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The Editorial Board manifests its deepest sorrow for the loss of

Professor Viktor Scharnitzky,

who died on 25th April, 2003, and has performed at highest niveau the task of the technical editor since 1957, the starting of our periodical.

(1930–2003)

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BILINEAR OPERATORS OVER VILENKIN GROUPS AND THEIR APPLICATIONS

By

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1. Introduction

It is well-known that multilinear operators have close connections with many problems of analysis such as the Cauchy integral on Lipschitz curves, and the compensated compactness in partial differential equations; see [15, 4] and [16]. It is their applicability and usability that prompt many authors to look into their boundedness in various spaces; see [5, 8, 9, 11, 10, 14, 17, 18] and [19].

In this paper, we consider multilinear operators over certain Vilenkin groups G. As applications, we obtain the factorization of Hardy spaces and the boundedness of the commutators generated by BMO functions and the Calderón–Zygmund operators or the fractional integral operators over G. Before stating our results, we establish some notation.

Throughout this paper, G denotes a bounded locally compact Vilenkin group, that is, G is a locally compact Abelian group containing a strictly decreasing sequence of compact open subgroups $\{G_n\}_{n=-\infty}^{\infty}$ such that

(a) $\bigcup\limits_{n=-\infty}^{\infty} G_n = G$ and $\bigcap\limits_{n=-\infty}^{\infty} G_n = \{0\}$.

(b) $\sup\{\text{order}(G_n/G_{n+1}) : n \in \mathbb{Z}\} \equiv B < \infty$.

Examples of such groups are described in ([7], §4.1.2). An additional example is the additive group of a local field; see [24].

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We choose Haar measures dx on G so that $|G_0| = 1$, where $|A|$ denotes the Haar measure of a measurable subset A of G. Let $|G_n| \equiv (m_n)^{-1}$ for each $n \in \mathbb{Z}$. Since $2m_n \le m_{n+1} \le Bm_n$ for each $n \in \mathbb{Z}$, it follows that

$$\sum_{n=k}^{\infty}(m_n)^{-\alpha} \le c(m_k)^{-\alpha} \tag{1.1}$$

and

$$\sum_{n=-\infty}^{k}(m_n)^{\alpha} \le c(m_k)^{\alpha} \tag{1.2}$$

for any $\alpha > 0$, $k \in \mathbb{Z}$, where c is a constant independent of k. Throughout this paper, c will always denote a constant which is independent of the main parameters, but may vary from line to line. We now define the function $d : G \times G \to R$ by $d(x, y) = 0$ if $x - y = 0$ and $d(x, y) = (m_n)^{-1}$ if $x - y \in G_n \setminus G_{n+1}$, then d defines a metric on $G \times G$ and the topology on G induced by this metric is the same as the original topology on G. For $x \in G$, we set $|x| = d(x, 0)$. Then $|x| = (m_n)^{-1}$ if and only if $x \in G_n \setminus G_{n+1}$. We now briefly recall the definitions of the spaces $\mathscr{S}(G)$ of test functions and $\mathscr{S}'(G)$ of distributions; for more details, see [24]. A function $\Phi : G \to C$ belongs to $\mathscr{S}(G)$ if there exist integers k, l, depending on Φ, so that $\operatorname{supp}\Phi \subset G_k$ and Φ is constant on the cosets of some subgroup G_l of G. A sequence $\{\Phi_n\}_1^\infty$ of functions in $\mathscr{S}(G)$ converges to $\Phi \in \mathscr{S}(G)$ if there exist integers k, l so that every Φ_n and Φ have supports in G_k and are constant on the cosets of G_l in G and if $\lim_{n\to\infty} \Phi_n(x) = \Phi(n)$ uniformly on G. The space of all continuous functionals on $\mathscr{S}(G)$ is denoted by $\mathscr{S}'(G)$.

DEFINITION 1.1. Let T be a linear operator mapping all continuous functions on G into measurable functions on G. We say that T is a Calderón–Zygmund operator if

(i) T can be extended to a bounded linear operator on $L^2(G)$;

(ii) There is a kernel $K(x)$ such that

$$Tf(x) = \int_{\operatorname{supp} f} K(x - y)f(y)dy$$

for all continuous functions f with compact supports and for all $x \notin \operatorname{supp} f$. Here K satisfies

(ii)$_1$ $|K(x)| \le c|x|^{-1}$ if $x \neq 0$;

(ii)$_2$ $|K(y) - K(x)| \leq c|x - y|/|x|^2$ if $|x - y| \leq |x|/2$.

The smallest constant satisfying (ii)$_1$, and (ii)$_2$ is called the Calderón–Zygmund constant of T. We denote it by c_T.

We first consider bilinear operator of the form

$$B(f, g)(x) = \sum_{\gamma=1}^{N} (T_\gamma^1 f)(x)(T_\gamma^2 g)(x), \qquad x \in G,$$

where $N \in \mathbb{N}$, T_γ^1 and T_γ^2 are Calderón–Zygmund operators on G. In what follows, we will denote the kernels of T_γ^1 and T_γ^2, respectively, by K_γ^1 and K_γ^2.

The following results on the boundedness of Calderón–Zygmund operators on G can be found in [21] and [22]. We remark that the authors have shown in [21] that (i) of Definition 1.1 can be replaced by five other equivalent conditions.

LEMMA 1.1. *Let T be a Calderón–Zygmund operator on G. Then,*

(i) ([21]) *If* $1 < p < \infty$, *then T is bounded on $L^P(G)$;*

(ii) ([22]) *If* $1/2 < p \leq 1$, *then T is bounded on $H^P(G)$.*

Moreover, the operator norm of T in both (i) *and* (ii) *depends only on B, p and c_T.*

Let $1/r = 1/p + 1/q$, and p, $q > r \geq 1$. Then by Hölder's inequality and the above lemma, we have

$$\begin{aligned}\|B(f,g)\|_{L^r(G)} &\leq \sum_{\gamma=1}^{N} \|(T_\gamma^2 f)(T_\gamma^2 g)\|_{L^r(G)} \leq \sum_{\gamma=1}^{N} \|T_\gamma^2 f\|_{L^p(G)} \|T_\gamma^2 g\|_{L^q(G)} \leq \\ &\leq c\|f\|_{H^p(G)} \|g\|_{L^p(G)}.\end{aligned}$$

Thus B continuously maps $H^P(G) \times H^q(G)$ into $L^r(G)$.

It is well-known that $H^p(G) = L^p(G)$ if $p > 1$; see [12]. The main purpose of this paper is to show that if

$$\int_G B(f, g)(x)\,dx = 0 \tag{1.3}$$

for all f, $g \in L^2(G)$ with compact supports, then we even have B bounded from $H^p(G) \times H^q(G)$ into $H^r(G)$ for $r \leq 1$.

Here are our main theorems.

THEOREM 1.1. *Let $p, q > 1$ and $1/r = 1/p + 1/q$. Assume that B satisfies* (1.3). *Then, if $1/2 < r \leq 1$, B can be extended to a bounded linear operator from $L^p(G) \times L^q(G)$ into $H^r(G)$.*

THEOREM 1.2. *Let $1/2 < p \leq 1$, $q > 1$, and $1/r = 1/p + 1/q$. Assume that B satisfies* (1.3). *Then, if $1/2 < r \leq 1$, B can be extended to a bounded linear operator from $H^p(G) \times L^q(G)$ into $H^r(G)$.*

For $1/2 < r \leq 1$ our theorems generalize Theorem 3 in [6]. We remark that our proofs of Theorems 1.1 and 1.2 also work for general k-linear operators of the same type. We acknowledge that some basic ideas on the proofs of our theorems are from [9].

We now give a brief outline of this paper. The proofs of Theorems 1.1 and 1.2 are given in next section. In Section 3, we generalize Theorems 1.1 and 1.2 to bilinear operators generated by the fractional integral operators and the Calderón–Zygmund operators; see Theorems 3.1, 3.2 and 3.3. The last section is devoted to some applications of our results. We obtain in Theorem 4.1 certain factorization of Hardy spaces. Theorem 1.1 enables us to prove in Corollary 4.1 the boundedness of commutators formed by the the Calderón–Zygmund operators and the $BMO(G)$ function. We end this section with Corollary 4.2 which characterises $BMO(G)$ functions by means of commutators generated by the fractional integral operators and the $BMO(G)$ function. Our results are $\mathbb{R}^n$-analogues of those in [1]; however, the proofs are totally different: see also [3] for the analogues on the simple martingales of the results in [1].

2. Proofs of Theorems 1.1 and 1.2

For $\alpha \in (0, 1)$ and $f \in \mathcal{S}(G)$, we define the fractional integral operator I_α of order α by

$$I_\alpha(f)(x) = c_\alpha \int_G \frac{f(y)}{|x-y|^{1-\alpha}} dy,$$

where c_α is a constant depending only on α.

The proofs of our theorems depend on the following two lemmas. Our first lemma is in Corollary 2 of ([13], p. 470).

LEMMA 2.1. *Let $0 < \alpha < 1$. Then I_α is bounded from $L^p(G)$ into $L^q(G)$ if $1/q = 1/p - \alpha$.*

We also need the following Kolmogorov's inequality; see [23].

LEMMA 2.2. *Assume that there is a constant $c(g)$ such that for each $\lambda > 0$,*

$$|\{x \in G : |g(x)| > \lambda\}| \leq c(g)/\lambda.$$

Then for any coset I of G having finite measure and any $\delta \in (0,1)$, we have

$$\int_I |g(x)|^\delta dx \leq \frac{1}{1-\delta}|I|^{1-\delta}[c(g)]^\delta.$$

We now give the proof of Theorem 1.1.

PROOF OF THEOREM 1.1. Let $\Delta_n = |G_n|^{-1}\chi G_n$. For $f \in \mathcal{S}'(G)$, we define

$$Mf(x) = \sup_{n\in\mathbb{Z}} |(\Delta_n * f)(x)|.$$

Let $f \in L^p(G)$ and $g \in L^q(G)$. By the results in [18], we need to show that

$$M(B(f,g)) \in L^r(G),$$

where

$$M(B(f,g)) = \sup_{n\in\mathbb{Z}} \left| \int_G \Delta_n(x-y)B(f,g)(y)dy \right|.$$

Write

$$\begin{aligned} B(f,g)(y) = {} & B(\chi_{G_n}(x-\cdot)f(\cdot), \chi_{G_n}(x-\cdot)g(\cdot))(y)+ \\ & +B(f,(1-\chi_{G_n}(x-\cdot))g(\cdot)) + B((1-\chi_{G_n}(x-\cdot))f(\cdot), g)- \\ & -B((1-\chi_{G_n}(x-\cdot))f(\cdot),(1-\chi_{G_n}(x-\cdot))g(\cdot)). \end{aligned}$$

Consider $B(f,(1-\chi_{G_n}(x-\cdot))g(\cdot))$ first. We have

$$\sup_{n\in\mathbb{Z}} \left| \int_G \Delta_n(x-y)B(f,(1-\chi_{G_n}(x-\cdot))g(\cdot))(y)dy \right| \leq$$

$$\leq c\sum_{\gamma=1}^N \sup_{n\in\mathbb{Z}} \int_G \Delta_n(x-y)|T_\gamma^1 f(y)||Y_\gamma^2((1-\chi_{G_n}(x-\cdot))g(\cdot))(y)-$$

$$-T_\gamma^2((1-\chi_{G_n}(x-\cdot))g(\cdot))(x)|dy+$$

$$+c\sum_{\gamma=1}^{N}\sup_{n\in\mathbb{Z}}\int_G \Delta_n(x-y)|T_\gamma^2 f(y)||T_\gamma^2((1-\chi_{G_n}(x-\cdot))g(\cdot))(x)|dy.$$

For simplicity, we write $\chi^1_{G_n}(y)=1-\chi_{G_n}(x-y)$.

By (1.2), we have

$$\sup_{n\in\mathbb{Z}}|T_\gamma^2(\chi^1_{G_n,x}g(y)-T_\gamma^2(\chi^1_{G_n,x}g)(x)|= \tag{2.1}$$

$$=\sup_{n\in\mathbb{Z}}\left|\int_G \left(K_\gamma^2(y-z)-K_\gamma^2(x-z)\right)\chi^1_{G_n,z}(z)g(z)dz\right|\leq$$

$$\leq c\sup_{n\in\mathbb{Z}}\int_{|x-z|\notin G_n}\frac{|x-y|}{|x-z|^2}\chi^1_{G_n,x}(z)g(z)|dz\leq$$

$$\leq c\sum_{l=-\infty}^{n-1} m_n^{-1}\gamma_{G_l\setminus G_{l+1}}\frac{1}{|x-z|^2}|\chi^1_{G_n,x}(z)g(z)|dz\leq$$

$$\leq cHL(g)(x)m_n^{-1}\sum_{l=-\infty}^{n-1} m_l\leq cHL(g)(x),$$

where HL is the Hardy–Littlewood maximal operator on G.

Therefore,

$$\sup_{n\in\mathbb{Z}}\left|f_G\Delta_n(x-y)B(f,\chi^1_{G_n,x}g)(y)dy\right|\leq$$

$$\leq cHL(g)(x)\sum_{\gamma=1}^{N}\int_G \Delta_n(x-y)|T_\gamma^1 f(y)|dy+$$

$$+c\sum_{\gamma=1}^{N}HL(T_\gamma^2 f)(x)\sup_{n\in\mathbb{Z}}|T_\gamma^2(\chi^1_{G_n,x}g)(x)|\leq$$

$$\leq c\sum_{\gamma=1}^{N}HL(T_\gamma^1 f)(x)HL(g)(x)+c\sum_{\gamma=1}^{N}HL(T_\gamma^1 f)(x)\sup_{n\in\mathbb{Z}}|T_\gamma^2(\chi^1_{G_n,x}g)(x)|.$$

Obviously,

$$T_\gamma^2(\chi^1_{G_n,x}g)(x) = \int\limits_G K_\gamma^2(x-y)(1-\chi_{G_n}(x-y))g(y)dy = (K^2_{\gamma,n} * g)(x),$$

where $K^2_{\gamma,n}(z) = K^2_\gamma(z)(1-\chi_{G_n}(z))$. It is easy to see that $K^2_{\gamma,n}(z)$ satisfies $(\text{ii})_1$, and $(\text{ii})_2$ of Definition 1.1 with the same Calderón–Zygmund constant as K^2_γ for each $n \in \mathbb{Z}$. It now follows from this that

$$\left(\int\limits_G \sup_{n\in\mathbb{Z}} \left|\Delta_n(x-y)B(f,\chi_{G_n,x}g)(y)dy\right|^r dx\right)^{1/r} \le$$

$$\le c\sum_{\gamma=1}^N \|HL(T^2_\gamma f)\|_{L^p(G)}\left(\|HL(g)\|_{L^q(G)} + \sup_{n\in\mathbb{Z}}\|K^2_{\gamma,n} * g\|_{L^q(G)}\right) \le$$

$$\le c\sum_{\gamma=1}^N \|T^1_\gamma f\|_{L^p(G)}\|g\|_{L^q(G)} \le c\|f\|_{L^p(G)}\|g\|_{L^q(G)},$$

which is a desirable estimate.

The estimate for $B((1-\chi_{G_n}(x-\cdot))f(\cdot), g)$ is similar and is omitted. For the last term $B((1-\chi_{G_n}(x-\cdot))f(\cdot), (1-\chi_{G_n}(x-\cdot))g(\cdot))$, we write

$$B(\chi^1_{G_n,x}f, \chi^1_{G_n,x}g)(y) =$$

$$= \sum_{\gamma=1}^N \left[T^2_\gamma(\chi_{G_n,x}f)(y) - T^1_\gamma(\chi^1_{G_n,x}f)(x)\right]\times\left[T^2_\gamma(\chi_{G_n,x}g)(y) - T^2_\gamma(\chi^1_{G_n,x}g)(x)\right] +$$

$$+\sum_{\gamma=1}^N T^1_\gamma(\chi^1_{G_n,x}f(y)T^2_\gamma(\chi_{G_n,x}g)(x) + \sum_{\gamma=1}^N T^2_\gamma(\chi^1_{G_n,x}f)(x)T^2_\gamma(\chi^1_{G_n,x}(g)(y) -$$

$$-\sum_{\gamma=1}^N T^1_\gamma(\chi^1_{G_n,x}f)(x)T^2_\gamma(\chi^1_{G_n,x}g)(x) \equiv A_1 + A_2 + A_3 + A_4.$$

By (2.1), we have

$$A_1 \le cHL(f)(x)HL(g)(x)$$

for all $x - y \in G_n$. Now Hölder's inequality gives the desired estimate for A_1. It is easy to see that A_2, A_3 and A_4 can be estimated as in the case for $B(f, \chi^1_{G_n,x}g)$.

We now turn to estimate $B(\chi_{G_n}(x-\cdot)f(\cdot),\chi_{G_n}(x-\cdot)g(\cdot))(y)$. We have

$$\sup_{n\in\mathbb{Z}}\left|\int_G \Delta_n(x-y)B(\chi_{G_n}(x-\cdot)f(\cdot),\chi_{G_n}(x-\cdot)g(\cdot))(y)dy\right| =$$

$$=\sup_{n\in\mathbb{Z}}\left|\int_G (\Delta_n(x-y)-\Delta_n(x))B(\chi_{G_n}(x-\cdot)f(\cdot),\chi_{G_n}(x-\cdot)g(\cdot))(y)dy\right| =$$

$$=\sup_{n\in\mathbb{Z}}\left|\int_G \sum_{\gamma=1}^{N}\chi_{G_n}(x-z)f(z)\{(T_\gamma^1)^*\left[\Delta_n(x-\cdot)T_\gamma^2(\chi_{G_n}(x-\cdot)g(\cdot))(\cdot)\right](z)\}dz\right|,$$

where $(T_\gamma^1)^*$ is the adjoint of T_γ^1.

Since $1/p+1/q=1/r<2$, we can choose $1<p_1<p$ and $1<q_1<q$ such that $1/p_1+1/q_1=1+\varepsilon$ for some $0<\varepsilon<1$. Let $1/p_1+1/p_1'=1$. Then $1p_1'=1/q_1-\varepsilon$. We first show

$$\left\|\sum_{\gamma=1}^{N}(T_\gamma^1)^*\left[\Delta_n(x-\cdot)T_\gamma^2(\chi_{G_n}(x-\cdot)g(\cdot))(\cdot)\right]\right\|_{L^{p_1'}(G)} \leq$$

$$\leq cm_n^{1+\varepsilon}\|\chi_{G_n}(x-\cdot)g(\cdot)\|_{L^{q_1}(G)}.$$

In fact, (1.3) implies that

$$0\int_G \sum_{\gamma=1}^{N} T_\gamma^1 f(y)T_\gamma^2(\chi_{G_n}(x-\cdot)g)(y)dy =$$

$$=\sum_{\gamma=1}^{N}\int_G f(y)(T_\gamma^1)^*\left[T_\gamma^2(\chi_{G_n}(x-\cdot)g)\right](y)dy.$$

It follows from the usual density argument that

$$\sum_{\gamma=1}^{N}(T_\gamma^1)^*\left[T_\gamma^2(\chi_{G_n}(x-\cdot)g)\right](y)=0 \quad \text{a.e. on } G.$$

Consequently, we have

$$\left|\sum_{\gamma=1}^{N}(T_\gamma^1)^*\left[\Delta_n(x-\cdot)T_\gamma^2(\chi_{G_n}(x-\cdot)g)\right](z)\right| =$$

$$= \left| \sum_{\gamma=1}^{N} \int_G \bar{K}_\gamma^1(z-y)[\Delta_n(x-y) - \Delta_n(x-z)] T_\gamma^2(\chi_{G_n}(x-\cdot)g)(y)dy \right| \le$$

$$\le c \sum_{\gamma=1}^{N} \int_G \frac{1}{|z-y|} |\Delta_n(x-y) - \Delta_n(x-z)| \, |T_\gamma^2(\chi_{G_n}(x-\cdot)g)(y)| dy.$$

Note that if $x - y \notin G_n$ or $x - z \notin G_n$, then

$$|y - z| = \max(|x-y|, |x-z|) > m_n^{-1}.$$

Thus, $|\Delta_n(x-y) - \Delta_n(x-z)| < m_n^{1+\varepsilon}|z-y|^\varepsilon$. Therefore,

$$\left| \sum_{\gamma=1}^{N} (T_\gamma^1)^* \left[\Delta_n(x-\cdot) T_\gamma^2(\chi_{G_n}(x-\cdot)g) \right] (z) \right| \le$$

$$\le c m_n^{1+\varepsilon} \sum_{\gamma=1}^{N} \int_G \frac{|T_\gamma^2(\chi_{G_n}(x-\cdot)g)(y)|}{|z-y|^{1-\varepsilon}} dy =$$

$$= c m_n^{1+\varepsilon} \sum_{\gamma=1}^{N} I_\varepsilon \left(|T_\gamma^2(\chi_{G_n}(x-\cdot)g)| \right)(z),$$

where I_ε is the fractional integral of order ε. By Lemma 2.1 and Lemma 1.1, we have

$$\left\| \sum_{\gamma=1}^{N} (T_\gamma^1)^* [\Delta_n(x-\cdot) T_\gamma^2(\chi_{G_n}(x-\cdot)g)] \right\|_{L^{p'_1}(G)} \le$$

$$\le c m_n^{1+\varepsilon} \sum_{\gamma=1}^{N} \left\| I_\varepsilon \left(|T_\gamma^2(\chi_{G_n}(x-\cdot)g)| \right) \right\|_{L^{p'_1}(G)} \le$$

$$\le c m_n^{1+\varepsilon} \sum_{\gamma=1}^{N} \| T_\gamma^2(\chi_{G_n}(x-\cdot)g) \|_{L^{q_1}(G)} \le c m_n^{1+\varepsilon} \| \chi_{G_n}(x-\cdot)g \|_{L^{q_1}(G)}.$$

Thus,

$$\sup_{n \in \mathbb{Z}} \left| \int_G \Delta_n(x-y) B(\chi_{G_n}(x-\cdot)f, \chi_{G_n}(x-\cdot)g)(y) dy \right| \le$$

$$\leq c\|\chi_{G_n}(x-\cdot)f\|_{L^{p_1}(G)} m_n^{1+\varepsilon} \|\chi_{G_n}(x-\cdot)g\|_{L^{q_1}(G)} \leq$$
$$\leq c m_n^{1+\varepsilon-1/p_1-1/q_1} [HL(|f|^{p_1}(x)]^{1/p_1} [HL(|g|^{q_1})(x)]^{1/q_1} \leq$$
$$\leq c[HL(|f|^{p_1})(x)]^{1/p_1} [HL(|g|^{q_1})(x)]^{1/q_1}.$$

Therefore, by the boundedness of HL, we have

$$\left\| \sup_{n\in\mathbb{Z}} \left| \int_G \Delta_n(x-y) B(\chi_{G_n}(x-\cdot)f, \chi_{G_n}(x-\cdot)g)(y)dy \right| \right\|_{L^r(G)} \leq$$
$$\leq c\|[HL(|f|^{p_1})]^{1/p_1} [HL(|g|^{q_1})]^{1/q_1}\|_{L^r(G)} \leq$$
$$\leq c\|[HL(|f|^{p_1})]^{1/p_1}\|_{L^p(G)} \|[HL(|g|^{q_1})]^{1/q_1}\|_{L^q(G)} \leq$$
$$\leq c\|f\|_{L^p(G)} \|g\|_{L^q(G)}.$$

This finishes the proof of Theorem 1.1. ∎

We now turn to the proof of Theorem 1.2.

PROOF OF THEOREM 1.2. Fix $1/2 < p \leq 1$ and $f \in H^p(G)$. By the results in [18], we know that $f = \sum_I \lambda_I a_I$, where $\sum_I |\lambda_I|^p < \infty$ and a_I is a (p,∞) atom supported by a coset I. That is,

(i) $\operatorname{supp} a \subset I$;

(ii) $\|a\|_{L^\infty(G)} \leq |I|^{-1/p}$;

(iii) $\int_G a(x)dx = 0$,

Let

$$E(x) = \sup_{n\in\mathbb{Z}} \left| \int_G \Delta_n(x-y) \sum_{\gamma=1}^{N} (T_\gamma^1 f)(y)(T_\gamma^2 g)(y)dy \right|,$$

where $g \in L^2(G)$ has a compact support. We shall show that

$$\left(\int_G E(x)^r dx \right)^{1/r} \leq c\|f\|_{H^p(G)} \|g\|_{L^p(G)}.$$

Suppose $I = \chi_I + G_{n_i}$. Let $I^* = \chi_I + G_{n_I - 3}$. Fix $l \in (1, q)$. Notice that

$$E(x) \leq \sum_{I^* \ni x} |\lambda_I| \sup_{n \in \mathbb{Z}} \left| \int_G \Delta_n(x-y) \sum_{\gamma=1}^{N} (T_\gamma^1 a_I)(y)(T_\gamma^2 g)(y) dy \right| +$$

$$+ \sum_{I^* \not\ni x} |\lambda_I \sup_{n \in \mathbb{Z}} \left| \int_G \Delta_n(x-y) \sum_{\gamma=1}^{N} (T_\gamma^1 a_I)(y)(T_\gamma^2 g)(y) dy \right| \equiv$$

$$\equiv E_1(x) + E_2(x).$$

We estimate $E_1(x)$ first. Let $1/l + 1/l' = 1$. Then,

$$\int_G E_1(x)^r dx \leq$$

$$\leq \sum_{\gamma=1}^{N} \int_G \left(\sum_{I^* \ni x} |\lambda_I| \sup_{n \in \mathbb{Z}} \int_G \sup_{n \in \mathbb{Z}} \int_G \Delta_n(x-y) |(T_\gamma^1 a_I)(y)| \, |(T_\gamma^2 g)(y)| dy \right)^r dy \leq$$

$$\leq c \sum_{\gamma=1}^{N} \int_G \left\{ \sum_{I^* \ni x} |\lambda_I| \left[HL(|T_\gamma^1 a_I|^{l'})(x) \right]^{1/l'})^p dx \right\}^{r/p} \times$$

$$\times \left\{ \int_G \left[HL(|T_\gamma^2 g|^l)(x) \right]^{q/l} dx \right\}^{r/q} \leq$$

$$\leq c \|g\|_{L^q}^r \sum_{\gamma=1}^{N} \left\{ \sum_I |\lambda_I|^p \int_{I^*} \left[HL(|T_\gamma^1 a_I|^{l'})(x) \right]^{p/l'} dx \right\}^{r/p}.$$

By Lemma 2.2, we have

$$\int_{I^*} \left[HL(|T_\gamma^1 a_I|^{l'})(x) \right]^{p/l'} dx \leq c|I|^{1-p/l'} \left(\int_G |T_\gamma^1 a_I(x)|^{l'} dx \right)^{p/l'} \leq$$

$$\leq c|I^*|^{1-p/l'} \left(\int_G |a_I(x)|^{l'} dx \right)^{p/l'} \leq c,$$

where c is a constant independent of I. Therefore,

$$\int_G E_1(x)^r dx \le c\|g\|^r_{L^q(G)} \left\{ \sum_I |\lambda_I|^p \right\}^{r/p}.$$

Thus,

$$\int_G E_1(x)^r dx \le c\|f\|^r_{H^p(G)}\|g\|^r_{L^q(G)},$$

which is a desirable estimate for $E_1(x)$.

