large families of pseudorandom binary sequences have been constructed first by L. Goubin, C. Mauduit, A. Sárközy [4]. I gave a further construction of this type in [6], [7].

Let p be an odd prime and g be a fixed primitive root modulo p , and for $(n, p) = 1$ $\text{ind } n$ denotes the index or discrete logarithm of n modulo p . Thus $\text{ind } n$ is defined as the unique integer with

$$(1) \quad g^{\text{ind } n} \equiv n \pmod{p},$$

and $1 \leq \text{ind } n \leq p - 1$. Using the discrete logarithm I introduced a new large family of pseudorandom sequences with strong pseudorandom properties in [6]. However the sequences in this family can be generated very slowly, so in [7] I slightly modified the construction so that the new family can be generated much faster. In this paper I will improve on results in [7]. Recently the discrete logarithm is used more and more frequently in cryptography. Z. Chen, S. Li and G. Xiao [3] generalized the pseudorandom constructions based on the notion of index using elliptic curves. Throughout this paper we will use the following

NOTATION. Let p be an odd prime, g be a fixed primitive root modulo p . Define $\text{ind } n$ by (1). Let $f(X) \in \mathbb{F}_p[X]$ be a polynomial of degree $k \geq 1$ which has no multiple roots. Moreover, let

$$m \mid p - 1$$

with $m \in \mathbb{N}$, and let x be coprime with m : $(x, m) = 1$.

The crucial idea of the new, faster construction defined in [7] was to reduce $\text{ind } n$ modulo m :

CONSTRUCTION 1. Let $\text{ind}^* n$ denote the following function: For all $1 \leq n \leq p - 1$, let

$$\text{ind } n \equiv x \cdot \text{ind}^* n \pmod{m} \quad \text{and} \quad 1 \leq \text{ind}^* n \leq m$$

($\text{ind}^* n$ exists since $(x, m) = 1$). Define the sequence $E_{p-1} = \{e_1, \dots, e_{p-1}\}$ by

$$(2) \quad e_n = \begin{cases} +1 & \text{if } 1 \leq \text{ind}^* f(n) \leq \frac{m}{2}, \\ -1 & \text{if } \frac{m}{2} < \text{ind}^* f(n) \leq m \text{ or } p \mid f(n). \end{cases}$$

Note that this construction also generalizes the Legendre symbol construction described in [4] and [5]. Indeed, in the special case $m = 2$, $x = 1$ the sequence e_n defined in (2) becomes

$$e_n = \begin{cases} +1 & \text{if } \left(\frac{f(n)}{p}\right) = -1, \\ -1 & \text{if } \left(\frac{f(n)}{p}\right) = 1 \text{ or } p \mid f(n). \end{cases}$$

In [7], I proved that this construction has good pseudorandom properties: each of the measures $W(E_{p-1})$, $C_k(E_{p-1})$ is less than $p^{1/2}(\log p)^c$ under certain conditions on the polynomial f . However in Theorem 3 in [7] for the correlation measure we obtained an upper bound which is optimal only for large m . Indeed, there the following was proved:

THEOREM A. *Suppose that m is even, or m is odd with $2m \mid p - 1$, and at least one of the following 4 conditions holds:*

- a) f is irreducible;
- b) If f has the factorization $f = \varphi_1^{\alpha_1} \varphi_2^{\alpha_2} \cdots \varphi_u^{\alpha_u}$ over \mathbb{F}_p where $\alpha_i \in \mathbb{N}$ and the φ_i 's are irreducible over \mathbb{F}_p , then there exists a β such that exactly one or two of φ_i 's have the degree β ;
- c) $\ell = 2$;
- d) $(4\ell)^k < p$ or $(4k)^\ell < p$.

Then

$$(3) \quad C_\ell(E_{p-1}) < 9k \ell 4^\ell p^{1/2} (\log p)^{\ell+1} + \frac{\ell! k^{\ell(\ell+1)}}{m^\ell} p.$$

If m is odd, then it is proved in section 4 in [7] that

$$C_\ell(E_{p-1}) \gg \frac{p}{m^\ell},$$

thus the second term in (3) can not be dropped completely. If m is even, I will improve on Theorem A under the further assumption that $f(x)$ has no multiple roots.

THEOREM 1. *Suppose that m is even, and at least one of the conditions a), b), c) and d) holds in Theorem A. Moreover we suppose that all conditions assumed in the Notation hold, in particular, $f(X) \in \mathbb{F}_p[x]$ has no multiple roots. Then*

$$(4) \quad C_\ell(E_{p-1}) < 9k \ell 4^\ell p^{1/2} (\log p)^{\ell+1}.$$

(4) is considerably sharper than (3) if

$$m^{2\ell+\varepsilon} \ll p.$$

Then the second term is much larger than the first term in (3). In particular, for $m = O(1)$ the second term is $\gg p$, so (3) becomes trivial, while our Theorem 1 still gives good upper bound.

2. Proof of Theorem 1

Consider any $\mathcal{D} = \{d_1, d_2, \dots, d_\ell\}$ with non-negative integers $d_1 < d_2 < \dots < d_\ell$ and positive integers M with $M + d_\ell \leq p - 1$. Then arguing as in the proof of Theorem 2 in [7] (formulas (11) and (12)) we have

$$\begin{aligned} |V(E_N, M, D)| &\leq 9k \ell 4^\ell p^{1/2} (\log p)^{\ell+1} \\ &\quad + \frac{2^\ell}{m^\ell} \sum_{\substack{1 \leq \delta_1, \dots, \delta_\ell < m, \\ f^{\delta_1}(n+d_1) \cdots f^{\delta_\ell}(n+d_\ell) \\ \text{is a perfect } m\text{-th power}}} (p-1) \left| \prod_{j=1}^{\ell} \left(\sum_{l_j=1}^{m/2} \chi^{\delta_j} \left(g^x \ell_j \right) \right) \right| \\ (5) \quad &= 9k \ell 4^\ell p^{1/2} (\log p)^{\ell+1} + \sum_2. \end{aligned}$$

We will prove that the sum \sum_2 is empty, which follows from the following lemma:

LEMMA 1. Suppose that the conditions of Theorem 1 hold. Then if $1 \leq \delta_1, \dots, \delta_\ell \leq m - 1$, and $f^{\delta_1}(n+d_1) \cdots f^{\delta_\ell}(n+d_\ell)$ is a perfect m -th power, then

$$m \mid \delta_1, \delta_2, \dots, \delta_\ell.$$

Indeed, this is a sharpened version of Lemma 4 in [7]. Assuming the further condition that $f(x)$ has no multiple roots in $\overline{\mathbb{F}}_p$ we will be able to prove this stronger result.

If \sum_2 is empty then by (5) we have

$$|V(E_N, M, D)| \leq 9k\ell 4^\ell p^{1/2} (\log p)^{\ell+1},$$

which proves Theorem 1.

Thus it remains to prove Lemma 1. We will need the following definition and lemma:

DEFINITION 1. Let \mathcal{A} and \mathcal{B} be multi-sets of the elements of \mathbb{Z}_p . If $\mathcal{A} + \mathcal{B}$ represents every element of \mathbb{Z}_p with multiplicity divisible by m , i.e., for all $c \in \mathbb{Z}_p$, the number of solutions of

$$a + b = c, \quad a \in \mathcal{A}, \quad b \in \mathcal{B}$$

(the a 's and b 's are counted with their multiplicities) is divisible by m , then the sum $\mathcal{A} + \mathcal{B}$ is said to have property P .

LEMMA 2. Let $\mathcal{A} = \{a_1, a_2, \dots, a_r\}$, $\mathcal{D} = \{d_1, d_2, \dots, d_\ell\} \subseteq \mathbb{Z}_p$. If one of the following two conditions holds:

(i) $\min\{r, \ell\} \leq 2$ and $\max\{r, \ell\} \leq p - 1$,

(ii) $(4\ell)^r \leq p$ or $(4r)^\ell \leq p$,

then there exist $c_1, \dots, c_\ell \in \mathbb{Z}_p$ and a permutation (q_1, \dots, q_ℓ) of (d_1, \dots, d_ℓ) such that for all $1 \leq i \leq \ell$

$$a + d = c_i \quad a \in \mathcal{A}, \quad d \in \mathcal{D}$$

has at least one solution, and the number of solutions is less than or equal to i . Moreover for all solutions $a \in \mathcal{A}$, $d \in \mathcal{D}$ we have $d \in \{q_1, q_2, \dots, q_i\}$, and $d = q_i$, $a = c_i - q_i$ is always a solution.

PROOF. This is Lemma 5 in [7].

Now we return to the proof of Lemma 1. The following equivalence relation was defined in [4] and also used in [6] and [7]: We will say that the polynomials $\varphi(X), \psi(X) \in \mathbb{F}_p[X]$ are equivalent, $\varphi \sim \psi$, if there is an $a \in \mathbb{F}_p$ such that $\psi(X) = \varphi(X + a)$. Clearly, this is an equivalence relation.

Write $f(X)$ as the product of irreducible polynomials over \mathbb{F}_p . Let us group these factors so that in each group the equivalent irreducible factors are collected. Consider a typical group $\varphi(X + a_1), \dots, \varphi(X + a_r)$. Then $f(X)$ is of the form $f(X) = \varphi(X + a_1) \cdots \varphi(X + a_r)g(X)$ where $g(X)$ has no irreducible factors equivalent with any $\varphi(X + a_i)$ ($1 \leq i \leq r$).

Let $h(X) = f^{\delta_1}(X + d_1) \cdots f^{\delta_\ell}(X + d_\ell)$ be a perfect m -th power where $1 \leq \delta_1, \dots, \delta_\ell < m$. Then writing $h(x)$ as the product of irreducible polynomials over \mathbb{F}_p , all the polynomials $\varphi(X + a_i + d_j)$ with $1 \leq i \leq r$, $1 \leq j \leq \ell$ occur amongst the factors. All these polynomials are equivalent, and no other irreducible factor belonging to this equivalence class will occur amongst the irreducible factors of $h(X)$.

Since distinct irreducible polynomials cannot have a common zero, each of the zeros of $h(X)$ is of multiplicity divisible by m , if and only if in each group, formed by equivalent irreducible factors $\varphi(X + a_i + d_j)$ of $h(X)$, every polynomial of form $\varphi(X + c)$ occurs with multiplicity divisible by m . In other words, writing $\mathcal{A} = \{a_1, \dots, a_r\}$, $\mathcal{D} = \{d_1, \dots, d_1, \dots, d_\ell, \dots, d_\ell\}$ where d_i has the multiplicity δ_i in \mathcal{D} ($h(X) = f^{\delta_1}(X + d_1) \cdots f^{\delta_\ell}(X + d_\ell)$ is a perfect m -th power), then for each group $\mathcal{A} + \mathcal{D}$ must possess property P .

Let \mathcal{D}' be the simple set version of \mathcal{D} , more exactly, let $\mathcal{D}' = \{d_1, \dots, d_\ell\}$. \mathcal{A} and \mathcal{D}' satisfy the conditions of Lemma 2. So by Lemma 2 for the sets \mathcal{A} and \mathcal{D}' we have the following: There exist $c_1, \dots, c_\ell \in \mathbb{Z}_p$ and a permutation $(q_1, \dots, q_\ell) = (d_{j_1}, \dots, d_{j_\ell})$ of (d_1, \dots, d_ℓ) such that if

$$a + d = c_i \quad a \in \mathcal{A}, d \in \mathcal{D}',$$

then we have

$$d \in \{q_1, \dots, q_i\} = \{d_{j_1}, \dots, d_{j_i}\}$$

and $d = q_i$, $a = c_i - q_i$ is a solution. Here (j_1, \dots, j_ℓ) is a permutation of $(1, \dots, \ell)$. Define the ρ_i 's by $\rho_i = \delta_{j_i}$ (so $(\rho_1, \dots, \rho_\ell) = (\delta_{j_1}, \dots, \delta_{j_\ell})$ is the same permutation of $(\delta_1, \dots, \delta_\ell)$ as the permutation $(q_1, \dots, q_\ell) = (d_{j_1}, \dots, d_{j_\ell})$ of (d_1, \dots, d_ℓ)). Returning to the multi-set case, using this notation we get that the number of the solutions

$$a + d = c_i \quad a \in \mathcal{A}, d \in \mathcal{D}$$

is of the form

$$\varepsilon_{i,1}\rho_1 + \varepsilon_{i,2}\rho_2 + \cdots + \varepsilon_{i,i}\rho_i$$

where $\varepsilon_{i,j} \in \{0, 1\}$ for $1 \leq j \leq i$ and $\varepsilon_{i,i} = 1$. (We study the number of solutions by multiplicity since \mathcal{D} is a multi-set.)

Since $\mathcal{A} + \mathcal{D}$ possesses property \mathcal{P} for all $1 \leq i \leq \ell$ we have

$$(6) \quad m \mid \varepsilon_{i,1}\rho_1 + \varepsilon_{i,2}\rho_2 + \cdots + \varepsilon_{i,i}\rho_i.$$

By induction on i we will prove that

$$(7) \quad m \mid \rho_i.$$

Indeed, for $i = 1$ by (6) and $\varepsilon_{1,1} = 1$ we get $m \mid \rho_1$. We will prove that if (7) holds for $i \leq j - 1$, then it also holds for $i = j$.

By the induction hypothesis we have

$$m \mid \rho_1, m \mid \rho_2, \dots, m \mid \rho_{j-1}.$$

Using this and (6) for $i = j$ we get:

$$m \mid \rho_j,$$

which was to be proved.

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**CORRECTIONS TO MY PAPER
“DISCRIMINATOR OR DUAL DISCRIMINATOR?”**

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0. Preliminaries

In one part of my original paper *Discriminator or dual discriminator?*, which appeared in *Algebra Universalis*, there were several inaccuracies, for various reasons. Here we give a corrected version, trying to make it as selfcontained as possible. To be able to keep in track with the original paper we follow the original numbering¹.

Firstly, we need some definitions:

DEFINITIONS. A ternary function $f : A^3 \rightarrow A$ will be called a semidiscriminator if for all $a', c' \in A$ for every choice $\{a, c\} = \{a', c'\}$ and for every $b \neq a$ ($b \in A$) we have either one of the following cases:

$$\begin{array}{lll} \text{case(d):} & f(a, a, c) = c & \text{and } f(a, b, c) = a, \\ \text{case(dd):} & f(a, a, c) = a & \text{and } f(a, b, c) = c. \end{array}$$

An algebra is called a semidiscriminator algebra if a term p yields a semidiscriminator in it. A ternary term p in a variety is a semidiscriminator term if it yields a semidiscriminator function in every subdirectly irreducible member of the variety.

A variety is called a semidiscriminator variety if it possesses a semidiscriminator term.

A variety of type 3 is called a pure semidiscriminator variety, if the operation itself is a semidiscriminator term.

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¹ This manuscript was sent to *Algebra Universalis* in 2003, but it went astray and was lost.

REMARKS. (1) If in the definition always case(d) holds then p is called a discriminator, and if always case(dd) then it is a dual discriminator. For a semidiscriminator term p , the term $q(x, y, z) = p(x, p(x, y, z), z)$ is a dual discriminator term, hence the variety is a dual discriminator variety. Thus, we can apply the results of [4] or [2]. In particular, in any algebra of the variety $c \equiv d$ ($\Theta(a, b)$) holds if and only if the equations

$$c = q(c, d, q(a, b, c)) = c \quad \text{and} \quad d = q(c, d, q(a, b, d))$$

are satisfied.

(2) The two cases in the definition do not exclude each other. It is trivial, that both cases are satisfied if and only if $a' = c'$.

(3) There is a one-to-one correspondence between semidiscriminator algebras and “two-colored” graphs: Let the edge, connecting a and c be red in case(d) and blue in case(dd). Every edge in the graph has exactly one color except the loops which have both colors.

3. Pure (Semi)discriminator varieties

We need a result of K. Baker ([1]) which is very simple for dual discriminator varieties:

PROPOSITION 3.2. *In any (dual) discriminator variety the intersection of finitely many principal congruences is a principal congruence.*

PROOF. Clearly, it is enough to prove our statement for two congruences. We shall give principal intersection polynomials in the sense of [1] (see, also, [4]). We claim, that

$$\Theta(a, b) \wedge \Theta(c, d) = \Theta(q(a, b, c), q(a, b, d)),$$

where $q(x, y, z)$ is a dual discriminator term. The right hand side is contained in the left hand side, since both $a = b$ and $c = d$ yields $q(a, b, c) = q(a, b, d)$ in every subdirectly irreducible algebra of the variety. Conversely, factorizing by the right hand side and consider the given congruence identity in a subdirectly irreducible algebra. We have $\Theta(a, b) \wedge \Theta(c, d) = 0$, the least congruence. If $\Theta(a, b) = 0$, we are done. Otherwise, $a \neq b$, therefore, $q(a, b, c) = c$ and $q(a, b, d) = d$, i.e. $\Theta(c, d) = 0$, by assumption. ■

Next, we are going to describe the pure semidiscriminator varieties by a set of identities.

