# ANNALES Universitatis Scientiarum BUDAPESTINENSIS de Rolando EÖTvÖs nominatae 

## SECTIO MATHEMATICA

TOMUS XLVIII.

REDIGIT<br>Á. CSÁSZÁR

ADIUVANTIBUS
L. BABAI, A. BENCZÚR, K. BEZDEK., M. BOGNÁR, K. BÖRÖCZKY, I. CSISZÁR, J. DEMETROVICS, GY. ELEKES, A. FRANK, J. FRITZ, E. FRIED, A. HAJNAL, G. HALÁSZ, A. IVÁNYI, A. JÁrAI, P. KACSUK, I. KÁTAI, E. KISS, P. KOMJÁTH, M. LACZKOVICH, L. LOVÁSZ, GY. MICHALETZKY,
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# ANNALES 

Universitatis Scientiarum Budapestinensis de Rolando EÖtvÖs nominatae

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SECTIO BIOLOGICA
    incepit anno MCMLVII
SECTIO CHIMICA
    incepit anno MCMLIX
SECTIO CLASSICA
    incepit anno MCMXXIV
SECTIO COMPUTATORICA
    incepit anno MCMLXXVIII
SECTIO GEOGRAPHICA
    incepit anno MCMLXVI
SECTIO GEOLOGICA
    incepit anno MCMLVII
SECTIO GEOPHYSICA ET METEOROLOGICA
    incepit anno MCMLXXV
SECTIO HISTORICA
        incepit anno MCMLVII
SECTIO IURIDICA
        incepit anno MCMLIX
SECTIO LINGUISTICA
        incepit anno MCMLXX
SECTIO MATHEMATICA
        incepit anno MCMLVIII
SECTIO PAEDAGOGICAET PSYCHOLOGICA
    incepit anno MCMLXX
SECTIO PHILOLOGICA
        incepit anno MCMLVII
SECTIO PHILOLOGICA HUNGARICA
    incepit anno MCMLXX
SECTIO PHILOLOGICA MODERNA
    incepit anno MCMLXX
SECTIO PHILOSOPHICAET SOCIOLOGICA
    incepit anno MCMLXII
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# ON THE MINIMAL GAPS BETWEEN PRODUCTS OF MEMBERS OF A SEQUENCE OF POSITIVE DENSITY 

By<br>CSABA SÁNDOR

(Received June 17, 2003)

## 1. Introduction

Throughout this paper we use the following notations: let $\mathbb{N}$ be the set of natural numbers. Let $c_{1}, c_{2}, \ldots$ denote positive real numbers. The cardinality of the finite set $S$ is denoted by $|S|$. If $\mathscr{A}$ is a given subset of $\mathbb{N}$ then we write $A(n)=|\mathscr{A} \cap\{1,2, \ldots, n\}|$. We call $\bar{d}\left(\mathscr{A}=\limsup _{n \rightarrow \infty} \frac{A(n)}{n}\right.$ and $\underline{d}(\mathscr{A})=\liminf _{n \rightarrow \infty} \frac{A(n)}{n}$ the upper asymptotic density and lower asymptotic density.

Let $\mathcal{A}$ be a given infinite subset of $\mathbb{N}$ with $\bar{d}(A)=\alpha>0$. Denote the set of products of the form $a_{i_{1}} a_{i_{2}} \ldots a_{i_{k}}$ with $a_{i j} \in A$ by $\mathscr{B}_{\mathscr{A}}^{(k)}=\left\{b_{1}<b_{2}<\ldots\right\}$. A. SARKÖZY asked the following (see [2]):

Is it true that for all $a>0$ there is a number $c=c(\alpha)$ depending only on $\alpha$ such that if $\mathscr{A} \subset \mathbb{N}$ is an infinite sequence whose lower asymptotic density $\underline{d}(\mathbb{A})$ is $>\alpha$, then $b_{n+1}^{(2)}-b_{n}^{(2)}<c$ holds for infinitely many $n$ ? If the answer is affirmative, does this hold with $c(\alpha) \leq \frac{1}{\alpha^{2}}$ ? (If $k>1$ is a natural number and $a_{n}=n k$, so $\alpha=\frac{1}{k}$ then $b_{n+1}^{(2)}-b_{n}^{(2)} \geq k^{2}$ for every $n$ so this upper bound would be sharp.)
G. BÉRCZI (see [1]) proved the existence of $c(\alpha)$ by showing that $c(\alpha) \leq$ $\leq \frac{c_{1}}{\alpha^{4}}$. We prove that $c(\alpha) \leq \frac{c_{2}}{\alpha^{3}}$.

[^0]THEOREM 1. Let $\mathscr{A}$ be a subset of natural numbers with $\bar{d}(\mathscr{A})=\alpha>0$. Let us denote the sequence of the numbers of the form $a_{i} a_{j}$ (with $a_{i}, a_{j} \in \mathcal{A}$ by $\mathscr{B}_{\mathscr{A}}=\left\{b_{1}, b_{2}, \ldots\right\},\left(b_{1}<b_{2}<\ldots\right)$. Then for infinitely many $n$ we have $b_{n+1}-b_{n}<\frac{c_{3}}{\alpha^{3}}$ with some absolute constant $c_{3}$.
G. BÉRCZI asked the similar question about $\mathscr{B}_{\mathscr{A}}^{(k)}$ :

Let $\mathcal{A}$ be an infinite sequence of natural numbers with $\underline{d}(\mathscr{A})>\alpha>0$ and $k \geq 2$ a natural number. It is a natural question what can we say about the distribution of the products $a_{i_{1}} a_{i_{2}} \ldots a_{i_{k}}$ where $a_{i_{j}} \in A$. Let $\mathscr{B}_{\mathbb{A}}^{(k)}$ be the set of these products as above. Is it true that there are infinitely many indices $n$ with $b_{n+1}^{(k)}-b_{n}^{(k)}<c(\alpha, k)$ where this constant depends only on $\alpha, k$ ? We have seen the case $k=2$ but unfortunately $I$ cannot say anything about the other values of $k$, but the conjecture is surely true for every $k$.

We will prove that this conjecture is true for small enough $k$.
THEOREM 2. For $k=2,3 \ldots 10,12$ and arbitrary $0<\alpha \leq 1$ there exist a $c(k, \alpha)$ such that for $\mathscr{A} \subset \mathbb{N}$ with $\bar{d}(\mathscr{A})=\alpha$ we have $b_{n+1}^{(k)}-b_{n}^{(k)}<c(\alpha, k)$ for infinitely many indices $n$.

## 2. Proofs

To prove Theorem 1 we will use a simple lemma about a subset of natural integers with positive upper density.

Lemma 1. Let $S \subset \mathbb{N}$ with $\bar{d}(S) \geq \alpha>0$. Then there exist $c_{4}$ and $c_{5}$ such that for some $0<l<\frac{c_{4}}{\alpha}$ we have $\bar{d}(S \cap(S+l))>\frac{c_{5}}{\alpha^{2}}$.

Proof of Lemma 1. It is sufficient to prove that for $S \subset \mathbb{N}$ with $\bar{d}(S) \geq \frac{1}{k}$ there exists $0<l<2 k$ for which $\bar{d}(S \cap(S+l)) \geq \frac{1}{2 k^{2}}$.

For $\bar{d}(S) \geq \frac{1}{k}$ there exists an infinite sequence $s_{1}<s_{2}<s_{2}<\ldots$ such that $S\left(2 k s_{t}\right)>\left(1-\frac{1}{8 k}\right) 2 s_{t}$. Then we have

$$
\left|\left\{(i, j): 0<a_{j}-a_{i}<2 k, a_{j}<2 k s_{t}\right\}\right| \geq \sum_{i-1}^{s_{t}}\binom{S(i 2 k)-S((i-1) 2 k)}{2}=
$$

$$
\begin{aligned}
& \quad=\frac{1}{2}\left(\sum_{i=1}^{s_{t}}\left(S(i 2 k)-S((i-1) s k)^{2}-\sum_{i=1}^{s_{t}}(S(i 2 k)-S((i-1) 2 k))\right) \geq\right. \\
& \geq \frac{1}{2}\left(\frac{\left(\sum_{i=1}^{s_{t}}(S(i 2 k)-S((i-1) 2 k))\right)^{2}}{s_{t}}-\sum_{i=1}^{s_{t}}(S(i 2 k)-S((i-1) 2 k)) \geq\right. \\
& \geq \frac{1}{2}\left(\frac{S\left(2 k s_{t}\right)^{2}}{s_{t}}-S\left(2 k s_{t}\right)\right) \geq \frac{1}{2}\left(\frac{\left(\left(1-\frac{1}{8 k}\right) 2 s_{t}\right)^{2}}{s_{t}}-\left(1-\frac{1}{8 k}\right) 2 s_{t}\right) \geq \\
& \geq\left(1-\frac{1}{2 k}\right) s_{t}
\end{aligned}
$$

Thus by the pigeon hole principle there exists $0<l<2 k$ for which

$$
\#\left\{(i, j): a_{j}-a_{i}=l, a_{j}<2 k s_{t}\right\} \geq \frac{\left(1-\frac{1}{2 k}\right) s_{t}}{2 k-1}=\frac{s_{t}}{2 k}
$$

which completes the proof.
Proof of Theorem 1. By Lemma 1 we have a $0<l \leq \frac{c_{4}}{\alpha}$ for which $\bar{d}(\mathscr{A} \cap(\mathscr{A}+l))>\frac{c_{5}}{\alpha^{2}}$. Let $\mathscr{C}=\mathscr{A} \cap(\mathscr{A}+l)$. Then $\bar{d}(\mathscr{C})>\frac{c_{5}}{\alpha^{2}}$. Using the lemma again we have a $0<m<\frac{c_{6}}{\alpha^{2}}$ such that $\bar{d}(\mathscr{C} \cap(\mathscr{C}+m))>\frac{c_{7}}{\alpha^{4}}$. So we get integers $0<l<\frac{c_{4}}{\alpha}, 0<m<\frac{c_{6}}{\alpha^{2}}$ such that

$$
\bar{d}(\mathscr{A} \cap(\mathscr{A}+l) \cap(\mathscr{A}+m) \cap(\mathscr{A}+l+m))>\frac{c_{7}}{\alpha^{4}}
$$

but for $a \in \mathscr{A} \cap(\mathscr{A}+l) \cap(\mathscr{A}+m)$ we have
and

$$
a, a-l, a-m, a-m-l \in \mathscr{A}
$$

$$
(a-l)(a-m)-a(a-m-l)=l m \leq \frac{c_{8}}{\alpha^{3}}
$$

which completes the proof.
Proof of Theorem 2. For $k=2,3, \ldots, 10,12$ there are integers $\left\{d_{1}, d_{2}\right.$, $\left.\ldots, d_{k}\right\},\left\{e_{1}, e_{2}, \ldots, e_{k}\right\}, f_{k}$ such that

$$
\left(x+d_{1}\right)\left(x+d_{2}\right) \ldots\left(x+d_{k}\right)-\left(x+e_{1}\right)\left(x+e_{2}\right) \ldots\left(x+e_{k}\right)=f_{k} \neq 0
$$

since this statement is equivalent to the solvability of the diophantine system

$$
d_{1}^{l}+d_{2}^{l}+\ldots+d_{k}^{l}=e_{1}^{l}+e_{2}^{l}+\ldots+e_{k}^{l} \text { for } l=1,2, \ldots, k-1
$$

(so called Prouhet-Tarry-Escott problem). This problem has nontrivial solution for $k=2,3, \ldots, 10,12$ (see [3]):

$$
\begin{aligned}
k=2: & \{0,2\},\{1,1\} \\
k=3: & \{1,2,6\},\{0,4,5\} \\
k=4: & \{0,4,7,11\},\{1,2,9,10\} \\
k=5: & \{1,2,10,14,18\},\{0,4,8,16,17\} \\
k=6: & \{0,4,9,17,22,26\},\{1,2,12,14,24,25\} \\
k=7: & \{-51,-33,-24,7,13,38,50\},\{51,33,24,-7,-13,-38,-50\} \\
k=8: & \{0,4,9,23,27,41,46,50\},\{1,2,11,20,30,39,48,49\} \\
k=9: & \{1,25,31,84,87,134,158,182,198\}, \\
& \{2,18,42,66,113,116,169,175,199\} \\
k=10: & \{-313,-301,-188,-100,-99,99,100,188,301,313\}, \\
& \{-308,-307,-180,-131,-71,71,131,180,307,308\} \\
k=12: & \{0,11,24,65,90,129,173,173,212,237,278,291,302\}, \\
& \{3,5,30,57,104,116,186,198,245,272,297,299\}
\end{aligned}
$$

Let $m_{k}=\min \left\{d_{1}, \ldots, d_{k}, e_{1}, \ldots, e_{k}\right\}, M_{k}=\max \left\{d_{1}, \ldots, d_{k}, e_{1}, \ldots, e_{k}\right\}$. By Szemerédi's theorem the condition $\bar{d}(\mathscr{A})=\alpha \geq 0$ implies that there is a positive integer $d<d(\alpha, k)$ such that for infinitely many positive integers $g_{n}$ we have $g_{n}+j d \in \mathscr{A}$ for $m_{k} \leq j \leq M_{k}$. Hence $g_{n}+d_{i} d$, $g_{n}+e_{i} d \in \mathscr{A}$ for $1 \leq i \leq k$ and

$$
\begin{aligned}
\left(g_{n}+d_{1} d\right)\left(g_{n}+d_{2} d\right) \ldots\left(g_{n}\right. & \left.+d_{k} d\right)-\left(g_{n}+e_{1} d\right)\left(g_{n}+e_{2} d\right) \ldots\left(g_{n}+e_{k} d\right)= \\
& =d^{k} f_{k}<c(\alpha, k),
\end{aligned}
$$

which proves the theorem.

## References

[1] G. BÉRCZI, On the distribution of products of members of a sequence with positive density, Per. Math. Hung., 44 (2002), 137-145.
[2] A. SÁrkÖzy, Unsolved problems in number theory, Per. Math. Hung., 42 (2001), 17-36.
[3] http://member.netease.com/chin/eslp/TarryPrb.htm

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# A SIMPLE TEST FOR INTEGER PROGRAMMING PROBLEMS WITH BOUNDED INTEGER VARIABLES 

By<br>PETER L. HAMMER, PÉTER KAS and BÉLA VIZVÁRI

(Received February 23, 2005)

## 1. Introduction

One of the effective methods of solving integer programming problems with bounded variables is implicit enumeration. The two main tools of this sort of algorithms are the enumeration tree, and the tests. The tests serve to prove that the set of feasible solutions of a branch of the enumeration tree is empty.

Tests are often based on the fact that in some cases it is very easy to prove that a certain polychedron is empty. Even the very first family of tests uses this sort of technique. Let $a_{1}, \ldots, a_{n}$, and $b$ real numbers. Then the set

$$
\left\{\mathbf{x} \in\{0,1\}^{n} \mid a_{1} x_{1}+\cdots+a_{n} x_{n} \leq b\right\}
$$

is empty if

$$
\begin{equation*}
\sum_{j=1}^{n} \min \left\{0, a_{j}\right\}>b \tag{1}
\end{equation*}
$$

as if (1) holds then even the polychedral

$$
\left\{\mathbf{x} \in \mathbb{R}^{n} \mid a_{1} x_{1}+\cdots+a_{n} x_{n} \leq b ; 0 \leq x_{j} \leq 1, j=1, \ldots, n\right\}
$$

is empty.
This paper is devoted to a similar result. All of the variables are nonnegative, bounded and integer. The constraints are equations. The result is a generalization of an earlier result of Hammer et al. [1] where the number of equations is restricted to 2 .

## 2. Manifestly infeasible consequences of system equations

Let $n$, and $k$ be positive integers. A finite $\operatorname{set} \mathscr{\mathscr { L }}$ of $n$-dimensional integer vectors is defined by the following set of constraints:

$$
\begin{align*}
a_{11} x_{1}+a_{12} x_{2}+\cdots+a_{1 n} x_{n} & =a_{10} \\
a_{21} x_{1}+a_{22} x_{2}+\cdots+a_{2 n} x_{n} & =a_{20}  \tag{2}\\
\cdots & \\
a_{k 1} x_{1}+a_{k 2} x_{2}+\cdots+a_{k n} x_{n}= & a_{k 0}
\end{align*}
$$

$$
\begin{equation*}
0 \leq x_{j} \leq u_{j}, \quad x_{j} \text { integer, } \quad j=1, \ldots, n \tag{3}
\end{equation*}
$$

where $a_{i j}(i=1, \ldots, k, j=1, \ldots, n)$ is a real number and $u_{j}(j=1, \ldots, n)$ is a positive integer.

DEFINITION 2.1. The system equations (2) is manifestly infeasible if there are real weights $w_{i}(i=1, \ldots, k)$ such that the equation obtained from (2) by these weights, i.e.

$$
\begin{equation*}
\sum_{j=1}^{n} a_{j} x_{j}=d f \sum_{j=1}^{n}\left(\sum_{i=1}^{k} a_{i j} w_{i}\right) x_{j}=a_{0}=d f \sum_{i=1}^{k} a_{i 0} w_{i} \tag{4}
\end{equation*}
$$

has the following property: either

$$
\begin{equation*}
\sum_{j: a_{j}>0} a_{j} u_{j}<a_{0} \tag{5}
\end{equation*}
$$

or

$$
\begin{equation*}
\sum_{j: a_{j}<0} a_{j} u_{j}>a_{0} \tag{6}
\end{equation*}
$$

Notice that if either (5) or (6) is satisfied then the system (2)-(3) does not have any feasible solution as the polychedral

$$
\left\{\mathbf{x} \in \mathbb{R}^{n} \mid a_{1} x_{1}+\cdots+a_{n} x_{n} \leq b ; 0 \leq x_{j} \leq 1, j=1, \ldots, n\right\}
$$

is empty.
Let $1 \leq l \leq k$ be a fixed index. Assume that there are weights proving the manifestly infeasibility of (2) such that $w_{l} \neq 0$. Then, without loss of generality, one can assume that $w_{l}=1$. If (4) is divided by $\left|w_{l}\right|$ then both (5), and (6) remain satisfied if the inequality was satisfied before the operation. If the new weight of equation $l$ is -1 then after multiplying (4) by -1 condition (5) is transformed into the form of (6) and vice versa.

THEOREM 2.1. There is a finite set $\mathcal{W}_{k} \subset \mathbb{R}^{k}$ such that if system (2) is manifestly infeasible then there is an element $\hat{\mathbf{w}} \in \mathcal{W}_{k}$ such that equation (4) formed by weight vector $\hat{\mathbf{w}}$ proves the manifestly infeasibility of (2).

Proof. Let $\mathbf{w} \in \mathbb{R}^{\mathbf{k}}$ be a weight vector that proves the manifestly infeasibility of (2). First suppose that $w_{k} \neq 0$. Then, without loss of generality, we may assume that $w_{k}=1$. Let $H_{j} \subset \mathbb{R}^{k-1}$ be the hyperplane defined by the equation

$$
\begin{equation*}
a_{1 j} y_{1}+a_{2 j} y_{j}+\cdots+a_{k-1, j} y_{k-1}+a_{k j}=0 \tag{7}
\end{equation*}
$$

i.e.

$$
H_{j}=\left\{\mathbf{y} \in \mathbb{R}^{k-1} \mid \mathbf{y} \text { satisfies (7) }\right\}
$$

The hyperplane $H_{j}$ exists if and only if $\exists i: 1 \leq i \leq k-1$ such that $a_{i j} \neq 0$, or $a_{k j}=0$. Let $\mathcal{g}$ be the set of the indices of the existing $H_{j}$ 's. Assume that $H_{j}$ exists. A point $\mathbf{y} \in \mathbb{R}^{k}$ represents an algebraic consequence of the system (2) of type

$$
\begin{equation*}
\sum_{l=1}^{n}\left(\sum_{i=1}^{k-1} a_{i l} y_{i}+a_{k l}\right) x_{l}=\sum_{i=1}^{k-1} a_{i 0} y_{i}+a_{k 0} \tag{8}
\end{equation*}
$$

It is easy to see that the coefficient of variable $x_{j}$ in equation (8) has the same sign on one side of $H_{j}$. Furthermore, the coefficient changes its sign if and only if the weight vector passes from one side of $H_{j}$ to the other one.

Assume that $\mathcal{g}=\emptyset$. Then $a_{i j}=0(i=1, \ldots, k-1, j=1, \ldots, n)$. If (2) is manifestly infeasible then either exists an index $i(1 \leq i \leq k-1)$ such that $a_{i 0} \neq 0$ and then $\mathbf{w}=\mathbf{i}^{\text {th }}$ unit vector proves the manifestly infeasibility, or the $k^{\text {th }}$ equation alone is manifestly infeasible and then the $k^{\text {th }}$ unit vector is an appropriate choice for $\mathbf{w}$.

Otherwise, i.e. if $\mathcal{g} \neq \emptyset$, the hyperplanes partition the space $\mathbb{R}^{k-1}$ into polyhedrons $P_{1}, \ldots, P_{m}$ such that in each polyhedron $P_{t}(t=1, \ldots, m)$ the sign of all coefficients remains the same.

Assume that the weight vector $\mathbf{w}$, which proves the manifestly infeasibility of (2) is an element of $P_{t}$. Without loss of generality, we may assume that condition (5) is satisfied. The case when condition (6) is fulfilled can
be discussed in a similar way. Let $\mathscr{L}$ be the set of nonnegative coefficients within $P_{t}$. Then the linear programming problem

$$
\begin{gather*}
\max \left(\sum_{i=1}^{k-1} a_{i 0} y_{i}-\sum_{j \in \mathscr{L}}\left(\sum_{i=1}^{k-1} a_{i j}\right) y_{j}+a_{k 0}-\sum_{j \in \mathscr{L}} a_{k j}\right)  \tag{9}\\
\mathbf{y} \in P_{t}
\end{gather*}
$$

has at least one feasible solution, namely $\mathbf{w}$, and either the optimal value is positive or the objective function is unbounded from above.

The well-known Motzkin theorem states that every polyhedron is the Minkowski sum the of the convex hull of finite many vectors and a cone spanned by finite many vectors, i.e. there are vectors $\mathbf{p}_{1}, \ldots, \mathbf{p}_{U_{t}}, \mathbf{q}_{1}, \ldots$ $\ldots, \mathbf{q}_{V_{t}} \in \mathbb{R}^{k-1}$ such that

$$
\begin{equation*}
P_{t}=\left\{\sum_{u=1}^{U_{t}} \mathbf{p}_{u} \lambda_{u}+\sum_{v=1}^{V_{t}} \mathbf{q}_{v} \mu_{v} \mid \sum_{u=1}^{U_{t}} \lambda_{u}=1, \lambda_{1}, \ldots, \lambda_{U_{t}}, \mu_{1}, \ldots, \mu_{V_{t}} \geq 0\right\} \tag{10}
\end{equation*}
$$

If an optimal solution exists then one of $\mathbf{p}_{1}, \ldots, \mathbf{p}_{U_{t}}$ is optimal. If the objective function is unbounded then it is unbounded along one of the directions $\mathbf{q}_{1}, \ldots, \mathbf{q}_{V_{t}}$. Thus, the set $\mathcal{W}_{k}$ can be chosen as

$$
\begin{equation*}
\mathcal{W}_{k}=\bigcup_{t=1}^{m}\left\{\mathbf{p}_{1}, \ldots, \mathbf{p}_{U_{t}}, \mathbf{q}_{1}, \ldots, \mathbf{q}_{V_{t}}\right\} \tag{11}
\end{equation*}
$$

If the assumption $w_{k} \neq 0$ made at the beginning of the proof does not hold, i.e. $w_{k}=0$, then the manifestly infeasibility can be proved without using the last equation. This equation can be omitted from the system and the other part of the proof can be repeated on the smaller system of equations.

Notice that if one of the $P_{t}$ 's has an extremal point then all $P_{t}$ 's have at least one extremal point. An extremal point is the intersection of $k-1$ linearly independent bounding hyperplanes of the polyhedron. In general, a nonempty polyhedron does not have an extremal point if and only if it contains a complete line. If an extremal point exists, then each line is intersected by at least one of the $k-1$ linearly independent hyperplanes. Algebrically the nonexistence of extremal points means that there is no $k-1$ linearly independent equation in the system (7).

Assume that there are extremal points. Then their number is at most

$$
\binom{n}{k-1}
$$

The extremal directions are the solutions of homogeneous version of system (7), i.e. the weight of the $k^{\text {th }}$ equation of (2) is 0 instead of 1 . Thus, their number is again at most

$$
\binom{n}{k-1}
$$

Therefore we can conclude that the number of elements of $\mathscr{W}_{k}$ is $O\left(n^{k-1}\right)$.
AN EXAMPLE. Let us consider the constraint set

$$
\begin{aligned}
x_{1}-\quad x_{4} & =1 \\
x_{2}-\quad & =1 \\
x_{3}-x_{4} & =1 \\
0 \leq x_{j} \leq 2, \quad x_{j} \text { integer, } j & =1,2,3,4 .
\end{aligned}
$$

By summing up the equations one obtains that

$$
\begin{equation*}
x_{1}+x_{2}+x_{3}-3 x_{4}=3 \tag{12}
\end{equation*}
$$

The system is obviously not manifestly infeasible as $\bar{x}_{1}=2, \bar{x}_{2}=2, \bar{x}_{3}=2$, $\bar{x}_{4}=1$ is a solution. On the other hand if $x_{4}$ is fixed to 2 then (12) becomes

$$
\begin{equation*}
x_{1}+x_{2}+x_{3}=9 \tag{13}
\end{equation*}
$$

Hence (5) is satisfied, i.e. the fact that the branch $x_{4}=2$ is empty, is proven. To get an equation of type (7) for $k=3$ and $j=1$, respectively $j=2$, one must have $y_{1}=0$, respectively $y_{2}=0$. Hence the intersection of the two lines which are now the $H_{1}$, and $H_{2}$ hyperplanes is $(0,0,1)$. If the procedure starts with $k=1$ or $k=2$ then the vectors of weight obtained are $(1,0,0)$, and $(0,1,0)$. The equations obtained by these weight if $x_{4}$ is fixed to 2 are

$$
x_{j}=3, \quad j=1,2,3
$$

Taking into account that $\forall j: x_{j} \leq 2$, the equations are manifestly infeasible.

## References

# [1] P. L. Hammer, M. W. Padberg and U. N. Peled: Constraint pairing in integer programming, INFOR - Canadian Journal of Operational Research and Information Processing 13 (1975), 68-81. 

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# GENERALIZATION OF A THEOREM ON ADDITIVE REPRESENTATION FUNCTIONS 

By<br>SÁNDOR KISS

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## 1. Introduction

Let $k \geq 2$ be a fixed integer and let $\mathscr{A}=\left\{a_{1}, a_{2}, \ldots\right\}\left(a_{1} \leq a_{2} \leq \ldots\right)$ be an infinite sequence of positive integers. For $n=0,1,2, \ldots$ let $R_{k}(n)$ denote the number of solutions of $a_{i_{1}}+a_{i_{2}}+\ldots+a_{i_{k}}=n, a_{i_{1}} \in \mathscr{A}, \ldots a_{i_{k}} \in \mathscr{A}$. If $s_{0}, s_{1}, \ldots$ is given sequence of real numbers then let $\Delta_{l} s_{n}$ denote the $l$-th difference of the sequence $s_{0}, s_{1}, s_{2}, \ldots$ defined by $\Delta_{1} s_{n}=s_{n+1}-s_{n}$ and $\Delta_{l} s_{n}=\Delta_{1}\left(\Delta_{l-1} s_{n}\right)$. It is well known and it is easy to see by induction that

$$
\begin{equation*}
\Delta_{l} s_{n}=\sum_{i=0}^{l}(-1)^{l-i}\binom{l}{i} s_{n+i} . \tag{1}
\end{equation*}
$$

We will consider the following problem : what condition is needed to guarantee that $\left|\Delta_{l} R_{k}(n)\right|$ cannot be bounded. P. Erdős, A. Sárközy and V. T. Sós proved in [1] that if $k=2, l=1$ and

$$
\begin{equation*}
\lim _{N \rightarrow \infty} \frac{B(\mathscr{A}, N)}{\sqrt{N}}=\infty \tag{2}
\end{equation*}
$$

then $\left|\Delta_{1} R_{2}(n)\right|=\left|R_{2}(n+1)-R_{2}(n)\right|$ cannot be bounded where $B(A, N)$ denotes the number of blocks formed by consecutive integers in $\mathscr{A}$ up to $N$, i.e.,

$$
B(\mathscr{A}, N)=\sum_{\substack{n \leq N \\ n \in \mathcal{A}, n-1 \notin \mathcal{A}}} 1,
$$

and they showed that assumption (2) is nearly best possible. I have proved in [2] that under condition (2) for arbitrary $l\left|\Delta_{l} R_{2}(n)\right|$ cannot be bounded either. One might like to extend this result to the case $k>2$. In this paper my goal is to prove the analog result for $k \geq 2$ and $l \leq k$ (but, unfortunately, I have not been able to settle the case $l>k$ ). In fact we will prove the following theorem:

THEOREM 1. If $k \geq 2$ is an integer and $\lim _{N \rightarrow \infty} \frac{B(\mathscr{A}, N)}{\sqrt[k]{N}}=\infty$, and $l \leq k$, then $\left|\Delta_{l} R_{k}(n)\right|$ cannot be bounded.

## 2. Proof of the theorem

Clearly it sufficies to prove the assertion of the theorem in the special case $l=k$. We prove by contradiction. Assume that contrary to the conclusion of the theorem there is a positive constant $C>0$ such that $\left|\Delta_{k} R_{k}(n)\right|<C$ for every $n$. Throughout the remaining part of the proof of the theorem we use the following notations: $N$ denotes a large integer. We write $e^{2 i \pi \alpha}=e(\alpha)$ and we put $r=e^{-1 / N}, z=r e(\alpha)$ where $\alpha$ is a real variable (so that a function of form $\mathrm{p}(\mathrm{z})$ is a function of the real variable $\alpha: p(z)=p(r e(\alpha))=P(\alpha))$. We write $f(z)=\sum_{j=1}^{\infty} z^{a_{j}}$. (By $r<1$, this infinite series and all the other infinite series in the remaining part of the proof are absolutely convergent).

We start out from the integral $I=\int_{0}^{1}|f(z)(1-z)|^{k} d \alpha$. We will give lower and upper bound for $I$. The comparison of these bounds will show that $\frac{B(A, N)}{\sqrt[k]{N}}$ is bounded which contradicts the assumption of the theorem. This contradiction will prove that our indirect assumption on $\left|\Delta_{k} R_{k}(n)\right|$ cannot hold which will complete the proof of the theorem.

First we will give a lower bound for $I$. We write $f(z)(1-z)=\sum_{n=1}^{\infty} b_{n} z^{n}$. Then for $n-1 \notin \mathscr{A}, n \in \mathscr{A}$ we have $b_{n}=1$, thus by the Hölder inequality and
the Parseval formula, we have

$$
\begin{aligned}
I^{2 / k} & =\left(\int_{0}^{1}|f(z)(1-z)|^{k} d \alpha\right)^{2 / k}\left(\int_{0}^{1} 1 d \alpha\right)^{1-2 / k} \geq \int_{0}^{1}|f(z)(1-z)|^{2} d \alpha \\
& =\int_{0}^{1}\left|\sum_{n=1}^{\infty} b_{n} z^{n}\right|^{2} d \alpha=\sum_{n=1}^{\infty} b_{n}^{2} r^{2 n} \geq r^{2 N} \sum_{\substack{n \leq N \\
n \in \mathscr{A}, n-1 \notin \mathcal{A}}} b_{n}^{2}=e^{-2} \sum_{\substack{n \leq N \\
n \in \mathcal{A}, n-1 \notin \mathcal{A}}} 1 \\
& =e^{-2} B(\mathscr{A}, N)
\end{aligned}
$$

whence

$$
I \geq e^{-k}(B(\mathscr{A}, N))^{k / 2}
$$

Now we will give an upper bound for $I$. By (1), our indirect assumption, the Cauchy inequality and the Parseval formula we have

$$
\begin{aligned}
I & =\int_{0}^{1}|f(z)(1-z)|^{k} d \alpha=\int_{0}^{1}\left|f^{k}(z)(1-z)^{k}\right| d \alpha=\int_{0}^{1}\left|\left(\sum_{j=1}^{\infty} z^{a_{j}}\right)^{k}(1-z)^{k}\right| d \alpha \\
& =\int_{0}^{1}\left|\left(\sum_{n=1}^{\infty} R_{k}(n) z^{n}\right)(1-z)^{k}\right| d \alpha=\int_{0}^{1}\left|\left(\sum_{n=1}^{\infty} R_{k}(n) z^{n}\right)\left(\sum_{i=0}^{k}(-1)^{i}\binom{k}{i} z^{i}\right)\right| d \alpha \\
& =\int_{0}^{1}\left|\sum_{m=1}^{\infty} \sum_{i=0}^{k}(-1)^{i}\binom{k}{i} R_{k}(m-i) z^{m}\right| d \alpha=\int_{0}^{1}\left|\sum_{m=1}^{\infty} \Delta_{k} R_{k}(m-k) z^{m}\right| d \alpha \\
& \leq\left(\int_{0}^{1}\left|\sum_{m=1}^{\infty} \Delta_{k} R_{k}(m-k) z^{m}\right|^{2} d \alpha\right)^{1 / 2}=\left(\sum_{m=1}^{\infty}\left|\Delta_{k} R_{k}(m-k)\right|^{2} r^{2 m}\right)^{1 / 2} \\
& \leq C\left(\sum_{m=1}^{\infty} r^{2 m}\right)^{1 / 2}=C\left(\frac{1}{1-r^{2}}\right)^{1 / 2} \leq C\left(\frac{1}{1-r}\right)^{1 / 2} \\
& =C\left(\frac{1}{\left.1-e^{-\frac{1}{N}}\right)^{1 / 2}<C \sqrt{2 N}}\right.
\end{aligned}
$$

since we have

$$
1-e^{-x}=x-\frac{x^{2}}{2!}+\frac{x^{3}}{3!}-\cdots>x-\frac{x^{2}}{2!}=x\left(1-\frac{x}{2}\right)>\frac{x}{2}
$$

for $0<x<1$.
Now we will complete the proof of the theorem. We have

$$
e^{-k}(B(\mathscr{A}, N))^{k / 2} \leq I<C \sqrt{2 N}
$$

hence

$$
\frac{B(\mathscr{A}, N)}{\sqrt[k]{N}}<e^{2} \sqrt[k]{2 C^{2}}
$$

This contradicts our assumption on $B(\mathscr{A}, N)$ which completes the proof of the theorem.

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# ON SOME SINGULARITY OF A HYPERCOMPLEX ANALYSIS 

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## Introduction

In [9] the author introduced a notion of an almost Clifford-type structure as a generalization of complex and quaternionic ones. In this paper we give examples of manifolds endowed with an almost Clifford-type structure. Moreover, using this structure we define a Clifford-type holomorphy and prove two characteristic and peculiar properties of Clifford-type holomorphic mappings.

## 1. A Clifford-type structure

Let $V$ be a real vector space.
DEFINITION 1.1. An almost Clifford-type structure $\mathscr{C}_{n}$ on $V$ is a set of $n$ almost complex structures $\left\{I_{1}, \ldots, I_{n}\right\}$ such that

$$
I_{\alpha} I_{\beta}+I_{\beta} I_{\alpha}=-2 \delta_{\alpha \beta} \mathrm{Id}, \quad \alpha, \beta=1, \ldots, n
$$

where Id stands for the identity endomorphism of $V, \delta$ denotes the „Kronecker delta", provided that any $I_{j}$ is not a product of two, four etc. of others from the set $\left\{I_{1}, \ldots, I_{n}\right\}$.

REMARK 1.1.
a) If $n=1$, then $\mathscr{C}_{1}=\{I\}$ with $I^{2}=-$ Id. This $\mathscr{C}_{1}$ is nothing but an almost complex structure. Recall that the standard form of an almost complex

[^1]structure looks as follows:
\[

I_{O}=\left($$
\begin{array}{cc}
0 & I \\
-I & 0
\end{array}
$$\right) \quad(I=\mathrm{Id})
\]

provided that $V$ has an even dimension (see, e.g. [7]).
b) If $n=2$, then $\mathscr{C}_{2}=\{I, J\}$ with $I^{2}=J^{2}=-\mathrm{Id}$ and $I J+J I=0$. Define $K:=I J$ then $I J K=-\mathrm{Id}$ and $K^{2}=-\mathrm{Id}$. Thus $\mathscr{C}_{2}$ is nothing but an almost quaternionic structure (see, e.g. [8]). The standard form of an almost quaternionic structure looks as follows:
$I_{O}=\left(\begin{array}{cccc}0 & I & 0 & 0 \\ -I & 0 & 0 & 0 \\ 0 & 0 & 0 & -I \\ 0 & 0 & I & 0\end{array}\right), J_{O}=\left(\begin{array}{cccc}0 & 0 & I & 0 \\ 0 & 0 & 0 & I \\ -I & 0 & 0 & 0 \\ 0 & -I & 0 & 0\end{array}\right), K_{o}=\left(\begin{array}{cccc}0 & 0 & 0 & I \\ 0 & 0 & -I & 0 \\ 0 & I & 0 & 0 \\ -I & 0 & 0 & 0\end{array}\right)$,
provided that $\operatorname{dim}_{\mathbf{R}} V=4 n$.
Note that any almost Clifford-type structure $\mathscr{C}_{n}=\left\{I_{1}, \ldots, I_{n}\right\}$ induces the following set of almost complex structures:
$I_{1}, \ldots, I_{n}$;
$I_{1} I_{2}, I_{1} I_{3}, \ldots, I_{1} I_{n}, I_{2} I_{3}, \ldots, I_{2} I_{n}, \ldots, I_{n-1} I_{n} ;$
(no triplet is an almost complex structure)
$I_{1} I_{2} I_{3} I_{4}, \ldots, I_{n-3} I_{n-2} I_{n-1} I_{n}$;
(no odd product is an almost complex structure)
etc.
Denote by $p_{n}^{\prime}$ the number of the above almost complex structures, then

$$
p_{n}^{\prime}= \begin{cases}\binom{n}{1}+\binom{n}{2}+\binom{n}{4}+\cdots+\binom{n}{n-1}, & \text { if } n \text { is odd, } \\ \binom{n}{1}+\binom{n}{2}+\binom{n}{4}+\cdots+\binom{n}{n}, & \text { if } n \text { is even. }\end{cases}
$$

By the straightforward calculations we get the following numbers:

$$
\begin{aligned}
& p_{1}^{\prime}=1, p_{2}^{\prime}=3, \\
& p_{5}^{\prime}=20, p_{6}^{\prime}=3, \quad p_{4}^{\prime}=11, \\
& p_{7}^{\prime}=70, \quad p_{8}^{\prime}=135 \quad \text { etc. }
\end{aligned}
$$

Denote by $V(n)$ a real vector space endowed with a Clifford-type structure $\mathscr{C}_{n}=\left\{I_{1}, \ldots, I_{n}\right\}$ then:

Theorem 1.1 [9]. We have

$$
\operatorname{dim}_{\mathbf{R}} V(n)=2^{n} \cdot s
$$

where $s>0$ is an integer.

## 2. Clifford-type numbers

In order to consider at the same time quaternions and octonions (even complex numbers) we introduce the "Clifford-type numbers".

Assume that $n=1,3,7$.
DEFINITION 2.1. Denote by $\mathscr{A}_{n}$ the field (skew field) of the "Clifford-type numbers". A typical element of $\mathscr{A}_{n}$ can be written as

$$
a \in A_{n}, \quad a:=x_{o}+e_{1} x_{1}+\ldots+e_{n} x_{n}, \quad x_{o}, x_{1}, \ldots, x_{n} \in \mathbf{R}, \quad n=1,3,7
$$

and we assume that the "Clifford-type units" $e_{1}, \ldots, e_{n}$ satisfy the relations:

$$
\begin{array}{r}
e_{k} \cdot e_{m}+e_{m} \cdot e_{k}=-2 \delta_{k m}, \quad k, m=1, \ldots, n \\
\left(e_{k}^{2}=-1, k=1, \ldots, n, \quad e_{k} \cdot e_{m}=-e_{m} \cdot e_{k}, \quad k \neq m, k, m=1, \ldots, n\right) \\
\left\{e_{k} \cdot e_{m}\right\}=\left\{ \pm e_{1}, \ldots, \pm e_{n}\right\}, \quad \overline{e_{k}}=-e_{k}, \quad\left(\overline{e_{k} \cdot e_{m}}=-e_{k} \cdot e_{m}, \quad k \neq m\right) \\
k, m=1, \ldots, n .
\end{array}
$$

We have also

$$
\overline{a \cdot b}=\bar{b} \cdot \bar{a}, \quad a, b \in \mathscr{A}_{n}
$$

## EXAMPLES 2.1.

1. $\mathbf{C}$ - complex numbers with the complex unit $e_{1}=i$ satisfying $i^{2}=-1$ and $\bar{i}=-i$.
(The complex multiplication is commutative and associative).
2. $\mathbf{H}$ - quaternions with the quaternionic units $e_{1}=i, e_{2}=j, e_{3}=k$ satisfying the conditions:

$$
\begin{gathered}
i^{2}=j^{2}=k^{2}=i j k=-1, \\
i j=k, \quad j k=i, \quad k i=j \\
\bar{i}=-i, \quad \bar{j}=-j, \quad \bar{k}=-k
\end{gathered}
$$

(The quaternionic multiplication is not commutative but it is associative).
3. $\mathbf{O}$ - octonions with the octonionic units $e_{1}, e_{2}, \ldots, e_{7}$ satisfying the conditions:

$$
\begin{aligned}
e_{k}^{2} & =-1, \quad k=1, \ldots, 7, \\
e_{k} e_{m}+e_{m} e_{k} & =0, \quad k \neq m, \quad k, m=1, \ldots, 7, \\
e_{1} e_{2} & =e_{3}, \quad e_{2} e_{3}=e_{1}, \quad e_{3} e_{1}=e_{2}, \quad e_{1} e_{4}=e_{5}, \\
e_{2} e_{4} & =e_{6}, \quad e_{3} e_{4}=e_{7}, \quad e_{5} e_{1}=e_{4} \quad \text { etc. }, \\
e_{1} e_{2} & =e_{3}, \quad e_{1} e_{3}=-e_{2}, \quad e_{1} e_{4}=e_{5}, \\
e_{1} e_{5} & =-e_{4}, \quad e_{1} e_{6}=-e_{7}, \quad e_{1} e_{7}=e_{6} \quad \text { etc. }
\end{aligned}
$$

and

$$
\overline{e_{k}}=-e_{k}
$$

(The octonionic multiplication is not commutative and it is not associative).
Let $\mathscr{A}_{n}^{p} \equiv \mathbf{R}^{(n+1) p}$ denote the "Clifford-type" Euclidean $p$-space with the coordinate $A=\left(a^{1}, \ldots, a^{n}\right)$, where

$$
a^{s}=x_{o}^{s}+e_{1} x_{1}^{s}+\cdots+e_{n} x_{n}^{s}, \quad s=1, \ldots, p
$$

Define the right multiplication $\bullet: \mathscr{A}_{n}^{p} \times \mathscr{A}_{n} \rightarrow \mathscr{A}_{n}^{p}$ by

$$
A \bullet a:=\left(a^{1} \cdot a, \ldots, a^{p} \cdot a\right), \quad A \in \mathscr{A}_{n}^{p}, a \in \mathscr{A}_{n}
$$

Let us emphasize that $\mathscr{A}_{n}^{p}$ can be identified with $\mathbf{R}^{(n+1) p}$ endowed with $n$ almost complex structures $I_{1}, \ldots, I_{n}$ satisfying the conditions:

$$
\begin{gathered}
I_{\alpha} I_{\beta}+I_{\beta} I_{\alpha}=-2 \delta_{\alpha \beta} \mathrm{Id} \\
I_{1} X:=e_{1} X, \quad \ldots, \quad I_{n} X=e_{n} X, \quad X \in \mathbf{R}^{(n+1) p}
\end{gathered}
$$

where Id stands for the identity mapping in $\mathbf{R}^{(n+1) p}$.
Define a bilinear form $\langle$,$\rangle on \mathscr{A}_{n}^{p}$ as follows:
if $A=\left(a^{1}, \ldots, a^{p}\right)$ and $B=\left(b^{1}, \ldots, b^{p}\right) \in \mathscr{A}_{n}^{p}$, then

$$
\begin{aligned}
\langle A, B\rangle & :=\frac{1}{2} \sum_{\alpha=1}^{p}\left(a^{\alpha} \overline{b^{\alpha}}+b^{\alpha} \overline{a^{\alpha}}\right) \\
& =\operatorname{Re}(A, B):=\operatorname{Re} \sum_{\alpha=1}^{p} a^{\alpha} \overline{b^{\alpha}}
\end{aligned}
$$

where

$$
\overline{a^{s}}:=x_{o}^{s}-e_{1} x_{1}^{s}-\cdots-e_{n} x_{n}^{s}, \quad s=1, \ldots, p
$$

Then $\langle A, B\rangle$ is an inner product of $\mathscr{A}_{n}^{p}$ considered as a $(n+1) p$-dimensional real vector space. Considering $\AA_{n}^{p}$ as a "Clifford-type" $p$-space we define the "Clifford-type symplectic product":

$$
(A, B):=\sum_{\alpha=1}^{p} a^{\alpha} \overline{b^{\alpha}}
$$

Note that we have the following relation:

$$
\langle A, B\rangle=\frac{1}{2}[(A, B)+(B, A)]
$$

Denote by $\operatorname{SP}_{n+1}(p)$ the set of all endomorphisms of $A_{n}^{p}$ which preserve the Clifford-type symplectic product (, ).
(In the quaternionic case $(n=3) \mathrm{SP}_{4}(p)$ is nothing but a well known group usual denoted by $\operatorname{Sp}(p)$.)

A norm of $A \in \mathscr{A}_{n}^{p}$ is defined by

$$
\|A\|^{2}:=(A, A)=\sum_{\beta=1}^{p} a^{\beta} \overline{a^{\beta}}
$$

Let us denote

$$
\operatorname{SP}_{n+1}(1):=\left\{a \in \mathcal{A}_{n},\|a\|=1\right\} \quad(p=1)
$$

Note

1. $\mathrm{SP}_{n+1}(1)$ is a group,
2. $\mathrm{SP}_{n+1}(p) \subseteq S O[(n+1) \cdot p]$.

## 3. Manifolds endowed with a Clifford-type structure

DEFINITION 3.1 [9]. A $[(n+1) \cdot p]$-dimensional Riemannian manifold $M$ is called a Clifford-type manifold if its holonomy group is a subgroup of $\mathrm{SP}_{n+1}(p) \times \mathrm{SP}_{n+1}(1)$.

EXAMPLES 3.1.

1. The basic examples of Clifford-type manifolds are "quaternionic" manifolds. Note that for $n=2\left(I_{1}, I_{2}\right)$ there are three almost complex structures on a given Riemannian manifold $(M, g)$, namely: $I_{1}, I_{2}, I_{3}:=I_{1} I_{2}$ and $\operatorname{dim}_{\mathbf{R}} M=$ $=2^{2}=4$. These manifolds are called almost quaternionic (see e.g. [3], [8]).

If $g$ is Hermitian for $I_{1}$ and $I_{2}$ then $(M, g)$ is called almost-quaternionicHermitian.

If the suitable fundamental 4-form is closed then an almost-quaternionicHermitian manifold is called almost-quaternionic-Kähler. The most important example of an almost-quaternionic-Kähler manifold is the quaternionic projective space $\mathbf{H P}^{n}$ with a standard metric (see e.g. [2], [11]).
2. More generally, in the case when the holonomy group of a given almost-quaternionic-Hermitian manifold $\left(M^{4 m}, g\right)$ is contained in the group $\operatorname{Sp}(m) \times$ $\times \operatorname{Sp}(1)$ then such a manifold is called quaternionic-Kähler (see e.g. [2]). Emphasize the important result by Berger [1] that a quaternionic-Kähler manifold (of dimension $4 n \geq 8$ ) is Einstein (Riemannian manifold of constant Ricci curvature). Moreover, quaternionic-Kähler manifolds whose dimension is a multiple of 8 are spin manifolds ([10]).

Some examples (but not called Clifford-type) of manifolds with a holonomy group contained in $\operatorname{Sp}(m), \operatorname{Sp}(m) \times \operatorname{Sp}(1)$ or $\operatorname{Spin}(n)$ one can find in [12].

EXAMPLE 3.2. Define a Clifford-type projective space $\mathscr{A}_{n} \mathbf{P}^{p}$ as a quotient of $\mathscr{A}_{n}^{p+1}$ by the group $\mathscr{A}_{n}^{*}$ of non-zero Clifford-type numbers:

$$
\mathscr{A}_{n} \mathbf{P}^{p}:=\mathscr{A}_{n}^{p+1} / \mathscr{A}_{n}^{*}, \quad \mathscr{A}_{n}^{*}:=\left\{a \in \mathscr{A}_{n}, a \neq 0\right\}
$$

acting by the right multiplication.
A point of $\mathscr{A}_{n} \mathbf{P}^{p}$ represents a Clifford-type line $\sigma$ in $\mathscr{A}_{n}^{p+1}$. The line $\sigma$ can be identified with the group $\mathrm{SP}_{n+1}(1)$. The subgroup of $\mathrm{SP}_{n+1}(p+1)$ stabilizing $\sigma$ is $\mathrm{SP}_{n+1}(p) \times \mathrm{SP}_{n+1}(1)$. This description gives $\mathscr{A}_{n} \mathbf{P}^{p}$ the structure of a symmetric space $\operatorname{SP}_{n+1}(p+1) / \mathrm{SP}_{n+1}(p) \times \mathrm{SP}_{n+1}(1)$ whose holonomy group equals $\mathrm{SP}_{n+1}(p) \times \mathrm{SP}_{n+1}(1)$.

Let us add an important note on the dimensions. Thus, for the complex projective space $\mathbf{C} \mathbf{P}^{p}$ we assume that $p=2 m \geq 4$, for the quaternionic projective space $\mathbf{H} \mathbf{P}^{p}$ we assume that $p=4 m \geq 8$ and for the octonionic projective space $\mathbf{O P}^{p}$ we have $p=8 m=16$.

## 4. A Clifford-type holomorphy

Take $n$ almost complex structures $I_{1}, \ldots, I_{n}$ defined on a real vector space $V$ satisfying the condition

$$
I_{\alpha} I_{\beta}+I_{\beta} I_{\alpha}=-2 \delta_{\alpha \beta} \mathrm{Id}, \quad \alpha, \beta=1, \ldots, n
$$

provided that any $I_{j}$ is not a product of two, four etc. of others from the set $\left\{I_{1}, \ldots, I_{n}\right\}$.

Let $X \in V, X \neq 0$, then the vector subspace $\tilde{V}$ of $V$ generated by

$$
X, I_{1} X, \ldots, I_{p_{n}^{\prime}} X
$$

according to the definition of the numbers $p_{n}^{\prime}$, has a dimension $p_{n}^{\prime}+1$ :

$$
\operatorname{dim} \tilde{V}=p_{n}^{\prime}+1
$$

## Example 4.1.

a) If $n=1$, then we have one almost complex structure, $p_{1}^{\prime}=1$ and $\operatorname{dim} \tilde{V}=$ $=\operatorname{dim} V=2,\left(V \cong \mathbf{R}^{2} \cong \mathbf{C}\right)$.
b) If $n=2$, then we have two almost complex structures, $p_{2}^{\prime}=3$ and $\operatorname{dim} \tilde{V}=\operatorname{dim} V=4,\left(V \cong \mathbf{R}^{4} \cong \mathbf{C} \oplus \mathbf{C} \cong \mathbf{H}\right)$.
c) If $n=3$, then we have three almost complex structures $I_{1}, I_{2}, I_{3}$ which generate a set of six almost complex structures on $V$, namely

$$
I_{1}, I_{2}, I_{3}, \quad I_{4}:=I_{1} I_{2}, \quad I_{5}:=I_{1} I_{3}, \quad I_{6}:=I_{2} I_{3} \quad\left[\left(I_{1} I_{2} I_{3}\right)^{2} \neq-\mathrm{Id}\right] .
$$

Thus $p_{3}^{\prime}=6$ and $\operatorname{dim} \tilde{V}=7$ etc.
DEFINITION 4.1. Let $V=\left(V, \mathscr{C}_{n}\right)$ and $W=\left(W, \tilde{\mathscr{C}}_{n}\right)$ be two real vector spaces equipped with two almost Clifford-type-Hermitian structures $\mathscr{C}_{n}=$ $=\left(I_{1}, \ldots, I_{n}\right)$ and $\tilde{\mathscr{C}}_{n}=\left(\tilde{I}_{1}, \ldots, \tilde{I}_{n}\right)$, respectively. Assume that $\Phi: V \rightarrow W$ is a smooth map. Then $\Phi$ is called Clifford-type holomorphic if the following condition:

$$
\begin{equation*}
\tilde{I}_{\alpha} \circ d \Phi=d \Phi \circ I_{\alpha} \quad \text { for } \quad \alpha=1, \ldots, n \tag{4.1}
\end{equation*}
$$

is fulfilled.

## Remark 4.2.

a) If $n=1$, then (4.1) reduces to the equation

$$
\tilde{I} \circ d \Phi=d \Phi \circ I .
$$

In this case the Clifford-type holomorphy is nothing but the holomorphy of complex analysis. Moreover, we have

$$
(\tilde{I} \circ d \Phi=d \Phi \circ I) \Leftrightarrow\left[\frac{1}{2}\left(\frac{\partial}{\partial x}+i \frac{\partial}{\partial y}\right)(u+i v)=0 \text { with } \Phi=u+i v\right] .
$$

b) If $n=2$, then (4.1) reduces to the system

$$
\left\{\begin{array}{l}
\tilde{I} \circ d \Phi=d \Phi \circ I, \\
\tilde{J} \circ d \Phi=d \Phi \circ J .
\end{array}\right.
$$

In this case the Clifford-type holomorphy is nothing but the quaternionic Q-holomorphy (see, e.g. [8]).

WARNING! It is important to emphasize that the Q-holomorphy and the Fueter regularity (see, e.g. [6], [8]) are two different notions, i.e.

$$
\left\{\begin{array}{c}
\tilde{I} \circ d \Phi=d \Phi \circ I \\
\tilde{J} \circ d \Phi=d \Phi \circ J
\end{array}\right\} \nLeftarrow\left[\frac{1}{4}\left(\frac{\partial}{\partial x^{o}}+i \frac{\partial}{\partial x^{1}}+j \frac{\partial}{\partial x^{2}}+k \frac{\partial}{\partial x^{3}}\right)\left(u^{o}+i u^{1}+j u^{2}+k u^{3}\right)=0\right]
$$

## 5. Some definitions

DEFINITION 5.1. Assume that $(M, g)$ is a $2^{n} \cdot s$-dimensional Riemannian manifold. An almost Clifford-type structure $\mathscr{C}_{n}$ on $(M, g)$ is defined as a covering $\left\{U^{i}\right\}$ of the manifold $M$ with a set of almost complex structures $\left\{I_{1}^{i}, \ldots, I_{n}^{i}\right\}$ on each $U^{i}$ such that

$$
I_{\alpha}^{i} I_{\beta}^{i}+I_{\beta}^{i} I_{\alpha}^{i}=-2 \delta_{\alpha \beta} \mathrm{Id}
$$

(provided that any $I_{j}$ is not a product of two, four etc. of others from the set $\left\{I_{1}, \ldots, I_{n}\right\}$ ) and ( $2^{n} \cdot s-1$ )-dimensional vector spaces of endomorphisms generated by

$$
I_{1}, \ldots, I_{n}, I_{1} I_{2}, \ldots, I_{n-1} I_{n}, \ldots, I_{1} I_{2} \cdots I_{n}
$$

i.e.