Next, we estimate $E_2(x)$. We have

$$E_2(x) \le \sum_{I^*\not\ni x} |\lambda_I| \sup_{n:\chi_I+G_n\not\ni x} \left| \int_G \Delta_n(x-y) \sum_{\gamma=1}^N (T^1_\gamma a_I)(y)(T^2_\gamma g)(y)dy \right| +$$

$$+ \sum_{I^*\not\ni x} |\lambda_I + \sup_{n:\chi_I+G_n\ni x} \left| \int_G \Delta_n(x-y) \sum_{\gamma=1}^N (T^1_\gamma a_I)(y)(T^2_\gamma g)(y)dy \right| \equiv$$

$$\equiv E_{21}(x) + E_{22}(x).$$

Let $\tilde{I} = \chi_I + G_{n_I-1}$. We further decompose $E_{21}(x)$ into

$$E_{21}(x) \le E^1_{21}(x) + E^2_{21}(x),$$

where

$$E^1_{21}(x) = \sum_{I^*\not\ni x} |\lambda_I| \sup_{n:\chi_I+G_n\not\ni x} \left| \int_{\tilde{I}} \Delta_n(x-y) \sum_{\gamma=1}^N (T^1_\gamma a_I)(y)(T^2_\gamma g)(y)dy \right|$$

and

$$E^2_{21}(x) = \sum_{I^*\not\ni x} |\lambda_I| \sup_{n:\chi_I+G_n\not\ni x} \left| \int_{G\setminus\tilde{I}} \Delta_n(x-y) \sum_{\gamma=1}^N (T^1_\gamma a_I)(y)(T^2_\gamma g)(y)dy \right|.$$

We claim that $E_{21}(x) = 0$. In fact, for $x \notin I^*$ and $\Delta_n(x-y)\neq 0$, we have $|y - x_I| = \max(|x-y|, |x-x_I|) = |x-x_I| > m^{-1}_{n_I-3}$ because $x - x_I \notin G_{n_l-3}$. Thus, $y \notin \tilde{I}$. Consequently, $\{y : \Delta_n(x-y)\neq 0\} \cap \tilde{I} = \Phi$, that is, $E^1_{21}(x) = 0$.

To estimate $E_{21}^2(x)$, we let $y \in G \setminus \tilde{I}$. We have

$$
\begin{aligned}
(2.2) \qquad |(T_\gamma^1 a_I)(y)| &= \left| \int_I K_\gamma^1(y-z) a_I(z)\,dz \right| = \\
&= \left| \int_I (K_\gamma^1(y-z) - K_\gamma^1(y-x_I)) a_I(z)\,dz \right| \le \\
&\le c \int_I \frac{|z-x_I|}{|y-x_I|^2} |a_I(z)|\,dz \le c \frac{|I|^{2-1/p}}{|y-x_I|}^2 .
\end{aligned}
$$

Note that if $\Delta_n(x-y) \neq 0$, then $|y-x_I| = \max(|x-x_I|, |x-y|) = |x-x_I|$. Therefore, we have

$$
\begin{aligned}
E_{21}^2(x) \le c \sum_{\gamma=1}^N \sum_{I^* \not\ni x} |\lambda_I| \sup_{n: \chi_I + G_n \not\ni x} \left(\int_{G \setminus \tilde{I}} \Delta_n(x-y) |(T_\gamma^1 a_I)(y)|^{l'} dy \right)^{1/l'} \times \\
\times \left(\int_G \Delta_n(x-y) |(T_\gamma^2 g)(y)|^l dy \right)^{1/l} \le \\
\le c \sum_{\gamma=1}^N \sum_{I^* \not\ni x} |\lambda_I| \frac{|I|^{2-1/p}}{|x-x_I|^2} \left[HL(|T_\gamma^2 g|^l)(x) \right]^{1/l} .
\end{aligned}
$$

It follows that

$$
\begin{aligned}
\int_G (E_{21}^2(x))^r dx \le c \sum_{\gamma=1}^N \int_G \left(\sum_{I^* \not\ni x} |\lambda_I| \frac{|I|^{2-1/p}}{|x-x_I|^2} \right)^r \left[HL(|T_\gamma^2 g|^l)(x) \right]^{r/l} dx \le \\
\le c \sum_{\gamma=1}^N \left\{ \int_G \left(\sum_{I^* \not\ni x} |\lambda_I| \frac{|I|^{2-1/p}}{|x-x_I|^2} \right)^p dx \right\}^{r/p} \times \\
\times \left\{ \int_G \left[HL(|T_\gamma^2 g)|^l)(x) \right]^{q/l} dx \right\}^{r/q} \le
\end{aligned}
$$

$$\leq c\|g\|^r_{L^q(G)} \left\{ \sum_I |\lambda_I|^p \int\limits_{x-x_I \notin G_{n_I-3}} \frac{|I|^{2p-1}}{|x-x_I|^{2p}} dx \right\}^{r/p} =$$

$$= c\|g\|^r_{L^q(G)} \left\{ \sum_{l=-\infty}^{n_I-4} \int\limits_{G_l \setminus G_{l+1}} \frac{|I|^{2p-2}}{|x-x_I|^{2p}} dx \right\}^{r/p} =$$

$$= c\|g\|^r_{L^q(G)} \left\{ \sum_I |\lambda_I|^p \left(\sum_{l=-\infty}^{n_I-4} m_l^{2p-1} \right) |I|^{2p-1} \right\}^{r/p} \leq$$

$$\leq c\|g\|^r_{L^q(G)} \left\{ \sum_I |\lambda_I|^p \right\}^{r/p}.$$

where the last inequality follows from (1.2) and $2p-1>0$. Thus,

$$\int\limits_G [E^2_{21}(x)]^r dx \leq c\|f\|^r_{H^p(G)} \|g\|^r_{L^q(G)}.$$

Next, we discuss $E_{22}(x)$. Let $s \in (0,1)$. It follows from (1.3) that

$$E_{22}(x) =$$

$$= \sum_{I^* \not\ni x} |\lambda_I| \sup_{n: \chi_I + G_n \ni x} \left| \int\limits_G (\Delta_n(x-y) - \Delta_n(x-x_I)) \sum_{\gamma=1}^N (T^1_\gamma a_I)(y)(T^2_\gamma(g)(y) dy) \right| \leq$$

$$\leq c \sum_{\gamma=1}^N \sum_{I^* \not\ni x} |\lambda_I| \sup_{n: \chi_I + G_n \ni x} \int\limits_G m_n^{1+s} |y-x_I|^s |(T^1_\gamma a_I)(y)(T^2_\gamma g)(y)| dy =$$

$$= c \sum_{\gamma=1}^N \sum_{I^* \not\ni x} |\lambda_I| \sup_{n: \chi_I + G_n \ni x} m_n^{1+s} \int\limits_{\tilde{I}} |y-x_I|^s |(T^1_\gamma a_I)(y)(T^2_\gamma g)(y)| dy +$$

$$+ c \sum_{\gamma=1}^N \sum_{I^* \not\ni x} |\lambda_I| \sup_{n: \chi_I + G_n \ni x} m_n^{1+s} \int\limits_{G \setminus \tilde{I}} |y-x_I|^s |(T^1_\gamma a_I)(y)(T^2_\gamma g)(y)| dy \equiv$$

$$\equiv E^1_{22}(x) + E^2_{22}(x).$$

We note that in $E_{22}^1(x)$, we have $|x-y| = \max(|y-x_I|, |x-x_I|) = |x-x_I| <$ $< m_n^{-1}$ because $y \in \tilde{I}$ and $x \notin \chi_I + G_{n_I-3}$. Therefore, by Hölder's inequality, we have

$$(2.3)\; E_{22}^1(x) \le c \sum_{\gamma=1}^{N} \sum_{I^* \not\ni x} |\lambda_I| \sup_{n:\chi_I+G_N\ni x} \int_{\tilde{I}} \left\{ m_n^{s+1/l'} |y-x_I|^s ||(T_\gamma^1 a_I)(y)| \right\} \times$$

$$\times \{ m_n^{1/l} |(T_\gamma^2 g)(y)| \} dy \le$$

$$\le c \sum_{\gamma=1}^{N} \sum_{I^* \not\ni x} |\lambda_I| \frac{|I^*|}{|x-x_I|^{s+1/l'}} \left(\int_G |(T_\gamma^1 a)(y)|^{l'} dy \right)^{1/l'} \times$$

$$\times \sup_{n:\chi_I+G_n\ni x} \left(\int_{|y-x|<m_n^{-1}} m_n |(T_\gamma^2 g)(y)|^l dy \right)^{1/l} \le$$

$$\le c \sum_{\gamma=1}^{N} \sum_{I^* \not\ni x} |\lambda_I| \frac{|I|^{s-1/p+1/l'}}{|x-x_I|^{s+1/l'}} \left[HL(|T_\gamma^2 g|^l)(x) \right]^{1/l}.$$

Since $r > 1/2$, we can choose $s \in (0,1)$ such that $(s+1/l')p > 1$. It is now easy to see that

$$\int_G [E_{22}^1(x)]^r dx \le c \|f\|_{H_p(G)}^r \|g\|_{L^q(G)}^r.$$

Finally, we come to the estimate of $E_{22}^2(x)$. Using (2.2), we easily obtain

$$E_{22}^2(x) \le$$

$$\le c \sum_{\gamma=1}^{N} \sum_{I^* \not\ni x} |\lambda_I| \sup_{n:\chi_I+G_n\ni x} m_n^{1+s} \int_{G\setminus\tilde{I}} \frac{|y-x_I|^s |I|^{2-1/p}}{|y-x_I|^2} |(T_\gamma^2 g)(y)| dy \le$$

$$\le c \sum_{\gamma=1}^{N} \sum_{I^* \not\ni x} |\lambda_I| \sup_{n:\chi_I+G_n\ni x} m_n^{1+s} \int_{\substack{y\in G\setminus\tilde{I} \\ y\in x+G_{n-1}}} \frac{|I|^{2-1/p}}{|y-x_I|^{2-s}} |(T_\gamma^2 g)(y)| dy +$$

$$+ c\sum_{\gamma=1}^{N}\sum_{I^*\not\ni x}|\lambda_I| \sup_{n:\chi_I+G_n\ni x} m_n^{1+s} \int\limits_{\substack{y\in G\setminus\tilde I\\ y\notin x+G_{n-1}}} \frac{|I|^{2-1/p}}{|y-x_I|^{2-s}}|(T_\gamma^2 g)(y)dy \equiv$$

$$\equiv F_1(x) + F_2(x).$$

For $F_1(x)$, we first have

$$m_n^{1+s} \int\limits_{\substack{y\in G\setminus\tilde I\\ y\in x+G_{n-1}}} \frac{|I|^{2-1/p}}{|y-x_I|^{2-s}}|(T_\gamma^2 g)(y)dy \leq$$

$$\leq m_n^{s+1/l'}|I|^{2-1/p}\left(\int\limits_{G\setminus\tilde I}\frac{1}{|y-x_I|^{(2-s)l'}}dy\right)^{1/l'} \times \left(m_n\int\limits_{x+G_{n-1}}|(T_\gamma^2 g)(y)|^l dy\right)^{1/l} \leq$$

$$\leq c\frac{|I|^{2-1/p+1/l'}}{|x-x_I|^{s+1/l'}}\left[HL(|T_\gamma^2 g|^l)(x)\right]^{1/l}.$$

Then, by a computation similar to (2.3), we obtain an estimate of $F_1(x)$ similar to that of $E_{22}^1(x)$. For $F_2(x)$, note that $|y-x_I| = \max(|y-x|, |x-x_I|) > |y-x|$. By Hölder's inequality, we have

$$|I|^{2-1/p}m_n^{1+s}\left(\int\limits_{G\setminus\tilde I}\frac{dy}{|y-x_I|^{2-s}}\right)^{1/l'}\left(m_n\int\limits_{y\notin x+G_{n-1}}\frac{|(T_\gamma^2 g)(y)|^l}{|y-x|^{2-s}}dy\right)^{1/l} \leq$$

$$\leq c\frac{|I|^{1-1/p+1/l+s/l']}}{|x-x_I|^{1+1/l+s/l'}}\left[HL(|T_\gamma^2 g|^l)(x)\right]^{1/l}.$$

Note that $(1+1/l+s/l')p = ((2-s)/l+s+1/l')p > (s+1/l')p > 2$. From this, we easily deduce a desirable estimate for $F_2(x)$.

This finishes the proof of Theorem 1.2. ∎

3. Some generalizations

In this section, I_α denote the standard fractional integral operator if $0 < \alpha < 1$ and the Calderón–Zygmund operators if $\alpha = 0$. For $x \in G$, let

$$B_\alpha(f,g)(x) = \sum_{\gamma=1}^{N} c_\gamma I_{\alpha_1^\gamma}(f)(x) I_{\alpha_2^\gamma}(g)(x), \tag{3.1}$$

where c_γ's are constants, $\alpha = \alpha_1^\gamma + \alpha_2^\gamma$ and $0 \le \alpha_1^\gamma, \alpha_2^\gamma < 1$ for $y \in \{1, \dots, N\}$ and $N \in \mathbb{N}$.

Theorem 1.1 and Theorem 1.2 have the following generalizations whose proofs are omitted as they involve only arguments similar to those of Theorem 1.1 and Theorem 1.2; see also [17].

THEOREM 3.1. *Let $p, q > 1$, $1/r = 1/p + 1/q - \alpha$, $\alpha = \alpha_1^\gamma + \alpha_2^\gamma$, $0 < \delta \le 1$, $0 \le \alpha_1^\gamma < 1/p$ and $0 < \alpha_2^\gamma < 1/q$ for $\gamma = 1, \dots, N$. Assume that*

$$\int_G B_\alpha(f,g)(x)dx = 0 \tag{3.2}$$

for all f, $g \in L^2(G)$ with compact supports. Then, for $\alpha + \delta \le 1$ and $1/(1+\delta) < r \le 1$, B_α can be extended to a bounded linear operator from $L^p(G) \times L^q(G)$ into $H^r(G)$.

THEOREM 3.2. *Let $0 < p \le 1$, $q > 1$, $1/r = 1/p + 1/q - \alpha$, $\alpha = \alpha_1^\gamma + \alpha_2^\gamma$, $1/2 < r \le 1$ and $0 \le \alpha_2^\gamma < 1/q$ for $\gamma = 1, \dots, N$. Assume that $B_\alpha(f,g)$ satisfies* (3.2). *Then B_α can be extended to a bounded linear operator from $H^p(G) \times L^q(G)$ into $H^r(G)$, if $0 \le \alpha < 1/q$ or $1/2 < p \le 1$.*

Our next theorem is more interesting.

THEOREM 3.3. *Let $0 < p, q \le 1$, $1/r = 1/p + 1/q - \alpha$, $\alpha = \alpha_1^\gamma + \alpha_2^\gamma$ and $0 < \alpha < 1$. Assume that $B_\alpha(f,g)$ satisfies* (3.2). *Then, for $1/2 < r \le 1$, B_α can be extended to a bounded linear operator from $H^p(G) \times H^q(G)$ into $H^r(G)$.*

PROOF. We consider the case α_1^γ, $\alpha_2^\gamma > 0$ only. The other cases are similar. Let $f \in H^p(G)$ and $g \in H^q(G)$. By the results in [18], we can write

$f = \sum_I \lambda_I a_I$ and $g = \sum_J \mu_J b_J$, where $\lambda_I,\ \mu_J > 0$, a_I's are (p,∞)-atoms and b_J's are (q,∞)-atoms. Define

$$S_\alpha(a_I, b_J)(x) = \sup_{n\in\mathbb{Z}} \left| \int_G \Delta_n(x-y) B_\alpha(a_I, b_J)(y)dy \right|.$$

Then

$$\begin{aligned} M(B_\alpha(f,g))(x) &\le \sum_{I,J} \lambda_I \mu_J S_\alpha(a_I, b_J)(x) = \\ &= \sum_{\substack{I,J \\ I^*\ni x, J^*\ni x}} \dots + \sum_{\substack{I,J \\ I^*\ni x, J^*\not\ni x}} \dots + \sum_{\substack{I,J \\ I^*\not\ni x, J^*\ni x}} \dots + \sum_{\substack{I,J \\ I^*\not\ni x, J^*\not\ni x}} \equiv \\ &\equiv A_1 + A_2 + A_3 + A_4, \end{aligned}$$

where I^*, J^* have the same meanings as in Section 2.

We first estimate $A_1(x)$. Let $1/r_1^\gamma = 1/p - \alpha_1^\gamma$ and $1/r_2^\gamma = 1/q - \alpha_2^\gamma$. Since $0 < \alpha < 1$ implies $1/(1-\alpha_2^\gamma) < 1/\alpha_1^\gamma$, we can choose l_γ satisfying $1/(1-\alpha_2^\gamma) < l_\gamma < 1/\alpha_1^\gamma$. Let $1/l'_\gamma + 1/l_\gamma = 1$, $1/r_1^\gamma - 1/p = -\alpha_1^\gamma = 1/l'_\gamma - 1/p_1^\gamma$ and $1/r_2^\gamma - 1/q = -\alpha_2^\gamma = 1/l_\gamma - 1/q_1^\gamma$. It is easy to see that $p_1^\gamma,\ q_1^\gamma > 1$ and $l'_\gamma > r_1^\gamma$, $l_\gamma > r_2^\gamma$. From Hölder's inequality, we deduce

$$\begin{aligned} S_\alpha(a_I, b_J)(x) &\le c \sup_{n\in\mathbb{Z}} \sum_{\gamma=1}^N \left| \int_G \Delta_n(x-y) I_{\alpha_1^\gamma} a_I(y) I_{\alpha_2^\gamma} b_J(y) dy \right| \le \\ &\le c \sup_{n\in\mathbb{Z}} \sum_{\gamma=1}^N \left(\int_G \Delta_n(x-y) |I_{\alpha_1^\gamma} a_I(y)|^{l'_\gamma} dy \right)^{1/l'_\gamma} \times \\ &\quad \times \left(\int_G \Delta_n(x-y) |I_{\alpha_2^\gamma} b_J(y)|^{l_\gamma} dy \right)^{1/l_\gamma} \le \\ &\le c \sum_{\gamma=1}^N [HL(|I_{\alpha_1^\gamma} a_I|^{l'_\gamma})(x)]^{1/l'_\gamma} [HL(|I_{\alpha_2^\gamma} b_J|^{l_\gamma})(x)]^{1/l_\gamma}. \end{aligned}$$

Therefore, by Hölder's inequality and Minkowski's inequality, we have

$$\int_G [A_1(x)]^r dx \le c \sum_{\gamma=1}^N \int_G \left\{ \sum_{\substack{I,J \\ I^* \cap J^* \ni x}} \lambda_I \mu_J [HL(|I_{\alpha_1^\gamma} a_I|^{l'_\gamma})(x)]^{1/l'_\gamma} \times \right.$$

$$\left. \times [HL(|I_{\alpha_2^\gamma} b_J|^{l_\gamma})(x)]^{1/l_\gamma} \right\} dx \le$$

$$\le c \sum_{\gamma=1}^N \int_G \left\{ \sum_{I: I^* \ni x} \lambda_I [HL(|I_{\alpha_1^\gamma} a_I|^{l'_\gamma})(x)]^{1/l'_\gamma} \right\}^r \times$$

$$\times \left\{ \sum_{J: J^* \ni x} \mu_J [HL(|I_{\alpha_2^\gamma} b_J|^{l_\gamma} (x)]^{1/l_\gamma} \right\}^r dx \le$$

$$\le c \sum_{\gamma=1}^N \left[\int_G \left\{ \sum_{I: I^* \ni x} \lambda_I [HL(|I_{\alpha_1^\gamma} a_I|^{l'_\gamma})(x)]^{1/l'_\gamma} \right\}^{r_1^\gamma} dx \right]^{r/r_1^\gamma} \times$$

$$\times \left[\int_G \left\{ \sum_{J: J^* \ni x} \mu_j [HL(|I_{\alpha_2^\gamma} a_I|^{l'_\gamma})(x)]^{r_1^\gamma / l_\gamma} \right\}^{r_2^\gamma} dx \right]^{r/r_2^\gamma} \le$$

$$\le c \sum_{\gamma=1}^N \left[\sum_I |\lambda_I|^p \left\{ \int_{I^*} [HL(|I_{\alpha_1^\gamma} a_I|^{l'_\gamma})(x)]^{r_1^\gamma / l'_\gamma} dx \right\}^{p/r_1^\gamma} \right]^{r/p} \times$$

$$\times \left[\sum_J |\mu_J|^q \left\{ \int_{J^*} [HL(|I_{\alpha_2^\gamma} b_J|^{l_\gamma})(x)]^{r_2^\gamma / l_\gamma} dx \right\}^{q/r_2^\gamma} \right]^{r/q} .$$

By Lemma 2.1, we have

$$\int_{I^*} [HL(|I_{\alpha_1^\gamma} a_I|^{l'_\gamma})(x)]^{r_1^\gamma / l'_\gamma} dx \le \tag{3.3}$$

$$\leq c|I^*|^{1-r_1^\gamma/l_\gamma'} \left[\int\limits_G |_{\alpha_1^\gamma} a_I(x)|^{l_\gamma'} dx\right]^{r_1^\gamma/l_\gamma'} \leq$$

$$\leq c|I|^{1-r_1^\gamma/l_\gamma'} \|a_I\|_{L^{p_1^\gamma}(G)}^{r_1^\gamma} \leq c,$$

where c is independent of I. Similarly, we have

$$\int\limits_{J^*} [HL(|I_{\alpha_2^\gamma} b_J|^{l_\gamma})(x)]^{r_2^\gamma/l_\gamma} dx \leq c.$$

Thus,

$$\int\limits_G [A_1(x)]^r dx \leq c \left\{\sum_I |\lambda_I|^p\right\}^{r/p} \left\{\sum_J |\mu_J|^q\right\}^{r/q}.$$

Therefore,

$$\int\limits_G [A_1(x)]^r dx \leq c\|f\|_{H^p(G)}^r \|g\|_{H^q(G)}^r.$$

For $A_2(x)$, let $J = \chi_J + G_{n_J}$. Then we have

$$S_\alpha(a_I, b_J)(x) \leq \sup_{n:\chi_J+G_n \not\ni x} \left|\int\limits_G \Delta_x(x-y) B_\alpha(a_I, b_J)(y)dy\right| +$$

$$+ \sup_{n:\chi_J+G_n \ni x} \left|\int\limits_G \Delta_n(x-y) B_\alpha(a_I, b_J)(y)dy\right| \equiv D_1(x) + D_2(x).$$

We consider $D_1(x)$ first. We have

$$D_1(x) \leq \sup_{n:\chi_J+G_n \not\ni x} \left|\int\limits_{\tilde{J}} \Delta_n(x-y) B_\alpha(a_I, b_J)(y)dy\right| +$$

$$+ \sup_{n:\chi_J+G_n \not\ni x} \left|\int\limits_{G\setminus\tilde{J}} \Delta_n(x-y) B_\alpha(a_I, b_J)(y)dy\right| \equiv D_{11}(x) + D_{12}(x).$$

Similar to the proof of $E_{21}^1(x)$, we can show that $D_{11}(x) = 0$. For $D_{12}(x)$, we first note that for $y \in G \setminus \tilde{J}$,

$$I_{\alpha_2^\gamma} b_J(y) = c \int\limits_J \frac{b_J(x)}{|y-z|^{1-\alpha_2^\gamma}} dz = \tag{3.4}$$

$$= c \int\limits_J \left[\frac{1}{|y-z|^{1-\alpha_2\gamma}} - \frac{1}{|y-x_J|^{1-\alpha_2^\gamma}} \right] b_J(z)dz = 0,$$

since $|y-z| = \max(|y-x_J|, |x_J - z|) = |y - x_J|$. Thus, $D_{12}(x) = 0$. Therefore $D_1(x) = 0$.

For $D_2(x)$, we let $s > 0$ and use (3.2) to obtain

$$D_2(x) = \sup_{n: \chi_J + G_n \ni x} \left| \int\limits_G \Delta_n(x-y) B_\alpha(a_I, b_J)(y)dy \right| \le$$

$$\le c \sup_{n: \chi_J + G_n \ni x} \sum_{\gamma=1}^N \int\limits_G |\Delta_n(x-y) - \Delta_n(x - x_J)|\, |I_{\alpha_1^\gamma} a_I(y)|\, |I_{\alpha_2^\gamma} b_J(y)|dy \le$$

$$\le c \sum_{\gamma=1}^N \sup_{n: \chi_J + G_n \ni x} \int\limits_G m_n^{1+s} |y-x_J|^s |I_{\alpha_1^\gamma} a_I(y)|\, |I_{\alpha_2^\gamma} b_J(y)|dy \le$$

$$\le c \sum_{\gamma=1}^N \sup_{n: \chi_J + G_n \ni x} \int\limits_{y \notin x + G_{n-2}} \ldots + c \sum_{\gamma=1}^N \sup_{n: \chi_J + G_n \ni x} \int\limits_{y \notin x + G_{n-2}} \ldots \equiv D_{21}(x) + D_{22}(x).$$

For $D_{21}(x)$, we note that $y \notin x + G_{n_2}$ and $x \in x_J + G_n$ imply $|y - x_J| = |y - x|$; thus $x \notin J^*$ further implies that $y \notin \tilde{J}$. Consequently we deduce from (3.4) that $D_{21}(x) = 0$.

Now, we estimate $D_{22}(x)$. We further decompose $D_{22}(x)$ into

$$D_{22}(x) \le c \sum_{\gamma=1}^N \sup_{n: \chi_J + G_n \ni x} \int\limits_{\tilde{J} \cap (x + G_{n-2})} m_n^{1+s} |y-x_J|^s |I_{\alpha_1^\gamma} a_I(y)|\, |I_{\alpha_2^\gamma} b_J(y)|dy +$$

$$+ c \sum_{\gamma=1}^N \sup_{n: \chi_J + G_n \ni x} \int\limits_{(G \setminus \tilde{J}) \cap (x + G_{n-2})} m_n^{1+s} |y-x_J|^s |I_{\alpha_1^\gamma} a_I(y)|\, |I_{\alpha_2^\gamma} b_J(y)|dy \equiv$$

$$\equiv D_{22}^1(x) + D_{22}^2(x).$$

Similar to the estimate for $D_{12}(x)$, we can easily show that $D_{22}^2(x) = 0$. So, we only need to estimate $D'_{22}(x)$. Since $y \in \tilde{J}$ and $x \notin J^*$, we have

$$m_n \le c/[y - x| = c/\max(|y - x_J|, |x - x_J|) = c/|x - x_J|.$$

From this and Lemma 2.1, we obtain

$$D_{22}^1(x) \le c \sum_{\gamma=1}^{N} |x - x_J|^{-1/l_\gamma - 2} \left(\sup_n m_n \int\limits_{x+G_{n-2}} |I_{\alpha_1^\gamma} a_I(y)|^{l'_\gamma} dy \right)^{1/l'_\gamma} \times$$

$$\times \left(\int\limits_{\tilde{J}} |y - x_J|^{l_\gamma s} |I_{\alpha_2 \gamma} b_J(y)|^{l_\gamma} dy \right)^{1/l_\gamma} \le$$

$$\le c \sum_{\gamma=1}^{N} |J|^s |x - x_J|^{-1/l_\gamma - s} \left[HL(|I_{\alpha_1^\gamma} a_I|^{l'_\gamma})(x) \right]^{1/l'_\gamma} \|I_{\alpha_2^\gamma} b_J\|_{L^{l_\gamma}(G)} \le$$

$$\le c \sum_{\gamma=1}^{N} |J|^s |x - x_J|^{-1/l_\gamma - s} \|b_J\|_{L^{q_1^\gamma}(G)} \left[HL(|I_{\alpha_1^\gamma} a_I|^{l'_\gamma})(x) \right]^{1/l'_\gamma} \le$$

$$\le c \sum_{\gamma=1}^{N} |J|^{s+1/l_\gamma - 1/r_2^\gamma} |x - x_J|^{-1/l_\gamma - 2} \left[HL(|I_{\alpha_1^\gamma} a_I|^{l'_\gamma})(x) \right]^{1/l'_\gamma}.$$

Note that $1/r_1^\gamma + 1/r_2^\gamma = 1/r$ and

$$\sum_{\gamma=1}^{N} \int\limits_{x \notin J^*} |J|^{(s+1/l_\gamma - 1/r_2^\gamma} |x - x_J|^{-(s+1/l_\gamma) r_2^\gamma} dx \le c,$$

if $s > 1/r_1^\gamma - 1/l'_\gamma$. Therefore,

$$\|D_{22}^1\|_{L^r(G)} \le c \|f\|_{H^p(G)} \|g\|_{H^q(G)}.$$

So far, we have obtained a desirable estimate for $A_2(x)$.