THEOREM 3.4. *Let \mathcal{S} be a variety of type 3, with the ternary operational symbol $p(x, y, z)$. \mathcal{S} is a pure semidisibrator variety with the semidisibrator $p(x, y, z)$ in every subdirectly irreducible algebra, if and only if the following hold:*

$$(1) \quad p(x, y, x) = x;$$

the term $q(x, y, z) = p(x, p(x, y, z), z)$ is the dual discriminator, i.e.,

$$1) \quad q(x, z, z) = z, \quad (q(x, y, x) = x), \quad q(x, x, z) = x,$$

$$(2) \quad 2) \quad q(x, y, q(x, y, z)) = q(x, y, z)$$

$$3) \quad q(x, y, p(u, v, z)) = p(q(x, y, u), q(x, y, v), q(x, y, z));$$

for the term $u = p(x, y, z)$ and for every v , we have

$$(3) \quad q(u, x, q(u, z, v)) = u;$$

$$(4) \quad x = q(x, y, q(p(x, x, y), p(x, y, y), x))$$

$$y = q(x, y, q(p(x, x, y), p(x, y, y), y));$$

$$(5) \quad p(x, x, y) = p(y, x, x).$$

PROOF. Suppose, $p(x, y, z)$ is a semidisibrator. (1) is obvious. (2) follows from the fact that q is the dual discriminator, therefore it satisfies the identities in [2]. Hence, \mathcal{S} is a dual discriminator variety with the dual discriminator q . Therefore, it is enough to verify the other identities for subdirectly irreducible algebras.

For the semidisibrator, we have $u \in \{x, z\}$, and (3) is satisfied in both cases.

For $x = y$, (4) holds, trivially. Suppose, $a \neq b$. Then, the definition of the semidisibrator yields $p(a, a, b) \neq p(a, b, b)$ in both cases (d) and (dd). Hence, we have the equality $q(p(a, a, b), p(a, b, b), c) = c$ for every c , yielding (4).

(5) holds in case $x = y$, clearly. Suppose, $a \neq b$. Then $p(a, b, b) \neq p(a, a, b)$ follows from the definition of the semidisibrator. Since in the definition $p(u, y, v)$ and $p(v, y, u)$ belong to the same case, $p(a, a, b) = a$ implies $p(b, b, a) = b$, which yields $p(b, a, a) = a$. The other case is “dual”.

Conversely, assume all the identities above hold. (2) implies, that \mathcal{S} is a dual discriminator variety. Therefore, it is enough to deal with algebras, where $q(x, y, z)$ is the dual discriminator. Choose any a, c in the algebra. $p(a, b, c) \in \{a, c\}$ follows from (1) if $a = c$. Next, let $a \neq c$. If the algebra has only two elements, then $d = p(a, b, c) \in \{a, c\}$ for every b in the algebra. Otherwise,

suppose $d \neq a$. Then, by (3), we have $q(d, c, b) = q(d, a, q(d, c, b)) = d$ for each b , i.e., $d = c$, since q is the dual discriminator. Hence, $p(a, b, c) \in \{a, c\}$.

Suppose, $a \neq b$. Then, by (4), we have

$$q(p(a, a, c), p(a, c, c), a) = a \quad \text{and} \quad q(p(a, a, c), p(a, c, c), c) = c.$$

This means, that if $p(a, a, c) = p(a, c, c) = d$, then $q(d, d, a) = q(d, d, c)$, yielding $a = c$. In other words, for distinct a and c , the elements $p(a, a, c)$ and $p(a, c, c)$ are distinct, as well.

Finally, (5) implies, that if $p(a, y, c)$ satisfies the condition of the first case in the definition, then so does $p(c, y, a)$. ■

4. Subvarieties of pure semidiscriminator varieties

Now, we turn to the description of some subvarieties of pure semidiscriminator varieties. Let \mathcal{S} be the semidiscriminator variety, defined by the identities in the previous section. Of course, every subvariety of \mathcal{S} is uniquely given by listing all the subdirectly irreducible members. Since we are in a congruence distributive variety, a class \mathbb{S} of subdirectly irreducible algebras of \mathcal{S} consists exactly of the subdirectly irreducible members of a subvariety, if and only if \mathbb{S} is closed under ultraproducts, subalgebras and homomorphic images. Since we are in a dual discriminator variety, subdirectly irreducibles are simple, hence, homomorphic images are not needed.

We make use of the one-to-one correspondence between the subdirectly irreducible members of \mathcal{S} and the two-colored graphs discussed in the previous section. It is clear, that the algebra-mappings are exactly the graph-mappings. Notice, however, that the image of an edge is red (blue) *only* if the original edge is red (blue). It will be easier, to use graph-theoretical concepts.

DEFINITION 4.1. *An n -element ($n > 2$) two-colored graph will be called a red cycle of length n , if for its vertices a_1, \dots, a_n , the edges*

$$(a_1, a_2), \dots, (a_{n-1}, a_n), (a_n, a_1)$$

are red and all the other edges are blue. Blue cycles of length n are defined dually. A two-colored graph contains a red (blue) cycle of length n if it contains a subgraph, which is a red (blue) cycle of length n .

PROPOSITION 4.2. *Let \mathbb{S} be a set of subdirectly irreducible elements of the variety \mathcal{S} . A finite subdirectly irreducible algebra of \mathcal{S} belongs to the*

subvariety generated by the elements of \mathbb{S} if and only if it is isomorphic to a subalgebra of some element of \mathbb{S} .

PROOF. The “if” part is trivial. Suppose the subdirectly irreducible algebra \mathfrak{S} belongs to the generated subvariety. As we mentioned at the beginning of this section, \mathfrak{S} must be a subalgebra of an ultraproduct of elements of \mathbb{S} . Let s_1, \dots, s_k be the elements of \mathfrak{S} , and $\mathbf{s}_1, \dots, \mathbf{s}_k$ elements of the direct product, which are mapped to the corresponding elements of \mathfrak{S} , when factoring by the ultrafilter. The subalgebra of the direct product, generated by $\mathbf{s}_1, \dots, \mathbf{s}_k$ is contained, obviously, in a direct product of subalgebras of elements of \mathbb{S} , such that \mathfrak{S} is a factor of this direct product by the original ultrafilter. List all the equalities $p(s_i, s_j, s_u) = s_v$ in \mathfrak{S} . Let π be the natural projection of the direct product to some components. Since we are in an ultraproduct, the relation $p(\pi(\mathbf{s}_i), \pi(\mathbf{s}_j), \pi(\mathbf{s}_u)) = \pi(\mathbf{s}_v)$ holds in almost all component, i.e., almost all components are isomorphic to \mathfrak{S} . ■

THEOREM 4.3. *Let $N^* = \{2 < n_1 < n_2 < \dots\}$ be any infinite sequence of integers. Let $\mathbb{S}(N^*)$ denote the class of subdirectly irreducible members of \mathcal{S} , which do not contain a, say, red cycle of length $n \in N^*$. Then, the subvariety $\mathcal{S}(N^*)$ generated by $\mathbb{S}(N^*)$ is not finitely based.*

The number of these varieties is continuum.

PROOF. If the subdirectly irreducible algebra \mathfrak{S} does not contain a red cycle, no subalgebra of \mathfrak{S} contains a red cycle. Then, by proposition 4.2, the subdirectly irreducible members of $\mathcal{S}(N^*)$ are exactly the elements of $\mathbb{S}(N^*)$.

Now, consider a finite set Σ of identities, valid in $\mathcal{S}(N^*)$. Let k be the number of variables in the members of Σ , and choose an $n \in N^*$ such that $n > k$. Let \mathfrak{C}_n be a red cycle of length n . By the definition of $\mathcal{S}(N^*)$, $\mathfrak{C}_n \notin \mathbb{S}(N^*)$. Take any k elements of \mathfrak{C}_n . Observe, that the subalgebra they generate does not contain other elements. Since this subalgebra contains no cycle, at all, it belongs to $\mathcal{S}(N^*)$. Thus, \mathfrak{C}_n satisfies all the identities in Σ . Therefore, Σ is not a basis for the identities of $\mathcal{S}(N^*)$, i.e., $\mathcal{S}(N^*)$ is not finitely based.

By proposition 4.2, distinct sequences define distinct varieties. Since the number of the given sequences is continuum, so is the number of constructed varieties.

Observe, that we not only proved that there is a continuum of non-finitely based subvarieties, but we, in some sense, also constructed them. ■

We finish this section by giving an equational basis for every variety $\mathcal{S}(N^*)$.

THEOREM 4.5. *For every n there is a finite system of identities, such that these identities hold in a subdirectly irreducible member of the pure semidiscriminator variety defined in Theorem 3.4 if and only if it contains – as a two-colored graph – no, say, red cycle of length n .*

This yields an equational base for every $\mathcal{S}(N^)$, finite if N^* is finite and infinite, otherwise.*

PROOF. Let $p(x, y, z)$ denote the semidiscriminator term, and let $n > 2$ be fixed. As the first element of desired finite system of identities we list all the identities of Theorem 3.4.

Suppose, the subdirectly irreducible element \mathfrak{S} of the variety contains a red cycle (a_1, \dots, a_n) of length n . Define the following operation on the integers:

$$i * j = \begin{cases} j & \text{if } |i - j| \equiv 1 \pmod{n}, \\ i & \text{otherwise.} \end{cases}$$

Using this operation, we have $p(a_i, a_i, a_j) = a_{i*j}$ ($1 \leq i, j \leq n$).

If \mathfrak{S} does not contain a red cycle of length n , then whenever these identities hold, the subalgebra consisting of a_1, \dots, a_n can not be proper, i.e., we must have $a_i = a_j$ for some $1 \leq i \neq j \leq n$. (As the cycles are simple, this equality holds for each $1 \leq i \neq j \leq n$.) In other words, the congruence $\alpha = \bigwedge \{\Theta(p(x_i, x_i, x_j)) = x_{i*j} \mid 1 \leq i, j \leq n\}$ implies, say, the congruence $\Theta(x_1, x_2)$. By Baker's theorem, α is a principal congruence of the form $\alpha = \Theta(f_n, g_n)$ where f_n, g_n are well-defined elements of the free algebra generated by x_1, \dots, x_n . Thus, a two-colored graph, containing no red cycle of length n , satisfies the identities

$$x_1 = q(x_1, x_2, q(f_n, g_n, x_1)) \quad \text{and} \quad x_2 = q(x_1, x_2, q(f_n, g_n, x_2)).$$

Conversely, if a_1, \dots, a_n is a red cycle in a colored graph, then the mapping $x_i \mapsto a_i$, for $1 \leq i \leq n$ sends f_n and g_n to the same element b , yielding $a_i = q(a_1, a_2, b) = b$ for $i \in \{1, 2, \dots, n\}$, a contradiction. ■

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A WEAKER FORM OF PRE- R_0

By

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1. Introduction

The notion of R_0 topological spaces is introduced by Shanin [13] in 1943. By definition, a topological space is R_0 if every open set contains the closure of each of its singletons. Later, Davis [4] rediscovered it and studied some properties of this weak separation axiom. Many researchers further investigated properties of R_0 topological spaces and many interesting results have been obtained in various contexts (see: [5], [7], [8], [11], [12]). In 1982, Mashhour et al. [10] introduced the notion of preopen sets which are also known under the name of locally dense sets [3] in the literature. Since then, this notion received wide usage in general topology. In 2000, Caldas et al. [1] introduced and investigated the fundamental properties of the separation axiom pre- R_0 . In this paper, we offer another characterization of pre- R_0 . We also introduce a new separation axiom called pre- R_T . It turns out that pre- R_T is weaker than pre- R_0 .

Throughout the paper (X, τ) (or simply X) will always denote a topological space. For a subset A of X , the closure, the interior and the complement of A in X are denoted by $\text{Cl}(A)$, $\text{Int}(A)$ and $X - A$, respectively. By $\text{PO}(X, \tau)$ and $\text{PC}(X, \tau)$ we denote the collection of all preopen sets and the collection of all preclosed sets of (X, τ) , respectively.

Since we shall require the following known notions, notations and some properties, we recall them in this section.

DEFINITION 1. Let A be a subset of a space (X, τ) . Then:

- (1) A is said to be preopen [10] if $A \subset \text{Int}(\text{Cl}(A))$.
- (2) A is said to be preclosed [10] if $X - A$ is preopen or equivalently if $\text{Cl}(\text{Int}(A)) \subset A$.
- (3) The intersection of all preclosed sets containing A is called the preclosure of A [6] and is denoted by $\text{pCl}(A)$.

LEMMA 1.1 (EL-DEEB ET AL. [6]). *Let (X, τ) be a space and A, B subsets of X . Then the following hold:*

- (1) $x \in \text{pCl}(A)$ if and only if $A \cap V \neq \emptyset$ for every $V \in \text{PO}(X, \tau)$ such that $x \in V$.
- (2) A is preclosed in (X, τ) if and only if $A = \text{pCl}(A)$.
- (3) $\text{pCl}(A) \subset \text{pCl}(B)$ if $A \subset B$.
- (4) $\text{pCl}(\text{pCl}(A)) = \text{pCl}(A)$.

2. Another Characterization of pre- R_0 spaces

DEFINITION 2. Let (X, τ) be a space and $A \subset X$. Then the pre-kernel of A [9], denoted by $\text{pKer}(A)$, is defined to be the set $\text{pKer}(A) = \cap\{G \in \text{PO}(X, \tau) \mid A \subset G\}$.

LEMMA 2.1 ([1]). *Let (X, τ) be a space, $A \subseteq X$ and $x, y \in X$. Then, the following hold:*

- (1) $y \in \text{pKer}(\{x\})$ if and only if $x \in \text{pCl}(\{y\})$.
- (2) $\text{pKer}(A) = \{x \in X \mid \text{pCl}(\{x\}) \cap A \neq \emptyset\}$.
- (3) $\text{pKer}(\{x\}) \neq \text{pKer}(\{y\})$ if and only if $\text{pCl}(\{x\}) \neq \text{pCl}(\{y\})$.

DEFINITION 3. A topological space (X, τ) is called a pre- R_0 space [1] if every preopen set contains the preclosure of each of its singletons equivalently $x \in \text{pCl}(\{y\})$ if and only if $y \in \text{pCl}(\{x\})$.

The following Lemma 2.2 is Theorem 3.8 in [1] and Lemma 2.3 is a special case of Corollary 3.9 in [1].

LEMMA 2.2. *A space (X, τ) is pre- R_0 if and only if for any $x \in X$, $\text{pCl}(\{x\}) \subset \text{pKer}(\{x\})$.*

LEMMA 2.3. *A space (X, τ) is pre- R_0 if and only if for any $x \in X$, $\text{pCl}(\{x\}) = \text{pKer}(\{x\})$.*

Since $\text{pCl}(\{x\})$ is the intersection of all preclosed sets containing x , Lemma 2.3 suggests a natural definition of pre- R_0 .

DEFINITION 4. A topological space (X, τ) is pre- R_0 if the intersection of all preopen sets containing x coincides with the intersection of all preclosed sets containing x .

THEOREM 2.4. *For a topological space (X, τ) , the following properties are equivalent:*

- (1) (X, τ) is a pre- R_0 space;
- (2) $\text{pKer}(\{x\}) \subset \text{pCl}(\{x\})$ for each $x \in X$.

PROOF. (1) \rightarrow (2): Suppose that (X, τ) is a pre- R_0 space. Let $y \in \text{pKer}(\{x\})$, then by Lemma 2.1(1) $x \in \text{pCl}(\{y\})$ and by Lemma 2.2 $x \in \text{pKer}(\{y\})$. Therefore, by Lemma 2.1(1) $y \in \text{pCl}(\{x\})$ and hence $\text{pKer}(\{x\}) \subset \text{pCl}(\{x\})$.

(2) \rightarrow (1): Let $x \in \text{pCl}(\{y\})$. Then by Lemma 2.1(1) $y \in \text{pKer}(\{x\})$. By (2) we obtain $y \in \text{pKer}(\{x\}) \subset \text{pCl}(\{x\})$. Therefore $x \in \text{pCl}(\{y\})$ implies $y \in \text{pCl}(\{x\})$. The converse is obvious. Now it follows from Definition 3 that (X, τ) is pre- R_0 . ■

Observe that Lemma 2.2 and Theorem 2.4 show the symmetry of pre- R_0 in another sense.

3. A weaker form of pre- R_0

We begin with the following definitions.

DEFINITION 5 ([2]). A topological space (X, τ) is said to be

- (1) pre- R_H if for any points x and y in X , $\text{pCl}(\{x\}) \neq \text{pCl}(\{y\})$ implies $\text{pCl}(\{x\}) \cap \text{pCl}(\{y\}) = \emptyset$, $\{x\}$ or $\{y\}$.
- (2) pre- R_D if for $x \in X$, $\text{pCl}(\{x\}) \cap \text{pKer}(\{x\}) = \{x\}$ implies that $\text{pD}(\{x\}) = \text{pCl}(\{x\}) \setminus \{x\}$ is preclosed, where $\text{pD}(\{x\})$ is the pre-derivate of $\{x\}$.