End $_{U^{i}}:=\left\{a^{1} I_{1}+\cdots+a^{n} I_{n}+a^{12} I_{1} I_{2}+\cdots+a^{1 \cdots n} I_{1} \cdots I_{n} ;\right.$

$$
\left.a^{1}, \ldots, a^{12}, \ldots, a^{1 \cdots n} \in \mathbf{R}\right\}
$$

are the same on all of the manifold.
EXAMPLE 5.1. Let $n=2$ and $s=1$, then the almost Clifford-type structure $\mathscr{C}_{2}$ is nothing but a quaternionic structure on a 4-dimensional Riemannian manifold.

DEfinition 5.2. A Riemannian metric $g$ is Clifford-type Hermitian if $g$ is Hermitian for each $I_{1}, \ldots, I_{n}$.

DEFINITION 5.3. a) A Riemannian manifold ( $M, g$ ) with an almost Clifford-type structure $\mathscr{C}_{n}$ is called an almost Clifford-type manifold,
b) An almost Clifford-type manifold $\left(M, g, \mathscr{C}_{n}\right)$ with $g$ Clifford-type Hermitian is called almost Clifford-type Hermitian.

REMARK 5.1. A real vector space $V$ with an almost Clifford-type structure $\mathscr{C}_{n}=\left\{I_{1}, \ldots, I_{n}\right\}$ can be turned into a Clifford vector space by defining a scalar multiplication by Clifford numbers as follows:

$$
\begin{aligned}
& \text { If } \mathbf{a} \in \mathscr{A}\left(e_{1}, \ldots, e_{n}\right):=\mathscr{A}_{n}^{C}, \text { then } \\
& \mathbf{a}= a^{o}+a^{1} e_{1}+\cdots+a^{n} e_{n}+ \\
&+a^{12} e_{1} e_{2}+\cdots+a^{1 n} e_{1} e_{n}+a^{23} e_{2} e_{3}+\ldots+a^{(n-1) n} e_{n-1} e_{n}+ \\
&+a^{123} e_{1} e_{2} e_{3}+\cdots+a^{(n-2)(n-1) n} e_{n-2} e_{n-1} e_{n}+\ldots+ \\
&+a^{12 \cdots(n-1)} e_{1} e_{2} \cdots e_{n-1}+\cdots+a^{23 \cdots n} e_{2} e_{3} \cdots e_{n}+a^{12 \cdots n} e_{1} e_{2} \cdots e_{n},
\end{aligned}
$$

where $a^{o}, a^{1}, \ldots, a^{n}, \ldots, a^{12 \cdots n} \in \mathbf{R}$. For $X \in V$ we have

$$
\begin{gathered}
\mathbf{a} \cdot X=\left(a^{o}+a^{1} e_{1}+\cdots+a^{n} e_{n}+a^{12} e_{1} e_{2}+\cdots+a^{12 \cdots n} e_{1} e_{2} \cdots e_{n}\right) \cdot X:= \\
:=a^{o} X+a^{1} I_{1} X+\cdots+a^{n} I_{n} X+a^{12} I_{1} I_{2} X+\cdots+a^{12 \cdots n} I_{1} I_{2} \cdots I_{n} X .
\end{gathered}
$$

DEFINITION 5.4. Take a point $a \in \mathbf{R}^{2^{n}} \cdot k$. A right Clifford line passing through the point $a[\mathscr{C}-\operatorname{line}(a)]$ is defined by

$$
\mathscr{C}-\operatorname{line}(a):=\left\{a^{\prime} \in \mathbf{R}^{2^{n} \cdot k} ; \quad a^{\prime}=a \cdot c, \quad c \in \mathscr{A}_{n}^{C}\right\} .
$$

## 6. Two properties of Clifford-type holomorphic maps

Assume that $n>1$.
THEOREM 6.1. Let $U$ be an open neighbourhood of the origin in $\mathbf{R}^{2^{n} \cdot s}$. Let $\Phi: \mathbf{R}^{2^{n}} \cdot s \rightarrow \mathbf{R}^{2^{n}}$ be a Clifford-type holomorphic map with respect to the almost Clifford-type structures $\left\{I_{1}, \ldots, I_{n}\right\}$ and $\left\{\tilde{I}_{1}, \ldots, \tilde{I}_{n}\right\}$, respectively. If $\Phi$ is singular at the origin then $\Phi$ is a constant map.

Proof. Denote by $\tilde{U} \subseteq U$ the intersection of $U$ with a right Clifford line, i.e. the right multiples by elements of $\mathcal{A}_{n}^{C}=\mathscr{A}\left(e_{1}, \ldots, e_{n}\right)$, here $\mathbf{R}^{2^{n} \cdot s}$ is identified with $\left(\mathscr{A}_{n}^{C}\right)^{s}$. Such a $2^{n} \cdot s$-dimensional submanifold is easily seen to be almost Clifford-type.

The singular set of $\Phi_{\mid \tilde{U}}$ is defined by $\operatorname{det}\left[d\left(\Phi_{\mid \tilde{U}}\right)\right]=0$ and it is a complex subvariety $N$ of $\tilde{U}$ in $I_{1}, \ldots, I_{n}$ complex structures. In particular, it cannot be isolated because it is non-null. At a regular point $x \in N$ the real tangent space of $N$ must be invariant under each $I_{1}, \ldots, I_{n}$. So, $N$ is $2^{n}$-dimensional at $x$ and thus at all points of $\tilde{U}$. Then $\Phi_{\mid \tilde{U}}$ is constant and because $U$ is the union of the right slices $\tilde{U}$, so $\Phi$ is also constant.

REMARK 6.1. A Clifford-type affine map from $\mathbf{R}^{2^{n}} \cdot s_{1}$ to $\mathbf{R}^{2^{n}} \cdot s_{2}$ is a map of the form $\Phi(X)=X \cdot A+B$, where $A \in \mathcal{M}\left(s_{1}, s_{2} ; \mathcal{A}_{n}^{C}\right)$ and $B \in \mathbf{R}^{2^{n} \cdot s_{2}}$. Identifying $\mathbf{R}^{2^{n} \cdot s_{1}}$ and $\mathbf{R}^{2^{n} \cdot s_{2}}$ with $\left(\mathscr{A}_{n}^{C}\right)^{s_{1}}$ and $\left(\mathscr{A}_{n}^{C}\right)^{s_{2}}$, then we have $\Phi(X)=$ $=\tilde{A} \cdot \tilde{X}+\tilde{B}$, where $\tilde{A}$ is the transpose matrix of $A$ with real $2^{n} \times 2^{n}$ submatrices replaced by Clifford numbers and where $\tilde{X}$ and $\tilde{B}$ are $s_{1}$ and $s_{2}$-tuples of Clifford numbers, respectively.

Proposition 6.1. Denote by $U$ and $V$ open sets in $\mathbf{R}^{2^{n} \cdot s_{1}}$ and $\mathbf{R}^{2^{n}} \cdot s_{2}$, resprctively. Let $\Phi: U \rightarrow V$ be Clifford-type holomorphic with respect to Clifford-type structures on $\mathbf{R}^{2^{n} \cdot s_{1}}$ and $\mathbf{R}^{2^{n} \cdot s_{2}}$, respectively. Then $\Phi$ is the restriction to $U$ of a Clifford-type affine map from $\mathbf{R}^{2^{n} \cdot s_{1}}$ to $\mathbf{R}^{2^{n}} \cdot s_{2}$.

Proof. Since the truth or falsity of the Proposition is not affected by translations of the range or domain, one can assume that $U$ and $V$ contain the origin and that $\Phi(0)=0$. Consider the map $X \rightarrow \Phi(X)-X d \Phi(0)$. Since $d \Phi(0) \in \mathcal{M}\left(s_{1}, s_{2} ; \mathcal{A}_{n}^{C}\right)$, so this function is Clifford-type holomorphic and by the construction its Jacobian is zero. Thus, composed with any projection $\pi_{i}: \mathbf{R}^{2^{n} \cdot s_{2}} \rightarrow \mathbf{R}^{2^{n}}$ defined by
$\pi_{i}\left(x_{1}^{1}, x_{1}^{2}, \ldots, x_{1}^{2^{n}}, \ldots, x_{i}^{1}, x_{i}^{2}, \ldots, x_{i}^{2^{n}}, \ldots, x_{s_{2}}^{1}, x_{s_{2}}^{2}, \ldots, x_{s_{2}}^{2^{n}}\right)$

$$
:=\left(x_{i}^{1}, x_{i}^{2}, \ldots, x_{i}^{2^{n}}\right), \quad i=1,2, \ldots, s_{2}
$$

one gets a constant map by Theorem 6.1. Thus, $\Phi(X)=X d \Phi(0)$.
EXAMPLE 6.1. The above theorem is not true for $n=1(s=1)$, i.e. in a complex analysis. Take for instance the map $\Phi: \mathbf{C} \rightarrow \mathbf{C}$ defined by $\Phi(z):=z^{2}$. It satisfies the assumption of the Theorem 6.1 but $\Phi$ is singular at 0 and it is not a constant map.

Putting $n=2$ we obtain the simplest, quaternionic form of Theorem 6.1, namely:

THEOREM 6.2. Let $U$ be an open neighbourhood of 0 in $\mathbf{R}^{4 s}$ and $\Phi: \mathbf{R}^{4 s} \rightarrow \mathbf{R}^{4}$ - a Clifford-type holomorphic map (quaternionic Q-holomorphic) with respect to the quaternionic structures $\left(I_{1}, I_{2}\right)$ and $\left(\tilde{I}_{1}, \tilde{I}_{2}\right)$, respectively. If $\Phi$ is singular at 0 then $\Phi$ is a constant map.

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# TIME-ANALYSIS OF PSEUDORANDOM BIT GENERATORS 

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## 1. Introduction

First we will recall several basic notions and definitions from cryptography. We will refer to the excellent monograph [1] repeatedly, all the quotations in the introduction are taken from [1]. The so called Vernam cipher, also described in [1], is a very simple but extremely important encrypting algorithm. Earlier it was used in form of the one-time pad, but nowadays it is much more frequently used in form of "... stream ciphers where the keystream is pseudorandomly generated from a smaller secret key, with the intent that the keystream appears random to a computationally bounded adversary". The algorithm constructing the keystream from the small secret key is called pseudorandom bit generator: "A pseudorandom bit generator (PRBG) is a deterministic algorithm which, given a truly random binary sequence of length $k$, outputs a binary sequence of length $l \gg k$ ( $l$ much greater than $k$ ) which "appears" to be random. The input to the PRBG is called the seed (produced by, say, coin toss, and passed the partner by public key cryptography) while the output of PRBG is called a pseudorandom (PR) bit sequence". The quality of the PRBG is usually characterized by using the terminology and tools of computational complexity: "A PRBG is said to pass the next-bit test if there is no polynomial-time algorithm which, on input of the first $l$ bits of an output sequence $s$, can predict the $(l+1)$ st bit of $s$ with probability significantly greater than $\frac{1}{2} \ldots$ A PRBG that passes the next-bit

[^2]test (possibly under some plausible but unproved mathematical assumption such as the intractability of factoring integers) is called a cryptographically secure pseudorandom bit generator (CSPRBG)."

However, recently this computational complexity approach has been criticized widely since it is of asymptotic nature only; it measures only the quality of the PRBG but not that of the individual sequences constructed; the non-existence of a polynomial time algorithm cannot be shown. Motivated by these problems, recently a more constructive approach has been developed which also avoids the use of any unproved hypotheses:

Consider a finite binary sequence $E_{N}=\left\{e_{1}, \ldots, e_{N}\right\} \in\{-1,+1\}^{N}$. In [2] Mauduit and Sárközy first introduced the following measures of pseudorandomness:

Write

$$
U\left(E_{N}, t, a, b\right)=\sum_{j=0}^{t-1} e_{a+j b}
$$

and, for $D=\left(d_{1}, \ldots, d_{k}\right)$ with non-negative integers $d_{1}<\cdots<d_{k}$,

$$
V\left(E_{N}, M, D\right)=\sum_{n=1}^{M} e_{n+d_{1}} e_{n+d_{2}} \cdots e_{n+d_{k}}
$$

Then the well-distribution measure of $E_{N}$ is defined as

$$
W\left(E_{N}\right)=\max _{a, b, t}\left|U\left(E_{N}, t, a, b\right)\right|=\max _{a, b, t}\left|\sum_{j=0}^{t-1} e_{a+j b}\right|
$$

where the maximum is taken over all $a, b, t \in \mathbb{N}$ such that $1 \leq a \leq a+$ $+(t-1) b \leq N$, while the correlation measure of order $k$ of $E_{N}$ is defined as

$$
C_{k}\left(E_{N}\right)=\max _{M, D}\left|V\left(E_{N}, M, D\right)\right|=\max _{M, D}\left|\sum_{n=1}^{M} e_{n+d_{1}} e_{n+d_{2}} \cdots e_{n+d_{k}}\right|
$$

where the maximum is taken over all $D=\left(d_{1}, \ldots, d_{k}\right)$ and $M$ such that $0 \leq d_{1}<\cdots<d_{k} \leq N-M$.

Then the sequence $E_{N}$ is considered as a "good" pseudorandom sequence if both these measures $W\left(E_{N}\right)$ and $C_{k}\left(E_{N}\right)$ (at least for "small" $k$ ) are "small" in terms of $N$ (in particular, both are $o(N)$ as $N \rightarrow \infty$ ). Indeed, later Cassaigne, Mauduit and Sárközy [6] showed that this terminology is
justified since for almost all $E_{N} \in\{-1,+1\}^{N}$, both $W\left(E_{N}\right)$ and $C_{k}\left(E_{N}\right)$ are less than $N^{\frac{1}{2}}(\log N)^{c}$.

Recently, large families of finite binary sequences have been constructed which possess "good" PR properties ("good" in this second, constructive sense).

In this paper my goal is to compare the two approaches described above from computational point of view. First in Section 2 I will present two of the most important PRBG's which are cryptographically secure in the computational complexity sense. Next in Section 3 I will present three constructions which are the best ones in the constructive sense. In Section 4 I will estimate and compare the running times of the 5 constructions in terms of the parameters involved. Finally, in Section 5 I will present the results of numerical computations in order to compare the 5 constructions for special values of the parameters.

## 2. The computational complexity approach: two cryptographically secure pseudorandom bit generators

In this section we present two especially important crypographically secure pseudorandom bit generators following [1]. The security of these generators relies on the presumed intractability of integer factorization. In the following algorithms there is a parameter $l$, which is the length of the pseudorandom binary sequence, and it is usually chosen as a power of the length of the dyadic representation of the modulus $\left(\sim \frac{\ln n}{\ln 2}\right)$.

### 2.1. RSA pseudorandom bit generator

The algorithm is the following:

1. Generate two secret RSA-type primes $p$ and $q$, and compute $n=p q$ and $\phi=(p-1)(q-1)$. Select a random integer $e, 1<e<\phi$, such that $\operatorname{gcd}(e, \phi)=1$.
2. Select a random integer $x_{0}$ (the seed) in the interval $[1, n-1]$.
3. For $i$ from 1 to $l$ do the following:
3.1. $x_{i}:=x_{i-1}^{e} \bmod n$.
3.2. Let $z_{i}$ be the least significant bit of $x_{i}$.
4. The output sequence is $z_{1}, z_{2}, \ldots, z_{l}$.

### 2.2. Blum-Blum-Shub pseudorandom bit generator

The algorithm is the following:

1. Generate two large secret random (and distinct) primes $p$ and $q$, each congruent to 3 modulo 4 , and compute $n=p q$.
2. Select a random integer $s$ (the seed) in the interval $[1, n-1]$ such that $\operatorname{gcd}(s, n)=1$, and compute $x_{0}:=s^{2} \bmod n$.
3. For $i$ from 1 to $l$ do the following:
3.1. $x_{i}:=x_{i-1}^{2} \bmod n$.
3.2. Let $z_{i}$ be the least significant bit of $x_{i}$.
4. The output sequence is $z_{1}, z_{2}, \ldots, z_{l}$.

We remark that the efficiency of these generators can be improved by extracting the $j$ least significant bits of $x_{i}$ in step 3.2., where $j=c \lg \lg n$ and $c$ is a constant. Provided that $n$ is sufficiently large, these modified generators are also cryptographically secure. Here we will analyse only the basic (unimproved) versions of the RSA and Blum-Blum-Shub pseudorandom bit generators.

## 3. The constructive approach: three constructions

As we mentioned in the introduction, a sequence $E_{N}$ is considered as a "good" pseudorandom sequence if the measures $W\left(E_{N}\right)$ and $C_{k}\left(E_{N}\right)$ (at least for small $k$ ) are "small" in terms of $N$.

It was shown in [2] that the Legendre symbol forms a "good" pseudorandom sequence. More exactly, let $p$ be an odd prime, $N=p-1$ and $E_{N}=\left\{e_{1}, \ldots, e_{N}\right\}$ is defined by $e_{n}=\left(\frac{n}{p}\right)$.

Then we have

$$
W\left(E_{N}\right) \ll p^{\frac{1}{2}} \log p \ll N^{\frac{1}{2}} \log N
$$

and

$$
C_{k}\left(E_{N}\right) \ll k p^{\frac{1}{2}} \log p \ll k N^{\frac{1}{2}} \log N .
$$

However, this construction produces only a "few" good binary sequences while in certain applications (e.g. in cryptography) one needs "large" families of "good" pseudorandom binary sequences. Now we will present three constructions of this type.

### 3.1. A further construction related to the Legendre symbol

In [3] Goubin, Mauduit and Sárközy extended the Legendre symbol construction to large families of binary sequences with good pseudorandom properties. The most important results in [3] can be combined in form of the following theorem:

Theorem 1. If $p$ is a prime number, $f(x) \in F_{p}[x]$ ( $F_{p}$ being the field of the modulo $p$ residue classes) has degree $k(>0)$ and no multiple zero in $\overline{F_{p}}$ (the algebraic closure of $F_{p}$ ), and the binary sequence $E_{p}=\left\{e_{1}, \ldots, e_{p}\right\}$ is defined by

$$
e_{n}= \begin{cases}\left(\frac{f(n)}{p}\right) & \text { for }(f(n), p)=1  \tag{1}\\ +1 & \text { for } p \mid f(n)\end{cases}
$$

then we have

$$
W\left(E_{p}\right)<10 k p^{\frac{1}{2}} \log p
$$

Morever, assume that for $l \in \mathbb{N}$ one of the following assumptions holds:
(i) $l=2$;
(ii) $l<p$, and 2 is a primitive root modulo $p$;
(iii) $(4 k)^{l}<p$.

Then we also have

$$
C_{l}\left(E_{p}\right)<10 k l p^{\frac{1}{2}} \log p
$$

Based on this theorem, they proposed the following algorithm for constructing pseudorandom binary sequences of a given length $p$ (where $p$ is prime):

Suppose a prime $p$ and an integer $L \in \mathbb{N}$ are given with

$$
L< \begin{cases}p & \text { for all } p, \\ \frac{p}{4} & \text { if } 2 \text { is not a primitive root modulo } p\end{cases}
$$

(and typically $L$ is "much smaller", than $\frac{p}{4}$, say, $L<p^{\frac{1}{4}}$ ). Suppose we want to control the correlation of order $l$ for all $l \leq L$.

Let $L \in \mathbb{N}$ be a "large" number but such that $k<\frac{\log p}{\log (4 L)}$ if 2 is not a primitive root modulo $p$. Write $t=\left[\frac{k}{2}\right]$. Then consider polynomial $g(x) \in$ $\in F_{p}[x]$ of the form

$$
g(x)=x^{k}+\sum_{i=0}^{t} a_{i} x^{i}
$$

where any coefficients $a_{i}$ can be chosen with $a_{i} \in \mathbb{Z}_{p}$ for $i=0,1, \ldots, t-1$ and $a_{t} \in \mathbb{Z}_{p} \backslash\{0\}$.

Let $d(x)$ denote the greatest common divisor of the polynomials $g(x)$ and $g^{\prime}(x)$, and compute

$$
f(x)=\frac{g(x)}{d(x)}=\frac{g(x)}{\left(g(x), g^{\prime}(x)\right)}
$$

Then computing the sequence $E_{p}$ defined as in Theorem 1, we obtain a "good" pseudorandom sequence $E_{p}$.

### 3.2. A construction using additive characters

Let $p$ be an odd prime number, $f(x) \in F_{p}[x]$, and define $E_{p}=\left\{e_{1}, \ldots, e_{p}\right\}$ by

$$
e_{n}= \begin{cases}+1 & \text { if } 0 \leq r_{p}(f(n))<\frac{p}{2}  \tag{2}\\ -1 & \text { if } \frac{p}{2} \leq r_{p}(f(n))<p\end{cases}
$$

where $r_{p}(n)$ denotes the unique $r \in\{0, \ldots, p-1\}$ such that $n \equiv r \bmod p$.
This sequence can be generated fast and the well-distribution measure and the correlations of "small" order are "small", which is formulated in the following theorem:

THEOREM 2. For $f \in F_{p}[x]$ of degree $d$ and $E_{p}=\left\{e_{1}, \ldots, e_{p}\right\}$ defined by (2), we have

$$
W\left(E_{p}\right) \ll d p^{\frac{1}{2}}(\log p)^{2}
$$

moreover, for $2 \leq l \leq d-1$ we have

$$
C_{l}\left(E_{p}\right) \ll d p^{\frac{1}{2}}(\log p)^{l+1}
$$

However we have to pay a certain price for the fast computability since the correlations of "large" order can be "large" as the corollary of the following theorem shows:

THEOREM 3. For any $k=2^{t}$ there exists a constant $c=c(k)>0$ such that if $p$ is a prime number large enough, $f \in F_{p}[x]$ is of degree $k$ and $E_{p}=\left\{e_{1}, \ldots, e_{p}\right\}$ is defined by (2), then

$$
\max _{\substack{T, M \\ 1 \leq T<T+M \leq p}}\left|\sum_{n=T}^{T+M} e_{n} e_{n+1} \cdots e_{n+k-1}\right| \geq c p
$$

COROLLARY 1. For any $k=2^{t}$, if $p$ is a prime number large enough, $f \in F_{p}[x]$ is of degree $k$ and $E_{p}=\left\{e_{1}, \ldots, e_{p}\right\}$ is defined by (2), then

$$
C_{k}\left(E_{p}\right) \gg p
$$

This construction is slightly weaker than the previous one which uses the Legendre symbol thus it provides a reasonable alternative to the previous construction only if it suffices to control the correlations of small order.

### 3.3. A construction using the multiplicative inverse

Define $r_{p}(n)$ as in Section 3.2. This construction is based on the use of the multiplicative inverse modulo $p$ :

THEOREM 4. Assume that $p$ is a prime number, $f(x) \in F_{p}[x]$ has degree $(0<) k(<p)$ and no multiple zero in $\overline{F_{p}}$. For $(a, p)=1$, denote the multiplicative inverse of a by $a^{-1}$ :

$$
a a^{-1} \equiv 1 \bmod p
$$

Define the binary sequence $E_{p}=\left\{e_{1}, \ldots, e_{p}\right\}$ by

$$
e_{n}= \begin{cases}+1 & \text { if }(f(n), p)=1, r_{p}\left(f(n)^{-1}\right)<\frac{p}{2}  \tag{3}\\ -1 & \text { if either }(f(n), p)=1, r_{p}\left(f(n)^{-1}\right)>\frac{p}{2} \text { or } p \mid f(n)\end{cases}
$$

Then we have

$$
W\left(E_{p}\right) \ll k p^{\frac{1}{2}}(\log p)^{2}
$$

Moreover, assume that $l \in \mathbb{N}$ with $2 \leq l \leq p$, and one of the following conditions holds:
(i) $l=2$;
(ii) $(4 k)^{l}<p$.

Then we also have

$$
C_{l}\left(E_{p}\right) \ll k l p^{\frac{1}{2}}(\log p)^{l+1}
$$

This sequence can be generated fast since we can calculate the multiplicative inverse quickly (in polinomial-time), but the nice sufficient condition in Theorem 1 that 2 is a primitive root modulo $p$ is missing now.

## 4. The theoretical running times of the considered pseudorandom bit generators

I will calculate the theoretical running time of the considered pseudorandom bit generators for fixed seed length.

In all of the considered five cases, first we choose one or two random numbers from a given interval, and then we look for the first primes after these random numbers. In case of the Blum-Blum-Shub PRBG we have to choose the first prime congruent to 3 modulo 4 , and in case of the construction related to the Legendre symbol we have to choose the first prime $p$, such that 2 is primitive root modulo $p$. Assuming that we have a primetable (in case of the Blum-Blum-Shub PRBG a table of the primes of form $4 k-1$, and in case of the Legendre symbol construction a table of the primes $p$, such that 2 is a primitive root modulo $p$, respectively), we may neglect the time needed for selecting the suitable primes.

As usually, we measure the running time of an algorithm in terms of the number of bit operations performed. We obtain the running time of a pseudorandom bit generator by adding the running times of the algorithms involved. We will neglect the time of having access to the memory, the shifting operations, the copying, the printing, etc.

### 4.1. RSA pseudorandom bit generator

Suppose that the length of the seed is $s$ and that we have already chosen two random primes, $p$ and $q$ of size $O\left(2^{s}\right)$. Then the time of calculating of the products $n=p \cdot q$ and $F=(p-1) \cdot(q-1)$, is $O((\log p)(\log q))$ [8, p. 4], i.e. $O\left(s^{2}\right)$.

Choosing a random number between 1 and $F=(p-1)(q-1)$, it will be relative prime to $F$ with probability $\frac{\varphi((p-1)(q-1))}{(p-1)(q-1)}$. So the running time of choosing $e$ (the number between 1 and $F$, such that relative prime to $F$ ) is
expected to be $O(\log (\log ((p-1)(q-1)))$ ), i.e. $O(\log s)$. (In the practice the integers in $(1, F)$ which are coprime with $F$ are usually given, we choose one of them randomly, and we neglect the time needed for this.)

We prepare the seed by coin toss and of course we determine the interval, from where we choose the primes, in such a way that the seed should be smaller than $n$.

We have to carry out step $3.1\left(x_{i}:=x_{i-1}^{e} \bmod n\right) t$ times (for example $t=s^{2}$ ), and the time of each step with using the Power Algorithm [7, p. 102] is $O\left((\log e)(\log n)^{2}\right)$, i.e. $O\left(t s^{3}\right)$. This is the most critical and slowest step in the algorithm. Then the complexity of taking the least significant bit of $x_{i}$ can be neglected, so the total running time of the RSA pseudorandom bit generator is $O\left(t s^{3}\right)$. In the special case of $t=s^{2}$, to be studied later, this is $O\left(s^{5}\right)$.

### 4.2. Blum-Blum-Shub pseudorandom bit generator

Suppose that the length of the seed is $s$ and that as we wrote it earlier we have already chosen two primes, $p$ and $q$ of size $O\left(2^{s}\right)$. We can calculate the product $n=p \cdot q$ in time $O\left(s^{2}\right)$ again as at the RSA PRBG.

We choose the limits of the interval here again so that the seed should be smaller than $n$. Let us denote the seed in the decimal system by $D$. We can calculate the greatest common divisor of $D$ and $n$ in time $O\left(s^{2}\right)$, and the initial element of the recursion $\left(x_{0}:=D^{2} \bmod n\right)$ in time $O\left((\log 2)(\log n)^{2}\right)$, i.e. $O\left(s^{2}\right)$. Here we use the Power Algorithm again, and so the time of the complete recursion is $t \cdot O\left(s^{2}\right)$, i.e. $O\left(t s^{2}\right)$. We take no notice of the time of the least significant bit's choice. So the total running time of the Blum-Blum-Shub pseudorandom bit generator is $O\left(t s^{2}\right)$. In the special case of $t=s^{2}$, to be studied later, this is $O\left(s^{4}\right)$.

### 4.3. The construction related to the Legendre symbol

Suppose that the length of the seed is $s$ and that we have already chosen a random prime $p$ of size $O\left(\left(\frac{s}{\log s}\right)^{4}\right)$ such that 2 is a primitive root modulo $p$ as we wrote at the beginning of Section 4 . We prepare the seed (the coefficients of the polinomial $f$ ) by coin toss. Since having the polinomial $f$ in special
form, we compute the derivative of $f$ in time $O(s \cdot \log s)$, then we determine the greatest common divisor of the polynomials $f$ and $f^{\prime}$ in time $O\left(s^{2}\right)$ by using Euclidean algorithm [7, p. 128]. For determining each pseudorandom bit first we compute the value of $x_{i}=f(i) \bmod p$ in time $O(s \cdot \log s)$ by using Horner's scheme [8, p. 4]. And if this value is not equal to zero, then we compute the Legendre symbol $\left(\frac{x_{i}}{p}\right)$ in time $O\left(\left(\log x_{i}\right)(\log p)\right)$ [7, p. 113], i.e. $O\left((\log s)^{2}\right)$. For constructing a pseudorandom binary sequence of length $t$ we have to perform this step $t$ times, so the total running time of this construction is $O(t \cdot s \cdot \log s)$. In the special case of $t=s^{2}$, to be studied later, this is $O\left(s^{3} \cdot \log s\right)$.

### 4.4. A construction using additive characters

Suppose that the length of the seed is $s$ and that we have already chosen a random prime $p$ of size $O\left(\left(\frac{s}{\log s}\right)^{4}\right)$. We prepare the seed (the coefficients of the polinomial $f$ ) by coin toss. In each step we compute the value of $x_{i}=f(i) \bmod p$ in time $O(s \cdot \log s)$ by using Horner's scheme [8, p. 4] again. We can decide in time $O(\log s)$ whether $x_{i}-\frac{p}{2}$ is negative or not. For constructing a pseudorandom binary sequence of length $t$ we perform these steps $t$ times, so the total running time of this construction is $O(t \cdot s \cdot \log s)$. In the special case of $t=s^{2}$, to be studied later, this is $O\left(s^{3} \cdot \log s\right)$.

### 4.5. A construction using the multiplicative inverse

This construction differs from the construction related to the Legendre symbol just in the last step, so when we have already determined the value of $x_{i}=f(i) \bmod p$ (in time $O(s \cdot \log s)$ ), we compute the multiplicative inverse of $x_{i}$ modulo $p$ in time $O\left((\log s)^{3}\right)$ [8, p. 19]. Finally we can decide whether this value is smaller than $\frac{p}{2}$ or not in time $O(\log s)$. For constructing a pseudorandom binary sequence of length $t$ we perform these steps $t$ times, so the total running time of this construction is $O(t \cdot s \cdot \log s)$. In the special case of $t=s^{2}$, to be studied later, this is $O\left(s^{3} \cdot \log s\right)$.

## 5. The running times of the different constructions and their pseudorandom measures

In Section 4 we estimated the theoretical running times by using Landau's big- $O$ notation. However, it might be troublesome to compute the numerical values of the constant implied by this big- $O$ notation. It may occur that in some cases this constant is so big that it influences the quality of the bounds obtained for the pseudorandom measures. Thus it is advisable to carry out numerical computations as well.

First I wrote a program for the two cryptographically secure pseudorandom bit generators mentioned in the second chapter, then for the three constructions mentioned in the third chapter.

While in the case of the RSA and the Blum-Blum-Shub PRBG we have to choose two secret random primes of the order of magnitude of $10^{19}$, and I constructed a pseudorandom binary sequence of length $t=s^{2}=16384$ (where $t, s$ are the parameters introduced in the previous section), while in case of the constructions mentioned in the third chapter only one random prime of the order of magnitude of $10^{5}$ is chosen. As a reasonable choice of the parameter $t$, I will set $t=s^{2}$ throughout all the numerical computations. For each of the five constructions, I used a seed of length 128 prepared by coin toss (in the case of the constructions described in the third chapter I fixed a few bits of the coefficients in order to reduce the number of the bits to be chosen to this common length 128), and I constructed a pseudorandom binary sequence of length $128^{2}=16384$. So in the case of the RSA and the Blum-Blum-Shub PRBG I ran the programs for 50 pairs of primes $p, q$ of order of magnitude of $10^{19}$ so that $p, q$ are close, while in case of the constructions mentioned in the third chapter for 50 primes of the order of magnitude of $10^{5}$, and I calculated the average running time for each of the pseudorandom bit generators. The results (the average, the best and the worst running times for the considered PRBGs) are in the next table:

|  | average | best | worst |
| :--- | :---: | ---: | ---: |
| RSA | 413.335 sec. | 373.9 sec. | 469.4 sec. |
| Blum-Blum-Shub | 69.14 sec. | 61.7 sec. | 94.7 sec. |
| Legendre symbol | 3.495 sec. | 3.4 sec. | 3.6 sec. |
| additive characters | 1.165 sec. | 1.1 sec. | 1.2 sec. |
| multiplicative inverse | 5.715 sec. | 5.3 sec. | 6.7 sec. |

The three constructions described in Chapter 3 are much faster than the RSA and Blum-Blum-Shub pseudorandom bit generators.

Then for each of the five constructions and for three different choices of the parameters I computed the PR measures by the same programs.

The next table shows the well-distribution measures and the correlation measures of order 2 for the binary sequences produced by the PRBGs studied:

|  | well-distribution | correlation |
| :--- | :---: | :---: |
| RSA | $187 ; 134 ; 163$ | $414 ; 394 ; 521$ |
| Blum-Blum-Shub | $199 ; 116 ; 117$ | $483 ; 464 ; 443$ |
| Legendre symbol | $163 ; 327 ; 137$ | $505 ; 446 ; 485$ |
| additive characters | $233 ; 194 ; 116$ | $6305 ; 3905 ; 5033$ |
| multiplicative inverse | $162 ; 214 ; 175$ | $449 ; 467 ; 570$ |

(The running times were between 3830.3 sec . and 6702.6 sec . for the welldistribution and between 2931 sec . and 4051.6 sec . for the correlation measure.)

The values of the PR measures are similar for the five constructions (but we can prove good upper bounds only for the ones studied in the third chapter), apart from that in case of the additive character construction the value of the correlation is greater by a factor about 10 than in the other cases. The reason of this is not quite clear; it is possible that this difference would disappear for greater values of $s$ or at least would become less significant.

Furthermore, one binary sequence out of 50 generated with the Blum-Blum-Shub PRBG and RSA PRBG, respectively, didn't pass one of the most basic statistic test, namely the serial test, which warns that before using the binary sequences obtained by these constructions it is advisable to test them.

We may conclude that the new constructions described in the third chapter are superior to the usual cryptographically secure pseudorandom bit generators at least in certain situations and it is worth to look for further "good" constructions.

## 6. Acknowledgements

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# LIFE AND WORK OF FRIGYES RIESZ 

By<br>ÁKOS CSÁSZÁR*

We commemorate this year the 125th anniversary of the birth of two brilliant researchers of Hungarian mathematics: Lipót Fejér and Frigyes Riesz (this is the Hungarian form of his Christian name, the better known form is the French one (Frédéric) as it has been used in most of the publications of Frigyes Riesz). Our purpose is to give a short survey on the life and work of this latter.

Frigyes Riesz was born on the 22nd of January 1880 in Győr, a city approximately on the middle way between Budapest and Vienna. His father was a physician; he gave a very careful education to all three of his sons and as a result of this two of them, Frigyes and his younger brother Marcell (Marcel in its more often used form) developed to excellent researchers in mathematical analysis.

Frigyes Riesz learned until the age of 18 years in his native town Győr. After this, he began to study in Switzerland, at the Eidgenössische Technische Hochschule in Zürich. However, after a few semesters, he remarked that the technical branches of his studies are much less interesting for him as mathematics, so that he decided to continue his studies from 1899 as a student of mathematics at the University of Budapest. In these years, there were two eminent professors of mathematics at this university, Gyula Kőnig and József Kürschák, and their influence surely helped the evolution of Frigyes Riesz. He also spent a year in Göttingen, under the influence of David Hilbert and Hermann Minkowski, and finally he finished his studies in 1902 in Budapest by earning a diploma of a high-school teacher in mathematics and physics

[^3]and very soon later a doctor's degree based on a dissertation on projective geometry.

In the following years, he worked as a high-school teacher in a town of northern Hungary, Lőcse (nowadays Levoča in Slovakia). After a short time, from 1908, he continued his teaching work in Budapest. He performed his research work in several directions: geometry, topology (in his formulation at that time, analysis situs), and other subjects. However, the most important feature of his work was the fact that he thoroughly studied the investigations of the French school of analysis, the works of Lebesgue, Borel, Fatou, etc. As a result, he began to publish from 1906 papers on questions of analysis in the spirit of this French school.

In fact, it is an imperishable merit of the history of mathematics in Hungary that we had a researcher, namely Frigyes Riesz, who, a few years after its birth, not only maked himself master of Lebesgue's theory of measure and integral, but who published in March of 1907 a classical work (Sur les systèmes orthogonaux de fonctions, Comptes Rendus (Paris), 144 (1907), 615-619) containing the proof of the so called Riesz-Fischer theorem.

In order to formulate the content of the Riesz-Fischer theorem, let us first consider the function class $L^{2}=L^{2}(a, b)$, i.e. the collection of all functions $f$ integrable in the sense of Lebesgue in the interval $[a, b]$ with the property that $f^{2}$ is integrable as well. Consider also an orthonormal sequence $\left(\varphi_{n}\right)$ in $[a, b]$, i.e. a sequence of functions $\varphi_{n} \in L^{2}$ such that

$$
\int_{a}^{b} \varphi_{n} \varphi_{m}=0
$$

whenever $n \neq m$ and

$$
\int_{a}^{b} \varphi_{n}^{2}=1
$$

for each $n$. The integrals in question surely exist because it is not difficult to prove that $f g$ is integrable whenever $f, g \in L^{2}$. (Here and in the following "integrable" always means "integrable in the Lebesgue sense".)

If $\left(\varphi_{n}\right)$ is such an orthonormal sequence and $f \in L^{2}$, we can speak of the orthonormal series or Fourier series of $f$; this is the series $\sum_{0}^{\infty} c_{n} \varphi_{n}$ composed with the coefficients

$$
c_{n}=\int_{a}^{b} f \varphi_{n}
$$

i.e. the Fourier coefficients of $f$ with respect to the sequence $\left(\varphi_{n}\right)$. These coefficients necessarily fulfil the Bessel inequality

$$
\sum_{0}^{\infty} c_{n}^{2} \leq \int_{a}^{b} f^{2}
$$

proved by a rather easy calculation.
Now the purpose of the Riesz-Fischer theorem is to answer the following question: Given a numerical sequence $\left(c_{n}\right)$, what is the condition of the existence of a function $f \in L^{2}$ such that the numbers $c_{n}$ coincide with the Fourier coefficients of $f$ ? The paper of Riesz quoted above states that the condition " $\sum_{0}^{\infty} c_{n}^{2}$ is convergent", necessary according to Bessel's inequality, is also sufficient.

The method of proof used by Riesz reduces the problem to the case of the classical orthonormal sequence composed, in the interval $[0,2 \pi]$, of

$$
1 / \sqrt{2 \pi}, \quad \cos n x / \sqrt{\pi}, \quad \sin n x / \sqrt{\pi} \quad(n=1,2, \ldots)
$$

and then proves an auxiliary theorem providing a certain condition for the integration of sequences:

If $f_{n} \in L^{2}(a, b)$ and $f_{n} \rightarrow f$ almost everywhere (a.e.) in [a, $\left.b\right]$, moreover the integrals

$$
\int_{a}^{b} f_{n}^{2}
$$

are bounded then $f \in L^{2}$ and

$$
\int_{a}^{b} f_{n} \rightarrow \int_{a}^{b} f
$$

A.e. means of course "with the exception of a set of measure zero".

The expression "Riesz-Fischer theorem" shows that the same result has been obtained, independently of Riesz, by the Austrian mathematician Ernst Fischer (Sur la convergence en moyenne, Comptes Rendus (Paris), 144 (1907), 1022-1024), published in the same Comptes Rendus in May, 1907, where the paper of Riesz appeared in March, 1907. The title of Fischer's paper shows that he uses another way for proving the theorem, namely the
concept of "convergence in mean". This is obtained if we introduce, for a function $f \in L^{2}(a, b)$, the norm

$$
\|f\|=\left(\int_{a}^{b} f^{2}\right)^{1 / 2}
$$

and define the convergence in mean of a sequence $\left(f_{n}\right) \in L^{2}$ to $f$ by the formula

$$
\left\|f_{n}-f\right\| \rightarrow 0
$$

It is not quite obvious that the result proved by Fischer is equivalent to the one proved by Riesz; however, for Riesz, this fact was well-known.

$$
* * *
$$

It was evident for the mathematicians in the first decade of the 20th century that the author of a paper considering so fundamental facts and methods of Lebesgue's theory of integration is worthy of being applied in the higher education. Consequently Frigyes Riesz obtained at the university of Kolozsvár (then in Transsylvania in East Hungary, now Cluj-Napoca in Roumania) an extraordinary (i.e. associate) professorship in 1912 and an ordinary (i.e. full) professorship in 1914. A few years later, in 1916, he was elected a corresponding member of the Hungarian Academy of Sciences.

The publications of Frigyes Riesz between 1905 and 1920 made clear that he is one of the fathers of a new branch of mathematics, called functional analysis nowadays. Although functional analysis had several roots in the 19th century, such as Fourier expansion of functions and spectral theory of some differential equations, its genesis could be put in the 20th century. Even at the end of the 19th century, linear algebra was very finite dimensional and dealt with $n$-tuples of real numbers, and Fréchet's thesis on metric spaces appeared in 1906. The crystalization of ideas was catalyzed by Lebesgue's integration theory. Many of the basic concepts of functional analysis were born in the setting of infinite dimensional function spaces. The intimate relation of the Lebesgue integral and infinite dimensional functional analysis is very transparent in the work of Frigyes Riesz.

The space $L^{2}$ of square integrable functions on an interval of the real line was the first infinite dimensional space on which functional analysis in the modern sense was studied. The Riesz-Fischer theorem, in Fischer's version, claims that the space $L^{2}$ is complete, that is, all Cauchy sequences
are convergent in the $L^{2}$-sense. Riesz's version gives a condition on Fourier coefficents, but Riesz was aware of the equivalence, as we told it above.

Another early paper of Riesz (Sur une espèce de géométrie analytique des sy-stèmes de fonctions sommables, Comptes Rendus (Paris), 144 (1907), 1409-1411) tells us that any bounded linear functional $A$ of $L^{2}$ is induced by an element $g$ of $L^{2}$ in the form of integration of the product:

$$
\begin{equation*}
A(f)=\int_{a}^{b} f g \tag{1}
\end{equation*}
$$

for some $g \in L^{2}$ if there exists a constant $M$ such that

$$
\int_{a}^{b}|f|^{2} \leq 1 \quad \text { implies } \quad|A(f)| \leq M .
$$

In 1910 Riesz published a paper in Hungarian (Integrálható függvények sorozatai (Sequences of integrable functions), Math. Phys. Lapok, 19 (1910), 165-182) in which he explains his understanding of $L^{2}$ and the above mentioned two results. In this paper he defines the weak topology on the space $L^{2}$ and shows that a bounded sequence contains a weakly convergent subsequence.

In our present language, (1) describes the dual of the $L^{2}$ space. The dual space was certainly a concept that should be attributed to Riesz. He defined the dual of $L^{2}$ in 1907, and in 1909 he dealt with the dual of the space of continuous functions (Sur les opérations fonctionnelles linéaires, Comptes Rendus (Paris), 149 (1909), 974-977). Let $C(a, b)$ denote the set of all continuous real-valued functions in the interval $[a, b]$. In 1903 Hadamard wanted to describe all linear functionals $U: C(a, b) \rightarrow \mathbb{R}$ such that $U f_{n} \rightarrow U f$ whenever $f_{n} \rightarrow f$ uniformly. He obtained a rather complicated result; Riesz reached a much simpler result by describing the continuous linear functionals $U$ by means of a Stieltjes integral. He proved that there exists a function $\alpha$ of bounded variation such that

$$
\begin{equation*}
U(f)=\int_{a}^{b} f(x) d \alpha(x) \tag{2}
\end{equation*}
$$

moreover, $\alpha$ is unique if $\alpha(a)=0$ and the left continuity of $\alpha$ is required.
Extending his work on the space $L^{2}$, Riesz devoted a fundamental paper to $L^{p}$ spaces in 1910 (Untersuchungen über Systeme integrierbarer Funktionen, Math. Annalen, 69 (1910), 449-497). $L^{p}$ is the set of all complex valued
measurable functions $f$ such that $|f|^{p}$ is integrable. He restricted himself to the case $p>1$ and extended the Hölder and Minkowski inequalities

$$
\begin{gathered}
\left|\sum_{k=1}^{n} a_{k} b_{k}\right| \leq\left(\sum_{k=1}^{n}\left|a_{k}\right|^{p}\right)^{1 / p}\left(\sum_{k=1}^{n}\left|b_{k}\right|^{q}\right)^{1 / q}\left(\frac{1}{p}+\frac{1}{q}=1\right) \\
\left(\sum_{k=1}^{n}\left|a_{k}+b_{k}\right|^{p}\right)^{1 / p} \leq\left(\sum_{k=1}^{n}\left|a_{k}\right|^{p}\right)^{1 / p}+\left(\sum_{k=1}^{n}\left|b_{k}\right|^{p}\right)^{1 / p}
\end{gathered}
$$

to measurable functions. If $f \in L^{p}$ and $g \in L^{q}$, then $f g$ is integrable and

$$
\left|\int_{a}^{b} f g\right| \leq\left(\int_{a}^{b}|f|^{p}\right)^{1 / p}\left(\int_{a}^{b}|g|^{q}\right)^{1 / q}
$$

Moreover, if $f, g \in L^{p}$, then $f+g \in L^{p}$ and

$$
\left(\int_{a}^{b}|f+g|^{p}\right)^{1 / p} \leq\left(\int_{a}^{b}|f|^{p}\right)^{1 / p}+\left(\int_{a}^{b}|g|^{p}\right)^{1 / p}
$$

Riesz also extended several definitions and results from the theory of $L^{2}$ spaces. He defined strong convergence in $L^{p}$ as $f_{n} \rightarrow f$ if and only if

$$
\int_{a}^{b}\left|f_{n}-f\right|^{p} \rightarrow 0
$$

He said that $f_{n} \rightarrow f$ weakly if

$$
\int_{a}^{t} f_{n} \rightarrow \int_{a}^{t} f
$$

for all $t$ in $[a, b]$. He showed that this is equivalent to

$$
\int_{a}^{b}\left(f-f_{n}\right) g \rightarrow 0 \quad \text { for all } \quad g \in L^{q}
$$

He proved the weak compactness of the unit ball of $L^{p}$ and he was particularly interested in the solution of the infinite system of linear equations

$$
\begin{equation*}
\int_{a}^{b} f_{i} \xi=c_{i} \tag{3}
\end{equation*}
$$

where $\xi$ is the unknown and the $f_{i}$ 's belong to $L^{q}$; the subscript $i$ can run over an arbitrary set, countable or not. One cannot give an easy condition for
the existence of the solution; however in the case when the functions $f_{i}$ run over all elements in $L^{q}$ then the condition is exactly the boundedness of the functional $A$ defined as $A\left(f_{i}\right)=c_{i}$, and he discovered that the dual of $L^{q}$ can be identified with $L^{p}$. By this, he obtained an example of what we call today reflexive Banach space.

Besides different papers, Riesz also wrote in these early years a monograph "Les systèmes d'équations linéaires à une infinité d'inconnues" (Paris, Gauthier-Villars, 1913), on problems related to the last mentioned one.

*     *         * 

By the end of World War I, a considerable part of Hungary was annexed to other countries; in particular, the city Kolozsvár went to Roumania and by this the Hungarian University had to end its activity. All professors, among them Frigyes Riesz and Alfréd Haar leaved Kolozsvár and continued their work in the city Szeged in the southern part of the remaining Hungary. There no university was active previously and, in particular, there was no scientific library. It was a remarkable success that Riesz and Haar founded a new periodical, Acta Scientiarum Mathematicarum or Acta Szeged in its short name, edited since 1922, and a lot of good journals accepted the exchange with it so that a mathematical library could be maintained. In this way, Szeged developed in a short time to an important centre of mathematical research.

Frigyes Riesz, in his research work, continued to reach various applications of Lebesgue's theory of integration. However, he was aware of the fact that Lebesgue's method for introducing the concept of the integral, based on the theory of measure, is rather long, and he looked almost immediately for other, shorter and easier, hence more elementary ways to reach the Lebesgue integral. He published a very short sketch of his method as early as 1912 (Sur quelques points de la théorie des fonctions sommables, Comptes Rendus (Paris), 154 (1912), 641-643). This method also furnished a short proof of Osgood's theorem having a quite classical character (Végtelen sorozatok integrálásáról, Math. Phys. Lapok 26 (1917), 67-73; Über Integration unendlicher Folgen, Jahresber. Deutsch. Math.-Vereinigung 26 (1918), 274278): if a sequence ( $f_{n}$ ) of continuous functions is bounded and converges to a continuous function $f$ in an interval $[a, b]$ then $\int_{a}^{b} f_{n} \rightarrow \int_{a}^{b} f$.

Finally, in 1920, a detailed paper dedicated to the elementary setting up of Lebesgue's integral theory has been published (Sur l'intégrale de Lebesgue, Acta Math. 42 (1920), 191-205). The method is based on a completely elementary particular case of Lebesgue measure, namely on the definition and simplest properties of the sets of measure zero (briefly null sets): a set $A \subset \mathbb{R}$
is a null set if and only if it can be covered, for an arbitrary $\varepsilon>0$, with a sequence of intervals $\left[a_{n}, b_{n}\right]$ such that $\sum_{1}^{\infty}\left(b_{n}-a_{n}\right)<\varepsilon$. A statement is true a.e. if it is true everywhere with the exception of the points of a null set.

The starting point is the, still quite elementary, definition of the integral of simple functions, i.e. functions in $[a, b]$ such that there is a decomposition of $[a, b]$ to finitely many pairwise disjoint subintervals in each of which the function is constant. For simple functions, the (Riemann) integral can be given by a finite sum.

Now a function $f$, bounded in $[a, b]$, is said to be integrable if and only if there exists a bounded sequence of simple functions $f_{n}$ (i.e. $\left|f_{n}\right| \leq M$ for some $M$ ) such that $f_{n} \rightarrow f$ a.e. in $[a, b]$. It can be shown that, under these conditions, the integrals $\int_{a}^{b} f_{n}$ converge to a limit depending only on $f$ (i.e. independent of the sequence); this limit defines $\int_{a}^{b} f$.

In order to define the integral of an unbounded function, let us first say that a (bounded or unbounded) function is measurable in $[a, b]$ if and only if it is the limit of an a.e. convergent sequence of simple functions. The integral $\int_{a}^{b} f$ of a measurable function $f$ is defined as the limit of the sequence $\int_{a}^{b} f_{c_{n}, d_{n}}$ where $\left(c_{n}\right)$ is an arbitrary sequence tending to $-\infty$ and $\left(d_{n}\right)$ one tending to $+\infty$, while $f_{c_{n}, d_{n}}$ is the function equal to $f(x)$ if $c_{n} \leq f(x) \leq d_{n}$, to $c_{n}$ if $f(x)<c_{n}$ and to $d_{n}$ if $f(x)>d_{n}$; the function is said to be integrable if and only if the above limit exists, is finite and independent of the choice of the sequences $\left(c_{n}\right)$ and $\left(d_{n}\right)$.

Based on these definitions, it is not difficult to deduce the usual properties of the integral (linearity, theorems on the integration of sequences of functions, etc.). It can be easily shown that, for a function integrable in the Riemann sense, the new integral exists and is equal to the Riemann integral.

In two papers (Sur le théorème de M . Egoroff et sur les opérations fonctionnelles linéaires, Acta Sci. Math. 1 (1922-23), 18-25; Elementarer Beweis des Egoroffschen Satzes, Monatshefte Math. Phys. 35 (1928), 243-248), Riesz presents an analysis of the role of Egoroff's theorem according to which a convergent sequence of measurable functions is uniformly convergent on a subset obtained by elimination of a subset of arbitrarily small measure, in the theory of the Lebesgue integral, and, in particular, he indicates the modifications necessary for extending the theorem for applications in the theory of the Lebesgue-Stieltjes integral; of course, the Stieltjes integral attired the attention of Riesz because, we have seen, it plays a decisive role in the integral representation of bounded linear operations in the space $C(a, b)$.

In three small papers (Sur les valeurs moyennes des fonctions, Journ. London Math. Soc. 5 (1930), 120-121; Sur une inégalité intégrale, Journ. London Math. Soc. 5 (1930), 122-128); Sur un théorème de maximum de MM. Hardy et Littlewood, Journ. London Math. Soc. 7 (1931), 10-13), parts of his letters written to G.H. Hardy, Riesz gives simple proofs for some integral inequalities; in general, the argument is based on the use of the function $m(y)=m(\{x \in[a, b]: f(x)<y\})$ associated with a function $f$ measurable in $[a, b]$, here $m$ denotes the Lebesgue measure. In the third paper, he uses the so called Riesz lemma.

The last mentioned result in Frigyes Riesz' rich work on Real Analysis does not touch the theory itself of Lebesgue integral but one of its famous applications; it is an elementary proof of the theorem of Lebesgue according to which a monotone function is differentiable a.e. (A monoton függvények differenciálhatóságáról, Mat. Fiz. Lapok 38 (1931), 125-131; Sur l'existence de la dérivée des fonctions monotones et sur quelques problèmes qui s'y rattachent, Acta Sci. Math. 5 (1930-32), 208-221; Sur l'existence de la dérivée des fonctions d'une variable réelle et des fonctions d'intervalle, Verhandl. Internat. Math. Kongr. Zürich (1932), 258-269). Riesz' proof is based on a completely elementary statement that was called in the literature, very soon after its publication, as Riesz lemma.

In its simplest form, i.e. formulated for continuous functions, the Riesz lemma is so elementary that its proof can be added to this historical survey.

RIESZ LEMMA. If $f$ is continuous in the interval $[a, b]$ and we denote by $H$ the subset composed of those points $x \in[a, b]$ for which there exists some point $x<x^{\prime} \leq b$ such that $f(x)<f\left(x^{\prime}\right)$ then $H$ is an open set, consequently we can write $H=\bigcup\left(a_{k}, b_{k}\right)$ (the number of the pairwise disjoint members being either finite or countably infinite) and we have $f\left(a_{k}\right) \leq f\left(b_{k}\right)$ for each $k$.