The estimate for A_3 is similar to that for A_2. If $s > 1/r_2^\gamma - 1/l_\gamma$, we can obtain the desired conclusion for A_3 just as we did for A_2.

Finally, we estimate A_4. Denote

$$\{n \in \mathbb{Z} : x_I + G_n \not\ni x, \ c_J + G_n \not\ni x\},$$

$$\{n \in \mathbb{Z} : x_I + G_n \not\ni x,\ c_J + G_n \ni x\},$$
$$\{n \in \mathbb{Z} : x_I + G_n \ni x,\ c_J + G_n \not\ni x\},$$
$$\{n \in \mathbb{Z} : x_I + G_n \ni x,\ c_J + G_n \ni x\},$$

respectively, by Λ_1, Λ_2, Λ_3, and Λ_4. Then we have

$$\sup_{n\in\mathbb{Z}} \left| \int_G \Delta_n(x-y) B_\alpha(a_I, b_J)(y)dy \right| \leq$$
$$\leq \sup_{\Lambda_1} |\ldots| + \sup_{\Lambda_2} |\ldots| + \sup_{\Lambda_3} |\ldots| + \sup_{\Lambda_4} |\ldots| \equiv$$
$$\equiv H_1(x) + H_2(x) + H_3(x) + H_4(x).$$

We first estimate $H_1(x)$. By (3.4), we have

$$H_1(x) \leq \sup_{\Lambda_1} \left| \int_{\tilde{I}\cap\tilde{J}} \Delta_n(x-y) \sum_{\gamma=1}^{N} c_\gamma I_{\alpha_1^\gamma} a_I(y) I_{\alpha_2^\gamma} b_J(y) dy \right|.$$

Note that $H(x)\neq 0$ only when $x - y \in G_n$. Thus, $m_n \leq 1/|x-y| = 1/|x-x_I|$ and similarly $m_n \leq 1/|x-x_J|$ since $y \in \tilde{I}\cap\tilde{J}$ and $x \notin I^* \cap J^*$. When $|J| \leq |I|$, noting that $\Delta_n(x-x_J) \equiv 0$ and by Lemma 2.1, we have

$$H_1(x) \leq \sup_{\Lambda_1} \left| \int_{\tilde{I}\cap\tilde{J}} (\Delta_n(x-y) - \Delta_n(x-x_J)) \sum_{\gamma=1}^{N} c_\gamma I_{\alpha_1^\gamma} a_I(y) I_{\alpha_2^\gamma} b_J(y) dy \right| \leq$$

$$\leq c \sum_{\gamma=1}^{N} \sup_{\Lambda_1} \int_{\tilde{I}\cap\tilde{J}} m_n^{2+s} |y-x_J|^{s+1} |I_{\alpha_1^\gamma} a_I(y))|\ |I_{\alpha_2^\gamma} b_J(y)| dy \leq$$

$$\leq c \sum_{\gamma=1}^{N} \sup_{\Lambda_1} \int_{\tilde{I}\cap\tilde{J}} |I|^{s_1+1} m_n^{1/l'_\gamma+s_1+1} |I_{\alpha_1^\gamma} a_I(y)| |J|^{s_2+1} m_n^{1/l_\gamma+s_2+1} |I_{\alpha_2^\gamma} b_J(y)| dy \leq$$

$$\leq c \sum_{\gamma=1}^{N} |x-x_J|^{-1/l'_\gamma - s_1 - 1} \left(\int_{\tilde{I}} |I|^{l'_\gamma(x_1+1)} |I_{\alpha_1^\gamma} a_I(y)|^{l'_\gamma} dy \right)^{1/l'_\gamma} \times$$

$$\times |x-x_J|^{-1/l'_\gamma - s_2 - 1} \left(\int_{\tilde{J}} |J|^{l_\gamma(s_2+1)} |I_{\alpha_2^\gamma} b_J(y)|^{l_\gamma} dy \right)^{1/l_\gamma} \leq$$

$$\leq c \sum_{\gamma=1}^{N} |I|^{1/l'_\gamma - 1/r_1^\gamma + s_1 + 1} |x - x_I|^{-1/l'_\gamma - s_1 - 1} \times$$
$$\times |J|^{1/l_\gamma - 1/r_2^\gamma + s_2 + 1} |x - x_J|^{-1/l_\gamma - s_2 - 1},$$

where l_γ and l'_γ are as before and $s = s_1 + s_2 + 1$, $s_1 > 1/r_1^\gamma - 1/l'_\gamma - 1$, and $s_2 > 1/r_2^\gamma - 1/l_\gamma - 1$. From this, we easily obtain a desirable estimate for $H_1(x)$.

If $|J| \geq |I|$, using $\Delta_n(x - x_I)$ instead of $\Delta_n(x - x_J)$ in the above estimate, we can obtain the same estimate for $H_1(x)$; and therefore, we have finished the estimate for $H_1(x)$.

Now, we estimate $H_4(x)$. First we suppose that $|J| \leq |I|$. By (3.2) and (3.4), we obtain

$$H_4(x) \leq c \sum_{\gamma=1}^{N} \sup_{\Lambda_4} \left| \int_G (\Delta_n(x - y) - \Delta_n(x - x_J)) I_{\alpha_1^\gamma} a_I(y) I_{\alpha_2^\gamma} b_J(y) dy \right| \leq$$
$$\leq c \sum_{\gamma=1}^{N} \sup_{\Lambda_4} \int_G m_n^{2+s} |y - x_J|^{s+1} |I_{\alpha_1^\gamma} a_I(y) I_{\alpha_2^\gamma} b_J(y)| dy \leq$$
$$\leq c \sum_{\gamma=1}^{N} \sup_{\Lambda_4} \int_{\tilde{I} \cap \tilde{J}} m_n^{2+s} |y - x_J|^{s+1} |I_{\alpha_1^\gamma} a_I(y) I_{\alpha_2^\gamma} b_J(y)| dy \leq$$
$$c \sum_{\gamma=1}^{N} |I|^{s_1+1} |x - x_I|^{-1/l'_\gamma - s_1 - 1} \left(\int_G |I_{\alpha_1^\gamma} a_I(y)|^{l'_\gamma} dy \right)^{1/l'_\gamma} \times$$
$$\times |J|^{s_2+1} |x - x_J|^{-1/l_\gamma - s_2 - 1} \left(\int_G |I_{\alpha_2^\gamma} b_J(y)|^{l_\gamma} dy \right)^{1/l_\gamma} \leq$$
$$\leq c \sum_{\gamma=1}^{N} |I|^{1/l'_\gamma - 1/r_1\gamma + s_1 + 1} |x - x_I|^{-1/l'_\gamma - s_1 - 1} \times$$
$$\times |J|^{1/l_\gamma - 1/r_2^\gamma + s_2 + 1} |x - x_J|^{-1/l_\gamma - s_2 - 1},$$

which is the desired estimate.

When $|I| < |J|$, the same estimates follow by symmetry. This finish the estimate for $H_4(x)$.

Since the estimate for $H_2(x)$ and $H_3(x)$ are similar, we only estimate $H_2(x)$. By (3.2) and (3.4), we have

$$H_2(x) \le c \sum_{\gamma=1}^{N} \sup_{\Lambda_2} \left| \int_G (\Delta_n(x-y) - \Delta_n(x-x_J)) I_{\alpha_1^\gamma} a_I(y) I_{\alpha_2^\gamma} b_J(y) dy \right| \le$$

$$\le c \sum_{\gamma=1}^{N} \sup_{\Lambda)_2} \int_G m_n^{2+s} |y-x_J|^{s+1} |I_{\alpha_1^\gamma} a_I(y) I_{\alpha_2^\gamma} b_J(y)| dy \le$$

$$\le c \sum_{\gamma=1}^{N} \sup_{\Lambda_2} \int_{\tilde{I}\cap\tilde{J}} m_n^{2+s} |y-x_J|^{s+1} |I_{\alpha_1^\gamma} a_I(y) I_{\alpha_2^\gamma} b_J(y)| dy.$$

We claim that if $|J| \le |I|$, then every I appearing in the last term satisfies $m_n^{-1} \ge |I|$; otherwise we would have

$$|x-x_J| = \max(|x-x_I|, |x_I-x_J|) = |x-x_I| \ge m_{n_I-3}^{-1} < m_{n_I}^{-1} = |I| > m_n^{-1},$$

which contradicts the fact $x \in x_I + G_n$. Thus, letting $x_1 \in \tilde{I} \cap \tilde{J}$, we have

$$|x-x_I| \le \max(|x-x_J|, |x_J-x_I|) \le \max(|x-x_J|, |x_J-x_1|, |x_1-x_I|) \le Bm_n^{-1}.$$

If $|I| \le |J|$, then $x \in x_J + G_n$ and $x \notin J^*$ imply that $|J| \le m_n^{-1}/8$. Thus, $|x-x_I| \le \max(|x-x_J|, |x_J-x_I|) \le Bm_n^{-1}$.

Therefore, in both cases, we have

$$H_2(x) \le c \sum_{\gamma=1}^{N} \int_{\tilde{I}\cap\tilde{J}} \left\{ |y-x_J|^{s_1+1} |x-x_I|^{-1/l'_\gamma - s_1 - 1} |I_{\alpha_1^\gamma} a_I(y)| \right\} \times$$

$$\times \left\{ |y-x_J|^{s_2+1} |x-x_J|^{-1/l_\gamma - s_2 - 1} |I_{\alpha_2^\gamma} b_J(y)| \right\} dy \le$$

$$\le c \sum_{\gamma=1}^{N} |x-x_I|^{-1/l'_\gamma - s_1 - 1} \left[\int_{\tilde{I}\cap\tilde{J}} (|y-x_J|^{s_1+1} |I_{\alpha_2^\gamma} a_I(y)|)^{l'_\gamma} \right]^{1/l'_\gamma} \le$$

$$\times\, |x - x_J|^{-1/l_\gamma - s_2 - 1} \left[\int\limits_{\tilde{I}\cap\tilde{J}} (|y - x_J|^{s_2+1} |I_{\alpha_2^\gamma} b_J(y)|)^{l_\gamma}\, dy \right]^{1/l_\gamma} \leq$$

$$\leq c \sum_{\gamma=1}^{N} |I|^{1/l'_\gamma - 1/r_1^\gamma + s_1 + 1} |x - x_I|^{-1/l'_\gamma - s_2 - 1} \times$$

$$\times\, |J|^{1/l_\gamma - 1/r_2^\gamma + s_2 + 1} |x - x_J|^{-1/l_\gamma - s_2 - 1},$$

which is what we want.

This finishes the proof of Theorem 3.3. ∎

4. Some applications

The following factorization of Hardy spaces is motivated by [6, 14]; see also [3].

THEOREM 4.1. *Let $0 < p, q \leq 1$ and let $1/2 < r \leq 1$ be such that $1/r = 1/p + 1/q - \alpha$ for some $0 < \alpha < 1$. Then for $f \in H^r(G)$, there exist sequences $\{g_i\} \in H^p(G)$ and $\{h_i\} \in H^q(G)$ such that*

$$f = \sum_{i=1}^{\infty} (h_i I_\alpha g_i - g_i I_\alpha h_i)$$

and

$$\|f\|_{H^r(G)} \simeq \inf \left\{ \sum_{i=1}^{\infty} \|g_i\|_{H^p(G)} \|h_i\|_{H^q(G)} : f = \sum_{i=1}^{\infty} (h_i I_\alpha g_i - g_i I_\alpha h_i) \right\}$$

PROOF. It suffices to prove the result for f being an r-atom whose support is contained in $x_0 + G_k$ for some $x_0 \in G$ and $k \in \mathbb{Z}$. Then $\|f\|_{L^\infty(G)} \leq c m_k^{1/r}$. Given $N \in \mathbb{N}$, choose y_0 and y_1 in G such that $|y_0 - x_0| > N m_k^{-1}$ and $|y_1 - x_0| > N m_{k-1}^{-1}$. Define the function h on G by $h(x) = N^{1-\alpha}(\chi_{y_0 + G_k} - - \chi_{y_1 + G_k})$. Then we have

$$|I_\alpha(h)(x)| = c_\alpha \left| \int\limits_G \frac{h(z)}{|x_0 - z|^{1-\alpha}} dz \right| =$$

$$= c_\alpha N^{1-\alpha} \left| \int\limits_{y_0+G_k} \frac{1}{|x_0 - z|^{1-\alpha}} dz - \int\limits_{y_1+G_k} \frac{1}{|x_0 - z|^{1-\alpha}} dz \right| \geq c m_k^{-\alpha},$$

where c is a constant depending only on α. Define the function g on G by $g(x) = -f(x)/I_\alpha h(x_0)$. Routine calculations show that

$$\|h\|_{H^q(G)} \leq c N^{1-\alpha} m_k^{-1/q}$$

and

$$\|g\|_{H^p(G)} \leq c m_k^{1/r+\alpha-1/p}.$$

Consequently we have

$$\|h\|_{H^q(G)} \|g\|_{H^p(G)} \leq c N^{1-\alpha}.$$

Following the argument given in ([14], p.443) we have

$$\|f - (h I_\alpha g - g I_\alpha h)\|_{H^r(G)} \leq$$

$$\leq c \left(\left\| \frac{f(I_\alpha h(x_0) - I_\alpha h)}{I_\alpha h(x_0)} \right\|_{H^r(G)} + \|h I_\alpha g\|_{H^r(G)} \right) \leq$$

$$\leq \begin{cases} c N^{-2+1/r}, & \text{if } 1/2 < r < 1, \\ c N^{-1} \log N, & \text{if } r = 1. \end{cases}$$

The result now follows by noting that both $N^{-2+1/r}$ and $N^{-1} \log N$ tend to zero as N tends to infinity. Using the atomic decomposition of $H^r(G)$, the factorization is complete. The norm equivalence follows from Theorem 3.3. ∎

Theorem 1.1 can be used to reobtain the following boundedness result of commutators on Vilenkin groups; see [6] and [21].

COROLLARY 4.1. *Let $b \in BMO(G)$ and T be a Calderón–Zygmund operator as in Section* 1. *Then the operator*

$$[b, T](f) = bT(f) - T(bf)$$

is bounded on $L^p(G)$ for any $p \in (1, \infty)$, and

$$\|[b, T](f)\|_{L^p(G)} < c \|f\|_{L^p(G)} \|b\|_{BM0(G)},$$

where c is independent of b and f.

PROOF. Let $1/q + 1/q = 1$. Obviously, for any $f \in L^p(G)$ and $g \in$ $\in L^q(G)$, $g(Tf) - f(T^* g)$ satisfies (1.3). By Theroem 1.1, we know that $g(Tf) - f(T^* g) \in H_1(G)$. Moreover,

$$\|g(Tf) - f(T^* g)\|_{H^1(G)} \leq c \|f\|_{L^p(G)} \|g\|_{L^q(G)}.$$

Using the duality between $H^1(G)$ and $BMO(G)$ on the Vilenkin group (see [2]), we obtain

$$\left|\int_G [b, T](f)(x)g(x)dx\right| = \left|\int_G b(x)[g(x)(Tf)(x) - f(x)(T^*g)(x)]dx\right| \leq$$
$$\leq \|b\|_{BMO(G)}\|g(Tf) - f(T^*g)\|_{H^1(G)} \leq$$
$$\leq c\|b\|_{BMO(G)}\|f\|_{L^p(G)}\|g\|_{L^q(G)}.$$

From this, it follows that

$$\|[b, T](f)\|_{L^p(G)} \leq c\|f\|)L^p(G)\|b\|_{BMO(G)}.$$

This finishes the proof of Corollary 4.1. ∎

A consequence of Theorem 4.1 and Theorem 3.1 is the following characterization of $BMO(G)$; see [14, 2, 1]. We leave the details of its proof to the readers.

COROLLARY 4.2. *Let $\alpha \in (0, 1)$ and let $1/q = 1/p - \alpha$, where $1 < p < 1/\alpha$. Then $b \in BMO(G)$ if and only if the commutator $[b, I_\alpha]$ is bounded from $L^p(G)$ into $L^q(G)$. Moreover, we have $\|[b, I_\alpha]\|_{L^p(G)\to L^q(G)} \simeq \|b\|_{BMO(G)}$.*

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UNIFORM CONVERGENT DISCRETE PROCESSES ON THE ROOTS OF FOUR KINDS OF CHEBYSHEV POLYNOMIALS

By

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1. Introduction

On an interval $I \subset \mathbb{R}$ one of the most natural discrete approximating tools is the Lagrange interpolation. However, as it was proved by G. Faber in 1914, there is *no* point system for which the corresponding sequence of Lagrange interpolatory polynomials would converge uniformly for all continuous functions. Then, it is natural to ask how to construct such processes which are uniformly convergent in suitable spaces of continuous functions.

One possibility of achieving this aim is to loosen the strict condition on degree of the interpolating polynomials, thus introducing free parameters to be suitable determined for the uniform convergence (see [14, Ch. II.], [3], [24], [19]). The success of a construction like this strongly depends on the matrix of nodes.

Another possibility to obtain uniformly convergent discrete processes is to replace the Lagrange interpolatory polynomials with suitable summations (see [1], [7], [5], [22], [21], [16], [17]).

In this paper we shall construct a wide class of discrete processes using summation and we shall investigate the uniform convergence of sequences of such operators in a suitable Banach space of continuous functions. Several interpolatory properties will also be investigated.

Many authors studied also the summability of different Fourier series (see e.g. [2], [4], [13], [9], [10], [11], [20], [26]–[29], [30] and the references therein.)

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2. A general construction of discrete processes

Let $I \subset \mathbb{R}$ be a bounded or unbounded interval and fix the natural numbers m and N. Consider a point system

$$X_N := \{x_{N,N} < x_{N-1,N} < \cdots < x_{1,N}\},$$

a discrete measure (or nonnegative weights)

$$\mu_N := \left\{\mu_{N,N}, \mu_{N-1,N}, \ldots, \mu_{1,N}\right\} \quad (\mu_{k,N} := \mu_N\{x_{k,N}\})$$

and a basis in $\mathcal{P}_m$ (the linear space of algebraic polynomials with real coefficients of degree not greater than m):

$$P_m := \{p_0, p_1, \ldots, p_m\}.$$

We investigate summation processes generated by a function Θ as defined below. Let us denote by Φ the set of *summation functions* $\Theta : [0, +\infty) \to \mathbb{R}$ satisfying the following requirements:

(i) $\operatorname{supp} \Theta \subset [0, 1]$,

(ii) $\lim\limits_{t \to 0+} \Theta(t) = \Theta(0) := 1$ and $\lim\limits_{t \to 1-} \Theta(t) = \Theta(1) := 0$

(iii) the limits

$$\Theta(t_0 \pm 0) := \lim_{t \to t_0 \pm 0} \Theta(t)$$

exist and finite in every $t_0 \in (0, +\infty)$,

(iv) for all $t \in \mathbb{R}$ the function value $\Theta(t)$ lies in the closed interval determined by $\Theta(t - 0)$ and $\Theta(t + 0)$.

The condition (iii) ensures that every $\Theta \in \Phi$ is Riemann integrable on $[0, 1]$ (see [16, p. 161]). Therefore Θ is continuous except at most countable points of $[0, 1]$.

If $f : I \to \mathbb{R}$ is an arbitrary function then let
(2.1)

$$(S^{\Theta}_{m,N} f)(x) := S^{\Theta}_{m,N}(f, X_N, \mu_N, P_m, x) := \sum_{l=0}^{m} \Theta\left(\frac{l}{m}\right) c_{l,N}(f) p_l(x) \quad (x \in I),$$

where

$$c_{l,N}(f) := c_{l,N}(f, X_N, \mu_N, P_m) := \sum_{k=1}^{N} f(x_{k,N}) p_l(x_{k,N}) \mu_{k,N}. \tag{2.2}$$

Using two arbitrary index sequences $(m_n, n \in \mathbb{N} := \{1, 2, \ldots\})$ and $(N_n, n \in \in \mathbb{N})$ we have a sequence of polynomials

$$\left(S^{\Theta}_{m_n,N_n} f, n \in \mathbb{N}\right) \tag{2.3}$$

for all $f : I \to \mathbb{R}$.

The following problem will be investigated: *Choose the parameters $X_{N_n}, \mu_{N_n}, P_{m_n}$ such that the sequence* (2.3) *tend uniformly to f in a suitable subspace of continuous functions for a fairly wide class of summation functions Θ.*

That special case when X_N is the roots of Jacobi polynomial $p_N^{(\alpha,\beta)}$, μ_N is the corresponding Cotes numbers and the basis is the Jacobi basis were investigated in [18].

In this paper we assume that $I = [-1, 1]$ and we shall choose the above parameters in other ways. Namely we shall construct point systems X_N using the roots of the four kinds of Chebyshev polynomials supplemented with some endpoints of $[-1, 1]$. The convergence will be considered in the Banach space $(C[-1, 1], \|\cdot\|_\infty)$, where $C[-1, 1]$ denotes the linear space of continuous functions defined on $[-1, 1]$ and

$$\|f\|_\infty := \max_{x \in [-1,1]} |f(x)| \qquad (f \in C[-1, 1]).$$

3. Processes on the roots of four kinds of Chebyshev polynomials

Fix $N \in \mathbb{N}$ and consider a point system $X_N \subset [-1, 1]$. The *index of the point* $x \in X_N$ is 1 if $x \in (-1, 1)$ and is $1/2$ if $x \in \{-1, 1\}$. The *index of the point system* X_N is the sum of the indices of its points. It will be denoted by $I_{X_N} =: I_N$. It is clear that $I_N = N, N - 1/2$ or $N - 1$ for any X_N.

Let us define the measure μ_N by

$$\mu_{k,N} := \begin{cases} \dfrac{1}{2I_N}, & \text{if } x_{k,N} \in \{-1, 1\} \\ \dfrac{1}{I_N}, & \text{if } x_{k,N} \in (-1, 1) \end{cases} \qquad (k = 1, 2, \ldots, N,\ N \in \mathbb{N}). \tag{3.1}$$

We will choose the basis in the following way

$$P_m := \{T_0, \sqrt{2}T_1, \sqrt{2}T_2, \ldots, \sqrt{2}T_m\} \subset \mathcal{P}_m,$$

where $T_l(x) := \cos(l \arccos x)$ $(l \in \mathbb{N}_0 := \{0, 1, 2, \ldots\})$ are the Chebyshev polynomials of the first kind.

For every $m, N \in \mathbb{N}$ the point system X_N uniquely determines the parameters (X_N, μ_N, P_m) of the sequence (2.3). Therefore in the sequel if we speak about an X_N-*system* then we think for the above defined parameters (X_N, μ_N, P_m).

We shall consider the following four X_N-systems.

CASE 1. *The system* $\mathbf{T}_N$*:*

$$\mathbf{T}_N := \left\{ x_{k,N} := \cos\frac{2k-1}{2N}\pi \;\middle|\; k = 1, 2, \dots, N \right\},$$

$I_N = I_{\mathbf{T}_N} = N$, $\mu_{k,N} = \frac{1}{N}$ *($k = 1, 2, \dots, N$).*

REMARK. $\mathbf{T}_N$ are the roots of T_N (the Chebyshev polynomial of the first kind).

CASE 2. *The system* $\mathbf{U}_N^{\pm}$*:*

$$\mathbf{U}_N^{\pm} := \left\{ x_{k,N} := \cos\frac{k-1}{N-1}\pi \;\middle|\; k = 1, 2, \dots, N \right\},$$

$I_N = I_{\mathbf{U}_N^{\pm}} = N - 1$,

$$\mu_{k,N} = \begin{cases} \dfrac{1}{2(N-1)}, & \text{if } k = 1 \text{ or } N \\[2ex] \dfrac{1}{N-1}, & \text{if } k = 2, \dots, N-1. \end{cases}$$

REMARK. $\mathbf{U}_N^{\pm}$ are the roots of U_{N-2} (the Chebyshev polynomial of the second kind) supplemented with the endpoints -1 and 1.

CASE 3. *The system* $\mathbf{V}_N^{-}$*:*

$$\mathbf{V}_N^{-} := \left\{ x_{k,N} := \cos\frac{2k-1}{2N-1}\pi \mid k = 1, 2, \dots, N \right\},$$

$I_N = I_{\mathbf{V}_N^{-}} = N - 1/2$,

$$\mu_{k,N} = \begin{cases} \dfrac{1}{2N-1}, & \text{if } k = N \\[2ex] \dfrac{2}{2N-1}, & \text{if } k = 1, \dots, N-1. \end{cases}$$

REMARK. $\mathbf{V}_N^{-}$ are the roots of V_{N-1} (the Chebyshev polynomial of the third kind) supplemented with -1.

CASE 4. *The system* $\mathbf{W}_N^+$*:*

$$\mathbf{W}_N^+ := \left\{ x_{k,N} := \cos \frac{2(k-1)}{2N-1}\pi \mid k = 1, 2, \ldots, N \right\},$$

$I_N = I_{\mathbf{W}_N^+} = N - 1/2,$

$$\mu_{k,N} = \begin{cases} \dfrac{1}{2N-1}, & \textit{if } k = 1 \\ \dfrac{2}{2N-1}, & \textit{if } k = 2, \ldots, N. \end{cases}$$

REMARK. $\mathbf{W}_N^+$ are the roots of W_{N-1} (the Chebyshev polynomial of the fourth kind) supplemented with 1.

The main goal of this paper is to give a sufficient condition with respect to the summation function Θ which guarantees the uniform convergence of (2.3) for the above four point systems (see Theorem 6.2). Moreover we shall investigate the interpolatory properties of (2.3) (see Theorems 5.1 and 5.4).

In the trigonometric case (when the fundamental point system is the equidistant one) similar results were proved in [16] and [17].

REMARK. In [8], J. C. Mason investigated some common (minimality) properties of four kinds of Chebyshev polynomials.