It follows from Lemma 2.2 and Theorem 2.4 that if X is not pre- R_0 , then there is some x such that $\text{pKer}(\{x\}) \setminus \text{pCl}(\{x\}) \neq \emptyset$, and there is some x such that $\text{pCl}(\{x\}) \setminus \text{pKer}(\{x\}) \neq \emptyset$. This suggests a new separation axiom. In the following definition, a set that contains at most one point is said to be degenerate.

DEFINITION 6. A topological space (X, τ) is called a pre- R_T space if for any x , both $\text{pKer}(\{x\}) \setminus \text{pCl}(\{x\})$ and $\text{pCl}(\{x\}) \setminus \text{pKer}(\{x\})$ are degenerate.

Obviously, every pre- R_0 space is pre- R_T . In general the converse may not be true.

EXAMPLE 3.1. Let $X = \{a, b, c, d\}$ and

$$\tau = \{\emptyset, \{a\}, \{a, b\}, \{c, d\}, \{a, c, d\}, X\}.$$

Then (X, τ) is pre- R_T but it is not pre- R_0 since $pCl(\{a\}) \setminus pKer(\{a\}) = \{b\}$.

EXAMPLE 3.2. Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, \{a\}, \{a, b\}, \{a, c\}, X\}$. Then (X, τ) is pre- R_D and pre- R_H but it is not pre- R_0 since $pCl(\{a\}) = X$ and $pKer(\{a\}) = \{a\}$.

EXAMPLE 3.3. Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, \{a\}, \{a, b\}, X\}$. Then (X, τ) is pre- T_0 but is not pre- R_T since $pCl(\{a\}) = X$ and $pKer(\{a\}) = \{a\}$.

THEOREM 3.4. *If (X, τ) is pre- R_T , then (X, τ) is pre- R_D .*

PROOF. Suppose that X is pre- R_T and denote

$$(x)_p = pCl(\{x\}) \cap pKer(\{x\}).$$

Then $pCl(\{x\}) = (x)_p \cup D$ and $pKer(\{x\}) = (x)_p \cup E$, where D and E are degenerate sets such that $D \not\subseteq pKer(\{x\})$ and $E \not\subseteq pCl(\{x\})$. If $(x)_p = \{x\}$, then $pCl(\{x\}) = \{x\} \cup D$ and $pKer(\{x\}) = \{x\} \cup E$. We prove that $pD(\{x\}) = pCl(\{x\}) \setminus \{x\} = D$ is a preclosed set. Let U be a preopen set containing $pKer(\{x\})$. Then $X \setminus U$ is a preclosed set, and $(X \setminus U) \cap pCl(\{x\}) = D$ or \emptyset . If $(X \setminus U) \cap pCl(\{x\}) = D$, then D is the intersection of two preclosed sets and hence D is also preclosed. If $(X \setminus U) \cap pCl(\{x\}) = \emptyset$, then $pCl(\{x\}) \subset U$ and $D \subset U$. Since $D \not\subseteq pKer(\{x\})$, there is a preopen set V such that $x \in V$ and $D \not\subseteq V$. Then $pCl(\{x\}) \cap (X \setminus V) = D$ is a preclosed set. Therefore $pD(\{x\})$ is preclosed whenever $(x)_p = \{x\}$. Hence X is pre- R_D . ■

THEOREM 3.5. *If (X, τ) is pre- R_T , then (X, τ) is pre- R_H .*

PROOF. Let X be pre- R_T and $x, y \in X$. Suppose $pCl(\{x\}) \neq pCl(\{y\})$ and there is an $a \in X$ such that $a \neq x, a \neq y$ but $a \in pCl(\{x\}) \cap pCl(\{y\})$. Then $a \in pCl(\{x\})$ and $a \in pCl(\{y\})$. Hence $x \in pKer(\{a\})$ and $y \in pKer(\{a\})$. Since $pKer(\{a\}) = (a)_p \cup E$, where E is a degenerate set and $E \not\subseteq pCl(\{a\})$, there are following four possible cases for $x, y \in pKer(\{a\})$:

- (i) $x \in (a)_p$ and $y \in (a)_p$. We have $x \in pCl(\{a\})$ and $y \in pCl(\{a\})$ and also $a \in pCl(\{x\})$ and $a \in pCl(\{y\})$. Hence $pCl(\{x\}) = pCl(\{y\}) = pCl(\{a\})$ which is a contradiction.
- (ii) $\{x\} = E$ and $y \in (a)_p$. We have $x \notin pCl(\{a\})$ and $y \in pCl(\{a\})$. Since $a \in pCl(\{y\})$, we have $pCl(\{y\}) = pCl(\{a\})$. We have two cases to consider between y and $pCl(\{x\})$. Case (1): $y \in pCl(\{x\})$. Then $pCl(\{y\}) = pCl(\{a\})$. Since $x \notin pCl(\{a\})$, $x \in X \setminus pCl(\{a\})$. Since $X \setminus pCl(\{a\})$ is preopen,

$\text{pKer}(\{x\}) \subset X \setminus \text{pCl}(\{a\})$ and hence $\text{pCl}(\{x\}) \setminus \text{pKer}(\{x\}) \supset \text{pCl}(\{a\}) \supset \{y, a\}$. Hence $\text{pCl}(\{x\}) \setminus \text{pKer}(\{x\})$ is not a degenerate set, a contradiction to the fact that X is pre- R_T . Case (2): $y \notin \text{pCl}(\{x\})$. Since $y \in \text{pCl}(\{a\})$ and $a \in \text{pCl}(\{x\})$, we have $y \in \text{pCl}(\{x\})$, a contradiction.

(iii) $x \in (a)_p$ and $\{y\} = E$. Similar to case (ii).

(iv) $\{x\} = \{y\} = E$. We have $\text{pCl}(\{x\}) = \text{pCl}(\{y\})$. But this is impossible. Therefore if $\text{pCl}(\{x\}) \neq \text{pCl}(\{y\})$, we have $\text{pCl}(\{x\}) \cap \text{pCl}(\{y\}) = \emptyset$, $\{x\}$ or $\{y\}$. It follows that X is pre- R_H . ■

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ON TRIPLES OF CONSECUTIVE LEGENDRE SYMBOLS

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In [1] Davenport proved that for any prime p large enough ($p > p(n)$) and $\delta = (\alpha_1, \dots, \alpha_n) \in \{1, -1\}^n$ there exists a $b \in N$ such that $(\frac{b+i}{p}) = \alpha_i$, $i = 1, \dots, n$. In [2] A. Weil improved on Davenport's theorem. These existence proofs estimate some character-sums and get a result for the number of solutions depending of p and n and are based on the statistical behaviour of the quadratic residues. The purpose of my paper is the determine explicit triples satisfying Davenport's theorem in case $n = 3$ or describe an algorithm how the get these triples from other ones. (The values of the triples may depend on the prime p .) The proof is based on combinatorial ideas and results concerning the quadratic residues.

THEOREM. *For an arbitrary prime $p > 17$ and $\delta = (\alpha_1, \alpha_2, \alpha_3) \in \{-1, 1\}^3$ there exists a well-defined set B_δ of 2-5 elements such that either*

$$(1) \quad \left(\frac{b+i}{p} \right) = \alpha_i \quad i = 1, 2, 3$$

for at least one element of B_δ or there is a short algorithm how to get B_δ from an other one.

NOTATION. $L(a_1, \dots, a_n) = (\alpha_1, \dots, \alpha_n)$ means that $(\frac{a_i}{p}) = \alpha_i$, $i = 1, \dots, n$.

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The following proof proves Davenport's theorem on a constructive way in the case $n = 3$. First we prove some lemmas. Lemma 3 almost proves Davenport's theorem in the case $n = 3$ based on lemmas 1 and 2. Lemma 2 is not necessary to prove the theorem. Instead of lemma 2 we find $c, d \in N$ such that $L(c, c+1, c+2) = (1, 1, 1)$ and $L(d, d+1, d+2) = (-1, -1, -1)$ using lemma 4.

LEMMA 1. If $p = 4k + 1$ and $L(a_1, a_2, a_3) = (\alpha_1, \alpha_2, \alpha_3)$ then

$$L(p - a_3, p - a_2, p - a_1) = (\alpha_3, \alpha_2, \alpha_1).$$

If $p = 4k - 1$ and $L(a_1, a_2, a_3) = (\alpha_1, \alpha_2, \alpha_3)$ then

$$L(p - a_3, p - a_2, p - a_1) = (-\alpha_3, -\alpha_2, -\alpha_1).$$

LEMMA 2. There exists a $c \in N$ such that $L(c, c+1, c+2) = (1, 1, 1)$ or $L(c, c+1, c+2) = (-1, -1, -1)$ if $p > 7$.

LEMMA 3. For the primes $p > 11$ all the triples $(\alpha_1, \alpha_2, \alpha_3) \in \{-1, 1\}^3$ occur if $p = 4k - 1$ and at least 4 triples occur if $p = 4k + 1$.

LEMMA 4. If $p = at + d$ and $(\frac{ai-d}{p}) = \alpha_i$, $i = 1, 2, 3$ then

$$(2) \quad \left(\frac{t+i}{p} \right) = \left(\frac{a}{p} \right) \alpha_i \quad i = 1, 2, 3.$$

PROOF OF THE LEMMAS.

1. It is true as $(\frac{p-a_i}{p}) = (\frac{-1}{p})(\frac{a_i}{p})$, $i = 1, 2, 3$. ■

2. $U = \sum_{1 \leq x \leq p-1} \left(\frac{x(x+1)}{p} \right) = \sum \left(\frac{2x(2x+2)}{p} \right) = \sum_{z \in \{0, 2, 3, \dots, p-2\}} \left(\frac{z^2-1}{p} \right) = -1$

as

$$(3) \quad z^2 - 1 \equiv u^2 \not\equiv 0 \pmod{p}$$

$\Leftrightarrow (z-u)(z+u) \equiv c \cdot c^{-1} \equiv 1 \pmod{p} \Leftrightarrow z \equiv (c+c^{-1})/2, u \equiv (c-c^{-1})/2$, $c \not\equiv \pm 1$.

z uniquely determines the pair (c, c^{-1}) as $s^2 - 2zs + 1 \equiv 0$ has at most two solutions for any fixed z (using the Viéte formulas) therefore (3) has exactly $\frac{p-3}{2}$ solutions.

$U = -1$ means that if $L(1, \dots, p-2) = (\beta_1, \dots, \beta_{p-2})$ then $\beta_i = \beta_{i+1}$ $\frac{p-3}{2}$ times and $\beta_i \neq \beta_{i+1}$ $\frac{p-1}{2}$ times. If $\beta_i = \beta_{i+1} = \beta_{i+2}$ is excluded then the

$\frac{p-3}{2}$ neighbouring residue pairs are disjunct and only 2 elements are missing. As there cannot be more identical neighbouring residue pairs these 2 elements are single as β_1 and β_{p-2} or are between 2 identical pairs beginning at z and $p-z$ for some z (by lemma 1). As $(\frac{1}{p}) = (\frac{4}{p}) = 1$ we have $L(1, 2, 3, 4, \dots) = (1, -1, -1, 1, 1, -1, \dots)$ or $(1, 1, -1, 1, 1, -1, -1, 1, 1, -1, \dots)$ for $p > 19$ depending on the place of the first single element. We get the contradiction by counting $(\frac{6}{p})$ and $(\frac{10}{p})$, resp. It can be verified that $p \notin [11, 23]$ either. ■

3. By lemma 2 there exist a b such that $(\frac{2b+2i}{p}) = (\frac{2}{p})(\frac{b+i}{p}) = \alpha$, $i = 1, 2, 3$ with $\alpha = 1$ or $\alpha = -1$. If $(\frac{2b+2i-1}{p}) \in \{\alpha, 0\}$ we multiply by 2 again. Sooner or later (multiplying by 2 repeatedly) we get a triple of residues $(1, -1, 1)$ or $(-1, 1, -1)$ as there does not exist a sequence of neighbouring identical residues of arbitrary length. For $p = 4k - 1$ we have both triples by lemma 1.

If we continue the triple $(1, 1, 1)$ or $(-1, -1, -1)$ we get at least one of the triples $(1, 1, -1)$, $(1, -1, -1)$, $(-1, -1, 1)$, $(-1, 1, 1)$. If $p = 4k + 1$ then by lemma 1 we get an other triple also. If $p = 4k - 1$ then $(1, 1, 1)$ and $(-1, -1, -1)$ both occur. Therefore $L(1, \dots, p-1) = (1, \dots, -1, -1, -1, \dots, 1, 1, 1, \dots, -1)$ or $(1, 1, 1, \dots, -1, -1, -1)$ or $(1, \dots, 1, 1, 1, \dots, -1, -1, -1, \dots, -1)$ and the triple $(-1, 1, -1)$ is a triple in $L(1, \dots, p-1)$. It is trivial that $(1, 1, -1)$ occurs. We prove that $(-1, -1, 1)$ occurs also and so by Lemma 1 we are ready. If $(-1, -1, 1)$ and $(-1, 1, 1)$ (its symmetric partner in Lemma 1) are missing then $L(1, \dots, p-1)$ has a special form: block of 1's of length $k (> 3)$, $-1, 1, -1, 1, \dots$ (alternating block), block of -1 's of length k . If $k < (p-1)/2$ then $L(p-3, p-1) = L((p-3)/2, (p-1)/2) = (1, -1)$ or $(-1, 1)$ is a contradiction. If $k = (p-1)/2$ then $L(p-5) = L(2)L((p-5)/2) = 1$ is a contradiction for $p > 11$. ■

4. It is true as $(\frac{a}{p})^2 (\frac{t+i}{p}) = (\frac{a}{p}) (\frac{at+ai-p}{p}) = (\frac{a}{p}) (\frac{ai-d}{p})$. ■

PROOF OF THE THEOREM. In the following table we consider the possible 8 cases depending on the values of $(\frac{2}{p})$, $(\frac{3}{p})$, $(\frac{5}{p})$.

	$(\frac{1}{p})$	$(\frac{2}{p})$	$(\frac{3}{p})$	$(\frac{4}{p})$	$(\frac{5}{p})$	$(\frac{6}{p})$	$(\frac{7}{p})$	$(\frac{8}{p})$	$(\frac{9}{p})$	$(\frac{10}{p})$
1.	+1	+1	+1	+1	+1	+1	± 1	+1	+1	+1
2.	+1	+1	+1	+1	-1	+1	± 1	+1	+1	-1
3.	+1	+1	-1	+1	+1	-1	± 1	+1	+1	+1
4.	+1	+1	-1	+1	-1	-1	± 1	+1	+1	-1
5.	+1	-1	+1	+1	+1	-1	± 1	-1	+1	-1
6.	+1	-1	+1	+1	-1	-1	± 1	-1	+1	+1
7.	+1	-1	-1	+1	+1	+1	± 1	-1	+1	-1
8.	+1	-1	-1	+1	-1	+1	± 1	-1	+1	+1

The triple $(1, 1, 1)$ does not occur in rows 4, 6, 8 only. Suitable triples which guarantee $(1, 1, 1)$ are in these cases as follows:

Row 4. If $L(17) = 1$ then $L(15, 16, 17) = (L(3)L(5), 1, 1) = (1, 1, 1)$ and if $L(17) = -1$ then $L(49, 50, 51) = (1, L(2), L(3)L(17)) = (1, 1, 1)$.

Row 6. If $L(13) = -1$ then $L(25, 26, 27) = ((1, L(2)L(13), L(3)^3) = (1, 1, 1)$.

If $L(7, 13) = (-1, 1)$ then $L(12, 13, 14) = (L(3)L(4), 1, L(2)L(7)) = (1, 1, 1)$. As $L(2, 3) = (-1, 1)$ p must have the form $24k + 11$ or $24k + 13$. If $p = 24k + 11 = 3 \cdot 8k + 11$ then by the choice $a = 3$, $t = 8k$, $d = 11$ in Lemma 4 we have $L(8k+1, 8k+2, 8k+3) = L(3) \cdot L(-8, -5, -2) = (1, 1, 1)$. If $p = 24k + 13 = 4(6k+2) + 5$ and $L(7) = 1$ then by the choice $a = 4$, $t = 6k+2$, $d = 5$ in Lemma 4 we have $L(6k+3, 6k+4, 6k+5) = L(4)L(-1, 3, 7) = (1, 1, 1)$.