Proof. The set $H$ can be empty; in this case we have nothing to prove. If $H \neq \emptyset$, it is clearly open by the continuity of $f$ so that the representation $H=\bigcup\left(a_{k}, b_{k}\right)$ is clearly possible. Fix a $k$ and consider $a_{k}<x<b_{k}$. Let $x_{0}$ be one of the points in the interval $[x, b]$ where the value of $f$ is maximal. Then $x \leq x_{0}<b_{k}$ is impossible since it would imply $x_{0} \in H$ and the existence of an $x_{0}<x^{\prime} \leq b$ satisfying $f\left(x_{0}\right)<f\left(x^{\prime}\right)$. Thus $b_{k} \leq x_{0} \leq b$ and then $f\left(b_{k}\right) \geq f\left(x_{0}\right)$ as $b_{k} \notin H$. On the other hand, $f(x) \leq f\left(x_{0}\right)$ by the choice of $x_{0}$, hence $f(x) \leq f\left(b_{k}\right)$ and, from the continuity of $f, x \rightarrow a_{k}$ implies $f\left(a_{k}\right) \leq f\left(b_{k}\right)$.

It is worth mentioning that the original proof due to Frigyes Riesz was slightly more complicated; the idea of applying the above point $x_{0}$ is due to his brother Marcell Riesz (1886-1969).

The proof of Lebesgue's theorem on the a.e. differentiability of a monotone function, based on the Riesz lemma, is nearly completely elementary because the covering of the sets that are of measure 0 according to the theorem with the help of a sequence of intervals having an arbitrary small sum of lengths is, using the lemma, quite automatical.

Frigyes Riesz himself was aware of the fact that this lemma can be used for proving further interesting theorems of Measure Theory (Sur les points de densité au sens fort, Fundam. Math. 22 (1934), 221-225) and even of Ergodic Theory (Az ergodikus elmélet néhány kérdéséről, Mat. Fiz. Lapok 49 (1942), 34-62; Sur la théorie ergodique, Comment. Math. Helv. 17 (1944-45), 221-239); On a recent generalization of G.D. Birkhoff's ergodic theorem, Acta Sci. Math. 11 (1948), 193-200).

The fact that Lebesgue's theorem on the differentiability of monotone functions obtained an elementary proof through the Riesz lemma, awoke in Frigyes Riesz the idea to present a new foundation of Lebesgue's integral theory, based precisely on the differentiability of monotone functions. In two papers (A Lebesgue-féle integrálról, mint a differenciálás múveletének megfordításáról, Mat. Fiz. Lapok 42 (1935), 1-24; Sur l'intégrale de Lebesgue comme l'opération inverse de la dérivation, Annali Pisa (2) 5 (1936), 191212), he describes this kind of setting up the theory.

The starting point is the following observation:
Theorem. Let $f \geq 0$ in the interval $[a, b]$ and suppose that there exists a function $F$, increasing in $[a, b]$ and satisfying $F^{\prime}(x)=f(x)$ a.e. in $(a, b)$. Then there exists, among these $F$, one for which the difference $F(b)-F(a)$ is the smallest possible.

After having proved the above remark, we can say that $f \geq 0$ is integrable in $[a, b]$ if and only if there is an $F$ with the above property, and define $\int_{a}^{b} f$ to be equal to $F(b)-F(a)$ for this extremal function $F$. A function $f$ of arbitrary sign is said to be integrable if and only if $f^{+}=\max (f, 0)$ and $f^{-}=$ $=-\min (f, 0)$ are integrable and then we define $\int_{a}^{b} f=\int_{a}^{b} f^{+}-\int_{a}^{b} f^{-}$. From these definitions, one can deduce without any difficulty the usual properties of the integral, e.g. the theorems on the integration of sequences of functions.

Besides the just mentioned results concerning Real Analysis, Riesz has maked important discoveries in several other branches of mathematics. In
topology, some of his early papers (A térfogalom genesise, Math. Phys. Lapok 15 (1906), 97-122; 16 (1907), 145-161; Die Genesis des Raumbegriffes, Math. Naturwiss. Berichte Ungarn 24 (1907), 309-353) can be considered as precursors of the theory of proximity spaces elaborated be Smirnov after the second war. He published an elegant proof of the fundamental theorem on Jordan curves (A Jordan-féle görbetételről, Mat. Term.tud. Ért. 57 (1938), 477-487); Sur le théorème de Jordan, Acta Sci. Math. 9 (1939), 154-162).

In the classical theory of analytic functions, Frigyes Riesz published a very important and often cited paper together with his brother Marcell (Über die Randwerte einer analytischen Funktion, Comptes Rendus 4. Congr. Math. Scand. Stockholm (1916), 27-44).

As a generalization for functions of two variables of the concept of a convex function of one variable, Frigyes Riesz introduced the concept of subharmonic functions and elaborated their theory in a series of papers (e.g. Über subharmonische Funktionen und ihre Rolle in der Funktionentheorie und in der Potentialtheorie, Acta Sci. Math. 2 (1924-26), 87-100). By this, he produced a very important contribution to potential theory.

We mentioned above that Riesz had also important results in ergodic theory.

In the year 1925/26, Frigyes Riesz was elected to the rector of the university in Szeged. In his inaugural lecture (Elemi módszerek a felső matematikában, Math. Phys. Lapok 32 (1925), 112-124), he demonstrates by examples chosen from higher mathematics that in this science adjectives like "higher", "involved", "difficult" have no constant value; what seems to be such today, may become "elementary", "simple", "easy" by tomorrow. In particular, in all papers of Frigyes Riesz, the reader is fascinated by the elegance of the methods and the effort of making everything as elementary as possible.

$$
* * *
$$

Riesz remained in Szeged during the years of World War II. Although he was not exempt, during the Nazi occupation of Hungary in 1944, from personal danger, he never lost his courage. Fortunately, Szeged was occupied by Russian troups in October 1944 without essential damage in the buildings. Riesz accomplished again, for a short period in the spring of 1945, the office of the rector of the university, but slightly afterwards he was invited to be a professor at the University of Budapest. He accepted the invitation, and so he tought in Budapest from the autumn semester in 1946. For myself, this fact
made possible to follow his mathematical courses in the last months of my studies, what produced for me an unforgettable experience.

During the years in Budapest, Riesz tried to summarize someway his results in Real Analysis and Functional Analysis. He wrote some big expository papers on the evolution of the concept of the integral (L'évolution de la notion d'intégrale depuis Lebesgue, Ann. Inst. Fourier 1 (1949), 29-42; Lebesgue integrálfogalmának fejlődése, Mat. Lapok 1 (1950), 79-90) and on the role of the null sets in real analysis (Nullahalmazok és szerepük az analízisben - Les ensembles de mesure nulle et leur rôle dans l'analyse, I. Magyar Mat. Kongr. Közl. (1952), 204-224). It is natural that his own ideas play a central role in all these summaries on central subjects of the field in question.

The last mentioned paper is the reproduction of the lecture held by Riesz at the first Hungarian Mathematical Congress organized in 1950 in Budapest and dedicated to the celebration of the 70th birthday of Riesz and Fejér. This birthday gave also an opportunity to the edition of a double volume of the Acta Szeged, containing a large number of contributions of first rank mathematicians from many parts of the world.

However, the most important summary of Riesz' results is the monograph "Leçons d’analyse fonctionnelle" appeared in 1952 (Budapest, Akadémiai Kiadó) written by Frigyes Riesz and his pupil Béla Szőkefalvi-Nagy. Without any exageration, we can say that this book had a determining success in the whole world and, in a short time, it was translated to English, German, Russian, etc. It begins with the setting up of Lebesgue's theory of integration using the method of Frigyes Riesz, as we sketched it above, and discusses an essential part of functional analysis including a part of the spectral theory of linear operators in Hilbert space.

In the years after the second war, Riesz could take pleasure in several kinds of appreciation. In 1945, he received the Grand Prize of the Hungarian Academy of Sciences, in 1949 and 1953 he got the Kossuth Prize (in those years, the most important scientific prize in Hungary). In 1948, he was elected corresponding member of the Academy of Sciences of Paris, and honorary member of the Bavarian Academy of Sciences in Munich, similarly to the Royal Physiographic Society in Lund (Sweden). He was honoured by the degree of doctor honoris causa from the University of Szeged in 1946, of Budapest in 1950, and of Paris in 1954. The Bolyai János Mathematical Society elected him its first Honorary President.

At the university, he held conscientiously his lectures, but, in the last years, he could only write to the blackboard with some difficulty so that this
task was assigned to one of his pupils (in a certain time to myself). He died on the 28th of February, 1956.

The collected papers of Frigyes Riesz, except the book written together with Béla Szőkefalvi-Nagy, have been edited in 1960 in two big volumes by the Publishing House of the Hungarian Academy of Sciences; my modest person had the role of editor.

The influence of his work on the development of mathematics did not fade in the past decades so that we can acknowledge him as one of the greatest mathematicians in the 20th century, and especially, one of the most brilliant researchers of Hungarian mathematics.

# THE THEORY OF SUBSPACES IN MIRON'S $O s c^{k} M$ 

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Dedicated to Professor Lajos Tamássy on the occasion of 80th birthday

## 1. Introduction

The first problem which we have during the learning and working on subspaces theory in osculator spaces of order $k$ was notation. Subspaces of tangent space, were mostly reduced on h-horizontal and v-vertical subspaces. It was difficult to determine which component of tensor is from which $v$ spaces. Because of that in this paper we used double indices for all tensors and quantities: the first indices $(A, B, C, \ldots=0,1, \ldots, k)$ denote the subspace horizontal, first, second or $k$-th vertical in which we are, the second index $(a, b, c, \ldots)$ denote the component in the subspace which run from 1 to $n$. In this way we avoid the use of symbols $\perp, h, v$ on different places.

Than we introduced the covariant derivative $\nabla_{X} Y$ where $X$ and $Y$ belong to any subspace and $\nabla_{X} Y$ has non-zero components in all subspaces. By reduction of this condition we obtain several special connections, one is the mentioned $d$-connection, which is used in most other papers.

Using the special transformation in $O s c^{k} M$ we obtain two complementary subspaces $O s c^{k} M_{1}$ and $O s c^{k} \boldsymbol{M}_{2}$, (Section 3) and in that way we solved the problem of tensors coordinate transformation in the complementary space of the subspace.

We construct the special adapted basis in $O s c^{k} M_{1}$ and $O s c^{k} M_{2}$ in such a way that the $A$-th vertical space in $O s c^{k} M$ is the direct sum of $A$-th vertical

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space of $O s c^{k} M_{1}$ and $O s c^{k} M_{2}$, and the same is valid for the dual spaces. This construction was not simple, because we have to determine the relations between the nonlinear connections of surrounding space and subspaces (see (3.44), (3.45)). The big help was the matrix representation of problem, which is first time used here. In such constructed adapted basis the decomposition of tensors is very simpler (see Section 7). The decomposition of curvature and torsion tensorv in direction of tangent space of subspace and complementary space in $O s c^{k} M$ is given.

## 2. Adapted basis in $T\left(O s c^{k} M\right)$ and $T^{*}\left(O s c^{k} M\right)$

Here $E=O s c^{k} M$ will be defined as a $(k+1) n$ dimensional $C^{\infty}$ manifold in which the transformations of form (2.3) are allowed. In some local chart $(U, \varphi)$ some point $y \in E$ has coordinates

$$
\begin{gathered}
\left(y^{0 a}, y^{1 a}, y^{2 a}, \ldots, y^{k a}\right)=\left(y^{A a}\right) \\
a, b, c, d, e, \ldots=1,2, \ldots, n, \quad A, B, C, D, \ldots=0,1,2, \ldots, k
\end{gathered}
$$

The following notations will be used:
(2.1) $\left[y^{\left(a^{\prime}\right)}\right]=\left[\begin{array}{c}y^{1 a^{\prime}} \\ y^{2 a^{\prime}} \\ \vdots \\ y^{k a^{\prime}}\end{array}\right], \quad\left[y^{(a)}\right]=\left[\begin{array}{c}y^{1 a} \\ y^{2 a} \\ \vdots \\ y^{\dot{k} a}\end{array}\right], \quad \partial_{A a}=\frac{\partial}{\partial y^{A a}}, \quad \partial_{a}=\partial_{0 a}=\frac{\partial}{\partial y^{0 a}}$;
(2.2) $\left[B_{(a)}^{\left(a^{\prime}\right)}\right]_{S}=\left[\begin{array}{ccccc}\partial_{0 a} y^{0 a^{\prime}} & 0 & 0 & \ldots & 0 \\ \partial_{0 a} y^{1 a^{\prime}} & \partial_{1 a} y^{1 a^{\prime}} & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ \partial_{0 a} y^{(k-1) a^{\prime}} & \partial_{1 a} y^{(k-1) a^{\prime}} & \partial_{2 a} y^{(k-1) a^{\prime}} & \ldots & \partial_{(k-1) a y^{(k-1) a^{\prime}}}\end{array}\right]$.

REMARK 2.1. We shall use the notation $\left[B_{(a)}^{\left(a^{\prime}\right)}\right]$ for the matrix, which has one column and one row more.

If in some other chart $\left(U^{\prime}, \varphi^{\prime}\right)$ the same point $y \in E$ has coordinates $\left(y^{0 a^{\prime}}, y^{1 a^{\prime}}, \ldots, y^{k a^{\prime}}\right)$, then in $U \cap U^{\prime}$ the allowable coordinate transformations are given by

$$
\begin{equation*}
\left[y^{\left(a^{\prime}\right)}\right]=\left[B_{(a)}^{\left(a^{\prime}\right)}\right]_{S}\left[y^{(a)}\right], \quad y^{0 a^{\prime}}=y^{0 a^{\prime}}\left(y^{0 a}\right) \tag{2.3}
\end{equation*}
$$

Theorem 2.2. The transformations of type (2.3) form a pseudo group.
Some nice example of the space $E$ can be obtained if the points $\left(x^{a}\right) \in$ $\in M, \operatorname{dim} M=n$ are considered as the points of the curve $x^{a}=x^{a}(t), t \in I$ and $y^{A a}, A=1,2, \ldots, k$ are determined by

$$
\begin{equation*}
y^{A a}=d_{t}^{A} y^{0 a}=d_{t}^{A} x^{a}, \quad d_{t}^{A}=\frac{d^{A}}{d t^{A}} . \tag{2.4}
\end{equation*}
$$

If in $U \cap U^{\prime}$ the equations $x^{a^{\prime}}(t)=x^{a^{\prime}}\left(x^{1}(t), x^{2}(t), \ldots, x^{n}(t)\right)$ are valid, then it is easy to see, that

$$
\begin{equation*}
y^{1 a^{\prime}}=d_{t}^{1} x^{a^{\prime}}, \quad y^{2 a^{\prime}}=d_{t}^{2} x^{a^{\prime}}, \ldots, y^{k a^{\prime}}=d_{t}^{k} x^{a^{\prime}} \tag{2.5}
\end{equation*}
$$

satisfy (2.3). In Miron's books and papers [16]-[22]

$$
y^{A a}=\frac{1}{A!} d_{t}^{A} x^{a}
$$

and it results, that the structure group is different from (2.3).
Let us introduce the notations

$$
\begin{equation*}
B_{a}^{a^{\prime}}={ }^{(0)} B_{a}^{a^{\prime}}=\partial_{a} y^{o a^{\prime}}, \quad{ }^{(A)} B_{a}^{a^{\prime}}=d_{t}^{A} B_{a}^{a^{\prime}}=\frac{d^{A} B_{a}^{a^{\prime}}}{d t} t^{A} \tag{2.6}
\end{equation*}
$$

$$
\begin{equation*}
\left[B_{(a)}^{\left(a^{\prime}\right)}\right]_{1}= \tag{2.7}
\end{equation*}
$$

$$
=\left[\begin{array}{ccccc}
\binom{0}{0}{ }^{(0)} B_{a}^{a^{\prime}} & 0 & 0 & \cdots & 0 \\
\binom{1}{0}^{(1)} B_{a}^{a^{\prime}} & \binom{1}{1}^{(0)} B_{a}^{a^{\prime}} & 0 & \cdots & 0 \\
\binom{2}{0}^{(2)} B_{a}^{a^{\prime}} & \binom{2}{1}^{(1)} B_{a}^{a^{\prime}} & \binom{2}{2}^{(0)} B_{a}^{a^{\prime}} & & 0 \\
\vdots & \vdots & \vdots & & \vdots \\
\binom{k-1}{0}^{(k-1)} B_{a}^{a^{\prime}} & \binom{k-1}{1}^{(k-2)} B_{a}^{a^{\prime}} & \binom{k-1}{2}^{(k-3)} B_{a}^{a^{\prime}} & \cdots & \binom{k-1}{k-1}^{(0)} B_{a}^{a^{\prime}}
\end{array}\right] .
$$

THEOREM 2.3. If $y^{A a}$ and $y^{A a^{\prime}}$ are determined by (2.4) and (2.5), then they satisfy the following equation:

$$
\begin{equation*}
\left[y^{\left(a^{\prime}\right)}\right]=\left[B_{(a)}^{\left(a^{\prime}\right)}\right]_{1}\left[y^{(a)}\right] . \tag{2.8}
\end{equation*}
$$

THEOREM 2.4. The partial derivatives of the variables and ${ }^{(A)} B_{a}^{a^{\prime}}$ are connected by the following formulae:

$$
\partial_{0 a} y^{0 a^{\prime}}=\partial_{1 a} y^{1 a^{\prime}}=\partial_{2 a} y^{2 a^{\prime}}=\cdots=\partial_{(k-1)} y^{(k-1) a^{\prime}}={ }^{(0)} B_{a}^{a^{\prime}}=B_{a}^{a^{\prime}},
$$

(2.9) $\partial_{0 a y}{ }^{A a^{\prime}}={ }^{(A)} B_{a}^{a^{\prime}}$,

$$
\begin{aligned}
\partial_{A a y} y^{(A+B) a^{\prime}} & =\binom{A+B}{1} \partial_{(A-1)} y y^{(A+B-1) a^{\prime}} \\
& =\cdots=\binom{A+B}{A} \partial_{0 a} y^{B a^{\prime}}=\binom{A+B}{A}{ }^{(B)} B_{a}^{a^{\prime}} .
\end{aligned}
$$

Theorem 2.5. The following relation is valid:

$$
\begin{align*}
{ }^{(A)} B_{a}^{a^{\prime}}=\left(\partial_{a}^{(A-1)} B_{b}^{a^{\prime}}\right) y^{1 b}+\binom{A-1}{1}\left(\partial_{a}^{(A-2)} B_{b}^{a^{\prime}}\right) y^{2 b}+\cdots+  \tag{2.10}\\
+\binom{A-1}{A-1}\left(\partial_{a}^{(0)} B_{b}^{a^{\prime}}\right) y^{A b}, \quad A=1,2, \ldots, k .
\end{align*}
$$

The natural basis $\bar{B}$ of $T(E)$ is

$$
\begin{equation*}
\bar{B}=\left\{\partial_{0 a}, \partial_{1 a}, \ldots, \partial_{k a}\right\} . \tag{2.11}
\end{equation*}
$$

If we introduce notations

$$
\left[\begin{array}{llll}
\partial_{(a)}
\end{array}\right]=\left[\begin{array}{llll}
\partial_{0 a} & \partial_{1 a} & \ldots & \partial_{k a} \tag{2.12}
\end{array}\right]
$$

$$
\left[\begin{array}{l}
\partial_{\left(a^{\prime}\right)}
\end{array}\right]=\left[\begin{array}{llll}
\partial_{0 a^{\prime}} & \partial_{1 a^{\prime}} & \ldots & \partial_{k a^{\prime}} \tag{2.13}
\end{array}\right]
$$

$$
\left[B_{(a)}^{\left(a^{\prime}\right)}\right]=\left[\begin{array}{ccc}
{\left[B_{a}^{a^{\prime}}\right]_{S^{\prime}}} & & 0  \tag{2.14}\\
\partial_{0 a} y^{k a^{\prime}} & \partial_{1 a} y^{k a^{\prime}} & \ldots \partial_{k a} y^{k a^{\prime}}
\end{array}\right]=\left[\begin{array}{cccc}
\partial_{0 a} y^{0 a^{\prime}} & 0 & & 0 \\
\partial_{0 a} y^{1 a^{\prime}} & \partial_{1 a} y^{1 a^{\prime}} & \ldots & 0 \\
\vdots & \vdots & & \vdots \\
\partial_{0 a y^{k a^{\prime}}} & \partial_{1 a} y^{k a^{\prime}} & \ldots & \partial_{k a} y^{k a^{\prime}}
\end{array}\right]
$$

$$
=\left[\begin{array}{cccc}
\left(\begin{array}{l}
0 \\
0
\end{array}{ }^{(0)} B_{a}^{a^{\prime}}\right. & 0 & \cdots & 0 \\
\binom{1}{0}{ }^{(1)} B_{a}^{a^{\prime}} & \binom{1}{1}^{(0)} B_{a}^{a^{\prime}} & \cdots & 0 \\
\vdots & \vdots & & \vdots \\
\binom{k}{0}^{(k)} B_{a}^{a^{\prime}} & \binom{k}{1}^{(k-1)} B_{a}^{a^{\prime}} & \cdots & \binom{k}{k}^{(0)} B_{a}^{a^{\prime}}
\end{array}\right]
$$

(see the last equation of (2.9)), then the transformation of $\bar{B}$ is given by

$$
\begin{equation*}
\left[\partial_{(a)}\right]=\left[\partial_{\left(a^{\prime}\right)}\right]\left[B_{(a)}^{\left(a^{\prime}\right)}\right] . \tag{2.15}
\end{equation*}
$$

The natural basis $\bar{B}^{*}$ of $T^{*}(E)$ is

$$
\begin{equation*}
\bar{B}^{*}=\left\{d y^{0 a}, d y^{1 a}, \ldots, d y^{k a}\right\} . \tag{2.16}
\end{equation*}
$$

If we use the notations

$$
\left[d y^{\left(a^{\prime}\right)}\right]=\left[\begin{array}{c}
d y^{0 a^{\prime}}  \tag{2.17}\\
d y^{1 a^{\prime}} \\
\vdots \\
d y^{k a^{\prime}}
\end{array}\right], \quad\left[d y^{(a)}\right]=\left[\begin{array}{c}
d y^{0 a} \\
d y^{1 a} \\
\vdots \\
d y^{k a}
\end{array}\right]
$$

then the elements of $\bar{B}^{*}$ with respect to (2.3) are transforming in the following way:

$$
\begin{equation*}
\left[d y^{\left(a^{\prime}\right)}\right]=\left[B_{(a)}^{\left(a^{\prime}\right)}\right]\left[d y^{(a)}\right] . \tag{2.18}
\end{equation*}
$$

We shall suppose that $\bar{B}^{*}$ and $\bar{B}$ are dual to each other, i.e.

$$
\begin{equation*}
\left[d y^{(a)}\right]\left[\partial_{(b)}\right]=\delta_{b}^{a} I \tag{2.19}
\end{equation*}
$$

The adapted basis $B^{*}$ of $T^{*}(E)$ is

$$
\begin{equation*}
B^{*}=\left\{\delta y^{0 a}, \delta y^{1 a}, \ldots, \delta y^{k a}\right\} \tag{2.20}
\end{equation*}
$$

If we introduce the notations

$$
\begin{align*}
{\left[\delta y^{(a)}\right]=} & {\left[\begin{array}{c}
\delta y^{0 a} \\
\delta y^{1 a} \\
\vdots \\
\delta y^{k a}
\end{array}\right], \quad\left[\delta y^{\left(a^{\prime}\right)}\right]=\left[\begin{array}{c}
\delta y^{0 a^{\prime}} \\
\delta y^{1 a^{\prime}} \\
\vdots \\
\delta y^{k a^{\prime}}
\end{array}\right], }  \tag{2.21}\\
{\left[M_{(b)}^{(a)}\right]=} & {\left[\begin{array}{ccccc}
\delta_{b}^{a} & 0 & 0 & \ldots & 0 \\
M_{0 b}^{1 a} & \delta_{b}^{a} & 0 & \ldots & 0 \\
M_{0 b}^{2 a} & M_{1 b}^{2 a} & \delta_{b}^{a} & \ldots & 0 \\
\vdots & \vdots & \vdots & & \vdots \\
M_{0 b}^{k a} & M_{1 b}^{k a} & M_{2 b}^{k a} & \ldots & \delta_{b}^{a}
\end{array}\right] }
\end{align*}
$$

then the adapted basis $B^{*}$ can be defined by

$$
\begin{equation*}
\left[\delta y^{(a)}\right]=\left[M_{(b)}^{(a)}\right]\left[d y^{(b)}\right] \tag{2.23}
\end{equation*}
$$

THEOREM 2.6. The necessary and sufficient conditions that $\delta y^{A a}$ are transformed as d-tensor fields, i.e.

$$
\delta y^{A a^{\prime}}=\frac{\partial y^{0 a^{\prime}}}{\partial y^{0 a}} \delta y^{A a}=B_{a}^{a^{\prime}} \delta y^{A a}, \quad A=0,1, \ldots, k
$$

are the following relations:

$$
\begin{equation*}
\left[M_{\left(b^{\prime}\right)}^{\left(a^{\prime}\right)}\right]\left[B_{(b)}^{\left(b^{\prime}\right)}\right]={ }^{(0)} B_{a}^{a^{\prime}}\left[M_{(b)}^{(a)}\right] . \tag{2.24}
\end{equation*}
$$

Let us denote the adapted basis of $T(E)$ by $B$, where

$$
\begin{equation*}
B=\left\{\delta_{0 a}, \delta_{1 a}, \ldots, \delta_{k a}\right\} \tag{2.25}
\end{equation*}
$$

If we use the notations

$$
\begin{align*}
& {\left[\delta_{(a)}\right]=\left[\begin{array}{llll}
\delta_{0 a} & \delta_{1 a} & \ldots & \delta_{k a}
\end{array}\right],\left[\begin{array}{llll}
\delta_{\left(a^{\prime}\right)}
\end{array}\right]=\left[\begin{array}{llll}
\delta_{0 a^{\prime}} & \delta_{1 a^{\prime}} & \ldots & \delta_{k a^{\prime}}
\end{array}\right],}  \tag{2.26}\\
& {\left[N_{(a)}^{(b)}\right]=\left[\begin{array}{ccccc}
\delta_{a}^{b} & 0 & 0 & \ldots & 0 \\
-N_{0 a}^{1 b} & \delta_{a}^{b} & 0 & & 0 \\
-N_{0 a}^{2 b} & -N_{1 a}^{2 b} & \delta_{a}^{b} & & 0 \\
\vdots & \vdots & \vdots & & \vdots \\
-N_{0 a}^{k b} & -N_{1 a}^{k b} & -N_{2 a}^{k b} & \ldots & \delta_{a}^{b}
\end{array}\right],}
\end{align*}
$$

then the adapted basis $B$ is defined by

$$
\begin{equation*}
\left[\delta_{(a)}\right]=\left[\partial_{(b)}\right]\left[N_{(a)}^{(b)}\right] \tag{2.28}
\end{equation*}
$$

THEOREM 2.7. The necessary and sufficient conditions that $\delta_{A a}$ are transformed as $d$-tensors fields, i.e.

$$
\delta_{A a^{\prime}}=\frac{\partial y^{0 a}}{\partial y^{0 a^{\prime}}} \delta_{A a}=B_{a^{\prime}}^{a} \delta_{A a}, \quad A=0,1, \ldots, k
$$

are the following relations:

$$
\begin{equation*}
\left[B_{(c)}^{\left(c^{\prime}\right)}\right]\left[N_{(a)}^{(c)}\right]=\left[N_{(a)}^{(b)}\right]{ }^{(0)} B_{a}^{a^{\prime}} \tag{2.29}
\end{equation*}
$$

THEOREM 2.8. The necessary and sufficient condition that $B^{*}$ be dual to $B$, i.e.

$$
\begin{equation*}
\left\langle\delta y^{B b}, \delta_{A a}\right\rangle=\delta_{A}^{B} \delta_{a}^{b} \tag{2.30}
\end{equation*}
$$

is that the matrices $\left[M_{(c)}^{(b)}\right]$ and $\left[N_{(a)}^{(c)}\right]$ are inverse matrices, i.e.

$$
\begin{equation*}
\left[M_{(c)}^{(b)}\right]\left[N_{(a)}^{(c)}\right]=\delta_{a}^{b} I \tag{2.31}
\end{equation*}
$$

Theorem 2.9. The elements of the bases $\bar{B}$ and $B$, further $\bar{B}^{*}$ and $B^{*}$ are connected by relations

$$
\begin{align*}
{\left[\partial_{(a)}\right] } & =\left[\delta_{(b)}\right]\left[M_{(a)}^{(b)}\right]  \tag{2.32}\\
{\left[d y^{(a)}\right] } & =\left[N_{(b)}^{(a)}\right]\left[\delta y^{(b)}\right] \tag{2.33}
\end{align*}
$$

The proof of Theorems 2.1-2.8 can be found in [5], [6], [9], [18], [19], [21], ...

Remark 2.10. The adapted bases $B^{\prime}$ and $B^{\prime *}$ introduced in [19], [20] and used in [17]-[23] has simpler form. They are very convenient for the definition of tangent structure $J$, in the spray theory etc. In [8] it is proved, that in the adapted bases $B$ and $B^{*}$ of type (3.22) and (3.23) the tangent space of $O s c^{k} M$ can be decomposed on orthogonal subspaces, what in $B^{\prime}$ and $B^{*}$ is valid only under some conditions.

## 3. The subspaces in $O s c^{k} M$

Here some special case of the general transformation (2.3) of $M$ will be considered, namely, when

$$
\begin{gather*}
y^{0 a}=y^{0 a}\left(u^{01}, \ldots, u^{0 m}, v^{0(m+1)}, \ldots, v^{0 n}\right)=y^{0 a}\left(u^{0 \alpha}, v^{0 \widehat{\alpha}}\right)  \tag{3.1}\\
a, b, c, \ldots=1,2, \ldots, n, \quad \alpha, \beta, \gamma, \delta, \ldots=1,2, \ldots, m \\
\widehat{\alpha}, \widehat{\beta}, \widehat{\gamma}, \widehat{\delta}, \ldots=m+1, \ldots, n
\end{gather*}
$$

and the new coordinates of the point $y$ in the base manifold $M$ with respect to another chart $\left(U^{\prime}, \varphi^{\prime}\right)$ are $\left(u^{01^{\prime}}, \ldots, u^{0 m^{\prime}}, v^{0(m+1)^{\prime}} \ldots v^{0 n^{\prime}}\right)$, where

$$
\begin{align*}
& u^{0 \alpha^{\prime}}=u^{0 \alpha^{\prime}}\left(u^{01}, \ldots, u^{0 m}\right), \quad v^{0 \widehat{\alpha}^{\prime}}=v^{0 \widehat{\alpha}^{\prime}}\left(v^{0(m+1)}, \ldots, v^{0 n}\right)  \tag{3.2}\\
& y^{0 a^{\prime}}=y^{0 a^{\prime}}\left(u^{01^{\prime}}, \ldots, u^{0 m^{\prime}}, v^{0(m+1)^{\prime}}, \ldots, v^{0 n^{\prime}}\right)=y^{0 a^{\prime}}\left(u^{0 \alpha^{\prime}}, v^{0 \widehat{\alpha}^{\prime}}\right) . \tag{3.3}
\end{align*}
$$

We shall use the notations

$$
\begin{array}{cl}
\partial_{\alpha}=\partial_{0 \alpha}=\frac{\partial}{\partial u^{0 \alpha}}, & \partial_{\widehat{\alpha}}=\partial_{0 \widehat{\alpha}}=\frac{\partial}{\partial v^{0 \widehat{\alpha}}}, \\
B_{\alpha}^{\alpha^{\prime}}=\partial_{0 \alpha} u^{0 \alpha^{\prime}}, & B_{\widehat{\alpha}}^{\widehat{\alpha}^{\prime}}=\partial_{0 \widehat{\alpha}} v^{0 \widehat{\alpha}^{\prime}},  \tag{3.4}\\
B_{\alpha}^{a}=\partial_{0 \alpha} y^{0 a}=\partial_{\alpha} x^{a}, & B_{\widehat{\alpha}}^{a}=\partial_{0 \widehat{\alpha}} y^{0 a}=\partial_{\widehat{\alpha}} x^{a} .
\end{array}
$$

If the transformation (3.1) is regular, then there exists an inverse transformation:

$$
\begin{equation*}
u^{0 \alpha}=u^{0 \alpha}\left(y^{0 a}\right), \quad v^{0 \widehat{\alpha}}=v^{0 \widehat{\alpha}}\left(y^{0 a}\right) \tag{3.5}
\end{equation*}
$$

In [9] is proved that if the Jacobian matrix $J$ has the form:

$$
J=\frac{D\left(y^{01}, \ldots, y^{0 n}\right)}{D\left(u^{01}, \ldots, u^{0 m}, v^{0(m+1)}, \ldots, v^{0 n}\right)}=\left[\left[B_{(\alpha)}^{(a)}\right]_{n \times m}\left[B_{(\hat{\alpha})}^{(a)}\right]_{n \times(n-m)}\right]
$$

then its inverse matrix is

$$
J^{-1}=\left[\begin{array}{l}
{\left[B_{(b)}^{(\beta)}\right]_{m \times n}} \\
{\left[B_{(b)}^{(\widehat{\beta})}\right]_{(n-m) \times n}}
\end{array}\right]
$$

As $y^{0 a}=x^{a}, u^{0 \alpha}=u^{\alpha}, v^{0 \widehat{\alpha}}=v^{\widehat{\alpha}}$, the explicit form of the above matrices are:

$$
\left.\begin{array}{l}
{\left[B_{(\alpha)}^{(a)}\right]=\left[\begin{array}{ccc}
\frac{\partial x^{1}}{\partial u^{1}} & \cdots & \frac{\partial x^{1}}{\partial u^{m}} \\
\vdots & & \vdots \\
\frac{\partial x^{n}}{\partial u^{1}} & \cdots & \frac{\partial x^{1}}{\partial u^{m}}
\end{array}\right]_{n \times m}\left[B_{(\widehat{\alpha})}^{(a)}\right.}
\end{array}\right]=\left[\begin{array}{ccc}
\frac{\partial x^{1}}{\partial v^{m+1}} & \cdots & \frac{\partial x^{1}}{\partial v^{n}}  \tag{3.6}\\
\vdots & & \vdots \\
\frac{\partial \dot{x}^{n}}{\partial v^{m+1}} & \cdots & \frac{\partial x^{n}}{\partial v^{n}}
\end{array}\right]_{n \times(n-m)} .\left[\begin{array}{cccc}
\frac{\partial u^{1}}{\partial x^{1}} & \cdots & \frac{\partial u^{1}}{\partial x^{n}} \\
\vdots & & \vdots \\
\frac{\partial u^{m}}{(\beta)} & \cdots & \frac{\partial u^{m}}{\partial x^{n}}
\end{array}\right]_{m \times n}\left[B_{(b)}^{(\widehat{\beta})}\right]=\left[\begin{array}{ccc}
\frac{\partial v^{m+1}}{\partial x^{1}} & \cdots & \frac{\partial v^{m+1}}{\partial x^{n}} \\
\vdots & & \vdots \\
\frac{\partial v^{n}}{\partial x^{1}} & \cdots & \frac{\partial v^{n}}{\partial x^{n}}
\end{array}\right]_{(n-m) \times n} .
$$

From $J^{-1} J=I$ we obtain

$$
\begin{gather*}
{\left[B_{(a)}^{(\beta)}\right]\left[B_{(\alpha)}^{(a)}\right]=\delta_{\alpha}^{\beta} I_{m \times m}, \quad\left[B_{(a)}^{(\widehat{\beta})}\right]\left[B_{(\alpha)}^{(a)}\right]=0_{(n-m) \times m}} \\
{\left[B_{(a)}^{(\beta)}\right]\left[B_{(\widehat{\alpha})}^{(a)}\right]=0_{m \times(n-m)}, \quad\left[B_{(a)}^{(\widehat{\beta})}\right]\left[B_{(\widehat{\alpha})}^{(a)}\right]=\delta_{\widehat{\alpha}} I_{(n-m) \times(n-m)},}  \tag{3.7}\\
{\left[B_{(\alpha)}^{(a)}\right]\left[B_{(b)}^{(\alpha)}\right]+\left[B_{(\widehat{\alpha})}^{(a)}\right]\left[B_{(b)}^{(\widehat{\alpha})}\right]=\delta_{b}^{a} I_{n \times n} .}
\end{gather*}
$$

From the above matrix equations we get

$$
\begin{equation*}
B_{a}^{\beta} B_{\alpha}^{a}=\delta_{\alpha}^{\beta}, \quad B_{a}^{\widehat{\beta}} B_{\alpha}^{a}=0, \quad B_{a}^{\beta} B_{\widehat{\alpha}}^{a}=0, \quad B_{a}^{\widehat{\beta}} B_{\widehat{\alpha}}^{a}=\delta_{\widehat{\alpha}}^{\widehat{\beta}}, \quad B_{\alpha}^{a} B_{b}^{\alpha}+B_{\widehat{\alpha}}^{a} B_{b}^{\widehat{\alpha}}=\delta_{b}^{a} \tag{3.8}
\end{equation*}
$$

We shall use the notations:

$$
\begin{gather*}
y^{1 a}=\frac{d y^{0 a}}{d t}, \quad \ldots, \quad y^{k a}=\frac{d^{k} y^{0 a}}{d t^{k}} \\
u^{1 \alpha}=\frac{d u^{0 \alpha}}{d t}, \quad, \ldots, \quad u^{k \alpha}=\frac{d^{k} u^{0 \alpha}}{d t^{k}}  \tag{3.9}\\
v^{1 \widehat{\alpha}}=\frac{d v^{0 \widehat{\alpha}}}{d t^{k}}, \quad \ldots, \quad v^{k \widehat{\alpha}}=\frac{d^{k} v^{0 \widehat{\alpha}}}{d t^{k}}
\end{gather*}
$$

In the base manifold we can construct two families of subspaces $M_{1}$ and $M_{2}$ given by equations
(3.10) $\quad M_{1}: y^{0 a}=y^{0 a}\left(u^{0 \alpha}, C^{0 \widehat{\alpha}}\right), \quad M_{2}: y^{0 a}=y^{0 a}\left(C^{0 \alpha}, v^{0 \widehat{\alpha}}\right)$,
where we suppose, that the functions appeared in (3.10) are $C^{\infty}$. The submanifolds $M_{1}$ and $M_{2}$ of $M$ induces subspaces $E_{1}=O s c^{k} M_{1}$ and $E_{2}=O s c^{k} M_{2}$ in $O s c^{k} M$. Some point $u \in E_{1}$ has coordinates $\left(u^{0 \alpha}, u^{1 \alpha}, \ldots, u^{k \alpha}\right)$ and some point $v \in E_{2}$ has coordinates $\left(v^{0 \widehat{\alpha}}, v^{1 \widehat{\alpha}}, \ldots, v^{k \widehat{\alpha}}\right)$. We have

$$
\operatorname{dim}\left(O s c^{k} \boldsymbol{M}_{1}\right)=(k+1) m, \quad \operatorname{dim}\left(O s c^{k} \boldsymbol{M}_{2}\right)=(k+1)(n-m)
$$

If we introduce the notations:

$$
\left[u^{(\alpha)}\right]=\left[\begin{array}{c}
u^{1 \alpha}  \tag{3.11}\\
u^{2 \alpha} \\
\vdots \\
u^{\dot{k} \alpha}
\end{array}\right], \quad\left[v^{(\widehat{\alpha})}\right]=\left[\begin{array}{c}
v^{1 \widehat{\alpha}} \\
v^{2 \widehat{\alpha}} \\
\vdots \\
v^{k} \widehat{\alpha}
\end{array}\right]
$$

(3.12) $\left[B_{(\alpha)}^{\left(\alpha^{\prime}\right)}\right]_{S}=\left[\begin{array}{cccc}\partial_{0 \alpha} u^{0 \alpha^{\prime}} & 0 & \cdots & 0 \\ \partial_{0 \alpha} u^{1 \alpha^{\prime}} & \partial_{1 \alpha} u^{1 \alpha^{\prime}} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ \partial_{0 \alpha} u^{(k-1) \alpha^{\prime}} & \partial_{1 \alpha} u^{(k-1) \alpha^{\prime}} & \cdots & \partial_{(k-1) \alpha} u^{(k-1) \alpha^{\prime}}\end{array}\right]$,
and $\left[B_{(\widehat{\alpha})}^{\left(\hat{\alpha}^{\prime}\right)}\right]_{S}$ obtained from (3.12) if $\alpha, \alpha^{\prime}, u$ are substituted by $\widehat{\alpha}, \widehat{\alpha}^{\prime}, v$ respectively, then the transformation group of $E_{1}$ and $E_{2}$ are given by (3.13) and (3.14):

$$
\begin{array}{ll}
{\left[u^{\left(\alpha^{\prime}\right)}\right]=\left[B_{(\alpha)}^{\left(\alpha^{\prime}\right)}\right]_{s}\left[u^{(\alpha)}\right]} & u^{0 \alpha^{\prime}}=u^{0 \alpha^{\prime}}\left(u^{0 \alpha}\right), \\
{\left[v^{\left(\hat{\alpha}^{\prime}\right)}\right]=\left[B_{(\widehat{\alpha})}^{\left(\hat{\alpha}^{\prime}\right)}\right]_{s}\left[v^{(\widehat{\alpha})}\right],} & v^{0 \hat{\alpha}^{\prime}}=v^{0 \hat{\alpha}^{\prime}}\left(v^{0 \widehat{\alpha}}\right) . \tag{3.14}
\end{array}
$$

The natural bases $\bar{B}_{1}$ of $T\left(E_{1}\right)$ and $\bar{B}_{2}$ of $T\left(E_{2}\right)$ are:

$$
\begin{equation*}
\bar{B}_{1}=\left\{\partial_{0 \alpha}, \partial_{1 \alpha}, \ldots, \partial_{k \alpha}\right\}, \quad \bar{B}_{2}=\left\{\partial_{0 \widehat{\alpha}}, \partial_{1 \widehat{\alpha}}, \ldots, \partial_{k \widehat{\alpha}}\right\} . \tag{3.15}
\end{equation*}
$$

and their dual bases $\bar{B}_{1}^{*}$ and $\bar{B}_{2}^{*}$ are:

$$
\begin{equation*}
\bar{B}_{1}^{*}=\left\{d u^{0 \alpha}, d u^{1 \alpha}, \ldots, d u^{k \alpha}\right\}, \quad \bar{B}_{2}^{*}=\left\{d v^{0 \widehat{\alpha}}, d v^{1 \widehat{\alpha}}, \ldots, d v^{k \widehat{\alpha}}\right\} . \tag{3.16}
\end{equation*}
$$

If we use the notations:
(3.17) $\left[\partial_{(\alpha)}\right]=\left[\begin{array}{llll}\partial_{0 \alpha} & \partial_{1 \alpha} & \ldots & \partial_{k \alpha}\end{array}\right], \quad\left[\begin{array}{lll}\partial_{(\widehat{\alpha})}\end{array}\right]=\left[\begin{array}{llll}\partial_{0 \widehat{\alpha}} & \partial_{1 \widehat{\alpha}} & \ldots & \partial_{k \widehat{\alpha}}\end{array}\right]$

$$
\begin{gather*}
{\left[d u^{(\alpha)}\right]=\left[\begin{array}{c}
d u^{0 \alpha} \\
d u^{1 \alpha} \\
\vdots \\
d u^{k \alpha}
\end{array}\right], \quad\left[d v^{\widehat{\alpha}}\right]=\left[\begin{array}{c}
d v^{0 \widehat{\alpha}} \\
d v^{1 \widehat{\alpha}} \\
\vdots \\
d v^{k \widehat{\alpha}}
\end{array}\right],}  \tag{3.18}\\
{\left[B_{(\alpha)}^{\left(\alpha^{\prime}\right)}\right]=\left[\begin{array}{cccc}
\partial_{0 \alpha} u^{0 \alpha^{\prime}} & 0 & \ldots & 0 \\
\partial_{0 \alpha} u^{1 \alpha^{\prime}} & \partial_{1 \alpha} u^{1 \alpha^{\prime}} & \cdots & 0 \\
\vdots & \vdots & & \vdots \\
\partial_{0 \alpha} u^{k \alpha^{\prime}} & \partial_{1 \alpha} u^{k \alpha^{\prime}} & \cdots & \partial_{k \alpha} u^{k \alpha^{\prime}}
\end{array}\right]} \tag{3.19}
\end{gather*}
$$

and $\left[B_{(\widehat{\alpha})}^{\left(\widehat{\alpha}^{\prime}\right)}\right]$ obtained from (3.19) if $\alpha, \alpha^{\prime}, u$ are substituted by $\widehat{\alpha}, \widehat{\alpha}^{\prime}, v$ respectively, then we have:

Theorem 3.1. The elements of $\bar{B}_{1}, \bar{B}_{2}, \bar{B}_{1}^{*}, \bar{B}_{2}^{*}$ under the coordinate transformations (3.13) and (3.14) are transforming in the following way:

$$
\begin{align*}
{\left[\partial_{(\alpha)}\right]=\left[\partial_{\left(\alpha^{\prime}\right)}\right]\left[B_{(\alpha)}^{\left(\alpha^{\prime}\right)}\right], } & {\left[\partial_{(\widehat{\alpha})}\right]=\left[\partial_{\left(\widehat{\alpha}^{\prime}\right)}\right]\left[B_{(\widehat{\alpha})}^{\left(\hat{\alpha}^{\prime}\right)}\right], }  \tag{3.20}\\
{\left[d u^{\left(\alpha^{\prime}\right)}\right]=\left[B_{(\alpha)}^{\left(\alpha^{\prime}\right)}\right]\left[d u^{(\alpha)}\right], } & {\left[d v^{\left(\widehat{\alpha}^{\prime}\right)}\right]=\left[B_{(\widehat{\alpha})}^{\left(\left(\hat{\alpha}^{\prime}\right)\right.}\right]\left[d v^{(\hat{\alpha})}\right] . }
\end{align*}
$$

The natural bases $\bar{B}_{1}$ and $\bar{B}_{1}^{*}, \bar{B}_{2}$ and $\bar{B}_{2}^{*}$ are dual to each other, i.e.

$$
\begin{equation*}
\left[d u^{(\alpha)}\right]\left[\partial_{(\beta)}\right]=\delta_{\beta}^{\alpha} I_{m \times m} \tag{3.22}
\end{equation*}
$$

$$
\begin{equation*}
\left[d v^{(\widehat{\alpha})}\right]\left[\partial_{\widehat{\beta})}\right]=\delta_{\widehat{\beta}}^{\widehat{\alpha}} I_{(n-m) \times(n-m)} . \tag{3.23}
\end{equation*}
$$

Let us denote by

$$
\begin{gather*}
{\left[M_{(\beta)}^{(\alpha)}\right]=\left[\begin{array}{ccccc}
\delta_{\beta}^{\alpha} & 0 & 0 & \ldots & 0 \\
M_{0 \beta}^{1 \alpha} & \delta_{\beta}^{\alpha} & 0 & \ldots & 0 \\
M_{0 \beta}^{2 \alpha} & M_{1 \beta}^{2 \alpha} & \delta_{\beta}^{\alpha} & \ldots & 0 \\
\vdots & \vdots & \vdots & & \vdots \\
M_{0 \beta}^{k \alpha} & M_{1 \beta}^{k \alpha} & M_{2 \beta}^{k \alpha} & \ldots & \delta_{\beta}^{\alpha}
\end{array}\right]}  \tag{3.24}\\
{\left[N_{(\alpha)}^{(\beta)}\right]=\left[\begin{array}{ccccc}
\delta_{\alpha}^{\beta} & 0 & 0 & \ldots & 0 \\
-N_{0 \alpha}^{1 \beta} & \delta_{\alpha}^{\beta} & 0 & \ldots & 0 \\
-N_{0 \alpha}^{2 \beta} & -N_{1 \alpha}^{2 \beta} & \delta_{\alpha}^{\beta} & \ldots & 0 \\
\vdots & \vdots & \vdots & & \vdots \\
-N_{0 \alpha}^{k \beta} & -N_{1 \alpha}^{k \beta} & -N_{2 \alpha}^{k \beta} & \ldots & \delta_{\alpha}^{\beta}
\end{array}\right]}
\end{gather*}
$$

and $\left[\boldsymbol{M}_{(\widehat{\beta})}^{(\widehat{\alpha})}\right],\left[N_{(\widehat{\alpha})}^{(\widehat{\beta})}\right]$ are obtained from (3.24), (3.25) if $\alpha, \beta$ are substituted by $\widehat{\alpha}, \widehat{\beta}$, then we can construct the adapted bases $B_{1}$ and $B_{1}^{*}$ of $T\left(E_{1}\right)$ and $T^{*}\left(E_{1}\right)$, further $B_{2}$ and $B_{2}^{*}$ of $T\left(E_{2}\right)$ and $T^{*}\left(E_{2}\right)$ respectively, where

$$
\begin{equation*}
B_{1}=\left\{\delta_{0 \alpha}, \delta_{1 \alpha}, \ldots, \delta_{k \alpha}\right\}, \quad B_{2}=\left\{\delta_{0 \widehat{\alpha}}, \delta_{1 \widehat{\alpha}}, \ldots, \delta_{k \widehat{\alpha}}\right\} \tag{3.26}
\end{equation*}
$$

(3.27) $B_{1}^{*}=\left\{\delta u^{0 \alpha}, \delta u^{1 \alpha}, \ldots, \delta u^{k \alpha}\right\}, \quad B_{2}^{*}=\left\{\delta v^{0 \widehat{\alpha}}, \delta v^{1 \widehat{\alpha}}, \ldots, \delta v^{k \widehat{\alpha}}\right\}$.

We shall use the following notations:

$$
\left[\begin{array}{llll}
\delta_{(\alpha)}
\end{array}\right]=\left[\begin{array}{llll}
\delta_{0 \alpha} & \delta_{1 \alpha} & \ldots & \delta_{k \alpha}
\end{array}\right], \quad\left[\begin{array}{lll}
\delta_{(\widehat{\alpha})}
\end{array}\right]=\left[\begin{array}{llll}
\delta_{0 \widehat{\alpha}} & \delta_{1 \widehat{\alpha}} & \ldots & \delta_{k \widehat{\alpha}} \tag{3.28}
\end{array}\right]
$$

$$
\left[\delta u^{(\alpha)}\right]=\left[\begin{array}{c}
\delta u^{0 \alpha}  \tag{3.29}\\
\delta u^{1 \alpha} \\
\vdots \\
\delta u^{k \alpha}
\end{array}\right], \quad\left[\delta v^{(\widehat{\alpha})}\right]=\left[\begin{array}{c}
\delta v^{0 \widehat{\alpha}} \\
\delta v^{1 \widehat{\alpha}} \\
\vdots \\
\delta v^{k \widehat{\alpha}}
\end{array}\right]
$$

DEFINITION 3.2. The adapted bases $B_{1}, B_{1}^{*}, B_{2}, B_{2}^{*}$ are defined by

$$
\begin{gather*}
{\left[\delta_{(\alpha)}\right]=\left[\partial_{(\beta)}\right]\left[N_{(\alpha)}^{(\beta)}\right], \quad\left[\delta_{(\widehat{\alpha})}\right]=\left[\partial_{(\widehat{\beta})}\right]\left[N_{(\widehat{\alpha})}^{(\widehat{\beta})}\right]}  \tag{3.30}\\
{\left[\delta u^{(\alpha)}\right]=\left[M_{(\beta)}^{(\alpha)}\right]\left[d u^{(\beta)}\right], \quad\left[\delta v^{(\widehat{\alpha})}\right]=\left[M_{(\widehat{\beta})}^{(\widehat{\alpha})}\right]\left[d v^{(\widehat{\beta})}\right],}
\end{gather*}
$$

where the notations (3.24)-(3.29) are used.

THEOREM 3.3. The necessary and sufficient condition that $\delta u^{A \alpha}\left(\delta v^{A \widehat{\alpha}}\right)$ are transformed as $d$-tensors, i.e.

$$
\begin{equation*}
\delta u^{A \alpha^{\prime}}=B_{\alpha}^{\alpha^{\prime}} \delta u^{A \alpha}, \quad\left(\delta v^{A \widehat{\alpha}^{\prime}}=B_{\widehat{\alpha}}^{\widehat{\alpha}^{\prime}} \delta v^{A \widehat{\alpha}}\right), \quad A=0,1, \ldots, k \tag{3.32}
\end{equation*}
$$

are

$$
\begin{equation*}
\left[\boldsymbol{M}_{\left(\beta^{\prime}\right)}^{\left(\alpha^{\prime}\right)}\right]\left[\boldsymbol{B}_{(\beta)}^{\left(\beta^{\prime}\right)}\right]=\boldsymbol{B}_{\alpha}^{\alpha^{\prime}}\left[\boldsymbol{M}_{(\beta)}^{(\alpha)}\right], \quad\left(\left[\boldsymbol{M}_{\left(\widehat{\beta}^{\prime}\right)}^{\left(\widehat{\alpha}^{\prime}\right)}\right]\left[B_{(\widehat{\beta})}^{\left(\widehat{\beta}^{\prime}\right)}\right]=B_{\widehat{\alpha}}^{\widehat{\alpha}^{\prime}}\left[\boldsymbol{M}_{(\widehat{\beta})}^{(\widehat{\alpha})}\right]\right) \tag{3.33}
\end{equation*}
$$

THEOREM 3.4. The necessary and sufficient condition that $\delta_{A \alpha}\left(\delta_{A \widehat{\alpha}}\right)$ are transformed as $d$-tensors, i.e.

$$
\begin{equation*}
\delta_{A \alpha^{\prime}}=B_{\alpha^{\prime}}^{\alpha} \delta_{A \alpha}, \quad\left(\delta_{A \widehat{\alpha}^{\prime}}=B_{\widehat{\alpha}^{\prime}}^{\widehat{\alpha}} \delta_{A \widehat{\alpha}}\right), \quad A=0,1, \ldots, k \tag{3.34}
\end{equation*}
$$

are
(3.35)

$$
\left[B_{(\gamma)}^{\left(\gamma^{\prime}\right)}\right]\left[N_{(\alpha)}^{(\gamma)}\right]=\left[N_{\left(\alpha^{\prime}\right)}^{\left(\gamma^{\prime}\right)}\right] B_{\alpha}^{\alpha^{\prime}}, \quad\left(\left[B_{(\widehat{\gamma})}^{\left(\widehat{\gamma}^{\prime}\right)}\right]\left[N_{(\widehat{\alpha})}^{(\widehat{\gamma})}\right]=\left[N_{\left(\widehat{\alpha}^{\prime}\right)}^{\left(\hat{\gamma}^{\prime}\right)}\right] B_{\widehat{\alpha}}^{\widehat{\alpha}^{\prime}}\right) .
$$

THEOREM 3.5. The necessary and sufficient conditions that $B_{1}^{*}$ be dual to $B_{1}$ and $B_{2}^{*}$ be dual to $B_{2}$ are the following equations:

$$
\begin{gather*}
{\left[M_{(\gamma)}^{(\beta)}\right]\left[N_{(\alpha)}^{(\gamma)}\right]=\delta_{\alpha}^{\beta} I_{m \times m}}  \tag{3.36}\\
{\left[\boldsymbol{M}_{(\widehat{\gamma})}^{(\widehat{\beta})}\right]\left[N_{(\widehat{\alpha})}^{(\widehat{\gamma})}\right]=\delta_{\widehat{\alpha}}^{\widehat{\beta}} I_{(n-m) \times(n-m)} .} \tag{3.37}
\end{gather*}
$$

THEOREM 3.6. The elements of bases $\bar{B}_{1}$ and $B_{1}$, further $\bar{B}_{1}^{*}$ and $B_{1}^{*}$ are connected by relations:

$$
\begin{align*}
{\left[\partial_{(\alpha)}\right] } & =\left[\delta_{(\beta)}\right]\left[M_{(\alpha)}^{(\beta)}\right]  \tag{3.38}\\
{\left[d u^{(\alpha)}\right] } & =\left[N_{(\beta)}^{(\alpha)}\right]\left[\delta u^{(\beta)}\right] \tag{3.39}
\end{align*}
$$

The above theorem is valid if $B_{1}, \alpha, \beta, u$ are substituted by $B_{2}, \widehat{\alpha}, \widehat{\beta}, v$, respectively.