4. Orthogonality relationships

Let us fix a natural number $N \in \mathbb{N}$ and consider an X_N-system (see Section 3). It is clear that the function

$$\langle f, g \rangle_{(X_N, \mu_N)} := \langle f, g \rangle_N := \sum_{k=1}^{N} f(x_{k,N}) g(x_{k,N}) \mu_{k,N} \tag{4.1}$$
$$(f, g \in C[-1, 1])$$

satisfies all properties of the scalar product but

$$\langle f, f \rangle_{(X_N, \mu_N)} = 0 \quad \Longrightarrow \quad f(x) = 0 \ (x \in [-1, 1]);$$

instead we have

$$\langle f, f \rangle_{(X_N, \mu_N)} = 0 \quad \Longleftrightarrow \quad f(x_{k,N}) = 0 \ (k = 1, 2, \ldots, N)$$

therefore it will be called *semi-scalar product*. Obviously

$$\|f\|_{(X_N, \mu_N)} := \|f\|_N := \sqrt{\langle f, f \rangle_N} = \left(\sum_{k=1}^{N} f^2(x_{k,N}) \mu_{k,N} \right)^{1/2} \tag{4.2}$$

is a semi-norm on $C[-1, 1]$.

Next we shall show that the Chebyshev polynomials of the first kind enjoy certain othogonality properties with respect to the semi scalar product (4.1) (cf. [12, pp. 53 and 54]).

LEMMA 4.1. *If* $X_N = \mathbf{T}_N$ *then*

$$\langle T_i, T_j \rangle_N = \langle T_i, T_j \rangle_{(\mathbf{T}_N, \mu_N)} = \frac{\triangle^+(i,j) + \triangle^-(i,j)}{2} \qquad (i,j \in \mathbb{N}_0),$$

where

$$\triangle^+(i,j) := \begin{cases} 1, & \text{if } (i+j)/(2N) \text{ is even} \\ -1, & \text{if } (i+j)/(2N) \text{ is odd} \\ 0, & \text{otherwise} \end{cases}$$

and

$$\triangle^-(i,j) := \begin{cases} 1, & \text{if } (i-j)/(2N) \text{ is even} \\ -1, & \text{if } (i-j)/(2N) \text{ is odd} \\ 0, & \text{otherwise.} \end{cases}$$

PROOF. From the identity

$$\cos\alpha + \cos 3\alpha + \cdots + \cos(2N-1)\alpha = \frac{\sin 2N\alpha}{2\sin\alpha} \qquad (\alpha \in \mathbb{R},\ N \in \mathbb{N})$$

we have

$$\frac{1}{N}\sum_{k=1}^{N} \cos(2k-1)\frac{l\pi}{2N} = \frac{\sin l\pi}{2N\sin\frac{l\pi}{2N}} = \begin{cases} 0, & \text{if } l \neq 2pN \\ (-1)^p, & \text{if } l = 2pN \end{cases}$$

(here p denotes an integer). Therefore the statement follows from the fact that for every $i,j \in \mathbb{N}_0$ the possible values of the scalar product

$$\langle T_i, T_j \rangle_N = \sum_{k=1}^{N} T_i(x_{k,N}) T_j(x_{k,N}) \mu_{k,N} = \frac{1}{N}\sum_{k=1}^{N} T_i(x_{k,N}) T_j(x_{k,N}) =$$

$$= \frac{1}{2N}\sum_{k=1}^{N}\left\{\cos(2k-1)\frac{(i+j)\pi}{2N} + \cos(2k-1)\frac{(i-j)\pi}{2N}\right\}$$

are $0,\ 1,\ -1,\ \frac{1}{2},\ -\frac{1}{2}$. ∎

LEMMA 4.2. *If* $X_N = \mathbf{U}_N^{\pm}$ *then*

$$\langle T_i, T_j \rangle_N = \langle T_i, T_j \rangle_{(\mathbf{U}_N^{\pm}, \mu_N)} = \frac{\triangle^+(i,j) + \triangle^-(i,j)}{2} \qquad (i,j \in \mathbb{N}_0),$$

where

$$\triangle^+(i,j) := \begin{cases} 1, & \textit{if } (i+j)/(2N-2) \textit{ is integer} \\ 0, & \textit{otherwise} \end{cases}$$

and

$$\triangle^-(i,j) := \begin{cases} 1, & \textit{if } (i-j)/(2N-2) \textit{ is integer} \\ 0, & \textit{otherwise.} \end{cases}$$

PROOF. It is easy to show that

$$\sum_{k=1}^{N} \mu_{k,N} \cos(k-1)\alpha =$$

$$= \frac{1}{N-1} \left\{ \frac{1}{2} + \sum_{k=2}^{N-1} \cos(k-1)\alpha + \frac{1}{2}\cos(N-1)\alpha \right\} = \frac{\sin(N-1)\alpha}{2(N-1)} \operatorname{ctg} \frac{\alpha}{2}$$

for all $\alpha \in \mathbb{R}$ and $N \in \mathbb{N}$. From it follows that

$$\sum_{k=1}^{N} \mu_{k,N} \cos(k-1)\frac{l\pi}{N-1} = \begin{cases} 1, & \text{if } l = 2p(N-1) \\ 0, & \text{otherwise} \end{cases}$$

for all $l \in \mathbb{N}_0$. In this case the corresponding scalar product is

$$\langle T_i, T_j \rangle_N = \sum_{k=1}^{N} T_i(x_{k,N}) T_j(x_{k,N}) \mu_{k,N} =$$

$$= \sum_{k=1}^{N} \mu_{k,N} \left\{ \cos(k-1)\frac{(i+j)\pi}{N-1} + \cos(k-1)\frac{(i-j)\pi}{N-1} \right\}.$$

Thus the possible values of $\langle T_i, T_j \rangle_N$ are $0,\ 1,\ \frac{1}{2}$ from which the statement follows. ∎

LEMMA 4.3. *If* $X_N = \mathbf{V}_N^{-}$ *then*

$$\langle T_i, T_j \rangle_N = \langle T_i, T_j \rangle_{(\mathbf{V}_N^{-}, \mu_N)} = \frac{\triangle^+(i,j) + \triangle^-(i,j)}{2} \qquad (i,j \in \mathbb{N}_0),$$

where

$$\triangle^{+}(i,j) := \begin{cases} 1, & \text{if } (i+j)/(2N-1) \text{ is even} \\ -1, & \text{if } (i+j)/(2N-1) \text{ is odd} \\ 0, & \text{otherwise} \end{cases}$$

and

$$\triangle^{-}(i,j) := \begin{cases} 1, & \text{if } (i-j)/(2N-1) \text{ is even} \\ -1, & \text{if } (i-j)/(2N-1) \text{ is odd} \\ 0, & \text{otherwise.} \end{cases}$$

PROOF. A simple calculation shows that from the identity

$$\cos\alpha + \cos 3\alpha + \cdots + \cos(2N-3)\alpha = \frac{\sin 2(N-1)\alpha}{2\sin\alpha} \quad (\alpha \in \mathbb{R},\ N \in \mathbb{N})$$

it follows that

$$\sum_{k=1}^{N} \mu_{k,N} \cos(2k-1)\frac{l\pi}{2N-1} =$$

$$= \frac{2}{2N-1}\left\{\sum_{k=1}^{N-1} \cos(2k-1)\frac{l\pi}{2N-1} + \frac{1}{2}\cos(2N-1)\frac{l\pi}{2N-1}\right\} =$$

$$= \begin{cases} (-1)^p, & \text{if } l = (2N-1)p \\ 0, & \text{otherwise} \end{cases}$$

for all $l \in \mathbb{N}_0$. Thus for $i,j \in \mathbb{N}_0$ the possible values of

$$\langle T_i, T_j\rangle_N = \sum_{k=1}^{N} T_i(x_{k,N})T_j(x_{k,N})\mu_{k,N} =$$

$$= \frac{1}{2}\sum_{k=1}^{N}\mu_{k,N}\left\{\cos(2k-1)\frac{(i+j)\pi}{2N-1} + \cos(2k-1)\frac{(i-j)\pi}{2N-1}\right\}$$

are $0,\ 1,\ -1,\ \frac{1}{2},\ -\frac{1}{2}$ from which the statement follows. ∎

LEMMA 4.4. *If* $X_N = \mathbf{W}_N^{+}$ *then*

$$\langle T_i, T_j\rangle_N = \langle T_i, T_j\rangle_{(\mathbf{W}_N^{+},\mu_N)} = \frac{\triangle^{+}(i,j) + \triangle^{-}(i,j)}{2} \qquad (i,j \in \mathbb{N}_0),$$

where

$$\triangle^+(i,j) := \begin{cases} 1, & \textit{if } (i+j)/(2N-1) \textit{ is integer} \\ 0, & \textit{otherwise.} \end{cases}$$

and

$$\triangle^+(i,j) := \begin{cases} 1, & \textit{if } (i+j)/(2N-1) \textit{ is integer} \\ 0, & \textit{otherwise.} \end{cases}$$

PROOF. Now we can use the identity

$$\frac{1}{2} + \sum_{k=2}^{N} \cos(k-1)\alpha = \frac{\sin(2N-1)\frac{\alpha}{2}}{2\sin\frac{\alpha}{2}} \quad (\alpha \in \mathbb{R},\ N \in \mathbb{N})$$

to show for all $l \in \mathbb{N}$

$$\sum_{k=1}^{N} \mu_{k,N} \cos(k-1)\frac{2l\pi}{2N-1} =$$

$$= \frac{2}{2N-1}\left\{\frac{1}{2} + \sum_{k=2}^{N}\cos(k-1)\frac{2l\pi}{2N-1}\right\} = \begin{cases} 1, & \text{if } l = (2N-1)p \\ 0, & \text{otherwise.} \end{cases}$$

Therefore for $i,j \in \mathbb{N}_0$ the values of

$$\langle T_i, T_j\rangle_N = \sum_{k=1}^{N} T_i(x_{k,N})T_j(x_{k,N})\mu_{k,N} =$$

$$= \frac{1}{2}\sum_{k=1}^{N}\mu_{k,N}\left\{\cos(k-1)\frac{2(i+j)\pi}{2N-1} + \cos(k-1)\frac{2(i-j)\pi}{2N-1}\right\}$$

are $0,\ 1,\ \frac{1}{2}$ from which the statement follows. ∎

From Lemmas 4.1–4.4 immediately follows that the polynomials T_0, T_1, …, T_{N-1} are orthogonal with respect to the semi-scalar product (4.1), that means, we have

THEOREM 4.5. *Let $N \geq 2$ be an integer and*

$$X_N := \mathbf{T}_N,\ \mathbf{U}_N^{\pm},\ \mathbf{V}_N^{-} \textit{ or } \mathbf{W}_N^{+}.$$

Then for $i,j = 0,1,2,\dots,N-1$ we have

$$\langle T_i, T_j\rangle_{(X_N,\mu_N)} = \begin{cases} 0, & \textit{if } i \neq j \\ \| T_i\|^2_{(X_N,\mu_N)}, & \textit{if } i = j, \end{cases} \tag{4.3}$$

where

$$\|T_i\|^2_{(X_N,\mu_N)} = \begin{cases} 1, & \text{if } i = 0 \\ \frac{1}{2}, & \text{if } i = 1, 2, \ldots, N-1 \end{cases} \tag{4.4}$$

for $X_N = \mathbf{T}_N,\ \mathbf{V}_N^-$ *or* $\mathbf{W}_N^+$ *and*

$$\|T_i\|^2_{(\mathbf{U}_N^{\pm},\mu_N)} = \begin{cases} 1, & \text{if } i = 0 \text{ or } i = N-1 \\ \frac{1}{2}, & \text{if } i = 1, 2, \ldots, N-2. \end{cases} \tag{4.5}$$

The polynomials $T_0, T_1, \ldots, T_{2I_N-1}$ also possess certain orthogonality relations. Namely, we have

THEOREM 4.6. *Let $N \geq 2$ be an integer and*

$$X_N := \mathbf{T}_N,\ \mathbf{U}_N^{\pm},\ \mathbf{V}_N^- \text{ or } \mathbf{W}_N^+.$$

Then for

$$i, j = 0, 1, 2, \ldots, 2I_N - 1, \quad i \neq j \quad \text{and} \quad i + j \neq 2I_N$$

we have

$$\langle T_i, T_j \rangle_{(X_N,\mu_N)} = 0.$$

PROOF. The statement is a direct consequence of Theorem 4.5 and the following "symmetry properties":

For every $j, l \in \mathbb{Z}$ we have

$$T_j(x) = (-1)^l\, T_{j+2lN}(x) \qquad (x \in \mathbf{T}_N), \tag{4.6}$$
$$T_j(x) = T_{j+2l(N-1)}(x) \qquad (x \in \mathbf{U}_N^{\pm}), \tag{4.7}$$
$$T_j(x) = (-1)^l\, T_{j+l(2N-1)}(x) \qquad (x \in \mathbf{V}_N^-), \tag{4.8}$$
$$T_j(x) = T_{j+l(2N-1)}(x) \qquad (x \in \mathbf{W}_N^+), \tag{4.9}$$

where $T_{-j} := T_j$ if $j \in \mathbb{N}$.

If $x_{k,N} = \cos \frac{2k-1}{2N}\pi =: \cos\vartheta_{k,N} \in \mathbf{T}_N$ then

$$T_{j+2lN}(x_{k,N}) = \cos(j + 2lN)\frac{2k-1}{2N}\pi =$$
$$= \cos(j\vartheta_{k,N} + (2k-1)l\pi) = (-1)^l\, T_j(x_{k,N})$$

which proves (4.6).

If $x_{k,N} = \cos \frac{k-1}{N-1}\pi =: \cos \vartheta_{k,N} \in \mathbf{U}_N^{\pm}$ then

$$T_{j+2l(N-1)}(x_{k,N}) = \cos(j + 2l(N-1))\frac{k-1}{N-1}\pi =$$
$$= \cos(j\vartheta_{k,N} + 2l(k-1)\pi) = T_j(x_{k,N})$$

which proves (4.7).

If $x_{k,N} = \cos \frac{2k-1}{2N-1}\pi \in \mathbf{V}_N^-$ then

$$T_{j+l(2N-1)}(x_{k,N}) = \cos(j + l(2N-1))\frac{2k-1}{2N-1}\pi =$$
$$= \cos\left[j\frac{2k-1}{2N-1}\pi + l(2k-1)\pi\right] = (-1)^l \cos j\frac{2k-1}{2N-1}\pi = (-1)^l T_j(x_{k,N}).$$

which proves (4.8).

If $x_{k,N} = \cos \frac{2(k-1)}{2N-1}\pi \in \mathbf{W}_N^+$ then

$$T_{j+l(2N-1)}(x_{k,N}) = \cos(j + l(2N-1))\frac{2(k-1)}{2N-1}\pi =$$
$$= \cos\left[j\frac{2(k-1)}{2N-1}\pi + 2(k-1)l\pi\right] = \cos j\frac{2(k-1)}{2N-1}\pi = T_j(x_{k,N}).$$

which proves (4.9). ∎

5. Interpolatory properties

Fix a natural number N, consider an X_N-system (see Section 3) and a summation function $\Theta \in \Phi$ (see Section 2). The polynomials $S^{\Theta}_{2I_N,N}f$ (see (2.1)) have degree $< 2I_N$ (cf. (ii)). It is clear that among them there are some which interpolate the function f at the points of X_N. Next we give a *necessary and sufficient* condition for the summation function Θ satisfying this requirement.

First we write the polynomials $S^{\Theta}_{2I_N,N}f$ in another form. From (2.1) we have

$$\left(S^{\Theta}_{2I_N,N}f\right)(x) = \sum_{l=0}^{2I_N} \Theta\left(\frac{l}{2I_N}\right) c_{l,N}(f) p_l(x) =$$

$$= \sum_{k=1}^{N} f(x_{k,N}) \left\{ \sum_{l=0}^{2I_N} \Theta\left(\frac{l}{2I_N}\right) p_l(x_{k,N}) p_l(x) \right\} \mu_{k,N}.$$

For $k = 1, 2, \ldots, N$ let us introduce the notation:

$$\ell_{k,N}^{\Theta}(x) := \ell_{k,N}^{\Theta}(X_N, \mu_N, x) := \sum_{l=0}^{2I_N} \Theta\left(\frac{l}{2I_N}\right) p_l(x_{k,N}) p_l(x) \mu_{k,N}.$$

Then

$$\left(S_{2I_N,N}^{\Theta} f\right)(x) = \sum_{k=1}^{N} f(x_{k,N}) \ell_{k,N}^{\Theta}(x). \tag{5.1}$$

Using a simple argument one can prove that the polynomials $S_{2I_N,N}^{\Theta} f$ (of degree $< 2I_N$) interpolates the function $f : [-1,1] \to \mathbb{R}$ at the points of X_N if and only if

$$\ell_{k,N}^{\Theta}(x_{j,N}) = \delta_{k,j} \qquad (j, k = 1, 2, \ldots, N). \tag{5.2}$$

Moreover, we state

THEOREM 5.1. *Let $N \geq 2$ be an integer, X_N be one of the point systems $\mathbf{T}_N$, $\mathbf{U}_N^{\pm}$, $\mathbf{V}_N^{-}$, $\mathbf{W}_N^{+}$, and let $\Theta \in \Phi$ be is a summation function. Then $S_{2I_N,N}^{\Theta} f$ interpolates the function $f : [-1,1] \to \mathbb{R}$ at the points of X_N if and only if*

$$\Theta\left(\frac{j}{2I_N}\right) + \Theta\left(1 - \frac{j}{2I_N}\right) = 1 \qquad (j = 0, 1, 2, \ldots, 2I_N) \tag{5.3}$$

and $\Theta(1/2)$ is arbitrary if $X_N = \mathbf{T}_N$.

PROOF. It is enough to show that

$$(5.2) \iff (5.3). \tag{5.4}$$

First we write the fundamental polynomials of the Lagrange interpolation with respect to the point system X_N in the basis $\{T_l\}$. They will be denoted by

$$\ell_{k,N}(x) := \ell_{k,N}(X_N, x) \qquad (k = 1, 2, \ldots, N).$$

These are the uniquely determined polynomials in $\mathcal{P}_{N-2}$ for which

$$\ell_{k,N}(x_{j,N}) = \delta_{k,j} \qquad (j, k = 1, 2, \ldots, N). \tag{5.5}$$

Obviously there are uniquely determined real numbers a_l $(l = 0, 1, \ldots, N-1)$ for which

$$\ell_{k,N}(x) = \sum_{l=0}^{N-1} a_l T_l(x) \qquad (x \in [-1, 1]).$$

Fix $i = 0, 1, \ldots, N-1$. Using (5.5) and (4.1) we have

$$\langle \ell_{k,N}, T_i \rangle_{(X_N, \mu_N)} = \sum_{l=1}^{N} \ell_{k,N}(x_{l,N}) T_i(x_{l,N}) \mu_{l,N} = T_i(x_{k,N}) \mu_{k,N}.$$

On the other hand by (4.3) we obtain that

$$\langle \ell_{k,N}, T_i \rangle_{(X_N, \mu_N)} = \sum_{l=0}^{N-1} a_l \langle T_l, T_i \rangle_{(X_N, \mu_N)} = a_i \| T_i \|^2_{(X_N, \mu_N)}.$$

Since $p_0 = T_0, p_l = \sqrt{2} T_l$ $(l \in \mathbb{N})$ thus by (4.4) and (4.5) we get

$$\begin{aligned} \ell_{k,N}(x) &= \sum_{l=0}^{N-1} \frac{T_l(x_{k,N}) T_l(x)}{\| T_l \|^2_{(X_N, \mu_N)}} \mu_{k,N} = \\ &= \sum_{l=0}^{N-1} p_l(x_{k,N}) p_l(x) \mu_{k,N} - \varepsilon_N \mu_{k,N} T_{N-1}(x_{k,N}) T_{N-1}(x), \end{aligned} \tag{5.6}$$

where

$$\varepsilon_N := \begin{cases} 0, & \text{if } X_N = \mathbf{T}_N,\ \mathbf{V}_N^- \text{ or } \mathbf{W}_N^+ \\ 1, & \text{if } X_N = \mathbf{U}_N^{\pm}. \end{cases} \tag{5.7}$$

Introduce $\alpha_l := \Theta\left(\frac{l}{2I_N}\right)$ and consider the following transformation of $\ell^{\Theta}_{k,N}$:

$$\begin{aligned} \ell^{\Theta}_{k,N}(x) = \sum_{l=0}^{2I_N} \alpha_l p_l(x_{k,N}) p_l(x) \mu_{k,N} &= \sum_{l=0}^{N-1} p_l(x_{k,N}) p_l(x) \mu_{k,N} + \\ + \sum_{l=0}^{N-1} (\alpha_l - 1 + \alpha_{2I_N - l}) p_l(x_{k,N}) p_l(x) \mu_{k,N} &+ \\ + \left\{ \sum_{l=N}^{2I_N} \alpha_l p_l(x_{k,N}) p_l(x) - \sum_{l=0}^{N-1} \alpha_{2I_N - l} p_l(x_{k,N}) p_l(x) \right\} \mu_{k,N} &= \end{aligned}$$

$$= \ell_{k,N}(x) + \sum_{l=0}^{N-1} (\alpha_l - 1 + \alpha_{2I_N-l}) p_l(x_{k,N}) p_l(x) \mu_{k,N} +$$

$$+ \sum_{l=N}^{2I_N} \alpha_l \left[p_l(x_{k,N}) p_l(x) - p_{2I_N-l}(x_{k,N}) p_{2I_N-l}(x) \right] \mu_{k,N} +$$

$$+ \varepsilon_N (1 - 2\alpha_{N-1}) T_{N-1}(x_{k,N}) T_{N-1}(x) \mu_{k,N} =:$$
$$=: \ell_{k,N}(x) + A_{k,N}(x) + B_{k,N}(x) + C_{k,N}(x).$$

From (4.6)–(4.9) it follows that

$$p_l(x_{k,N}) p_l(x_{j,N}) = p_{2I_N-l}(x_{k,N}) p_{2I_N-l}(x_{j,N})$$
$$(l = 0, 1, \ldots, 2I_N,\ x_{k,N}, x_{j,N} \in X_N = \mathbf{T}_N, \mathbf{U}_N^{\pm}, \mathbf{V}_N^{-} \text{ or } \mathbf{W}_N^{+})$$

and thus

$$B_{k,N}(x_{j,N}) = 0 \qquad (k, j = 1, 2, \ldots, N).$$

This means that

$$\ell_{k,N}^{\Theta}(x_{j,N}) = \delta_{k,j} + A_{k,N}(x_{j,N}) + C_{k,N}(x_{j,N}) \quad (j, k = 1, 2, \ldots, N),$$

i.e. $\ell_{k,N}^{\Theta}(x_{j,N}) = \delta_{j,k}$ for $k, j = 1, 2, \ldots, N$ if and only if

$$A_{k,N}(x_{j,N}) + C_{k,N}(x_{j,N}) = 0 \qquad (j, k = 1, 2, \ldots, N).$$

Therefore the polynomial $A_{k,N} + C_{k,N}$ of degree at most $(N-1)$ has N distinct roots. Consequently it is the zero polynomial. Since

$$p_l(x_{k,N}) \neq 0 \qquad (l = 0, 1, \ldots, N-1,\ x_{k,N} \in X_N)$$

thus $\alpha_l + \alpha_{2I_N-l} = 1$ $(l = 0, 1, \ldots, N-1)$ which proves the statement. ∎

Our next aim to obtain Hermite–Fejér type interpolation polynomials by a suitable summation function (see Theorem 5.4). We shall need the following two lemmas.

LEMMA 5.2. *Let $N \geq 2$ be an integer and*

$$X_N := \mathbf{T}_N,\ \mathbf{U}_N^{\pm},\ \mathbf{V}_N^{-} \text{ or } \mathbf{W}_N^{+}.$$

Then for $l = 1, 2, \ldots, N-1$ we have

$$(2I_N - l) p_l(x) p_l'(y) = -l\, p_{2I_N-l}(x) p_{2I_N-l}'(y) \tag{5.8}$$
$$(x \in X_N,\quad y \in X_N \cap (-1, 1)).$$

PROOF. From (4.6)–(4.9) we obtain that

$$p_l(x) = \begin{cases} -p_{2I_N-l}(x), & \text{if } x \in \mathbf{T}_N \text{ or } \mathbf{V}_N^- \\ p_{2I_N-l}(x), & \text{if } x \in \mathbf{U}_N^\pm \text{ or } \mathbf{W}_N^+. \end{cases} \tag{5.9}$$

Now we prove that

$$p_l'(y) = \frac{l}{2I_N - l} \begin{cases} p_{2I_N-l}'(y), & \text{if } y \in \mathbf{T}_N \text{ or } \mathbf{V}_N^- \setminus \{-1\} \\ -p_{2I_N-l}'(y), & \text{if } y \in \mathbf{U}_N^\pm \setminus \{\pm 1\} \text{ or } \mathbf{W}_N^+ \setminus \{1\}. \end{cases} \tag{5.10}$$

Obviously (5.9) and (5.10) $\Longrightarrow$ (5.8).

To verify (5.10) let $y := \cos\vartheta_{j,N} \in X_N \cap (-1, 1)$. Then

$$p_{2I_N-l}'(y) = \sqrt{2}T_{2I_N-l}'(y) = \sqrt{2}\cos'[(2I_N - l)\arccos y] =$$
$$= \sqrt{2}(2I_N - l)\frac{\sin(2I_N - l)\vartheta_{j,N}}{\sin\vartheta_{j,N}}.$$

(We remark that $\sin\vartheta_{j,N} \neq 0$ because $\vartheta_{j,N} \in (0, \pi)$.)