Row 8. If $L(7) = -1$ then $L(14, 15, 16) = (1, 1, 1)$. If $L(13) = -1$ then $L(24, 25, 26) = (1, 1, 1)$. As $L(2, 3) = (-1, -1)$ then p must have the form $24k + 5$ or $24k - 5$. If $p = 24k + 5 = 6 \cdot 4k + 5$ and $L(7, 13) = (1, 1)$ then by the choice $a = 6$, $t = 4k$, $d = 5$ in Lemma 4 we have $L(4k+1, 4k+2, 4k+3) = L(6)L(1, 7, 13) = (1, 1, 1)$. If $p = 24k - 5 = 3(8k - 2) + 1$ then by the choice $a = 3$, $t = 8k - 2$, $d = 1$ in Lemma 4 we have $L(8k-1, 8k, 8k+1) = L(3)L(2, 5, 8) = (1, 1, 1)$.

If $p = 4k - 1$ then $L(a, a+1, a+2) = (1, 1, 1)$ implies

$$L(p-a-2, p-a-1, p-a) = (-1, -1, -1)$$

by Lemma 1.

If $p = 4k + 1$ then let p_1 denote the least prime for which $L(p_1) = -1$ and let s denote the element in $\{2p_1 - d, -p_1 - d\}$. If $L(2) = 1$ we have $L(p_1z - d) = L(p_1)L(t+z) = -1$.

If $p = 4k + 1$ and $L(2) = -1$ then the following triples guarantee $(-1, -1, -1)$:

Row 5. p must have the form $24k + 13$ as $p = 4k + 1$ and $L(2, 3) = (-1, 1)$. By the choice $a = 2, t = 12k + 6, d = 1$ in Lemma 4 we have

$$L(12k + 7, 12k + 8, 12k + 9) = L(2)L(1, 3, 5) = (-1, -1, -1).$$

Row 6. If $L(7) = -1$ then $L(5, 6, 7) = (-1, -1, -1)$. If $L(7, 13) = (1, -1)$ then $L(13, 14, 15) = (-1, -1, -1)$.

p must have the form $24k + 13$ as $p = 4k + 1$ and $L(2, 3) = (-1, 1)$. If $L(7, 13) = (1, 1)$ then by the form $p = 6(4k - 1) + 19$ ($a = 6, t = 4k - 1, d = 19$ in Lemma 4) we have

$$L(4k, 4k + 1, 4k + 2) = L(6)L(-13, -7, -1) = (-1, -1, -1).$$

Row 7. If $L(11) = -1$ then $L(10, 11, 12) = (-1, -1, -1)$. p must have the form $24k + 5$ as $p = 4k + 1$ and $L(2, 3) = (-1, -1)$. If $L(7) = 1$ then by the form $p = 3(8k + 1) + 2$ in Lemma 4 we have $L(8k + 2, 8k + 3, 8k + 4) = L(3)L(1, 4, 7) = (-1, -1, -1)$. Considering $L(5) = 1$ we have $p = 120m + 29$ or $p = 120m - 19$. If $L(7) = -1$ then by the form $p = 15 \cdot 8m + 29$ in Lemma 4 we have $L(8m + 1, 8m + 2, 8m + 3) = L(15)L(-14, 1, 16) = (-1, -1, -1)$. If $L(11) = 1$ then by the form $p = 10(12m - 4) + 21$ in Lemma 4

$$L(12m - 3, 12m - 2, 12m - 1) = L(10)L(-11, -1, 9) = (-1, -1, -1).$$

Row 8. The construction is similar to the case 7. If $L(19) = -1$ then $L(18, 19, 20) = (-1, -1, -1)$. p must have the form $24k + 5$ (as $p = 4k + 1$ and $L(2, 3) = (-1, -1)$) furthermore considering $L(5) = -1$ $p = 120m + 53$ or $p = 120m - 43$. If $L(7) = 1$ then by the form $p = 3(8k + 1) + 2$ in Lemma 4 we have $L(8k + 2, 8k + 3, 8k + 4) = L(3)L(1, 4, 7) = (-1, -1, -1)$. If $L(17, 19) = (1, 1)$ in the case $p = 2(12k + 9) - 13$ by Lemma 4 we have $L(12k + 10, 12k + 11, 12k + 12) = L(2)L(15, 17, 19) = (-1, -1, -1)$. If $L(7, 17) = (-1, -1)$ then in the case $p = 10(12m + 4) + 13$ by Lemma 4 $L(12m + 5, 12m + 6, 12m + 7) = L(10)L(-3, 7, 17) = (-1, -1, -1)$. If $L(7) = -1$ then in the case $p = 15(8m - 6) - 47$ by Lemma 4 we have

$$L(8m - 5, 8m - 4, 8m - 3) = L(15)L(-32, -17, 2) = (-1, -1, -1).$$

For $p = 4k - 1$ we use the ideas of lemma 3 to derive the further B_δ 's from $B_{(1,1,1)}$ and $B_{(-1,-1,-1)}$. The proof works for $p = 4k + 1$ also with the exceptional cases $\delta \in \{(1, -1, 1), (-1, 1, -1)\}$ if $L(2) = -1$. Considering the table we need to construct triples for $(-1, 1, -1)$ in case 6 if $L(7) = -1$ and for $(1, -1, 1)$ in case 7 if $L(7) = -1$ only. If $L(7, 13) = (-1, -1)$ then

$L(13, 14, 16) = (-1, 1, -1)$ and if $L(7, 13) = (-1, 1)$ then $L(24, 25, 26) = (-1, 1, -1)$ in row 6. If $L(7) = -1$ then $L(14, 15, 16) = (1, -1, 1)$ in row 7. ■

REMARK. A further construction for $(1, -1, 1)$ if $L(2) = 1$: If p_1 is the minimal prime such that $L(p_1) = -1$ then $L(p_1 - 1, p_1, p_1 + 1) = (1, -1, 1)$.

The proof works for all $p > 19$. The theorem is valid for the primes 11 and 19 either. For 13 and 17, e.g. $(1, 1, 1)$ is missing.

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**STABILIZATION OF SOLUTIONS TO A NONLINEAR SYSTEM
MODELLING FLUID FLOW IN POROUS MEDIA**

By

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1. Introduction

In this paper we investigate a system of nonlinear equations that models fluid flow through porous medium. A porous medium, roughly speaking, is a solid medium with lots of tiny holes, for example limestone. Such medium consists of two parts, the solid matrix and the holes. If the fluid carries dissolved chemical species, a variety of chemical reactions can occur that can change the porosity. This process was modelled by J. Logan, M. R. Petersen, T. S. Shores in [10] by the following system of equations in one dimension:

$$(1) \quad \omega(t, x)u_t(t, x) = \\ = \alpha \cdot (|v(t, x)|u_x(t, x))_x + K(\omega(t, x))p_x(t, x)u_x(t, x) - ku(t, x)g(\omega(t, x))$$

$$(2) \quad \omega_t(t, x) = bu(t, x)g(\omega(t, x))$$

$$(3) \quad (K(\omega(t, x))p_x(t, x))_x = bu(t, x)g(\omega(t, x)),$$

$$(4) \quad v(t, x) = -K(\omega(t, x))p_x(t, x), \quad t > 0, x \in (0, 1),$$

with initial and boundary conditions

$$u(0, x) = u_0(x), \quad \omega(0, x) = \omega_0(x) \quad x \in (0, 1),$$

$$u(t, 0) = u_1(t), \quad u_x(t, 1) = 0 \quad t > 0,$$

$$p(t, 0) = 1, \quad p(t, 1) = 0 \quad t > 0$$

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where ω is the porosity, u is the concentration of the dissolved solute, p is the pressure, v is the velocity, further, α , k , b are given constants, K and g are given real functions. For the details of making this model and on flow in such media see [7, 10] and the references there. In what follows, we investigate the following system of nonlinear differential equations:

$$(5) \quad D_t \omega(t, x) = f(t, x, \omega(t, x), u(t, x)), \quad \omega(0, x) = \omega_0(x),$$

$$(6) \quad \begin{aligned} & D_t u(t, x) - \sum_{i=1}^n D_i [a_i(t, x, \omega(t, x), u(t, x), Du(t, x), \mathbf{p}(t, x), D\mathbf{p}(t, x))] + \\ & + a_0(t, x, \omega(t, x), u(t, x), Du(t, x), \mathbf{p}(t, x), D\mathbf{p}(t, x)) = \\ & = g(t, x, \omega(t, x)), \quad u(0, x) = 0, \end{aligned}$$

$$(7) \quad \begin{aligned} & - \sum_{i=1}^n D_i [b_i(t, x, \omega(t, x), u(t, x), \mathbf{p}(t, x), D\mathbf{p}(t, x))] + \\ & + b_0(t, x, \omega(t, x), u(t, x), \mathbf{p}(t, x), D\mathbf{p}(t, x)) = h(t, x, \omega(t, x), u(t, x)) \end{aligned}$$

with boundary conditions homogenous Dirichlet or Neumann, for example

$$\begin{aligned} & \sum_{i=1}^n a_i(t, x, \omega(t, x), u(t, x), Du(t, x), \mathbf{p}(t, x), D\mathbf{p}(t, x)) v_i = 0, \\ & \mathbf{p}(t, x) = 0 \quad x \in \partial \Omega, t > 0, \end{aligned}$$

where v is the unit normal along the boundary. (The variable \mathbf{p} is written by boldface letter for the purpose of distinguishing it from exponents p_1, p_2). Moreover, if $\partial \Omega = S_1 \cup S_2$ where $S_1 \cap S_2 = \emptyset$, then we can pose different boundary conditions on the elements of the partition. That is the case in the model (1)–(4) where the partitions are the endpoints of the interval $[0, 1]$. Clearly, we can assume the boundary conditions to be homogeneous by subtracting a suitable function from the unknown function.

The above system was investigated in [4] and it was shown there that due to some assumptions (that will be introduced later in this paper) the solution ω of (2) is positive thus we can divide equation (1) by ω . Hence the above system is a generalization of the model (1)–(4). Observe that for fixed u equation (5) is an ordinary differential equation with respect to the function ω ; for fixed ω and \mathbf{p} equation (6) is a parabolic equation with respect to the function u ; and for fixed ω and u equation (7) is an elliptic equation with respect to the function \mathbf{p} . This shows that the above system is a hybrid evolutionary/elliptic problem, thus theorems of „usual” systems on

partial differential equations do not work. In [4] existence of weak solutions to the above system were proved in $(0, T) \times \Omega$ (where $0 < T < \infty$) by using the theory of operators of monotone type. In the following we consider the equations in $(0, \infty) \times \Omega$. First we define the weak forms then we prove (under suitable conditions) existence of weak solutions, the boundedness of the solutions in appropriate norms in the time interval $(0, \infty)$, further, we show the stabilization of the solutions, in the sense that they converge to the stationary solutions as $t \rightarrow \infty$. We use some results and arguments of [11, 12, 13]. Finally, we give some examples.

2. Notation

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with the uniform C^1 regularity property (see [1]), further, $2 \leq p_1, p_2 < \infty$ be real numbers. For $0 < T < \infty$ let $Q_T := (0, T) \times \Omega$, and let $Q_\infty := (0, \infty) \times \Omega$. Denote by $W^{1,p_i}(\Omega)$ the usual Sobolev space with the norm

$$\|v\|_{W^{1,p_i}(\Omega)} = \left(\int_{\Omega} (|v|^{p_i} + \sum_{j=1}^n |D_j v|^{p_i}) \right)^{\frac{1}{p_i}}$$

where D_j denotes the distributional derivative with respect to the j -th variable (later we use the notation $D = (D_1, \dots, D_n)$). In addition, let V_i be a closed linear subspace of the space $W^{1,p_i}(\Omega)$ which contains $W_0^{1,p_i}(\Omega)$ (the closure of $C_0^\infty(\Omega)$ in $W^{1,p_i}(\Omega)$), and let $L^{p_i}(0, T; V_i)$ be the Banach space of measurable functions $u: (0, T) \rightarrow V_i$ such that $\|u\|_{V_i}^{p_i}$ is integrable and the norm is given by

$$\|u\|_{L^{p_i}(0, T; V_i)} = \left(\int_0^T \|u(t)\|_{V_i}^{p_i} dt \right)^{\frac{1}{p_i}}.$$

The dual space of $L^{p_i}(0, T; V_i)$ is $L^{q_i}(0, T; V_i^*)$ where $\frac{1}{p_i} + \frac{1}{q_i} = 1$ and V_i^* is the dual space of V_i . In what follows, we use the notation $X_i^T := L^{p_i}(0, T; V_i)$. The pairing between V_i^* and V_i is denoted by $\langle \cdot, \cdot \rangle$, the pairing between $L^{q_i}(0, T; V_i^*)$ and $L^{p_i}(0, T; V_i)$ is denoted by $[\cdot, \cdot]$, further, $D_t u$ stands for the distributional derivative (with respect to the variable t)

of a function $u \in L^{p_i}(0, T; V_i)$. It is well known (see [14]) that if $u \in L^{p_i}(0, T; V_i)$, $D_t u \in L^{q_i}(0, T; V_i^*)$ then $u \in C([0, T], L^2(\Omega))$ so that $u(0)$ makes sense. Denote by $L_{\text{loc}}^{p_i}(0, \infty; V_i)$ the space of measurable functions $u: (0, \infty) \rightarrow V_i$ such that for every $0 < T < \infty$, $u|_{(0, T)} \in L^{p_i}(0, T; V_i)$. If for every $0 < T < \infty$ there exists $D_t(u|_{(0, T)}) \in L^{q_i}(0, T; V_i^*)$ then $D_t u \in X_i^* = L_{\text{loc}}^{q_i}(0, \infty; V_i^*)$. Finally, let $L_{\text{loc}}^\infty(Q_\infty)$ denote the space of functions $\omega: Q_\infty \rightarrow \mathbb{R}$ such that $\omega|_{Q_T} \in L^\infty(Q_T)$ for every $0 < T < \infty$.

3. The problem in $(0, \infty)$

3.1. Assumptions

We formulate some assumptions in order to prove existence of weak solutions. In the following ξ , (ξ_0, ξ) , (η_0, η) refer for variables ω , (u, Du) , $(\mathbf{p}, D\mathbf{p})$, respectively.

- A1. Functions $a_i: Q_\infty \times \mathbb{R} \times \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \rightarrow \mathbb{R}$ ($i = 0, \dots, n$) have the Carathéodory property, i.e., they are measurable in $(t, x) \in Q_\infty$ for every $(\xi, \xi_0, \zeta, \eta_0, \eta) \in \mathbb{R} \times \mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$ and continuous in $(\xi, \xi_0, \zeta, \eta_0, \eta) \in \mathbb{R} \times \mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$ for a.a. $(t, x) \in Q_\infty$.
- A2. There exist a continuous function $c_1: \mathbb{R} \rightarrow \mathbb{R}^+$ and a function $k_1 \in L^{q_1}(\Omega)$ such that

$$\begin{aligned} |a_i(t, x, \xi, \xi_0, \zeta, \eta_0, \eta)| &\leq \\ &\leq c_1(\xi) \left(|\xi_0|^{p_1-1} + |\zeta|^{p_1-1} + |\eta_0|^{\frac{p_2}{q_1}} + |\eta|^{\frac{p_2}{q_1}} + k_1(x) \right), \end{aligned}$$

for a.a. $(t, x) \in Q_\infty$ and every $(\xi, \xi_0, \zeta, \eta_0, \eta) \in \mathbb{R} \times \mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$ ($i = 0, \dots, n$).

- A3. There exists a constant $C > 0$ such that for a.a. $(t, x) \in Q_\infty$ and every $(\xi, \xi_0, \zeta, \eta_0, \eta), (\tilde{\xi}, \xi_0, \tilde{\zeta}, \eta_0, \eta) \in \mathbb{R} \times \mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$

$$\begin{aligned} \sum_{i=1}^n \left(a_i(t, x, \xi, \xi_0, \zeta, \eta_0, \eta) - a_i(t, x, \tilde{\xi}, \xi_0, \tilde{\zeta}, \eta_0, \eta) \right) (\xi_i - \tilde{\xi}_i) &\geq \\ &\geq C \cdot |\zeta - \tilde{\zeta}|^{p_1}. \end{aligned}$$

- A4. There exist a constant $c_2 > 0$, a continuous function $\gamma: \mathbb{R} \rightarrow \mathbb{R}$ and a function $k_2 \in L^1(\Omega)$ such that

$$\sum_{i=0}^n a_i(t, x, \xi, \zeta_0, \xi, \eta_0, \eta) \zeta_i \geq c_2 (|\xi_0|^{p_1} + |\xi|^{p_1}) - \gamma(\xi) k_2(x)$$

for a.a. $(t, x) \in Q_\infty$ and every $(\xi, \zeta_0, \xi, \eta_0, \eta) \in \mathbb{R} \times \mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$.