Now we want to obtain the relations between the adapted bases $B$ and $B^{\prime}$, where

$$
\begin{align*}
B & =\left\{\delta_{0 a}, \delta_{1 a}, \ldots, \delta_{k a}\right\} \\
B^{\prime} & =B_{1} \cup B_{2}=\left\{\delta_{0 \alpha}, \delta_{0 \widehat{\alpha}}, \delta_{1 \alpha}, \delta_{1 \widehat{\alpha}}, \ldots, \delta_{k \alpha}, \delta_{k \widehat{\alpha}}\right\} \tag{3.40}
\end{align*}
$$

further between $B^{*}$ and $B^{* *}$, where

$$
\begin{align*}
B^{*} & =\left\{\delta y^{0 a}, \delta y^{1 a}, \ldots, \delta y^{k a}\right\}  \tag{3.41}\\
B^{*} & =B_{1}^{*} \cup B_{2}^{*}=\left\{\delta u^{0 \alpha}, \delta v^{0 \widehat{\alpha}}, \delta u^{1 \alpha}, \delta v^{1 \widehat{\alpha}}, \ldots, \delta u^{k \alpha}, \delta v^{k \widehat{\alpha}}\right\}
\end{align*}
$$

The adapted basis $B, B^{\prime}, B^{*}, B^{* *}$ are functions of

$$
\left[M_{(b)}^{(a)}\right], \quad\left[N_{(b)}^{(a)}\right], \quad\left[M_{(\beta)}^{(\alpha)}\right], \quad\left[N_{(\beta)}^{(\alpha)}\right], \quad\left[M_{(\widehat{\beta})}^{(\widehat{\alpha})}\right], \quad\left[N_{(\widehat{\beta})}^{(\widehat{\alpha})}\right]
$$

which have to satisfy the conditions given in previous text. It is clear, that the adapted bases are not uniquely determined.

For the easier calculations we want to obtain such adapted bases, for which the following relations are valid.

$$
\begin{array}{cc}
\delta_{A a}=B_{a}^{\alpha} \delta_{A \alpha}+B_{a}^{\widehat{\alpha}} \delta_{A \widehat{\alpha}}, & A=0,1, \ldots, k \\
\delta y^{A a}=B_{\alpha}^{a} \delta u^{A \alpha}+B_{\widehat{\alpha}}^{a} \delta v^{A \widehat{\alpha}}, & A=0,1, \ldots, k \tag{3.43}
\end{array}
$$

If elements of $B, B^{*}, B^{\prime}$ and $B^{* *}$ satisfy (3.42) and (3.43), then $B^{*}$ is dual to $B^{\prime}$ if $B_{1}^{*}$ is dual to $B_{1}$ and $B_{2}^{*}$ is dual to $B_{2}$, i.e.

$$
\begin{aligned}
\left\langle\delta y^{A a}, \delta_{B b}\right\rangle & =\left\langle B_{\alpha}^{a} \delta u^{A \alpha}+B_{\widehat{\alpha}}^{a} \delta v^{A \widehat{\alpha}}, B_{b}^{\beta} \delta_{B \beta}+B_{b}^{\widehat{\beta}} \delta_{B \widehat{\beta}}\right\rangle \\
& =B_{\alpha}^{a} B_{b}^{\beta} \delta_{B}^{A} \delta_{\beta}^{\alpha}+B_{\widehat{\alpha}}^{a} B_{b}^{\widehat{\beta}} \delta_{B}^{A} \delta_{\widehat{\beta}}^{\widehat{\alpha}}=\delta_{B}^{A}\left(B_{\alpha}^{a} B_{b}^{\alpha}+B_{\widehat{\alpha}}^{a} B_{b}^{\widehat{\alpha}}\right)=\delta_{B}^{A} \delta_{b}^{a}
\end{aligned}
$$

From $u^{\alpha^{\prime}}=u^{\alpha^{\prime}}\left(u^{\alpha}\right)=u^{\alpha^{\prime}}\left(u^{\alpha}\left(x^{a}\right)\right)=u^{\alpha^{\prime}}\left(u^{\alpha}\left(x^{a}\left(x^{a^{\prime}}\right)\right)\right)$ we have

$$
B_{a^{\prime}}^{\alpha^{\prime}}=\frac{\partial u^{\alpha^{\prime}}}{\partial x^{a^{\prime}}}=\frac{\partial u^{\alpha^{\prime}}}{\partial u^{\alpha}} \frac{\partial u^{\alpha}}{\partial x^{a}} \frac{\partial x^{a}}{\partial x^{a^{\prime}}}=B_{\alpha}^{\alpha^{\prime}} B_{a}^{\alpha} B_{a^{\prime}}^{a}
$$

From $x^{a^{\prime}}=x^{a^{\prime}}\left(x^{a}\right)=x^{a^{\prime}}\left(x^{a}\left(u^{\alpha}, v^{\widehat{\alpha}}\right)\right)=x^{a^{\prime}}\left(x^{a}\left(u^{\alpha}\left(u^{\alpha^{\prime}}\right), v^{\widehat{\alpha}}\left(v^{\widehat{\alpha}^{\prime}}\right)\right)\right)$ we have

$$
B_{\alpha^{\prime}}^{a^{\prime}}=\frac{\partial x^{a^{\prime}}}{\partial u^{\alpha^{\prime}}}=\frac{\partial x^{a^{\prime}}}{\partial x^{a}} \frac{\partial x^{a}}{\partial u^{\alpha}} \frac{\partial u^{\alpha}}{\partial u^{\alpha^{\prime}}}=B_{a}^{a^{\prime}} B_{\alpha}^{a} B_{\alpha^{\prime}}^{\alpha}
$$

The above two equations are valid if $\alpha, u$ are substituted by $\widehat{\alpha}, v$.

From these equations it follows

$$
\begin{aligned}
\delta_{A a^{\prime}} & =B_{a^{\prime}}^{a} \delta_{A a}=B_{a^{\prime}}^{a}\left(B_{a}^{\alpha} \delta_{A \alpha}+B_{a}^{\widehat{\alpha}} \delta_{A \widehat{\alpha}}\right) \\
& =B_{a^{\prime}}^{a}\left(B_{a}^{\alpha} B_{\alpha}^{\alpha^{\prime}} \delta_{A \alpha^{\prime}}+B_{a}^{\widehat{\alpha}} B_{\widehat{\alpha}}^{\widehat{\alpha}^{\prime}} \delta_{A \widehat{\alpha}^{\prime}}\right)=B_{a^{\prime}}^{\alpha^{\prime}} \delta_{A \alpha^{\prime}}+B_{a^{\prime}}^{\widehat{\alpha}^{\prime}} \delta_{A \widehat{\alpha}^{\prime}} \\
\delta y^{A a^{\prime}} & =B_{a}^{a^{\prime}} \delta y^{A a}=B_{a}^{a^{\prime}}\left(B_{\alpha}^{a} \delta u^{A \alpha}+B_{\widehat{\alpha}}^{a} \delta v^{A \widehat{\alpha}}\right) \\
& =B_{a}^{a^{\prime}}\left(B_{\alpha}^{a} B_{\alpha^{\prime}}^{\alpha} \delta u^{A \alpha^{\prime}}+B_{\widehat{\alpha}}^{a} B_{\widehat{\alpha}{ }^{\prime}}^{\widehat{\alpha}} \delta v^{A \widehat{\alpha}^{\prime}}\right)=B_{\alpha^{\prime}}^{a^{\prime}} \delta u^{A \alpha^{\prime}}+B_{\widehat{\alpha}^{\prime}}^{a^{\prime}} \delta v^{A \widehat{\alpha}^{\prime}},
\end{aligned}
$$

which shows that (3.42), (3.43) are tensor equations.
THEOREM 3.7. The adapted bases $B, B^{*}, B^{\prime}$ and $B^{* *}$ satisfy (3.42) and (3.43) if different $[M]$ and $[N]$ are connected by

$$
\begin{equation*}
\left[M_{(b)}^{(a)}\right]\left[B_{(\beta)}^{(b)}\right]=\left[B_{(\alpha)}^{(a)}\right]\left[M_{(\beta)}^{(\alpha)}\right], \quad\left[M_{(b)}^{(a)}\right]\left[B_{(\widehat{\beta})}^{(b)}\right]=\left[B_{(\widehat{\alpha})}^{(a)}\right]\left[M_{(\widehat{\beta})}^{(\widehat{\alpha})}\right] \tag{3.44}
\end{equation*}
$$

and

$$
\begin{equation*}
\left[B_{(b)}^{(\beta)}\right]\left[N_{(a)}^{(b)}\right]=\left[N_{(\alpha)}^{(\beta)}\right]\left[B_{(a)}^{(\alpha)}\right] \quad\left[B_{(b)}^{(\widehat{\beta})}\right]\left[N_{(a)}^{(b)}\right]=\left[N_{(\widehat{\alpha})}^{(\widehat{\beta})}\right]\left[B_{(a)}^{(\widehat{\alpha})}\right] \tag{3.45}
\end{equation*}
$$

The proof of this important theorem is given in [10].
THEOREM 3.8. If the adapted basis $B_{1}$ is dual to $B_{1}^{*}$ (formed by $\left[N_{(\beta)}^{(\alpha)}\right]$ and $\left[M_{(\beta)}^{(\alpha)}\right]$ respectively), $B_{2}$ is dual to $B_{2}^{*}$ (formed by $\left[N_{(\widehat{\beta})}^{(\widehat{\alpha})}\right]$ and $\left[M_{(\widehat{\beta})}^{(\widehat{\alpha})}\right]$ respectively), then $B$ is dual to $B^{*}$ (formed by $\left[N_{(b)}^{(a)}\right]$ and $\left[M_{(b)}^{(a)}\right]$ respectively), if and only if (3.44) and (3.45) are satisfied.

PROOF. If we the first equation in (3.44) multiply with $\left[B_{(c)}^{(\beta)}\right]$ and second with $\left[B_{(c)}^{(\widehat{\beta})}\right]$ and then add this two equation we got

$$
\begin{aligned}
{\left[M_{(b)}^{(a)}\right]\left(\left[B_{(\beta)}^{(b)}\right]\left[B_{(c)}^{(\beta)}\right]+\right.} & {\left.\left[B_{(\widehat{\beta})}^{(b)}\right]\left[B_{(c)}^{(\widehat{\beta})}\right]\right)=} \\
& =\left[B_{(\alpha)}^{(a)}\right]\left[M_{(\beta)}^{(\alpha)}\right]\left[B_{(c)}^{(\beta)}\right]+\left[B_{(\widehat{\alpha})}^{(a)}\right]\left[M_{(\widehat{\beta})}^{(\widehat{\alpha})}\right]\left[B_{(c)}^{(\widehat{\beta})}\right] .
\end{aligned}
$$

Now using (3.7) we have

$$
\begin{equation*}
\left[M_{(c)}^{(a)}\right]=\left[B_{(\alpha)}^{(a)}\right]\left[M_{(\beta)}^{(\alpha)}\right]\left[B_{(c)}^{(\beta)}\right]+\left[B_{(\widehat{\alpha})}^{(a)}\right]\left[M_{(\widehat{\beta})}^{(\widehat{\alpha})}\right]\left[B_{(c)}^{(\widehat{\beta})}\right] . \tag{3.46}
\end{equation*}
$$

If we the first equation in (3.45) multiply with $\left[B_{(\beta)}^{(c)}\right]$ and second with $\left[B_{(\widehat{\beta})}^{(c)}\right]$ and then add this two equation we got

$$
\left.\left.\left.\begin{array}{rl}
\left(\left[B_{(\beta)}^{(c)}\right]\left[B_{(b)}^{(\beta)}\right]+\left[B_{(\widehat{\beta})}^{(c)}\right]\right.
\end{array}\right]\left[B_{(b)}^{(\widehat{\beta})}\right]\right)\left[N_{(a)}^{(b)}\right]=\right] .\left[B_{(\beta)}^{(c)}\right]\left[N_{(\alpha)}^{(\beta)}\right]\left[B_{(a)}^{(\alpha)}\right]+\left[B_{(\widehat{\beta})}^{(c)}\right]\left[N_{(\widehat{\alpha})}^{(\widehat{\beta})}\right]\left[B_{(a)}^{(\widehat{\alpha})}\right] .
$$

Now using (3.7) we have

$$
\begin{equation*}
\left[N_{(a)}^{(c)}\right]=\left[B_{(\beta)}^{(c)}\right]\left[N_{(\alpha)}^{(\beta)}\right]\left[B_{(a)}^{(\alpha)}\right]+\left[B_{(\widehat{\beta})}^{(c)}\right]\left[N_{(\widehat{\alpha})}^{(\widehat{\beta})}\right]\left[B_{(a)}^{(\widehat{\alpha})}\right] \tag{3.47}
\end{equation*}
$$

From the (3.46) and (3.47) using (3.7), (3.36) and (3.37) it follows

$$
\begin{aligned}
{\left[N_{(a)}^{(c)}\right][ } & \left.M_{(d)}^{(a)}\right]=\left(\left[B_{(\beta)}^{(c)}\right]\left[N_{(\alpha)}^{(\beta)}\right]\left[B_{(a)}^{(\alpha)}\right]+\left[B_{(\widehat{\beta})}^{(c)}\right]\left[N_{(\widehat{\alpha})}^{(\widehat{\beta})}\right]\left[B_{(a)}^{(\widehat{\alpha})}\right]\right) \\
& \cdot\left(\left[B_{(\gamma)}^{(a)}\right]\left[M_{(\delta)}^{(\gamma)}\right]\left[B_{(d)}^{(\delta)}\right]+\left[B_{(\widehat{\gamma})}^{(a)}\right]\left[M_{(\widehat{\delta})}^{(\widehat{\gamma})}\right]\left[B_{(d)}^{(\widehat{\delta})}\right]\right)= \\
= & {\left[B_{(\beta)}^{(c)}\right]\left[N_{(\alpha)}^{(\beta)}\right] \delta_{\gamma}^{\alpha}\left[M_{(\delta)}^{(\gamma)}\right]\left[B_{(d)}^{(\delta)}\right]+} \\
& +\left[B_{(\widehat{\beta})}^{(c)}\right]\left[N_{(\widehat{\alpha})}^{(\widehat{\beta})}\right] \delta_{\widehat{\gamma}}^{\widehat{\alpha}}\left[M_{(\widehat{\delta})}^{(\widehat{\gamma})}\right]\left[B_{(d)}^{(\widehat{\delta})}\right]= \\
= & {\left[B_{(\beta)}^{(c)}\right]\left[N_{(\alpha)}^{(\beta)}\right]\left[M_{(\delta)}^{(\alpha)}\right]\left[B_{(d)}^{(\delta)}\right]+\left[B_{(\widehat{\beta})}^{(c)}\right]\left[N_{(\widehat{\alpha})}^{(\widehat{\beta})}\right]\left[M_{(\widehat{\delta})}^{(\widehat{\alpha})}\right]\left[B_{(d)}^{(\widehat{\delta})}\right]=} \\
= & {\left[B_{(\beta)}^{(c)}\right]\left[B_{(d)}^{(\beta)}\right]+\left[B_{(\widehat{\beta})}^{(c)}\right]\left[B_{(d)}^{(\widehat{\beta})}\right]=\delta_{d}^{c} I . }
\end{aligned}
$$

The proof in the opposite direction is obvious.
THEOREM 3.9. If the adapted basis $B_{1}$ and $B_{1}^{*}$ (formed by $\left[N_{(\beta)}^{(\alpha)}\right]$ and $\left[M_{(\beta)}^{(\alpha)}\right]$ respectively), are dual to each other and further (3.33) and (3.35) are satisfied, then $B_{1}^{\prime}$ and $B_{1}^{\prime *}$ (formed by $\left[N_{\left(\beta^{\prime}\right)}^{\left(\alpha^{\prime}\right)}\right]$ and $\left[M_{\left(\beta^{\prime}\right)}^{\left(\alpha^{\prime}\right)}\right]$ respectively), are also dual to each other. Similar statement is valid for $B_{2}$ and $B_{2}^{*}$.

PROOF. If we multiply the first equation in (3.33) by $\left[B_{\left(\gamma^{\prime}\right)}^{(\beta)}\right]$ and the first equation in (3.35) by $B_{\delta^{\prime}}^{\alpha}$ we get

$$
\begin{gathered}
{\left[M_{\left(\beta^{\prime}\right)}^{\left(\alpha^{\prime}\right)}\right]\left[B_{(\beta)}^{\left(\beta^{\prime}\right)}\right]\left[B_{\left(\gamma^{\prime}\right)}^{(\beta)}\right]=B_{\alpha}^{\alpha^{\prime}}\left[M_{(\beta)}^{(\alpha)}\right]\left[B_{\left(\gamma^{\prime}\right)}^{(\beta)}\right] \Rightarrow} \\
\Rightarrow\left[M_{\left(\gamma^{\prime}\right)}^{\left(\alpha^{\prime}\right)}\right]=B_{\alpha}^{\alpha^{\prime}}\left[M_{(\beta)}^{(\alpha)}\right]\left[B_{\left(\gamma^{\prime}\right)}^{(\beta)}\right] \\
{\left[B_{(\gamma)}^{\left(\gamma^{\prime}\right)}\right]\left[N_{(\alpha)}^{(\gamma)}\right] B_{\delta^{\prime}}^{\alpha}=\left[N_{\left(\alpha^{\prime}\right)}^{\left(\gamma^{\prime}\right)}\right] B_{\alpha}^{\alpha^{\prime}} B_{\delta^{\prime}}^{\alpha} \Rightarrow\left[B_{(\gamma)}^{\left(\gamma^{\prime}\right)}\right]\left[N_{(\alpha)}^{(\gamma)}\right] B_{\delta^{\prime}}^{\alpha}=\left[N_{\left(\delta^{\prime}\right)}^{\left(\gamma^{\prime}\right)}\right]}
\end{gathered}
$$

Now using (3.36) we have

$$
\begin{gathered}
{\left[M_{\left(\gamma^{\prime}\right)}^{\left(\alpha^{\prime}\right)}\right]\left[N_{\left(\delta^{\prime}\right)}^{\left(\gamma^{\prime}\right)}\right]=} \\
=B_{\alpha}^{\alpha^{\prime}}\left[M_{(\beta)}^{(\alpha)}\right]\left[B_{\left(\gamma^{\prime}\right)}^{(\beta)}\right]\left[B_{(\gamma)}^{\left(\gamma^{\prime}\right)}\right]\left[N_{(\alpha)}^{(\gamma)}\right] B_{\delta^{\prime}}^{\alpha}= \\
=B_{\alpha}^{\alpha^{\prime}} B_{\delta^{\prime}}^{\alpha}=\delta_{\delta^{\prime}}^{\alpha^{\prime}} I
\end{gathered}
$$

## 4. The linear connection on $T(E)$. The induced connection

Definition 4.1. The generalized linear connection

$$
\nabla: T(E) \times T(E) \rightarrow T(E), \quad \nabla:(X, Y) \rightarrow \nabla_{X} Y
$$

is the linear connection which in the adapted basis $B=\left\{\delta_{A a}\right\}$ is given by (4.1)

$$
\begin{aligned}
\nabla_{\delta_{A a}} \delta_{B b} & =\Gamma_{B b A a}^{C c} \delta_{C c} \\
& =\Gamma_{B b A a}^{0 c} \delta_{0 c}+\Gamma_{B b A a}^{1 c} \delta_{1 c}+\cdots+\underline{\Gamma_{B b A a}^{B c} \delta_{B c}}+\cdots+\Gamma_{B b A a}^{k c} \delta_{k c}
\end{aligned}
$$

and the summation is going over both kinds of indices $(A, B, C=0,1, \ldots, k$, $a, b, c=1,2, \ldots, n)$.

DEFINITION 4.2. The $d$-connection (distinguished connection) is such a linear connection for which in (4.1) remain only the underlined terms, the other are equal to zero.

If we denote by $T_{0}(E), T_{1}(E), \ldots, T_{k}(E)$ the subspaces of $T(E)$ spanned by $\left\{\delta_{0 a}\right\},\left\{\delta_{1 a}\right\}, \ldots,\left\{\delta_{k a}\right\}$ respectively, we see that the $d$-connection preserves $T_{0}(E), T_{1}(E), \ldots, T_{k}(E)$. More precisely $\nabla_{X} Y$ and $Y$ belong to the same subspace, one of $T_{0}(E), T_{1}(E), \ldots, T_{k}(E)$ for every $X \in T(E)$.

Definition 4.3. The generalized linear connection is strongly distinguished (s.d) connection if

$$
\begin{gathered}
\nabla_{\delta_{A a}} \delta_{B b}=0 \quad \text { for } A \neq B, \\
\nabla_{\delta_{A a}} \delta_{A b}=\Gamma_{A b A a}^{A c} \delta_{A c} .
\end{gathered}
$$

For the $s . d$-connection $X, Y$ and $\nabla_{X} Y$ belong to the same subspace of $T(E)$, one of $T_{0}(E), T_{1}(E), \ldots, T_{k}(E)$.

Now we shall consider the generalized linear connection. If $X$ and $Y$ are two vector fields in $T(E)$, i.e. $X=X^{A a} \delta_{A a}, Y=Y^{B b} \delta_{B b}$, then $\nabla_{X} Y=\nabla_{X^{A a} \delta_{A a}} Y^{B b} \delta_{B b}$. Using the properties of linear connection $\nabla$, we get

$$
\begin{align*}
\nabla_{X} Y & =X^{A a}\left(\delta_{A a} Y^{B b}\right) \delta_{B b}+X^{A a} Y^{B b} \Gamma_{B b A a}^{C c} \delta_{C c} \\
& =X^{A a}\left(\delta_{A a} Y^{C c}+\Gamma_{B b A a}^{C c} Y^{B b}\right) \delta_{C c}=\left(Y_{\mid A a}^{C c}\right) X^{A a} \delta_{C c} . \tag{4.2}
\end{align*}
$$

where $Y_{\mid A a}^{C c}=\delta_{A a} Y^{C c}+\Gamma_{B b A a}^{C c} Y^{B b}$ is the covariant derivative of $Y^{C c}$ in the direction of $\delta_{A a}$.

Theorem 4.4. If $T$ is arbitrary tensor field defined on $T(E) \otimes T^{*}(E)$ and $X$ a vector field defined on $T(E)$ we have

$$
\begin{equation*}
\nabla_{X} T=\left(T_{C c \mid A a}^{B b} X^{A a}\right) \delta_{B b} \otimes \delta y^{C c}, \tag{4.3}
\end{equation*}
$$

where

$$
\begin{equation*}
T_{C c \mid A a}^{B b}=\delta_{A a} T^{B b}{ }_{C c}+T^{D d}{ }_{C c} \Gamma_{D d A a}^{B b}-T^{B b}{ }_{D d} \Gamma_{C c}^{D d} . \tag{4.4}
\end{equation*}
$$

As before in (4.3) and (4.4) the summation is going over all kinds of indices.

DEfintion 4.5. The induced connection, also denoted by $\nabla$, is the action of generalized connection $\nabla$ on the basis $B^{\prime}$ and it is defined by

$$
\begin{align*}
& \nabla_{\delta_{A \alpha}} \delta_{B \beta}=\Gamma_{B \beta A \alpha}^{C \gamma} \delta_{C \gamma}+\Gamma_{B \beta A \alpha}^{C \widehat{\gamma}} \delta_{C \widehat{\gamma}}, \\
& \nabla_{\delta_{A \alpha}} \delta_{B \widehat{\beta}}=\Gamma_{B \widehat{\beta} A \alpha}^{C \gamma} \delta_{C \gamma}+\Gamma_{B \widehat{\beta} A \alpha}^{C \widehat{\gamma}} \delta_{C \widehat{\gamma}},  \tag{4.5}\\
& \nabla_{\delta_{A \widehat{\alpha}}} \delta_{B \beta}=\Gamma_{B \beta A \widehat{\alpha}}^{C \gamma} \delta_{C \gamma}+\Gamma_{B \beta A \widehat{\alpha}}^{C \widehat{~}} \delta_{C \widehat{\gamma}}, \\
& \nabla_{\delta_{A \widehat{\alpha}} \delta_{B \widehat{\beta}}=\Gamma_{B \widehat{\beta} A \widehat{\alpha}}^{C \gamma} \delta_{C \gamma}+\Gamma_{B \widehat{\beta} A \widehat{\alpha}}^{C \widehat{~}} \delta_{\widehat{\gamma}} .} .
\end{align*}
$$

Let us denote by $T_{0}\left(E_{1}\right), T_{1}\left(E_{1}\right), \ldots, T_{k}\left(E_{1}\right)$, the subspaces of $T\left(E_{1}\right)$ spanned by $\left\{\delta_{0 \alpha}\right\},\left\{\delta_{1 \alpha}\right\}, \ldots,\left\{\delta_{k \alpha}\right\}$ respectively and by $T_{0}\left(E_{2}\right), T_{1}\left(E_{2}\right)$, $\ldots, T_{k}\left(E_{2}\right)$, the subspaces of $T\left(E_{2}\right)$ spanned by $\left\{\delta_{0 \widehat{\alpha}}\right\},\left\{\delta_{1 \widehat{\alpha}}\right\}, \ldots,\left\{\delta_{k \widehat{\alpha}}\right\}$ respectively.

We have:

$$
\begin{gathered}
T\left(E_{i}\right)=T_{0}\left(E_{i}\right) \oplus T_{1}\left(E_{i}\right) \oplus \cdots \oplus T_{k}\left(E_{i}\right) \quad i=1,2 \\
T(E)=T\left(E_{1}\right) \oplus T\left(E_{2}\right)=T_{0}(E) \oplus T_{1}(E) \oplus \cdots \oplus T_{k}(E)
\end{gathered}
$$

DEfinition 4.6. The induced connection defined by (4.5) is almost $d$ connection if it preserves $T_{B}(E)=T_{B}\left(E_{1}\right) \oplus T_{B}\left(E_{2}\right), B=0,1,2, \ldots, k$, i.e. $\nabla_{X} Y$ and $Y$ belong to the same $T_{B}(E)$ for $\forall X \in T(E)$. It is given by (4.5) if we put everywhere $C=B$ (no summation over $B$ ), the other coefficients are equal to zero.

The induced connection is $d$-connection if it preserves $T_{B}\left(E_{1}\right)$ and $T_{B}\left(E_{2}\right)$, i.e. $\nabla_{X} Y$ and $Y$ belong both to $T_{B}\left(E_{1}\right)$ or $T_{B}\left(E_{2}\right)$ for $\forall X \in T(E)$. The induced $d$-connection is given by

$$
\begin{array}{ll}
\nabla_{\delta_{A \alpha}} \delta_{B \beta}=\Gamma_{B \beta A \alpha}^{B \gamma} \delta_{B \gamma}, & \nabla_{\delta_{A \alpha}} \delta_{B \widehat{\beta}}=\Gamma_{B \widehat{\beta} A \alpha}^{B \widehat{\gamma}} \delta_{B \widehat{\gamma}}  \tag{4.6}\\
\nabla_{\delta_{A \widehat{\alpha}}} \delta_{B \beta}=\Gamma_{B \beta A \widehat{\alpha}}^{B \gamma} \delta_{B \gamma}, & \nabla_{\delta_{A \widehat{\alpha}}} \delta_{A \widehat{\beta}}=\Gamma_{B \widehat{\beta} A \widehat{\alpha}}^{B \widehat{\gamma}} \delta_{B \widehat{\gamma}}
\end{array}
$$

(no summation over $B$ ). The other connection coefficients are equal to zero.
DEFINITION 4.7. The induced connection defined by (4.5) is almost $s . d$ connection if it is given by (4.5), where we put $A=B=C$. The other connection coefficients are equal to zero.

For this connections $X, Y$ and $\nabla_{X} Y$ belong to the same $T_{B}(E)(B=$ $=0,1,2, \ldots, k)$.

The induced connection is $s . d$-connection if $X, Y$ and $\nabla_{X} Y$ belong to the same $T_{B}\left(E_{1}\right)$ or $T_{B}\left(E_{2}\right)$, i.e.

$$
\begin{array}{ll}
\nabla_{\delta_{A \alpha}} \delta_{A \beta}=\Gamma_{A \beta A \alpha}^{A \gamma} \delta_{A \gamma}, & \nabla_{\delta_{A \alpha}} \delta_{A \widehat{\beta}}=0 \\
\nabla_{\delta_{A \widehat{\alpha}}} \delta_{A \widehat{\beta}}=\Gamma_{A \widehat{\beta} A \widehat{\alpha}}^{A \widehat{\gamma}} \delta_{A \widehat{\gamma}}, & \nabla_{\delta_{A \widehat{\alpha}}} \delta_{A \beta}=0
\end{array}
$$

THEOREM 4.8. All induced connection coefficients are the corresponding projections of connection coefficients defined in the surrounding space, only eight types of them: $\Gamma_{B \beta 0 \alpha}^{B \gamma}, \Gamma_{B \beta 0 \alpha}^{B \widehat{\gamma}}, \Gamma_{B \widehat{\beta} 0 \alpha}^{B \gamma}, \Gamma_{B \widehat{\beta} 0 \alpha}^{B \widehat{\gamma}}, \Gamma_{B \beta 0 \widehat{\alpha}}^{B \gamma}, \Gamma_{B \beta 0 \widehat{\alpha}}^{B \widehat{\gamma}}$,
$\Gamma_{B \widehat{\beta} 0 \widehat{\alpha}}^{B \gamma}, \Gamma_{B \widehat{\beta} 0 \widehat{\alpha}}^{B \widehat{\gamma}}$ have different relations to the connection coefficients in the ambient space, i.e.

$$
\begin{aligned}
\Gamma_{B y A x}^{C z} & =\Gamma_{B b A a}^{C c} B_{x}^{a} B_{y}^{b} B_{c}^{z}, \quad \text { for } B \neq C \text { and every } A=0,1,2, \ldots k \\
\Gamma_{B y A x}^{B z} & =\Gamma_{B A a}^{B c} B_{x}^{a} B_{y}^{b} B_{c}^{z}, \quad \text { for every } A \neq 0 \\
\Gamma_{B y 0 x}^{B z} & =\Gamma_{B b 0 a}^{B c} B_{x}^{a} B_{y}^{b} B_{c}^{z}-B_{b}^{z} B_{y x}^{b}, \quad \text { for } A=0
\end{aligned}
$$

where $x \in\{\alpha, \widehat{\alpha}\}, y \in\{\beta, \widehat{\beta}\}, z \in\{\gamma, \widehat{\gamma}\}$.
THEOREM 4.9. If the generalized linear connection $\nabla$ acts on the surrounding space as $d$-connection, then the induced connection on subspaces are almost d-connections.

THEOREM 4.10. If the generalized linear connection $\nabla$ acts on the surrounding space as $s . d$-connection, then the induced connections on subspaces are almost $s . d$-connections.

The proof of Theorems 4.1-4.4 can be found in [11].

## 5. Integrability conditions and Lie brackets

For the further consideration it is useful to introduce the following notation:

$$
\begin{aligned}
& \delta_{A a}^{(B)}= \begin{cases}0 & \text { for } B<A, \\
\partial_{A a} & \text { for } B=A, \\
\partial_{A a}-N_{A a}^{(A+1) c} \partial_{(A+1) c}-\cdots-N_{A a}^{B c} \partial_{B c} & \text { for } A<B \leq k .\end{cases} \\
& K_{0 a 0 b}^{B d}=\left(\delta_{0 b}^{(B)} N_{0 a}^{B d}-\delta_{0 a}^{(B)} N_{0 b}^{B d}\right)+\cdots+ \\
& +\left(\delta_{0 b}^{(2)} N_{0 a}^{2 c}-\delta_{0 a}^{(2)} N_{0 b}^{2 c}\right) M_{2 c}^{B d}+\left(\delta_{0 b}^{(1)} N_{0 a}^{1 c}-\delta_{0 a}^{(1)} N_{0 b}^{1 c}\right) M_{1 c}^{B d} . \\
& K_{A a}^{B d}{ }_{A b}= \\
& =\left(\delta_{A b}^{(B)} N_{A a}^{B d}-\delta_{A a}^{(B)} N_{A b}^{B d}\right)+\left(\delta_{A b}^{(B-1)} N_{A a}^{(B-1) c}-\delta_{A a}^{(B-1)} N_{A b}^{(B-1) c}\right) M_{(B-1) c}^{B d}+ \\
& +\cdots+\left(\delta_{A b}^{(2 A+1)} N_{A a}^{(2 A+1) c}-\delta_{A a}^{(2 A+1)} N_{A b}^{(2 A+1) c}\right) M_{(2 A+1) c}^{B d}+ \\
& +\left(\delta_{A b}^{(2 A)} N_{A a}^{2 A c}-\delta_{A a}^{(2 A)} N_{A b}^{2 A c}\right) M_{2 A c}^{B d} .
\end{aligned}
$$

THEOREM 5.1. For $k=2 l-1\left(T(E)=T\left(\right.\right.$ Osc $\left.\left.^{2 l-1} M\right)\right), T_{l}(E),{ }_{l+1}(E)$, $\ldots, T_{2 l-1}(E)$ are integrable distributions. $T_{H}(E)=T_{0}(E)$ is integrable distribution if and only if $K_{0 a 0 b}^{B d}=0, \forall B=1,2, \ldots, 2 l-1 . T_{A}(E)$, $A=1,2, \ldots, l-1$ are integrable distributions if and only if $K_{A a}^{B d} A b=0$, $\forall B=2 A, 2 A+1, \ldots, 2 l-1$.

THEOREM 5.2. For $k=2 l\left(T(E)=T\left(\right.\right.$ Osc $\left.\left.^{2 l} M\right)\right), T_{l+1}(E), T_{l+2}(E), \ldots$, $T_{2 l}(E)$ are integrable distributions. $T_{H}(E)=T_{0}(E)$ is integrable distribution if and only if $K_{0 a 0 b}^{B d}=0, \forall B=1,2, \ldots, 2 l . T_{A}(E), A=1,2, \ldots, l$ are integrable distributions if and only if $K_{A a}^{B d} A b=0, \forall B=2 A, 2 A+1, \ldots, 2 l$.

THEOREM 5.3. In Osc ${ }^{k} M$ we have:
(a) For $A \leq B<k, A+B \leq k$ we have

$$
\begin{equation*}
\left[\delta_{A a}, \delta_{B b}\right]=\bar{K}_{A a B b}^{(A+B) c} \partial_{(A+B) c}+\ldots+\bar{K}_{A a B b}^{k c} \partial_{k c} \tag{5.1}
\end{equation*}
$$

where we apply
$\bar{K}_{0 a 0 b}^{0 c}=0, \quad \bar{K}_{A a B b}^{C c}=\delta_{B b}^{(C-A)} N_{A a}^{C c}-\delta_{A a}^{(C-B)} N_{B b}^{C c}, \quad C=A+B, \ldots, k$.
For $A+B=k$ (5.1) reduces to

$$
\left[\delta_{A a}, \delta_{B b}\right]=\bar{K}_{A a B b}^{k c} \partial_{k c}=\left(\delta_{B b}^{(B)} N_{A a}^{k c}-\delta_{A b}^{(A)} N_{B a}^{k c}\right) \partial_{k c}
$$

For $A=0, B=k$ the above equation has the form

$$
\left[\delta_{0 a}, \delta_{k b}\right]=\bar{K}_{0 a k b}^{k c} \partial_{k c}=\delta_{k b}^{(k)} N_{0 a}^{k c} \partial_{k c}=\partial_{k b} N_{0 a}^{k c} \partial_{k c}
$$

(b) For $A+B>k$ we find

$$
\left[\delta_{A a}, \delta_{B b}\right]=0 \text { is equivalent to } \bar{K}_{A a B b}^{C c}=0, \quad C=0,1, \ldots, k
$$

In (5.1) the Lie bracket $\left[\delta_{A a}, \delta_{B b}\right]$ was expressed in the basis $\bar{B}$. In the basis $B$ we have:

THEOREM 5.4. The Lie brackets in the basis $B$ have the form

$$
\begin{equation*}
\left[\delta_{A a}, \delta_{B b}\right]=\sum_{C=A+B}^{k} K_{A a B b}^{C c} \delta_{C c} \tag{5.2}
\end{equation*}
$$

where we use the notation:

$$
\begin{aligned}
K_{A a B b}^{(A+B) c} & =\bar{K}_{A a B b}^{(A+B) c}, \\
K_{A a B b}^{(A+B+1) c} & =\bar{K}_{A a B b}^{(A+B+1) c}+\bar{K}_{A a B b}^{(A+B) d} M_{(A+B) d}^{(A+B+1) c}, \\
& \vdots \\
K_{A a B b}^{k c} & =\bar{K}_{A a B b}^{k c}+\bar{K}_{A a B b}^{(k-1) d} M_{(k-1) d}^{k c}+\cdots+\bar{K}_{A a B b}^{(A+B) d} M_{(A+B) d}^{k c} .
\end{aligned}
$$

For $A+B=k$ we find

$$
\left[\delta_{A a}, \delta_{B b}\right]=K_{A a B b}^{k c} \delta_{k c}
$$

where we use the fact

$$
K_{A a B b}^{k c}=\bar{K}_{A a B b}^{k c}, \quad\left[\delta_{A a}, \delta_{B b}\right]=0 \text { for } A+B>k
$$

The proof of Theorems 5.1-5.4 can be found in [5].

## 6. The torsion and curvature tensors

The torsion tensor $T(X, Y)$ is usually defined as follows:

$$
\begin{equation*}
T(X, Y)=\nabla_{X} Y-\nabla_{Y} X-[X, Y] \tag{6.1}
\end{equation*}
$$

If $X$ and $Y$ expressed in the basis $B$ have forms

$$
X=X^{A a} \delta_{A a}, \quad Y=Y^{B b} \delta_{B b}
$$

then using linearity of $\nabla$ and (6.1) we get the local expression

$$
\begin{equation*}
T(X, Y)=T_{B b A a}^{C c} X^{A a} Y^{B b} \delta_{C c} \tag{6.2}
\end{equation*}
$$

where the components are given by

$$
\begin{equation*}
T_{B b A a}^{C c}=\Gamma_{B b A a}^{C c}-\Gamma_{A a B b}^{C c}-K_{A a B b}^{C c}, \tag{6.3}
\end{equation*}
$$

$$
\begin{equation*}
\left[\delta_{A a}, \delta_{B b}\right]=K_{A a B b}^{C c} \delta_{C c} \tag{6.4}
\end{equation*}
$$

The curvature tensor for the generalized connection $\nabla$ is usually defined as follows:

$$
\begin{equation*}
R(X, Y) Z=\nabla_{X} \nabla_{Y} Z-\nabla_{Y} \nabla_{X} Z-\nabla_{[X, Y]} \tag{6.5}
\end{equation*}
$$

If the notations $X=X^{A a} \delta_{A a}, Y=Y^{B b} \delta_{B b}$ and $Z=Z^{C c} \delta_{C c}$ are used, then we get the local expression

$$
\begin{equation*}
R(X, Y) Z=R_{C c B b A a}^{D d} Z^{C c} Y^{B b} X^{A a} \delta_{D d} \tag{6.6}
\end{equation*}
$$

where the components are given by

$$
\begin{gather*}
R_{C c B b A a}^{D d}=K_{C c B b A a}^{D d}-K_{A a B b}^{E e} \Gamma_{C c E e}^{D d}  \tag{6.7}\\
K_{C c B b A a}^{D d}=\left(\delta_{A a} \Gamma_{C c B b}^{D d}+\Gamma_{C c B b}^{E e} \Gamma_{E e A a}^{D d}\right)-(A a / B b) \tag{6.8}
\end{gather*}
$$

Theorem 5.4 has consequences on the above formulae. From (5.2), (6.3) and (6.4) we have:

TheOrem 6.1. In $\operatorname{Osc}^{k} \boldsymbol{M}$ the torsion tensor has the form ([5])

$$
\begin{align*}
& T_{B b A a}^{C c}=\Gamma_{B b A a}^{C c}-\Gamma_{A a B b}^{C c} \quad \text { for } C<A+B<k \text { or } A+B>k \text { and }  \tag{6.9}\\
& T_{B b A a}^{C c}=\Gamma_{B b A a}^{C c}-\Gamma_{A a B b}^{C c}-K_{A a B b}^{C c} \quad \text { for } A+B \leq C \leq k
\end{align*}
$$

Using (5.2) we obtain:
THEOREM 6.2. In $O s c^{k} M$ the curvature tensor has the form ([5])

$$
\begin{array}{ll}
R_{C c B b A a}^{D d}=K_{C c B b A a}^{D d} & \text { for } A+B>k \text { and }  \tag{6.10}\\
R_{C c B b A a}^{D d}=K_{C c B b A a}^{D d}-\sum_{E=A+B}^{k} K_{A a B b}^{E e} \Gamma_{C c E e}^{D d} \quad \text { for } 0<A+B \leq k
\end{array}
$$

THEOREM 6.3. If the connection $\nabla$ in $O s c^{k} M$ is $d$-connection (i.e. $\Gamma_{B b A a}^{C c}=0$ for $B \neq C$ ) the components of the torsion tensor have the form: (6.11)

$$
\begin{aligned}
T_{B b A a}^{C c} & =0 \quad \text { for } C \neq B, C \neq A, C<A+B<k \text { or } A+B>k \\
T_{B b A a}^{B c} & =\Gamma_{B b A a}^{B c}, \quad T_{B b A a}^{B c}=\Gamma_{B b A a}^{B c} \quad \text { for } A \neq B, A \neq 0, B \neq 0 \\
T_{B b A a}^{C c} & =-K_{A a B b}^{C c} \quad \text { for } A+B<C \leq k \\
T_{B b 0 a}^{B c} & =\Gamma_{B b 0 a}^{B c}-K_{0 a B b}^{B c} \quad \text { for } B \neq 0 \\
T_{0 b A a}^{A c} & =-\Gamma_{A a 0 b}^{A c}-K_{A a 0 b}^{A c} \quad \text { for } A \neq 0 \\
T_{A b A a}^{A c} & =\Gamma_{A b A a}^{A c}-\Gamma_{A a A b}^{A c}
\end{aligned}
$$

THEOREM 6.4. For the strongly distinguished connection (all connection coefficients are equal to zero except $\Gamma_{A b A a}^{A c}$ ) the components of the torsion
are given by

$$
\begin{align*}
& T_{B b A a}^{C c}=-K_{A a B c}^{C c} \quad \text { for } A+B \leq C \leq k \\
& T_{A b A a}^{A c}=\Gamma_{A b A a}^{A c}-\Gamma_{A a A b}^{A c} \tag{6.12}
\end{align*}
$$

all other connection coefficients are equal to zero.

THEOREM 6.5. If the connection $\nabla$ in $\operatorname{Osc} c^{k} M$ is $d$-connection $\left(\Gamma_{B b A a}^{C c}=\right.$ $=0$ for $B \neq C$ ) the components of the curvature tensor have the form

$$
K_{C c B b A a}^{D d}=0 \quad \text { for } C \neq D
$$

$$
\begin{equation*}
K_{C c B b A a}^{C d}=\left(\delta_{A a} \Gamma_{C c B b}^{C d}+\Gamma_{C c B b}^{C e} \Gamma_{C e A a}^{C d}\right)-(A a / B b) \tag{6.13}
\end{equation*}
$$ (no summation over $C$ )

$$
\begin{align*}
& R_{C c B b A a}^{C d}=K_{C c B b A a}^{C d} \quad \text { for } A+B>k  \tag{6.14}\\
& R_{C c B b A a}^{C d}=K_{C c B b A a}^{C d}-\sum_{E=A+B}^{k} K_{A a B b}^{E e} \Gamma_{C c E e}^{C d} \quad \text { for } A+B \leq k,
\end{align*}
$$

other coefficients of the curvature tensor are equal to zero.

THEOREM 6.6. If the connection $\nabla$ in $O s c^{k} M$ is $s . d$-connection (all connection coefficients are equal to zero except $\Gamma_{A b A a}^{A c}$ ) the components of the curvature tensor have the form

$$
\begin{aligned}
& K_{A c A b A a}^{A d}=\left(\delta_{A a} \Gamma_{A c A b}^{A d}+\Gamma_{A c A b}^{A e} \Gamma_{A e A a}^{A d}\right)-(a / b), \\
& R_{A c A b A a}^{A d}=K_{A c A b A a}^{A d},
\end{aligned}
$$

all other components of the curvature tensor are equal to zero.

## 7. Decomposition of the torsion and curvature tensor

THEOREM 7.1. The torsion tensor in the base $B^{\prime}$ and $B^{* *}$ can be written as follows

$$
\begin{align*}
T(X, Y)= & \left(T_{B \beta A \alpha}^{C \gamma} X^{A \alpha} Y^{B \beta} \delta_{C \gamma}+T_{B \beta A \widehat{\alpha}}^{C \gamma} X^{A \widehat{\alpha}} Y^{B \beta} \delta_{C \gamma}+\right. \\
& \left.+T_{B \widehat{\beta} A \alpha}^{C \gamma} X^{A \alpha} Y^{B \widehat{\beta}} \delta_{C \gamma}+T_{B \widehat{\beta}}^{C \gamma}{ }_{A \widehat{\alpha}}^{A \widehat{\alpha}} X^{B \widehat{\beta}} \delta_{C \gamma}\right)+  \tag{7.1}\\
& +\left(T_{B \beta}^{C \widehat{\gamma}} A_{A \alpha}^{A \alpha} X^{B \beta} \delta_{C \widehat{\gamma}}+T_{B \beta A \widehat{\alpha}}^{C \widehat{\gamma}} X^{A \widehat{\alpha}} Y^{B \beta} \delta_{C \widehat{\gamma}}+\right. \\
& \left.+T_{B \widehat{\beta} A \alpha}^{C \widehat{\gamma}} X^{A \alpha} Y^{B \widehat{\beta}} \delta_{C \widehat{\gamma}}+T_{B \widehat{\beta} A \widehat{\alpha}}^{C \widehat{\alpha}} X^{A \widehat{\alpha}} Y^{B \widehat{\beta}} \delta_{C \widehat{\gamma}}\right)
\end{align*}
$$

and

$$
\begin{array}{ll}
T_{B \beta A \alpha}^{C \gamma}=T_{B b A a}^{C c} B_{\alpha}^{a} B_{\beta}^{b} B_{\gamma}^{c}, & T_{B \beta A \widehat{\alpha}}^{C \gamma}=T_{B b A a}^{C c} B_{\widehat{\alpha}}^{a} B_{\beta}^{b} B_{\gamma}^{c} \\
T_{B \widehat{\beta} A \alpha}^{C \gamma}=T_{B b A a}^{C c} B_{\alpha}^{a} B_{\widehat{\beta}}^{b} B_{\gamma}^{c}, & T_{B \widehat{\beta} A \widehat{\alpha}}^{C \gamma}=T_{B b A a}^{C c} B_{\widehat{\alpha}}^{a} B_{\widehat{\beta}}^{b} B_{\gamma}^{c} . \tag{7.2}
\end{array}
$$

The above formulae are valid if $\gamma$ is substituted by $\widehat{\gamma}$ everywhere.
Proof. Using (3.42) and (3.8) we have

$$
X=X^{A a} \delta_{A a}=X^{A a}\left(B_{a}^{\alpha} \delta_{A \alpha}+B_{a}^{\widehat{\alpha}} \delta_{A \widehat{\alpha}}\right)=X^{A \alpha} \delta_{A \alpha}+X^{A \widehat{\alpha}} \delta_{A \widehat{\alpha}}
$$

where

$$
\begin{align*}
& X^{A \alpha}=X^{A a} B_{a}^{\alpha} \\
& X^{A \widehat{\alpha}}=X^{A a} B_{a}^{\widehat{\alpha}} \tag{7.3}
\end{align*}
$$

From the above it follows
i.e.

$$
X^{A \alpha} B_{\alpha}^{b}+X^{A \widehat{\alpha}} B_{\widehat{\alpha}}^{b}=X^{A a}\left(B_{a}^{\alpha} B_{\alpha}^{b}+B_{a}^{\widehat{\alpha}} B_{\widehat{\alpha}}^{b}\right)=X^{A a} \delta_{a}^{b}=X^{A b},
$$

$$
\begin{align*}
& X^{A a}=X^{A \alpha} B_{\alpha}^{a}+X^{A \widehat{\alpha}} B_{\widehat{\alpha}}^{a} \\
& Y^{B b}=Y^{B \beta} B_{\beta}^{b}+Y^{B \widehat{\beta}} B_{\widehat{\beta}}^{b} \tag{7.4}
\end{align*}
$$

The substitution of (3.42) and (7.4) into (6.2) results

$$
\begin{align*}
T(X, Y)=T_{B b A a}^{C c}\left(X^{A \alpha} B_{\alpha}^{a}+X^{A \widehat{\alpha}} B_{\widehat{\alpha}}^{a}\right)\left(Y^{B \beta} B_{\beta}^{b}+Y^{B \widehat{\beta}} B_{\widehat{\beta}}^{b}\right)  \tag{7.5}\\
\cdot\left(B_{c}^{\gamma} \delta_{C \gamma}+B_{c}^{\widehat{\gamma}} \delta_{C \widehat{\gamma}}\right)
\end{align*}
$$

The explicit form of (7.5) results (7.1) and (7.2).

THEOREM 7.2. The Gauss-Codazzi equations for the Osc ${ }^{k} M$ which give the relation between the curvature tensor in the bases $B$ and $B^{*}$, further $B^{\prime}$ and $B^{* *}$ can be expressed in the following way:

$$
\begin{align*}
& R(X, Y, Z)=R_{C c}^{D d}{ }^{D b} A a^{C c} Y^{B b} X^{A a} \delta_{D d}= \\
& \quad+R_{C \gamma B \beta A \alpha}^{D \delta} Z^{C \gamma} Y^{B \beta} X^{A \alpha} \delta_{D \delta}+R_{C \gamma B \beta A \widehat{\alpha}}^{D \delta} Z^{C \gamma} Y^{B \beta} X^{A \widehat{\alpha}} \delta_{D \delta}+ \\
& \quad+R_{C \gamma B \widehat{\beta} A \alpha}^{D \delta} Z^{C \gamma} Y^{B \widehat{\beta}} X^{A \alpha} \delta_{D \delta}+R_{C \widehat{\gamma} B \beta A \alpha}^{D \delta} Z^{C \widehat{\gamma}} Y^{B \beta} X^{A \alpha} \delta_{D \delta}+ \\
& \quad+R_{C \gamma B \widehat{\beta} A \widehat{\alpha}}^{D \delta} Z^{C \gamma} Y^{B \widehat{\beta}} X^{A \widehat{\alpha}} \delta_{D \delta}+R_{C \widehat{\gamma} B \beta A \widehat{\alpha}}^{D \delta} Z^{C \widehat{\gamma}} Y^{B \beta} X^{A \widehat{\alpha}} \delta_{D \delta}+ \\
& \quad+R_{C \widehat{\gamma} B \widehat{\beta} A \alpha}^{D \delta} Z^{C \widehat{\gamma}} Y^{B \widehat{\beta}} X^{A \alpha} \delta_{D \delta}+R_{C \widehat{\gamma} B \widehat{\beta} A \widehat{\alpha}}^{D \delta} Z^{C \widehat{\gamma}} Y^{B \widehat{\beta}} X^{A \widehat{\alpha}} \delta_{D \delta}+  \tag{7.6}\\
& \quad+R_{C \gamma B \beta A \alpha}^{D \widehat{\delta}} Z^{C \gamma} Y^{B \beta} X^{A \alpha} \delta_{D \widehat{\delta}}+R_{C \gamma B \beta A \widehat{\alpha}}^{D \widehat{\delta}} Z^{C \gamma} Y^{B \beta} X^{A \widehat{\alpha}} \delta_{D \widehat{\delta}}+ \\
& \quad+R_{C \gamma B \widehat{\beta} A \alpha}^{D \widehat{\delta}} Z^{C \gamma} Y^{B \widehat{\beta}} X^{A \alpha} \delta_{D \widehat{\delta}}+R_{C \widehat{\gamma} B \beta A \alpha}^{D \widehat{\delta}} Z^{C \widehat{\gamma}} Y^{B \beta} X^{A \alpha} \delta_{D \widehat{\delta}}+ \\
& \quad+R_{C \gamma B \widehat{\beta} A \widehat{\alpha}}^{D \widehat{\delta}} Z^{C \gamma} Y^{B \widehat{\beta}} X^{A \widehat{\alpha}} \delta_{D \widehat{\delta}}+R_{C \widehat{\gamma} B \beta A \widehat{\alpha}}^{D \widehat{\delta}} Z^{C \widehat{\gamma}} Y^{B \beta} X^{A \widehat{\alpha}} \delta_{D \widehat{\delta}}+ \\
& \quad+R_{C \widehat{\gamma} B \widehat{\beta} A \alpha}^{D \widehat{\delta}} Z^{C \widehat{\gamma}} Y^{B \widehat{\beta}} X^{A \alpha} \delta_{D \widehat{\delta}}+R_{C \widehat{\gamma} B \widehat{\beta} A \widehat{\alpha}}^{D \widehat{\delta}} Z^{C \widehat{\gamma}} Y^{B \widehat{\beta}} X^{A \widehat{\alpha}} \delta_{D \widehat{\delta}}
\end{align*}
$$

where

$$
\begin{align*}
R_{C \gamma B \beta A \alpha}^{D \delta} & =R_{C c B b A a}^{D d} B_{c}^{\gamma} B_{\beta}^{b} B_{\alpha}^{a} B_{d}^{\delta}, \\
R_{C \gamma B \beta A \widehat{\alpha}}^{D \delta} & =R_{C c B b A a}^{D d} B_{c}^{\gamma} B_{\beta}^{b} B_{\widehat{\alpha}}^{a} B_{d}^{\delta}, \\
R_{C \gamma B \widehat{\beta} A \alpha}^{D \delta} & =R_{C c B b A a}^{D d} B_{c}^{\gamma} B_{\widehat{\beta}}^{b} B_{\alpha}^{a} B_{d}^{\delta}, \\
R_{C \widehat{\gamma} B \beta A \alpha}^{D \delta} & =R_{C c B b A a}^{D d} B_{\widehat{\gamma}}^{c} B_{\beta}^{b} B_{\alpha}^{a} B_{d}^{\delta}, \\
R_{C \gamma B \widehat{\beta} A \widehat{\alpha}}^{D \delta} & =R_{C c B b A a}^{D d} B_{c}^{\gamma} B_{\widehat{\beta}}^{b} B_{\widehat{\alpha}}^{a} B_{d}^{\delta},  \tag{7.7}\\
R_{C \widehat{\gamma} B \beta A \widehat{\alpha}}^{D \delta} & =R_{C c B b A a}^{D d} B_{\widehat{\gamma}}^{c} B_{\beta}^{b} B_{\widehat{\alpha}}^{a} B_{d}^{\delta}, \\
R_{C \widehat{\gamma} B \widehat{\beta} A \alpha}^{D \delta} & =R_{C c B b A a}^{D d} B_{\widehat{\gamma}}^{c} B_{\widehat{\beta}}^{b} B_{\alpha}^{a} B_{d}^{\delta}, \\
R_{C \widehat{\gamma} B \widehat{\beta} A \widehat{\alpha}}^{D \delta} & =R_{C c B b A a}^{D d} B_{\widehat{\gamma}}^{c} B_{\widehat{\beta}}^{b} B_{\widehat{\alpha}}^{a} B_{d}^{\delta}
\end{align*}
$$

The equations (7.7) are valid if $\delta$ is substituted by $\widehat{\delta}$ everywhere.