A simple calculation shows that

$$\sin(2I_N - l)\vartheta_{j,N} = \begin{cases} \sin l\vartheta_{j,N}, & \text{if } y \in \mathbf{T}_N \text{ or } \mathbf{V}_N^- \setminus \{-1\} \\ -\sin l\vartheta_{j,N}, & \text{if } y \in \mathbf{U}_N^\pm \setminus \{\pm 1\} \text{ or } \mathbf{W}_N^+ \setminus \{1\}. \end{cases}$$

Therefore

$$p_{2I_N-l}'(y) = \frac{\sqrt{2}(2I_N - l)}{\sin\vartheta_{j,N}} \begin{cases} \sin l\vartheta_{j,N}, & \text{if } y \in \mathbf{T}_N \text{ or } \mathbf{V}_N^- \setminus \{-1\} \\ -\sin l\vartheta_{j,N}, & \text{if } y \in \mathbf{U}_N^\pm \setminus \{\pm 1\} \text{ or } \mathbf{W}_N^+ \setminus \{1\} \end{cases} =$$
$$= \frac{2I_N - l}{l} \begin{cases} p_l'(y), & \text{if } y \in \mathbf{T}_N \text{ or } \mathbf{V}_N^- \setminus \{-1\} \\ -p_l'(y), & \text{if } x \in \mathbf{U}_N^\pm \setminus \{\pm 1\} \text{ or } \mathbf{W}_N^+ \setminus \{1\} \end{cases}$$

which proves (5.10). ∎

LEMMA 5.3. *Let $N \geq 2$ be an integer and*

$$X_N := \mathbf{T}_N,\ \mathbf{U}_N^\pm,\ \mathbf{V}_N^- \ \textit{or}\ \mathbf{W}_N^+.$$

Then for a fixed $x_{k,N} \in X_N$ the polynomial

$$R_k(x) := \sum_{l=1}^{2I_N-1} b_l p_l(x_{k,N}) p_l'(x)$$

satisfies the requirements

(5.11) $$R_k(x_{j,N})=0 \qquad (\forall\, x_{j,N}\in X_N\cap(-1,1))$$

if and only if

(5.12) $$l b_l=(2I_N-l)b_{2I_N-l} \qquad (l=1,2,\dots,N-1).$$

PROOF. Let $X_N := \mathbf{T}_N$, $\mathbf{V}_N^-$ or $\mathbf{W}_N^+$ and consider the following transformation of R_k:

$$R_k(x)=\sum_{l=1}^{N-1} b_l p_l(x_{k,N})p_l'(x)+\sum_{l=N}^{2I_N-1} b_l p_l(x_{k,N})p_l'(x)=$$

$$=\sum_{l=1}^{N-1} b_l p_l(x_{k,N})p_l'(x)+\sum_{l=1}^{N-1} b_{2I_N-l}p_{2I_N-l}(x_{k,N})p_{2I_N-l}'(x).$$

(Here we used that $p_N(x_{k,N})=0$ if $x_{k,N}\in\mathbf{T}_N$.) By (5.8) we get

$$R_k(x_{j,N})=\sum_{l=1}^{N-1}\left(b_l-\frac{2I_N-l}{l}b_{2I_N-l}\right)p_l(x_{k,N})p_l'(x_{j,N})$$

for all $x_{j,N}\in X_N\cap(-1,1)$. Therefore from (5.11) we obtain that the polynomial

$$A_k(x):=\sum_{l=1}^{N-1}\left(b_l-\frac{2I_N-l}{l}b_{2I_N-l}\right)p_l(x_{k,N})p_l'(x)$$

of degree at most $(N-2)$ has at least $(N-1)$ distinct roots. Consequently $A_k(x)=0$ $(x\in\mathbb{R})$. Since

$$p_l(x_{k,N})\neq 0 \qquad (l=1,2,\dots,N-1,\ x_{k,N}\in X_N)$$

thus

$$b_l-\frac{2I_N-l}{l}b_{2I_N-l}=0 \qquad (l=1,2,\dots,N-1)$$

which proves (5.12) in these cases.

The proof is similar if $X_N=\mathbf{U}_N^\pm$. In this case

$$p_{N-1}'(x_{j,N})=0 \qquad (\forall\, x_{j,N}\in\mathbf{U}_N^\pm\setminus\{\pm1\})$$

and thus for $x_{j,N}\in\mathbf{U}_N^\pm\setminus\{\pm1\}$ we have

$$R_k(x_{j,N})=\sum_{l=1}^{N-2}\left(b_l-\frac{2I_N-l}{l}b_{2I_N-l}\right)p_l(x_{k,N})p_l'(x_{j,N}),$$

i.e. the polynomial

$$\sum_{l=1}^{N-2}\left(b_l-\frac{2I_N-l}{l}b_{2I_N-l}\right)p_l(x_{k,N})p_l'(x)$$

of degree $\leq N-3$ has at least $(N-2)$ distinct roots. So we get (5.12) in this case, too. ∎

THEOREM 5.4. *Let $N\geq 2$ be an integer,*

$$X_N := \mathbf{T}_N,\ \mathbf{U}_N^{\pm},\ \mathbf{V}_N^{-}\ \text{or}\ \mathbf{W}_N^{+}$$

and

$$\Theta_F(t) := \begin{cases} 1-t, & \text{if}\ 0\leq t\leq 1 \\ 0, & \text{if}\ t>1. \end{cases} \tag{5.13}$$

Then for every function $f:[-1,1]\to\mathbb{R}$ the polynomial $S_{2I_N,N}^{\Theta_F}f$ is the unique element of $\mathcal{P}_{2I_N-1}$ satisfying the following Hermite–Fejér type interpolation conditions

$$\left(S_{2I_N,N}^{\Theta_F}f\right)(x_{j,N})=f(x_{j,N})\qquad (x_{j,N}\in X_N), \tag{5.14}$$

$$\left(S_{2I_N,N}^{\Theta_F}f\right)'(x_{j,N})=0\qquad (x_{j,N}\in X_N\cap(-1,1)). \tag{5.15}$$

PROOF. By (2.1) we have

$$\left(S_{2I_N,N}^{\Theta_F}f\right)(x)=\sum_{l=0}^{2I_N}\Theta_F\left(\frac{l}{2I_N}\right)c_{l,N}(f)p_l(x)=$$
$$=\sum_{k=1}^{N}f(x_{k,N})\left\{\sum_{l=0}^{2I_N}\Theta_F\left(\frac{l}{2I_N}\right)p_l(x_{k,N})p_l(x)\right\}\mu_{k,N}.$$

Obviously the summation function Θ_F satisfies (5.3) therefore conditions (5.14) hold.

Moreover

$$l\Theta_F\left(\frac{l}{2I_N}\right)=(2I_N-l)\Theta_F\left(\frac{2I_N-l}{2I_N}\right)\qquad (l=1,2,\ldots,N-1)$$

thus by Lemma 5.3 we have

$$\sum_{l=0}^{2I_N}\Theta_F\left(\frac{l}{2I_N}\right)p_l(x_{k,N})p_l'(x_{j,N})=0$$

for all $x_{j,N}\in X_N\cap(-1,1)$ which proves (5.15). ∎

6. Convergence

In this Section we shall show that if the Fourier transform of the summation function Θ is Lebesgue integrable on $\mathbb{R}^+ := [0, +\infty)$ then the sequence (2.3) tends to f uniformly on $[-1, 1]$ for all $f \in C[-1, 1]$.

Denote by $L^1(\mathbb{R}^+)$ the ususal linear space of measurable functions $g : \mathbb{R}^+ + \to \mathbb{R}$ for which the Lebesgue integral $\int_{\mathbb{R}^+} |g|$ is finite. The function

$$\|g\|_{L^1(\mathbb{R}^+)} := \int_0^{+\infty} |g(x)|dx \qquad (g \in L^1(\mathbb{R}^+))$$

is a norm on $L^1(\mathbb{R}^+)$ and $\left(L^1(\mathbb{R}^+), \|\cdot\|_{L^1(\mathbb{R}^+)}\right)$ is a Banach space.

The Fourier transform of $g \in L^1(\mathbb{R}^+)$ is defined by

$$\hat{g}(x) := \frac{1}{2\pi} \int_0^{+\infty} g(t)\cos(tx)dt \qquad (x \in \mathbb{R}^+).$$

In general, the Fourier transform of a function from $L^1(\mathbb{R}^+)$ does not belong to the space $L^1(\mathbb{R}^+)$. The verification of $\hat{g} \in L^1(\mathbb{R}^+)$ is not always easy, but the following sufficient condition is known:

THEOREM 6.1 ([10, p. 176]). *If $g : \mathbb{R}^+ \to \mathbb{R}$ is a continuous function supported in $[0, 1]$ and $g \in \mathrm{Lip}\,\beta$ $(\beta > 1/2)$ on $[0, 1]$ then $\hat{g} \in L^1(\mathbb{R}^+)$.*

We prove

THEOREM 6.2. *Let X_N be one of the point systems $\mathbf{T}_N$, $\mathbf{U}_N^{\pm}$, $\mathbf{V}_N^{-}$, $\mathbf{W}_N^{+}$. Suppose that*

$$m_n \to +\infty \ \ (n \to +\infty) \ \ \text{and} \ \ m_n \leq 2I_{N_n} \ (n \in \mathbb{N}),$$

moreover $\Theta \in \Phi$ is a summation function. If $\hat{\Theta} \in L^1(\mathbb{R}^+)$ then the sequence $S^{\Theta}_{m_n,N_n} f$ $(n \in \mathbb{N})$ uniformly converges on $[-1, 1]$ to f for all $f \in C[-1, 1]$.

PROOF. We shall use the Banach–Steinhaus Theorem. The polynomials

$$p_0 := T_0, \qquad p_l := \sqrt{2}T_l \ \ (l \in \mathbb{N}) \tag{6.1}$$

form a closed system for the space $\left(C[-1,1], \|\cdot\|_\infty\right)$, therefore we have to show that

(6.2) $$\left\| S^{\Theta}_{m_n,N_n} p_j - p_j \right\|_\infty \to 0 \quad (n \to +\infty)$$

for every fixed $j \in \mathbb{N}_0$, moreover the norms of the operators

$$\mathcal{S}^{\Theta}_{m_n,N_n} : \left(C[-1,1], \|\cdot\|_\infty\right) \to \mathcal{P}_{m_n} \subset \left(C[-1,1], \|\cdot\|_\infty\right),$$

$$\mathcal{S}^{\Theta}_{m_n,N_n} f := S^{\Theta}_{m_n,N_n} f \qquad (f \in C[-1,1]),$$

i.e. the sequence of real numbers

$$\left\| \mathcal{S}^{\Theta}_{m_n,N_n} \right\| := \sup_{0 \not\equiv f \in C[-1,1]} \frac{\left\| S^{\Theta}_{m_n,N_n} f \right\|_\infty}{\|f\|_\infty} = \sup_{x \in [-1,1]} \sum_{k=1}^{N_n} \left| \ell^{\Theta}_{k,N_n}(x) \right| =$$

$$= \sup_{x \in [-1,1]} \sum_{k=1}^{N_n} \left| \sum_{l=0}^{m_n} \Theta\left(\frac{l}{m_n}\right) p_l(x_{k,N_n}) p_l(x) \right| \mu_{k,N_n}$$

is uniformly bounded; i.e. there exists $c > 0$ independent of n such that

(6.3) $$\left\| \mathcal{S}^{\Theta}_{m_n,N_n} \right\| \le c \qquad (n \in \mathbb{N}).$$

To verify (6.2), let us fix $j \in \mathbb{N}_0$ and assume that n is so large that $\min\{m_n, N_n\} > j$. Then by Theorem 4.6 we have

$$\left(S^{\Theta}_{m_n,N_n} p_j\right)(x) = \sum_{l=0}^{m_n} \Theta\left(\frac{l}{m_n}\right) \langle p_j, p_l \rangle_{N_n} p_l(x) =$$

$$= \Theta\left(\frac{j}{m_n}\right) \|p_j\|^2_{N_n} p_j(x) + \Theta\left(\frac{2I_N - j}{m_n}\right) \langle p_j, p_{2I_{N_n}-j} \rangle_{N_n} p_{2I_{N_n}-j}(x).$$

Using Theorem 4.5 we obtain that

$$S^{\Theta}_{m_n,N_n} p_j - p_j = \left(\Theta\left(\frac{j}{m_n}\right) - 1\right) p_j +$$

$$+ \Theta\left(1 - \frac{j}{m_n} + \frac{2I_{N_n} - m_n}{m_n}\right) \langle p_j, p_{2I_{N_n}-j} \rangle_{N_n} p_{2I_{N_n}-j}.$$

Since $\Theta \in \Phi$ (see Section 2) and $m_n \le 2I_{N_n}$ $(n \in \mathbb{N})$ thus

$$\Theta\left(\frac{j}{m_n}\right) \to \Theta(0) = 1 \qquad (n \to +\infty)$$

and

$$\Theta\left(1-\frac{j}{m_n}+\frac{2I_{N_n}-m_n}{m_n}\right)\to\Theta(1)=0 \qquad (n\to\infty)$$

therefore (6.2) follows from the above relations.

Next we prove (6.3). Let $x =: \cos\vartheta$ ($\vartheta\in[0,\pi]$), $x_{k,N} =: \cos\vartheta_{k,N}$ ($k=1,2,\dots,N$). Since $p_0=T_0$, $p_l=\sqrt{2}T_l$ ($l=1,2,\dots$) thus for every $m,N\in\mathbb{N}$ we have

$$\sum_{l=0}^{m}\Theta\left(\frac{l}{m}\right)p_l(x_{k,N})p_l(x)=1+2\sum_{l=0}^{m}\Theta\left(\frac{l}{m}\right)T_l(x_{k,N})T_l(x)=$$

$$=1+2\sum_{l=0}^{m}\Theta\left(\frac{l}{m}\right)\cos l\vartheta_{k,N}\cos l\vartheta=$$

$$=\frac{1}{2}+\sum_{l=0}^{m}\Theta\left(\frac{l}{m}\right)\cos l(\vartheta+\vartheta_{k,N})+\frac{1}{2}+\sum_{l=0}^{m}\Theta\left(\frac{l}{m}\right)\cos l(\vartheta-\vartheta_{k,N})=:$$

$$=: D_m^{\Theta}(\vartheta+\vartheta_{k,N})+D_m^{\Theta}(\vartheta-\vartheta_{k,N}).$$

Therefore

$$\left\|\mathcal{S}_{m,N}^{\Theta}\right\|=\max_{\vartheta\in[0,\pi]}\sum_{k=1}^{N}\left|D_m^{\Theta}(\vartheta+\vartheta_{k,N})+D_m^{\Theta}(\vartheta-\vartheta_{k,N})\right|\mu_{k,N}\le$$

$$\le\max_{\vartheta\in[0,\pi]}\sum_{k=1}^{N}\left|D_m^{\Theta}(\vartheta+\vartheta_{k,N})\right|\mu_{k,N}+\max_{\vartheta\in[0,\pi]}\sum_{k=1}^{N}\left|D_m^{\Theta}(\vartheta-\vartheta_{k,N})\right|\mu_{k,N}.$$

Since

$$\max_{\vartheta\in[0,\pi]}\sum_{k=1}^{N}\left|D_m^{\Theta}(\vartheta\pm\vartheta_{k,N})\right|\mu_{k,N}\le C\left(1+2\frac{m}{N}\pi\right)\|D_m^{\Theta}\|_1,$$

where

$$\|D_m^{\Theta}\|_1:=\frac{1}{2\pi}\int_{-\pi}^{\pi}|D_m^{\Theta}(t)|dt$$

(see [16, (26)]) and

$$2\sup_{n\in\mathbb{N}}\|D_{m_n}^{\Theta}\|_1=\|\hat{\Theta}\|_{L^1(\mathbb{R}^+)}$$

(see [16, (27)]) thus condition $\hat{\Theta}\in L^1(\mathbb{R}^+)$ ensures (6.3). ∎

7. Applications

7.1. *Lagrange interpolation.* On the interval $[-1,1]$ one of the most natural discrete approximating tools is the Lagrange interpolation.

Throughout this section we assume that the interpolatory point system is

$$X_N := \mathbf{T}_N,\ \mathbf{U}_N^{\pm},\ \mathbf{V}_N^{-} \text{ or } \mathbf{W}_N^{+} \qquad (N \in \mathbb{N}) \tag{7.1}$$

(see Section 3). Denote, as usual,

$$(L_N f)(x) := L_N(f, X_N, x) := \sum_{k=1}^{N} f(x_{k,N})\ell_{k,N}(x) \qquad (x \in [-1,1],\ N \in \mathbb{N})$$

the Lagrange interpolatory polynomial of degree $\le N-1$ based on the nodes (7.1), i.e.

$$\ell_{k,N}(x) := \ell_{k,N}(X_N, x) \qquad (x \in [-1,1],\ k = 1,2,\dots,N,\ N \in \mathbb{N})$$

are the fundamental polynomials of the Lagrange interpolation.

Using (5.6) and (2.2) we have

$$\begin{aligned}(L_N f)(x) &= \sum_{k=1}^{N} f(x_{k,N})\ell_{k,N}(x) = \\ &= \sum_{k=1}^{N} f(x_{k,N}) \left\{ \sum_{l=0}^{N-1} p_l(x_{k,N})p_l(x) - \frac{\varepsilon_N}{2} p_{N-1}(x_{k,N})p_{N-1}(x) \right\} \mu_{k,N} = \\ &= \sum_{l=0}^{N-1} c_{l,N}(f)p_l(x) - \frac{\varepsilon_N}{2} c_{N-1,N}(f)p_{N-1}(x) = \sum_{l=0}^{N-1} \gamma_{l,N} p_l(x),\end{aligned} \tag{7.2}$$

where

$$\gamma_{l,N}(f) := \begin{cases} c_{l,N}(f), & \text{if } l = 0,1,\dots,N-2 \\ c_{N-1,N}(f)\{1 - \frac{\varepsilon_N}{2}\}, & \text{if } l = N-1. \end{cases} \tag{7.3}$$

The polynomials $L_N f$ $(N \in \mathbb{N})$ can be obtained as special cases of (2.1). Indeed, let

$$\Theta_L(t) := \begin{cases} 1, & \text{if } t \in [0, 1/2) \\ \frac{1}{2}, & \text{if } t = \frac{1}{2} \\ 0, & \text{if } t \in (1/2, +\infty). \end{cases}$$

Then

$$\left(S^{\Theta_L}_{2I_N,N}f\right)(x) = \sum_{l=0}^{2I_N} \Theta_L\left(\frac{l}{2I_N}\right) c_{l,N}(f)p_l(x) =$$

$$= \sum_{k=1}^{N} f(x_{k,N}) \left\{ \sum_{l=0}^{N-1} p_l(x_{k,N})p_l(x) - \frac{\varepsilon_N}{2} p_{N-1}(x_{k,N})p_{N-1}(x) \right\} \mu_{k,N},$$

i.e.

$$(L_N f)(x) = \left(S^{\Theta_L}_{2I_N,N}f\right)(x) \qquad (x \in [-1,1],\ N \in \mathbb{N}).$$

It is known that (see Faber's Theorem) that $L_N f$ $(N \in \mathbb{N})$ generally does not tend uniformly in $[-1,1]$ to f for all $f \in C[-1,1]$.

Using Theorems 6.1 and 6.2 one can easily construct *a lot of discrete processes* which are uniformly convergent in the *whole* interval $[-1,1]$. In the following Parts we shall discuss only some of them. It is important to note that the corresponding polynomials have very simple explicit forms.

7.2. *Arithmetic means of Lagrange interpolation.* Let

$$\left(L_{m,N}f\right)(x) := L_{m,N}(f, X_N, x) := \sum_{l=0}^{m} \gamma_{l,N}(f)p_l(x) \tag{7.4}$$
$$(x \in [-1,1],\ m = 0,1,\dots,N-1,\ N \in \mathbb{N})$$

(see (7.3)) and consider the following arithmetic means of the Lagrange interpolation:

$$(\sigma_N f)(x) := \sigma_N(f, X_N, x) := \frac{1}{N} \sum_{m=0}^{N-1} \left(L_{m,N}f\right)(x) \tag{7.5}$$
$$(x \in [-1,1],\ N \in \mathbb{N}).$$

THEOREM 7.1. *Let X_N be one of the point systems $\mathbf{T}_N$, $\mathbf{U}_N^{\pm}$, $\mathbf{V}_N^{-}$, $\mathbf{W}_N^{+}$ $(N \in \mathbb{N})$. Then for every $f \in C[-1,1]$ the sequence $\sigma_N f$ $(N \in \mathbb{N})$ tends to f uniformly on the whole interval $[-1,1]$.*

PROOF. From (7.3) and (7.4) we have

$$\left(L_{m,N}f\right)(x) = \sum_{l=0}^{m} c_{l,N}(f)p_l(x) \qquad (m = 0,1,\dots,N-2)$$

$$\left(L_{N-1,N}f\right)(x) = \sum_{l=0}^{N-1} c_{l,N}(f)p_l(x) - \frac{\varepsilon_N}{2} c_{N-1,N}(f)p_{N-1}(x).$$

Let us define the summation function

$$\Theta_F(t) := \begin{cases} 1-t, & \text{if } t \in [0,1] \\ 0, & \text{if } t \in (1,+\infty). \end{cases}$$

Then

$$(\sigma_N f)(x) = \sum_{l=0}^{N-1} \left(1 - \frac{l}{N}\right) c_{l,N}(f) p_l(x) - \frac{\varepsilon_N}{2N} c_{N-1,N}(f) p_{N-1}(x) =$$

$$= \left(S_{N,N}^{\Theta_F} f\right)(x) - \frac{\varepsilon_N}{2N} c_{N-1,N}(f) p_{N-1}(x).$$

Since $\hat{\Theta}_F \in L^1(\mathbb{R}^+)$ (see Theorem 6.1) thus from Theorem 6.2 we obtain that

$$\lim_{N\to+\infty} \left\| f - S_{N,N}^{\Theta_F} f \right\|_\infty = 0$$

for all $f \in C[-1,1]$ which proves the statement if $X_N = \mathbf{T}_N, \mathbf{V}_N^-$ or $\mathbf{W}_N^+$ (see (5.7)).

If $X_N = \mathbf{U}_N^\pm$ (i.e. $\varepsilon_N = 1$) then

$$c_{N-1,N}(f) = \sum_{k=1}^{N} f(x_{k,N}) p_{N-1}(x_{k,N}) \mu_{k,N} =$$

$$\sqrt{2} \sum_{k=1}^{N} f(x_{k,N}) \cos[(k-1)\pi] \mu_{k,N} = \sqrt{2} \sum_{k=1}^{N} (-1)^{k-1} f(x_{k,N}) \mu_{k,N}.$$

From it follows that

$$|c_{N-1,N}(f)| \le c\|f\|_\infty \qquad (N \in \mathbb{N}),$$

i.e.

$$\lim_{N\to+\infty} \left\| \frac{\varepsilon_N}{2N} c_{N-1,N}(f) p_{N-1} \right\|_\infty = 0$$

which proves the statement for $X_N = \mathbf{U}_N^\pm$. ∎

REMARK. Theorem 7.1 is a discrete version of the fundamental Fejér's theorem about $(C, 1)$ summability of Fourier series. In the trigonometric case the analogue result due to J. Marcinkiewicz [7] and to S. N. Bernstein [1]. If $X_N = \mathbf{T}_N$ then Theorem 7.1 follows from the Theorem of [21].

CONJECTURE. If in Theorem 7.1 X_N is the roots of the orthogonal polynomial U_N, V_N or W_N then the uniform convergence is true *only* on the compact intervals of $(-1, 1)$.

COROLLARY 7.2. *If* $X_N = \mathbf{T}_N, \mathbf{U}_N^{\pm}, \mathbf{V}_N^{-}$ *or* $\mathbf{W}_N^{+}$, $\lim\limits_{n\to+\infty} = +\infty$ *and* $m_n \le 2I_{N_n}$ $(n \in \mathbb{N})$ *then*

$$\lim_{n\to+\infty} \left\| S_{m_n,N_n}^{\Theta_F} f - f \right\|_\infty = 0 \qquad (f \in C[-1,1]).$$

REMARK. $S_{m_n,N_n}^{\Theta_F} f$ can also be considered as certain arithmetic mean of the Lagrange interpolation.

7.3. *Grünwald–Rogosinski type processes.* Let us consider the summation function

$$\Theta_G(t) := \begin{cases} \cos t\frac{\pi}{2}, & \text{if } t \in [0,1] \\ 0, & \text{if } t \in (1,+\infty). \end{cases}$$

For its Fourier transform we obtain

$$\hat{\Theta}_G(x) = \frac{\sin(x-\pi/2)}{2(x^2-(\pi/2)^2)} \qquad (x \in \mathbb{R}^+)$$

and thus $\hat{\Theta}_G \in L^1(\mathbb{R}^+)$.

Theorem 6.2 immediately yields

COROLLARY 7.3. *If* $X_N = \mathbf{T}_N, \mathbf{U}_N^{\pm}, \mathbf{V}_N^{-}$ *or* $\mathbf{W}_N^{+}$, $\lim\limits_{n\to+\infty} m_n = +\infty$ *and* $m_n \le 2I_{N_n}$ $(n \in \mathbb{N})$ *then*

$$\lim_{n\to+\infty} \left\| S_{m_n,N_n}^{\Theta_G} f - f \right\|_\infty = 0 \qquad (f \in C[-1,1]).$$

A simple calculation shows that

$$\left(S_{[I_N],N}^{\Theta_G} f\right)(x) = \frac{1}{2}\left\{\left(\mathcal{L}_N f\right)\left(\vartheta + \frac{\pi}{2[I_N]}\right) + \left(\mathcal{L}_N f\right)\left(\vartheta - \frac{\pi}{2[I_N]}\right)\right\},$$

where

$$\left(\mathcal{L}_N f\right)(\vartheta) := (L_N f)(\cos\vartheta).$$

Therefore Corollary 7.3 contains the Grünwald's theorem about the Rogosinski type average of Lagrange interpolation based on the roots of T_N (see [5, Theorem]).

REMARK. In [25] M. S. Webster obtained similar result for the roots of Chebyshev polynomials of the second kind U_N. He proved that the uniform convergence is true only in any closed subinterval of $(-1,1)$. Later

G. I. Natanson (see [15, p. 481]) and P. Vértesi [22] generalized these results for Jacobi roots and for any inrterval $[a, b] \subset (-1, 1)$. Our Corollary 7.3 states that on the four point system X_N the Grünwald–Rogosinski type average of the corresponding Lagrange interpolation polynomials are uniformly convergent on the *whole* interval $[-1, 1]$.

7.4. *Hermite–Fejér type interpolation.* Let $N \geq 2$ be an integer and $X_N = \mathbf{T}_N, \mathbf{U}_N^{\pm}, \mathbf{V}_N^{-}$ or $\mathbf{W}_N^{+}$. Theorem 5.4 states that for every function $f \in C[-1, 1]$ there exists a uniquely determined polynomial $H_N(f, X_N, x)$ of degree $\leq 2I_N - 1$ such that

$$H_N(f, X_N, x_{j,N}) = f(x_{j,N}) \qquad (x_{j,N} \in X_N)$$
$$H_N'(f, X_N, x_{j,N}) = 0 \qquad (x_{j,N} \in X_N \cap (-1, 1).$$

Moreover

$$H_N(f, X_N, x) = \left(S_{2I_N,N}^{\Theta_F} f\right)(x) = \sum_{l=0}^{2I_N} \left(1 - \frac{l}{2I_N}\right) c_{l,N}(f) p_l(x),$$

where $c_{l,N}(f)$ is given by (2.2). Since

$$\hat{\theta}_F(x) = \frac{1}{2\pi} \left(\frac{\sin(x/2)}{x/2}\right)^2 \qquad (x \in \mathbb{R}^+)$$

belongs to $L^1(\mathbb{R}^+)$ thus from Theorem 6.2 we have

COROLLARY 7.4. *If* $X_N = \mathbf{T}_N, \mathbf{U}_N^{\pm}, \mathbf{V}_N^{-}$ *or* $\mathbf{W}_N^{+}$ *then*

$$\lim_{N \to +\infty} \|H_N(f, X_N, \cdot) - f\|_\infty = 0 \qquad (f \in C[-1, 1]).$$

7.5. *De la Vellée Poussin–Erdős type interpolation.* Fix a number $\alpha \in$ $\in (0, 1)$ and let

$$\Theta_\alpha(t) := \begin{cases} 1, & \text{if } t \in [0, \frac{1-\alpha}{2}] \\ -\frac{1}{\alpha}(x - \frac{1+\alpha}{2}), & \text{if } t \in [\frac{1-\alpha}{2}, \frac{1+\alpha}{2}] \\ 0, & \text{if } t \in (\frac{1+\alpha}{2}, +\infty). \end{cases}$$

An easy calculation shows that

$$\hat{\Theta}_\alpha(x) = \frac{1}{2(1-\alpha)\pi} \frac{\sin^2(x/2) - \sin^2(1+\alpha)x}{(x/2)^2} \qquad (x \in \mathbb{R}^+).$$

Consequently $\hat{\Theta}_\alpha \in L^1(\mathbb{R}^+)$. Moreover

$$\Theta_\alpha(t) + \Theta_\alpha(1-t) = 1 \qquad (t \in [0,1]).$$

Thus from Theorem 5.1 and 6.2 we have

COROLLARY 7.5. *Fix a number $\alpha \in (0,1)$ and let $X_N = \mathbf{T}_N, \mathbf{U}_N^{\pm}, \mathbf{V}_N^{-}$ or $\mathbf{W}_N^{+}$. Then the degree of the polynomial $S_{2I_N,N}^{\Theta_\alpha} f$ is $\le I_N(1+\alpha)$ and it interpolates the function f at the points of X_N. Moreover*

$$\lim_{N\to+\infty} \left\| S_{2I_N,N}^{\Theta_\alpha} f - f \right\|_\infty = 0 \qquad (f \in C[-1,1]).$$

We also have the following Erdős type result:

COROLLARY 7.6. *If $X_N = \mathbf{T}_N, \mathbf{U}_N^{\pm}, \mathbf{V}_N^{-}$ or $\mathbf{W}_N^{+}$ then to every $f \in$ $\in C[-1,1]$ and $\alpha > 0$ there exists a sequence of polynomials Q_N $(N \in \mathbb{N})$ such that*

(i) the degree of Q_N is $\le N(1+\alpha)$ $(N \in \mathbb{N})$,

(ii) Q_N interpolates f at the points of X_N,

(iii) $(Q_N, N \in \mathbb{N})$ tends to f uniformly in $[-1,1]$.