- B1. Functions $b_i: Q_\infty \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{n+1} \rightarrow \mathbb{R}$ ($i = 0, \dots, n$) have the Carathéodory property, i.e., they are measurable in $(t, x) \in Q_\infty$ for every $(\xi, \zeta_0, \eta_0, \eta) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{n+1}$ and continuous in $(\xi, \zeta_0, \eta_0, \eta) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{n+1}$ for a.a. $(t, x) \in Q_\infty$.
- B2. There exist a continuous function $\hat{c}_1: \mathbb{R} \rightarrow \mathbb{R}^+$ and a function $\hat{k}_1 \in L^{q_2}(\Omega)$ such that

$$|b_i(t, x, \xi, \zeta_0, \eta_0, \eta)| \leq \hat{c}_1(\xi) \left(|\eta_0|^{p_2-1} + |\eta|^{p_2-1} + |\zeta_0|^{\frac{p_1}{q_2}} + \hat{k}_1(x) \right)$$

for a.a. $(t, x) \in Q_\infty$ and every $(\xi, \zeta_0, \eta_0, \eta) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{n+1}$ ($i = 0, \dots, n$).

- B3. There exists a constant $\hat{C} > 0$ such that for a.a. $(t, x) \in Q_\infty$ and every $(\xi, \zeta_0, \eta_0, \eta), (\tilde{\xi}, \tilde{\zeta}_0, \tilde{\eta}_0, \tilde{\eta}) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{n+1}$

$$\begin{aligned} \sum_{i=0}^n (b_i(t, x, \xi, \zeta_0, \eta_0, \eta) - b_i(t, x, \tilde{\xi}, \tilde{\zeta}_0, \tilde{\eta}_0, \tilde{\eta})) (\eta_i - \tilde{\eta}_i) &\geq \\ &\geq \hat{C} \cdot (|\eta_0 - \tilde{\eta}_0|^{p_2} + |\eta - \tilde{\eta}|^{p_2}). \end{aligned}$$

- B4. There exist a constant $\hat{c}_2 > 0$, a continuous function $\hat{\gamma}: \mathbb{R} \rightarrow \mathbb{R}$ and a function $\hat{k}_2 \in L^1(\Omega)$ such that

$$\sum_{i=0}^n b_i(t, x, \xi, \zeta_0, \eta_0, \eta) \eta_i \geq \hat{c}_2 (|\eta_0|^{p_2} + |\eta|^{p_2}) - \hat{\gamma}(\xi) (|\zeta_0|^{p_1} + \hat{k}_2(x))$$

for a.a. $(t, x) \in Q_\infty$ and every $(\xi, \zeta_0, \eta_0, \eta) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{n+1}$.

- F1. Function $f: Q_\infty \times \mathbb{R}^2 \rightarrow \mathbb{R}$ is of Carathéodory type, i.e., it is measurable in $(t, x) \in Q_\infty$ for every fixed $(\xi, \zeta_0) \in \mathbb{R}^2$ and continuous in $(\xi, \zeta_0) \in \mathbb{R}^2$ for a.a. $(t, x) \in Q_\infty$. Further, for every bounded set $I \subset \mathbb{R}$ there exists a continuous function $K_1: \mathbb{R} \rightarrow \mathbb{R}^+$ such that

- (i) $|K_1(\zeta_0)| \leq d_1 |\zeta_0|^{\frac{p_1}{q_2}} + d_2$ for every $\zeta_0 \in \mathbb{R}$, with some nonnegative constants d_1, d_2 (depending on I),

(ii) for a.a. $(t, x) \in Q_\infty$ and every $(\xi, \zeta_0), (\tilde{\xi}, \zeta_0) \in I \times \mathbb{R}$,

$$|f(t, x, \xi, \zeta_0) - f(t, x, \tilde{\xi}, \zeta_0)| \leq K_1(\zeta_0) \cdot |\xi - \tilde{\xi}|.$$

F2. There exists a continuous function $K_2: \mathbb{R} \rightarrow \mathbb{R}^+$ such that for a.a. $(t, x) \in Q_\infty$ and all $(\xi, \zeta_0), (\tilde{\xi}, \zeta_0) \in \mathbb{R}^2$

$$|f(t, x, \xi, \zeta_0) - f(t, x, \tilde{\xi}, \zeta_0)| \leq K_2(\xi) \cdot |\zeta_0 - \tilde{\zeta}_0|.$$

F3. There exists $\omega^* \in L^\infty(\Omega)$ such that for a.a. $(t, x) \in Q_\infty$ and every $(\xi, \zeta_0) \in \mathbb{R}^2$, $(\xi - \omega^*(x)) \cdot f(t, x, \xi, \zeta_0) \leq 0$.

G1. Function $g: Q_\infty \times \mathbb{R} \rightarrow \mathbb{R}$ is of Carathéodory type, i.e., it is measurable in $(t, x) \in Q_\infty$ for every $\xi \in \mathbb{R}$ and continuous in $\xi \in \mathbb{R}$ for a.a. $(t, x) \in Q_\infty$.

G2. There exist a continuous function $c_3: \mathbb{R} \rightarrow \mathbb{R}^+$ and a function $k_3 \in L^{q_1}(\Omega)$ such that

$$|g(t, x, \xi)| \leq c_3(\xi)k_3(x)$$

for a.a. $(t, x) \in Q_\infty$ and every $\xi \in \mathbb{R}$.

H1. Function $h: Q_\infty \times \mathbb{R}^2 \rightarrow \mathbb{R}$ is of Carathéodory type, i.e., it is measurable in $(t, x) \in Q_\infty$ for every $(\xi, \zeta_0) \in \mathbb{R}^2$ and continuous in $(\xi, \zeta_0) \in \mathbb{R}^2$ for a.a. $(t, x) \in Q_\infty$.

H2. There exist a continuous function $c_4: \mathbb{R} \rightarrow \mathbb{R}^+$ and a function $k_4 \in L^{q_2}(\Omega)$ such that

$$|h(t, x, \xi, \zeta_0)| \leq c_4(\xi) \left(|\zeta_0|^{\frac{p_1}{q_2}} + k_4(x) \right)$$

for a.a. $(t, x) \in Q_\infty$ and every $(\xi, \zeta_0) \in \mathbb{R}^2$.

3.2. Weak form

If the above assumptions are satisfied, we may define the weak form of the system (5)–(7) in Q_∞ . First for arbitrary

$$\omega \in L_{\text{loc}}^\infty(Q_\infty), u \in L_{\text{loc}}^{p_1}(0, \infty; V_1), \mathbf{p} \in L_{\text{loc}}^{p_2}(0, \infty; V_2) \text{ and } t \in (0, \infty)$$

define functionals

$$\begin{aligned} [A(\omega, u, p)](t) &\in V_1^*, [B(\omega, u, \mathbf{p})](t) \in V_2^*, \\ [G(\omega)](t) &\in V_1^*, [H(\omega, u)](t) \in V_2^* \end{aligned}$$

by

$$\begin{aligned}
& \langle [A(\omega, u, p)](t), v_1 \rangle := \\
& = \int_{\Omega} \sum_{i=1}^n a_i(t, x, \omega(t, x), u(t, x), Du(t, x), \mathbf{p}(t, x), D\mathbf{p}(t, x)) D_i v_1(x) dx + \\
& \quad + \int_{\Omega} a_0(t, x, \omega(t, x), u(t, x), Du(t, x), \mathbf{p}(t, x), D\mathbf{p}(t, x)) v_1(x) dx, \\
& \langle [B(\omega, u, \mathbf{p})](t), v_2 \rangle := \\
& = \int_{\Omega} \sum_{i=1}^n b_i(t, x, \omega(t, x), u(t, x), \mathbf{p}(t, x), D\mathbf{p}(t, x)) D_i v_2(x) dx + \\
& \quad + \int_{\Omega} b_0(t, x, \omega(t, x), u(t, x), \mathbf{p}(t, x), D\mathbf{p}(t, x)) v_2(x) dx, \\
& \langle [G(\omega)](t), v_1 \rangle := \int_{\Omega} g(t, x, \omega(t, x)) v_1(x) dx, \\
(8) \quad & \langle [H(\omega, u)](t), v_2 \rangle := \int_{\Omega} h(t, x, \omega(t, x), u(t, x)) v_2(x) dx
\end{aligned}$$

for all $v_1 \in V_1$ and $v_2 \in V_2$. Further, for every $0 < T < \infty$ we introduce operators

$$\begin{aligned}
A_T: L^\infty(Q_T) \times X_1^T \times X_2^T \rightarrow (X_1^T)^*, \quad B_T: L^\infty(Q_T) \times X_1^T \times X_2^T \rightarrow (X_2^T)^*, \\
G_T: L^\infty(Q_T) \rightarrow (X_1^T)^*, \quad H_T: L^\infty(Q_T) \times X_1^T \rightarrow (X_2^T)^*
\end{aligned}$$

such that for all $w_1 \in X_1^T$ and $w_2 \in X_2^T$,

$$\begin{aligned}
(9) \quad & [A_T(\omega, u, \mathbf{p}), w_1] = \int_0^T \langle [A(\omega, u, \mathbf{p})](t), w_1(t) \rangle dt, \\
& [B_T(\omega, u, \mathbf{p}), w_2] = \int_0^T \langle [B(\omega, u, \mathbf{p})](t), w_2(t) \rangle dt, \\
& [G_T(\omega), w_1] = \int_0^T \langle [G(\omega)](t), w_1(t) \rangle dt, \\
& [H_T(\omega, u), w_2] = \int_0^T \langle [H(\omega, u)](t), w_2(t) \rangle dt.
\end{aligned}$$

In addition, let us introduce the linear operators $L_T : D(L_T) \rightarrow (X_1^T)^*$ by the formula

$$D(L_T) = \{u \in X_1^T : D_t u \in (X_1^T)^*, u(0) = 0\}, \quad L_T u = D_t u.$$

Now we are ready to define the weak form of (5)–(7), namely, we say that $\omega \in L_{\text{loc}}^\infty(Q_\infty)$, $u \in L_{\text{loc}}^{p_1}(0, \infty; V_1)$, $\mathbf{p} \in L_{\text{loc}}^{p_2}(0, \infty; V_2)$ are weak solutions of the system (5)–(7) if for every $0 < T < \infty$,

$$(10) \quad \omega(t, x) = \omega_0(x) + \int_0^t f(s, x, \omega(s, x), u(s, x)) ds, \quad \text{a.e. in } Q_T$$

$$(11) \quad L_T(u|_{Q_T}) + A_T(\omega|_{Q_T}, u|_{Q_T}, \mathbf{p}|_{Q_T}) = G_T(\omega|_{Q_T})$$

$$(12) \quad B_T(\omega|_{Q_T}, u|_{Q_T}, \mathbf{p}|_{Q_T}) = H_T(\omega|_{Q_T}, u|_{Q_T}).$$

It is well known (see e.g. [9]) that one obtains the above weak forms by considering sufficiently smooth solutions and then using Green's theorem, after that one considers the equations in the spaces X_i . It is clear that if the boundary condition is homogenous Neumann than $V_i = W^{1,p_i}(\Omega)$ (since the boundary term vanishes in Green's theorem) and if we have homogeneous Dirichlet boundary condition then $V_i = W_0^{1,p_i}(\Omega)$ (in order to eliminate the boundary terms in Green's theorem). Further, if we have a partition then for example in our one dimensional equation (1) with homogenous boundary conditions $V_1 = \{v \in W^{1,p_1}(0, 1) : v(t, 0) = 0\}$, and in addition $V_2 = W_0^{1,p_2}(0, 1)$. In the next section we prove that the earlier introduced assumptions imply existence of solutions of the above system.

3.3. Existence of weak solutions

THEOREM 1. *Suppose that conditions A1–A4, B1–B4, F1–F3, G1–G2, H1–H2 are satisfied. Then for all $\omega_0 \in L^\infty(\Omega)$ there exist solutions $\omega \in L^\infty(Q_\infty)$, $u \in L_{\text{loc}}^{p_1}(0, \infty; V_1)$, $\mathbf{p} \in L_{\text{loc}}^{p_2}(0, \infty; V_2)$ of problem (10)–(12).*

PROOF. The main idea is the following. By [4] for every $0 < T < \infty$ there exist solutions in Q_T . By using a diagonal process we can choose weakly convergent subsequences from the solutions and then prove that the limit functions are solutions in Q_∞ .

Let (T_k) be a monotone increasing sequence of positive numbers such that $T_k \rightarrow +\infty$. Then by Theorem 2.1 in [4], for every T_k we have $\omega_k \in L^\infty(Q_{T_k})$, $u_k \in L^{p_1}(0, T_k; V_1)$, $\mathbf{p}_k \in L^{p_2}(0, T_k; V_2)$, such that

$$(13) \quad \omega_k(t, x) = \omega_0(x) + \int_0^t f(s, x, \omega_k(s, x), u_k(s, x)) ds, \quad (t, x) \in Q_{T_k}$$

$$(14) \quad L_{T_k} u_k + A_{T_k}(\omega_k, u_k, \mathbf{p}_k) = G_{T_k}(\omega_k)$$

$$(15) \quad B_{T_k}(\omega_k, u_k, \mathbf{p}_k) = H_{T_k}(\omega_k, u_k).$$

(We omit the notation $|_{Q_{T_m}}$ if it is not confusing, since the operators and the norms contain the information about the space). By similar arguments as in Proposition 2.2 of the cited paper, it is easy to see that

$$(16) \quad \|\omega_k\|_{L^\infty(Q_{T_m})} \leq \|\omega_0\|_{L^\infty(\Omega)} + \|\omega^*\|_{L^\infty(\Omega)}.$$

Now by following the proof of Theorem 2.1 in [4] word for word and by using the boundedness of the sequence (ω_k) in $L^\infty(Q_{T_m})$ for all m one obtains that for fixed $m \in \mathbb{N}$, $(u_k|_{Q_{T_m}})$, $(L_{T_m} u_k|_{Q_{T_m}})$ and $(\mathbf{p}_k|_{Q_{T_m}})$ are bounded in $X_1^{T_m}$, $(X_1^{T_m})^*$ and $X_2^{T_m}$, respectively.

Now let $m = 1$. Since (u_k) , $(L_{T_1} u_k)$, (\mathbf{p}_k) are bounded sequences in reflexive Banach spaces $X_1^{T_1}$, $(X_1^{T_1})^*$, $X_2^{T_1}$, respectively, there exist subsequences $(u_{1,k}) \subset (u_k)$, $(\mathbf{p}_{1,k}) \subset (\mathbf{p}_k)$ and functions $u_{1,*} \in X_1^{T_1}$, $\mathbf{p}_{1,*} \in X_2^{T_1}$ such that

$$u_{1,k} \rightarrow u_{1,*} \text{ weakly in } X_1^{T_1},$$

$$L_{T_1} u_{1,k} \rightarrow L_{T_1} u_{1,*} \text{ weakly in } (X_1^{T_1})^*,$$

$$\mathbf{p}_{1,k} \rightarrow \mathbf{p}_{1,*} \text{ weakly in } X_2^{T_1}.$$

If $(u_{m-1,k})_{k \geq m-1}$ is given then

$$(u_{m-1,k})_{k \geq m-1}, (L_{T_{m-1}} u_{m-1,k})_{k \geq m-1}, (\mathbf{p}_{m-1,k})_{k \geq m-1}$$

are bounded in reflexive spaces $X_1^{T_{m-1}}$, $(X_1^{T_{m-1}})^*$, $X_2^{T_{m-1}}$, thus there exist subsequences $(u_{m,k}) \subset (u_{m-1,k})$, $(\mathbf{p}_{m,k}) \subset (\mathbf{p}_{m-1,k})$ and functions $u_{m,*} \in X_1^{T_m}$, $\mathbf{p}_{m,*} \in X_2^{T_m}$ such that

$$u_{m,k} \rightarrow u_{m,*} \text{ weakly in } X_1^{T_m},$$

$$L_{T_m} u_{m,k} \rightarrow L_{T_m} u_{m,*} \text{ weakly in } (X_1^{T_m})^*,$$

$$\mathbf{p}_{m,k} \rightarrow \mathbf{p}_{m,*} \text{ weakly in } X_2^{T_m}.$$

It is easy to see that for each fixed $l < m$ the above weak convergences hold in $X_1^{T_l}$, $(X_1^{T_l})^*$, $X_2^{T_l}$, respectively, which yields $u_{m,*}|_{Q_{T_l}} = u_{l,*}$ and $\mathbf{p}_{m,*}|_{Q_{T_l}} = u_{l,*}$ for $l < m$. Consequently there exist unique functions $u:(0,\infty) \rightarrow V_1$, $\mathbf{p}:(0,\infty) \rightarrow V_2$ such that $u|_{Q_{T_m}} = u_{m,*}$, $\mathbf{p}|_{Q_{T_m}} = \mathbf{p}_{m,*}$ and $u_{m,*} \in D(L_{T_m})$ for every $m \in \mathbb{N}$. This means that $u \in L_{\text{loc}}^{p_1}(0,\infty; V_1)$ and $\mathbf{p} \in L_{\text{loc}}^{p_2}(0,\infty; V_2)$. Consider the sequences $(u_k) = (u_{k,k})$, $(\mathbf{p}_k) = (\mathbf{p}_{k,k})$ and the appropriate sequence (ω_k) . Observe that $u_k \rightarrow u$ weakly in $X_1^{T_m}$, $\mathbf{p}_k \rightarrow \mathbf{p}$ weakly in $X_2^{T_m}$. Thus by the well known embedding theorem (see [9]) we may assume that $u_k \rightarrow u$ in $L^{p_1}(Q_{T_m})$. Then Propositions 2.2, 2.3 in [4] yield that for every m there exists $\omega_{m,*} \in L^\infty(Q_{T_m})$ such that $(\omega_k) \rightarrow \omega_{m,*}$ a.e. in Q_{T_m} and

$$\omega_{m,*}(t,x) = \omega_0(x) + \int_0^t f(s,x,\omega_{m,*}(s,x),u_{m,*}(s,x)) ds, \quad (t,x) \in Q_{T_m}.$$