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## ON SOME CLASS OF HYPERSURFACES IN EUCLIDEAN SPACES

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Dedicated to the memory of Professor Barbara Rabijewska

## 1. Introduction

Let $M$ be a hypersurface immersed isometrically in a semi-Riemannian space of constant curvature $N_{s}^{n+1}(c)$, with signature $(s, n+1-s), n \geq 4$. In [16](Proposition 5.1) it is proved that if on $\mathcal{U}_{H} \subset M$ the second fundamental tensor $H$ of $M$ satisfies

$$
\begin{equation*}
H^{3}=\operatorname{tr}(H) H^{2}+\psi H+\rho g, \tag{1}
\end{equation*}
$$

for some functions $\psi$ and $\rho$ on $\mathcal{U}_{H}$, then on this set we have
(a)

$$
\begin{aligned}
R \cdot C= & Q(S, R)-\frac{(n-2) \widetilde{\kappa}}{n(n+1)} Q(g, R) \\
& +\alpha_{2} Q(S, G)+\frac{\rho}{n-2} Q(H, G)
\end{aligned}
$$

(b) $C \cdot R=\frac{n-3}{n-2} Q(S, R)+\alpha_{1} Q(g, R)+\alpha_{2} Q(S, G)$,

$$
\text { (c) } \begin{align*}
C \cdot C= & \frac{n-3}{n-2} Q(S, R)+\alpha_{1} Q(g, C)  \tag{2}\\
& -\frac{\alpha_{2}}{n-2} Q(S, G)+\frac{(n-3) \rho}{(n-2)^{2}} Q(H, G), \\
\alpha_{1}= & \frac{1}{n-2}\left(\frac{\kappa}{n-1}+\varepsilon \psi-\frac{\left(n^{2}-3 n+3\right) \widetilde{\kappa}}{n(n+1)}\right),
\end{align*}
$$

$$
\alpha_{2}=-\frac{(n-3) \widetilde{\kappa}}{(n-2) n(n+1)}
$$

where $\widetilde{\kappa}$ is the scalar curvature of the ambient space. For precise definitions of the symbols used we refer to Sections 2 and 3 of [12] (see also [14] and [15]). Hypersurfaces in $N_{s}^{n+1}(c), n \geq 4$, satisfying (1) and some other curvature conditions were recently investigated in [15]. In [16](Theorem 5.1) it was proved that if (2)(b) holds on the set $V$ of all points of $U_{H} \subset M$ of a hypersurface $M$ in $N_{s}^{n+1}(c), n \geq 4$, at which $R \cdot S \neq \frac{\widetilde{\kappa}}{n(n+1)} Q(g, S)$ then (1) holds on $V$. We say that the conditions (2)(a), (2)(b) or (2)(c), as well as other conditions of this kind are conditions of pseudosymmetry type. We refer to [4] for a survey of results related to manifolds, and in particular to hypersurfaces, satisfying pseudosymmetry type conditions. We mention that hypersurfaces $M$ in $N_{s}^{n+1}(c), n \geq 4$, having the tensor $R$. . $C$ expressed by a linear combination of the tensors $Q(S, R), Q(g, R)$ and $Q(S, G)$ were investigated in [14] (see also [12]). Among other things in [14] it was shown that on hypersurfaces $M$ satisfying such condition we have $R \cdot S=\frac{\widetilde{\kappa}}{n(n+1)} Q(g, S)$ on $U_{H} \subset M$, and in a consequence $M$ is a Ricci-pseudosymmetric manifold.

In the case, when the ambient space is a semi-Euclidean space $\mathbb{E}_{s}^{n+1}$, with signature $(s, n+1-s), n \geq 4$, (2) yields
(a) $\quad R \cdot C=Q(S, R)+\frac{\rho}{n-2} Q(H, G)$,
(b) $C \cdot R=\frac{n-3}{n-2} Q(S, R)+\alpha_{1} Q(g, R)$,
(3)
(c) $C \cdot C=\frac{n-3}{n-2} Q(S, R)+\alpha_{1} Q(g, C)+\frac{(n-3) \rho}{(n-2)^{2}} Q(H, G)$, $\alpha_{1}=\frac{1}{n-2}\left(\frac{\kappa}{n-1}+\varepsilon \psi\right), \quad \alpha_{2}=0$.

Further, when on $U_{H}$ we have

$$
\begin{equation*}
H^{3}=\operatorname{tr}(H) H^{2}-\frac{\varepsilon \kappa}{n-1} H+\rho g \tag{4}
\end{equation*}
$$

i.e. (1) with

$$
\begin{equation*}
\psi=-\frac{\varepsilon \kappa}{n-1} \tag{5}
\end{equation*}
$$

then (3) turns into
(a) $R \cdot C=Q(S, R)+\frac{\rho}{n-2} Q(H, G)$,
(b) $C \cdot R=\frac{n-3}{n-2} Q(S, R)$,
(c)

$$
\begin{equation*}
C \cdot C=\frac{n-3}{n-2} Q(S, R)+\frac{(n-3) \rho}{(n-2)^{2}} Q(H, G) \tag{6}
\end{equation*}
$$

Investigations on the problem of the equivalence of the conditons Ricci-semisymmetry $(R \cdot S=0)$ and semisymmetry $(R \cdot R=0)$ on hypersurfaces in $\mathbb{E}_{s}^{n+1}$, named the problem of P. J. Ryan (see [1], [10] and references therein), lead among others things to quasi-Einstein hypersurfaces satisfying

$$
\begin{equation*}
H^{3}=\operatorname{tr}(H) H^{2}-\frac{\varepsilon \kappa}{n-1} H \tag{7}
\end{equation*}
$$

i.e. (4) with $\rho=0$ (see e.g. [10], Theorem 1.1). Clearly, such hypersurfaces satisfy (6)(b). We refer to [1], [10] and [11] for examples of hypersurfaces in $\mathbb{E}_{s}^{n+1}$ satisfying (7) (see also [12], Example 5.1(i)). We mention that hypersurfaces in $\mathbb{E}_{s}^{n+1}, n \geq 4$, satisfying (6)(b) were investigated in [3]. We also refer to [2] for results related to hypersurfaces in $\mathbb{E}_{s}^{n+1}, n \geq 4$, having the tensor $C \cdot R$ expressed by a linear combination of some tensors.

In section 2 of this paper we consider hypersurfaces $M$ in $N_{S}^{5}(c)$ satisfying (1) on $\mathcal{U}_{H} \subset M$. We prove that on this set we have (see Proposition 2.1)

$$
\text { (a) } \quad \psi=-\frac{1}{2}\left((\operatorname{tr}(H))^{2}-\operatorname{tr}\left(H^{2}\right)\right), \quad \rho=0
$$

$$
\begin{equation*}
\text { (b) } \quad R \cdot R=\frac{\widetilde{\kappa}}{20} Q(g, R) \tag{8}
\end{equation*}
$$

Thus, in particular, $M$ is a pseudosymmetric manifold. If the ambient space is $\mathbb{E}_{S}^{5}$ and if (1) and (5) hold on $\mathcal{U}_{H}$ then on this set we have $\rho=\psi=\kappa=0$ and $R \cdot R=0$ (Proposition 2.2). In section 3 (see Example 3.1) we construct some tubes in Euclidean spaces $\mathbb{E}^{n+1}, n \geq 5$. We show that (4), with nonzero function $\rho$, holds at some points of that tubes. We also recall some known examples of hypersurfaces satisfying (1) with $\rho=0$. Recently examples of hypersurfaces $M$ in an Euclidean space $\mathbb{E}^{n+1}, n \geq 5$, satisfying (1), with nonzero function $\rho$, were found in [17].

## 2. 4-dimensional hypersurfaces satisfying (1)

Proposition 2.1. If $M$ is a hypersurface in $N_{s}^{5}(c)$ satisfying (1) on $\mathcal{U}_{H} \subset M$ then (8) holds on this set.

Proof. Applying (1) in the equation (10) of [9] we obtain on $\mathcal{U}_{H}$

$$
\begin{align*}
& \operatorname{tr}(H)\left(H_{l k} H_{i j}-H_{l j} H_{i k}\right)+\alpha G_{l i j k} \\
& \quad-\left(H_{i j} H_{l k}^{2}+H_{l k} H_{i j}^{2}-H_{i k} H_{j l}^{2}-H_{j l} H_{i k}^{2}\right) \\
& \quad-g_{i j}\left((\psi-\beta) H_{l k}+\rho g_{l k}\right)-g_{l k}\left((\psi-\beta) H_{i j}+\rho g_{i j}\right)  \tag{9}\\
& \quad+g_{j l}\left((\psi-\beta) H_{i k}+\rho g_{i k}\right)+g_{i k}\left((\psi-\beta) H_{l j}+\rho g_{l j}\right)=0
\end{align*}
$$

where $\alpha$ and $\beta$ are defined by (11) and (12) of [9], respectively. Next, transvecting (9) with $g^{i j}$ we find

$$
\begin{aligned}
&(n-1)(\alpha-2 \rho) g_{l k}+\operatorname{tr}(H)^{2} H_{l k}-\operatorname{tr}(H) H_{l k}^{2}+2 H_{l k}^{3}-\operatorname{tr}(H) H_{l k}^{2} \\
&-\operatorname{tr}\left(H^{2}\right) H_{l k}+(n-2)(\beta-\psi) H_{l k}+\operatorname{tr}(H)(\beta-\psi) g_{l k}=0
\end{aligned}
$$

Applying in this (1) we get

$$
\begin{align*}
H^{3}= & \operatorname{tr}(H) H^{2}-\frac{1}{2}\left((\operatorname{tr}(H))^{2}-\operatorname{tr}\left(H^{2}\right)+2(\beta-\psi)\right) H \\
& -\frac{1}{2}(\operatorname{tr}(H)(\beta-\psi)+3(\alpha-2 \rho)) g . \tag{10}
\end{align*}
$$

Comparing the right hand sides of (1) and (10) we obtain

$$
\begin{align*}
\psi & =-\frac{1}{2}\left((\operatorname{tr}(H))^{2}-\operatorname{tr}\left(H^{2}\right)+2(\beta-\psi)\right) \\
\rho & =-\frac{1}{2}(\operatorname{tr}(H)(\beta-\psi)+3(\alpha-2 \rho)) \tag{11}
\end{align*}
$$

(9), by transvection with $H_{h}^{l}$, yields

$$
\begin{aligned}
& H_{i j}\left(\frac{\operatorname{tr}(H)}{2} H_{h k}^{2}-H_{h k}^{3}+(\beta-\psi) H_{h k}\right)-H_{i k}\left(\frac{\operatorname{tr}(H)}{2} H_{h j}^{2}-H_{h j}^{3}+(\beta-\psi) H_{h j}\right) \\
& +H_{h k}^{2}\left(\frac{\operatorname{tr}(H)}{2} H_{i j}-H_{i j}^{2}+(\beta-\psi) g_{i j}\right)-H_{h j}^{2}\left(\frac{\operatorname{tr}(H)}{2} H_{i k}-H_{i k}^{2}+(\beta-\psi) g_{i k}\right) \\
& \quad=(2 \rho-\alpha)\left(g_{i j} H_{h k}-g_{i k} H_{h j}\right)
\end{aligned}
$$

Symmetrizing this in $h$ and $i$ and using (1) we find

$$
(\alpha-\rho) Q(g, H)+(\beta-\psi) Q\left(g, H^{2}\right)=0
$$

Since the tensors $Q(g, H)$ and $Q\left(g, H^{2}\right)$ are linearly independent at every point of $\mathcal{U}_{H}$, the last relation implies $\alpha=\rho$ and $\beta=\psi$. Applying this in (11) we obtain (8)(a). Now, in view of Proposition 3.2 of [8], we get $R \cdot S=\frac{\widetilde{\kappa}}{20} Q(g, S)$. But this, in view of Proposition 4.1 of [9], implies (8)(b), completing the proof.

Proposition 2.2. Let $M$ be a hypersurface in $\mathbb{E}_{s}^{5}$ satisfying (1) on $\mathcal{U}_{H} \subset$ $\subset M$.
(i) On $U_{H}$ we have $R \cdot R=0, \rho=0$ and

$$
\begin{equation*}
\psi=-\frac{\varepsilon}{2} \kappa \tag{12}
\end{equation*}
$$

(ii) Moreover, if (5) holds on $\mathcal{U}_{H}$ then $\psi=\kappa=0$.

Proof. The first two equations of our assertion are an immediate consequence of Proposition 2.1. Applying to (8)(a) the identity

$$
\begin{equation*}
\kappa=\varepsilon\left((\operatorname{tr}(H))^{2}-\operatorname{tr}\left(H^{2}\right)\right) \tag{13}
\end{equation*}
$$

which holds on any hypersurface in $\mathbb{E}_{s}^{n+1}$ (see e.g. [12], section 2), we obtain (12).
(ii) The second assertion follows immediately from (8)(a) and (12).

EXAMPLE 2.1. (i) Examples of hypersurfaces $M$ in $N_{s}^{5}(c)$ satisfying on $\mathcal{U}_{H} \subset M$ the equation (1), with $\rho=0$, are given in [13]. Precisely, on the hypersurface $M$ defined in Example 5.1 (resp. Example 5.2) of [13] we have $H^{3}=\operatorname{tr}(H) H^{2}, \psi=\rho=0$, resp., $H^{3}=\psi H, \psi=\frac{1}{2} \operatorname{tr}\left(H^{2}\right), \operatorname{tr}(H)=\rho=0$. The second type hypersurfaces are also mentioned in Example 3.2 of this paper.
(ii) In Example 3.3(ii) of [10] a 4-dimensional semisymmetric manifold is defined. That manifold is a warped product of an 1-dimensional manifold and a manifold isometric to the 3-dimensional Cartan hypersurface. As it was stated in Example 4.3 of [10] that warped product can be locally realized as a hypersurface $M$ in $\mathbb{E}_{S}^{5}$ satisfying on $\mathcal{U}_{H} \subset M$ the relations $H^{3}=\psi H$, $\psi=\frac{1}{2} \operatorname{tr}\left(H^{2}\right), \operatorname{tr}(H)=\rho=0$. That hypersurfaces is also mentioned in Example 3.2 of this paper.
(iii) An example of a hypersurface $M$ in $\mathbb{E}_{s}^{n+1}, n \geq 4$, satisfying the relations $R \cdot R=0, \kappa=0, H^{3}=\operatorname{tr}(H) H^{2}$ and $\psi=\rho=0$ on $\mathcal{U}_{H} \subset M$ is given in [11] (see examples 4.1 and 5.1 of that paper).

## 3. Examples

Let $\bar{M}$ be a $(p+q)$-dimensional submanifold in $\mathbb{E}^{n+1}, n>p+q, n \geq 5$, and let $\xi$ be a unit vector field normal to $\bar{M}$ such that the shape operator $\mathscr{A}_{\xi}$ determined by $\xi$ has at every point of $\bar{M}$ exactly two distinct nonzero constant principal curvatures $\mu_{1}$ and $\mu_{2}$, with multiplicities $p$ and $q$, respectively. Further, let $\Phi_{t}$ be a tube of radius $t>0$ over the submanifold $\bar{M}$. It is known that the hypersurface $\Phi_{t}$ has at every point $(x, \xi), x \in \bar{M}$, three principal curvatures ([5], Theorem 3.2, p. 131)

$$
\begin{equation*}
\lambda_{1}=\frac{\mu_{1}}{1-t \mu_{1}}, \quad \lambda_{2}=\frac{\mu_{2}}{1-t \mu_{2}}, \quad \lambda_{3}=-\frac{1}{t} \tag{14}
\end{equation*}
$$

with multiplicities $p, q$ and $n-p-q$, respectively. Evidently, we assume that

$$
\begin{equation*}
t \mu_{1} \neq 1 \quad \text { and } \quad t \mu_{2} \neq 1 \tag{15}
\end{equation*}
$$

Clearly, at points $(x, \xi), x \in \bar{M}$, of $\Phi_{t}$ we have

$$
\begin{equation*}
\operatorname{tr}(H)-\left(\lambda_{1}+\lambda_{2}+\lambda_{3}\right)=(p-1) \lambda_{1}+(q-1) \lambda_{2}+(n-p-q-1) \lambda_{3} . \tag{16}
\end{equation*}
$$

We assume that at a point $(x, \xi), x \in \bar{M}$, of $\Phi_{t}$ we have

$$
\begin{equation*}
\operatorname{tr}(H)=\lambda_{1}+\lambda_{2}+\lambda_{3} \tag{17}
\end{equation*}
$$

We note that if at such point we have $p=q=1$ then (16) implies $\lambda_{3}=0$, a contradiction. If $p=n-p-q=1$ then (16) implies $\lambda_{2}=0$, and by (14) $\mu_{2}=0$, a contradiction. If $q=n-p-q=1$ then (16) implies $\lambda_{1}=0$ and by (14) $\mu_{1}=0$, a contradiction. Therefore, at most one of the numbers $p, q$ or $n-p-q$ can be equal to 1 at every point $(x, \xi), x \in \bar{M}$, of $\Phi_{t}$ at which (17) holds.

Further, (17) by making use of (14) and (16) is equivalent at $(x, \xi)$ to

$$
\begin{equation*}
(n-3) \mu_{1} \mu_{2} t^{2}-\left((n-q-2) \mu_{1}+(n-p-2) \mu_{2}\right) t+(n-p-q-1)=0 \tag{18}
\end{equation*}
$$

We set

$$
\begin{align*}
w= & w(t)=a_{1} t^{2}+b_{1} t+c_{1}, t \in(0,+\infty) \\
a_{1}= & (n-3) \mu_{1} \mu_{2}, \quad b_{1}=-(n-q-2) \mu_{1}-(n-p-2) \mu_{2} \\
c_{1}= & n-p-q-1 \\
\Delta_{1}= & b_{1}^{2}-4 a_{1} c_{1}=\mu_{1}^{2}\left((n-p-2)^{2} \mu^{2}+2((n-q-2)(n-p-2)\right. \\
& \left.-2(n-3)(n-p-q-1)) \mu+(n-q-2)^{2}\right) \tag{19}
\end{align*}
$$

$$
\begin{aligned}
\Delta_{2}= & 4\left(((n-q-2)(n-p-2)-2(n-3)(n-p-q-1))^{2}\right. \\
& \left.-(n-p-2)^{2}(n-q-2)^{2}\right) \\
= & -16(n-3)(p-1)(q-1)(n-p-q-1),
\end{aligned}
$$

where $\mu=\frac{\mu_{2}}{\mu_{1}}$. Evidently, $w(t)=0$, for some $t \in(0,+\infty)$, if and only if $\Delta_{1} \geq 0$. Clearly, $\Delta_{1} \geq 0$ if and only if $\Delta_{2} \leq 0$.

We consider now the following cases: $p=1$ or $q=1$ or $n-p-q=1$.
If $p=1, q \geq 2$ and $n-q \geq 3$ then (18) yields

$$
\begin{equation*}
t=\frac{n-q-2}{n-3} \mu_{2}^{-1} \tag{20}
\end{equation*}
$$

provided that $\mu_{2}>0$. Applying (20) to (14) we find

$$
\begin{align*}
& \lambda_{1}=\frac{(n-3) \mu_{1} \mu_{2}}{(n-3) \mu_{2}-(n-q-2) \mu_{1}} \\
& \lambda_{2}=\frac{(n-3) \mu_{2}}{q-1}  \tag{21}\\
& \lambda_{3}=-\frac{(n-3) \mu_{2}}{n-q-2}
\end{align*}
$$

provided that $(n-q-2) \mu_{1} \neq(n-3) \mu_{2}$. Using (13), with $\varepsilon=1$, (17) and (21) we get

$$
\begin{aligned}
\frac{\kappa}{n-1}+\psi= & \frac{1}{n-1}\left((\operatorname{tr}(H))^{2}-\operatorname{tr}\left(H^{2}\right)\right)-\left(\lambda_{1} \lambda_{2}+\lambda_{1} \lambda_{3}+\lambda_{2} \lambda_{3}\right) \\
= & -\frac{1}{n-1}\left((p-1) \lambda_{1}^{2}+(q-1) \lambda_{2}^{2}+(n-p-q-1) \lambda_{3}^{2}\right) \\
& -\frac{n-3}{n-1}\left(\lambda_{1} \lambda_{2}+\lambda_{1} \lambda_{3}+\lambda_{2} \lambda_{3}\right) \\
= & -\frac{(n-3)^{3}(n-2 q-1)}{(n-1)(q-1)(n-q-2)} \frac{\mu_{1} \mu_{2}^{2}}{(n-3) \mu_{2}-(n-q-2) \mu_{1}} \\
= & \frac{(n-3)(n-2 q-1)}{(n-1)(q-1)} \lambda_{1} \lambda_{3} .
\end{aligned}
$$

We obtain similar results for $q=1, p \geq 2$ and $n-p \geq 3$, as well as for $p \geq 2, q \geq 2$ and $n-p-q \geq 2$.

We consider the case: $n=p+q+1, p \geq 2$ and $q \geq 2$. Now (18) yields

$$
\begin{equation*}
t=\frac{(n-q-2) \mu_{1}+(n-p-2) \mu_{2}}{(n-3) \mu_{1} \mu_{2}} \tag{23}
\end{equation*}
$$

provided that the right hand side of (23) is $>0$. Applying (23) to (14) we get

$$
\begin{align*}
& \lambda_{1}=-\frac{(n-3) \mu_{1} \mu_{2}}{(p-1)\left(\mu_{1}-\mu_{2}\right)} \\
& \lambda_{2}=\frac{(n-3) \mu_{1} \mu_{2}}{(q-1)\left(\mu_{1}-\mu_{2}\right)}  \tag{24}\\
& \lambda_{3}=-\frac{(n-3) \mu_{1} \mu_{2}}{(p-1) \mu_{1}+(q-1) \mu_{2}}
\end{align*}
$$

provided that $(p-1) \mu_{1}+(q-1) \mu_{2} \neq 0$. Now using (13), (17) and (24) we find

$$
\begin{align*}
\frac{\kappa}{n-1}+\psi= & -\frac{1}{n-1}\left((p-1) \lambda_{1}^{2}+(q-1) \lambda_{2}^{2}+(n-p-q-1) \lambda_{3}^{2}\right) \\
& -\frac{n-3}{n-1}\left(\lambda_{1} \lambda_{2}+\lambda_{1} \lambda_{3}+\lambda_{2} \lambda_{3}\right) \\
= & \frac{(n-3)^{3}(p-q)}{(n-1)(p-1)(q-1)} \frac{\mu_{1}^{2} \mu_{2}^{2}}{\left(\mu_{1}-\mu_{2}\right)\left((p-1) \mu_{1}+(q-1) \mu_{2}\right)}  \tag{25}\\
= & \frac{(n-3)(p-q)}{n-1} \lambda_{1} \lambda_{3}=\frac{(n-3)(n-2 q-1)}{n-1} \lambda_{1} \lambda_{3} .
\end{align*}
$$

Clearly, if $p=q$ then $\frac{\kappa}{n-1}+\psi=0$.
We consider now some tube assuming that $n=p+q+1$ and $p=q \geq 2$.
EXAMPLE 3.1. (i) (cf. [7]) Let $S^{p}\left(r_{1}\right)$ be a $p$-dimensional sphere in $\mathbb{E}^{p+1}$, $p \geq 2$, with radius $r_{1}>0$. Similarly, let $S^{q}\left(r_{2}\right)$ be a $q$-dimensional sphere in $\mathbb{E}^{q+1}, q \geq 2$, with radius $r_{2}>0$ and let $r_{1}^{2}+r_{2}^{2}=1$. We consider the standard product embedding of spheres $S^{p}\left(r_{1}\right)$ and $S^{q}\left(r_{2}\right)$

$$
S^{p}\left(r_{1}\right) \times S^{q}\left(r_{2}\right) \rightarrow S^{p+q+1}(1) \rightarrow \mathbb{E}^{p+1} \times \mathbb{E}^{q+1}=\mathbb{E}^{p+q+2}
$$

Let $f_{0}, f_{1}, \ldots, f_{p}$ be an orthonormal frame field for $\mathbb{E}^{p+1}$ such that $f_{0}$ is normal to $S^{p}\left(r_{1}\right)$. Similarly, let $f_{p+1}, f_{p+2}, \ldots, f_{p+q+1}$ be an orthonormal frame field for $\mathbb{E}^{q+1}$ such that $f_{p+q+1}$ is normal to $S^{q}\left(r_{2}\right)$. We define now a frame field $e_{0}, e_{1}, \ldots, e_{p+q}, f_{p+q+1}$ for $\mathbb{E}^{p+q+2}$ as follows $e_{0}=r_{1} f_{0}+r_{2} f_{p+q+1}, e_{i}=f_{i}$, $i=1, \ldots, p+q, e_{p+q+1}=r_{2} f_{0}-r_{1} f_{p+q+1}$. We can check that $e_{0}$ is normal to $S^{p+q+1}(1)$ and $e_{p+q+1}$ to $S^{p}\left(r_{1}\right) \times S^{q}\left(r_{2}\right)$. Let $\mathscr{A}_{e_{0}}$ and $\mathscr{A}_{e_{p+q+1}}$ be the shape operators with respect to $e_{0}$ and $e_{p+q+1}$, respectively. We have

$$
\mathscr{A}_{e_{0}}=\operatorname{Id}, \quad \mathscr{A}_{e_{p+q+1}}=\operatorname{diag}\left(\mu_{1}, \ldots, \mu_{1}, \mu_{2}, \ldots, \mu_{2}\right)
$$

where $\mu_{1}=\sqrt{r_{1}^{-2}-1}$ and $\mu_{2}=-\sqrt{r_{2}^{-2}-1}$, occurs $p$ times and $q$ times, respectively. We note that $\mu_{1} \mu_{2}=-1$. We set $r_{1}=\cos \phi$ and $r_{2}=\sin \phi$, where $\phi \in\left(0, \frac{\pi}{2}\right)$.
(ii) Let $\mathscr{A}_{\eta_{1}}$ and $\mathscr{A}_{\eta_{2}}$ be the shape operators with respect to unit vector fields $\eta_{1}$ and $\eta_{2}$ normal to $S^{p}\left(r_{1}\right) \times S^{q}\left(r_{2}\right)$ in $\mathbb{E}^{p+q+2}$ such that $\eta_{1}=\cos \alpha e_{0}+$ $+\sin \alpha e_{p+q+1}$ and $\eta_{2}=-\sin \alpha e_{0}+\cos \alpha e_{p+q+1}$. where $\alpha \in\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. We have

$$
\begin{align*}
& \mathscr{A}_{\eta_{1}}=\operatorname{diag}\left(\frac{\cos (\phi-\alpha)}{\cos \phi}, \ldots, \frac{\cos (\phi-\alpha)}{\cos \phi}, \frac{\sin (\phi-\alpha)}{\sin \phi}, \ldots, \frac{\sin (\phi-\alpha)}{\sin \phi}\right) \\
& \mathscr{A}_{\eta_{2}}=\operatorname{diag}\left(\frac{\sin (\phi-\alpha)}{\cos \phi}, \ldots, \frac{\sin (\phi-\alpha)}{\cos \phi},-\frac{\cos (\phi-\alpha)}{\sin \phi}, \ldots,-\frac{\cos (\phi-\alpha)}{\sin \phi}\right) . \tag{26}
\end{align*}
$$

The multiplicities of the principal curvatures of $\mathscr{A}_{\eta_{1}}$ are equal to $p$ and $q$, respectively. An analogous statement is true for $\mathscr{A}_{\eta_{2}}$.
(iii) Let $p=q \geq 2, \xi=e_{2 p+1}$ and $\phi \in\left(0, \frac{\pi}{4}\right)$. Thus $r_{1}>r_{2}$. Since $\mu_{1}=\tan \phi$ and $\mu_{2}=-\cot \phi$, (23) turns into

$$
\begin{equation*}
t=\cot 2 \phi>0 \tag{27}
\end{equation*}
$$

Further, (24) yields

$$
\begin{aligned}
& \lambda_{1}=-\lambda_{2}=-\sin 2 \phi \\
& \lambda_{3}=-\tan 2 \phi
\end{aligned}
$$

We consider now the tube $\Phi_{t}$ over the submanifold $\bar{M}=S^{p}\left(r_{1}\right) \times S^{p}\left(r_{2}\right)$ in $\mathbb{E}^{2(p+1)}$ of radius $t$ defined by (27). With respect to the above considerations we can state that (1) holds at all points $(x, \xi) \in \mathcal{U}_{H} \subset \Phi_{t}, x \in \bar{M}$. In addition, (25) yields $\psi=-\frac{\kappa}{n-1}$. It means that (4) holds at all points $(x, \xi)$, $x \in \bar{M}$.
(iv) Let $\Phi_{t}$ be the tube defined in (iii) and let $\xi=e_{2 p+1}$. Using now (24) and (26) we compute principal curvatures $\lambda_{11}, \lambda_{12}, \lambda_{13}$ and $\lambda_{21}, \lambda_{22}, \lambda_{23}$ of the tube $\Phi_{t}$ at all points $\left(x, \eta_{1}\right) \in \Phi_{t}$ and $\left(x, \eta_{2}\right) \in \Phi_{t}, x \in \bar{M}$, respectively.

Namely, we have

$$
\begin{aligned}
& \lambda_{11}=\frac{\cos (\phi-\alpha)}{\cos \phi-\cot 2 \phi \cos (\phi-\alpha)} \\
& \lambda_{12}=\frac{\sin (\phi-\alpha)}{\sin \phi-\cot 2 \phi \sin (\phi-\alpha)} \\
& \lambda_{13}=\lambda_{23}=-\tan 2 \phi \\
& \lambda_{21}=\frac{\sin (\phi-\alpha)}{\cos \phi-\cot 2 \phi \sin (\phi-\alpha)} \\
& \lambda_{22}=-\frac{\cos (\phi-\alpha)}{\sin \phi+\cot 2 \phi \cos (\phi-\alpha)}
\end{aligned}
$$

(v) Let $\Phi_{t}$ be the tube defined in (iii) and let $\xi=\eta_{1}$ and $0<\alpha<\phi<\frac{\pi}{4}$. From (26) we have $\mu_{11}=\frac{\cos (\phi-\alpha)}{\cos \phi}$ and $\mu_{12}=\frac{\sin (\phi-\alpha)}{\sin \phi}$. Now (23) turns into

$$
\begin{equation*}
t=\frac{\sin (2 \phi-\alpha)}{\sin 2(\phi-\alpha)} \tag{28}
\end{equation*}
$$

Further, (24) yields $\lambda_{11}=-\lambda_{12}=-\frac{\sin 2(\phi-\alpha)}{\sin \alpha}, \lambda_{13}=-\frac{\sin 2(\phi-\alpha)}{\sin (2 \phi-\alpha)}$. We consider now the tube $\Phi_{t}$ over the submanifold $\bar{M}=S^{p}\left(r_{1}\right) \times S^{p}\left(r_{2}\right)$ in $\mathbb{E}^{2(p+1)}$ of radius $t$ defined by (28). With respect to the above considerations we can state that (1) holds at all points $(x, \xi)$ of $\Phi_{t}, x \in \bar{M}$. Since $p=q$ and $r_{1}>r_{2}$, (25) yields $\psi=-\frac{\kappa}{n-1}$. Thus (4) holds at all points $(x, \xi)$ of $\Phi_{t}$, $x \in \bar{M}$. Finally, we also compute principal curvatures at all points $\left(x, \eta_{2}\right)$ of $\Phi_{t} x \in \bar{M}$. We have $\mu_{21}=\frac{\sin (\phi-\alpha)}{\cos \phi}$ and $\mu_{22}=-\frac{\cos (\phi-\alpha)}{\sin \phi}$. Now, by making use of (24) and (28), at the points $\left(x, \eta_{2}\right)$ we get $\lambda_{23}=\lambda_{13}$ and

$$
\begin{aligned}
& \lambda_{21}=\frac{\sin 2(\phi-\alpha)}{2 \cos \phi \cos (\phi-\alpha)-\sin (2 \phi-\alpha)} \\
& \lambda_{22}=-\frac{\sin 2(\phi-\alpha)}{2 \sin \phi \sin (\phi-\alpha)+\sin (2 \phi-\alpha)}
\end{aligned}
$$

EXAMPLE 3.2 It is known that every Cartan hypersurface $M$ in the $(n+1)$ dimensional standard sphere $S^{n+1}(1)$ have exactly three distinct principal curvatures $\lambda=$ const. $>0,-\lambda$ and 0 with multiplicities $p, p$ and $p$, respectively, where $p=\frac{n}{3}$ and $n=3,6,12$ or 24 . These hypersurfaces lead to a family of hypersurfaces $M$ in $\mathbb{E}^{n+1}$ having exactly three distinct principal curvatures $\sqrt{\gamma},-\sqrt{\gamma}$ and 0 with multiplicities $\frac{n-p}{3}, \frac{n-p}{3}$ and $\frac{n+2 p}{3}$, respectively,
where $p \geq 1, n-p=3,6,12$ or 24 and $\gamma$ is a positive function on $M$ ([10], Example 4.3). In [6](Section 6) the notion of a generalized Cartan hypersurface was introduced. That hypersurfaces, by definition, are some tubes $M$ in $S^{n+1}(1), n \geq 3$. They have exactly three principal curvatures $\gamma,-\gamma$ and 0 with multiplicities 1,1 and $n-2$, respectively, where $\gamma$ is a function on $M$. The presented above hypersurfaces $M$ satisfy on $\mathcal{U}_{H} \subset M$ the equation $H^{3}=\psi H$, where $\psi$ is some function on $\mathcal{U}_{H}$.

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# MAXIMUM PRINCIPLE AND NON-NEGATIVITY PRESERVATION IN LINEAR PARABOLIC PROBLEMS 

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## 1. Introduction

The maximum principle is a widely used tool in the theory of partial differential equations. Although the maximum principle is usually used in the proof of the uniqueness and stability of the investigated problems, it is closely related to the qualitative characterization of mathematical models on the continuous level. Roughly speaking, this means that the value of the solution on the whole domain can be estimated by the values of the solution on the boundary and the values of the source function on the domain. In typical formulation, as a consequence of the maximum principle, when the source term is absent from the equation we get that the maximum (and the minimum) of the solution is attained at the parabolic boundary. Another important qualitative property is the non-negativity preservation property which says the following: in case of the non-negativity of the given initial, boundary and source functions the solution is also non-negative.

The maximum principle for the heat equation with constant coefficients (in any dimension) can be formulated as follows:

$$
\begin{gather*}
\frac{\partial u}{\partial t}-\sum_{k=1}^{d} \frac{\partial^{2} u}{\partial x_{k}^{2}}=f, \quad(t, x) \in(0, T) \times \Omega  \tag{1.1}\\
u=g \quad \text { on } \quad[0, T) \times \partial \Omega, \quad \text { and }\left.\quad u\right|_{t=0}=u_{0} \quad \text { in } \Omega, \tag{1.2}
\end{gather*}
$$

where $\Omega$ is a polygonal domain in $\mathbb{R}^{d}$ with boundary $\partial \Omega, T>0$ and $f, g$ and $u_{0}$ are given sufficiently smooth functions. For the homogeneous problem (i.e., for $f=0$ ) the solution of the problem (1.1)-(1.2) $u(t, x)$ takes its
minimum and maximum at the parabolic boundary, namely on $([0, T) \times$ $\times \partial \Omega) \bigcup(\{0\} \times \Omega)$. (See, e.g. [3], [13].) The maximum principle for the non-homogenous problem is investigated in [5].

Our aim is to extend the maximum principle for a more general form of equation.

We note that the maximum principle is closely related to the non-negativity preservation property, too. This means that for the case $g, f \geq 0$ and $u_{0} \geq 0$ the non-negativity of the solution $u \geq 0$ is also guaranteed. Obviously, these principles express some fundamental physical principle, too.

In our work we will consider the more general parabolic problem, namely we will consider the equation

$$
\begin{gather*}
\frac{\partial u}{\partial t}-\sum_{m=1}^{d}\left(\frac{\partial}{\partial x_{m}}\left(k(t, x) \frac{\partial u}{\partial x_{m}}\right)\right)+c(t, x) u=f  \tag{1.3}\\
u=g \quad \text { on } \quad[0, T) \times \partial \Omega, \quad \text { and }\left.\quad u\right|_{t=0}=u_{0} \quad \text { in } \Omega . \tag{1.4}
\end{gather*}
$$

Throughout the paper we assume that the functions $k, c$ and $f$ are sufficiently smooth given functions and the following ellipticity conditions are satisfied:

ASSUMPTIONS (A1).

$$
\begin{equation*}
0<k_{0} \leq k(t, x) \leq k_{1}<\infty, \quad 0<c_{0} \leq c(t, x) \leq c_{1}<\infty \tag{1.5}
\end{equation*}
$$

for all $(t, x) \in Q_{T}$. (Here $k_{i}$ and $c_{i}(i=0,1)$ are given constants.)
We note that under the above assumptions the problem is well-posed.
For the case $c=0$ the maximum principle is investigated in [6], where the definition of the maximum principle is a generalization of the homogeneous case. Clearly, with the more special choice $k=1, c=0$ and $f=0$ we regain the problem (1.1)-(1.2).

However, for this problem the question about the discrete analogue of these results is not investigated. Intensive work is being done in this field. For the elliptic problems we recall the papers [1], [2] and [11]. For the discrete analogue of the maximum principle in the simplified time-dependent (parabolic) problems (one-dimensional heat equation) we refer to the papers [4], [5], [7] and [8]. The maximum norm contractivity is investigated in [10]. The discrete maximum principle has close relation to the discrete non-negativity preservation property, too and the latter property is also investigated in several works (see e.g. [5], [6] and [9]).

The paper is organized as follows.

In Section 2 we formulate the maximum principle and non-negativity preservation for the problem (1.3)-(1.4). In Section 3 we prove the equivalence of the maximum principle and the non-negativity preservation properties. In Section 4 we prove a two-sided estimate for the solution of equation (1.3). On the base of this result, in Section 5 we prove the basic theorem of the paper, namely, the condition for both the maximum principle and the non-negativity preservation in the above problem. We finish the paper with some conclusions and further plans.

## 2. Maximum principle and non-negativity preservation for the general problem

In the sequel we will use the following notations.

$$
\begin{equation*}
Q_{t}:=(0, t) \times \Omega, \quad \Gamma_{t}=(\partial \Omega \times(0, t)) \cup\{\Omega \times\{0\}\} \tag{2.6}
\end{equation*}
$$

(Clearly, $\Gamma_{T}$ means the parabolic boundary of the domain of the problem (1.3)-(1.4).)

Let us consider the problem (1.3)-(1.4). First, we formulate the maximum principle and the non-negativity preservation for this problem. (C.f. [6] and [12].)

DEFINITION 2.1. We say that (1.3)-(1.4) satisfies the maximum principle if for its solution $u(t, x)$ the relation

```
\(\min \left\{0 ; \min _{\Gamma_{t_{1}}} u\right\}+t_{1} \min \left\{0 ; \min _{Q_{t_{1}}} f\right\} \leq u\left(t_{1}, x\right) \leq\)
        \(\leq \max \left\{0 ; \max _{\Gamma_{t_{1}}} u\right\}+t_{1} \max \left\{0 ; \max _{Q_{t_{1}}} f\right\}\)
```

holds for all $x \in \Omega$ and fixed $t_{1} \in(0, T)$.
REMARK 2.2. The relation (2.7) does not imply that the boundary maximum principle is valid, i.e., in case $f=0$ the solution of the problem (1.3)(1.4) takes its largest positive and smallest negative values on the parabolic boundary $\Gamma_{t_{1}}$ for any $t_{1} \in(0, T)$. (E.g., when $u_{0}=g=1$, then the maximum principle (2.7) only states that the values of the solution are between zero and one. Since for $c \neq 0$ the function $u=1$ is not the solution of this problem, hence it follows that the maximum and the minimum are not taken on the boundary.) This means that Definition 2.1 is not a generalization of the maximum principle formulated in [3], p.51. However, we note that this maximum principle can be applied to prove the uniqueness of the solution:
when $u_{0}=g=0$, then the maximum principle implies that $u=0$ on the whole domain.

DEFINITION 2.3. We say that (1.3)-(1.4) satisfies the non-negativity preservation principle when the relations $f \geq 0, g \geq 0$ and $u_{0} \geq 0$ imply

$$
\begin{equation*}
u(t, x) \geq 0 \tag{2.8}
\end{equation*}
$$

for all $(t, x) \in Q_{T}$.
In the following, under condition A1 we examine the relation between the maximum principle and the non-negativity preservation.

## 3. Equivalence of the maximum principle and non-negativity preservation

The following theorem proves the equivalence between the maximum principle and the non-negativity preservation.

THEOREM 3.1. Under condition A1 (see (1.5)) the solution of the problem (1.3)-(1.4) satisfies the maximum principle if and only if the non-negativity preservation property is valid.

Proof. We show that the non-negativity preservation is a necessary and sufficient condition of the maximum principle.

- First, using indirect proof, we show the necessity.

Assume that the solution of (1.3)-(1.4) does not satisfy the non-negativity preservation condition, i.e., for some functions $u_{0} \geq 0, f \geq 0$ and $g \geq 0$ there exists a $\left(t_{0}, x_{0}\right) \in Q_{T}$ such that $u\left(t_{0}, x_{0}\right)<0$. Then, with the choice $t=t_{0}$ the left-hand side of (2.7) implies the relation

$$
\begin{equation*}
\min \left\{0 ; \min _{\Gamma_{t_{0}}} u\right\}+t_{0} \min \left\{0 ; \min _{Q_{t_{0}}} f\right\} \leq u\left(t_{0}, x\right) \tag{3.9}
\end{equation*}
$$

for all $x \in \Omega$. Due to our assumption, $u$ is non-negative on the parabolic boundary $\Gamma_{t_{0}}$ and $f$ is also non-negative on $Q_{t_{0}}$. Hence, the left-hand side of (3.9) is equal to zero. Therefore, with the choice $x=x_{0}$ (3.9) implies the inequality $u\left(t_{0}, x_{0}\right) \geq 0$, which leads to contradiction and proves the necessity.

- We show the sufficiency, i.e., that the non-negativity preservation property implies the maximum principle.

We construct a new function $\bar{u}$ as follows:

$$
\begin{equation*}
\bar{u}:=u-\min \left\{0, \min _{\Gamma_{t_{1}}} u\right\}-t \min \left\{0, \min _{Q_{t_{1}}} f\right\} \equiv u-A-t B \tag{3.10}
\end{equation*}
$$

where the notations

$$
\begin{align*}
A & :=\min \left\{0, \min _{\Gamma_{t_{1}}} u\right\}  \tag{3.11}\\
B & :=\min \left\{0, \min _{Q_{t_{1}}} f\right\}
\end{align*}
$$

were used. Clearly, $A \leq 0, B \leq 0$.
(Here the function u is the solution of the problem (1.3)-(1.4)). Then, as one can easily verify, the relations

$$
\begin{gather*}
\frac{\partial u}{\partial t}=\frac{\partial \bar{u}}{\partial t}+B  \tag{3.12}\\
\frac{\partial}{\partial x_{m}}\left(k(t, x) \frac{\partial u}{\partial x_{m}}\right)=\frac{\partial}{\partial x_{m}}\left(k(t, x) \frac{\partial \bar{u}}{\partial x_{m}}\right) \tag{3.13}
\end{gather*}
$$

are valid for all $m=1,2, \ldots d$.
Substituting (3.12) and (3.13) into (1.3), we get

$$
\begin{equation*}
\left(\frac{\partial \bar{u}}{\partial t}+B\right)-\sum_{m=1}^{d}\left(\frac{\partial}{\partial x_{m}}\left(k(t, x) \frac{\partial \bar{u}}{\partial x_{m}}\right)\right)+c(t, x)(\bar{u}+A+t B)=f \tag{3.14}
\end{equation*}
$$

Then equation (3.14) can be written as

$$
\begin{equation*}
\frac{\partial \bar{u}}{\partial t}-\sum_{m=1}^{d}\left(\frac{\partial}{\partial x_{m}}\left(k(t, x) \frac{\partial \bar{u}}{\partial x_{m}}\right)\right)+c(t, x) \bar{u}=F(t, x) \tag{3.15}
\end{equation*}
$$

where we used the notation

$$
\begin{equation*}
F(t, x)=f(t, x)-B-c(t, x)(A+t B) \tag{3.16}
\end{equation*}
$$

Now, we consider the problem (3.15) with the Dirichlet boundary condition

$$
\begin{equation*}
\left.\bar{u}\right|_{\Gamma_{t_{1}}}=\left.u\right|_{\Gamma_{t_{1}}}-A-t B=\left.u\right|_{\Gamma_{t_{1}}}-\min \left\{0, \min _{\Gamma_{t_{1}}} u\right\}-t B \tag{3.17}
\end{equation*}
$$

and with the initial condition

$$
\begin{equation*}
\left.\bar{u}\right|_{t=0}=\left.u\right|_{t=0}-A=u_{0}-A \tag{3.18}
\end{equation*}
$$

Since $\left.u\right|_{\Gamma_{1}}-\min \left\{0, \min _{\Gamma_{t_{1}}} u\right\} \geq 0$ and $-t B \geq 0$, therefore

$$
\begin{equation*}
\left.\bar{u}\right|_{\Gamma_{t_{1}}} \geq 0 \tag{3.19}
\end{equation*}
$$

Due to the definition of $A$ in (3.11), clearly

$$
\begin{equation*}
\left.\bar{u}\right|_{t=0} \geq 0 \tag{3.20}
\end{equation*}
$$

On the other hand, for the right side of (3.15), according to (3.16), we have the estimation

$$
\begin{equation*}
F(t, x)=f(t, x)-\min \left\{0, \min _{\mathbf{Q}_{t_{1}}} f\right\}-c(t, x)(A+t B) \geq 0 \tag{3.21}
\end{equation*}
$$

Hence, on the base of the relations (3.19), (3.20) and (3.21), the nonnegative preservation assumption implies that

$$
\begin{equation*}
\bar{u}(t, x) \geq 0 \tag{3.22}
\end{equation*}
$$

holds for all $(t, x) \in Q_{T}$. Using the definition of the function $\bar{u}$ in (3.10), the relation (3.22) results in

$$
\begin{equation*}
u(t, x) \geq \min \left\{0, \min _{\Gamma_{t_{1}}} u\right\}+t \min \left\{0, \min _{\mathbf{Q}_{t_{1}}} f\right\} \tag{3.23}
\end{equation*}
$$

which proves the left-hand side of (2.7). The right-hand side is proved in the same manner.

## 4. Auxiliary estimate for the solution of the problem

In this section we give an upper and lower estimation for the solution of equation (1.3).

## THEOREM 4.1. The relation

$$
\begin{gather*}
\sup _{\lambda>0}\left(e^{\lambda t_{1}} \min \left\{\min _{\Gamma_{t_{1}}} u e^{-\lambda t} ; \frac{1}{\lambda+c_{1}} \min _{Q_{t_{1}}} f e^{-\lambda t}\right\}\right) \leq u\left(x, t_{1}\right) \leq  \tag{4.24}\\
\leq \inf _{\lambda>0}\left(e^{\lambda t_{1}} \max \left\{\max _{\Gamma_{t_{1}}} u e^{-\lambda t} ; \frac{1}{\lambda+c_{0}} \max _{Q_{t_{1}}} f e^{-\lambda t}\right\}\right)
\end{gather*}
$$

holds for any solution of the equation (1.3).
Proof. We introduce the new function

$$
\begin{equation*}
\hat{u}(t, x) \equiv u(t, x) e^{-\lambda t} \tag{4.25}
\end{equation*}
$$

Hence, due to the obvious relation

$$
\begin{equation*}
u(t, x)=e^{\lambda t} \hat{u}(t, x) \tag{4.26}
\end{equation*}
$$

we have

$$
\begin{gather*}
\frac{\partial u}{\partial t}=e^{\lambda t}\left(\lambda \hat{u}+\frac{\partial \hat{u}}{\partial t}\right) ; \quad \frac{\partial u}{\partial x_{m}}=e^{\lambda t} \frac{\partial \hat{u}}{\partial x_{m}} \\
\frac{\partial}{\partial x_{m}}\left(k(t, x) \frac{\partial u}{\partial x_{m}}\right) \equiv e^{\lambda t} \frac{\partial}{\partial x_{m}}\left(k(t, x) \frac{\partial \hat{u}}{\partial x_{m}}\right) \tag{4.27}
\end{gather*}
$$

for all $m=1,2, \ldots d$. Substituting (4.26) and (4.27) into equation (1.3), we get:

$$
\begin{equation*}
\frac{\partial \hat{u}}{\partial t}-\sum_{m=1}^{d} \frac{\partial}{\partial x_{m}}\left(k(t, x) \frac{\partial \hat{u}}{\partial x_{m}}\right)+(c(t, x)+\lambda) \hat{u}=f e^{-\lambda t} . \tag{4.28}
\end{equation*}
$$

Since $\hat{u}$ is a continuous function on $\bar{Q}_{t_{1}}$, it has a maximum (and minimum) on $\bar{Q}_{t_{1}}$. This means that there exists a point $\left(x_{0}, t_{0}\right) \in \bar{Q}_{t_{1}}$ such that $\hat{u}(t, x) \leq$ $\leq \hat{u}\left(x_{0}, t_{0}\right)$ for all $(t, x) \in \bar{Q}_{t_{1}}$.

- First we assume that this point belongs to the parabolic boundary, i.e., $\left(x_{0}, t_{0}\right) \in \Gamma_{t_{1}}$. Then, due to the obvious relation

$$
\hat{u}(t, x) \leq \hat{u}\left(x_{0}, t_{0}\right)=\max _{\Gamma_{t_{1}}} \hat{u}
$$

for all $(t, x) \in \bar{Q}_{t_{1}}$, we get the estimation

$$
\max _{Q_{t_{1}}} \hat{u} \leq \max _{\Gamma_{t_{1}}} \hat{u}
$$

- Assume now that $\left(x_{0}, t_{0}\right) \in Q_{t_{1}}$, i.e.,

$$
\begin{equation*}
\max _{Q_{t_{1}}} \hat{u}=\hat{u}\left(x_{0}, t_{0}\right) \tag{4.29}
\end{equation*}
$$

Since the function $\hat{u}$ is the classical solution therefore it is differentiable twice w.r.t. $x$ and once w.r.t. $t$. This implies that the relations

$$
\begin{equation*}
\frac{\partial \hat{u}}{\partial t}\left(x_{0}, t_{0}\right) \geq 0, \quad \frac{\partial \hat{u}}{\partial x_{m}}\left(x_{0}, t_{0}\right)=0 \tag{4.30}
\end{equation*}
$$

and, due to the maximum at the point $\left(x_{0}, t_{0}\right)$,

$$
\frac{\partial^{2} \hat{u}\left(x_{0}, t_{0}\right)}{\partial x_{m}^{2}} \leq 0
$$

holds. Hence, an easy computation shows the validity of the relation

$$
\begin{equation*}
\sum_{m=1}^{d} \frac{\partial}{\partial x_{m}}\left(k \cdot \frac{\partial \hat{u}}{\partial x_{m}}\right)\left(x_{0}, t_{0}\right) \leq 0 \tag{4.31}
\end{equation*}
$$

For the equation (4.28) at the point $\left(x_{0}, t_{0}\right)$, we obtain

$$
\begin{align*}
& \frac{\partial \hat{u}}{\partial t}\left(x_{0}, t_{0}\right)-\sum_{m=1}^{d} \frac{\partial}{\partial x_{m}}\left(k \cdot \frac{\partial \hat{u}}{\partial x_{m}}\right)\left(x_{0}, t_{0}\right)=  \tag{4.32}\\
& \quad=e^{-\lambda t_{0}} f\left(x_{0}, t_{0}\right)-\left(c\left(x_{0}, t_{0}\right)+\lambda\right) \hat{u}\left(x_{0}, t_{0}\right)
\end{align*}
$$

which, on base of the inequalities (4.30) and (4.31), yields the estimate

$$
e^{-\lambda t_{0}} f\left(x_{0}, t_{0}\right)-\left(c\left(x_{0}, t_{0}\right)+\lambda\right) \hat{u}\left(x_{0}, t_{0}\right) \geq 0
$$

Hence, we get

$$
\begin{align*}
\hat{u}\left(x_{0}, t_{0}\right) & \leq \frac{e^{-\lambda t_{0}} f\left(x_{0}, t_{0}\right)}{c\left(x_{0}, t_{0}\right)+\lambda} \leq \frac{e^{-\lambda t_{0}} f\left(x_{0}, t_{0}\right)}{c_{0}+\lambda} \\
& \leq \frac{1}{c_{0}+\lambda} \max _{Q_{t_{1}}}\left(e^{-\lambda t} f(t, x)\right) \tag{4.33}
\end{align*}
$$

Since the function $\hat{u}$ takes its maximum at the point $\left(x_{0}, t_{0}\right)$, therefore the estimation (4.33) shows the validity of the following inequality:

$$
\begin{equation*}
\max _{Q_{t_{1}}} \hat{u} \leq \frac{1}{c_{0}+\lambda} \max _{Q_{t_{1}}}\left(e^{-\lambda t} f(t, x)\right) \tag{4.34}
\end{equation*}
$$

Clearly, the estimates of the two different cases, namely (4.29) and (4.34) together imply that

$$
\begin{equation*}
\max _{Q_{t_{1}}} \hat{u} \leq \max \left\{\max _{\Gamma_{t_{1}}} \hat{u}, \frac{1}{c_{0}+\lambda} \max _{Q_{t_{1}}}\left(e^{-\lambda t} f(t, x)\right)\right\} \tag{4.35}
\end{equation*}
$$

Due to the definition of the function $\hat{u}$ in (4.25), from (4.35) we obtain that

$$
u\left(x, t_{1}\right) \leq e^{\lambda t_{1}} \max \left\{\max _{\Gamma_{t_{1}}}\left(u e^{-\lambda t}\right), \frac{1}{c_{0}+\lambda} \max _{Q_{t_{1}}}\left(e^{-\lambda t} f(t, x)\right)\right\}
$$

which holds for any $\lambda>0$. Therefore, taking the infimum by $\lambda$, we get

$$
u\left(x, t_{1}\right) \leq \inf _{\lambda>0}\left(e^{\lambda t_{1}} \max \left\{\max _{\Gamma t_{1}} u e^{-\lambda t}, \frac{1}{\lambda+c_{0}} \max _{Q_{1}} f \mathbf{e}^{-\lambda t}\right\}\right)
$$

which proves the right-hand side of (4.24).
The other part of the inequality in (4.24) is proved in the same manner.