In 1943, P. Erdős [3, Theorem 1] proved the above statement if the interpolatory point system is such that the fundamental polynomials of Lagrange interpolation are uniformly bounded. For our four point systems X_N the polynomials Q_N have *very simple* explicit forms. Namely

$$Q_N(x) = \left(S_{2I_N,N}^{\Theta_\alpha} f \right)(x) \qquad (x \in [-1,1],\ N \in \mathbb{N}).$$

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DISCRETE CROUZEIX–VELTE DECOMPOSITIONS ON NON-EQUIDISTANT RECTANGULAR GRIDS

By

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The difference approximation of the Stokes problem on staggered non-equidistant grids and for finite difference and finite volume schemes is investigated in two dimensions. A full description of the discrete Crouzeix–Velte decomposition is given in the case of a non-equidistant grid for Shortley–Weller approximation and in the case of second order difference approximation and homogeneous Dirichlet boundary conditions.

1. Introduction

The Crouzeix–Velte decomposition, introduced in [3], and, independently in [14] (see also [4]), can be regarded as an $(H_0^1)^n$ equivalent of the well-known Helmholtz decomposition for vector functions in $(L_2)^n$. This decomposition contains, besides the subspaces of rotation-free and divergence-free vector functions, a third orthogonal subspace consisting of biharmonic $(H_0^1)^n$ functions, which are neither rotation-free nor divergence-free. This decomposition can be used to determine the optimal constant in the so-called inf-sup condition for the Stokes problem. It is known that the eigenvalues of the Schur complement operator $S = -\operatorname{div}(\Delta)^{-1}\operatorname{grad}$ lie in $[0, 1]$, and the eigenvectors to eigenvalues in $(0, 1)$ span the third Velte subspace. The inf-sup constant is the square root of the smallest among these latter eigenvalues, that is this optimal constant can be characterized by the third subspace alone [11].

For the numerical solution of the Stokes problem and its connection to the inf-sup problem, see [1], [7], [2]. If using finite difference methods for approximating the Stokes equations and if the discrete scheme admits a discrete Crouzeix–Velte decomposition, then the same conclusions as in the continuous case can be drawn. In this case both discrete rotation-free and discrete divergence-free functions exist and their subspaces are in similar relation as the corresponding subspaces of $(H_0^1)^n$ and there is a third orthogonal subspace consisting of discrete biharmonic vector functions (or harmonic functions, for the pressure space, respectively). The optimal constant of the discrete inf-sup condition can hence be computed on this much smaller third space (see [11]). The optimal inf-sup constant is useful not only in error estimates for the numerical schemes but in the determination of the optimal iteration parameters of some numerical methods for solving the corresponding linear systems. In [13] the Uzawa and Arrow–Hurwitz iterations are investigated and are shown to reach the third Crouzeix–Velte subspace after at most 2 steps in the sense that then the error of the iterative solution belongs to that subspace. Using the fact that in the harmonic Crouzeix–Velte pressure subspace the spectrum of the Schur complement is closer and can be bounded through estimates of the inf-sup constant, new optimal iteration parameters for both methods have been calculated. Also in [13] an improved convergence estimate is derived for the conjugate gradient method using the estimates of the inf-sup constant. This method can be also restricted to the third Crouzeix–Velte subspace. Then, the dimension of the latter subspace is an upper bound on the maximal number of steps. This dimension is connected to the number of boundary points that is usually lower (by one power) than the full number of unknowns. For iterative methods based on space decomposition see also [15].

The aim of the present paper is to investigate the well-known staggered grid approximation of the Stokes-problem and to prove the existence of the discrete Crouzeix–Velte decomposition in the case of a non-equidistant grid for the Shortley–Weller approximation, for a finite volume scheme and in the case of second order approximation and homogeneous Dirichlet boundary conditions. The first order staggered grid approximation in a special case was investigated in [11]. This result has been generalized in [6] to general two- and three-dimensional domains.

The outline of the paper is as follows. In Section 2, the necessary notations as well as the Crouzeix–Velte decomposition in the continuous case are introduced and the discrete Stokes problem and discrete Crouzeix–Velte decompositions are described. In Section 3, the case of a non-equidistant grid is investigated and it is shown that the discrete Crouzeix–Velte decomposition exists for the Shortley–Weller approximation only if the grid spacing is changing linearly (Section 3.1). It is also shown that using the finite volume method on a rectangular grid, the discrete Crouzeix–Velte decomposition exists without any condition on the grid spacing (Section 3.2). In Section 4, all details of the discrete Crouzeix–Velte decomposition are given using the second order finite difference method. Finally, in Section 5 we show some computational results. Here it becomes visible that — from the point of view of the convergence rate of Uzawa- and conjugate gradient-like methods for the iterative solution of the corresponding discrete Stokes problems — it is worth using non-equidistant grids.

2. The Stokes problem and the Crouzeix–Velte decomposition

Let Ω be a bounded, simply-connected open domain in $\mathbb{R}^n$, $n = 2, 3$, and denote by $V := (H_0^1(\Omega))^n$ the Sobolev space of vector functions

$$u(x) = (u_1(x), \dots, u_n(x))^T$$

defined for $x = (x_1, \dots, x_n) \in \Omega$, with generalized derivatives in $(L_2(\Omega))^n$ and with zero boundary values in the sense of traces on the Lipschitz-continuous boundary $\partial\Omega$ of Ω.

For a given $f \in (L_2(\Omega))^n$ and denoting by $L_{2,0}(\Omega)$ the subspace of $L_2(\Omega)$ of square integrable functions with zero integral over Ω, consider the following first-kind Stokes problem in variational formulation:

$$a(u, v) + b(v, p) = (f, v)_0, \quad \text{for all } v \in V, \tag{1}$$

$$b(u, q) = 0, \quad \text{for all } q \in L_{2,0}(\Omega). \tag{2}$$

where

$$(f, v) := \sum_{i=1}^{n} (f_i, v_i) = \int_\Omega \sum_{i=1}^{n} f_i(x) v_i(x) dx, \tag{3}$$

$$a(u, v) := \int_\Omega \sum_{i,j=1}^{n} \frac{\partial u_i}{\partial x_j} \frac{\partial v_i}{\partial x_j} dx, \tag{4}$$

$$b(u, p) := -(\operatorname{div} u, p). \tag{5}$$

The problem consists in finding a velocity vector $u \in V$ and a pressure $p \in L_{2,0}(\Omega)$. It is well known (see [7], [2]) that this problem is solvable and its solution depends in stable way on the data since the so-called inf-sup condition is satisfied:

(6) $$\sup_{u\neq 0} b^2(u,p)/a(u,u) \geq \beta_0^2\|p\|^2 \quad \text{for all } 0\neq p \in L_{2,0}; \quad \beta_0 = \text{const} > 0.$$

The following identity is well-known for any $w,\ v \in V$ in both the two-or three-dimensional case:

(7) $$a(w,v) := (\operatorname{div} w, \operatorname{div} u) + (\operatorname{rot} w, \operatorname{rot} v).$$

In the three-dimensional case $\operatorname{rot} w$ is defined as usual and in two-dimensional case $\operatorname{rot} w$ is defined as the scalar $\frac{\partial w_2}{\partial x_1} - \frac{\partial w_1}{\partial x_2}$ often also denoted by $\operatorname{curl} w$. On the basis of (7), the following orthogonal decomposition of V was derived in [14]:

(8) $$(H_0^1(\Omega))^n = V = V_0 \oplus V_1 \oplus V_\beta,$$

where

(9) $$V_0 = \ker \operatorname{div} = \{w \in V,\ \operatorname{div} w = 0\},$$

(10) $$V_1 = \ker \operatorname{rot} = \{w \in V,\ \operatorname{rot} w = 0\}.$$

The third orthogonal subspace V_β has been characterized in [14] and in Lemma 1 in [11] as consisting of the solutions $u = u(p)$ of the variational problem

(11) $$a(u,v) := b(v,p) \qquad \text{for all } v \in V$$

for harmonic $p \in L_{2,0}$. The space L_2 is decomposed similarly (see [11]) into three orthogonal subspaces:

(12) $$L_2 = P = P_0 \oplus P_1 \oplus P_\beta,$$

where

(13) $$P_0 := \ker \operatorname{grad}, \quad P_1 = \operatorname{div} \ker \operatorname{rot} = \operatorname{div} V_1, \quad P_\beta = \operatorname{div} V_\beta.$$

Here P_0 is the one-dimensional space of functions constant on Ω, P_β consists of harmonic functions (see [14]). Both decomposition (8) and (12) are called Crouzeix–Velte decomposition.

After discretization by finite element or finite difference methods, the Stokes problem (1) takes the following form:

(14) $$\begin{pmatrix} A_h & B_h^T \\ B_h & 0 \end{pmatrix} \begin{pmatrix} u \\ p \end{pmatrix} = \begin{pmatrix} f \\ 0 \end{pmatrix}.$$

Here A_h corresponds to the vector Laplace operator and is a symmetric positive definite matrix of dimension $n_h \times n_h$ where n_h denotes the number of velocity degrees of freedom; the 0 block is of dimension $m_h \times m_h$, with the number m_h of pressure degrees of freedom; B_h corresponds to the negative divergence operator and is of dimension $m_h \times n_h$. Further, u, p and f denote the coefficient vectors of velocities and pressure and of the projection of the force vector, respectively.

3. The staggered grid approximation on a nonequidistant grid

Now we consider the well-known staggered-grid approximation where Ω is the unit square subdivided by a non-equidistant grid. The cell midpoints are pressure nodes, the pressure vector is denoted by p_h and its components by p_{ij}, with $i, j = 1, \ldots, n-1$. The area of the cell of p_{ij} is $h_{1,i+1/2}h_{2,j+1/2}$. The sides of the cells contain as their midpoints the velocity nodes: nodes of the u-components of the velocity are on the east-west sides, nodes of the v-components are on the north-south sides, and there are $(n-1)n$ such nodes of each velocity component (including the boundary nodes). The velocity components are denoted by u_{ij}, $i = 1, \ldots, n$, $j = 1, \ldots, n-1$, and by v_{ij}, $i = 1, \ldots, n-1$, $j = 1, \ldots, n$. Here the u_{ij} with $i = 1$ and $i = n$ are the boundary values of u_h; the v_{ij} with $j = 1$ and $j = n$ are the boundary values of v_h. For the approximation of the Stokes problem we need the discrete divergence operator and the discrete vector Laplace operator. Moreover, we will define also the discrete rotation operator.

3.1. The finite difference approximation on a staggered grid

First we use the finite difference method to approximate the Stokes problem. To simplify the expressions, the following notations will be introduced:

$$\tilde{h}_{1,i+1/2} := \frac{h_{1,i-1/2} + 2h_{1,i+1/2} + h_{1,i+3/2}}{4}$$

$$\tilde{h}_{2,j+1/2} := \frac{h_{2,j-1/2} + 2h_{2,j+1/2} + h_{2,j+3/2}}{4}$$

$$\tilde{h}_{1,i} := \frac{h_{1,i-1/2} + h_{1,i+1/2}}{2}$$

$$\tilde{h}_{2,j} := \frac{h_{2,j-1/2} + h_{2,j+1/2}}{2}$$

For pressure vectors p_h, q_h and velocity vectors $\vec{u}_h = (u_h, v_h)^T$, $\vec{w}_h = (r_h, S_h)^T$ the following discrete scalar products and the corresponding norms are introduced:

$$(p_h, q_h)_{0,h} := \sum_{i,j=1}^{n-1} p_{ij} q_{ij} h_{1,i+1/2} h_{2,j+1/2}, \tag{15}$$

$$\|p_h\|_{0,h}^2 := (p_h, p_h)_{0,h},$$

$$(p_h, q_h)_{0,\tilde{h}} := \sum_{i=1}^{n-2} \sum_{j=1}^{n} p_{ij} q_{ij} \tilde{h}_{1,i+1} \tilde{h}_{2,j} + \tag{16}$$

$$+ \sum_{j=2}^{n-1} p_{0,j} q_{0,j} \tilde{h}_{1,1} \tilde{h}_{2,j} + \sum_{j=2}^{n-1} p_{n-1,j} q_{n-1,j} \tilde{h}_{1,n} \tilde{h}_{2,j},$$

$$\|p_h\|_{0,\tilde{h}}^2 := (p_h, p_h)_{0,\tilde{h}},$$

$$(\vec{u}_h, \vec{w}_h)_{0,h} := \sum_{i=2}^{n-1} \sum_{j=1}^{n-1} u_{ij} r_{ij} \tilde{h}_{1,i} \tilde{h}_{2,j+1/2} + \tag{17}$$

$$+ \sum_{i=1}^{n-1} \sum_{j=2}^{n-1} v_{ij} s_{ij} \tilde{h}_{1,i+1/2} \tilde{h}_{2,j},$$

$$\|\vec{u}_h\|_{0,h}^2 := (\vec{u}_h, \vec{u}_h)_{0,h}.$$

The space $\mathbb{R}^{m_h}$ with the scalar product (15) and corresponding norm will be called the pressure space and denoted by P_h; similarly, the velocity space $\vec{V}_h$ is the space $\mathbb{R}^{n_h}$, $n_h := 2(n-2)(n-1)$, with the scalar product (17) and the corresponding norm — taking into account that the boundary values of the velocity components are zero.

The divergence is approximated as follows:

$$(\mathrm{div}_h, \vec{u}_h)_{ij} := \frac{u_{i+1,j} - u_{ij}}{h_{1,i+1/2}} + \frac{v_{i,j+1} - v_{ij}}{h_{2,j+1/2}}, \tag{18}$$

where $\vec{u}_h := (u_h, v_h)^T$ and $i = 1, \ldots, n-1$, $j = 1, \ldots, n-1$. The matrix corresponding to the mapping $-\operatorname{div} h$ from the velocity space into the pressure space is denoted by B_h. For the approximation of the discrete Laplace operator we continue the grid by two lines for u-nodes: one above the square at a distance $h_{2,n+1/2}/2$ and one below the square at a distance $h_{2,1/2}/2$. Further we continue the grid by two lines for v-nodes: one left to the square at a distance $h_{1,1/2}/2$ and one right to the square at a distance $h_{1,n+1/2}/2$. Putting zero values into the u- or v-nodes on these lines and using the usual Shortley–Weller approximation for the discrete Laplace operator (see, e.g. [10]) in all inner velocity nodes, we get the following first order approximation:

$$\Delta_h \vec{u}_h = (\Delta_h u_h, \Delta_h v_h)^T,$$

$$(\Delta_h u_h)_{ij} := \frac{1}{\tilde{h}_{1,i}} \left(\frac{u_{i+1,j} - u_{ij}}{h_{1,i+1/2}} - \frac{u_{ij} - u_{i-1,j}}{h_{1,i-1/2}} \right) + $$
$$+ \frac{1}{\tilde{h}_{2,j+1/2}} \left(\frac{u_{i,j+1} - u_{ij}}{\tilde{h}_{2,j+1}} - \frac{u_{ij} - u_{i,j-1}}{\tilde{h}_{2,j}} \right),$$

$$2 \le i \le n-1, \quad 1 \le j \le n-1, \tag{19}$$

$$(\Delta_h v_h)_{ij} := \frac{1}{\tilde{h}_{1,i+1/2}} \left(\frac{v_{i+1,j} - v_{ij}}{\tilde{h}_{1,i+1}} - \frac{v_{ij} - v_{i-1,j}}{\tilde{h}_{1,i}} \right) + $$
$$+ \frac{1}{\tilde{h}_{2,j}} \left(\frac{v_{i,j+1} - v_{ij}}{h_{2,j+1/2}} - \frac{v_{ij} - v_{i,j-1}}{h_{2,j-1/2}} \right),$$
$$1 \le i \le n-1, \quad 2 \le j \le n-1.$$

The matrix corresponding to the mapping $-\Delta_h$ from the velocity space into itself is denoted by A_h and is positive definite. Finally, we define the discrete rotation as follows:

$$(\mathrm{rot}_h\, \vec{u}_h)_{ij} := \frac{v_{i+1,j} - v_{ij}}{\tilde{h}_{1,i+1}} - \frac{u_{i+1,j} - u_{i+1,j-1}}{\tilde{h}_{2,j}}, \tag{20}$$

and we introduce the following notation: C_h for the matrix of the operator $\mathrm{rot}\, h$.

THEOREM 1.

$$(A_h \vec{u}_h, \vec{u}_h)_{0,h} = \|B_h \vec{u}_h\|^2_{0,h} + \|C_h \vec{u}_h\|^2_{0,\tilde{h}} \tag{21}$$

holds for all vectors $\vec{u}_h := (u_h, v_h)^T \in \vec{V}_h$ *if and only if*

$$h_{1,i+1/2} = \frac{h_{1,i-1/2} + h_{1,i+3/2}}{2}$$

and

$$h_{2,j+1/2} = \frac{h_{2,j-1/2} + h_{2,j+3/2}}{2}.$$

PROOF. We apply partial summation to $(A_h\vec{u}_h, \vec{u}_h)_{0,h}$ and then continue the grid functions u_h and v_h by zero onto the grid of the whole two-dimensional plane and extend the summation in all expressions to all integer i, j. Then we get:

$$\begin{aligned}
(A_h\vec{u}_h, \vec{u}_h)_{0,h} &= \sum_{i,j} \left(\frac{u_{i+1,j} - u_{ij}}{h_{1,i+1/2}}\right)^2 h_{1,i+1/2}\tilde{h}_{2,j+1/2} + \\
&\quad + \sum_{i,j} \left(\frac{u_{i,j+1} - u_{ij}}{\tilde{h}_{2,j+1}}\right)^2 \tilde{h}_{1,i}\tilde{h}_{2,j+1} + \\
&\quad + \sum_{i,j} \left(\frac{v_{i+1,j} - v_{ij}}{\tilde{h}_{1,i+1}}\right)^2 \tilde{h}_{1,i+1}\tilde{h}_{2,j} + \\
&\quad + \sum_{i,j} \left(\frac{v_{i,j+1} - v_{ij}}{h_{2,j+1/2}}\right)^2 h_{2,j+1/2}\tilde{h}_{1,i+1/2} = \\
&= \sum_{i,j} \left(\frac{\tilde{h}_{2,j+1/2}}{h_{1,i+1/2}}(u_{i+1,j} - u_{ij})^2 + \frac{\tilde{h}_{1,i}}{\tilde{h}_{2,j+1}}(u_{i,j+1} - u_{ij})^2 + \right. \\
&\quad \left. + \frac{\tilde{h}_{2,j}}{\tilde{h}_{1,i+1}}(v_{i+1,j} - v_{ij})^2 + \frac{\tilde{h}_{1,i+1/2}}{h_{2,j+1/2}}(v_{i,j+1} - v_{ij})^2 \right).
\end{aligned}$$

$\|B_h\vec{u}_h\|_{0,h}^2$ may be written as follows:

$$\begin{aligned}
\|B_h\vec{u}_h\|_{0,h}^2 &= \sum_{i,j} \left(\frac{u_{i+1,j} - u_{ij}}{h_{1,i+1/2}} + \frac{v_{i,j+1} - v_{ij}}{h_{2,j+1/2}}\right)^2 h_{1,i+1/2}h_{2,j+1/2} = \\
&= \sum_{i,j} \left(\frac{h_{2,j+1/2}}{h_{1,i+1/2}}(u_{i+1,j} - u_{ij})^2 + \frac{h_{1,i+1/2}}{h_{2,j+1/2}}(v_{i,j+1} - v_{ij})^2 + \right. \\
&\quad \left. + 2(u_{i+1,j} - u_{ij})(v_{i,j+1} - v_{ij}) \right).
\end{aligned}$$

$\|C_h\vec{u}_h\|^2_{0,\tilde{h}}$ can be written as

$$\|C_h\vec{u}_h\|^2_{0,\tilde{h}} = \sum_{i,j}\left(\frac{-(u_{i+1,j}-u_{i+1,j-1})}{\tilde{h}_{2,j}} + \frac{v_{i+1,j}-v_{ij}}{\tilde{h}_{1,i+1}}\right)^2 \tilde{h}_{1,i+1}\tilde{h}_{2,j}$$
$$= \sum_{i,j}\left(\frac{\tilde{h}_{1,i+1}}{\tilde{h}_{2,j}}(u_{i+1,j}-u_{i+1,j-1})^2 + \frac{\tilde{h}_{2,j}}{\tilde{h}_{1,i+1}}(v_{i+1,j}-v_{ij})^2 - \right.$$
$$\left. - 2(u_{i+1,j}-u_{i+1,j-1})(v_{i+1,j}-v_{ij})\right).$$

Performing some index shifts we get:

$$2\sum_{i,j}(u_{i+1,j}-u_{ij})(v_{i,j+1}-v_{ij}) =$$
$$= 2\sum_{i,j}(u_{i+1,j}v_{i,j+1} - u_{i+1,j}v_{ij} - u_{ij}v_{i,j+1} + u_{ij}v_{ij}) =$$
$$= 2\sum_{i,j}(u_{i+1,j-1}v_{ij} - u_{i+1,j}v_{ij} - u_{i+1,j-1}v_{i+1,j} + u_{i+1,j}v_{i+1,j}) =$$
$$= 2\sum_{i,j}(u_{i+1,j}-u_{i+1,j-1})(v_{i+1,j}-v_{ij}).$$

Namely, to get from the second to the third line in the above formulae, in the fourth term we have replaced i by $i+1$, in the first term j by $j-1$, and in the third term both indices have been shifted. Applying both shifts also to the expression $\sum_{i,j}(u_{i,j+1}-u_{ij})^2$ in $(A_h\vec{u}_h,\vec{u}_h)_{0,h}$, we find

$$(A_h\vec{u}_h,\vec{u}_h)_{0,h} - \|B_h\vec{u}_h\|^2_{0,h} - \|C_h\vec{u}_h\|^2_{0,\tilde{h}} =$$
$$= \sum_{i,j}\left(\frac{h_{2,j-1/2}-2h_{2,j+1/2}+h_{2,j+3/2}}{4h_{1,i+1/2}}(u_{i+1,j}-u_{ij})^2 + \right.$$
$$\left. + \frac{h_{1,i-1/2}-2h_{1,i+1/2}+h_{1,i+3/2}}{4h_{2,j+1/2}}(v_{i,j+1}-v_{ij})^2\right).$$

Using the assumptions on the step lengths, we get

$$(A_h\vec{u}_h,\vec{u}_h)_{0,h} - \|B_h\vec{u}_h\|^2_{0,h} - \|C_h\vec{u}_h\|^2_{0,\tilde{h}} = 0. \qquad \blacksquare$$

REMARK 1. It can be shown that $\dim(V_{h0}) = (n-2)^2$, $\dim(V_{h1}) = (n-3)^2$ and $\dim(V_{h,\beta}) = 4n-9$, where n denotes the number of grid points (including corner points) along a side of the square. Namely, to compute $\dim(V_{h0}) = \dim\ker\operatorname{div}_h$ we count the number of conditions to get $\|\operatorname{div}_h \vec{u}_h\|^2_{0,h} = 0$. Excluding the very last cell (which is depending on the other cells) we have to require $\operatorname{div}_h \vec{u}_h = 0$ in all remaining cells, that is, in $(n-1)^2-1$ points. Then $\dim\ker\operatorname{div}_h = n_h - ((n-1)^2-1) = (n-2)^2$ where $n_h = 2(n-1)(n-2)$ denotes the number of velocity degrees of freedom. Similarly, to compute $\dim(V_{h1}) = \dim\ker\operatorname{rot}_h$ we count the number of conditions to get $\|\operatorname{rot}_h \vec{u}_h\|^2_{0,h} = 0$, starting from the boundary of the grid and proceeding to the center. Excluding the corners of the square and the very last cell corner in the center of the grid, in all other cell corners we have to require $\operatorname{rot}_h \vec{u}_h = 0$, that is in n^2-5 points. Then $\dim\ker\operatorname{rot}_h = n_h-(n^2-5) = (n-3)^2$. Therefore $\dim(V_{h,\beta}) = n_h - (n-2)^2 - (n-3)^2 = 4n-9$.

(We remark that basis functions for $\ker\operatorname{rot}_h$ and $\ker\operatorname{div}_h$ have been described in [8], and for $\ker\operatorname{div}_h$ also in [5] and [2].)

REMARK 2. Although A_h is not a symmetric matrix, it is symmetric in the sense of the scalar product (17). This means that (21) can be described in matrix terms as follows:

$$\begin{aligned}(D_A A_h\vec{u}_h,\vec{u}_h) &= (D_B B_h\vec{u}_h, B_h\vec{u}_h) + (D_C C_h\vec{u}_h, C_h\vec{u}_h) = \\ &= (B_h^T D_B B_h\vec{u}_h,\vec{u}_h) + (C_h^T D_C C_h\vec{u}_h,\vec{u}_h).\end{aligned} \tag{22}$$

where $(.\,,.)$ is the Euclidean scalar product and D_A, D_B, D_C are diagonal matrices corresponding to (17), (15) and (16):

$$(D_A u_h)_{k,k} = u_{i,j}\tilde{h}_{1,i}\tilde{h}_{2,j+1/2},$$
$$k = (n-1)(i-2)+j, \qquad 2 \le i \le n-1,\ 1 \le j \le n-1,$$
$$(D_A v_h)_{k,k} = v_{i,j}\tilde{h}_{1,i+1/2}\tilde{h}_{2,j},$$
$$k = (n-2)(i-1)+j-1, \qquad 1 \le i \le n-1,\ 2 \le j \le n-1,$$
$$(D_B p_h)_{k,k} = p_{i,j}h_{1,i+1/2}h_{2,j+1/2},$$
$$k = (n-1)(i-1)+j, \qquad 1 \le i,j \le n-1$$
$$(D_C p_h)_{k,k} = p_{i,j}\tilde{h}_{1,i+1}\tilde{h}_{2,j},$$
$$k = n-2+n(i-1)+j, \qquad 1 \le i \le n-2,\ 1 \le j \le n;$$
$$k = j-1, \qquad i = 0,\ 2 \le j \le n-1;$$
$$k = (n-2)(n+1)+j-1, \qquad i = n-1,\ 2 \le j \le n-1.$$

From (22) we get:

$$D_A A_h = B_h^T D_B B_h + C_h^T D_C C_h. \tag{23}$$

It follows that $D_A A_h$ is a symmetric matrix and can be written in the following form:

$$D_A A_h =: A = B + C, \tag{24}$$

where $B = \tilde{B}^T \tilde{B}$ and $C = \tilde{C}^T \tilde{C}$ with the notation $\tilde{B} = D_B^{1/2} B_h$, $\tilde{C} = D_C^{1/2} C_h$.