Since for every fixed $u_{m,*}$ the solution of the above equation is unique, further, functions $(u_{m,*})$ are the restrictions of the function u to Q_{T_m} , it follows that there exists a unique $\omega \in L_{\text{loc}}^\infty(Q_\infty)$ such that $\omega_{m,*} = \omega|_{Q_{T_m}}$ for every m and

$$\omega(t,x) = \omega_0(x) + \int_0^t f(s,x,\omega(s,x),u(s,x)) ds \quad (t,x) \in Q_\infty.$$

By similar arguments as in Proposition 2.2 in [4] it is easy to see that $\omega \in L^\infty(Q_\infty)$. Now fix $m \in \mathbb{N}$. We conclude from the above arguments that

$$\begin{aligned} \omega_k &\rightarrow \omega \text{ a.e. in } Q_{T_m} \\ u_k &\rightarrow u \text{ weakly in } X_1^{T_m}, \text{ strongly in } L^{p_1}(Q_{T_m}), \text{ a.e. in } Q_{T_m}; \\ L_{T_m} u_k &\rightarrow L_{T_m} u \text{ weakly in } (X_1^{T_m})^*; \\ \mathbf{p}_k &\rightarrow \mathbf{p} \text{ weakly in } X_2^{T_m}. \end{aligned}$$

By applying word for word step 3 of the proof of Theorem 2.1 in [4], by the above convergences it is not difficult to show that (for a suitable subsequence) $u_k \rightarrow u$ strongly in $X_1^{T_m}$, $\mathbf{p}_k \rightarrow \mathbf{p}_{m,*}$ strongly in $X_2^{T_m}$ and

$$\begin{aligned} L_{T_m} u|_{Q_{T_m}} + A_{T_m}(\omega|_{Q_{T_m}}, u|_{Q_{T_m}}, \mathbf{p}|_{Q_{T_m}}) &= G_{T_m}(\omega|_{Q_{T_m}}) \\ B_{T_m}(\omega|_{Q_{T_m}}, u|_{Q_{T_m}}, \mathbf{p}|_{Q_{T_m}}) &= H_{T_m}(\omega|_{Q_{T_m}}, u|_{Q_{T_m}}). \end{aligned}$$

This means that ω, u, \mathbf{p} are solutions in $(0, \infty)$, so the proof of the theorem is complete. ■

4. Boundedness of solutions

In this section we show that under some further assumptions, the solutions of (10)–(12) are bounded in appropriate norms in the time interval $(0, \infty)$. First suppose the following.

B4*. There exist a constant $\hat{c}_2 > 0$, a continuous function $\hat{\gamma}: \mathbb{R} \rightarrow \mathbb{R}$ and a function $\hat{k}_2 \in L^1(\Omega)$ such that

$$\sum_{i=0}^n b_i(t, x, \xi, \xi_0, \eta_0, \eta) \eta_i \geq \hat{c}_2 (|\eta_0|^{p_2} + |\eta|^{p_2}) - \hat{\gamma}(\xi) (|\xi_0|^2 + \hat{k}_2(x))$$

for a.a. $(t, x) \in Q_\infty$ and every $(\xi, \xi_0, \eta_0, \eta) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{n+1}$.

H2*. There exist a continuous function $c_4: \mathbb{R} \rightarrow \mathbb{R}^+$ and a function $k_4 \in L^{q_2}(\Omega)$ such that

$$|h(t, x, \xi, \xi_0)| \leq c_4(\xi) \left(|\xi_0|^{\frac{2}{q_2}} + k_4(x) \right)$$

for a.a. $(t, x) \in Q_\infty$ and every $(\xi, \xi_0) \in \mathbb{R}^2$.

THEOREM 2. *Assume that conditions A1–A4, B1–B3, B4*, F1–F3, G1–G2, H1, H2* are fulfilled and let ω, u, \mathbf{p} be solutions of (10)–(12). Then $\omega \in L^\infty(Q_\infty)$, $u \in L^\infty(0, \infty; L^2(\Omega))$, $\mathbf{p} \in L^\infty(0, \infty; V_2)$.*

PROOF. In Theorem 1 we have verified that $\omega \in L^\infty(Q_\infty)$ (which was a trivial consequence of (16)). In the following let $y(t) = \|u(t, \cdot)\|_{L^2(\Omega)}^2$. First note that $u \in C([0, T], L^2(\Omega))$ thus y is continuous in $[0, T]$ (see, e.g., [14]). We show that y is bounded in $(0, \infty)$. Since u is a solution of (11) for all $0 < T < \infty$, thus for arbitrary $0 < T_1 < T_2 < \infty$ we have

$$(17) \quad \begin{aligned} & \int_{T_1}^{T_2} \langle D_t u(t), u(t) \rangle dt + \int_{T_1}^{T_2} \langle [A(\omega, u, \mathbf{p})](t), u(t) \rangle dt = \\ & = \int_{T_1}^{T_2} \langle [G(\omega)](t), u(t) \rangle dt. \end{aligned}$$

By $\omega \in L^\infty(Q_\infty)$ and G2, $\|[G(\omega)](t)\|_{V_1^*}$ is bounded, thus by Young's inequality we obtain for $\varepsilon > 0$ that

$$\begin{aligned} & \int_{T_1}^{T_2} \langle [G(\omega)](t), u(t) \rangle dt \leq \\ & \leq \int_{T_1}^{T_2} \left(\frac{\varepsilon^{p_1}}{p_1} \|u(t)\|_{V_1}^{p_1} + \frac{1}{q_1 \varepsilon^{q_1}} \|[G(\omega)](t)\|_{V_1^*} \right) dt \leq \\ & \leq \int_{T_1}^{T_2} \left(\frac{\varepsilon^{p_1}}{p_1} \|u(t)\|_{V_1}^{p_1} + c(\varepsilon) \right) dt. \end{aligned}$$

By using the relation $\int_{T_1}^{T_2} \langle D_t u(t), u(t) \rangle dt = y(T_2) - y(T_1)$ (see [14]) and condition A4 on the left hand side of (17), further, by applying the above estimates with sufficiently small ε on the right hand side it follows

$$\begin{aligned} & \frac{1}{2} (y(T_1) - y(T_2)) + \frac{1}{2} c_2 \int_{T_1}^{T_2} \|u(t)\|_{V_1}^{p_1} dt \leq \\ & \leq \text{const} \int_{T_1}^{T_2} \int_{\Omega} (|\gamma(\omega(t, x)) k_2(x)| + 1) dx dt. \end{aligned}$$

Since $\omega \in L^\infty(Q_\infty)$ and γ is continuous, the integrand (with respect to variable t) is bounded, thus $\text{const} \cdot (T_2 - T_1)$ is an upper bound of the right hand side. By the continuous embedding $L^2(\Omega) \trianglerightarrow V_1$,

$$\|u(t)\|_{L^2(\Omega)} \leq \text{const} \cdot \|u(t)\|_{V_1},$$

hence

$$(18) \quad y(T_2) - y(T_1) + \text{const} \int_{T_1}^{T_2} y(t)^{\frac{p_1}{2}} dt \leq \text{const} \cdot (T_2 - T_1).$$

Now we show that the above inequality implies the boundedness of y . Suppose the contrary. Then for every $M > 0$ there exist $0 < T_1 < T_2 < \infty$ such that $y(T_2) = M$ and $M - 1 \leq y(t) \leq M$ if $T_1 \leq t \leq T_2$. By choosing these T_1, T_2 , from (18) we obtain

$$\text{const} \cdot (T_2 - T_1)(M - 1) \leq \text{const} \cdot (T_2 - T_1)$$

for every $M > 0$. This is impossible, thus y is bounded.

It remains to show that $\mathbf{p} \in L^\infty(0, \infty; V_2)$. The proof goes the same way as the previous part (moreover it is simpler since there is no derivative with respect to t), from $B(\omega, u, \mathbf{p}) = H(\omega, u)$, by using conditions B4*, H2* and the boundedness of ω we obtain

$$\|p(t)\|_{V_2}^{p_2} \leq \text{const} \left(\|u(t)\|_{L^2(\Omega)}^2 + 1 \right).$$

In the previous part of the proof we have shown that $y(t) = \|u(t)\|_{L^2(\Omega)}^2$ is bounded thus the above inequality implies the desired $\mathbf{p} \in L^\infty(0, \infty; V_2)$. The proof of Theorem 2 is complete. ■

5. Stabilization of solutions

In this section we consider a special case of the system (10)–(12), namely, let $p_1 = p_2 = p$ (thus $q_1 = q_2 = q$, $V_1 = V_2 = V$), $h = f$ and consider the following problem: for every $0 < T < \infty$

$$(19) \quad \omega(t, x) = \omega_0(x) + \int_0^t f(s, x, \omega(s, x), u(s, x)) ds, \quad (t, x) \in Q_T$$

$$(20) \quad L_T(u|_{Q_T}) + A_T(\omega|_{Q_T}, u|_{Q_T}, \mathbf{p}|_{Q_T}) = G_T(\omega|_{Q_T})$$

$$(21) \quad B_T(\omega|_{Q_T}, u|_{Q_T}, \mathbf{p}|_{Q_T}) = F_T(\omega|_{Q_T}, u|_{Q_T}).$$

The above operator F_T is defined through formula (9) with $H = F$, where operator F is given by (8) with $h = f$. We note that in the original model (1)–(4) $h = f$ thus the above system is also the generalization of it. In what follows, we show stabilization of the solutions of the above system. That is, we prove the convergence (in some sense) of solutions as $t \rightarrow \infty$ to the solutions of the stationary system. We need some further assumptions:

- A5. There exist Carathéodory functions $a_{i,\infty}: \Omega \times \mathbb{R} \times \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \rightarrow \mathbb{R}$ ($i = 0, \dots, n$), i.e., they are measurable in $x \in \Omega$ for every $(\xi, \xi_0, \zeta, \eta_0, \eta) \in \mathbb{R} \times \mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$ and continuous in $(\xi, \xi_0, \zeta, \eta_0, \eta) \in \mathbb{R} \times \mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$

for a.a. $x \in \Omega$. Further, for a.a. $x \in \Omega$ and every $(\xi_0, \zeta, \eta_0, \eta) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n$, $\xi^* \in \mathbb{R}$,

$$\lim_{\substack{t \rightarrow \infty \\ \xi \rightarrow \xi^*}} a_i(t, x, \xi, \xi_0, \zeta, \eta_0, \eta) = a_{i, \infty}(x, \xi^*, \xi_0, \zeta, \eta_0, \eta).$$

- B5. There exist Carathéodory functions $b_{i, \infty}: \Omega \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{n+1} \rightarrow \mathbb{R}$ ($i = 0, \dots, n$), i.e., they are measurable in $x \in \Omega$ for every $(\xi, \zeta_0, \eta_0, \eta) \in \mathbb{R} \times \mathbb{R}^{n+1}$ and continuous in $(\xi, \zeta_0, \eta_0, \eta) \in \mathbb{R} \times \mathbb{R}^{n+1}$ for a.a. $x \in \Omega$. Further, for a.a. $x \in \Omega$ and every $(\xi_0, \eta_0, \eta) \in \mathbb{R} \times \mathbb{R}^{n+1}$, $\xi^* \in \mathbb{R}$,

$$\lim_{\substack{t \rightarrow \infty \\ \xi \rightarrow \xi^*}} b_i(t, x, \xi, \xi_0, \eta_0, \eta) = b_{i, \infty}(x, \xi^*, \xi_0, \eta_0, \eta).$$

- AB There exists a positive constant \mathcal{C} such that for a.a. $(t, x) \in Q_\infty$, every $\xi \in \mathbb{R}$ and $(\xi_0, \zeta, \eta_0, \eta)$, $(\tilde{\xi}_0, \tilde{\zeta}, \tilde{\eta}_0, \tilde{\eta}) \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$,

$$\begin{aligned} & \sum_{i=0}^n \left(a_i(t, x, \xi, \xi_0, \zeta, \eta_0, \eta) - a_i(t, x, \xi, \tilde{\xi}_0, \tilde{\zeta}, \tilde{\eta}_0, \tilde{\eta}) \right) (\xi_i - \tilde{\xi}_i) + \\ & + \sum_{i=0}^n \left(b_i(t, x, \xi, \xi_0, \eta_0, \eta) - b_i(t, x, \xi, \tilde{\xi}_0, \tilde{\eta}_0, \tilde{\eta}) \right) (\eta_i - \tilde{\eta}_i) \geq \\ & \geq \mathcal{C} \cdot \left(|\xi_0 - \tilde{\xi}_0|^p + |\zeta - \tilde{\zeta}|^p + |\eta_0 - \tilde{\eta}_0|^p + |\eta - \tilde{\eta}|^p \right). \end{aligned}$$

- F1/(i*) Suppose (instead of condition F1/(i)) that there exist nonnegative constants d_1, d_2 such that $|K_1(\zeta_0)| \leq d_1 |\zeta_0|^{\frac{2}{q}} + d_2$ for every $\zeta_0 \in \mathbb{R}$.

- F4. There exists a positive constant m such that $(\xi - \omega^*(x))f(t, x, \xi, \zeta_0) \leq -m(\xi - \omega^*(x))^2$ for a.a. $(t, x) \in Q_\infty$ and every $(\xi, \zeta_0) \in \mathbb{R}^2$.

- G3. There exists a Carathéodory function $g_\infty: \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ such that for a.a. $x \in \Omega$ and every $\xi^* \in \mathbb{R}$,

$$\lim_{\substack{t \rightarrow \infty \\ \xi \rightarrow \xi^*}} g(t, x, \xi) = g_\infty(x, \xi^*).$$

Now introduce operators

$$A_\infty: L^\infty(\Omega) \times V \times V \rightarrow V^*,$$

$$B_\infty: L^\infty(\Omega) \times V \times V \rightarrow V^*,$$

$$G_\infty: L^\infty(\Omega) \rightarrow V^*$$

by

$$\begin{aligned}
& \langle A_\infty(\omega, u, \mathbf{p}), v \rangle := \\
& = \int_{\Omega} \sum_{i=1}^n a_{i,\infty}(x, \omega(x), u(x), Du(x), \mathbf{p}(x), D\mathbf{p}(x)) D_i v(x) dx + \\
& \quad + \int_{\Omega} a_{0,\infty}(x, \omega(x), u(x), Du(x), \mathbf{p}(x), D\mathbf{p}(x)) v(x) dx, \\
& \langle B_\infty(\omega, u, \mathbf{p}), v \rangle := \\
& = \int_{\Omega} \sum_{i=1}^n b_{i,\infty}(x, \omega(x), u(x), \mathbf{p}(x), D\mathbf{p}(x)) D_i v(x) dx + \\
& \quad + \int_{\Omega} b_{0,\infty}(x, \omega(x), u(x), \mathbf{p}(x), D\mathbf{p}(x)) v(x) dx, \\
& \langle G_\infty(\omega), v \rangle := \int_{\Omega} g_\infty(x, \omega(x)) v(x) dx,
\end{aligned}$$

where $v \in V$.

THEOREM 3. *Assume that conditions A1–A5, B1–B3, B4*, B5, AB, F1–F4, F1/(f*), G1–G3 are satisfied (with $p = p_1 = p_2$). Then there exist $u_\infty \in V$, $\mathbf{p}_\infty \in V$ such that for the solutions ω , u , \mathbf{p} of problem (19)–(21),*

$$\omega(t, \cdot) \rightarrow \omega^* \text{ in } L^\infty(\Omega), \quad u(t) \rightarrow u_\infty \text{ in } L^2(\Omega), \quad \int_{t-1}^{t+1} \|u(s) - u_\infty\|_V^p ds \rightarrow 0,$$

$\int_{t-1}^{t+1} \|\mathbf{p}(s) - \mathbf{p}_\infty\|_V^p ds \rightarrow 0$, $[F(\omega, u)](t) \rightarrow 0$ in V^* , $[G(\omega)](t) \rightarrow G_\infty(\omega^*)$ in V^* as $t \rightarrow \infty$, further,

$$\begin{aligned}
A_\infty(\omega^*, u_\infty, \mathbf{p}_\infty) &= G_\infty(\omega^*) \\
B_\infty(\omega^*, u_\infty, \mathbf{p}_\infty) &= 0.
\end{aligned}$$

REMARK 4. Precisely, by the convergence $s(t) \rightarrow 0$ as $t \rightarrow \infty$ where $s: \mathbb{R}^+ \rightarrow M$ is a measurable function, and M is a normed space, we mean that for all $\varepsilon > 0$ there exists t_0 such that $\|s(t)\|_M \leq \varepsilon$ for a.a. $t > t_0$.