## 5. Maximum principle for the general problem

The results of the previous section imply the following:
Theorem 5.1. Assume that the functions $f, g$ and $u_{0}$ in the problem (1.3)-(1.4) are are given, sufficiently smooth non-negative functions. Then the solution $u(t, x)$ of this problem is also non-negative.

Proof. Since $u(t, x)$ is the solution of the equation (1.3), we can apply Theorem 4.1, namely the estimation (4.24). Under our assumptions, the left-hand side of the inequality (4.24) is non-negative, which proves the statement.

We can summarize our results in the following basic proposition, which is an obvious consequence of Theorem 3.1 and Theorem 4.1.

Theorem 5.2. Assume that the problem (1.3)-(1.4) is well-posed. Then under the Assumptions (A1), for the solution $u(t, x)$ both the non-negativity preservation and the maximum principle properties are valid.

## 6. Conclusions

In this paper we showed that in the initial-boundary value problem for the general class of linear parabolic equations two properties, namely, the non-negativity preservation and the maximum principle, are equivalent. We also gave those conditions under which these properties can be guaranteed for this problem.

Our plan for the future is to formulate those conditions under which the above basic qualitative properties are preserved for the different numerical methods (finite difference and finite element methods).

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# BUTTERFLY THEOREMS IN THE HYPERBOLIC PLANE 

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We start with recalling the Butterfly Theorem in the Euclidean plane, [4]: Let the complete quadrangle $A B C D$ be inscribed into a circle $c$. Let the line $p$ perpendicular to diameter $s$ intersects the sides $A B, C D, A C, B D, A D, B C$ of the quadrangle at the points $T, T^{\prime}, Q, Q^{\prime}, R, R^{\prime}$, respectively. If the point $L=p \cap s$ is midpoint of the segment $R R^{\prime}$, then $L$ is midpoint of the segments $T T^{\prime}$ and $Q Q^{\prime}$.

We will prove the theorem above in the hyperbolic plane. Our the most important motivation is proving the analogue of the following theorem, [4]: Butterfly points associated with the given quadrangle inscribed into a circle are located on an equilateral hyperbola.

## 1. Introduction

Let $a$ be the absolute conic for the Cayley-Klein model of the hyperbolic plane (H-plane) represented by a circle of classical Euclidean plane. Interior points of the absolute conic are called real points, exterior points are ideal points and the points of the absolute conic are called absolute points.

Definition of the collineation in the H-plane is analogous to the definition of the collineation in the Euclidean plane. If the collineation is determined by the center $O$ and the axis $o$, absolute polar of the point $O$, moreover by the pair of corresponding points $A$ and $A^{\prime}$ as intersection points of the absolute and a ray $z \in(O)$, we talk about harmonic homology. This harmonic homology that maps the absolute conic onto itself, is analogous to the line
reflection or the point reflection in Euclidean plane [2], [3]. If the center of a harmonic homology is ideal point, its absolute polar is real line perpendicular to every ray of collineation. Therefore, the intersection point of the axis and ray of collineation is midpoint of the segment that has as end points pair of mapped points. It follows that axis of collineation is perpendicular bisector of every segment whose ending points are corresponding points, so described mapping is analogous to line symmetry in the Euclidean plane. If the center of collineation is real point, axis is ideal line, and this mapping is analogous to point symmetry in the Euclidean plane.

## 2. Butterfly Theorems

There are many variants of Butterfly Theorem in the Euclidean plane, [1], [4], [5]. Here some analogous theorems in the hyperbolic plane are proved. There are three classes of circles in the H-plane, and the absolute conic is one of them, so different variants of Butterfly Theorem are expected. However, it is shown that proofs do not depend on the class of circle into which a complete quadrangle is inscribed. The only difference is between the case when a quadrangle is inscribed into the absolute conic and the case when it is inscribed into a circle. First two theorems in case of the absolute will be proved.

THEOREM 1. Let the complete quadrangle $A B C D$ be inscribed into the absolute conic $a$ of the $H$-plane. By $E, F$ and $G$ the vertices of the diagonal triangle are denoted. If the line $p$ through $G$ (or $E$ or $F$ ) intersects the pairs of opposite sides of the quadrangle at points $T, T^{\prime}$ and $Q, Q^{\prime}$, respectively, then $G$ is midpoint of the segments $T T^{\prime}$ and $Q Q^{\prime}$, (Fig. 1).

Proof. It has to be shown that there exists harmonic homology which maps $T$ onto $T^{\prime}$ and $Q$ onto $Q^{\prime}$. It is obvious that collineation $(G, g)$ satisfies the required conditions. $A$ is mapped onto $C$ and $B$ is mapped onto $D$. Therefore, the quadrangle is mapped onto itself. Side $A B$ of the quadrangle is mapped onto $C D$ and side $A D$ onto $B C$, hence, $T, T^{\prime}$ and $Q, Q^{\prime}$ are pairs of corresponding points. The absolute conic is mapped onto itself. The theorem can be proved analogously for every line $p$ passing through $E$ or $F$.

This theorem is stronger then many theorems in the Euclidean plane because $p$ can be any line through the diagonal point of the quadrangle, while in the Euclidean case $p$ has to be perpendicular to the diameter of circle that passes through a diagonal point, [1]. Therefore, the analogue of this theorem


Figure 1. The Butterfly Theorem for a quadrangle inscribed into the absolute conic and the line passing through a diagonal point
in the Euclidean plane does not exist. This is because the absolute conic is considered as a circle concentric to every circle in the H-plane and every line can be regarded as diameter of absolute, but also as its axis. The stronger statement is formulated in the following theorem, analogues to [5]:

THEOREM 2. Let the complete quadrangle $A B C D$ be inscribed into the absolute conic a of the H-plane. Let the line $p$ intersects the pairs of opposite sides $A D, B C, A C, B D$ and $A B, C D$ of the quadrangle at the points $R, R^{\prime}$, $Q, Q^{\prime}$ and $T, T^{\prime}$, respectively. If $L \in p$ is the midpoint of any of the following segments $R R^{\prime}, Q Q^{\prime}$ and $T T^{\prime}$, then it is the midpoint of them all.

Proof. Let $P$ be absolute pole of the line $p$ and let $S$ be absolute pole of the connecting line $L P=s$, (Fig. 2). It can be assumed that $L$ is the midpoint of the segment $R R^{\prime}$. It follows that there exists line reflection that maps $R$ onto $R^{\prime}$ and lives $L$ fixed. Obviously line reflection $(S, s)$ has that property. Therefore, $\left(R R^{\prime}, S L\right)=-1$ holds. Since equality $\left(K K^{\prime}, S L\right)=-1$, where $K$ and $K^{\prime}$ are absolute points of the line $p$, also holds, it can be concluded that $L$ and $S$ are fixed points of involution determined on the line $p$ by the pencil of conics $A B C D$. Pairs of intersection points $T, T^{\prime}$ and $Q, Q^{\prime}$ between line $p$ and degenerated conics of the pencil are pairs of points of that involution. This leads to equalities $\left(T T^{\prime}, S L\right)=-1$ and $\left(Q Q^{\prime}, S L\right)=-1$. Hence, line reflection $(S, s)$ maps $T$ and $Q$ onto $T^{\prime}$ and $Q^{\prime}$, respectively. The theorem is proved.


Figure 2. The Butterfly Theorem for the quadrangle inscribed into the absolute conic

The previous theorems deal with the quadrangle inscribed into absolute conic. The following theorems assume that the quadrangle is inscribed into a circle of the H-plane.

THEOREM 3. Let the complete quadrangle $A B C D$ be inscribed into a real or ideal hypercycle $c$ of the H-plane, (Fig. 3). Let the line s connect the center $O$ of $c$ with one of the three diagonal points of the quadrangle, say $G$. Let the line $p$ through $G$ perpendicular to $s$ intersects the pairs of opposite sides of the quadrangle at points $T, T^{\prime}$ and $Q, Q^{\prime}$, respectively. Point $G$ is midpoint of the segments $T T^{\prime}$ and $Q Q^{\prime}$.

Proof. The pole of the line $s$ is denoted by $S$. Let $(O, o)$ be collineation with center $O$ and axis $o$ that maps the absolute conic $a$ onto our hypercycle $c$. The quadrangle $A B C D$ is mapped onto the quadrangle $A_{a} B_{a} C_{a} D_{a}$ inscribed into the absolute conic $a$ and diagonal points $E, F, G$ are mapped onto diagonal points $E_{a}, F_{a}, G_{a}$ of new quadrangle. Since, the absolute polar lines of $S$, $E_{a}$ and $F_{a}$ pass through the point $G_{a}$, those points are collinear. So, points $S, E$ and $F$ are collinear, too. On the other hand, lines $A B, C D, F G$ and $F E$ passing through $F$ make harmonic quadruple of lines. After intersecting this lines with the line $p$, harmonic quadruple of points $T, T^{\prime}, G, S$ is obtained. Therefore, $T$ is mapped onto $T^{\prime}$ under the line reflection $(S, s)$. That the collineation ( $S, s$ ) maps $Q$ onto $Q^{\prime}$ can be shown analogously.

In the statement of the theorem it was assumed that the quadrangle is inscribed into a hypercycle. That case is shown on Fig. 3. It is obvious that


Figure 3. The Butterfly Theorem for the quadrangle inscribed into a circle (ideal hypercycle) and the line passing through a diagonal point $G$
assumption that circle touches the absolute at a pair of different real points wasn't used. The same statement holds for quadrangle inscribed into cycle or horocycle.

THEOREM 4. Let the complete quadrangle $A B C D$ be inscribed into a circle $c$ of the H-plane, (Fig. 4). Let the line $p$ perpendicular to diameter $s$ intersects the sides $A B, C D, A C, B D, A D, B C$ of the quadrangle at the points $T, T^{\prime}, Q, Q^{\prime}, R, R^{\prime}$, respectively. If the point $L=p \cap s$ is midpoint of the segment $R R^{\prime}$, then $L$ is midpoint of the segments $T T^{\prime}$ and $Q Q^{\prime}$.

Proof. Let $S$ be the pole of the given diameter $s$ of the circle $c$ and let $L$ be intersection point of two perpendicular lines $s$ and $p$. If $L$ bisects the segment $R R^{\prime}$, then there exists line reflection that maps $R$ onto $R^{\prime}$ and leaves $L$ fixed. Line reflection $(S, s)$ has this property. Since the line $p$ is conjugate to the line $s$, pole $P$ of line $p$ lies on $s$ and connecting line $P S$ is polar of the point $L$.

Furthermore, there exists collineation $(O, o)$ that maps absolute conic $a$ onto the circle $c$. Let the points $A, B, C, D$ of the circle be mapped onto the points $A_{a}, B_{a}, C_{a}, D_{a}$ of the absolute conic $a$, let the points $T, T^{\prime}, Q, Q^{\prime}, R, R^{\prime}$


Figure 4. The Butterfly Theorem for the quadrangle inscribed into a circle (real hypercycle)
be mapped onto $T_{a}, T_{a}^{\prime}, Q_{a}, Q_{a}^{\prime}, R_{a}, R_{a}^{\prime}$ and let the line $p$ be mapped onto $p_{a}$. This collineation and equality $\left(R R^{\prime}, S L\right)=-1$ imply $\left(R a R^{\prime} a, S L a\right)=-1$. By Theorem 2 equalities $\left(T_{a} T_{a}^{\prime}, S L_{a}\right)=-1$ and $\left(Q_{a} Q_{a}^{\prime}, S L_{a}\right)=-1$ are obtained. Therefore, $\left(T T^{\prime}, S L\right)=-1$ and $\left(Q Q^{\prime}, S L\right)=-1$ hold. Then $T, T^{\prime}$ and $Q, Q^{\prime}$ are pairs of mapped points of line reflection $(S, s)$, as it had to be proved.

Point $L$, that on some line $p$ bisects the segments formed by intersections of the pairs of opposite sides of the quadrangle $A B C D$ with the line $p$, is called butterfly point of the quadrangle $A B C D$ and line $p$ is called butterfly line.

## 3. Butterfly Curve Theorem

Till now it is shown that every quadrangle $A B C D$ inscribed into a circle (hypercycle, cycle or horocycle) of the hyperbolic plane has at least three butterfly points. These are the diagonal points of the quadrangle that have
the butterfly property on the lines perpendicular to diameters passing through them. Someone can asks himself is there more butterfly points, how many of them are there and how they can be constructively determined. The following theorem gives the answers for these questions.

THEOREM 5. Let the complete quadrangle $A B C D$ be inscribed into a circle $c$ with center $O$ (Fig. 5) and let $s$ be any line through $O$. There is butterfly point $L$ of the quadrangle $A B C D$ on the line $s$.


Figure 5. The construction of the butterfly point

Proof. Let $S$ be pole of the diameter $s$ of the circle $c$. Let $T$ be intersection point of any side of the quadrangle, say $A B$, and mirror image of the opposite side $C D$ with respect to the line $s$. Let $p$ be line through $T$ perpendicular to $s$. The intersection point $L$ of $s$ and $p$ has obviously the property of a butterfly point on the line $p$ because $L$ bisects the segment $T T^{\prime}$, where $T^{\prime}$ is intersection point between the line $s$ and the side $C D$.

As there is a butterfly point $L$ on each diameter of the circle $c$, an infinite number of butterfly points is associated with the given quadrangle.


Figure 6. The butterfly curve
THEOREM 6. For a given complete quadrangle $A B C D$ inscribed into a circle $c$ of the H-plane, all butterfly points are located on a second order curve.

Proof. Let $L$ be butterfly point lying on its butterfly line $p$ and on the diameter $s$ (Figure 6). Pole $S$ of the line $s$ is the harmonic conjugate of $L$ with respect to the points of intersection with any pair of opposite sides, say $T=A B \cap p$ and $T^{\prime}=C D \cap p$. After connecting these points with the diagonal point $F=A B \cap C D$ a harmonic quadruple of lines is obtained. For any given line $s$ the line $F L$ is the harmonic conjugate of $F S$, perpendicular to $s$, with respect to $A B$ and $C D$. Hence, all butterfly points are located on the second order curve $h$ that can be produced by a projective mapping between the pencils $(O)$ and $(F)$ of the lines.

The butterfly curve $h$ passes through the tree diagonal points of the quadrangle and through the center $O$. Point $O$ has the property of the butterfly point on the line perpendicular to the line that is fourth harmonic conjugate of the line $F O$ with respect to $A B$ and $C D$. One of two midpoints of each of the six sides of the quadrangle is also located on the butterfly curve. The construction of the butterfly point on the side $A B$, midpoint of that side, is
shown on Figure 6, too. It can be obtained as intersection point of side $A B$ and diameter perpendicular to $A B$.

The above theorem is analogous to the theorem in the Euclidean plane that says that butterfly points associated with the given quadrangle inscribed into a circle are located on an equilateral hyperbola, [4].

REMARK. It should be noticed that beside the butterfly point $L$ the point $S$ bisects all of the segments that are determined on the line $p$ by the pairs of the opposite sides of the quadrangle $A B C D$. All these points $S$ are located on the axis $o$ of the circle $c$ into which complete quadrangle is inscribed, and that is why they were not studied in more detail in this paper.

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# SOME INTEGRAL VERSIONS OF ALMOST SURE LIMIT THEOREMS 

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## 1. Introduction

The aim of our paper is to present some integral versions of almost sure limit theorems.

We will denote by $\xrightarrow{d}$ the convergence in distribution by $\xrightarrow{w}$ the weak convergence of measures, by $\mu_{\zeta}$ the distribution of the random variable $\zeta$. Let $\mathscr{B}(\mathbb{R})$ denote the $\sigma$-algebra of the Borel subsets of $\mathbb{R}$.

Let $\xi_{n}, n \in \mathbb{N}$, be a sequence of random variables defined on the probability space $(\Omega, \mathscr{A}, \mathbb{P})$. Usual limit theorems deal with the convergence $\zeta_{n} \xrightarrow{d} \zeta$, as $n \rightarrow \infty$. Consider the sequence of measures

$$
Q_{n}^{*}\left[\zeta_{n}\right](\omega)=\frac{1}{D_{n}} \sum_{k=1}^{n} d_{k} \delta_{\zeta_{k}(\omega)}, \quad \omega \in \Omega, \quad n \in \mathbb{N} .
$$

Here $\delta_{x}$ is the unit mass at the point $x$. In several cases we have

$$
Q_{n}^{*}\left[\zeta_{n}\right](\omega) \xrightarrow{w} \mu_{\zeta}, \quad \text { as } \quad n \rightarrow \infty,
$$

for almost all $\omega \in \Omega$. This is the general shape of the almost sure limit theorems.

Classical papers in the almost sure limit theory are Brosamler [2], Schatte [10] and Lacey and Philipp [7]. For the methods and results of this theory see e.g. Berkes and Csáki [1], Major [8], Fazekas and Rychlik [5], and Móri [9].

In our paper, instead of the random sequence $\xi_{n}, n \in \mathbb{N}$, we shall consider the random process $X(t)=X(t, \omega), t \in[1, \infty)$, and instead of $Q_{n}^{*}[\zeta](\omega)$ we shall consider the integral

$$
Q_{T, \omega}(A)=\frac{1}{D(T)} \int_{1}^{T} \delta_{X(t, \omega)}(A) d(t) d t, \quad A \in \mathscr{B}(\mathbb{R})
$$

Our first result (Theorem 2.1) is an a.s. limit theorem with Poisson limit law. Theorem 3.1 contains a result where the limit law is Gaussian. This theorem can be applied for Poisson, Wiener and Ornstein-Uhlenbeck processes (see Corollary 3.1). The proofs are based on a general theorem of Chuprunov and Fazekas [3]. We remark that in [2] Lévy's formula was applied to obtain results for processes with independent stationary increments. However, in our case Kolmogorov's formula offers a simpler proof.

We will denote by $N\left(m, \sigma^{2}\right)$ a centered Gaussian law with expectation $m$ and variance $\sigma^{2}$. By $\pi$ we will denote a standard Poisson random variable (i.e. $\mathbb{E} \pi=1$ ). We will use the convention that $\sum_{i \in \emptyset} a_{i}=0$. Let $\mathbb{I}_{A}$ denote the indicator function of the set $A$.

## 2. Convergence to Poisson distribution

Let $\xi_{i}, i \in \mathbb{N}$, be independent random variables uniformly distributed on $[0,1]$. Set

$$
X^{\prime}(t)=\sum_{i=1}^{[t]} \mathbb{I}_{\left[0, \frac{1}{t}\right]}\left(\xi_{i}\right), \quad 1 \leq t
$$

Here [ t ] denotes the integer part of $t$.
Let $f(t), t \geq 1$, be a positive function such that the function

$$
\begin{equation*}
\frac{f(t)}{t^{\beta}} \text { is increasing for some } \beta>0 . \tag{2.1}
\end{equation*}
$$

Theorem 2.1. Let $X(t)=X^{\prime}(f(t)), 1 \leq t$. Then

$$
\frac{1}{\log (T)} \int_{1}^{T} \delta_{X(t, \omega)} \frac{d t}{t} \stackrel{w}{\longrightarrow} \mu_{\pi}, \quad \text { as } T \rightarrow \infty,
$$

for almost all $\omega \in \Omega$.

## 3. Convergence to Gaussian law

Let $V(t), t>0$, be a centered homogeneous (infinitely divisible) random process with independent increments and with finite variance. Using the Kolmogorov representation (see [6], sect. 18), we can assume that its characteristic function is

$$
\begin{align*}
\varphi_{V(t)}(x) & =\mathbb{E}\left(e^{i x V(t)}\right) \\
& =\exp \left(t\left\{\int_{-\infty}^{+\infty}\left(e^{i x y}-1-i x y\right) \frac{1}{y^{2}} d K(y)\right\}\right), \tag{3.1}
\end{align*}
$$

$x \in \mathbb{R}$. Here $K(y)$ is an increasing bounded function such that $K(-\infty)=0$.
Let $f(t)$ be a function with property (2.1). Consider the processes

$$
\begin{equation*}
X(t)=\frac{V(f(t))}{(f(t))^{1 / 2}}, \quad 0 \leq t \tag{3.2}
\end{equation*}
$$

Theorem 3.1. Let the $X(t)$ be defined by (3.2), where $V(t)$ is a process with characteristic function (3.1) and $f(t)$ is a function with property (2.1). Then

$$
\frac{1}{\log (T)} \int_{1}^{T} \delta_{\frac{V(f(t),())}{\sqrt{f(t)}}} \frac{d t}{t} \xrightarrow{w} N(0, K(\infty)), \quad \text { as } \quad T \rightarrow \infty
$$

for almost all $\omega \in \Omega$.
Corollary 3.1. (a) Let $\pi(t), 0 \leq t$, be the standard Poisson process (i.e. $\mathbb{E} \pi(t)=t$ ). We have

$$
\frac{1}{\log (T)} \int_{1}^{T} \delta_{\frac{\pi(f(t),()-f(t)}{\sqrt{f(t)}}} \frac{d t}{t} \xrightarrow{w} N(0,1), \quad \text { as } \quad T \rightarrow \infty
$$

for almost all $\omega \in \Omega$.
(b) Let $W(t)$ be the standard Wiener process. We have

$$
\frac{1}{\log (T)} \int_{1}^{T} \delta_{\frac{W(f(t), \omega)}{\sqrt{f(t)}}} \frac{d t}{t} \xrightarrow{w} N(0,1), \quad \text { as } \quad T \rightarrow \infty,
$$

for almost all $\omega \in \Omega$.
(c) Let $U(t)$ be the Ornstein-Uhlenbeck process. Then $U(t)$ has the representation $U(t)=C e^{-m t / 2} W\left(e^{m t}\right), t>0$, where $C, m>0$ and $W(t)$
is the standard Wiener process. Put $f(t)=e^{m t}$. Since $\frac{f(t)}{t}=\frac{e^{m t}}{t}, 1 \leq t$, is an increasing function, by (b) we have

$$
\frac{1}{\log (T)} \int_{1}^{T} \delta_{U(t, \omega)} \frac{d t}{t} \xrightarrow{w} N\left(0, C^{2}\right), \quad \text { as } \quad T \rightarrow \infty
$$

for almost all $\omega \in \Omega$.

## 4. The proofs of the theorems

We will use the next result (see Chuprunov and Fazekas [3], Theorem 2.1).

THEOREM A. Let $(\boldsymbol{B}, \varrho)$ be a complete separable metric space, denote by $\mathscr{B}(\boldsymbol{B})$ the $\sigma$-algebra of the Borel sets of $\boldsymbol{B}$. Let $X(t), t \geq 0$, be a random process defined on $(\Omega, \mathscr{A}, \mathbb{P})$ with values in $\boldsymbol{B}$ such that the function $X(t)=$ $=X(t, \omega),(t, \omega) \in[1, \infty) \times \Omega$ is (Borel) measurable. Let $\log _{+} x=\log x$, if $1 \leq x$, and $\log _{+} x=0$, if $x<1$.

Assume that there exist $C<\infty, \varepsilon>0$, an increasing sequence of positive numbers $c_{n}$ with $\lim _{n \rightarrow \infty} c_{n}=\infty, c_{n+1} / c_{n}=O(1)$, moreover, there exists a strictly increasing unbounded sequence of nonnegative numbers $v_{n}$ such that for each pair $(l, k)$, with $l<k, l, k \in \mathbb{N}$, there exists a $\boldsymbol{B}$-valued random process $X_{l k}(t), v_{k} \leq t<v_{k+1}$, with the following properties. For $l<k\left\{X(t): v_{l} \leq t<v_{l+1}\right\}$ and $\left\{X_{l k}(t): v_{k} \leq t<v_{k+1}\right\}$ are independent families of random variables, moreover, for all $t$ with $v_{k} \leq t<v_{k+1}$

$$
\begin{equation*}
\mathbb{E} \varrho\left(X(t), X_{l k}(t)\right) \leq C\left(\frac{c_{l}}{c_{k}}\right)^{\beta} \tag{4.1}
\end{equation*}
$$

Suppose that there exists a decreasing positive function $d(t), v_{1} \leq t$, with $\int_{v_{k}}^{v_{k+1}} d(t) d t \leq \log \left(c_{k+1} / c_{k}\right)$ for each $k$, and $\int_{v_{1}}^{\infty} d(t) d t=\infty$. Set $D(T)=\int_{v_{1}}^{T} d(t) d t$ and

$$
Q_{T, \omega}(A)=\frac{1}{D(T)} \int_{v_{1}}^{T} \delta_{X(t, \omega)}(A) d(t) d t, \quad A \in \mathscr{B}(\boldsymbol{B})
$$

Then for any probability distribution $\mu$ on the Borel $\sigma$-algebra $\mathscr{B}(\boldsymbol{B})$ the following two statements are equivalent

$$
\begin{equation*}
Q_{T, \omega} \xrightarrow{w} \mu, \quad \text { as } \quad T \rightarrow \infty, \text { for almost all } \omega \in \Omega \tag{4.2}
\end{equation*}
$$

$$
\begin{equation*}
\frac{1}{D(T)} \int_{v_{1}}^{T} \mu_{X(t)} d(t) d t \stackrel{w}{\longrightarrow} \mu, \quad \text { as } \quad T \rightarrow \infty \tag{4.3}
\end{equation*}
$$

Above $v_{T}=\frac{1}{D(T)} \int_{v_{1}}^{T} \mu_{X(t)} d(t) d t$ is a short notation of the measure $v_{T}$ with $v_{T}(A)=\frac{1}{D(T)} \int_{v_{1}}^{T} \mu_{X(t)}(A) d(t) d t$.

The following corollaries are also included in [3].
Corollary 4.1. Let $c_{n}, v_{n}, d(t)$, and $D(t)$ be the same as in Theorem $A$. Let $X(t)$ be a random process satisfying conditions of Theorem A. Assume that there exists a random element $\zeta$ in $(\boldsymbol{B}, \varrho)$ such that

$$
\begin{equation*}
X(t) \xrightarrow{d} \xi, \quad \text { as } t \rightarrow \infty \tag{4.4}
\end{equation*}
$$

Then we have

$$
Q_{T, \omega} \xrightarrow{w} \mu_{\zeta}, \text { as } \quad T \rightarrow \infty
$$

for almost all $\omega \in \Omega$.
Corollary 4.2. Consider the real valued case. Let $\varphi_{X(t)}(x)$ and $\varphi_{\mu}(x)$ be the characteristic functions of $X(t)$ and $\mu$, respectively. Then, in Theorem $A$, condition (4.3) can be substituted with the following

$$
\begin{equation*}
\lim _{T \rightarrow \infty} \frac{1}{D(T)} \int_{v_{1}}^{T} \varphi_{X(t)}(x) d(t) d t=\varphi_{\mu}(x), \quad x \in \mathbb{R} \tag{4.5}
\end{equation*}
$$

REMARK 4.1. The conditions of Theorem A are valid if $v_{k}=k^{\alpha}, c_{k}=$ $=k^{\beta}, k=1,2, \ldots$, where $0<\alpha, 0<\beta$ are fixed. $d(t)=\frac{1}{t}, t \geq 1$, and $D(T)=\log (T), T \geq 1$.

Proof of Theorem 2.1. The distribution of $X(t)$ is binomial with parameters $n=[f(t)]$ and $p=1 / f(t)$. Therefore $X(t) \xrightarrow{d} \pi$, as $t \rightarrow \infty$.

Let

$$
X_{l k}^{\prime}(t)=\sum_{i=l+1}^{[t]} \mathbb{I}_{\left[0, \frac{1}{t}\right]}\left(\xi_{i}\right), \quad l<k \leq t<k+1
$$

Let $X_{l k}(t)=X_{l k}^{\prime}(f(t))$. If $l<k$, then $\{X(t): l \leq t<l+1\}$ and $\left\{X_{l k}(t): k \leq t<k+1\right\}$ are independent families. Moreover, for all $k \leq t<k+1$,

$$
\mathbb{E}\left|X(t)-X_{l k}(t)\right|=\frac{f(l)}{f(t)} \leq\left(2 \frac{l}{k}\right)^{\beta}
$$

The result follow from Theorem A .

Proof of Theorem 3.1. Let us define the process

$$
X_{l k}(t)=\frac{V(f(t))-V(f(l+1))}{(f(t))^{1 / 2}}, \quad l<k \leq t
$$

Then, for $l<k,\{X(t): l \leq t<l+1\}$ and $\left\{X_{l k}(t): k \leq t<k+1\right\}$ are independent families. Applying the well-known formula for the variance of the infinitely divisible law (see [6], sect. 18), we get for all $k \leq t<k+1$

$$
\begin{aligned}
\mathbb{E}\left|X(t)-X_{l k}(t)\right| & \leq \sqrt{\mathbb{E}\left(X(t)-X_{l k}(t)\right)^{2}}=\sqrt{\frac{\mathbb{E}[V(f(l+1))]^{2}}{f(t)}} \\
& =\sqrt{\frac{f(l+1)}{f(t)} K(\infty)} \leq \sqrt{K(\infty)}\left(\frac{2 l}{k}\right)^{\beta / 2}
\end{aligned}
$$

By the convergence criterion of [6] sect $19, X(t) \xrightarrow{d} N(0, K(\infty))$, as $t \rightarrow$ $\rightarrow \infty$. Therefore the proof is complete.

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# ON GENERALIZING THE OGY METHOD 

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## 1. Introduction

Working with discrete dynamic systems, which is driven by some recurrent relation, plays a central and important role in various branches of applied mathematics, especially in economics and physics. From both theoretical and practical point of view the characterization, and hence our comprehension about some economic or physical processes are fundamentally based on the modelling dynamic systems we develop. However, many dynamic systems that are defined by appropriate recurrence relations are chaotic by nature, which essentially limits their straightforward applicability. In order to utilize systems that have chaotic, i.e. unpredictable behavior and comprehensively employ them in economic and physical modelling, we have to introduce some control on them keeping their trajectories in predetermined unstable fixed points.

The $O G Y$ method, originally proposed by Ott, Grebogi and Yorke in [4], formulates a powerful methodology for controlling dynamic systems of chaotic behavior. In the literature the framework of the OGY method has extensively been analyzed in several applications and different environment, and in a large number of cases the method was shown to be a completely adaptable and excellently working instrument [2,3,4].

In this paper we shall generalize and give a mathematically firm footing to the OGY method to arbitrary dimensions, and deduce the exact necessary conditions for its applicability. The control of the OGY method, i.e. the choice of its controlling parameters, can be desired in such a way that the linear approximation of the system is embedded in a higher dimensional space. The sets of the eigenvalues and eigenvectors of this higher dimensional
system are completely explored, and we will prove that they are actually independent from the particular form of the control.

Many chaotic dynamic systems have infinite number of unstable periodic orbits. The OGY method can be used to stabilize one of them, if some natural conditions are satisfied. In particular, the OGY method can also be used to stabilize a periodic system about any of its unstable fixed point [3].

As usual, the method is discussed in the case when an unstable fixed point is stabilized. Let us consider the following recurrence relation

$$
\mathbf{r}_{m+1}=\mathbf{f}\left(\mathbf{r}_{m}, \mathbf{p}_{m}\right)
$$

where $\mathbf{r}_{m}$ is the state vector of the system, and $\mathbf{p}_{m}$ is the vector of the controlling parameters ( $m=1,2, \ldots$ ).

Starting from an initial vector $\mathbf{r}_{1}$, at each iteration $m$ we try to shift the orbit towards the so-called stable manifold by determining an appropriate value of the vector $\mathbf{p}_{m}$.

In the following sections we shall analyze the OGY method for system control in every detail from a mathematical aspect and present its fundamental properties. We will proceed as follows. In Section 2 we shall formulate the OGY method in arbitrary finite dimensional space, and give the necessary conditions for its applicability. In Section 3 we shall compute the eigenvalues of the local linearization of the controlled system, and show that they are independent from the particular form of the control. Finally, in Section 4, as an illustration of our framework, we will analyze and control the famous logistic map. In particular, we will show that the OGY method can always stabilize the unsteady nonzero fixed point, although no stable eigenvalue of the system exists; furthermore, we shall also investigate the case when the system is in a periodic state of length 2 , and point out that the traditional OGY method, when bounded, does not work if the bound is too small.

## 2. The generalized OGY method

The idea of the OGY method can be used in arbitrary high dimensions as follows.

Let us consider the following recurrence relation

$$
\begin{equation*}
\mathbf{r}_{m+1}=\mathbf{f}\left(\mathbf{r}_{m}, \mathbf{p}_{m}\right), \quad m=1,2, \ldots, \tag{1}
\end{equation*}
$$

where $\mathbf{r}_{m}=\left(r_{1}^{(m)}, \ldots, r_{n}^{(m)}\right) \in \mathbb{R}^{n}$ is the vector of state variables, $\mathbf{p}_{m}=$ $=\left(p_{1}^{(m)}, \ldots, p_{q}^{(m)}\right) \in \mathbb{R}^{q}$ is the parameter vector, and $\mathbf{f}=\left(f_{1}, \ldots, f_{n}\right): \mathbb{R}^{n+q} \rightarrow$ $\rightarrow \mathbb{R}^{n}$.

The generalized OGY method is based on the following assumptions $\left(a_{1}\right)-\left(a_{6}\right)$.
$\left(a_{1}\right)$ The chaotic solution of the nonlinear dynamic system (1) may have even infinite number of unstable periodic orbits.
$\left(a_{2}\right)$ In a neighbourhood of a periodic orbit the evolution of the system can be approximated by an appropriate local linearization of the system equation.
$\left(a_{3}\right)$ Small perturbations of the parameter vector $\mathbf{p}$ of the system can shift the chaotic orbit towards the so-called stable manifold of the chosen periodic orbit.
$\left(a_{4}\right)$ Suppose that we have a fixed point $\mathbf{r}_{0}=\left(r_{1}^{(0)}, \ldots, r_{n}^{(0)}\right)$ with a fixed parameter vector $\mathbf{p}_{0}=\left(p_{1}^{(0)}, \ldots, p_{q}^{(0)}\right)$ such that

$$
\begin{equation*}
\mathbf{r}_{0}=\mathbf{f}\left(\mathbf{r}_{0}, \mathbf{p}_{0}\right) \tag{2}
\end{equation*}
$$

and this fixed point is unstable.
Assumptions ( $a_{5}$ ) and ( $a_{6}$ ) require some more mathematical explanation, and they will be presented later after the necessary analysis has been finished.

Let $\mathbf{r}_{m}$ and $\mathbf{p}_{m}$ be close enough to $\mathbf{r}_{0}$ and $\mathbf{p}_{0}$ as required in $\left(a_{2}\right)$, so that we can write the Taylor series of $\mathbf{f}$ about the point $\left(\mathbf{r}_{0}, \mathbf{p}_{0}\right)$ as

$$
\mathbf{f}\left(\mathbf{r}_{m}, \mathbf{p}_{m}\right)=\mathbf{f}\left(\mathbf{r}_{0}, \mathbf{p}_{0}\right)+\mathbf{f}^{\prime}\left(\mathbf{r}_{0}, \mathbf{p}_{0}\right)\left[\begin{array}{l}
\mathbf{r}_{m}-\mathbf{r}_{0} \\
\mathbf{p}_{m}-\mathbf{p}_{0}
\end{array}\right]+O\left(\left\|\mathbf{r}_{m}-\mathbf{r}_{0}\right\|^{2},\left\|\mathbf{p}_{m}-\mathbf{p}_{0}\right\|^{2}\right)
$$

Hence, neglecting all the higher order terms, we have

$$
\mathbf{f}\left(\mathbf{r}_{m}, \mathbf{p}_{m}\right)=\mathbf{r}_{0}+\mathbf{f}^{\prime}\left(\mathbf{r}_{0}, \mathbf{p}_{0}\right)\left[\begin{array}{l}
\mathbf{r}_{m}-\mathbf{r}_{0}  \tag{3}\\
\mathbf{p}_{m}-\mathbf{p}_{0}
\end{array}\right] .
$$

The next point of the orbit is determined by (1), that is,

$$
\begin{equation*}
\mathbf{r}_{m+1}=\mathbf{f}\left(\mathbf{r}_{m}, \mathbf{p}_{m}\right) \tag{4}
\end{equation*}
$$

It follows from $\left(a_{3}\right)$ that we can shift the chaotic orbit towards a stable manifold by determining $\mathbf{p}_{m}$, i.e. we can try to control system (1) such that its orbit approaches the unstable fixed point $\mathbf{r}_{0}$. The method of determining $\mathbf{p}_{m}$ is based on the theory of covariant and contravariant vectors.

Denote

$$
\mathbf{J}=\frac{\partial \mathbf{f}\left(\mathbf{r}_{0}, \mathbf{p}_{0}\right)}{\partial \mathbf{r}}=\left[\begin{array}{ccc}
\frac{\partial f_{1}\left(\mathbf{r}_{0}, \mathbf{p}_{0}\right)}{\partial r_{1}} & \ldots & \frac{\partial f_{1}\left(\mathbf{r}_{0}, \mathbf{p}_{0}\right)}{\partial r_{n}} \\
\vdots & & \vdots \\
\frac{\partial f_{n}\left(\mathbf{r}_{0}, \mathbf{p}_{0}\right)}{\partial r_{1}} & \ldots & \frac{\partial f_{n}\left(\mathbf{r}_{0}, \mathbf{p}_{0}\right)}{\partial r_{n}}
\end{array}\right]
$$

and

$$
\mathbf{w}_{i}=\frac{\partial \mathbf{f}\left(\mathbf{r}_{0}, \mathbf{p}_{0}\right)}{\partial p_{i}}=\left[\frac{\partial f_{1}\left(\mathbf{r}_{0}, \mathbf{p}_{0}\right)}{\partial p_{i}}, \ldots, \frac{\partial f_{n}\left(\mathbf{r}_{0}, \mathbf{p}_{0}\right)}{\partial p_{i}}\right]^{T}, \quad i=1, \ldots, q
$$

Let $\mathbf{W}$ be the matrix, in which the $i$-th column is vector $\mathbf{w}_{i},(i=1, \ldots, q)$, that is,

$$
\mathbf{W}=\frac{\partial \mathbf{f}\left(\mathbf{r}_{0}, \mathbf{p}_{0}\right)}{\partial \mathbf{p}}=\left[\mathbf{w}_{1}, \ldots, \mathbf{w}_{q}\right]
$$

$\left(a_{5}\right)$ Assume that matrix $\mathbf{J}$ has $n$ real eigenvalues, which determine $n$ linearly independent eigenvectors of $\mathbb{R}^{n}$.

We note that if the eigenvalues of $\mathbf{J}$ are different, then this assumption will be satisfied.

Denote $\lambda_{1}, \ldots, \lambda_{k}$ the stable and $\mu_{1}, \ldots, \mu_{l}$ the unstable eigenvalues of matrix $\mathbf{J}$ with multiplicities, i.e. $\left|\lambda_{i}\right|<1,(i=1, \ldots, k)$ and $\left|\mu_{j}\right| \geq 1$, $(j=1, \ldots, l)$, where $k+l=n$. We note that it may happen that $k=0$ or $l=0$. From $\left(a_{5}\right)$ we have that there exist $n$ linearly independent eigenvectors of $\mathbf{J}$, which, without loss of generality, can be taken normalized. Let us denote them by $\mathbf{u}_{1}, \ldots, \mathbf{u}_{k}, \mathbf{v}_{1}, \ldots, \mathbf{v}_{l}$, where

$$
\begin{align*}
& \mathbf{J} \mathbf{u}_{i}=\lambda_{i} \mathbf{u}_{i},\left\|\mathbf{u}_{i}\right\|^{2}=\mathbf{u}_{i}^{T} \mathbf{u}_{i}=1, \quad i=1, \ldots, k \\
& \mathbf{J}_{j}=\mu_{j} \mathbf{v}_{j},\left\|\mathbf{v}_{j}\right\|^{2}=\mathbf{v}_{j}^{T} \mathbf{v}_{j}=1, \quad j=1, \ldots, l \tag{5}
\end{align*}
$$

It follows immediately that the subspace generated by vectors $\mathbf{u}_{1}, \ldots, \mathbf{u}_{k}$ approximates the stable manifold, while the subspace generated by $\mathbf{v}_{1}, \ldots, \mathbf{v}_{l}$ approximates the direct sum of the unstable and centre manifolds of the fixed point $\mathbf{r}_{0}$ in the domain of linearity.

Let

$$
\begin{align*}
& \mathbf{U}=\left[\mathbf{u}_{1}, \ldots, \mathbf{u}_{k}\right] \\
& \mathbf{V}=\left[\mathbf{v}_{1}, \ldots, \mathbf{v}_{l}\right] \tag{6}
\end{align*}
$$

be the matrices, in which the $i$-th $(i=1, \ldots, k)$ and $j$-th $(j=1, \ldots, l)$ column is vector $\mathbf{u}_{i}$ and $\mathbf{v}_{j}$, respectively, and let

$$
\mathbf{P}=\left[\begin{array}{ll}
\mathbf{U} & \mathbf{V} \tag{7}
\end{array}\right]=\left[\mathbf{u}_{1}, \ldots, \mathbf{u}_{k}, \mathbf{v}_{1}, \ldots, \mathbf{v}_{l}\right] .
$$

It is obvious, that matrix $\mathbf{P}$ is invertible, i.e.

$$
\operatorname{det}(\mathbf{P}) \neq 0
$$

Hence, let

$$
\begin{equation*}
\mathbf{R}=\left(\mathbf{P}^{-1}\right)^{T} \tag{8}
\end{equation*}
$$

furthermore, let $\mathbf{g}_{r}(r=1, \ldots, k)$ denote the $r$-th column of $\mathbf{R}$, and let $\mathbf{h}_{s}$ ( $s=1, \ldots, l$ ) denote the $(k+s)$-th column of $\mathbf{R}$, i.e.

$$
\begin{equation*}
\mathbf{R}=\left[\mathbf{g}_{1}, \ldots, \mathbf{g}_{k}, \mathbf{h}_{1}, \ldots, \mathbf{h}_{l}\right] \tag{9}
\end{equation*}
$$

From

$$
\mathbf{R}^{T} \mathbf{P}=\mathbf{I}_{n}
$$

where $\mathbf{I}_{n}$ denotes the $n \times n$ unit matrix, and from (7) and (9) we obtain

$$
\begin{array}{rlrl}
\mathbf{g}_{r}^{T} \mathbf{u}_{i} & =\delta_{i r}, & & \mathbf{g}_{r}^{T} \mathbf{v}_{j}=0  \tag{10}\\
\mathbf{h}_{s}^{T} \mathbf{u}_{i} & =0, & \mathbf{h}_{s}^{T} \mathbf{v}_{j}=\delta_{j s}
\end{array}
$$

where $i, r=1, \ldots, k$ and $j, s=1, \ldots, l$, and where $\delta_{a b}$ stands for the Kronecker delta, i.e. $\delta_{a b}=1$ if $a=b$, and $\delta_{a b}=0$ if $a \neq b$.

Since the vectors $\mathbf{u}_{1}, \ldots, \mathbf{u}_{k}, \mathbf{v}_{1}, \ldots, \mathbf{v}_{l}$ form a basis of $\mathbb{R}^{n}$, all vectors in $\mathbb{R}^{n}$ can uniquely be written as their linear combination, that is,

$$
\forall \mathbf{x} \in \mathbb{R}^{n} \exists!\left(\alpha_{1}, \ldots, \alpha_{k}, \beta_{1}, \ldots, \beta_{l}\right) \in \mathbb{R}^{n}: \mathbf{x}=\sum_{i=1}^{k} \alpha_{i} \mathbf{u}_{i}+\sum_{j=1}^{l} \beta_{j} \mathbf{v}_{j}
$$

However, from (10) we find that

$$
\begin{array}{ll}
\mathbf{g}_{r}^{T} \mathbf{x}=\alpha_{r}, & r=1, \ldots, k, \\
\mathbf{h}_{s}^{T} \mathbf{x}=\beta_{s}, & s=1, \ldots, l
\end{array}
$$

thus we can conclude that any vector $\mathbf{x} \in \mathbb{R}^{n}$ can be written in the form

$$
\begin{equation*}
\mathbf{x}=\sum_{i=1}^{k}\left(\mathbf{g}_{i}^{T} \mathbf{x}\right) \mathbf{u}_{i}+\sum_{j=1}^{l}\left(\mathbf{h}_{j}^{T} \mathbf{x}\right) \mathbf{v}_{j} . \tag{11}
\end{equation*}
$$

Notice that if we consider $\mathbf{u}_{1}, \ldots, \mathbf{u}_{k}, \mathbf{v}_{1}, \ldots, \mathbf{v}_{l}$ as covariant vectors, then $\mathbf{g}_{1}, \ldots, \mathbf{g}_{k}, \mathbf{h}_{1}, \ldots, \mathbf{h}_{l}$ will be their corresponding contravariant vectors.

Hence, from (3) and (4) we have the following recurrence relation

$$
\mathbf{r}_{m+1}=\mathbf{r}_{0}+\mathbf{J}\left(\mathbf{r}_{m}-\mathbf{r}_{0}\right)+\mathbf{W}\left(\mathbf{p}_{m}-\mathbf{p}_{0}\right)
$$

Introducing the notations $\delta \mathbf{r}_{m}=\mathbf{r}_{m}-\mathbf{r}_{0}$ and $\delta \mathbf{p}_{m}=\mathbf{p}_{m}-\mathbf{p}_{0}$ we have

$$
\begin{equation*}
\delta \mathbf{r}_{m+1}=\mathbf{J} \delta \mathbf{r}_{m}+\mathbf{W} \delta \mathbf{p}_{m} \tag{12}
\end{equation*}
$$

Now, using (12), (11) and (5) we can write

$$
\delta \mathbf{r}_{m+1}=\mathbf{J} \delta \mathbf{r}_{m}+\mathbf{W} \delta \mathbf{p}_{m}=\mathbf{J}\left(\sum_{i=1}^{k}\left(\mathbf{g}_{i}^{T} \delta \mathbf{r}_{m}\right) \mathbf{u}_{i}+\sum_{j=1}^{l}\left(\mathbf{h}_{j}^{T} \delta \mathbf{r}_{m}\right) \mathbf{v}_{j}\right)+\mathbf{W} \delta \mathbf{p}_{m}
$$

$$
\begin{aligned}
& =\sum_{i=1}^{k}\left(\mathbf{g}_{i}^{T} \delta \mathbf{r}_{m}\right) \mathbf{J} \mathbf{u}_{i}+\sum_{j=1}^{l}\left(\mathbf{h}_{j}^{T} \delta \mathbf{r}_{m}\right) \mathbf{J} \mathbf{v}_{j}+\mathbf{W} \delta \mathbf{p}_{m} \\
& =\sum_{i=1}^{k} \lambda_{i}\left(\mathbf{g}_{i}^{T} \delta \mathbf{r}_{m}\right) \mathbf{u}_{i}+\sum_{j=1}^{l} \mu_{j}\left(\mathbf{h}_{j}^{T} \delta \mathbf{r}_{m}\right) \mathbf{v}_{j}+\mathbf{W} \delta \mathbf{p}_{m}
\end{aligned}
$$

We want to shift the chaotic orbit towards the stable manifold, so we choose the parameter vector $\mathbf{p}_{m}$ such that

$$
\mathbf{h}_{s}^{T} \delta \mathbf{r}_{m+1}=0, \quad s=1, \ldots, l
$$

Then, from (13) and (10) we obtain

$$
\begin{align*}
0=\mathbf{h}_{s}^{T} \delta \mathbf{r}_{m+1} & =\sum_{i=1}^{k} \lambda_{i}\left(\mathbf{g}_{i}^{T} \delta \mathbf{r}_{m}\right)\left(\mathbf{h}_{s}^{T} \mathbf{u}_{i}\right)+\sum_{j=1}^{l} \mu_{j}\left(\mathbf{h}_{j}^{T} \delta \mathbf{r}_{m}\right)\left(\mathbf{h}_{s}^{T} \mathbf{v}_{j}\right)+\mathbf{h}_{s}^{T} \mathbf{W} \delta \mathbf{p}_{m} \\
& =\mu_{s}\left(\mathbf{h}_{s}^{T} \delta \mathbf{r}_{m}\right)+\mathbf{h}_{s}^{T} \mathbf{W} \delta \mathbf{p}_{m} \tag{14}
\end{align*}
$$

for any $s=1, \ldots, l$. Let

$$
\begin{equation*}
\mathbf{H}=\left[\mathbf{h}_{1}, \ldots, \mathbf{h}_{l}\right] \tag{15}
\end{equation*}
$$

be the matrix composed of the vectors $\mathbf{h}_{1}, \ldots, \mathbf{h}_{l}$, and let

$$
\mathbf{D}=\operatorname{diag}\left(\mu_{1}, \ldots, \mu_{l}\right)
$$

be a diagonal matrix of size $l \times l$, where the $j$-th diagonal element is $\mu_{j}$ ( $j=1, \ldots, l$ ).

Using (14) we have the following matrix equation

$$
\begin{equation*}
\mathbf{D} \mathbf{H}^{T} \delta \mathbf{r}_{m}+\mathbf{H}^{T} \mathbf{W} \delta \mathbf{p}_{m}=\mathbf{0} \tag{16}
\end{equation*}
$$

We note that since $\mathbf{H} \in \mathbb{R}^{n \times l}$ and $\mathbf{W} \in \mathbb{R}^{n \times q}$, therefore $\mathbf{H}^{T} \mathbf{W} \in \mathbb{R}^{l \times q}$.
In any case of control it is important that the controlled system has freedom to some "sufficient degree". In traditional control theory this property is described by the theorems of controllability. In engineering applications the system must be designed in such a way that the appropriate conditions are satisfied. In the case of economics we have to assume that sufficient number of factors are under control. For the OGY method the appropriate necessary condition is summarized in the next assumption.
$\left(a_{6}\right)$ Assume that $l=q$ and $\mathbf{H}^{T} \mathbf{W}$ is an invertible matrix.
We note that in the case $q=1$, i.e. when the dimension of the control in dynamic system (1) is 1 , assumption $\left(a_{6}\right)$ requires that $l=1$, i.e. the
dimension of the subspace approximating the non-stable manifolds about the fixed point $\mathbf{r}_{0}$ is also 1 ; furthermore, now $\mathbf{H}=H \in \mathbb{R}$ and $\mathbf{W}=W \in \mathbb{R}$, and applying $\left(a_{6}\right)$ we have that $\mathbf{H}^{T} \mathbf{W}=H W \neq 0$.

In general, if $l=q$, then $\mathbf{H}^{T} \mathbf{W}$ is an invertible matrix only if both matrices $\mathbf{H}$ and $\mathbf{W}$ have full column rank. However, from the definition of matrix $\mathbf{H}$ it follows that the columns of $\mathbf{H}$ are linearly independent, that is, $\mathbf{H}$ has full column rank. Hence, we can conclude that if $l=q$ then $\mathbf{H}^{T} \mathbf{W}$ is an invertible matrix only if $\mathbf{W}$ has full column rank.

From (16) we have

$$
\begin{equation*}
\delta \mathbf{p}_{m}=-\left(\mathbf{H}^{T} \mathbf{W}\right)^{-1} \mathbf{D} \mathbf{H}^{T} \delta \mathbf{r}_{m} \tag{17}
\end{equation*}
$$

Formula (17) describes the way how the parameter vector at iteration $m$ should be chosen. Essentially, this formula is applied in the generalized OGY method.

From the results presented above we obtain the following theorem.
THEOREM 1. Under the assumptions $\left(a_{1}\right)-\left(a_{6}\right)$ there is a parameter vector $\mathbf{p}_{m}$ such that the trajectory of the recurrence relation (1) is shifted towards the stable manifold; namely, equations (14) are satisfied.

We note that the difference $\delta \mathbf{r}_{m+1}=\mathbf{r}_{m+1}-\mathbf{r}_{0}$ is shifted into the subspace generated by the vectors $\mathbf{u}_{1}, \ldots, \mathbf{u}_{k}$, which approximates the stable manifold of the fixed point $\mathbf{r}_{0}$ in the domain of linearity. Indeed, from the equations

$$
\begin{equation*}
\mathbf{h}_{s}^{T} \mathbf{u}_{i}=0, \quad i=1, \ldots, k, s=1, \ldots, l \tag{18}
\end{equation*}
$$

it follows that

$$
\begin{equation*}
\mathbf{h}_{s}^{T} \delta \mathbf{r}_{m+1}=0, \quad s=1, \ldots, l \tag{19}
\end{equation*}
$$

Furthermore, since $k+l=n$, we have that the following relationship also holds

$$
\begin{equation*}
\left\langle\mathbf{u}_{1}, \ldots, \mathbf{u}_{k}\right\rangle=\left\langle\mathbf{h}_{1}, \ldots, \mathbf{h}_{l}\right\rangle^{\perp} \tag{20}
\end{equation*}
$$

where $\left\langle\mathbf{u}_{1}, \ldots, \mathbf{u}_{k}\right\rangle$ denotes the subspace generated by vectors $\mathbf{u}_{1}, \ldots, \mathbf{u}_{k}$, and $\left\langle\mathbf{h}_{1}, \ldots, \mathbf{h}_{l}\right\rangle^{\perp}$ stands for the orthogonal complement of the subspace generated by vectors $\mathbf{h}_{1}, \ldots, \mathbf{h}_{l}$. Now, using (19) and (20) we get

$$
\delta \mathbf{r}_{m+1} \in\left\langle\mathbf{h}_{1}, \ldots, \mathbf{h}_{l}\right\rangle^{\perp}=\left\langle\mathbf{u}_{1}, \ldots, \mathbf{u}_{k}\right\rangle
$$

REMARK 1. Let us take a closer look at two special cases of the generalized OGY method.
(i) If $l=0$ then there is no need to shift the chaotic orbit towards the stable manifold in a neighborhood of the fixed point $\mathbf{r}_{0}$, because the stable manifold is $\mathbb{R}^{n}$ itself, and any trajectory starting from an appropriate neighborhood of $\mathbf{r}_{0}$ converges to $\mathbf{r}_{0}$ irrespectively of the choice of $\mathbf{p}_{m}$. Thus, we do not need to iteratively readjust $\mathbf{p}_{m}$, but can constantly keep it $\mathbf{p}_{0}$.
(ii) If $l=n$ then in addition to $l=q$ assumption ( $a_{6}$ ) requires that $\mathbf{W}$ is an invertible matrix. In this case the vector $\delta \mathbf{r}_{m+1}$ has to be shifted to zero, because only the origin as a linear space approximates the stable manifold of the fixed point $\mathbf{r}_{0}$ in the domain of linearity.