Using the staggered grid approximation based on the finite volume method (see the following subsection) we get similarly:

$$D_{\underline{A}} A_{\underline{h}} = B_h^T D_B B_h + C_h^T D_C C_h. \tag{25}$$

where the diagonal matrix $D_{\underline{A}}$ corresponds to the scalar product (32).

Let us mention that using finite element methods — if the discrete Crouzeix–Velte decomposition exists — we can get the following:

$$A_h = B_h^T M_h^{-1} B_h + C_h^T M_h^{-1} C_h. \tag{26}$$

where M_h is the mass matrix (See [6]).

REMARK 3. Remark 1 and 2 mean that a proper Crouzeix–Velte decomposition of the velocity and the pressure space into three nontrivial parts exists, if $n > 3$ (see [12]).

3.2. The staggered grid approximation based on the finite volume method

Now we use the finite volume method (sometimes also called box method) to approximate the Stokes problem ([9]). For this approximation Ω is subdivided into $\Omega = \cup \Omega_{ij}$ rectangular cells. For the approximation of $(\Delta_{\underline{h}} u_h)_{ij}$ we choose the subdivision where the midpoint of Ω_{ij} is u_{ij}. The expression (Δu) is integrated over Ω_{ij} and the Gauss–Ostrogradskij formula is used for transformation of the second order derivatives:

$$\int_{\Omega_{ij}} \operatorname{div}(\operatorname{grad} u) dx_1 dx_2 = \int_{\Gamma_{ij}} (\operatorname{grad} u) \vec{n} ds = \sum_{k=1}^{4} \vec{n}_k \int_{\gamma_k} (\operatorname{grad} u) ds, \tag{27}$$

where n is the normal vector of Γ_{ij}. Using suitable low order quadraturc formulae for approximation of the integrals in (27), $(\Delta_{\underline{h}} u_h)_{ij}$ may be written as follows:

$$(28)\quad (\Delta_{\underline{h}} u_h)_{ij} = \left(\frac{-(u_{ij} - u_{i,j-1})}{\tilde{h}_{2,j}} \tilde{h}_{1,i} + \frac{u_{i+1,j} - u_{ij}}{h_{1,i+1/2}} h_{2,j+1/2} + \right.$$
$$\left. + \frac{u_{i,j+1} - u_{ij}}{\tilde{h}_{2,j+1}} \tilde{h}_{1,i} - \frac{u_{ij} - u_{i-1,j}}{h_{1,i-1/2}} h_{2,j+1/2} \right) \frac{1}{h_{2,j+1/2}} \frac{1}{\tilde{h}_{1,i}} =$$
$$= \frac{1}{\tilde{h}_{1,i}} \left(\frac{u_{i+1,j} - u_{ij}}{h_{1,i+1/2}} - \frac{u_{ij} - u_{i-1,j}}{h_{1,i-1/2}} \right) +$$
$$+ \frac{1}{h_{2,j+1/2}} \left(\frac{u_{i,j+1} - u_{ij}}{\tilde{h}_{2,j+1}} - \frac{u_{ij} - u_{i,j-1}}{\tilde{h}_{2,j}} \right),$$

$$(29)\qquad 2 \le i \le n-1, \quad 1 \le j \le n-1.$$

For approximation of $(\Delta_{\underline{h}} v_h)_{ij}$ the domain Ω is subdivided in a different way. In this case the midpoint of Ω_{ij} sub-domain is v_{ij}. Similar to (28) we get:

$$(30)\qquad (\Delta_{\underline{h}} v_h)_{ij} := \frac{1}{h_{1,i+1/2}} \left(\frac{v_{i+1,j} - v_{ij}}{\tilde{h}_{1,i+1}} - \frac{v_{ij} - v_{i-1,j}}{\tilde{h}_{1,i}} \right) +$$
$$+ \frac{1}{\tilde{h}_{2,j}} \left(\frac{v_{i,j+1} - v_{ij}}{h_{2,j+1/2}} - \frac{v_{ij} - v_{i,j-1}}{h_{2,j-1/2}} \right),$$
$$1 \le i \le n-1, \quad 2 \le j \le n-1$$

and

$$(31)\qquad \Delta_{\underline{h}} \vec{u}_h = (\Delta_{\underline{h}} u_h, \Delta_{\underline{h}} v_h)^T$$

is the discrete vector Laplace operator. For approximation of $(\mathrm{div}_h\, \vec{u}_h)_{ij}$ we use the subdivision of Ω which is determined by the original grid and for $(\mathrm{rot}_h\, \vec{u}_h)_{ij}$ another subdivision where the midpoint of Ω_{ij} is a grid point of the original grid. Integrating the corresponding equations over Ω_{ij} and using suitable quadrature formulae we get the same approximation for $(\mathrm{div}_h\, \vec{u}_h)_{ij}$ and $(\mathrm{rot}_h\, \vec{u}_h)_{ij}$ as in (18) and (20). Instead of (17) we introduce the scalar product and corresponding norm as follows:

$$(32)\qquad (\vec{u}_h, \vec{w}_h)_{0,h^*} := \sum_{i=2}^{n-1} \sum_{j=1}^{n-1} u_{ij} r_{ij} \tilde{h}_{1,i} h_{2,j+1/2} +$$

$$+\sum_{i=1}^{n-1}\sum_{j=2}^{n-1} v_{ij}\, s_{ij}\, h_{1,i+1/2}\tilde{h}_{2,j},$$

(33) $$\|\vec{u}_h\|^2_{0,h^*} := (\vec{u}_h, \vec{u}_h)_{0,h^*}.$$

Using the scalar products and norms (15), (16) and (32), and the approximations (28), (30), (18) and (20), we obtain:

THEOREM 2.

$$(A_{\underline{h}}\vec{u}_h, \vec{u}_h)_{0,h^*} = \|B_h\vec{u}_h\|^2_{0,h} + \|C_h\vec{u}_h\|^2_{0,\tilde{h}}.$$

where let $A_{\underline{h}}$ *be defined as* $-\Delta_{\underline{h}}$ *and* B_h, C_h *are the same as in Theorem* 1.

PROOF. Similarly to the proof of Theorem 1 we apply partial summation to $(A_{\underline{h}}\vec{u}_h, \vec{u}_h)_{0,h^*}$ and then continue the grid functions u_h and v_h by zero onto the whole plane. Then there follows:

$$\begin{aligned}
(A_{\underline{h}}\vec{u}_h, \vec{u}_h)_{0,h^*} &= \sum_{i,j}\left(\frac{u_{i+1,j}-u_{ij}}{h_{1,i+1/2}}\right)^2 h_{1,i+1/2}h_{2,j+1/2}+\\
&\quad+\sum_{i,j}\left(\frac{u_{i,j+1}-u_{ij}}{\tilde{h}_{2,j+1}}\right)^2 \tilde{h}_{1,i}\tilde{h}_{2,j+1}+\\
&\quad+\sum_{i,j}\left(\frac{v_{i+1,j}-v_{ij}}{\tilde{h}_{1,i+1}}\right)^2 \tilde{h}_{1,i+1}\tilde{h}_{2,j}+\\
&\quad+\sum_{i,j}\left(\frac{v_{i,j+1}-v_{ij}}{h_{2,j+1/2}}\right)^2 h_{2,j+1/2}h_{1,i+1/2} =\\
&= \sum_{i,j}\left(\frac{h_{2,j+1/2}}{h_{1,i+1/2}}(u_{i+1,j}-u_{ij})^2 + \frac{\tilde{h}_{1,i}}{\tilde{h}_{2,j+1}}(u_{i,j+1}-u_{ij})^2+\right.\\
&\quad\left.+\frac{\tilde{h}_{2,j}}{\tilde{h}_{1.i+1}}(v_{i+1,j}-v_{ij})^2 + \frac{h_{1,i+1/2}}{h_{2,j+1/2}}(v_{i,j+1}-v_{ij})^2\right).
\end{aligned}$$

The further steps of the proof are the same as in the proof of Theorem 1. ∎

4. Second order staggered grid approximation

Now we consider the staggered grid approximation on an equidistant grid. That is, Ω is the unit square subdivided by a square grid into $m_h := (n-1)^2$ cells of area h^2 each, $h := 1/(n-1)$. For approximation of the Laplace operator with first order, in [11], as usual for staggered grids, the grid has been supplemented by four lines at a distance $h/2$ from the boundary of the square, and fictitious zero values have been put into the u or v nodes on these lines, and then the standard five-point approximation of the Laplace operator has been used near the boundary as well. (The supplementary values are needed also for the standard approximation of rot.)

To get a second order approximation, we take the usual five-point approximation in the inner cells. The corresponding formulae can be obtained by simplifying (19) to our present case of an equidistant grid:

$$
\begin{aligned}
\Delta_{\overline{h}}\vec{u}_h &= (\Delta_{\overline{h}}u_h, \Delta_{\overline{h}}v_h)^T,\\
(\Delta_{\overline{h}}u_h)_{ij} &:= \frac{u_{i+1,j} - 2u_{ij} + u_{i-1,j}}{h^2} + \frac{u_{i,j+1} - 2u_{ij} + u_{i,j-1}}{h^2},\\
&\qquad 2 \le i \le n-1, \quad 2 \le j \le n-2,\\
(\Delta_{\overline{h}}v_h)_{ij} &:= \frac{v_{i+1,j} - 2v_{ij} + v_{i-1,j}}{h^2} + \frac{v_{i,j+1} - 2v_{ij} + v_{i,j-1}}{h^2},\\
&\qquad 2 \le i \le n-2, \quad 2 \le j \le n-1.
\end{aligned}
\tag{34}
$$

For the boundary cells, to get a second order approximation, we put now zero values into additional u or v nodes on the original boundary. These additional points are at a distance of $h/2$ from the nearest u or v point. Therefore, the approximation in the boundary cells will be different from that in the inner cells, but both are of Shortley–Weller type:

$$
\begin{aligned}
(\Delta_{\overline{h}}u_h)_{i,1} &:= \frac{u_{i+1,1} - 2u_{i,1} + u_{i-1,1}}{h^2} + \frac{1}{h}\left(\frac{u_{i,2} - u_{i,1}}{h} - \frac{u_{i,1} - u_{i,0}}{h/2}\right),\\
(\Delta_{\overline{h}}u_h)_{i,n-1} &:= \frac{u_{i+1,n-1} - 2u_{i,n-1} + u_{i-1,n-1}}{h^2} +\\
&\quad + \frac{1}{h}\left(\frac{u_{i,n} - u_{i,n-1}}{h/2} - \frac{u_{i,n-1} - u_{i,n-2}}{h}\right), \quad 2 \le i \le n-1,\\
(\Delta_{\overline{h}}v_h)_{,j} &:= \frac{1}{h}\left(\frac{v_{2,j} - v_{1,j}}{h} - \frac{v_{1,j} - v_{0,j}}{h/2}\right) + \frac{v_{1,j+1} - 2v_{1,j} + v_{1,j-1}}{h^2},
\end{aligned}
$$

$$(\Delta_{\overline{h}} v_h)_{n-1,j} := \frac{1}{h}\left(\frac{v_{n,j}-v_{n-1,j}}{h/2} - \frac{v_{n-1,j}-v_{n-2,j}}{h}\right) + \\ + \frac{v_{n-1,j+1}-2v_{n-1,j}+v_{n-1,j-1}}{h^2}, \qquad 2 \le j \le n-1.$$

Here $u_{i,0}$, $u_{i,n}$, $v_{0,j}$, $v_{n,j}$ are the additional zero values on the boundary. The discrete divergence is the same as in (18):

$$(\mathrm{div}_{\overline{h}}\, \vec{u}_h)_{ij} := \frac{u_{i+1,j}-u_{ij}}{h} + \frac{v_{i,j+1}-v_{ij}}{h}, \qquad 1 \le i,\ j \le n-1,$$

and the discrete rotation is:

$$\begin{aligned}
(\mathrm{rot}_{\overline{h}}\, \vec{u}_h)_{ij} &:= \frac{v_{i+1,j}-v_{ij}}{h} - \frac{u_{i+1,j}-u_{i+1,j-1}}{h}, \\
&\qquad\qquad 1 \le i \le n-2,\ 2 \le j \le n-1, \\
(\mathrm{rot}_{\overline{h}}\, \vec{u}_h)_{i,1} &:= \frac{v_{i+1,1}-v_{i,1}}{h} - \frac{u_{i+1,1}-u_{i+1,0}}{h/2}, \\
(\mathrm{rot}_{\overline{h}}\, \vec{u}_h)_{i,n} &:= \frac{v_{i+1,n}-v_{i,n}}{h} - \frac{u_{i+1,n}-u_{i+1,n-1}}{h/2}, \\
&\qquad 1 \le i \le n-2, \\
(\mathrm{rot}_{\overline{h}}\, \vec{u}_h)_{0,j} &:= \frac{v_{1,j}-v_{0,j}}{h/2} - \frac{u_{1,j}-u_{1,j-1}}{h}, \\
(\mathrm{rot}_{\overline{h}}\, \vec{u}_h)_{n-1,j} &:= \frac{v_{n,j}-v_{n-1,j}}{h/2} - \frac{u_{n,j}-u_{n,j-1}}{h}, \\
&\qquad 2 \le j \le n-1.
\end{aligned} \tag{35}$$

The discrete L_2 scalar products (15) and (17) simplify to

$$(p_h, q_h)_{0,h} := \sum_{i,j=1}^{n-1} p_{ij} q_{ij} h^2, \tag{36}$$

$$(\vec{u}_h, \vec{w}_h)_{0,h} := \sum_{i=2}^{n-1}\sum_{j=1}^{n-1} u_{ij} r_{ij} h^2 + \sum_{i=1}^{n-1}\sum_{j=2}^{n-1} v_{ij} s_{ij} h^2, \tag{37}$$

and the corresponding norms are now

$$\|p_h\|_{0,h}^2 := (p_h, p_h)_{0,h}, \qquad \|\vec{u}_h\|_{0,h}^2 := (\vec{u}_h, \vec{u}_h)_{0,h}.$$

THEOREM 3. *For the staggered grid approximation with second order approximation,*

$$
\begin{aligned}
&(A_{\overline{h}}\vec{u}_h, \vec{u}_h)_{0,h} - \|B_{\overline{h}}\vec{u}_h\|_{0,h}^2 - \|C_{\overline{h}}\vec{u}_h\|_{0,h}^2 = \\
&= -\sum_{i=2}^{n-1} 2(u_{i,1}^2 + u_{i,n-1}^2) - \sum_{j=2}^{n-1} 2(v_{1,j}^2 + v_{n-1,j}^2)
\end{aligned}
\tag{38}
$$

where $B_{\overline{h}} := -\operatorname{div}_{\overline{h}}$ *and* $C_{\overline{h}} := \operatorname{rot}_{\overline{h}}$.

PROOF. Similarly to the proof of Theorem 1 we apply partial summation to $(A_{\overline{h}}\vec{u}_h, \vec{u}_h)_{0,h}$ and then continue the grid functions u_h and v_h by zero onto the grid of the whole two-dimensional plane. Taking into account that

$$u_{i,0},\ u_{i,n},\ v_{0,j},\ v_{n,j},\ u_{1,j},\ u_{n,j},\ v_{i,1},\ v_{i,n}$$

are zero if $1 \le i,\ j \le n-1$, we may write:

$$
\begin{aligned}
(A_{\overline{h}}\vec{u}_h, \vec{u}_h)_{0,h} = &\sum_{i,j} \Big((u_{i+1,j} - u_{ij})^2 + (u_{i,j+1} - u_{ij})^2 + \\
&+ (v_{i+1,j} - v_{ij})^2 + (v_{i,j+1} - v_{ij})^2 \Big) + \\
&+ \sum_{i=2}^{n-1} (u_{i,1}^2 + u_{i,n-1}^2) + \sum_{j=2}^{n-1} (v_{1,j}^2 + v_{n-1,j}^2), \\
\|B_{\overline{h}}\vec{u}_h\|_{0,h}^2 = &\sum_{i,j} \Big((u_{i+1,j} - u_{ij})^2 + (v_{i,j+1} - v_{ij})^2 + \\
&+ 2(u_{i+1,j} - u_{ij})(v_{i,j+1} - v_{ij}) \Big), \\
\|C_{\overline{h}}\vec{u}_h\|_{0,h}^2 = &\sum_{i,j} \Big((u_{i+1,j} - u_{i+1,j-1})^2 + (v_{i+1,j} - v_{ij})^2 - \\
&- 2(u_{i+1,j} - u_{i+1,j-1})(v_{i+1,j} - v_{ij}) \Big) + \\
&+ \sum_{i=1}^{n-2} (3u_{i+1,1}^2 + 3u_{i+1,n-1}^2) + \sum_{j=2}^{n-1} (3v_{1,j}^2 + 3v_{n-1,j}^2),
\end{aligned}
$$

Performing the same index shifts as during the proof of Theorem 1, we get the result of Theorem 3. ∎

REMARK 1. Introducing the notation $\tilde{A}_{\overline{h}}$, where

$$(\tilde{A}_{\overline{h}}\vec{u}_h, \vec{u}_h)_{0,h} = (A_{\overline{h}}\vec{u}_h, \vec{u}_h)_{0,h} + \sum_{i=2}^{n-1} 2(u_{i,1}^2 + u_{i,n-1}^2) + \sum_{j=2}^{n-1} 2(v_{1,j}^2 + v_{n-1,j}^2).$$

$\tilde{A}_{\overline{h}}$ is a symmetric positive definite matrix, together with $A_{\overline{h}}$. Since $\dim(V_{h0}) = (n-2)^2$, $\dim(V_{h1}) = (n-3)^2$ and $\dim(V_{h,\beta}) = 4n-9$, a proper Crouzeix–Velte decomposition exists in this case as well, for $n > 3$. In this case the algebraic decomposition (see [12]) exists not for the matrices $A_{\overline{h}}$, $(B_{\overline{h}}^T B_{\overline{h}})$, $(C_{\overline{h}}^T C_{\overline{h}})$, but for $\tilde{A}_{\overline{h}}$, $(B_{\overline{h}}^T B_{\overline{h}})$, $(C_{\overline{h}}^T C_{\overline{h}})$. Let us mention that in [3] there appears an analytical counterpart of (38). ∎

5. Numerical results

Below we show some computational results for the Stokes problem using the staggered grid approximation based on the finite volume method (28)–(31), (18), (20), on a non-equidistant grid in the unit square. Choosing different nonequidistant grid spacings we have maximized the rate-of convergence for Uzawa-like methods, that is minimized $q_{U_z} := (\overline{\lambda}_h - \underline{\lambda}_h)/(\overline{\lambda}_h + \underline{\lambda}_h)$, where $\underline{\lambda}_h$ and $\overline{\lambda}_h$ are the smallest and the largest of the eigenvalues different from 0 and 1 of the discrete inf-sup problem, i.e. of $S_h = B_h A_{\underline{h}}^{-1} B_h^T$. (Then $[\underline{\lambda}_h, \overline{\lambda}_h]$ contains the eigenvalues corresponding to V_β and $\beta_h = \sqrt{\underline{\lambda}_h}$ is the discrete inf-sup constant.) We used the Matlab eig function to calculate the eigenvalues and the Matlab fmins function for minimization.

We found that the optimal grid is not an equidistant one, but a grid which is condensing in the center and coarser near the boundary of the unit square. Based on preliminary numerical experiments with arbitrary non-equidistant grids we chose a symmetrical grid with the same non-equidistant grid spacing in both directions. From the experimental results we found that the more condensed the grid is in the center, the smaller q_{U_z} is. For the optimal grid we chose the grid spacing in the center as follows: $h_{1,n/2} := \frac{1}{n-1}10^{-2}$. In Table 1 we show the optimal $\underline{\lambda}_h$ and $\overline{\lambda}_h$ denoted by $\underline{\lambda}_{h,*}$ and $\overline{\lambda}_{h,*}$ in the case of $n = 11, 17, 23$. Because of the huge computational time for the grid optimization in the case of $n = 31, 51$ the optimal $\underline{\lambda}_h$ and $\overline{\lambda}_h$ were not calculated. In these cases $\underline{\lambda}_{h,*}$ and $\overline{\lambda}_{h,*}$ were calculated on a grid which is obtained by interpolating the optimal grid for the case of $n = 23$. We give also the smallest and the largest eigenvalues different from 0 and 1 in the case of an equidistant grid, denoted by $\underline{\lambda}_{h,eq}$ and $\overline{\lambda}_{h,eq}$ and in the practically used case when the grid is coarser in the center and condensing near the boundary of the unit square, denoted by $\underline{\lambda}_{h,pr}$ and $\overline{\lambda}_{h,pr}$. In this case the grid spacings are determined by the following expression:

$$h_{1,i-1/2} =$$
$$= 0.111737 - 0.006259i(1-i) - 0.000289(i(1-i))^2 - 0.000011(i(1-i))^3,$$

where $2 \leq i \leq \frac{n+1}{2}$. The grid here is symmetrical also with the same nonequidistant grid spacing in both directions. In the table, n denotes the number of grid points — including corner points — along a side of the square.

Table 1

n	11	17	23	31	51
$\underline{\lambda}_{h,*}$	0.6667	0.6667	0.6667	0.6667	0.6667
$\overline{\lambda}_{h,*}$	0.9999	0.9999	0.9999	0.9999	0.9999
q_{*,U_z}	0.2	0.2	0.2	0.2	0.2
$q_{*,CG}$	0.101	0.101	0.101	0.101	0.101
$\underline{\lambda}_{h,eq}$	0.4016	0.3489	0.3226	0.3022	0.2766
$\overline{\lambda}_{h,eq}$	0.8538	0.8545	0.8549	0.8552	0.8555
q_{eq,U_z}	0.3602	0.4202	0.4521	0.4778	0.5114
$q_{eq,CG}$	0.1864	0.2203	0.239	0.2544	0.275
$\underline{\lambda}_{h,pr}$	0.3507	0.2497	0.2168	0.2031	0.1933
$\overline{\lambda}_{h,pr}$	0.8429	0.8463	0.8469	0.847	0.847
q_{pr,U_z}	0.4124	0.5443	0.5923	0.6132	0.6284
$q_{pr,CG}$	0.2158	0.296	0.328	0.3426	0.3534

From these experimental results we can conclude that compared with the equidistant grid the use of such non-equidistant grids can economize between 58 and 140 percent of the computational work when iterating with an Uzawa-type method and between 36 and 78 percent for conjugate gradient-type methods, where

$$q_{CG} := \left(\sqrt{\overline{\lambda_h}} - \sqrt{\underline{\lambda_h}}\right) / \left(\sqrt{\overline{\lambda_h}} + \sqrt{\underline{\lambda_h}}\right).$$

Compared with the practically used grid we can economize between 82 and 246 percent of the computational work for Uzawa-type methods and between 50 and 120 percent for conjugate gradient-type methods. Moreover, these numbers of gain in percent are increasing together with n. Finally, according to the table, the conjugate gradient-like methods are approximately one and a half times faster than the Uzawa-like ones, on the optimized grids.

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ON THE WEIGHTED LEBESGUE FUNCTIONS FOR GENERAL EXPONENTIAL WEIGHTS

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1. Introduction

In [1] and [2] we established estimates for weighted Lebesgue functions for a class of general exponential weights which includes non-even weights. Here we use different methods to prove some new results on the weighted Lebesgue constants.

Let $I = (c, d)$, $-\infty \le c < 0 < d \le \infty$. Let $w := \exp(-Q)$, where $Q : I \to \mathbb{R}$ is continuous and convex in I, and such that all moments

$$\int_I x^n w^2(x)dx, \ n = 0, 1, 2, \ldots,$$

converge. Corresponding to $w^2(x)$ we form a sequence of orthonormal polynomials

$$p_n(x) := p_n(w^2, x) := \gamma_n x^n + \ldots, \ \gamma_n > 0$$

satisfying

$$\int_I p_n p_m w^2 = \delta_{mn}, \ m, n = 0, 1, 2, \ldots.$$

The zeros of $p_n(x)$ are denoted by

$$c < y_{nn} < y_{n-1,n} < \ldots < y_{2n} < y_{1n} < d \quad (y_{kn} := y_{kn}(w^2))$$

arranged in increasing order.

Let $f : I \to \mathbb{R}$ be a continuous function such that

$$\lim_{x \to c+} |f(x)|w(x) = 0 = \lim_{x \to d-} |f(x)|w(x).$$

Let $L_n[f] \in \mathcal{P}_{n-1}$ denote the Lagrange interpolation polynomial to f at the zeros of $p_n(x)$. Here $\mathcal{P}_{n-1}$ denotes the set of all algebraic polynomials of degree at most $n-1$. Then

$$L_n[f](y_{kn}) = f(y_{kn}),\ 1 \leq k \leq n,$$

and it admits the representation

$$L_n[f](x) = \sum_{k=1}^{n} f(y_{kn}) l_{kn}(x), \tag{1.1}$$

where

$$l_{kn}(x) := \frac{p_n(x)}{p_n'(y_{kn})(x - y_{kn})},\ 1 \leq k \leq n, \tag{1.2}$$

are the fundamental polynomials associated with the zeros of $p_n(x)$. We define the *fundamental polynomials of weighted Lagrange interpolation* by

$$u_{kn}(x) := u_{kn}(w^2, x) := \frac{(p_n w)(x)}{(p_n w)'(y_{kn})(x - y_{kn})}. \tag{1.3}$$

We define the *nth weighted Lebesgue function* by

$$\Lambda_n(w, U_n(w^2), x) := \sum_{k=1}^{n} |l_{kn}(x)| \frac{w(x)}{w(y_{kn})} = \sum_{k=1}^{n} |u_{kn}(x)| \tag{1.4}$$

where $U_n(w^2) := \{y_{kn} : 1 \leq k \leq n\}$.

Our main objective here is to estimate (1.4). In [1] and [2] we established bounds for $\Lambda_n(w, U_n(w^2), x)$ using different methods. The methods we use here are similar to those used by Vértesi [9] (for Freud type weights), Szili and Vértesi [7], [8] (for Erdős type weights and for exponential weights on $[-1, 1]$).

First, we introduce our class of weights. To do this we need the notion of a quasi-increasing function on an interval I: we say that a function $f : (0, d) \to \mathbb{R}$ is *quasi-increasing* if there exists $C > 0$ such that

$$f(x) \leq C f(y),\ 0 < x \leq y < d.$$

Obviously, a monotone increasing function is quasi-increasing. Similarly, we may define the notion of a *quasi-decreasing* function. Following is our class of weights.