PROOF. First we show that $\omega(t, \cdot) \rightarrow \omega^*$ in $L^\infty(\Omega)$ as $t \rightarrow \infty$. Fix $x \in \Omega$ and assume that $\omega_0(x) > \omega^*(x)$. By using similar arguments as in the proof of Proposition 2.5 in [4] we obtain that $\omega(t, x) > \omega^*(x)$ for $t > 0$. Then conditions F3 and F4 yield $f(t, x, \omega(t, x), u(t, x)) \leq -m(\omega(t, x) - \omega^*(x))$. Since ω is absolutely continuous, it is a.e. differentiable, so

$$\omega'(t, x) = f(t, x, \omega(t, x), u(t, x)) \leq -m(\omega(t, x) - \omega^*(x)).$$

By the positivity of $\omega - \omega^*$ we obtain

$$\frac{\omega'(t, x)}{\omega(t, x) - \omega^*(x)} \leq -m$$

hence

$$\omega(t, x) - \omega^*(x) \leq \omega_0(x)e^{-mt}.$$

When $\omega_0(x) < \omega^*(x)$ one has the estimate $-\omega_0(x)e^{-mt} \leq \omega(t, x) - \omega^*(x)$, thus $\|\omega(t, \cdot) - \omega^*(\cdot)\|_{L^\infty(\Omega)} \leq \|\omega_0\|_{L^\infty(\Omega)}e^{-mt}$. This means that $\omega(t, \cdot) \rightarrow \omega^*(\cdot)$ in $L^\infty(\Omega)$ as $t \rightarrow \infty$. Note, that this implies that $\omega(t, x) \rightarrow \omega^*(x)$ for a.a. $x \in \Omega$ as $t \rightarrow \infty$.

Now we prove that $[F(\omega, u)](t) \rightarrow 0$ in V^* as $t \rightarrow \infty$. Clearly, by conditions F1, F1/(i*), F3,

$$\begin{aligned} \| [F(\omega, u)](t) \|_{V^*}^q &\leq \int_{\Omega} |f(t, x, \omega(t, x), u(t, x))|^q dx \\ &\leq \int_{\Omega} |K_1(u(t, x))|^q |\omega(t, x) - \omega^*(x)|^q dx \\ &\leq \text{const} \cdot \left(\|u(t)\|_{L^2(\Omega)}^2 + \text{const} \right) \cdot \left(\|\omega(t, \cdot) - \omega^*\|_{L^\infty(\Omega)}^q \right). \end{aligned}$$

The multiplicator of the term $\|\omega(t, \cdot) - \omega^*\|_{L^\infty(\Omega)}$ is bounded by the previous section, thus the right hand side of the above inequality tends to 0 as $t \rightarrow \infty$.

Now we verify that $[G(\omega)](t) \rightarrow G_\infty(\omega^*)$ in V^* . Observe that

$$\| [G(\omega)](t) - G_\infty(\omega^*) \|_{V_2^*}^q \leq \int_{\Omega} |g(t, x, \omega(t, x)) - g_\infty(x, \omega^*)|^q dx$$

and the integrand is a.e. convergent to 0 in Ω as $t \rightarrow \infty$ by G3. Further, by G2, G3, $|g_\infty(x, \omega)| \leq c_3(\omega)k_3(x)$, hence $|g(t, x, \omega(t, x)) - g_\infty(x, \omega^*)|^q \leq (\|c_3(\omega)\|_{L^\infty(Q_\infty)} + \|c_3(\omega^*)\|_{L^\infty(\Omega)}) |k_3(x)|^q$ and the right hand side is integrable, thus Lebesgue's theorem implies that the above integral converges to 0 as $t \rightarrow \infty$.

Now we show that problem

$$(22) \quad A_\infty(\omega^*, u_\infty, \mathbf{p}_\infty) = G_\infty(\omega^*)$$

$$(23) \quad B_\infty(\omega^*, u_\infty, \mathbf{p}_\infty) = 0$$

has a unique solution $u_\infty \in V, \mathbf{p}_\infty \in V$ (for fixed ω^*). First observe that functions $a_{i,\infty}, b_{i,\infty}, g_\infty$ satisfy conditions A1–A4, B1–B4, G1–G2, H1–H2 (with variables x instead of (t, x)). Then by the using the idea of the main theorem of [4] (successive approximation) it is easy to see that there exist solutions $u_\infty \in V, \mathbf{p}_\infty \in V$ of problem (22)–(23) (the proof goes almost word for word, but it is simpler since there is no ω in the above problem). Uniqueness follows from condition AB, indeed if u_1, \mathbf{p}_1 and u_2, \mathbf{p}_2 are solutions then

$$\begin{aligned} 0 &= \langle A_\infty(\omega^*, u_1, \mathbf{p}_1) - A_\infty(\omega^*, u_2, \mathbf{p}_2), u_1 - u_2 \rangle + \\ &\quad + \langle B_\infty(\omega^*, u_1, \mathbf{p}_1) - B_\infty(\omega^*, u_2, \mathbf{p}_2), \mathbf{p}_1 - \mathbf{p}_2 \rangle \geq \\ &\geq \mathcal{C} \cdot \left(\|u_1 - u_2\|_V^{p_1} + \|\mathbf{p}_1 - \mathbf{p}_2\|_V^p \right) \end{aligned}$$

which implies $u_1 = u_2$ and $\mathbf{p}_1 = \mathbf{p}_2$.

In order to show the desired convergences we prove inequality for u and \mathbf{p} . From equations (19)–(21) and (22)–(23) we obtain

$$\begin{aligned} &\langle D_t(u(t) - u_\infty), u(t) - u_\infty \rangle + \langle [A(\omega, u, \mathbf{p})](t) - A_\infty(\omega^*, u_\infty, \mathbf{p}_\infty), u(t) - u_\infty \rangle + \\ &\quad + \langle [B(\omega, u, \mathbf{p})](t) - B_\infty(\omega^*, u_\infty, \mathbf{p}_\infty), \mathbf{p}(t) - \mathbf{p}_\infty \rangle = \\ (24) \quad &= \langle [G(\omega)](t) - G_\infty(\omega^*), u(t) - u_\infty \rangle + \langle [F(\omega, u)](t), \mathbf{p}(t) - \mathbf{p}_\infty \rangle. \end{aligned}$$

Observe that the first term on the left hand side of the above equation equals to $\frac{1}{2}y'(t)$ where $y(t) = \int\limits_{\Omega} (u(t) - u_\infty)^2$ (note that y is bounded in $[0, \infty)$ by

Theorem 2). Further, for the second and third terms of the above equation we have by condition AB and Young's inequality

$$\begin{aligned} &\langle [A(\omega, u, \mathbf{p})](t) - A_\infty(\omega^*, u_\infty, \mathbf{p}_\infty), u(t) - u_\infty \rangle + \\ &\quad + \langle [B(\omega, u, \mathbf{p})](t) - B_\infty(\omega^*, u_\infty, \mathbf{p}_\infty), \mathbf{p}(t) - \mathbf{p}_\infty \rangle = \\ &= \langle [A(\omega, u, \mathbf{p})](t) - [A(\omega, u_\infty, \mathbf{p}_\infty)](t), u(t) - u_\infty \rangle + \\ &\quad + \langle [B(\omega, u, \mathbf{p})](t) - [B(\omega, u_\infty, \mathbf{p}_\infty)](t), \mathbf{p}(t) - \mathbf{p}_\infty \rangle + \\ &\quad + \langle [A(\omega, u_\infty, \mathbf{p}_\infty)](t) - A_\infty(\omega^*, u_\infty, \mathbf{p}_\infty), u(t) - u_\infty \rangle + \\ &\quad + \langle [B(\omega, u_\infty, \mathbf{p}_\infty)](t) - B_\infty(\omega^*, u_\infty, \mathbf{p}_\infty), \mathbf{p}(t) - \mathbf{p}_\infty \rangle \geq \\ &\geq \mathcal{C} \cdot \left(\|u(t) - u_\infty\|_V^p + \|\mathbf{p}(t) - \mathbf{p}_\infty\|_V^p \right) - \frac{\varepsilon^p}{p} \|u(t) - u_\infty\|_V^p - \end{aligned}$$

$$(25) \quad -\frac{\varepsilon^p}{p} \|\mathbf{p}(t) - \mathbf{p}_\infty\|_V^p - \frac{1}{q\varepsilon^q} \|[(A(\omega, u_\infty, \mathbf{p}_\infty))(t) - A_\infty(\omega^*, u_\infty, \mathbf{p}_\infty)]\|_{V^*}^q - \frac{1}{q\varepsilon^q} \|[B(\omega, u_\infty, \mathbf{p}_\infty)](t) - B_\infty(\omega^*, u_\infty, \mathbf{p}_\infty)\|_{V^*}^q$$

with $\varepsilon > 0$. On the right hand side of (24) by Young's inequality we obtain

$$(26) \quad \begin{aligned} & |\langle [G(\omega)](t) - G_\infty(\omega^*), u(t) - u_\infty \rangle + \langle [F(\omega, u)](t), \mathbf{p}(t) - \mathbf{p}_\infty \rangle| \leq \\ & \leq \frac{\varepsilon^p}{p} \|u(t) - u_\infty\|_V^p + \frac{\varepsilon^p}{p} \|\mathbf{p}(t) - \mathbf{p}_\infty\|_V^p + \\ & + \frac{1}{q\varepsilon^q} \|[(G(\omega))(t) - G_\infty(\omega^*)]\|_{V^*}^q + \frac{1}{q\varepsilon^q} \|[F(\omega, u)](t)\|_{V^*}^q \end{aligned}$$

where the last two terms (as we have verified earlier) tend to 0 as $t \rightarrow \infty$. We show that last two terms of the right hand side on (25) converges to 0 as $t \rightarrow \infty$. Clearly,

$$\begin{aligned} & \|[(A(\omega, u_\infty, \mathbf{p}_\infty))(t) - A_\infty(\omega^*, u_\infty, \mathbf{p}_\infty)]\|_{V^*}^q \leq \\ & \leq \sum_{i=0}^n \int_{\Omega} |a_i(t, \cdot, \omega(t, \cdot), u_\infty, Du_\infty, \mathbf{p}_\infty, D\mathbf{p}_\infty) - \\ & - a_{i,\infty}(\omega^*, u_\infty, Du_\infty, \mathbf{p}_\infty, D\mathbf{p}_\infty)|^q. \end{aligned}$$

The integrand on the right hand side is a.e. convergent in Ω as $t \rightarrow \infty$ by condition A5 and since $\omega(t, x) \rightarrow \omega^*(x)$ for a.a. $x \in \Omega$. Further, by conditions A2, A5,

$$\begin{aligned} & |a_i(t, \cdot, \omega(t, \cdot), u_\infty, Du_\infty, \mathbf{p}_\infty, D\mathbf{p}_\infty) - a_{i,\infty}(\omega^*, u_\infty, Du_\infty, \mathbf{p}_\infty, D\mathbf{p}_\infty)|^q \leq \\ & \leq \text{const} \cdot (\|c_1(\omega)\|_{L^\infty(Q_\infty)} + \|c_1(\omega^*)\|_{L^\infty(Q_\infty)}) \cdot \\ & \cdot (|u_\infty|^p + |Du_\infty|^p + |\mathbf{p}_\infty|^p + |D\mathbf{p}_\infty|^p + \|k_1\|_{L^q(\Omega)}) \end{aligned}$$

where the right hand side is integrable in $L^1(\Omega)$, so by Lebesgue's theorem we obtain $\|[(A(\omega, u_\infty, \mathbf{p}_\infty))(t) - A_\infty(\omega, u_\infty, \mathbf{p}_\infty)]\|_{V^*}^q \rightarrow 0$ as $t \rightarrow \infty$. The convergence of the last term in (25) can be proved similarly.

Now, by choosing sufficiently small ε in (24) and by using (25), (26) and the above convergences we obtain

$$(27) \quad y'(t) + \text{const} \cdot \|u(t) - u_\infty\|_V^p + \text{const} \cdot \|\mathbf{p}(t) - \mathbf{p}_\infty\|_V^p \leq \varphi(t)$$

where $\varphi(t) \rightarrow 0$ as $t \rightarrow \infty$ and the constants are positive. By the continuous embedding $L^p(\Omega) \hookrightarrow L^2(\Omega)$

$$y'(t) + \text{const} \cdot y(t)^{\frac{p}{2}} + \text{const} \cdot \|\mathbf{p}(t) - \mathbf{p}_\infty\|_V^p \leq \varphi(t).$$

It is not difficult to show that this inequality implies $\lim_{t \rightarrow \infty} y(t) = 0$ (see the proof of Theorem 2 in [13]).

By integrating (27) over $(T - 1, T + 1)$ we obtain

$$\begin{aligned} & y(T+1) - y(T-1) + \text{const} \cdot \int_{T-1}^{T+1} \|u(t) - u_\infty\|_V^p dt + \\ & + \text{const} \int_{T-1}^{T+1} \|\mathbf{p}(t) - \mathbf{p}_\infty\|_V^p dt \leq \int_{T-1}^{T+1} \varphi(t) dt. \end{aligned}$$

Clearly the right hand side tends to 0, and by the convergence of $y(t)$ the first difference on the left hand side tends to 0, too, which yields the desired convergences. The proof of stabilization is complete. ■

6. Examples

Now we give some examples of functions which fulfil conditions of the previous theorems. Consider the following functions

$$\begin{aligned} (28) \quad a_i(t, x, \xi, \zeta_0, \xi, \eta_0, \eta) &= \\ &= \mathcal{P}(t, x) P(\xi) Q(\eta_0, \eta) \xi_i |\xi|^{p_1-2} + \tilde{\mathcal{P}}(t, x) \tilde{P}(\xi) \tilde{Q}(\eta_0, \eta) \xi_i |\xi|^{r_1-1}, \\ & \text{for } i = 1, \dots, n, \end{aligned}$$

$$\begin{aligned} (29) \quad a_0(t, x, \xi, \zeta_0, \xi, \eta_0, \eta) &= \\ &= \mathcal{P}(t, x) P(\xi) Q(\eta_0, \eta) \xi_0 |\xi_0|^{p_1-2} + \tilde{\mathcal{P}}_0(t, x) \tilde{P}_0(\xi) \tilde{Q}(\eta_0, \eta) \xi_0 |\xi_0|^{r_1-1}, \end{aligned}$$

$$\begin{aligned} (30) \quad b_i(t, x, \xi, \zeta_0, \eta_0, \eta) &= \\ &= \mathcal{R}(t, x) R(\xi) S(\zeta_0) \eta_i |\eta|^{p_2-2} + \tilde{\mathcal{R}}(t, x) \tilde{R}(\xi) \tilde{S}(\zeta_0) \eta_i |\eta|^{r_2-1}, \\ & \text{for } i = 1, \dots, n, \end{aligned}$$

$$\begin{aligned} (31) \quad b_0(t, x, \xi, \zeta_0, \eta_0, \eta) &= \\ &= \mathcal{R}(t, x) R(\xi) S(\zeta_0) \eta_0 |\eta|^{p_2-2} + \tilde{\mathcal{R}}(t, x) \tilde{R}(\xi) \tilde{S}(\zeta_0) \eta_0 |\eta_0|^{r_2-1}, \end{aligned}$$

where

E1 a) $\mathcal{P}, \tilde{\mathcal{P}}, \tilde{\mathcal{P}}_0 \in L^\infty(Q_\infty)$, $P, \tilde{P}, \tilde{P}_0 \in C(\mathbb{R})$, $Q, \tilde{Q} \in C(\mathbb{R}^{n+1})$. Further, there exists constants $\alpha, \beta > 0$ such that $\mathcal{P}(t, x) P(\xi) Q(\eta_0, \eta) \geq \alpha$,

$$|Q(\eta_0, \eta)| \leq \beta, \text{ and } \tilde{Q}(\eta_0, \eta) \leq \beta \cdot \left(|(\eta_0, \eta)|^{\frac{p_2(p_1-1-r_1)}{p_1}} + 1 \right),$$

$\tilde{P}(t, x)\tilde{P}(\xi)\tilde{Q}(\eta_0, \eta) \geq 0$ for a.a. $(t, x) \in Q_\infty$ and every $(\xi, \eta_0, \eta) \in \mathbb{R} \times \mathbb{R}^{n+1}$ where $0 \leq r_1 < p_1 - 1$.