SUMMARY 1. Assume that we have a dynamic system described by a recurrence relation (1), and we are in possession of a certain degree of freedom to control its trajectories at each iteration $m$ by appropriately choosing a free parameter vector $\mathbf{p}_{m}(m=1,2, \ldots)$. Furthermore, let us assume that there is a fixed point $\mathbf{r}_{0}$ of system (1) such that (2) holds for some $\mathbf{p}_{0}$, and let $\mathbf{r}_{1}$ be an initial point wherefrom we start iterating an orbit of (1). In these circumstances we would like to control the orbit $\mathbf{r}_{m}$ of the system by recurrently selecting the parameters $\mathbf{p}_{m}$ to force it into the fixed point $\mathbf{r}_{0}$. It should be noted that by the time the system determines $\mathbf{r}_{m+1}$ we need to provide the control $\mathbf{p}_{m}$ to modify the original orbit of $\mathbf{r}_{m}$ and shift it towards $\mathbf{r}_{0}$.

In this kind of settings, let they arise from theoretical or practical applications, the framework of the OGY method presents a useful and powerful tool for controlling the original behavior of (chaotic) systems. We shall distinguish between two kinds of OGY control.

THE TRADITIONAL OGY METHOD. Let $\varepsilon>0$ be a fixed positive number. The control is applied only if $\left\|\delta \mathbf{p}_{m}\right\|=\left\|\mathbf{p}_{m}-\mathbf{p}_{0}\right\|<\varepsilon$ holds. If $\left\|\delta \mathbf{p}_{m}\right\| \geq \varepsilon$ then the system freely follows a trajectory, and we wait for the moment when it approaches $\mathbf{r}_{0}$ "close enough", so that the control can be switched on.
Hence, from (1) applying (17) we can mathematically formulate the traditional OGY method as follows.

$$
\begin{aligned}
& \mathbf{r}_{m+1}= \begin{cases}\mathbf{f}\left(\mathbf{r}_{m}, \mathbf{p}_{0}-\left(\mathbf{H}^{T} \mathbf{W}\right)^{-1} \mathbf{D} \mathbf{H}^{T}\left(\mathbf{r}_{m}-\mathbf{r}_{0}\right)\right) & \text { if }\left\|\mathbf{p}_{m}-\mathbf{p}_{0}\right\|<\varepsilon \\
\mathbf{f}\left(\mathbf{r}_{m}, \mathbf{p}_{0}\right) & \text { if }\left\|\mathbf{p}_{m}-\mathbf{p}_{0}\right\| \geq \varepsilon \\
& m=1,2, \ldots\end{cases} \\
&
\end{aligned}
$$

Here we assume and actually utilize the fact that the system is chaotic; therefore any trajectory will sooner or later be arbitrarily close to the
fixed point. That is, at some iteration $m \in \mathbb{N}$ there will be a controlling parameter vector $\mathbf{p}_{m}$ such that $\delta \mathbf{p}_{m}<\varepsilon$, thus the control will be under an a priori defined level. This assumption is satisfied if chaotic equipments are stabilized, but might be violated if a periodic system must be stabilized about a fixed point.
However, we should be aware of the fact, that in many applications the bound $\varepsilon$ is an exogenous variable, which is not in direct connection with the controlled system. For instance, if the system is a market of a certain commodity (e.g. currency, oil or wheat) or financial instrument (e.g. security or bond) which the central bank in power wants to keep at a steady price, then the value $\varepsilon$ will represent the budget constraint, i.e. the amount of money that can be used at each period for stabilization.
Furthermore, concerning economic systems it might happen that simply there is no time to wait for the actual event, when the trajectory is close enough to the fixed point. To be able to refer to this case, which we have to encounter very often when applying the OGY framework, we introduce another notion of the OGY method.

The pure OGY method. The adaptation of this kind of OGY control is based on the idea of stabilizing the system with "brute force". That is, in this consideration we compute the controlling parameters $\mathbf{p}_{m}$ at each iteration $m=1,2, \ldots$, and apply them to the system regardless of their amplitude $\left\|\delta \mathbf{p}_{m}\right\|=\left\|\mathbf{p}_{m}-\mathbf{p}_{0}\right\|$. Hence, when stabilizing a system with the pure OGY method, we do not consider any threshold value $\varepsilon>0$, which plays a central role in using the traditional OGY method with respect to switching on and off the control, but always keep the control switched on.
Mathematically, by using (1) and (17), the pure OGY method can simply be formulated as the following recurrence relation.

$$
\mathbf{r}_{m+1}=\mathbf{f}\left(\mathbf{r}_{m}, \mathbf{p}_{0}-\left(\mathbf{H}^{T} \mathbf{W}\right)^{-1} \mathbf{D} \mathbf{H}^{T}\left(\mathbf{r}_{m}-\mathbf{r}_{0}\right)\right), \quad m=1,2, \ldots
$$

Since in real-world applications the size of $\left\|\delta \mathbf{p}_{m}\right\|$ is usually related to the consumption of some resources, which is required to be covered, in practice the straightforward adaptation of the pure OGY control may not fully succeed in the object of system stabilization. However, even though the outcome of the pure OGY method can be ideal or speculative and might be viewed as some hypothetical orbit of the system equipped with a full controlling capability, it can represent an important benchmark of the total controllability of the system, which makes its introduction adjusted.

In the next section we shall formulate the exact equations of the so-called controlled system of (1), and compute the eigenvalues of its local linearization. In particular, we shall prove that these eigenvalues are independent from matrix $\mathbf{W}$.

## 3. On the eigenvalues of the linearized controlled system

In the following we shall assume that the control is switched on, and formula (17) is applied at each iteration. We note that this is the case after finite many iterations if the traditional method successfully stabilizes a chaotic system. However, it may not always be the case. If the distance of the trajectory from the desired target point is at least $\varepsilon>0$ after any finite number of iteration, then the controlling method will actually never be switched on, and thus the system will not be stabilized about its fixed point.

Hence, in case of efficient control the system and its control, i.e. the choice of the parameter vector, can be unified in a higher dimensional dynamic system. Furthermore, this higher dimensional system represents the application of the OGY method on the particular system (1).

Let us consider the following dynamic system of $n+l$ dynamic variables

$$
\begin{gather*}
\mathbf{r}_{m+1}=\mathbf{f}\left(\mathbf{r}_{m}, \mathbf{p}_{m}\right) \\
\mathbf{p}_{m}-\mathbf{p}_{0}=-\left(\mathbf{H}^{T} \mathbf{W}\right)^{-1} \mathbf{D} \mathbf{H}^{T}\left(\mathbf{r}_{m}-\mathbf{r}_{0}\right), \tag{2}
\end{gather*}
$$

where $\mathbf{r}_{0}=\mathbf{f}\left(\mathbf{r}_{0}, \mathbf{p}_{0}\right)$ is the unstable fixed point of system (1) with parameter vector $\mathbf{p}_{0}$. We call this system the controlled system of (1), and refer to the columns of matrix $\mathbf{W}=\left[\mathbf{w}_{1}, \ldots, \mathbf{w}_{q}\right]$ as the control vectors. It is obvious, that as we change the control vectors of system (21), its behavior of motion changes as well. As the main result of this section we shall prove that in a neighbourhood of the fixed point $\mathbf{r}_{0}$ the eigenvalues of the local linearization of the controlled system (21) are independent from the choice of $\mathbf{w}_{1}, \ldots, \mathbf{w}_{l}$. In particular, we will show that all of the eigenvalues of (21) are stable; namely, their absolute values are less than 1 . What is more, we will also obtain that not all of the eigenvectors of the linearized system actually depend on the control vectors.

First, let us formulate system (21) in the form

$$
\mathbf{r}_{m+1}=\mathbf{f}\left(\mathbf{r}_{m}, \mathbf{p}_{m}\right)
$$

$$
\begin{equation*}
\mathbf{p}_{m+1}-\mathbf{p}_{0}=-\left(\mathbf{H}^{T} \mathbf{W}\right)^{-1} \mathbf{D} \mathbf{H}^{T}\left(\mathbf{f}\left(\mathbf{r}_{m}, \mathbf{p}_{m}\right)-\mathbf{r}_{0}\right) . \tag{22}
\end{equation*}
$$

The matrix of the local linearization of this controlled system in a neighbourhood of the fixed point $\mathbf{r}_{0}$ with parameter vector $\mathbf{p}_{0}$ is

$$
\mathbf{M}=\left[\begin{array}{cc}
\mathbf{J} & \mathbf{W}  \tag{23}\\
-\mathbf{T}^{-1} \mathbf{D} \mathbf{H}^{T} \mathbf{J} & -\mathbf{T}^{-1} \mathbf{D} \mathbf{T}
\end{array}\right] \in \mathbb{R}^{(n+l) \times(n+l)},
$$

where

$$
\begin{equation*}
\mathbf{T}=\mathbf{H}^{T} \mathbf{W} \tag{24}
\end{equation*}
$$

Note that assumption $\left(a_{6}\right)$ requires that matrix $\mathbf{T}$ is invertible.
Let

$$
\widehat{\mathbf{P}}=\left[\begin{array}{cc}
\mathbf{P} & \mathbf{0} \\
\mathbf{0} & \mathbf{I}_{l}
\end{array}\right] \in \mathbb{R}^{(n+l) \times(n+l)}
$$

be a matrix, of which upper left $n \times n$ submatrix is $\mathbf{P}$, defined in (7), lower right $l \times l$ submatrix is the unit matrix $\mathbf{I}_{l}$, and every other elements are zero.
From (7) it is obvious that $\widehat{\mathbf{P}}$ is an invertible matrix, and

$$
\widehat{\mathbf{P}}^{-1}=\left[\begin{array}{cc}
\mathbf{P}^{-1} & \mathbf{0} \\
\mathbf{0} & \mathbf{I}_{l}
\end{array}\right] .
$$

Using the rule of multiplying hypermatrices, we find

$$
\widehat{\mathbf{P}}^{-1} \mathbf{M} \widehat{\mathbf{P}}=\left[\begin{array}{cc}
\mathbf{P}^{-1} \mathbf{J P} & \mathbf{P}^{-1} \mathbf{W}  \tag{25}\\
-\mathbf{T}^{-1} \mathbf{D} \mathbf{H}^{T} \mathbf{J P} & -\mathbf{T}^{-1} \mathbf{D T}
\end{array}\right]
$$

Let

$$
\mathbf{C}=\operatorname{diag}\left(\lambda_{1}, \ldots, \lambda_{k}\right)
$$

be the diagonal matrix having its $i$-th diagonal element equall to $\lambda_{i}(i=$ $=1, \ldots, k$ ). Then, from (5) we have

$$
\mathbf{J P}=\mathbf{P}\left[\begin{array}{cc}
\mathbf{C} & \mathbf{0}  \tag{26}\\
\mathbf{0} & \mathbf{D}
\end{array}\right]
$$

that is,

$$
\mathbf{P}^{-1} \mathbf{J P}=\left[\begin{array}{cc}
\mathbf{C} & \mathbf{0}  \tag{27}\\
\mathbf{0} & \mathbf{D}
\end{array}\right] .
$$

Let

$$
\mathbf{G}=\left[\mathbf{g}_{1}, \ldots, \mathbf{g}_{k}\right]
$$

be the matrix of vectors $\mathbf{g}_{1}, \ldots, \mathbf{g}_{k}$. Then, from (7) and (8) we have

$$
\mathbf{P}^{-1}=\mathbf{R}^{T}=\left[\begin{array}{ll}
\mathbf{G} & \mathbf{H}
\end{array}\right]^{T}=\left[\begin{array}{l}
\mathbf{G}^{T} \\
\mathbf{H}^{T}
\end{array}\right]
$$

and we obtain

$$
\mathbf{P}^{-1} \mathbf{W}=\left[\begin{array}{c}
\mathbf{G}^{T} \mathbf{W}  \tag{28}\\
\mathbf{T}
\end{array}\right]
$$

Furthermore, from (6) and (15) using (10) we have

$$
\mathbf{H}^{T} \mathbf{U}=\mathbf{0}, \quad \mathbf{H}^{T} \mathbf{V}=\mathbf{I}_{l}
$$

thus from (7) and (26) we obtain

$$
\begin{align*}
\mathbf{H}^{T} \mathbf{J P} & =\mathbf{H}^{T} \mathbf{P}\left[\begin{array}{cc}
\mathbf{C} & \mathbf{0} \\
\mathbf{0} & \mathbf{D}
\end{array}\right]=\mathbf{H}^{T}\left[\begin{array}{ll}
\mathbf{U} & \mathbf{V}
\end{array}\right]\left[\begin{array}{cc}
\mathbf{C} & \mathbf{0} \\
\mathbf{0} & \mathbf{D}
\end{array}\right] \\
& =\left[\begin{array}{ll}
\mathbf{H}^{T} \mathbf{U} & \mathbf{H}^{T} \mathbf{V}
\end{array}\right]\left[\begin{array}{cc}
\mathbf{C} & \mathbf{0} \\
\mathbf{0} & \mathbf{D}
\end{array}\right]=\left[\begin{array}{ll}
\mathbf{0} & \mathbf{I}_{l}
\end{array}\right]\left[\begin{array}{cc}
\mathbf{C} & \mathbf{0} \\
\mathbf{0} & \mathbf{D}
\end{array}\right]=\left[\begin{array}{ll}
\mathbf{0} & \mathbf{D}
\end{array}\right] . \tag{29}
\end{align*}
$$

Hence, substituting (27), (28) and (29) into (25) we can write

$$
\widehat{\mathbf{P}}^{-1} \mathbf{M} \widehat{\mathbf{P}}=\left[\begin{array}{ccc}
\mathbf{C} & \mathbf{0} & \mathbf{G}^{T} \mathbf{W}  \tag{30}\\
\mathbf{0} & \mathbf{D} & \mathbf{T} \\
\mathbf{0} & -\mathbf{T}^{-1} \mathbf{D}^{2} & { }_{-\mathbf{T}^{-1} \mathbf{D T}}
\end{array}\right] \in \mathbb{R}^{(k+l+l) \times(k+l+l)} .
$$

Let us denote

$$
\widetilde{\mathbf{P}}=\left[\begin{array}{ccc}
\mathbf{I}_{k} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & -\mathbf{D}^{-1} \mathbf{T} & \mathbf{D}^{-1} \mathbf{T} \\
\mathbf{0} & \mathbf{I}_{l} & \mathbf{0}
\end{array}\right] \in \mathbb{R}^{(k+l+l) \times(k+l+l)} .
$$

Then, as can easily be checked, matrix $\widetilde{\mathbf{P}}$ is invertible, and for its inverse

$$
\widetilde{\mathbf{P}}^{-1}=\left[\begin{array}{ccc}
\mathbf{I}_{k} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{I}_{l} \\
\mathbf{0} & \mathbf{T}^{-1} \mathbf{D} & \mathbf{I}_{l}
\end{array}\right]
$$

Finally, let us introduce the notation

$$
\mathbf{Q}=\widehat{\mathbf{P}} \widetilde{\mathbf{P}}
$$

Then, using (30) we obtain

$$
\begin{aligned}
\mathbf{Q}^{-1} \mathbf{M} \mathbf{Q} & =\widetilde{\mathbf{P}}^{-1} \widehat{\mathbf{P}}^{-1} \mathbf{M} \widehat{\mathbf{P}} \widetilde{\mathbf{P}} \\
& =\widetilde{\mathbf{P}}^{-1}\left[\begin{array}{ccc}
\mathbf{C} & \mathbf{0} & \mathbf{G}^{T} \mathbf{W} \\
\mathbf{0} & \mathbf{D} & \mathbf{T} \\
\mathbf{0} & -\mathbf{T}^{-1} \mathbf{D}^{2} & -\mathbf{T}^{-1} \mathbf{D T}
\end{array}\right] \widetilde{\mathbf{P}}=\left[\begin{array}{ccc}
\mathbf{C} & \mathbf{G}^{T} \mathbf{W} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & -\mathbf{T}^{-1} \mathbf{D T} \\
\mathbf{0} & \mathbf{0} & \mathbf{0}
\end{array}\right] .
\end{aligned}
$$

Now, let us compute the characteristic polynomial of matrix $\mathbf{M}$

$$
\chi_{\mathbf{M}}=\operatorname{det}\left(\mathbf{M}-x \mathbf{I}_{n+l}\right),
$$

where $\mathbf{I}_{n+l}$ denotes the $(n+l) \times(n+l)$ unit matrix.

Then, since the characteristic polynomial remains unchanged under similarity transformations (conjugations), we have

$$
\chi_{\mathbf{M}}=\chi_{\mathbf{Q}^{-1} \mathbf{M} \mathbf{Q}}=\operatorname{det}\left(\mathbf{Q}^{-1} \mathbf{M} \mathbf{Q}-x \mathbf{I}_{n+l}\right)=x^{2 l} \prod_{i=1}^{k}\left(\lambda_{i}-x\right) .
$$

Hence, from the results above we can state the following theorem.

THEOREM 2. The eigenvalues of the matrix of the local linearization of the controlled system (22) in a neighbourhood of its fixed point $\mathbf{r}_{0}$ with parameter vector $\mathbf{p}_{0}$ are computed by

$$
\begin{gathered}
x_{i}=\lambda_{i}, \quad i=1, \ldots, k \\
x_{k+j}=0, \quad j=1, \ldots, 2 l
\end{gathered}
$$

In particular, the eigenvalues of the local linearization $\mathbf{M}$ of the controlled system only depend on the stable eigenvalues of the local linearization $\mathbf{J}$ of the uncontrolled system; that is, they are independent from its non-stable eigenvalues and the local linearization $\mathbf{W}$ of the control itself.

In the following we shall calculate the eigenvectors corresponding to eigenvalues $x_{1}, \ldots, x_{k}$. Let

$$
\hat{\mathbf{u}}_{i}=\left[\begin{array}{c}
\mathbf{u}_{i} \\
\mathbf{0}
\end{array}\right] \in \mathbb{R}^{n+l}, \quad i=1, \ldots, k
$$

Then, from (10) and (15) we have

$$
\mathbf{H}^{T} \mathbf{u}_{i}=0, \quad i=1, \ldots, k
$$

and hence, from (5) and (23) we obtain

$$
\mathbf{M} \hat{\mathbf{u}}_{i}=\left[\begin{array}{c}
\mathbf{J} \mathbf{u}_{i} \\
-\mathbf{T}^{-1} \mathbf{D} \mathbf{H}^{T} \mathbf{J} \mathbf{u}_{i}
\end{array}\right]=\left[\begin{array}{c}
\lambda_{i} \mathbf{u}_{i} \\
\mathbf{0}
\end{array}\right]=x_{i} \hat{\mathbf{u}}_{i}, \quad i=1, \ldots, k .
$$

Especially, if $k \geq 1$, i.e. there exist stable eigenvalues of matrix $\mathbf{J}$, then there exist $k$ eigenvectors of $\mathbf{M}$ which are independent from the choice of $\mathbf{W}$.

## 4. Illustration

In this section we shall apply the OGY method to control and stabilize the logistic map. Furthermore, we will demonstrate the adaptability of the method by choosing a fixed point which is not unstable but part of a cycle.

Let us consider the following one-dimensional dynamic system

$$
\begin{equation*}
r_{m+1}=a r_{m}\left(1-r_{m}\right), \quad m=1,2, \ldots, \tag{31}
\end{equation*}
$$

where $a \in \mathbb{R}, a \geq 3$ is an arbitrary constant.
This is the famous logistic map. We shall control this system through controlling parameters $p_{m}$ in the following way

$$
\begin{equation*}
r_{m+1}=\operatorname{ar}_{m}\left(1-r_{m}\right)+w p_{m}, \quad m=1,2, \ldots, \tag{32}
\end{equation*}
$$

where $w \neq 0$. Let

$$
r_{0}=\frac{a-1}{a}, \quad p_{0}=0 .
$$

Then, it can easily be checked that

$$
a r_{0}\left(1-r_{0}\right)=r_{0}
$$

that is, $r_{0}$ is a fixed point of the system (31), and ( $r_{0}, p_{0}$ ) is a fixed point of the controlled system (32). Let

$$
f(r, p)=a r(1-r)+w p,
$$

then, using the notations of Section 2, we have

$$
J=\frac{\partial f\left(r_{0}, p_{0}\right)}{\partial r}=2-a, \quad W=\frac{\partial f\left(r_{0}, p_{0}\right)}{\partial p}=w .
$$

From our results we know that the eigenvalue of the controlled system (32) is independent from value $w$. Furthermore, from $a \geq 3$ it follows that $J=$ $=\lambda_{1}=2-a<-1,\left|\lambda_{1}\right|>1$, that is, $r_{0}$ is an unstable fixed point.

It can easily be verified that in this case the OGY-controlled dynamic system (21) can be formulated as

$$
\begin{align*}
& r_{m+1}=a r_{m}\left(1-r_{m}\right)+w p_{m} \\
& p_{m}=\frac{a-2}{w}\left(r_{m}-\frac{a-1}{a}\right), \tag{3}
\end{align*}
$$

where the computation of the controlling values $p_{m}$ represents the specific application of the OGY method in the case of system (32).

Calculating the Taylor series of the function $f(r, p)$ about $\left(r_{0}, p_{0}\right)$ we obtain

$$
\begin{aligned}
\operatorname{ar}(1-r)+w p & =r_{0}+a\left(1-2 r_{0}\right)\left(r-r_{0}\right)+w\left(p-p_{0}\right)-a\left(r-r_{0}\right)^{2} \\
& =\frac{a-1}{a}+(2-a)\left(r-\frac{a-1}{a}\right)+w p-a\left(r-\frac{a-1}{a}\right)^{2}
\end{aligned}
$$

hence, the controlled system (33) can be written as

$$
\begin{equation*}
r_{m+1}=\frac{a-1}{a}-a\left(r_{m}-\frac{a-1}{a}\right)^{2} \tag{34}
\end{equation*}
$$

In the following we shall show that the logistic map under the OGY control, represented by the 2 -dimensional dynamic system (33), has a stable fixed point at $r_{0}$. Indeed, $r_{0}$ is a stable fixed point of (33) if and only if there exists $N_{0} \in \mathbb{N}$ such that

$$
\left|r_{m+1}-r_{0}\right|<\left|r_{m}-r_{0}\right|, \quad \forall m \geq N_{0}
$$

which, since system (33) can simply be expressed by the recurrence relation (34), is actually equivalent to

$$
a\left(r_{m}-r_{0}\right)^{2}<\left|r_{m}-r_{0}\right|, \quad \forall m \geq N_{0}
$$

that is,

$$
\left|r_{m}-\frac{a-1}{a}\right|<\frac{1}{a}, \quad \forall m \geq N_{0}
$$

Now let us assume that

$$
\left|r_{1}-\frac{a-1}{a}\right|<\frac{1}{a}
$$

Then, there exists $\gamma<1$ such that

$$
\left|r_{1}-\frac{a-1}{a}\right|=\frac{1}{a} \gamma
$$

hence, using (34), we have that

$$
\begin{equation*}
\left|r_{m}-\frac{a-1}{a}\right|=\frac{1}{a} \gamma^{2^{m-1}} \longrightarrow m \rightarrow \infty 0 \tag{35}
\end{equation*}
$$

which implies that the fixed point $r_{0}$ is stable.
In what follows, we will explore the cycles of system (31) of length 2. In order to find them, we need to solve the following equation

$$
\begin{equation*}
a(\operatorname{ar}(1-r))(1-\operatorname{ar}(1-r))=r \tag{36}
\end{equation*}
$$

It can easily be verified that

$$
\begin{aligned}
& a(\operatorname{ar}(1-r))(1-\operatorname{ar}(1-r))-r= \\
& \quad=-r(a r-(a-1))\left(a^{2} r^{2}-a(a+1) r+(a+1)\right)
\end{aligned}
$$

thus we find that system (31) has a single cycle of length 2 , and it consists of the following points

$$
r^{(1)}=\frac{a+1+\sqrt{(a+1)(a-3)}}{2 a}, \quad r^{(2)}=\frac{a+1-\sqrt{(a+1)(a-3)}}{2 a}
$$

(since $a \geq 3$, they indeed define a cycle of (31)). Furthermore, we can easily check that

$$
\left|r^{(1)}-\frac{a-1}{a}\right|<\frac{1}{a}
$$

also holds. Let us assume that for the initial point $r_{1}=r^{(1)}$. Then, without any control the system (31) will not make the orbit of $r_{1}$ converge to the fixed point $r_{0}$ but keep it in a cycle such that

$$
r_{2 m-1}=r^{(1)}, \quad r_{2 m}=r^{(2)}, \quad m=1,2, \ldots
$$

However, as we have pointed out in (35), adapting the OGY control to this system, we will have a 2-dimensional dynamic system defined by (33), which will stabilize the orbit of $r_{1}$ and force it into the unstable fixed point $r_{0}$.

We recall that in the traditional OGY method there is a fixed positive constant $\varepsilon>0$, and we apply the control if $\left|\delta p_{m}\right|=\left|p_{m}\right|<\varepsilon$. If $\left|p_{m}\right| \geq \varepsilon$, then we let the system freely follow a trajectory waiting for the moment it approaches $r_{0}$ "close enough", when we switch the OGY control on. On the other hand, in the pure OGY method we do not consider any value $\varepsilon>0$, but always hold the control switched on.

In the following we shall show that applying the traditional OGY method we cannot always stabilize the fixed point. Indeed, let the initial point $r_{1}$ be an arbitrary point of $\mathbb{R}$ for which

$$
\left|r_{1}-r_{0}\right| \geq \max \left\{\left|r^{(1)}-r_{0}\right|,\left|r^{(2)}-r_{0}\right|\right\}
$$

(for instance, either $r_{1}=r^{(1)}$ or $r_{1}=r^{(2)}$ is a feasible choice), and let

$$
\varepsilon=\frac{a-2}{|w|} \min \left\{\left|r^{(1)}-r_{0}\right|,\left|r^{(2)}-r_{0}\right|\right\}>0
$$

Then, we can see that employing the traditional OGY method we cannot stabilize the fixed point $r_{0}$, because

$$
r_{2 m-1}=r_{1}, \quad r_{2 m}=\operatorname{ar}_{1}\left(1-r_{1}\right), \quad m=1,2, \ldots
$$

hence

$$
\left|p_{m}\right|=\left|\frac{a-2}{w}\left(r_{m}-r_{0}\right)\right| \geq \frac{a-2}{|w|} \min \left\{\left|r^{(1)}-r_{0}\right|,\left|r^{(2)}-r_{0}\right|\right\}=\varepsilon
$$

for any $m=1,2, \ldots$, which actually means that we will never switch the control on.

However, as we obtained in (35), using the pure OGY method we are always able to force any orbit of (32) into the fixed point $r_{0}$.

## 5. Conclusions

In this paper we presented the framework of the OGY method from mathematical point of view and reintroduced its notions in connection with dynamic systems of arbitrary finite dimensions. We generally formulated the exact necessary conditions for the applicability of the method, and provided a complete analysis of OGY-controlled systems. Especially, to put our results into practise we demonstrated the capabilities of the OGY method by controlling the classical logistic map.

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# GENERALIZED CLOSED SETS 

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## 0. Introduction

Let $X \neq \emptyset$ be a set and $\tau$ a topology on $X, \mathfrak{m}$ a minimal structure on $X$ in the sense of [3], [4], [5] or [6], i.e. $\mathfrak{m} \subset \exp X$ (where $\exp X$ denotes the power set of $X$ ) and $\emptyset, X \in \mathfrak{m}$. In the paper [4], a set $A \subset X$ is said to be $m g^{*}$-closed iff $A \subset U \in \mathfrak{m}$ implies $c A \subset U$ (where $c A$ denotes the $\tau$-closure of $A$ ).

According to [1], the concept of a topology on $X$ has a generalization called generalized topology (briefly GT): this is a subset $\mu$ of $\exp X$ such that $\emptyset \in \mu$ and every union of elements of $\mu$ belongs to $\mu$. Also the concept of a minimal structure admits a slight generalization: let us say that $\mathfrak{m}$ is a minimality on $X$ iff $\mathfrak{m} \subset \exp X$ and $\emptyset \in \mathfrak{m}$.

Now the purpose of the present paper is to show that almost all results on $m g^{*}$-closed sets contained in [4] remain valid if the concept of $m g^{*}$-closed set is replaced by the more general concept of $\mu \mathfrak{m} g$-closed set, where a subset $A$ of $X$ is said to be $\mu \mathfrak{m g}$-closed for a GT $\mu$ and a minimality $\mathfrak{m}$ on $X$ iff $A \subset U \in \mathfrak{m}$ implies $c_{\mu} A \subset U$; here $c_{\mu} A$ is the $\mu$-closure of $A$, i.e. the intersection of all $\mu$-closed sets containing $A$, a set $F$ being $\mu$-closed iff $X-F \in \mu$.

[^4]
## 1. Preliminaries

Let $\mu$ be a GT on $X$ and $A \subset X$. The elements of $\mu$ are called $\mu$-open, their complements $\mu$-closed. According to [2], the union of all $\mu$-open sets contained in $A \subset X$ is denoted by $i_{\mu} A$ and the intersection of all $\mu$-closed sets containing $A$ by $c_{\mu} A$. Clearly $i_{\mu} A \subset A \subset c_{\mu} A$ and, by [2], both $i_{\mu}$ and $c_{\mu}$ are monotone and idempotent. Moreover, $c_{\mu}(X-A)=X-i_{\mu} A$.

## 2. Minimalities

Similarly, if $\mathfrak{m}$ is a minimality on $X$ in the above sense, let us say that a set $M \in \mathfrak{m}$ is $\mathfrak{m}$-open and the complement $X-M \mathfrak{m}$-closed. For $A \subset X$, let $i_{\mathfrak{m}} A$ denote the union of all $\mathfrak{m}$-open sets contained in $A$ and $c_{\mathfrak{m}} A$ the intersection of all $\mathfrak{m}$-closed sets containing $A$. Observe that $\emptyset$ is $\mathfrak{m}$-open and $X$ is $\mathfrak{m}$-closed. Clearly $i_{\mathfrak{m}} A \subset A \subset c_{\mathfrak{m}} A$ and both $i_{\mathfrak{m}}$ and $c_{\mathfrak{m}}$ are monotone, i.e.

$$
\begin{equation*}
A \subset B \quad \text { implies } \quad i_{\mathfrak{m}} A \subset i_{\mathfrak{m}} B \quad \text { and } \quad c_{\mathfrak{m}} A \subset c_{\mathfrak{m}} B \tag{2.1}
\end{equation*}
$$

further, for each $A \subset X$,

$$
\begin{equation*}
c_{\mathfrak{m}}(X-A)=X-i_{\mathfrak{m}} A \tag{2.2}
\end{equation*}
$$

Moreover, if $\boldsymbol{M}$ is $\mathfrak{m}$-open then $i_{\mathfrak{m}} \boldsymbol{M}=\boldsymbol{M}$ and $c_{\mathfrak{m}}(X-M)=X-M$.
Obviously, a GT is a minimality $\mathfrak{m}$ such that every union of $\mathfrak{m}$-open sets is $\mathfrak{m}$-open.

## 3. $\mu$-m-generalized closed sets

Let $\mu$ be a GT and $\mathfrak{m}$ a minimality on $X$. We shall say that a set $A \subset X$ is $\mu$-m-generalized closed (briefly $\mu \mathfrak{m g}$-closed) iff $A \subset U \in \mathfrak{m}$ implies $c_{\mu} A \subset U$. The complement of a $\mu \mathfrak{m} g$-closed set is called $\mu \mathfrak{m} g$-open.

Proposition 3.1 If $A$ is $\mu$-closed then $A$ is $\mu \mathfrak{m} g$-closed.
Proposition 3.2 If $A \in \mathfrak{m}$ is $\mu \mathfrak{m g}$-closed then $A$ is $\mu$-closed.
Proof. $U=A$ can be chosen in the definition of a $\mu \mathfrak{m} g$-closed set, so $c_{\mu} A=A$.

Proposition 3.3 If $A$ is $\mu \mathfrak{m g}$-closed and $A \subset B \subset c_{\mu} A$ then $B$ is $\mu \mathfrak{m g}$-closed.

Proof. If $B \subset U \in \mathfrak{m}$ then $A \subset U \in \mathfrak{m}$ so that $c_{\mu} A \subset U$ and $c_{\mu} B \subset c_{\mu} A \subset U$.

If $\mathfrak{m}$ is a minimality on $X$ and $A \subset X$, let us call $\mathfrak{m}$-frontier of $A$ the set $c_{\mathfrak{m}} A \cap c_{\mathfrak{m}}(X-A)$.

Proposition 3.4 If $A$ is $\mu \mathfrak{m} g$-closed and $A \subset U \in \mathfrak{m}$ then the $\mathfrak{m}$-frontier $F$ of $U$ is contained in $i_{\mu}(X-A)$.

Proof. By hypothesis, $c_{\mu} A \subset U$. Then $F=c_{\mathfrak{m}} U-U$ since $c_{\mathfrak{m}}(X-$ $-U)=X-U$. Therefore $F \subset X-U \subset X-c_{\mu} A=i_{\mu}(X-A)$.

The following statements arise by taking the complements of the sets considered. Let again $\mu$ be a GT and $\mathfrak{m}$ a minimality on $X$.

Proposition 3.5 $A$ set $A \subset X$ is $\mu \mathfrak{m g}$-open iff $F \subset i_{\mu} A$ whenever $F \subset A$ and $F$ is $\mathfrak{m}$-closed. Every $\mu$-open set is $\mu \mathfrak{m g}$-open. If $A$ is $\mu \mathfrak{m g}$-open and $\mathfrak{m}$-closed then $A$ is $\mu$-open. If $A$ is $\mu \mathfrak{m} g$-open and $i_{\mu} A \subset B \subset A$ then $B$ is $\mu \mathrm{m} g$-open.

PROOF. 3.1, 3.2, 3.3, 3.4.
For the statements in this section, see [4], Proposition 4.1, Proposition 4.3, Proposition 4.4, Proposition 4.6, Corollary 4.1.

## 4. Characterizations of $\mu \mathfrak{m g}$-closed sets

Let again $\mu$ be a GT and $\mathfrak{m}$ a minimality on $X$.
THEOREM 4.1 $A$ set $A \subset X$ is $\mu \mathfrak{m g}$-closed iff $c_{\mu} A \cap F=\emptyset$ whenever $A \cap F=\emptyset$ and $F$ is $\mathfrak{m}$-closed.

Proof. Put $U=X-F$.
THEOREM 4.2 Let $\mathfrak{m}$ be a $G T$ and $\mu \subset \mathfrak{m}$. Then $A$ is $\mu \mathfrak{m g}$-closed iff $c_{\mu} A-A$ does not contain any $\mathfrak{m}$-closed set $F \neq \emptyset$.

Proof. Let $A$ be $\mu \mathfrak{m} g$-closed. If $A=\emptyset$ then $U=\emptyset \in \mathfrak{m}, A \subset U$ implies $c_{\mu} A=\emptyset$. If $A \neq \emptyset$, then $X-F \in \mathfrak{m}$ and $F \subset c_{\mu} A-A$ imply $A \subset X-F$, hence $c_{\mu} A \subset X-F$ so that $F \subset c_{\mu} A \subset X-F$ which is impossible if $F \neq \emptyset$.

Now suppose that $A$ is not $\mu \mathfrak{m} g$-closed. Then $c_{\mu} A-U \neq \emptyset$ for a suitable $U \in \mathfrak{m}$ such that $A \subset U$. Thus $X-U$ is $\mathfrak{m}$-closed and, by hypothesis, the $\mu$-closed set $c_{\mu} A$ is $\mathfrak{m}$-closed as well. $\mathfrak{m}$ being a GT, $c_{\mu} A-U$ is $\mathfrak{m}$-closed and $\emptyset \neq c_{\mu} A-U \subset c_{\mu} A-A$.

THEOREM 4.3 When $\mu \subset \mathfrak{m}$ and $\mathfrak{m}$ is a GT then $A \subset X$ is $\mu \mathfrak{m} g$-closed iff $c_{\mu} A-A$ is $\mu \mathfrak{m} g$-open.

Proof. Suppose that $A$ is $\mu \mathfrak{m} g$-closed. If $F \subset c_{\mu} A-A$ is $\mathfrak{m}$-closed then by 4.2 $F=\emptyset$ and $F \subset i_{\mu}\left(c_{\mu} A-A\right)$. By $3.5 c_{\mu} A-A$ is $\mu \mathfrak{m} g$-open.

Conversely let $A \subset U \in \mathfrak{m}$. Then $c_{\mu} A \cap(X-U) \subset c_{\mu} A-A$; suppose that $c_{\mu} A-A$ is $\mu \mathfrak{m} g$-open. By hypothesis, $c_{\mu} A \cap(X-U)$ is $\mathfrak{m}$-closed so that $c_{\mu} A \cap(X-U) \subset i_{\mu}\left(c_{\mu} A-A\right)$ by 3.5. As $i_{\mu}\left(c_{\mu} A-A\right) \subset c_{\mu} A \cap i_{\mu}(X-$ $-A)=c_{\mu} A \cap\left(X-c_{\mu} A\right)=\emptyset$, we have $c_{\mu} A \cap(X-U)=\emptyset$ and $c_{\mu} A \subset U$ shows that $A$ is $\mu \mathfrak{m} g$-closed.

THEOREM 4.4 Suppose that $\mathfrak{m}$ is a GT. Then $A \subset X$ is $\mu \mathfrak{m g}$-closed iff $c_{\mathfrak{m}}\{x\} \cap A \neq \emptyset$ whenever $x \in c_{\mu} A$.

Proof. Let $A$ be $\mu \mathfrak{m} g$-closed and $x \in c_{\mu} A$. Assume $c_{\mathfrak{m}}\{x\} \cap A=\emptyset$. By hypothesis, $c_{\mathfrak{m}}\{x\}$ is $\mathfrak{m}$-closed and $A \subset X-c_{\mathfrak{m}}\{x\} \in \mathfrak{m}$ so that $c_{\mu} A \subset X-$ $-c_{\mathfrak{m}}\{x\} \subset X-\{x\}$. This is impossible, hence necessarily $c_{\mathfrak{m}}\{x\} \cap A \neq \emptyset$.

Now suppose that $A$ is not $\mu \mathfrak{m} g$-closed. Then, for a suitable $U \in \mathfrak{m}$, $A \subset U$ and $c_{\mu} A-U \neq \emptyset$. Choose $x \in c_{\mu} A-U$. Clearly $X-U$ is $\mathfrak{m}$-closed so that $c_{\mathfrak{m}}\{x\} \subset X-U$ and $c_{\mathfrak{m}}\{x\} \cap A \subset c_{\mathfrak{m}}\{x\} \cap U=\emptyset$. Thus there is $x \in c_{\mu} A$ such that $c_{\mathfrak{m}}\{x\} \cap A=\emptyset$.

The following corollary is proved by using complementary sets:
Corollary 4.5 Let $\mu \subset \mathfrak{m}$ and $\mathfrak{m}$ be a GT. Then the following statements are equivalent:
a) $A \subset X$ is $\mu \mathfrak{m g}$-open;
b) $A-i_{\mu} A$ does not contain any non-empty $\mathfrak{m}$-closed set;
c) $A-i_{\mu} A$ is $\mu \mathfrak{m} g$-open;
d) $c_{\mathfrak{m}}\{x\} \cap(X-A) \neq \emptyset$ when $x \in A-i_{\mu} A$.

Proof. 4.2, 4.3, 4.4.
The statements in this section correspond to [4], Theorem 5.1, Theorem 5.2, Theorem 5.3, Theorem 5.4, Corollary 5.1.

## 5. Properties of mappings

Consider again a GT $\mu$ and a minimality $\mathfrak{m}$ on $X$, further a GT $v$ and a minimality $\mathfrak{n}$ on $Y$. It is quite natural to say that a function $f: X \rightarrow Y$ is $(\mathfrak{m}, \mathfrak{n})$-continuous iff $f^{-1}(N) \in \mathfrak{m}$ whenever $N \in \mathfrak{n}$. However, this definition is more restricted than the corresponding one in [4]. Clearly $f$ is ( $\mathfrak{m}, \mathfrak{n}$ )continuous iff $f^{-1}\left(N^{\prime}\right)$ is $\mathfrak{m}$-closed whenever $N^{\prime}$ is $\mathfrak{n}$-closed.

On the other side, $f$ is said to be $(\mathfrak{m}, \mathfrak{n})$-closed iff $f(M)$ is $\mathfrak{n}$-closed whenever $M$ is $\mathfrak{m}$-closed. As each GT is a minimality, we have also a definition for $(\mu, v)$-continuous and $(\mu, v)$-closed functions. The concept of $(\mu, v)$-continuity can be found also in [1].

Lemma 5.1 $A$ function $f: X \rightarrow Y$ is $(\mathfrak{m}, \mathfrak{n})$-closed iff, for $B \subset Y$ and $f^{-1}(B) \subset U \in \mathfrak{m}$, there exists $V \in \mathfrak{n}$ such that $B \subset V$ and $f^{-1}(V) \subset U$.

Proof. Suppose that $f$ is $(\mathfrak{m}, \mathfrak{n})$-closed, $B \subset Y$ and $f^{-1}(B) \subset U \in \mathfrak{m}$. Consider $V=Y-f(X-U)$. Then $V \in \mathfrak{n}, B \subset V$ and $f^{-1}(V) \subset U$. In fact, $y \in B-V$ would imply $y=f(x), x \in X-U$ and $x \in f^{-1}(B) \subset U$ which is impossible, further $z \in X, f(z) \in V$ implies $z \notin X-U, z \in U$.

Conversely, let $F \subset X$ be $\mathfrak{m}$-closed, $f(F)=B$. Then $F \subset f^{-1}(B)$, $f^{-1}(Y-B) \subset X-F \in \mathfrak{m}$. Assume that there exists $V \in \mathfrak{n}$ satisfying $Y-B \subset V$ and $f^{-1}(V) \subset X-F$. Then $Y-V \subset B=f(F) \subset Y-V$, i.e. $f(F)=Y-V$ is $\mathfrak{n}$-closed.

THEOREM 5.2 Iff $: X \rightarrow Y$ is $(\mu, v)$-closed and $(\mathfrak{m}, \mathfrak{n})$-continuous then $f(A)$ is $\nu \mathfrak{n g}$-closed whenever $A$ is $\mu \mathfrak{m} g$-closed.

Proof. Let $A$ be $\mu \mathfrak{m} g$-closed and $f(A) \subset V \in \mathfrak{n}$. Then $A \subset f^{-1}(V) \in$ $\in \mathfrak{m}$ by hypothesis. Therefore $c_{\mu} A \subset f^{-1}(V)$, hence $f\left(c_{\mu} A\right) \subset V . c_{\mu} A$ being $\mu$-closed and $f$ being $(\mu, v)$-closed, $f\left(c_{\mu} A\right)$ is $v$-closed. Clearly $f(A) \subset$ $\subset f\left(c_{\mu} A\right)$ so that $c_{v}(f(A)) \subset f\left(c_{\mu} A\right) \subset V$. Therefore $f(A)$ is $v \mathfrak{n g}$-closed.

THEOREM 5.3 If $: X \rightarrow Y$ is $(\mu, v)$-continuous and $(\mathfrak{m}, \mathfrak{n})$-closed then $f^{-1}(B)$ is $\mu \mathfrak{m} g$-closed whenever $B \subset Y$ is $v \mathfrak{n g}$-closed.

Proof. Let $B$ be $v \mathfrak{n} g$-closed and $f^{-1}(B) \subset U \in \mathfrak{m}$. By 5.1, there is $V \in \mathfrak{n}$ satisfying $B \subset V$ and $f^{-1}(V) \subset U$. By hypothesis, $c_{\nu} B \subset V$. Now $c_{\nu} B$ is $v$-closed so that $f^{-1}\left(c_{\nu} B\right)$ is $\mu$-closed. Therefore $f^{-1}(B) \subset f^{-1}\left(c_{\nu} B\right)$
implies $c_{\mu}\left(f^{-1}(B)\right) \subset f^{-1}\left(c_{v} B\right) \subset f^{-1}(V) \subset U$. By this, $f^{-1}(B)$ is $\mu \mathfrak{m} g$-closed.

The statements in this section correspond to [4], Lemma 6.2, Theorem 6.1 and Theorem 6.2.

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# ON CONNECTIONS BETWEEN THE STOKES-SCHUR AND THE FRIEDRICHS OPERATOR, WITH APPLICATIONS TO THE INF-SUP PROBLEM 

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## 1. Introduction

The Schur complement operator of the Stokes problem on a plane domain $\Omega$ is

$$
\begin{equation*}
\mathscr{S}=\operatorname{div} \Delta_{0}^{-1} \operatorname{grad}: L_{2}(\Omega) \rightarrow L_{2}(\Omega) \tag{1}
\end{equation*}
$$

where div and grad denote the usual divergence and gradient, $\Delta_{0}$ denotes the vector Laplace operator corresponding to homogeneous Dirichlet boundary values and $L_{2}(\Omega)$ is the usual Hilbert space of square integrable functions on $\Omega$. Properties of (1) has been given in [10], [11]. Its eigenvalues are important for stability and error estimates connected with the Stokes problem. Particularly important is the least positive eigenvalue, which is the so-called inf-sup constant of the domain. Preliminary work on different properties this constant and related eigenvalue problems has been done in [3], [4], [6]-[8], [16] and [30]-[32]. However, explicit values of the inf-sup constant for special domains are known only in a few cases: for the circle, the annulus [6], and the ellipse, see [16], and for an infinite strip - assuming periodicity along the strip [20], and, in the three-dimensional case, for the sphere [30]. Some lower and upper bounds for inf-sup constants of several domains are derived in [28], the use of this knowledge for the solution of corresponding discretized boundary value problems is shown in [29].

The Schur complement operator turns out to be closely related (see [9]) to the Friedrichs operator

$$
\begin{equation*}
\mathscr{F}=\mathscr{P} \circ \mathscr{C}: A L_{2}(\Omega) \rightarrow A L_{2}(\Omega) \tag{2}
\end{equation*}
$$

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where $A L_{2}(\Omega)$ is that subspace of $L_{2}(\Omega)$ which consists of square integrable analytic functions on $\Omega$ and is called the Bergman space of the domain. $\mathscr{P}$ is the orthogonal projection of $L_{2}(\Omega)$ onto $A L_{2}(\Omega)$ and $\mathscr{C}$ denotes the conjugacy operator. $\mathscr{F}$ is the underlying operator of the Friedrichs inequality, see [14]. It was extensively studied in [23], [24].

In Section 2 we formulate some definitions and preliminary results and investigate the connection between the Schur complement and the Friedrichs operator of the domain. In Section 3 the dependence of the operators on the domain is given in terms of the conformal mapping of the unit disc onto the domain. In Section 4 we examine some spectral properties of the operators using their correspondence and derive bounds for the inf-sup constants of special domains and calculate explicitly a few examples, too. In the last section we calculate some examples for domains with corners. This case is practically important and theoretically interesting since such results have been unknown until now.

## 2. The Friedrichs operator and the Schur complement

Let $\Omega$ be a simply connected Jordan domain with rectifiable boundary $\partial \Omega . L_{2}(\Omega)$ denotes the space of square integrable functions on $\Omega$ with respect to the usual planar Lebesque measure $d A=d x d y$. The inner product of $f, g \in L_{2}(\Omega)$ is defined by $(f, g):=\int_{\Omega} f \bar{g} d A$, and the norm of a function $f \in L_{2}(\Omega)$ is $\|f\|:=\left(\int_{\Omega}|f|^{2} d A\right)^{1 / 2} . L_{2,0}(\Omega) \subset L_{2}(\Omega)$ is the subspace of functions with zero integral on $\Omega$. Denote by $W^{1,2}(\Omega)$ the Sobolev space of functions defined on $\Omega$ with generalized derivative in $L_{2}(\Omega)$ and let $W_{0}^{1,2}(\Omega)$ be the subspace of $W^{1,2}(\Omega)$ with zero boundary values in the sense of traces on $\partial \Omega$ (see e.g. [1]). The Hardy space of $\Omega$ will be denoted by $H^{2}(\Omega)$ (see [13]). The Bergman space $A L_{2}(\Omega)$ of complex analytic functions in $\Omega$ which belong to $L_{2}(\Omega)$ is a Hilbert space with a reproducing kernel $K(z, \zeta)$ called Bergman kernel of the domain $\Omega$, that is we have for all $f \in A L_{2}(\Omega)$

$$
\begin{equation*}
f(z)=\int_{\Omega} K(z, \zeta) f(\zeta) d A(\zeta) \tag{3}
\end{equation*}
$$

The Bergman kernel is analytic in its first variable, conjugate analytic in the second variable and has the property $K(z, \zeta)=\overline{K(\zeta, z)}$. The Bergman kernel of the unit disc $D$ is

$$
K_{D}(z, \zeta)=\frac{1}{\pi(1-z \bar{\zeta})^{2}}
$$

and we have the transformation formula

$$
\begin{equation*}
K_{D}(z, \zeta)=g^{\prime}(z) K(g(z), g(\xi)) \overline{g^{\prime}(\zeta)} \tag{4}
\end{equation*}
$$

under the conformal mapping $g: D \rightarrow \Omega$ (for further details we refer to [5]).
The (conjugate-linear) Friedrichs operator (5) of the domain $\Omega$ can be expressed as an integral transformation using the Bergman kernel:

$$
\begin{equation*}
\mathscr{F}: A L_{2}(\Omega) \rightarrow A L_{2}(\Omega), \quad \mathscr{F}(f)(z)=\int_{\Omega} K(z, \zeta) \overline{f(\zeta)} d A(\zeta) \tag{5}
\end{equation*}
$$

$\mathscr{F}$ is the underlying operator of an eigenvalue problem studied in [14] because

$$
(f, \mathscr{F} g)=(g, \mathscr{F} f)=\int_{\Omega} f g d A \text { for } f, g \in A L_{2}(\Omega)
$$

We also have $\|\mathscr{F}(f)\| \leq\|f\|$. The square of the Friedrichs operator $\mathscr{F}^{2}$ is a complex linear, self-adjoint and contractive operator on the Bergman space:

$$
0 \leq \mathscr{H}^{2} \leq \mathscr{F}
$$

Its square root is referred to as the modulus of $\mathscr{F}$, see [23].
In the present paper - as also in [23] - the boundary of the domain $\Omega$ is assumed to be sufficiently regular in the sense that the space $W^{1,2}(\Omega)$ has traces in $L_{2}(\partial \Omega)$. This is certainly fulfilled for domains with piecewise $C^{2}$ regular boundary. To establish a connection between the Friedrichs and the Schur complement operators we need:

Proposition 1 (COROLLARY 2.5 IN [24]). Let the boundary of the plane domain $\Omega$ be sufficiently regular in the sense that the space $W^{1,2}(\Omega)$ has traces in $L_{2}(\partial \Omega)$. The solution to the Dirichlet problem:

$$
\Delta u=0 \text { in } \Omega ; \quad u(\zeta)=\bar{\xi} f(\xi), \text { for } \zeta \in \partial \Omega
$$

is $u(z)=\overline{G(z)}+h(z)$, where $G \in H^{2}(\Omega)$ is the primitive function of $\mathscr{H}(f)$ and the function $h \in H^{2}(\Omega)$ is defined by

$$
h(z)=\frac{1}{2 \pi i} \int_{\partial \Omega} \frac{\bar{\xi} f(\zeta)-\overline{G(\zeta)}}{\zeta-z} d \zeta
$$

Using this representation we reestablish the following theorem already announced in [9] where, however, the smoothness of the boundary has not been specified. In the following $\mathscr{\mathscr { V }}$ denotes the identity and $\mathscr{C}$ the conjugacy operator.

ThEOREM 2. Let $\Omega$ be a domain with a boundary as in Proposition 1. Then we have

$$
\begin{equation*}
2 \mathscr{S}=\mathscr{F}-\mathscr{C} \circ \mathscr{F} \tag{6}
\end{equation*}
$$

for the Friedrichs operator $\mathscr{F}$ and the Schur complement operator $\mathscr{S}$ of the domain.

PROOF. In the proof we use the notation $\partial_{z}=\frac{1}{2}\left(\frac{\partial}{\partial x}-i \frac{\partial}{\partial y}\right)$ and $\partial_{\bar{z}}=$ $=\frac{1}{2}\left(\frac{\partial}{\partial x}+i \frac{\partial}{\partial y}\right)$, and then for a complex valued function $u=u_{1}+i u_{2}$ the divergence is given by

$$
\operatorname{div} u=\frac{\partial u_{1}}{\partial x}+\frac{\partial u_{2}}{\partial y}=2 \operatorname{Re} \partial_{z} u
$$

and for a real valued function $p$ the (complex valued) gradient is expressed as

$$
\operatorname{grad} p=\frac{\partial p}{\partial x}+i \frac{\partial p}{\partial y}=2 \partial_{\bar{z}} p
$$

Assume $f \in A L_{2}(\Omega)$ and set $u_{0}(z)=\frac{1}{2} z \overline{f(z)}, p_{R}(z)=2 \operatorname{Re} f(z)$. There follows $\Delta u_{0}=\nabla p_{R}$ in $\Omega$ (see [32]) and

$$
\varphi_{p_{R}(z)}=\operatorname{div} u_{0}(z)=2 \operatorname{Re} \partial_{z} u_{0}(z)=\operatorname{Re} \overline{f(z)}=\frac{1}{2} p_{R}
$$

but, in general, $u_{0}$ does not fulfil the homogeneous boundary condition. Let us solve the Dirichlet problem

$$
\Delta H=0 \text { in } \Omega, H(z)=\bar{z} f(z) \text { for } z \in \partial \Omega
$$

By Proposition 1 we have $H=\bar{G}+h$, where

$$
G^{\prime}(z)=\mathscr{F}(f)(z)=\int_{\Omega} K(z, \zeta) \overline{f(\zeta)} d A(\zeta)
$$

Now $u=u_{0}-\frac{1}{2} \bar{H}$ satisfies $\Delta u=\nabla p_{R}$ in $\Omega$ with homogeneous boundary values. Further we have for the divergence

$$
\operatorname{div} u(z)=\operatorname{div} u_{0}(z)-\operatorname{Re} \partial_{z} \overline{H(z)}=\frac{1}{2} p_{R}(z)-\operatorname{Re} G^{\prime}(z)
$$

which gives

$$
\begin{equation*}
\mathscr{S}_{p_{R}}(z)=\frac{1}{2} p_{R}(z)-\operatorname{Re} \mathscr{F}(f)(z) \tag{7}
\end{equation*}
$$

Analogously, with -if instead of $f$, we obtain for $p_{I}=2 \operatorname{Im} f$

$$
\begin{equation*}
\mathscr{S}_{p_{I}}(z)=\frac{1}{2} p_{I}(z)+\operatorname{Im} \mathscr{F}(f)(z) \tag{8}
\end{equation*}
$$

Combining the equalities (7) and (8) gives for $2 f=p_{R}+i p_{I}$

$$
2 \varphi(f)(z)=f(z)-\overline{\mathscr{F}(f)(z)}
$$

which proves the theorem.
REMARK 3. Equations (7) and (8) show that the deviation of the operator $\mathscr{S}$ from $\frac{1}{2} \mathscr{F}$ is due to the boundary of the domain.

REMARK 4. Equation (6) can be rearranged to

$$
\begin{equation*}
\mathscr{F}-2 \mathscr{S}=\mathscr{C} \circ \mathscr{F} \tag{9}
\end{equation*}
$$

Properties (compactness, spectrum, etc.) of the operator $\mathscr{J}-2 \mathscr{\text { are investi- }}$ gated in [11], and of $\mathscr{F}$ defined by (5) in [23] and [24]. Due to Theorem 2, several known properties of the Friedrichs operator carry over to the Schur complement operator and vice versa.

REMARK 5. Using the conformal map of the unit disc onto the domain $\Omega$, the operator $\mathscr{F}$ (and then also $\mathscr{\mathscr { S }}$ ) can be represented by an infinite matrix, see [32], [33]. This facilitates the investigation of the relationship between the operators and the domain $\Omega$ via its conformal map. If the conformal mapping $g$ of the unit disc $D$ onto $\Omega$ is of the form

$$
\begin{equation*}
g(z)=\sum_{m=0}^{\infty} a_{m} z^{m} \tag{10}
\end{equation*}
$$

then, computing the quantities

$$
\begin{equation*}
s_{n}=\frac{1}{\pi} \int_{D} \bar{z}^{n} \frac{g^{\prime}(z)}{\overline{g^{\prime}(z)}} d A(z) \tag{11}
\end{equation*}
$$

the matrix representant of $\mathscr{F}$ is $-2 \mu_{S}$, where

$$
\begin{equation*}
\mathcal{M}_{S}=\left(s_{k, \ell}\right)_{k, l=0}^{\infty} \tag{12}
\end{equation*}
$$

and $s_{k, \ell}=-\frac{k+1}{2} s_{k+\ell}$. This matrix is a Hankel-type matrix of the entries (11) multiplied by a diagonal matrix, see [33].

## 3. Domain dependence of the operators

Let $g: D \rightarrow \Omega$ and $\tilde{g}: D \rightarrow \tilde{\Omega}$ be conformal mappings of the unit disc $D$ onto the domains $\Omega$ and $\tilde{\Omega}$ and set $\tilde{g}=\eta \circ g$. Define the entries $s_{k}$ and $\tilde{s}_{k}$ by (11).

Lemma 6. Let $g$ and $\tilde{g}$ be the conformal maps of $D$ onto $\Omega$ and $\tilde{\Omega}$, respectively. Then we have for $k=0,1,2, \ldots$

$$
\begin{equation*}
\left|\tilde{s}_{k}-s_{k}\right| \leq \frac{4}{k+2} \sup _{z \in D}\left|\sin \left(\arg \tilde{g}^{\prime}(z)-\arg g^{\prime}(z)\right)\right| . \tag{13}
\end{equation*}
$$

Proof. Introduce the set $D_{\rho}=\{z:|z|<\rho\}$, and put

$$
s_{k}(\rho)=\frac{1}{\pi} \int_{D_{\rho}} \frac{\bar{z}^{k}}{} \frac{g^{\prime}(z)}{\overline{g^{\prime}(z)}} d A(z), \quad \tilde{s}_{k}(\rho)=\frac{1}{\pi} \int_{D_{\rho}} \bar{z}^{k} \frac{\tilde{g}^{\prime}(z)}{\tilde{g}^{\prime}(z)} d A(z)
$$

Compute the difference using $\tilde{g}^{\prime}(z)=\eta^{\prime}(g(z)) \cdot g^{\prime}(z)$

$$
\tilde{s}_{k}(\rho)-s_{k}(\rho)=\frac{1}{\pi} \int_{D_{\rho}} \bar{z}^{k} \frac{g^{\prime}(z)}{\overline{g^{\prime}(z)}}\left(\frac{\eta^{\prime}(g(z))}{\overline{\eta^{\prime}(g(z))}}-1\right) d A(z) .
$$

Estimate this by taking the absolute value and using $e^{2 i \alpha}-1=2 i e^{i \alpha} \sin \alpha$ :

$$
\left|\tilde{s}_{k}(\rho)-s_{k}(\rho)\right| \leq \frac{4 \rho^{k+2}}{k+2} \sup _{z \in D_{\rho}}\left|\sin \left(\arg \tilde{g}^{\prime}(z)-\arg g^{\prime}(z)\right)\right|
$$

since $\arg \eta^{\prime}(g(z))=\arg \tilde{g}^{\prime}(z)-\arg g^{\prime}(z)$. Now we get (13) by taking the limit for $\rho \rightarrow 1$.

Remark 7. If we define a norm for the matrix $\mu_{S}$ by

$$
\left\|\mu_{S}\right\|_{*}=\sup _{j, \ell}\left|\frac{j+\ell+2}{2(j+\ell)+2}\left[\mu_{S}\right]_{j, \ell}\right|=\max _{k \geq 0}\left|\frac{k+2}{4} s_{k}\right|,
$$

then by Lemma 6 we have in this norm

$$
\begin{equation*}
\left\|\tilde{M}_{S}-\mathcal{M}_{S}\right\|_{*} \leq \sup _{z \in D}\left|\sin \left(\arg \tilde{g}^{\prime}(z)-\arg g^{\prime}(z)\right)\right| \tag{14}
\end{equation*}
$$

which shows the continuous dependence of the entries of the matrix $\mathcal{M}_{S}$ (and also of the Friedrichs operator of the domain) on the argument of the derivative of the conformal mapping. (Using Theorem 2 we have also the continuous dependence of the Schur complement.) We can further improve
this result by using another norm for the operator, for example induced by the $L_{2}$ norm.

To achieve this we start by transforming to polar coordinates in (5). Let be $p=(f \circ g) \cdot g^{\prime} \in A L_{2}(D)$ for $f \in A L_{2}(\Omega)$. This is a unitary mapping: $\|f\|_{\Omega}=\|p\|_{D}$.

Now we have instead of (5):

$$
\begin{equation*}
\mathscr{F}_{D}: A L_{2}(D) \rightarrow A L_{2}(D), \quad \mathscr{F}_{D} p(z)=\int_{D} K_{D}(z, \zeta) \frac{g^{\prime}(\xi)}{g^{\prime}(\zeta)} \overline{p(\xi)} d A(\xi), \tag{15}
\end{equation*}
$$

where we also have used the transformation formula (4) for the Bergman kernel. Similarly to this we can define $\tilde{\mathscr{F}}_{D}$ for the domain $\tilde{\Omega}=\tilde{g}(D)$. Subtract these two definitions:

$$
\left(\tilde{\mathscr{F}}_{D}-\mathscr{F}_{D}\right) p(z)=\int_{D} K_{D}(z, \zeta) \frac{g^{\prime}(\xi)}{g^{\prime}(\xi)}\left(\frac{\eta^{\prime}(g(\xi))}{\overline{\eta^{\prime}(g(\xi))}}-1\right) \overline{p(\xi)} d A(\xi)
$$

Multiply this by the conjugate of an arbitrary $q \in A L_{2}(D)$ and integrate over $D$, change the integrations with respect to the variables $z$ and $\zeta$ and use the reproducing property (3) of the Bergman kernel along with $K_{D}(z, \zeta)=$ $=\overline{K_{D}(\zeta, z)}$ :

$$
\begin{aligned}
& \int_{D}\left(\tilde{\mathscr{F}}_{D}-\mathscr{F}_{D}\right) p \bar{q} d A= \\
&=\int_{D} \frac{\overline{p(\zeta)}}{\underline{g^{\prime}(\zeta)}} \overline{g^{\prime}(\zeta)} \\
&\left(\frac{\eta^{\prime}(g(\zeta))}{\overline{\eta^{\prime}(g(\zeta))}}-1\right)\left(\overline{\int_{D} K_{D}(\zeta, z) q(z) d A(z)}\right) d A(\zeta)
\end{aligned}
$$

There follows

$$
\left(\left(\tilde{\mathscr{F}}_{D}-\mathscr{F}_{D}\right) p, q\right)_{D}=\int_{D} \overline{p(\xi) q(\zeta)} \frac{g^{\prime}(\zeta)}{g^{\prime}(\zeta)}\left(e^{2 i \arg \eta^{\prime}(g(z))}-1\right) d A(\zeta) .
$$

We estimate the right-hand side by the Cauchy-Schwarz inequality and obtain

$$
\left|\left(\left(\tilde{\mathscr{F}}_{D}-\mathscr{F}_{D}\right) p, q\right)_{D}\right| \leq 2 \sup _{z \in D} \mid \sin \left(\arg \eta^{\prime}(g(z)) \mid \cdot\|p\|_{D} \cdot\|q\|_{D} .\right.
$$

Set $q=\left(\tilde{\mathscr{H}}_{D}-\mathscr{F}_{D}\right) p$, change the coordinates again and simplify:

$$
\|(\tilde{\mathscr{F}}-\mathscr{F}) f\|_{\Omega} \leq 2 \sup _{\Omega}\left|\sin \left(\arg \eta^{\prime}\right)\right| \cdot\|f\|_{\Omega} .
$$

Therefore we have in the operator norm

$$
\begin{equation*}
\|\tilde{\mathscr{F}}-\mathscr{F}\| \leq 2 \sup _{\Omega}\left|\sin \left(\arg \eta^{\prime}\right)\right| . \tag{16}
\end{equation*}
$$

In this way, we have obtained the following:

THEOREM 8. Let $\eta$ be the conformal map of the domain $\Omega$ onto $\tilde{\Omega}$. For the Friedrichs operators of these domains there follows (16).

REMARK 9. Equations (14) and (16) have the same structure (but with different operator norm) because of $\arg \eta^{\prime}=\arg \tilde{g}^{\prime}-\arg g^{\prime}$. The angle of the unit tangent vector $\tau$ to the conformal image of the circle $C_{\rho}=\{z \in D:|z|=\rho\}$ at the point $g(z)$ is $\arg \left(i z g^{\prime}(z)\right)$.

$$
\arg \tilde{g}^{\prime}(z)-\arg g^{\prime}(z)=\arg \frac{\tilde{g}^{\prime}(z)}{g^{\prime}(z)}=\arg \frac{i z \tilde{g}^{\prime}(z)}{i z g^{\prime}(z)}=\arg \tilde{\tau}-\arg \tau
$$

So there follows

$$
\|\tilde{\mathscr{F}}-\mathscr{F}\| \leq 2 \sup _{\Omega}|\sin (\arg \tilde{\tau}-\arg \tau)| .
$$

If, further, the derivative of $\eta$ is continuous up to the boundary (that is if $\partial \Omega$ has a continuous tangent at every point) and $\eta^{\prime} \neq 0$ for the mapping in the closure of $\Omega$, then we can restrict the supremum in Theorem 8 to the boundary of $\Omega$ and we obtain

$$
\begin{equation*}
\|\tilde{\mathscr{F}}-\mathscr{F}\| \leq 2 \max _{\partial \Omega}\left|\sin \left(\arg \eta^{\prime}\right)\right| \tag{17}
\end{equation*}
$$

and

$$
\begin{equation*}
\|\tilde{\mathscr{F}}-\mathscr{F}\| \leq 2 \max _{\partial \Omega}|\sin (\arg \tilde{\tau}-\arg \tau)| . \tag{18}
\end{equation*}
$$

So among domains with continuous tangent to the boundary the operator norm of the Friedrichs operator depends continuously on the shape of the domain.

The use of the conformal mapping reveals also a connection between the matrix representant $\mathcal{M}_{S}$ of the Friedrichs operator (and so the operator itself) and the domain.

LEMMA 10. Let be $M \geq 1$ integer. For the quantities (11) we have $s_{k}=0$ for $k \geq M$ iff $g$ is a polynomial of order $M$. In this case $\mathcal{M}_{S} \in \mathbb{C}^{M \times M}$ is of $\operatorname{rank} M$ and $s_{M-1}=\frac{a_{M}}{\bar{a}_{1}} \neq 0$.

Proof. By (11) the assumption $s_{k}=0$ for $k \geq M$ implies

$$
\int_{D} \bar{z}^{M+\ell} \frac{g^{\prime}(z)}{\overline{g^{\prime}(z)}} d A(z)=0
$$

for $\ell=0,1,2, \ldots$ Combining these equalities with arbitrary complex numbers $\left\{\bar{h}_{\ell}\right\}_{\ell=0}^{\infty}, \sum_{\ell=0}^{\infty}\left|h_{\ell}\right|^{2}<\infty$, we have

$$
\int_{D} \bar{z}^{M} \overline{h(z)} \frac{g^{\prime}(z)}{\overline{g^{\prime}(z)}} d A(z)=0
$$

where $h(z)=\sum_{\ell=0}^{\infty} h_{\ell} z^{\ell}$. Now we set $h(z)=z^{\ell} g^{\prime}(z)$ for $\ell=0,1,2, \ldots$. There follows

$$
\int_{D} \bar{z}^{M+\ell} g^{\prime}(z) d A(z)=0
$$

From (10) this is equivalent to $a_{M+\ell+1}=0$. The coefficients of (10) vanish for $m>M, g$ is a polynomial of order $M$. The converse statement follows from the definition (11). Further let be $g(z)=a_{0}+a_{1} z+\ldots+a_{M} z^{M}$. From (11) follows also $s_{M-1}=\frac{a_{M}}{\bar{a}_{1}} \neq 0$. Because of its structure, $\mathcal{M}_{S}$ is therefore of rank $M$.

REMARK 11. If the conformal mapping function $g$ is a polynomial of degree $M$ then the image of the unit circle is a lemniscate of degree $M$, that is the locus of those points whose distances from $M$ given points (the foci of the lemniscate) form a constant product.

We can generalize the statement of the previous lemma and characterize those mappings for which the matrix $\mu_{S}$ is infinite but of finite rank.

Lemma 12. Let $g$ be given by (10). The matrix $\mathcal{M}_{S}$ is of finite rank iff $g$ is a fractional rational transformation.

Proof. Suppose first that $\operatorname{rank}\left(\mathcal{M}_{S}\right)=\rho+1$ for some integer $\rho \geq 0$. From the structure of $\mu_{S}$ there follows that the quantities (11) fulfil a certain recursion in the form

$$
\begin{equation*}
s_{\rho+\ell+1}=\sum_{n=0}^{\rho} \bar{\alpha}_{n} s_{\ell+n} \tag{19}
\end{equation*}
$$

for all $\ell=0,1,2, \ldots$, where $\left\{\alpha_{n}\right\}_{n=0}^{\rho}$ are fixed complex constants. Substitute the integrals (11) into (19). There follows

$$
\int_{D} \bar{z}^{\rho+\ell+1} \frac{g^{\prime}(z)}{\overline{g^{\prime}(z)}} d A(z)=\int_{D} z^{\ell} \overline{\alpha(z)} \frac{g^{\prime}(z)}{\overline{g^{\prime}(z)}} d A(z)
$$

for all $\ell=0,1,2, \ldots$, where $\alpha(z)=\alpha_{0}+\alpha_{1} z+\ldots+\alpha_{\rho} z^{\rho}$. We combine these equalities again like in Lemma 10 and get

$$
\begin{equation*}
\int_{D} \bar{z}^{\rho+1} \overline{h(z)} \frac{g^{\prime}(z)}{\overline{g^{\prime}(z)}} d A(z)=\int_{D} \overline{\alpha(z) h(z)} \frac{g^{\prime}(z)}{\overline{g^{\prime}(z)}} d A(z) \tag{20}
\end{equation*}
$$

where $h(z)=h_{0}+h_{1} z+\ldots$ is an arbitrary function. Setting $h(z)=z^{\ell} g^{\prime}(z)$ for $\ell=0,1,2, \ldots$ implies

$$
\int_{D} \bar{z}^{\rho+\ell+1} g^{\prime}(z) d A(z)=\int_{D} \bar{z}^{\ell} \overline{\alpha(z)} g^{\prime}(z) d A(z)
$$

or, equivalently to this,

$$
\begin{equation*}
a_{\rho+\ell+2}=\sum_{n=0}^{\rho} \bar{\alpha}_{n} a_{n+\ell+1} . \tag{21}
\end{equation*}
$$

This is a recursion similar to (19). Now we multiply (21) by $z^{\rho+\ell+2}$ and sum up over $\ell=0,1,2, \ldots$. Using (10) results into

$$
g(z)-\sum_{n=0}^{\rho+1} a_{n} z^{n}=\sum_{n=0}^{\rho} \bar{\alpha}_{n} z^{\rho+1-n}\left[g(z)-\sum_{m=0}^{n} a_{m} z^{m}\right]
$$

Solving this equation for $g(z)$ gives a fractional rational function of the form

$$
\begin{equation*}
g(z)=a_{0}+\frac{\sum_{n=1}^{\rho+1}\left[a_{n}-\sum_{m=0}^{n-1} a_{m} \bar{\alpha}_{\rho+1-n+m}\right] z^{n}}{1-\sum_{n=1}^{\rho+1} \bar{\alpha}_{\rho+1-n} z^{n}} \tag{22}
\end{equation*}
$$

The proof of the converse statement is merely the same calculation into the opposite direction.

REMARK 13. The matrix $\mathcal{M}_{S}$ and so the examined operators $\mathscr{F}$ and $\mathscr{\mathscr { S }}$ are of finite rank for a special class of domains described in Lemma 10 and Lemma 12. Such domains are called (classical) quadrature domains and there is extensive research on such domains, see [2], [23] [24] and [27] and the references given there. A bounded domain in the complex plane is called a (classical) quadrature domain if there exist finitely many points $z_{1}, z_{2}, \ldots$ $\ldots, z_{m} \in \Omega$ and coefficients $c_{k j} \in \mathbb{C}$ so that

$$
\begin{equation*}
\int_{\Omega} f d A=\sum_{k=1}^{m} \sum_{j=0}^{n_{k}-1} c_{k j} f^{(j)}\left(z_{k}\right) \tag{23}
\end{equation*}
$$

for all integrable analytic functions $f$ in $\Omega$. The identity (23) is then called a quadrature identity and the integer $n=\sum_{k=1}^{m} n_{k}$ is the order of the quadrature identity (assuming $c_{k, n_{k}-1} \neq 0$ ). In a generalized concept the ellipse, the annulus, the infinite strip and the wedge are also quadrature domains, see [26]. The connection of the Friedrichs operator and the domain is also investigated in [23] and [24].

REMARK 14. A reformulation of (20) implies

$$
\int_{D} \overline{\left(z^{\rho+1}-\alpha(z)\right)} \cdot \overline{h(z)} \frac{g^{\prime}(z)}{\overline{g^{\prime}(z)}} d A(z)=0
$$

for an arbitrary $h \in A L_{2}(D)$. This reveals in connection with (15) that

$$
\begin{equation*}
z^{\ell}\left(z^{\rho+1}-\alpha(z)\right) \in \operatorname{ker} \mathscr{F}_{D} \tag{24}
\end{equation*}
$$

for $\ell=0,1,2, \ldots$, that is the kernel of $\mathscr{F}_{D}$ has codimension $\rho+1$. This gives

$$
\operatorname{codim}(\operatorname{ker} \mathscr{F})=\operatorname{codim}(\operatorname{ker}(\mathscr{F}-2 \mathscr{S}))=\rho+1
$$

using the unitary correspondence between $\mathscr{F}$ and $\mathscr{F}_{D}$ and Theorem 2. In case the mapping $g$ is of the form (22) then there exists a unique function $S$, called Schwarz function, holomorphic in a neighbourhood of the boundary $\partial \Omega$ with $S(w)=\bar{w}$ for $w \in \partial \Omega$, see [27]. Especially we have by setting $w=g(z)$

$$
(S \circ g)(z)=\overline{g\left(\frac{1}{\bar{z}}\right)}
$$

Substituting $\frac{1}{\bar{z}}$ instead of $z$ into (22) implies

$$
(S \circ g)(z)=\bar{a}_{0}+\frac{\sum_{k=0}^{\rho}\left[\bar{a}_{\rho+1-k}-\sum_{m=0}^{\rho-k} \bar{a}_{m} \alpha_{k+m}\right] z^{k}}{z^{\rho+1}-\alpha(z)} .
$$

The Schwarz function of the domain transformed into polar coordinates is again a rational function and its denominator spans the kernel of the investigated operators in the sense of (24). So we have achieved also a full description of $\operatorname{ker} \mathscr{F}$ and $\operatorname{ker}(\mathscr{F}-2 \mathscr{Y})$ for quadrature domains of arbitrary order.

THEOREM 15. Let $\Omega$ be a quadrature domain of order $\rho+1, \rho \geq 0$. Then the kernel of the associated operators $\mathscr{F}$ and $\mathscr{\mathscr { S }}-2 \mathscr{Y}$ has codimension $\rho+1$ and is spanned by the denominator of the Schwarz function of the domain in the sense of (24).

LEMMA 16. Let $g$ map the unit disc onto a domain of finite area. Then at least one of the quantities (11) is not zero for $k=0,1, \ldots$.

Proof. Suppose conversely that

$$
s_{k}=\frac{1}{\pi} \int_{D} \bar{z}^{k} \frac{g^{\prime}(z)}{\overline{g^{\prime}(z)}} d A(z)=0
$$

for all $k=0,1, \ldots$ Similarly to the proof of Lemma 10 we have

$$
\int_{D} \overline{h(z)} \frac{g^{\prime}(z)}{\underline{g^{\prime}(z)}} d A(z)=0
$$

for an arbitrary function $h(z) \in A L_{2}(D)$. Because $g(D)$ has finite area, we have $g^{\prime} \in A L_{2}(D)$ and we can substitute $h(z)=z^{\ell} g^{\prime}(z)$ for an arbitrary integer $\ell \geq 0$. There follows

$$
\int_{D} \bar{z}^{\ell} g^{\prime}(z) d A(z)=0
$$

and this implies $a_{\ell+1}=0$. That is $g^{\prime}(z)=0$ which is not possible.
Remark 17. If the domain is not of finite area then Lemma 16 is not true, see for example the mapping of the unit disc onto the halfplane or onto the exterior of a parabola (see [33]), where all the quantities (11) are zero. More generally, the matrix $\mathcal{M}_{S}$ and also the Friedrichs operator and the Schur complement is zero (so even simpler as in case of a disc) for the so called null quadrature domains. This class consists of half-planes, exteriors of ellipses, exteriors of parabolas and some degenerate cases, see [25]. (In general the Friedrichs operator preserves constant functions on domains with finite area but for an infinite $\Omega$ the constant functions are not in $L_{2}(\Omega)$.)

## 4. Spectral properties of the operators

We use again Theorem 2 to obtain information about the eigenvalues of the Schur complement and so about the inf-sup constant of the domain, which is defined by

$$
\beta_{0}^{2}(\Omega):=\inf _{0 \neq p \in L_{2,0}} \sup _{0 \neq u \in\left(W_{0}^{1,2}\right)^{2}} \frac{(\operatorname{div} u, p)^{2}}{(u, u)_{1}(p, p)},
$$

where $(u, v)_{1}=\int_{\Omega} \sum_{i, j=1}^{2} \frac{\partial u_{i}}{\partial x_{j}} \frac{\partial v_{i}}{\partial x_{j}} d A(x)$ is the scalar product of the Sobolev space $\left(W_{0}^{1,2}\right)^{2}$ of vector functions, see [15].

Combining (7) and (8) gives $2 \mathscr{\varphi}(f)=f-\overline{\mathscr{F}(f)}$ and $2 \mathscr{S}(\bar{f})=\bar{f}-\mathscr{H}(f)$ for $f$ analytic in the domain. These equations imply

$$
4 \varphi^{2}(f)=2 \varphi(f-\overline{\mathscr{F}(f)})=2 \varphi(f)-2 \varphi(\overline{\mathscr{F}(f)})=f-2 \overline{\mathscr{F}(f)}+\mathscr{F}^{2}(f),
$$

and therefore we have

$$
4 \varphi(f)-4 \varphi^{2}(f)=f-\mathscr{F}^{2}(f) .
$$

Hence there follows

$$
4(\mathscr{F}-\mathscr{Y}) \mathscr{Y}=\mathscr{F}-\mathscr{F}^{2}
$$

which implies further

$$
\mathscr{H}^{2}=(\mathscr{F}-2 \varphi)^{2} .
$$

This identity establishes also a connection between the eigenvalues of the Schur complement operator and the modulus of the Friedrichs operator. If $v$ is an eigenvalue of $\mathscr{F}$ (and hence $-\frac{1}{2} v$ an eigenvalue of $\mathcal{M}_{S}$, see Remark 5), then there follows that $|v| \leq 1$ and that both $\frac{1 \pm|v|}{2}$ are eigenvalues of the Schur complement operator, see also [33]. The value 1 is an eigenvalue of $\mathscr{F}$ because the Friedrichs operator of a domain with finite area preserves the constant functions, i.e.

$$
\mathscr{F} 1(z)=\int_{\Omega} K(z, \zeta) \overline{1} d A(\zeta)=1 .
$$

According to [14], in case the boundary does not have any corners, the spectrum of $\mathscr{F}$ is discrete and the modulus of all other eigenvalues is less than 1 . Similarly the spectrum of $\mathscr{S}$ is

$$
\sigma(\mathscr{Y}) \subseteq\{0\} \cup\left[\beta_{0}^{2}(\Omega), 1-\beta_{0}^{2}(\Omega)\right] \cup\{1\}
$$

where $\beta_{0}(\Omega)>0$ is the inf-sup constant of the domain.
Especially important is the eigenvalue

$$
\nu_{2}:=\max \{|v|: v \in \sigma(\mathscr{F}) \backslash\{1\}\}
$$

of the Friedrichs operator. This is the constant in the Friedrichs inequality

$$
\begin{equation*}
\left|\int_{\Omega} w^{2} d A\right| \leq \gamma \int_{\Omega}|w|^{2} d A \text { for all } w \in A L_{2}(\Omega), \int_{\Omega} w d A=0, \tag{25}
\end{equation*}
$$

that is $\gamma=v_{2}$, and $\nu_{2}$ determines the inf-sup constant, too:

$$
\begin{equation*}
\beta_{0}^{2}(\Omega)=\frac{1-\nu_{2}}{2} . \tag{26}
\end{equation*}
$$

The lemmas in the previous section and Remark 17 imply the following.

Corollary 18. The Schur complement operator $\mathscr{S}$ has the spectrum $\left\{0, \frac{1}{2}, 1\right\}$ iff the domain is a quadrature domain of order less than 2.

REMARK 19. Theorem 15 implies further that $\operatorname{ker} \mathscr{F}$ is 1 co-dimensional for domains concerned in Corollary 18.

The question whether the spectrum of the examined operators characterize the domain in the general case is answered in [23]. Although this is true for quadrature domains of order one (i.e. discs) and two but it is not true in case of order three.

PROPOSITION 20 (PROPOSITION 4.9 IN [23]). There exists a continuous family of quadrature domains of order three with the same Friedrichs operator (up to unitary equivalence) and such that no two domains in the family are related by an affine transformation of $\mathbb{C}$.
"Affine transformation" means here a function $w \mapsto a w+b$ for some $a, b \in \mathbb{C}$.

REMARK 21. So the inf-sup constant also does not characterize the domain, even not up to affine transformations of the plane.

To estimate the inf-sup constant we can also use the concept of conformal mapping. We start with an alternative to (25):

$$
\begin{equation*}
\int_{\Omega} u^{2} d A \leq \Gamma_{\Omega} \int_{\Omega} v^{2} d A \text { for all } \int_{\Omega} u d A=0 \tag{27}
\end{equation*}
$$

where $u$ and $v$ are conjugate harmonic functions in $L_{2}(\Omega)$. We have

$$
\begin{equation*}
\gamma=\frac{\Gamma_{\Omega}-1}{\Gamma_{\Omega}+1} \tag{28}
\end{equation*}
$$

see [14]. If $\Omega=g(D)$ and $u, v$ are conjugate harmonic functions in $D$, then $u \circ\left(g^{-1}\right)$ and $v \circ\left(g^{-1}\right)$ are conjugate harmonic functions in $\Omega$. So we have as in [17]:

$$
\begin{equation*}
\Gamma_{\Omega} \leq \frac{\int_{D} u^{2}\left|g^{\prime}\right|^{2} d A}{\int_{D} v^{2}\left|g^{\prime}\right|^{2} d A} \leq \frac{\sup _{D}\left|g^{\prime}\right|^{2} \int_{D} u^{2} d A}{\inf _{D}\left|g^{\prime}\right|^{2} \int_{D} v^{2} d A}=\frac{\sup _{D}\left|g^{\prime}\right|^{2}}{\inf _{D}\left|g^{\prime}\right|^{2}}=\left(\frac{\sup _{\partial D}\left|g^{\prime}\right|}{\inf _{\partial D}\left|g^{\prime}\right|}\right)^{2} \tag{29}
\end{equation*}
$$

because $\Gamma_{D}=1$ (and so $\int_{D} u^{2} d A=\int_{D} v^{2} d A$ if $u(0)=\int_{D} u d A=0$ ) and using the maximum principle. However this estimation is useable only if $\Omega$ does not have any corners (in the presence of corners $\sup _{\partial D}\left|g^{\prime}\right|^{2}=\infty$ )
and if $\Omega$ does not have any internal cusps (in case of internal cusps on $\partial \Omega$, $\inf _{\partial D}\left|g^{\prime}\right|^{2}=0$ ).

The inequality (29) can be continued if the domain is further specified. Let $\Omega=g(D)$ be a star-shaped domain with respect to the origin, that is $g(0)=0$. Let the boundary $\partial \Omega$, which is described in polar coordinates $(r, \varphi)$ by

$$
r=f(\varphi), 0 \leq \varphi<2 \pi
$$

have a continuous tangent. Set $e^{i \theta} \in \partial D$ and

$$
g\left(e^{i \theta}\right)=f(\varphi(\theta)) e^{i \varphi(\theta)}
$$

where $\varphi=\varphi(\theta)$ is the boundary correspondence function: $\varphi(\theta)=\arg g\left(e^{i \theta}\right)$. By differentiation with respect to $\theta$ we obtain

$$
i e^{i \theta} g^{\prime}\left(e^{i \theta}\right)=\dot{f}(\varphi) e^{i \varphi} \varphi^{\prime}(\theta)+i f(\varphi) e^{i \varphi} \varphi^{\prime}(\theta)
$$

where $\dot{f}(\varphi)=\frac{d f}{d \varphi}$. Now we have for $z=e^{i \theta}$

$$
\begin{equation*}
z g^{\prime}(z)=g(z)\left(1-i \frac{\dot{f}(\varphi)}{f(\varphi)}\right) \varphi^{\prime}(\theta) \tag{30}
\end{equation*}
$$

Take the absolute value and substitute this into (29):

$$
\Gamma_{\Omega} \leq \frac{\sup _{0 \leq \theta<2 \pi}\left(1+\left(\frac{\dot{f}(\varphi)}{f(\varphi)}\right)^{2}\right) \varphi^{\prime}(\theta)^{2}}{\inf _{0 \leq \theta<2 \pi}\left(1+\left(\frac{\dot{f}(\varphi)}{f(\varphi)}\right)^{2}\right) \varphi^{\prime}(\theta)^{2}} \cdot \frac{\sup _{z \in \partial D}|g(z)|^{2}}{\inf _{z \in \partial D}|g(z)|^{2}}
$$

which can be simplified to

$$
\Gamma_{\Omega} \leq \sup _{0 \leq \varphi<2 \pi}\left(1+Q^{2}(\varphi)\right) \frac{\sup _{0 \leq \theta<2 \pi} \varphi^{\prime}(\theta)^{2}}{\inf _{0 \leq \theta<2 \pi} \varphi^{\prime}(\theta)^{2}} \cdot \frac{\sup _{z \in \partial D}|g(z)|^{2}}{\inf _{z \in \partial D}|g(z)|^{2}}
$$

where the notation $Q(\varphi):=\frac{\dot{f}(\varphi)}{f(\varphi)}$ is also used. If the domain $\Omega$ has a continuous tangent, then

$$
\begin{equation*}
1 \leq c_{\partial \Omega}:=\sup _{0 \leq \varphi<2 \pi}\left(1+Q^{2}(\varphi)\right) \frac{\sup _{0 \leq \theta<2 \pi} \varphi^{\prime}(\theta)^{2}}{\inf _{0 \leq \theta<2 \pi} \varphi^{\prime}(\theta)^{2}}<\infty \tag{31}
\end{equation*}
$$

and $c_{\partial \Omega}$ depends only on the shape of $\partial \Omega$. We further have

$$
\frac{\sup _{z \in \partial D}|g(z)|^{2}}{\inf _{z \in \partial D}|g(z)|^{2}}=\frac{R^{2}}{r^{2}}
$$

where $0<r \leq|g(z)| \leq R$, i.e. $\partial \Omega$ is contained in the annulus whose inner and outer radii are $r$ and $R$. Therefore, for such domains we have

$$
\begin{equation*}
\Gamma_{\Omega} \leq c_{\partial \Omega} \frac{R^{2}}{r^{2}} \tag{32}
\end{equation*}
$$

Substituting (28) into (26), we obtain

$$
\begin{equation*}
\beta_{0}^{2}(\Omega)=\frac{1}{1+\Gamma_{\Omega}} \tag{33}
\end{equation*}
$$

and we also get

$$
\begin{equation*}
\beta_{0}^{2}(\Omega) \geq \frac{1}{1+c_{\partial \Omega} \frac{R^{2}}{r^{2}}} \geq \frac{1}{2 c_{\partial \Omega}} \cdot \frac{r^{2}}{R^{2}} \tag{34}
\end{equation*}
$$

REMARK 22. The constant $c_{\partial \Omega}$ in (31) could be computed if the derivative of the boundary correspondence function were known. For example in case $\Omega$ is the unit disc, we have the conformal mapping $g(z)=z$ and so $\varphi(\theta)=\theta$. There follows $c_{\partial D}=1$ and the estimation $\beta_{0}^{2}(D) \geq \frac{1}{2}$, where we also have used $r=R$. (In fact $\beta_{0}^{2}(D)=\frac{1}{2}$ for the disc, see e.g. [33].) For other domains we can use for example Symm's equation to obtain an estimate for the derivative of the boundary correspondence function. This equation is

$$
\log |w|=\frac{1}{2 \pi} \int_{\partial \Omega} s(\omega(s)) \log |w-\omega(s)| d \omega(s), \quad w \in \partial \Omega
$$

where $\omega=\omega(s)$ is a parametrization of $\partial \Omega$ by the arc length parameter $s$. Now the function $s(\omega)$ is the reciprocal of the derivative of the boundary correspondence function.

REMARK 23. The estimation (34) is similar to an estimation mentioned in [12]. However the constant used therein is not given explicitly. The estimation (34) has the additional advantage that it explicitly relates the used constant $c_{\partial \Omega}$ to the shape of the domain boundary.

REMARK 24. We also compare (32) to the estimation

$$
\begin{equation*}
\Gamma_{\Omega} \leq \Gamma_{H P}:=\sup _{0 \leq \varphi<2 \pi}\left(|Q(\varphi)|+\sqrt{1+Q^{2}(\varphi)}\right)^{2} \tag{35}
\end{equation*}
$$

given in [16] for the constant $\Gamma_{\Omega}$. Assume $0<m \leq\left|\varphi^{\prime}(\theta)\right| \leq M$ for $0 \leq \theta<2 \pi$ and simplify (31):

$$
c_{\partial \Omega} \leq \frac{M^{2}}{m^{2}} \sup _{0 \leq \varphi<2 \pi}\left(1+Q^{2}(\varphi)\right)
$$

By rearranging this we obtain

$$
\frac{m^{2}}{M^{2}} c_{\partial \Omega} \leq \sup _{0 \leq \varphi<2 \pi}\left(1+Q^{2}(\varphi)\right) \leq c_{\partial \Omega}
$$

which gives

$$
\frac{m^{2}}{M^{2}} c_{\partial \Omega} \leq \Gamma_{H P} \leq 4 c_{\partial \Omega} .
$$

## 5. Domains with corners

As proved in [14] the spectrum of (5) (and then also of (1)) is discrete iff the domain does not have any corners except internal cusps. In case of corners there appears a continuous spectrum. The simplest example for such a domain is a wedge:

$$
W_{\alpha}:=\left\{z \in \mathbb{C}:|\arg z| \leq \frac{\alpha}{2}\right\}
$$

This domain has a corner at the origin and hence the derivative of the associated conformal map has a singularity on the unit circle. A simple calculation of the quantities (11) shows

$$
s_{2 k}^{\left(W_{\alpha}\right)}=\frac{\sin \alpha}{\alpha}\left(\frac{1}{2 k+1}-\frac{1}{2 k+3}\right) ; \quad s_{2 k+1}^{\left(W_{\alpha}\right)}=0
$$

for $k=0,1, \ldots$, and further

$$
M_{S}^{\left(W_{\alpha}\right)}=\frac{\sin \alpha}{\alpha} M_{S}^{\left(W_{0}\right)}
$$

where the set $W_{0}:=\left\{z \in \mathbb{C}:|\operatorname{Im} z| \leq \frac{\pi}{2}\right\}$ is an infinite strip. Therefore we have a similar equality for all eigenvalues and there follows

$$
1-2 \beta_{0}^{2}\left(W_{\alpha}\right)=\frac{\sin \alpha}{\alpha}\left(1-2 \beta_{0}^{2}\left(W_{0}\right)\right)
$$

By $0<\beta_{0}^{2}\left(W_{0}\right) \leq \frac{1}{2}$ (see e.g. [23]) we obtain

$$
\frac{1}{2}\left(1-\left|\frac{\sin \alpha}{\alpha}\right|\right) \leq \beta_{0}^{2}\left(W_{\alpha}\right) \leq \frac{1}{2}
$$

Moreover in [23] it is proved that the infinite strip and the wedge are quadrature domains in the generalized sense and the concerned operators have only continuous spectra.

We investigate another example which is the conformal map

$$
\begin{equation*}
g(z)=(1-z)^{\omega} \tag{36}
\end{equation*}
$$

$0<\omega \leq 2$ of the unit disc onto a ("drop shaped") finite domain $\Omega^{(\omega)}$ with smooth boundary except at $g(1)=0$, where it has an inner angle $\omega \pi$ for $\omega \neq 1$. If $\omega=1$ then the domain is the translated unit disc. (In case $\omega=2$ the domain is a cardioid and it has an interior cusp at the origin.) Using the series expansions

$$
g(z)=\sum_{k=0}^{\infty} \frac{\Gamma(k-\omega)}{\Gamma(-\omega) \Gamma(k+1)} z^{k} \text { and } \frac{1}{g^{\prime}(z)}=\sum_{k=0}^{\infty} \frac{-\Gamma(k-1+\omega)}{\omega \Gamma(\omega-1) \Gamma(k+1)} z^{k}
$$

for (36) we calculate for $k=0,1,2, \ldots$

$$
s_{k}^{(\omega)}=\frac{\sin (\omega \pi)}{\omega \pi}\left(\frac{1}{(k-\omega+1)}-\frac{1}{(k-\omega+2)}\right) .
$$

Set $\mathscr{F}_{D}^{(\omega)}$ for the transformed Friedrichs operator of the domain with the associated mapping (36). A simple calculation with the associated infinite matrices shows

$$
\left(\mathscr{F}_{D}^{(1+\omega)}\left(\boldsymbol{M}_{z} p\right), \boldsymbol{M}_{z} p\right)=\frac{\omega}{1+\omega}\left(\mathscr{F}_{D}^{(\omega)} p, M_{z} p\right) .
$$

for $p \in L_{2}(D)$ and $0 \leq \omega \leq 1$, where $M_{z}$ denotes the multiplication by the variable $z$. For $p(z)=\sum_{k=0}^{\infty} p_{k} z^{k} \in A L_{2}(D)$ we have

$$
\|p\|^{2}=\sum_{k=0}^{\infty} \frac{1}{k+1}\left|p_{k}\right|^{2} \text { and }\left\|M_{z} p\right\|^{2}=\sum_{k=0}^{\infty} \frac{1}{k+2}\left|p_{k}\right|^{2}
$$

There follows $\left\|M_{z} p\right\|^{2} \leq\|p\|^{2} \leq 2\left\|M_{z} p\right\|^{2}$. Using this along with the Cauchy-Schwarz inequality, gives:

$$
\frac{\left|\left(\mathscr{F}_{D}^{(1+\omega)}\left(M_{z} p\right), M_{z} p\right)\right|}{\left\|M_{z} p\right\|^{2}} \leq\left|\frac{\sqrt{2} \omega}{1+\omega}\right| \cdot \frac{\left\|\mathscr{F}_{D}^{(\omega)} p\right\|}{\|p\|}
$$

According to $\left\|\mathscr{F}^{(\omega)}\right\| \leq 1$ in operator norm we obtain

$$
\begin{equation*}
\left\|\mathscr{H}^{(1+\omega)}\right\| \leq\left|\frac{\sqrt{2} \omega}{1+\omega}\right|<1 \tag{37}
\end{equation*}
$$

for $0 \leq \omega \leq 1$, which implies

$$
\begin{equation*}
\beta_{0}^{2}\left(\Omega^{(1+\omega)}\right) \geq \frac{1}{2}\left(1-\left|\frac{\sqrt{2} \omega}{1+\omega}\right|\right) \tag{38}
\end{equation*}
$$

REMARK 25. The inequality (37) holds also for $1-\sqrt{2} \leq \omega<0$ but then the domain $\Omega^{(\omega)}$ is infinite (with smooth boundary) although its Friedrichs operator also satisfies the norm estimate $\left\|\mathscr{F}^{(\omega)}\right\| \leq 1$. Therefore the estimation (38) is also valid for all domains $\Omega^{(1+\omega)}$ with $1-\sqrt{2} \leq \omega \leq 0$.

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## ON THE LINEAR COMPLEXITY OF BINARY SEQUENCES

By<br>\section*{ÁGNES ANDICS}

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## 1. Introduction

Let us denote the linear complexity profile of a binary sequence $\left(s_{n}\right)$ by $L\left(s_{n}, N\right)$ for every integer $N \geq 2$. This function $L\left(s_{n}, N\right)$ is defined as the shortest length $L$ of a linear recursion

$$
s_{n+L}=c_{L-1} s_{n+L-1}+c_{L-2} s_{n+L-2}+\cdots+c_{0} s_{n} \quad(0 \leq n \leq N-L-1)
$$

which is satisfied by this sequence $\left(s_{n}\right)$. The linear complexity profile is an important cryptographic characteristic of bitsequences. A high linear complexity profile is a desirable feature of sequences used for cryptographic purposes.

The correlation measure of order $k$ was introduced by Mauduit and Sárközy [5] as another measure of pseudorandomness of binary sequences. This measure can be defined as

$$
C_{k}\left(s_{n}\right)=\max _{M, D}\left|\sum_{n=0}^{M-1}(-1)^{s_{n+d_{1}}+s_{n+d_{2}}+\cdots+s_{n+d_{k}}}\right|
$$

(see [5] for a different but equivalent definition), where the maximum is taken over all $D=\left(d_{1}, d_{2}, \ldots, d_{k}\right)$ (where $0 \leq d_{1}<\cdots<d_{k}$ are given integers) and $M \in \mathbb{N}$ such that $M+d_{k} \leq N$, where $N$ is the length of the sequence $\left(s_{n}\right)$.

Brandstätter and Winterhof [1] gave a lower bound on the linear complexity profile in terms of the correlation measure. They proved the following

Theorem 1.1. Let $\left(s_{n}\right)$ be a $T$-periodic binary sequence. For $2 \leq N \leq$ $\leq T-1$,

$$
L\left(s_{n}, N\right) \geq N-\max _{1 \leq k \leq L\left(s_{n}, N\right)+1} C_{k}\left(s_{n}\right) .
$$

Using this inequality they gave (partly) new lower bounds on the linear complexity profile of certain sequences. For example, they applied this theorem to study the linear complexity profile of the sequences studied in [7, 2, 3].

1. Corollary. Let us define the sequence $\left(d_{n}\right)$ for $1 \leq n \leq p$ by

$$
d_{n}= \begin{cases}0, & \text { if } 1 \leq \text { ind } n \leq \frac{p-1}{2} \\ 1, & \text { if } \frac{p+1}{2} \leq \text { ind } n \leq p-1, \text { or } p \mid n\end{cases}
$$

where ind a denotes the discrete logarithm of a modulo $p$ to the base $g$

$$
g^{\text {ind } a} \equiv a \quad \bmod p, \quad 1 \leq \text { ind } a \leq p-1
$$

whenever $p$ is a fixed odd prime, $g$ is a fixed primitive root modulo $p$, and $\operatorname{gcd}(a, p)=1$.

Then the linear complexity profile of this sequence $\left(d_{n}\right)$ satisfies

$$
L\left(d_{n}, N\right)=\Omega\left(\frac{\log \left(N / p^{\frac{3}{4}}\right)}{\log \log p}\right) \quad(2 \leq N \leq p-1)
$$

2. Corollary. Define the Legendre sequence $\left(l_{n}\right)$ as

$$
l_{n}= \begin{cases}1, & \text { if }\left(\frac{n}{p}\right)=-1 \\ 0, & \text { otherwise }\end{cases}
$$

where $(\dot{\bar{p}})$ is the Legendre symbol of the finite field $\mathbb{F}_{p}$ for $p>2$ prime.
The linear complexity profile of this sequence $\left(l_{n}\right)$ satisfies

$$
L\left(l_{n}, N\right)=\Omega\left(\frac{N}{p^{\frac{1}{2}} \log p}\right) \quad(2 \leq N \leq p-1)
$$

In this paper first we will present a further application of Theorem 1.1. Next, in Section 3 we will compute the linear complexity of the special sequences described above for certain random values of the parameters involved in order to compare the values of the linear complexity to the theoretical bounds proved. In Section 4 we will prove a further inequality between the linear complexity and the correlation of order 2. Finally, in Section 5 we will present computational evidences which seem to indicate that the inequality in Section 4 is nearly the best possible.

## 2. A further application

In [4] Mauduit and Sárközy presented the following construction. For $p$ prime, denote the least non-negative residue of $n$ modulo $p$ by $r_{p}(n)$ (i.e., $n \equiv r \bmod p$, where $r \in\{0, \ldots, p-1\}$ ), and for $\operatorname{gcd}(a, p)=1$ denote the multiplicative inverse of $a$ by $a^{-1}$

$$
a a^{-1} \equiv 1(\bmod p) .
$$

Let the polynomial $f(x) \in F_{p}[x]$ have degree $d$ (where $0<d<p$ ), and assume that $f(x)$ has no multiple zero in $\bar{F}_{p}$. They defined the sequence $E_{p}=\left(e_{1}, \ldots, e_{p}\right) \in\{-1,+1\}^{p}$ for $1 \leq n \leq p$ by

$$
e_{n}= \begin{cases}+1, & \text { if }(f(n), p)=1, r_{p}\left(f(n)^{-1}\right)<\frac{p}{2} \\ -1, & \text { if either }(f(n), p)=1, r_{p}\left(f(n)^{-1}\right)>\frac{p}{2}, \text { or } p \mid f(n)\end{cases}
$$

It is easy to see that by the transformation $s_{n}=\frac{e_{n}+1}{2}$ we can get a $0-1$ binary sequence.

For this construction they proved the following:
THEOREM 2.1. Assume that one of the following two conditions holds $(k \in \mathbb{N}, 2 \leq k \leq p)$ :
a) $k=2$,
b) $(4 d)^{k}<p$;
then we have

$$
C_{k}\left(E_{p}\right) \ll d k p^{\frac{1}{2}}(\log p)^{k+1} .
$$

By using this theorem and Theorem 1.1 we will prove the following:
3. Corollary. The linear complexity profile of the sequence $\left(e_{n}\right)$ defined above satisfies

$$
L\left(e_{n}, N\right)=\Omega\left(\frac{\log \left(N / d p^{\frac{1}{2}}\right)}{\log \log p}\right), \quad(2 \leq N \leq p-1)
$$

Proof. Let $L=L\left(e_{n}, N\right)$ for every $2 \leq N \leq p-1$. Then from Theorem 1.1 and Theorem 2.1 we get

$$
N \leq L+\max _{1 \leq k \leq L+1} C_{k}\left(e_{p}\right)=L+O\left(d L p^{\frac{1}{2}}(\log p)^{L+2}\right)=O\left(d L p^{\frac{1}{2}}(\log p)^{L+2}\right)
$$

We may assume that $L \leq \frac{\log p}{\log \log p}$, since otherwise the result is trivial.

Thus, $N=O\left(d p^{\frac{1}{2}}(\log p)^{L+3}\right)$, from which the result is clear with the following computation

$$
\begin{gathered}
N<c d p^{\frac{1}{2}}(\log p)^{L+3} \\
\frac{N}{c d p^{\frac{1}{2}}}<(\log p)^{L+3} \\
\log \left(\frac{N}{c d p^{\frac{1}{2}}}\right)<2 L \log \log p \\
L>\frac{\log \left(N / c d p^{\frac{1}{2}}\right)}{\log \log p}=\frac{\log \left(N / d p^{\frac{1}{2}}\right)}{\log \log p}-\frac{\log c}{\log \log p}=\frac{\log \left(N / d p^{\frac{1}{2}}\right)}{\log \log p}-o(1)
\end{gathered}
$$

## 3. Numerical values of the linear complexity

One might like to see how far the theoretical bounds for the linear complexity proved above are from the actual value of it. It seems to be very difficult to prove anything in this direction; the best what one can do is to compute numerical values of the linear complexity and to compare them to the theoretical bounds.

The following three tables contain the results of linear complexity computations for large primes for the sequence based on the discrete logarithm, the Legendre sequence and the sequence defined in the previous section by using the multiplicative inverse, respectively. The values of the linear complexity have been computed by the Berlekamp-Massey algorithm. In each of the cases, $p$ denotes the prime used for generating the sequence, $L$ denotes the linear complexity of the generated sequence.

TABLE 1. values for the discrete logarithm sequence

| $p$ | $L$ | $\left\lfloor\frac{p}{2}\right\rfloor$ |
| :---: | :---: | :---: |
| 7919 | 3959 | 3959 |
| 30773 | 15386 | 15386 |
| 101203 | 50601 | 50601 |
| 101771 | 50885 | 50885 |
| 104659 | 52329 | 52329 |
| 500177 | 250088 | 250088 |

TABLE 2. values for the Legendre sequence

| $p$ | $L$ | $\left\lfloor\frac{p}{2}\right\rfloor$ |
| :---: | :---: | :---: |
| 7919 | 3959 | 3959 |
| 30773 | 15386 | 15386 |
| 101203 | 50601 | 50601 |
| 101771 | 50885 | 50885 |
| 104659 | 52329 | 52329 |
| 201101 | 100551 | 100550 |
| 500177 | 250088 | 250088 |
| 501077 | 250539 | 250538 |
| 536999 | 268499 | 268498 |
| 1011277 | 505639 | 505638 |
| 1020101 | 510051 | 510050 |

TABLE 3. values for the sequence defined in [4] by using multiplicative inverse

| $p$ | $d$ | $L$ | $\left\lfloor\frac{p+1}{4}\right\rfloor$ |
| :---: | :---: | :---: | :---: |
| 500069 | 25 | 125019 | 125017 |
| 500069 | 26 | 125017 | 125017 |
| 500069 | 27 | 125018 | 125017 |
| 501077 | 25 | 125270 | 125269 |
| 501077 | 26 | 125270 | 125269 |
| 501077 | 27 | 125271 | 125269 |
| 1002773 | 31 | 250694 | 250693 |
| 1002773 | 34 | 250694 | 250693 |
| 1002773 | 37 | 250693 | 250693 |
| 1006613 | 31 | 251654 | 251653 |
| 1006613 | 34 | 251654 | 251653 |
| 1006613 | 37 | 251655 | 251653 |
| 2000717 | 31 | 500180 | 500179 |
| 2007557 | 31 | 501892 | 501889 |

In each of these cases the numerical value of the linear complexity is very close to the half of the length of the sequence (which is the value of the linear complexity for a truly random sequence). This seems to indicate that the theoretical bounds are very far from the best possible values.

## 4. A new inequality

If we want to use the inequality in Theorem 1.1 to estimate the linear complexity then we also need upper bounds for the correlation. However, in most of the applications it is difficult to estimate the correlations of large order, thus, one might like to show another lower bound for the linear complexity where only correlations of small order, possibly only the correlation of order 2 , are used. Next we will present such an inequality.

Theorem 4.1. Let $L$ denote the linear complexity of the sequence $S_{N}$ of length $N$. Then the following holds:

$$
2^{L} \geq N-C_{2}\left(S_{N}\right)
$$

Proof. $S_{N}$ denotes the sequence $S_{N}=s_{1}, s_{2}, \ldots, s_{N}$. We use the theory of linear feedback shift registers (LFSRs). According to the feedback polynomial of an LFSR $C(D)=1+c_{1} D+c_{2} D^{2}+\cdots+c_{L} D^{L}$, the $(n+L+1)$ st element of the sequence $S_{N}$ is given by the following recursion

$$
s_{n+L+1}=c_{1} s_{n+L}+c_{2} s_{n+L-1}+\cdots+c_{L} s_{n+1} \bmod 2,
$$

for every $n=0,1, \ldots, N-L-1$. It is obvious that the maximal period of the sequence generated by this LFSR is $2^{L}$ (the feedback polynomial has degree $L$ ). Consider two arbitrary subsequences of length $L$ of the sequence $S_{N}$ :

$$
s_{u+1}, s_{u+2}, \ldots, s_{u+L} ; \quad s_{v+1}, s_{v+2}, \ldots, s_{v+L} \quad\left(0 \leq u<v \leq 2^{L}\right) .
$$

If these sequences are bit by bit equal then $s_{m}=s_{m+v-u}$ for every $m=u+1$, $u+2, \ldots$ and the correlation of order 2 of the sequence $S_{N}$ is

$$
C_{2}\left(S_{N}\right) \geq \sum_{m=u+1}^{N-(v-u)} e_{m} e_{m+v-u}=N-(v-u)-u=N-v,
$$

where $v \leq 2^{L}$.
This theorem gives strong bound only for $L \approx \log N$. If $L$ is greater than $\log N$, there are sequences for which the correlation of order 2 is small. It is difficult to prove this in general. We showed for some randomly chosen parameters that there exist sequences $S_{N}$ for which the linear complexity $L$ is $\lceil 2 \cdot \log N\rceil$ and $C_{2}\left(S_{N}\right)$ is small (i.e., $C_{2}\left(S_{N}\right)$ is much closer to $\sqrt{N}$ than to $N$ ). We chose the length $N$ of the sequences such that the running time
of a computation is around 5 minutes. We used the following recursion to generate the sequences

$$
s_{n+1}=c_{1} s_{n+1}+c_{2} s_{n+2}+\cdots+c_{L} s_{n+L} \bmod 2
$$

In our case $L=\lceil 2 \cdot \log N\rceil$, so we prepared two random sequences of length $L$ by coin tossing: one for containing the coefficients, $C$ and one for the initial state (the input to the recursion), $I$. The next table contains the investigated sequences and results for $N=10000$ in the first 3 rows and for $N=100000$ in the last 2 rows.

| $(C, I)$ | $C_{2}\left(S_{N}\right)$ |
| :---: | :---: |
| $(10000010010110000110100011,00000110110000000101110010)$ | 360 |
| $(11100101111000001001000001,00111110100011011100000011)$ | 433 |
| $(11111011000100000001001101,11100100001000011010110000)$ | 385 |
| $(11000100000001000010111011111011,10000101110010000001011101010010)$ | 1248 |
| $(10000110100000100001100010000001,1110110010000000010001101111000)$ | 1290 |

As the table shows, the correlation of order 2 indicated in the last column is much closer to $\sqrt{N}$ (100 and 316.23) than to $N(10000$ and 100000) - the values are around $\sqrt{N}$ times a small constant - , that allows us to state that the correlation of order 2 is small when the linear complexity is around $2 \cdot \log N$.

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