- b) Functions $\mathcal{R}, \tilde{\mathcal{R}} \in L^\infty(Q_\infty)$, $R, \tilde{R}, S, \tilde{S} \in C(\mathbb{R})$. Further, there exists constants $\alpha, \beta > 0$ such that $\mathcal{R}(t, x)R(\xi)S(\zeta_0) \geq \alpha$, $|S(\zeta_0)| \leq \beta$, and $\tilde{S}(\zeta_0) \leq \beta \cdot \left(|\zeta_0|^{\frac{p_1(p_2-1-r_2)}{p_2}} + 1 \right)$, $\tilde{\mathcal{R}}(t, x)\tilde{R}(\xi)\tilde{S}(\zeta_0) \geq 0$ for a.a. $(t, x) \in Q_\infty$ and every $(\xi, \zeta_0) \in \mathbb{R} \times \mathbb{R}$ where $0 \leq r_2 < p_2 - 1$.

Assuming E1/a, E1/b one can easily verify that functions (28)–(30) fulfil conditions A1–A4, B1–B4 (see, e.g., [4]). Further, if we make a minor

modification in condition E1/b, namely let $\tilde{S}(\zeta_0) \leq \beta \left(|\zeta_0|^{\frac{2(p_2-1-r_2)}{p_2}} + 1 \right)$

for every $\zeta_0 \in \mathbb{R}^{n+1}$, then the above functions satisfy conditions A1–A4, B1–B3, B4*.

Now consider the following functions for $i = 0, \dots, n$

$$(32) \quad a_i(t, x, \xi, \zeta_0, \xi, \eta_0, \eta) = \mathcal{P}(t, x)P(\xi)\xi_i|(\zeta_0, \xi, \eta_0, \eta)|^{p-2},$$

$$(33) \quad b_i(t, x, \xi, \zeta_0, \eta_0, \eta) = \mathcal{R}(t, x)R(\xi)\eta_i|(\zeta_0, \eta_0, \eta)|^{p-2}$$

where

- E2. Functions $\mathcal{P}, \mathcal{R} \in L^\infty(Q_\infty)$, $P, R \in C(\mathbb{R})$ and there exists a positive constant α such that $\mathcal{P}(t, x)P(\xi) \geq \alpha$, $\mathcal{R}(t, x)R(\xi) \geq \alpha$ for a.a. $(t, x) \in Q_\infty$ and every $\xi \in \mathbb{R}$. Further, there exist functions $\mathcal{P}_*, \mathcal{R}_* \in L^\infty(\Omega)$ such that $\lim_{t \rightarrow \infty} \mathcal{P}(t, x) = \mathcal{P}_*(x)$ and $\lim_{t \rightarrow \infty} \mathcal{R}(t, x) = \mathcal{R}_*(x)$.

THEOREM 5. Suppose $2 \leq p \leq 4$ and E2, then the above (32)–(33) functions satisfy conditions A1–A5, B1–B3, B4*, B5, AB with $p_1 = p_2 = p$.

PROOF. It is easy to see that all the conditions instead of AB are satisfied. For the proof of AB see [5]. ■

If we consider functions

$$a_i(t, x, \xi, \zeta_0, \xi, \eta_0, \eta) = \xi_i|(\zeta_0, \xi)|^{p-2} + \mathcal{P}(t, x)P(\xi)\xi_i|(\zeta_0, \xi, \eta_0, \eta)|^{r-2},$$

$$b_i(t, x, \xi, \zeta_0, \eta_0, \eta) = \eta_i|(\eta_0, \eta)|^{p-2} + \mathcal{R}(t, x)R(\xi)\eta_i|(\zeta_0, \eta_0, \eta)|^{r-2}$$

where $1 \leq r \leq 4$ and E2 hold then it is easy to see that these functions satisfy conditions A1–A5, B1–B3, B4*, B5, AB with $p_1 = p_2 = p \geq \max\{2, r\}$.

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“CONTINUITY SPACE” = QUASI-UNIFORM SPACE

By

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ABSTRACT. In Amer. Math. Monthly **95** (1988), pp. 89–97 (and in subsequent – as well as previous – publications) Ralph Kopperman has presented a concept he calls “continuity space”. In these structures one can define notions of limit, neighborhood and (even uniform) continuity. By establishing the equivalence of this concept with that of quasi-uniform space, one makes the constructs of either theory available to the other and so can transfer results from one to the other – e.g., that every topology can be induced from a continuity structure is seen to be Császár’s (1960) theorem that every topology is quasi-uniformizable.

A quasi-uniformity (on set X) is a filter on $X \times X$, refined by its filter of relative composites, all of whose elements contain the diagonal.

A typical example is a (strictly speaking “pseudo-”, although this term will be suppressed) quasi-metric, which is a (real) non-negative valued function d on $X \times X$, zero on the diagonal and satisfying the triangle inequality – the filter is that of the inverse images by d of initial intervals. These are the most general quasi-uniformities with countable base, as follows from Kelley’s 6.12 Metrization Lemma. As a consequence, every quasi-uniformity is the supremum of those induced by its quasi-uniformly continuous quasi-metrics.

To formulate the triangle inequality one requires an addition and an order. Accordingly, postulate a p.o. commutative semigroup (i.e. addition order preserving) with an identity which is the smallest element 0; the set P of

non-zero elements, closed for addition, will be taken down-directed (hence a filterbase) and refined by the filterbase of pairwise sums. Then a “continuity space” is (after stripping away some of Kopperman’s superfluous baggage) a set equipped with an abstract quasi-metric with values in such a structure.

By taking (as was done in the real-valued case) the inverse images of the “intervals” $[0, p]$, one obtains a filterbase for a quasi-uniformity, whose derived convergence constructs coincide with those derived from the abstract quasi-metric. To show, conversely, that every quasi-uniform space can be construed as a continuity space, represent it as a subspace of a product of quasi-metrics via the above representation as a quasi-metric supremum.

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A THEOREM ON INTEGRABILITY OF TRIGONOMETRIC SERIES

By

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*(Received March 19, 2007)***1. Introduction**

The basic results on integrability theorems for trigonometric series deal with monotone or positive coefficients and functions. Their generalizations weakened the monotonicity conditions gradually. Now we present a further generalization.

Recently, in [3], we already proved the following result. For notions and notations, please, see Section 2.

THEOREM A. *Let $\lambda \in \gamma GB VS$,*

$$(1.1) \quad g(x) := \sum_{n=1}^{\infty} b_n \sin nx$$

and

$$(1.2) \quad f(x) := \sum_{n=1}^{\infty} a_n \cos nx.$$

(i) If $0 < \alpha \leq 1$ and

$$(1.3) \quad \sum_{n=1}^{\infty} n^{\alpha-1} \gamma_n < \infty,$$

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then $x^{-\alpha}g(x) \in L := L(0, \pi)$.

(ii) If $0 < \alpha < 1$ and (1.3) holds, then $x^{-\alpha}f(x) \in L$.

In this note we will replace the function $x^{-\alpha}$ by a more general function $\psi(x)$ which belongs to one of the classes defined by Yung-Ming Chen [2] (see also H. P. Mulholland [4]).

2. Notions and notations

We preserve the following notations throughout.

1. By φ we mean either an f or a g ; the coefficients associated with φ are denoted by $\lambda := \{\lambda_n\}$.

2. A null-sequence $c := \{c_n\}$ ($c_n \rightarrow 0$) belongs to the class $\gamma GB VS$ if

$$\sum_{n=m}^{2m} |\Delta c_n| \leq K(c)\gamma_m, \quad (\Delta c_n := c_n - c_{n+1}), \quad m = 1, 2, \dots$$

holds, where $K(c)$ is a positive constant and $\gamma := \{\gamma_n\}$ is a given sequence of nonnegative numbers.

3. A not identically zero nonnegative function $\psi(x)$ defined on $(0, \infty)$ belongs to the set Ψ if for some $\varepsilon > 0$ $x^{1-\varepsilon}\psi(x)$ is nondecreasing and $x^\varepsilon\psi(x)$ is nonincreasing as x increases.

4. We shall use the following notation: $L \ll R$ if there exists a positive constant K such that $L \ll KR$, but not necessarily the same K at each occurrence.

3. New theorem

Our result reads as follows.

THEOREM. *Let $\psi(x) \in \Psi$ and $\lambda \in \gamma GB VS$. If*

$$(3.1) \quad \sum_{n=1}^{\infty} \psi\left(\frac{1}{n}\right) \frac{1}{n} \gamma_n < \infty,$$

then $\psi(x)\varphi(x) \in L$.

4. Lemma

We need the following lemma, which was proved in [3] implicitly.

LEMMA. *If $\lambda := \{\lambda_n\} \in \gamma GB VS$, then*

$$\sum_{k=n}^{\infty} |\Delta \lambda_k| \ll \gamma_n + \sum_{k=\frac{n}{2}}^{\infty} k^{-1} \gamma_k, \quad n = 1, 2, \dots$$

5. Proof of Theorem

Our proof combines the methods of R. P. Boas, Jr. [1] and Yung-Ming Chen [2] with ours [3].

First we consider the cosine series, that is, if $\lambda_n = a_n$ and $\varphi(x) = f(x)$. Then we have

$$(5.1) \quad \begin{aligned} \frac{1}{2}f(x) &= \frac{1}{\sin x} \sum_{n=0}^{\infty} a_{n+1} [\sin(n+2) - \sin nx] \\ &= \frac{1}{\sin x} \left[-a_2 \sin x + a_1 \sin 2x + \sum_{n=3}^{\infty} (a_{n-1} - a_{n+1}) \sin nx \right]. \end{aligned}$$

Since $\psi(x) \in \Psi$, thus for the functions

$$Q_n(x) := \psi\left(\frac{x}{n}\right) \psi\left(\frac{1}{n}\right)^{-1}, \quad n = 1, 2, \dots$$

we obtain, by an easy consideration, that

$$(5.2) \quad x^{-\varepsilon} \leq Q_n(x) \leq x^{\varepsilon-1} \quad (0 < x < 1), \quad x^{\varepsilon-1} \leq Q_n(x) \leq x^{-\varepsilon} \quad (x > 1).$$

Utilizing (5.2) we can estimate the following integral:

$$\begin{aligned}
 (5.3) \quad & \int_0^\pi \psi(x)x^{-1}|\sin nx|dx = \psi\left(\frac{1}{n}\right) \int_0^\pi Q_n(nx)x^{-1}|\sin nx|dx = \\
 & = \psi\left(\frac{1}{n}\right) \int_0^{n\pi} Q_n(x)x^{-1}|\sin x|dx = \\
 & = \psi\left(\frac{1}{n}\right) \left(\int_0^1 + \int_1^{n\pi} \right) Q_n(x)x^{-1}|\sin x|dx \ll \\
 & \ll \psi\left(\frac{1}{n}\right) \left\{ \int_0^1 x^{\varepsilon-1}dx + \int_1^\infty x^{-1-\varepsilon}dx \right\} \ll \\
 & \ll \psi\left(\frac{1}{n}\right).
 \end{aligned}$$

Since $\psi(x) = Q_1(x)\psi(1)$, thus, by (5.2),

$$(5.4) \quad \int_0^\pi \psi(x)dx \ll 1$$

also holds.

Consequently, (5.1), (5.3) and (5.4) imply that

$$(5.5) \quad \int_0^\pi \psi(x)|f(x)|dx \ll 1 + \sum_{n=1}^\infty |\Delta a_n| \psi\left(\frac{1}{n}\right).$$

Thus, in order to prove that $\psi(x)f(x)$ is integrable, by (3.1), we have only to verify that

$$(5.6) \quad \sum_{n=1}^\infty |\Delta a_n| \psi\left(\frac{1}{n}\right) \ll \sum_{n=1}^\infty \psi\left(\frac{1}{n}\right) \frac{1}{n} \gamma_n.$$

First we show that

$$(5.7) \quad \psi\left(\frac{1}{n}\right) \ll \sum_{k=1}^n \psi\left(\frac{1}{k}\right) \frac{1}{k} \ll \psi\left(\frac{1}{n}\right).$$

Since $\left\{ \left(\frac{1}{k} \right)^\varepsilon \psi \left(\frac{1}{k} \right) \right\}$ is a nondecreasing sequence, thus

$$\begin{aligned} \sum_{k=1}^n \psi \left(\frac{1}{k} \right) \frac{1}{k} &= \sum_{k=1}^n \left(\frac{1}{k} \right)^\varepsilon \psi \left(\frac{1}{k} \right) k^{\varepsilon-1} \leq \\ &\leq \left(\frac{1}{n} \right)^\varepsilon \psi \left(\frac{1}{n} \right) \sum_{k=1}^n k^{\varepsilon-1} \ll \\ &\ll \psi \left(\frac{1}{n} \right). \end{aligned}$$

On the other hand, since $\left\{ \left(\frac{1}{k} \right)^{1-\varepsilon} \psi \left(\frac{1}{k} \right) \right\}$ is a nonincreasing sequence, we obtain that

$$\sum_{k=1}^n \psi \left(\frac{1}{k} \right) \frac{1}{k} \geq \sum_{k=\frac{n}{2}}^n \psi \left(\frac{1}{k} \right) \frac{1}{k} \gg \psi \left(\frac{1}{n} \right),$$

herewith (5.7) is proved.

Using the first inequality of (5.7) and Abel's transformation we get that

$$\begin{aligned} (5.8) \quad \sum_{n=1}^{\infty} |\Delta a_n| \psi \left(\frac{1}{n} \right) &\ll \sum_{n=1}^{\infty} |\Delta a_n| \sum_{k=1}^n \psi \left(\frac{1}{k} \right) \frac{1}{k} \\ &= \sum_{k=1}^{\infty} \psi \left(\frac{1}{k} \right) \frac{1}{k} \sum_{n=k}^{\infty} |\Delta a_n|. \end{aligned}$$

Hereafter we can utilize our Lemma, later an Abel's transformation and (3.1) deliver the following inequalities:

$$\begin{aligned} (5.9) \quad \sum_{n=2}^{\infty} \psi \left(\frac{1}{n} \right) \frac{1}{n} \sum_{k=n}^{\infty} |\Delta a_k| &\ll \sum_{n=2}^{\infty} \psi \left(\frac{1}{n} \right) \frac{1}{n} \left(\gamma_n + \sum_{k=\frac{n}{2}}^{\infty} k^{-1} \gamma_k \right) \\ &\ll 1 + \sum_{k=1}^{\infty} k^{-1} \gamma_k \sum_{n=1}^{2k} \psi \left(\frac{1}{n} \right) \frac{1}{n} \\ &\ll \sum_{k=1}^{\infty} k^{-1} \gamma_k \psi \left(\frac{1}{k} \right) < \infty. \end{aligned}$$

Summing up, (3.1), (5.5), (5.6), (5.8) and (5.9) prove that $\psi(x)f(x) \in L$.

Next we verify that $\psi(x)g(x) \leq L$ also holds.

If $\varphi(x) = g(x)$ and $\lambda_n = b_n$, then instead of (5.1) we shall start with the following equality, using the function $c(x) := 1 - \cos x$:

$$(5.10) \quad \begin{aligned} \frac{1}{2}g(x) &= \frac{1}{\sin x} \sum_{n=0}^{\infty} b_{n+1} [c((n+2)x) - c(nx)] \\ &= \frac{1}{\sin x} \left[-b_2 c(x) + b_1 c(2x) + \sum_{n=3}^{\infty} (b_{n-1} - b_{n+1}) c(nx) \right]. \end{aligned}$$

By (5.4) the functions $\psi(x)c(jx)/\sin x$ ($j = 1, 2, \dots$) are integrable, therefore it is enough to show the existence of the following integral:

$$(5.11) \quad \begin{aligned} \int_0^{\pi} \psi(x)x^{-1} \sum_{n=3}^{\infty} |b_{n-1} - b_{n+1}| c(nx) dx &= \\ &= \sum_{n=3}^{\infty} |b_{n-1} - b_{n+1}| \int_0^{\pi} \psi(x)x^{-1} c(nx) dx. \end{aligned}$$

As in (5.3), using again (5.2), we obtain that

$$\begin{aligned} \int_0^{\pi} \psi(x)x^{-1} c(nx) dx &= \psi\left(\frac{1}{n}\right) \int_0^{\pi} Q_n(nx)x^{-1} c(nx) dx = \\ &= \psi\left(\frac{1}{n}\right) \int_0^{n\pi} Q_n(x)x^{-1} c(x) dx \\ &\ll \psi\left(\frac{1}{n}\right) \left\{ \int_0^1 x^{\varepsilon-1} dx + \int_1^{\infty} x^{-1-\varepsilon} dx \right\} \ll \\ &\ll \psi\left(\frac{1}{n}\right). \end{aligned}$$

Thus, by (5.10) and (5.11), we get

$$\int_0^{\pi} \psi(x)|g(x)| dx \ll 1 + \sum_{n=1}^{\infty} |\Delta b_n| \psi\left(\frac{1}{n}\right).$$

Hereafter the proof is similar to that of the cosine series, therefore we omit it.

The proof of our theorem is complete. ■

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