DEFINITION. Let $I := (c, d)$ $(-\infty \leq c < 0 < d \leq \infty)$ and $w := \exp(-Q)$, where $Q : I \to [0, \infty)$ satisfies the following properties:

(a) Q' is continuous in I and $Q(0) = 0$;

(b) Q'' exists and is positive in $I \setminus \{0\}$;

(c) $\lim_{t \to c+} Q(t) = \infty = \lim_{t \to d-} Q(t)$.

(d) The function

$$T(t) := \frac{tQ'(t)}{Q(t)}, \qquad t \in I \setminus \{0\} \tag{1.5}$$

is quasi-increasing in $(0, d)$ and quasi-decreasing in $(c, 0)$, with $T(t) \geq \Lambda > 1$, $t \in I \setminus \{0\}$.

(e) There exists $C_1 > 0$ such that

$$\frac{Q''(x)}{|Q'(x)|} \leq C_1 |Q'(x)| Q(x), \; a.e. \; x \in I \setminus \{0\}.$$

(f) There exists a compact subinterval J of the open interval I, and $C_2 > 0$ such that

$$\frac{Q''(x)}{|Q'(x)|} \geq C_2 |Q'(x)| Q(x), \; a.e. \; x \in I \setminus J.$$

Then we write $w \in \mathcal{F}(C^2+)$.

REMARK. The simplest example of the above definition is when $I = \mathbb{R}$ and

$$Q(x) = \begin{cases} x^\alpha, & \text{if } x \in [0, \infty) \\ |x|^\beta, & \text{if } x \in (-\infty, 0) \end{cases}$$

where $\alpha, \beta > 1$. Here it is easy to see that for the function $T(x)$ defined by (1.5) we have

$$T(x) = \begin{cases} \alpha, & \text{if } x \in [0, \infty) \\ \beta, & \text{if } x \in (-\infty, 0). \end{cases}$$

A more general example is

$$Q(x) = Q_{k,l,\alpha,\beta}(x) = \begin{cases} \exp_k(x^\alpha) - \exp_k(0), & \text{if } x \in [0, \infty) \\ \exp_l(|x|^\beta) - \exp_l(0), & \text{if } x \in (-\infty, 0) \end{cases}$$

where $k, l \geq 0$, $\alpha, \beta > 1$. Here $\exp_k$ denotes the kth iterated exponential: $\exp_0(x) = x$ and $\exp_k(x) = \exp(\exp(\ldots \exp(x)))$, $k \geq 1$. See [3] for further discussion of $\mathcal{F}(C^2+)$ and other examples.

2. Notations

In the sequel, $C, C_1, C_2, c, c_1, c_2, \ldots$, will denote positive constants independent of x, k and n. The same symbol does not necessarily represent the same constant in different occurrences.

If (A_n) and (B_n) are real sequences, then we write $A_n \sim B_n$ if there exist $C_1, C_2 > 0$ such that $C_1 \leq \frac{A_n}{B_n} \leq C_2$, $(n \to \infty)$.

For $w \in \mathcal{F}(C^2+)$ and $n \in \mathbb{N}$ we define the *Mhaskar–Rahmanov–Saff numbers* $a_{\pm n}(w) =: a_{\pm n}$ to be the roots of the system of equations

$$\frac{1}{\pi}\int_{a_{-n}}^{a_n} \frac{x\,Q'(x)}{\sqrt{(x-a_{-n})(a_n-x)}}dx = n,$$

$$\frac{1}{\pi}\int_{a_{-n}}^{a_n} \frac{Q'(x)}{\sqrt{(x-a_{-n})(a_n-x)}}dx = 0.$$

The significance of $a_{\pm n}$ lies partly in the identity

$$\max_{x\in I}|(Pw)(x)| = \max_{x\in[a_{-n},a_n]}|(Pw)(x)| \tag{2.1}$$

valid for all polynomials P of degree at most n (cf. [3, Theorem 1.8]). For more on $a_{\pm n}$, see chapters 1 and 3 in [3].

For a fixed $w \in \mathcal{F}(C^2+)$ and for $n \in \mathbb{N}$ we set

$$\beta_n := \frac{a_n + a_{-n}}{2}, \qquad \delta_n := \frac{a_n + |a_{-n}|}{2} \tag{2.2}$$

$$\eta_{\pm n} := \left(n\,T(a_{\pm n})\sqrt{\frac{|a_{\pm n}|}{\delta_n}}\right)^{-2/3}, \tag{2.3}$$

$$D_{\pm n} := T(a_{\pm n})\frac{\delta_n}{|a_{\pm n}|}, \quad D_n^* := \max\{D_{-n}, D_n\}, \tag{2.4}$$

$$\varphi_n(x) := \begin{cases} \frac{1}{n}\frac{|x-a_{-2n}|\,|x-a_{2n}|}{\sqrt{(|x-a_{-n}|+|a_{-n}|\eta_{-n})(|x-a_n|+a_n\eta_n)}}, & \text{if } x \in [a_{-n}, a_n] \\ \varphi_n(a_n), & \text{if } x \in (a_n, d) \\ \varphi_n(a_{-n}), & \text{if } x \in (c, a_{-n}). \end{cases} \tag{2.5}$$

For our weights the restricted range inequality (2.1) can be sharpened (see [3, Theorem 1.9(a) and (1.50)]). Let $w \in \mathcal{F}(C^2+)$ and $M > 0$. Then there exist $C > 0$, $n_0 \in \mathbb{N}$ independent of P and n such that for $n \geq n_0$ and $P \in \mathcal{P}_n$

$$\max_{x \in I} |(Pw)(x)| \leq C \max_{x \in I_{M,n}} |(Pw)(x)|, \tag{2.6}$$

where

$$I_{M,n} := [a_{-n}(1 - M\eta_{-n}), a_n(1 - M\eta_n)]. \tag{2.7}$$

The fundamental properties of orthonormal polynomials $p_n(w^2, x)$ for the weight $w \in \mathcal{F}(C^2+)$ were proved by Levin and Lubinsky in [3]. P. Vértesi [12] supplemented results with respect to the distribution of the roots of $p_n(w^2, x)$. Fix a weight $w \in \mathcal{F}(C^2+)$ and let us define the linear transformations

$$x = \delta_n t + \beta_n, \quad x \in [a_{-n}, a_n], \qquad t = \frac{x - \beta_n}{\delta_n}, \quad t \in [-1, 1]$$

$$t = \cos\vartheta, \quad \vartheta \in [0, \pi].$$

For every $n \in \mathbb{N}$, let

$$t_{kn} := \frac{y_{kn} - \beta_n}{\delta_n} =: \cos\vartheta_{kn}, \quad 1 \leq k \leq n \tag{2.8}$$

$$(y_{kn} := y_{kn}(w^2), t_{kn} := t_{kn}(w^2), \vartheta_{kn} := \vartheta_{kn}(w^2), 1 \leq k \leq n, n \in \mathbb{N}).$$

t_{kn} $(k = 1, 2, \ldots, n)$ are called *normalized roots* of $p_n(w^2)$.

If

$$y_{n+1,n} := a_{-n}, \; y_{0n} := a_n,$$
$$t_{n+1,n} := -1, t_{0n} := 1, \vartheta_{0n} := 0 \; \text{ and } \; \vartheta_{n+1,n} := \pi$$

then we have

$$0 = \vartheta_{0n}(w^2) < \vartheta_{1n}(w^2) < \ldots < \vartheta_{nn}(w^2) < \vartheta_{n+1,n}(w^2) = \pi.$$

3. Results

THEOREM 1. *If $w \in \mathcal{F}(C^2+)$ then the weighted Lebesgue constants satisfy*

$$\max_{x \in I} \Lambda_n(w, U_n(w^2), x) \sim (nD_n^*)^{1/6} \qquad (n \in \mathbb{N}). \tag{3.1}$$

Let the points

$$v_{0n} := v_{0n}(w^2) \quad \text{and} \quad v_{n+1,n} := v_{n+1,n}(w^2)$$

satisfy

$$\begin{aligned} &v_{0n} - y_{1n} = \kappa_+ a_n \eta_n, \quad \kappa_+ > 0 \ \text{ fixed}, \\ &|(p_n w)(x)| \sim |(p_n w)'(y_{1n})||x - y_{1n}| \quad (x \in [y_{1n}, v_{0n}]), \end{aligned} \tag{3.2}$$

and

$$\begin{aligned} &y_{nn} - v_{n+1,n} = \kappa_- |a_{-n}| \eta_{-n}, \quad \kappa_- > 0 \ \text{ fixed}, \\ &|(p_n w)(x)| \sim |(p_n w)'(y_{nn})||x - y_{nn}| \quad (x \in [v_{n+1,n}, y_{nn}]). \end{aligned} \tag{3.3}$$

The existence of such κ_+, κ_- will be proved later (see Lemma 3).

THEOREM 2. *Let* $w \in \mathcal{F}(C^2+)$ *and*

$$V_n(w^2) := U_n(w^2) \cup \{v_{n+1,n}, v_{0n}\}.$$

Then we have

$$\max_{x \in I} \Lambda_n(w, V_n(w^2), x) \sim \log n \qquad (n \in \mathbb{N}). \tag{3.4}$$

THEOREM 3. *For every weight* $w \in \mathcal{F}(C^2+)$ *there esists* $C > 0$ *such that*

$$H_n(w, x) := \sum_{k=1}^{n} u_{kn}^2(w, U_n(w^2), x) \leq C \tag{3.5}$$

for all $x \in I$ *and* $n \in \mathbb{N}$.

4. Proofs

Our main tool is the following lemma which supplements the results with respect to the distribution of the roots of $p_n(w^2, x)$ proved by Levin and Lubinsky in [3].

LEMMA 1. *Let* $w \in \mathcal{F}(C^2+)$. *Then for any constants* $0 < c_1 < \sqrt{D_n}$ *and* $0 < c_2 < \sqrt{D_{-n}}$, *and for* $n \in \mathbb{N}$, *we have*

(a) *if* $\vartheta_{kn} \in \left(0, \frac{\pi}{2}\right]$ *then*

$$\vartheta_{kn} \sim \begin{cases} \left(\dfrac{k}{nD_n}\right)^{1/3}, & \text{if } 1 \leq k \leq c_1 \frac{n}{\sqrt{D_n}}\ ; \\ \dfrac{k}{n}, & \text{otherwise} \end{cases} \tag{4.1}$$

(b) *if* $\vartheta_{kn} \in \left[\frac{\pi}{2}, \pi\right)$ *then*

$$\pi - \vartheta_{kn} \sim \begin{cases} \left(\dfrac{K}{nD_{-n}}\right)^{1/3}, & \text{if } 1 \le K \le c_2\dfrac{n}{\sqrt{D_{-n}}} \\ \dfrac{K}{n}, & \text{otherwise} \end{cases}; \tag{4.2}$$

(c)

$$\vartheta_{k+1,n} - \vartheta_{kn} \sim \begin{cases} \dfrac{1}{(nD_n)^{1/3}(k+1)^{2/3}}, & \text{if } 0 \le k \le c_1\dfrac{n}{\sqrt{D_n}} \\ \dfrac{1}{n}, & \text{if } \dfrac{c_1 n}{\sqrt{D_n}} \le k \le n - \dfrac{c_2 n}{\sqrt{D_{-n}}} \\ \dfrac{1}{(nD_{-n})^{1/3}K^{2/3}}, & \text{if } 1 \le K \le c_2\dfrac{n}{\sqrt{D_{-n}}} \end{cases}; \tag{4.3}$$

(d)

$$\vartheta_{k+1,n} - \vartheta_{kn} \sim \begin{cases} \dfrac{\vartheta_{kn}}{k}, & \text{if } \vartheta_{kn}, \vartheta_{k+1,n} \in \left(0, \frac{\pi}{2}\right] \\ \dfrac{\pi - \vartheta_{kn}}{K}, & \text{if } \vartheta_{kn}, \vartheta_{k+1,n} \in \left[\frac{\pi}{2}, \pi\right) \end{cases}. \tag{4.4}$$

Here $K := n - k + 1$. *Moreover,*

(e) *for every fixed* $A > 0$

$$\vartheta_{kn} \sim \vartheta_{[Ak]n}, \quad \vartheta_{kn}, \vartheta_{[Ak]n} \in \left(0, \frac{\pi}{2}\right), \tag{4.5}$$

and

(f) *for* $\vartheta_{kn}, \vartheta_{jn} \in \left(0, \frac{\pi}{2}\right)$, *we have*

$$|\vartheta_{jn}^2 - \vartheta_{kn}^2| \sim \begin{cases} \vartheta_{jn}^2, & \text{if } 1 \le k \le \frac{j}{2} \\ \vartheta_{jn}^2 \frac{|k-j|}{j}, & \text{if } \frac{j}{2} \le k \le 2j \\ \vartheta_{kn}^2, & \text{if } 2j \le k. \end{cases} \tag{4.6}$$

PROOF. Since for every $w \in \mathcal{F}(C^2+)$ there exist $\varepsilon > 0$, $C > 0$ and $n_0 \in \mathbb{N}$ such that

$$D_n = \frac{T_n \delta_n}{a_n} \le Cn^{2-\varepsilon} \qquad (n \ge n_0)$$

(see [3, (3.38)]) thus

$$\frac{n}{\sqrt{D_n}} \ge Cn^{\varepsilon/2} \to +\infty \quad \text{as} \quad n \to +\infty. \tag{4.7}$$

Similarly we have

$$\frac{n}{\sqrt{D_{-n}}} \geq Cn^{\varepsilon/2} \to +\infty \quad \text{as} \quad n \to +\infty. \tag{4.8}$$

(a), (b) and (c) were proved in [12] and (d) follows from these.

The proof of (e) and (f) is a word-by-word repetition of the proof of (3.5) and (3.6) of [7], so we omit the details. ∎

REMARK. Observe that from (a) it also follows that there exists $c > 0$ independent of n such that

$$1 \leq k \leq cn \quad \text{for } \vartheta_{kn} \in \left(0, \frac{\pi}{2}\right]. \tag{4.9}$$

Moreover, there exists $c > 0$ independent of n such that

$$1 \leq K = n + 1 - k \leq cn \quad \text{for } \vartheta_{kn} \in \left[\frac{\pi}{2}, \pi\right) \tag{4.10}$$

(see (b)).

LEMMA 2. *If* $w \in \mathcal{F}(C^2+)$ *then we have*

$$y_{k-1,n} - y_{kn} \sim y_{kn} - y_{k+1,n} \quad (k = 1, 2, \ldots, n); \tag{4.11}$$

$$|y_{jn} - y_{kn}| \sim \delta_n \sin\left(\frac{\vartheta_{jn} + \vartheta_{kn}}{2}\right) |\vartheta_{jn} - \vartheta_{kn}| \qquad (j, k = 1, 2, \ldots, n); \tag{4.12}$$

$$y_{kn} - y_{k+1,n} \sim \delta_n (\sin\vartheta_{kn})(\vartheta_{k+1,n} - \vartheta_{kn}) \quad (k = 0, 1, \ldots, n); \tag{4.13}$$

$$|(p_n w)'(y_{kn})| \sim \frac{1}{\delta_n^{3/2}} \frac{1}{(\sin\vartheta_{kn})^{3/2}} \frac{1}{\vartheta_{k+1,n} - \vartheta_{k,n}} \qquad (k = 1, 2, \ldots, n); \tag{4.14}$$

PROOF. (4.11) follows from [3, (1.110) and (12.20)].

Using (2.8) we have

$$y_{jn} - y_{kn} = \delta_n(\cos\vartheta_{jn} - \cos\vartheta_{kn}) = 2\delta_n \sin\frac{\vartheta_{jn} + \vartheta_{kn}}{2} \sin\frac{\vartheta_{kn} - \vartheta_{jn}}{2}$$

which yields (4.12). From (4.5) and (4.12) we obtain (4.13).

To verify (4.14) first observe that

$$|(p_n w)'(y_{kn})| = |(p_n' w)(y_{kn})| \sim$$
$$\sim \varphi_n^{-1}(y_{kn})(|y_{kn} - a_{-n}||y_{kn} - a_n|)^{-\frac{1}{4}} \sim$$
$$\sim (y_{kn} - y_{k+1,n})^{-1}(|y_{kn} - a_{-n}||y_{kn} - a_n|)^{-\frac{1}{4}}.$$

Here we have used Theorem 1.19 (a) and (e) in [3]. Next, we continue this as follows: since

$$|y_{kn} - a_{-n}||y_{kn} - a_n| = \delta_n^2(1 - \cos^2\vartheta_{kn}) = \delta_n^2 \sin^2\vartheta_{kn} \tag{4.15}$$

thus for every $k = 1, 2, \dots, n$ we have

$$|(p_n w)'(y_{kn})| \sim (\delta_n(\sin\vartheta_{kn})(\vartheta_{k+1,n} - \vartheta_{kn}))^{-1}(\delta_n^2 \sin^2\vartheta_{kn})^{-1/4} =$$
$$= \delta_n^{-3/2}(\sin\vartheta_{kn})^{-3/2}(\vartheta_{k+1,n} - \vartheta_{kn})^{-1}$$

which is (4.14). ∎

For $x \in I$ let us denote by y_{jn} (one of) the closest node(s) to x (shortly $x \approx y_{jn}$) from $U_n(w^2) = \{y_{kn} \mid k = 1, 2, \dots, n\}$.

LEMMA 3. *If* $w \in \mathcal{F}(C^2+)$ *then*

(a)

$$\max_{x \in I} |u_{kn}(x)| \sim 1 \qquad (n \in \mathbb{N}). \tag{4.16}$$

(b) *Moreover, there exist* $\kappa_+ > 0$, $\kappa_- > 0$ *such that uniformly for* x *and* n, *we have*

$$|(p_n w)(x)| \sim |(p_n w)'(y_{jn})||x - y_{jn}| \qquad (x \approx y_{jn} \in [v_{n+1,n}, v_{0n}],\ n \in \mathbb{N}), \tag{4.17}$$

where

$$v_{0n} := y_{1n} + \kappa_+ a_n \eta_n, \qquad v_{n+1,n} := y_{nn} - \kappa_- |a_{-n}| \eta_{-n}.$$

(c)

$$|u_{kn}(x)| \le c \frac{(\sin\vartheta_{kn})^{3/2}}{\sqrt{\sin\vartheta_{jn}}} \frac{|\vartheta_{k+1,n} - \vartheta_{kn}|}{\sin\frac{\vartheta_{jn}+\vartheta_{kn}}{2} \sin\frac{|\vartheta_{jn}-\vartheta_{kn}|}{2}} \qquad (x \in [v_{n+1,n}, v_{0n}],\ k = 1, 2, \dots, n,\ n \in \mathbb{N}). \tag{4.18}$$

PROOF. (a) The relation (4.16) is Theorem 1.19(c) in [3].

(b) If $x \in [y_{nn}, y_{1n}]$ then (4.17) is Theorem 1.19(d) in [3].

Next we show that there exists $\kappa_+ > 0$ such that

$$|(p_n w)(x)| \sim |(p_n w)'(y_{1n})||x - y_{1n}| \qquad (x \in [y_{1n}, v_{0n}],\ n \in \mathbb{N}). \tag{4.19}$$

(The existence of κ_- can be proved similarly.)

First we observe that by (4.16) we have

$$|(p_n w)(x)| \le C|(p_n w)'(y_{1n})(x - y_{1n})| \tag{4.20}$$

for *all* $x \in I$ and $n \in \mathbb{N}$.

Now let $\kappa_+ > 0$. It will be fixed later. By [3, Theorem 1.19(e)], (4.13) and Lemma 1 we have

$$\varphi_n(y_{1n}) \sim y_{1n} - y_{2n} \sim \delta_n \vartheta_{1n}^2 \sim \frac{\delta_n}{(nD_n)^{2/3}} = a_n \eta_n \tag{4.21}$$

(see also (2.3) and (2.4)).

From Theorem 5.7(b) and (1.50) of [3] we obtain that

$$\varphi_n(x) \sim \varphi_n(y_{1n}) \qquad (x \in [y_{1n}, v_{0n}],\ n \in \mathbb{N}). \tag{4.22}$$

We shall need the Markov–Bernstein inequality for $w \in \mathcal{F}(C^2+)$ (see [3, Theorem 1.15 and (1.50)]) which states that there exists $C > 0$ such that for $n \ge 1$ and $P \in \mathcal{P}_n$

$$|(Pw)'(x)\varphi_n(x)| \le C \max_{x \in I} |(Pw)(x)| \qquad (x \in I).$$

Using this with $Pw = u_{1n}$ we have from (4.16), (4.21) and (4.22)

$$|u_{1n}'(s)| \le \frac{C}{\varphi_n(y_{1n})} \le \frac{C_1}{a_n \eta_n} \qquad (s \in [y_{1n}, v_{0n}]).$$

Thus

$$|u_{1n}'(s)(x - y_{1n})| \le C_1 \kappa_+ \qquad (y_{1n} \le s \le x \le v_{0n} = y_{1n} + \kappa_+ a_n \eta_n).$$

Hence, if $x \in [y_{1n}, v_{0n}]$, we have for some s between y_{1n} and x,

$$|u_{1n}(x)| = |u_{1n}(y_{1n}) + u_{1n}'(s)(x - y_{1n})| \ge$$
$$\ge 1 - |u_{1n}'(s)(x - y_{1n})| \ge 1 - C_1 \kappa_+ \ge \frac{1}{2}$$

if $0 < \kappa_+ \le \frac{1}{2C_1}$. Thus

$$|u_{1n}(x)| = \left| \frac{(p_n w)(x)}{(p_n w)'(y_{1n})(x - y_{1n})} \right| \ge \frac{1}{2}$$

for $x \in [y_{1n}, v_{0n}]$ and $n \in \mathbb{N}$. Combining this and (4.20) we obtain ((4.19) which proves the statement (b).

(c) By (4.17) we get

$$\begin{aligned} |u_{kn}(x)| &= \left| \frac{(p_n w)(x)}{(p_n w)'(y_{kn})(x - y_{kn})} \right| \sim \\ &\sim \left|(p_n w)'(y_{jn})(p_n w)'(y_{kn})\right| \left| \frac{x - y_{jn}}{x - y_{kn}} \right| \le \\ &\le \left|(p_n w)'(y_{jn})(p_n w)'(y_{kn})\right| \left| \frac{y_{j-1,n} - y_{jn}}{y_{jn} - y_{kn}} \right|. \end{aligned} \tag{4.23}$$

Therefore (4.18) follows from Lemma 2. ∎

4.2. Proof of Theorem 1

Since

$$\begin{gathered} v_{0n} - y_{1n} \sim a_n \eta_n \sim a_n - y_{1n}, \\ y_{nn} - v_{n+1,n} \sim |a_{-n}| \eta_{-n} \sim y_{nn} - a_{-n} \end{gathered}$$

(see (3.2), (3.3) and [3, Theorem 1.19(f)]) thus from (2.6) it follows that it is enough to estimate the weighted Lebesgue function $\Lambda_n(w, U_n(w^2), x)$ on the interval $x \in [v_{n+1,n}, v_{0n}]$.

Fix a weight $w \in \mathcal{F}(C^2+)$ and $n \in \mathbb{N}$. First suppose that $x \approx y_{jn} \in$ $\in [\beta_n, v_{0n}]$ (i.e. $\vartheta_{jn} \in (0, \frac{\pi}{2}]$). Let

$$\begin{aligned} A_n &:= \{k \mid y_{kn} \in [\beta_n, a_n)\} = \left\{k \mid 0 < \vartheta_{kn} \le \frac{\pi}{2}\right\}, \\ B_n &:= \{k \mid y_{kn} \in (a_{-n}, \beta_n)\} = \left\{k \mid \frac{\pi}{2} < \vartheta_{kn} < \pi\right\}. \end{aligned} \tag{4.24}$$

Then

$$\Lambda_n(w, U_n(w^2), x) = \sum_{k=1}^{n} |u_{kn}(x)| = \sum_{k \in A_n} |u_{kn}(x)| + \sum_{k \in B_n} |u_{kn}(x)|. \tag{4.25}$$

In order to estimate (4.25), we distinguish several cases.

CASE 1. Let $x \approx y_{jn} \in [\beta_n, v_{0n}]$ and $k \in A_n$. Then $\vartheta_{jn}, \vartheta_{kn} \in \left(0, \frac{\pi}{2}\right]$ thus by (4.18), (4.4), (4.6) and (4.16) we have

$$|u_{kn}(x)| \leq C \begin{cases} \frac{1}{k}\left(\frac{\vartheta_{kn}}{\vartheta_{jn}}\right)^{5/2}, & \text{if } 1 \leq k \leq j/2 \\ \frac{1}{|k-j|+1}, & \text{if } j/2 \leq k \leq 2j \\ \frac{1}{k}\left(\frac{\vartheta_{kn}}{\vartheta_{jn}}\right)^{1/2}, & \text{if } 2j \leq k, k \in A_n. \end{cases} \tag{4.26}$$

Then we obtain that

$$\sum_{k \in A_n} |u_{kn}(x)| \leq$$

$$\leq C \left\{ \sum_{k=1}^{\frac{j}{2}} \frac{1}{k}\left(\frac{\vartheta_{kn}}{\vartheta_{jn}}\right)^{5/2} + \sum_{k=j/2}^{2j} \frac{1}{|k-j|+1} + \sum_{k \geq 2j, k \in A_n} \frac{1}{k}\sqrt{\frac{\vartheta_{kn}}{\vartheta_{jn}}} \right\}.$$

If $k \in A_n$ then $1 \leq k \leq cn$ with a constant $0 < c < 1$ independent of n (see (4.1)).

CASE 1(a). If $1 \leq j \leq \frac{c_1 n}{\sqrt{D_n}}$, then using (4.1) we obtain that

$$\sum_{k \in A_n} |u_{kn}(x)| \leq C \left\{ \sum_{k=1}^{\frac{j}{2}} \frac{1}{k}\left(\frac{k}{j}\right)^{5/6} + \log(2j) + \sum_{k=2j}^{c_1 n D_n^{-\frac{1}{2}}} \frac{1}{k}\left(\frac{k}{j}\right)^{1/6} + \right.$$

$$\left. + \sum_{k=c_1 n D_n^{-\frac{1}{2}}}^{[cn]} \frac{1}{k}\sqrt{\frac{k}{n}}\left(\frac{nD_n}{j}\right)^{1/6} \right\} \leq C_1 \left\{ \left(\frac{1}{j}\right)^{5/6} \sum_{k=1}^{\frac{j}{2}} \frac{1}{k^{1/6}} + \log(2j) + \right.$$

$$\left. + \frac{1}{j^{1/6}} \sum_{k=2j}^{\frac{c_1 n}{\sqrt{D_n}}} \frac{1}{k^{5/6}} + \left(\frac{nD_n}{j}\right)^{1/6} \frac{1}{\sqrt{n}} \sum_{k=\frac{c_1 n}{\sqrt{D_n}}}^{[cn]} \frac{1}{\sqrt{k}} \right\} \leq$$

$$\le C_2\left\{1+\log(2j)+\frac{1}{j^{1/6}}\left(\frac{n}{\sqrt{D_n}}\right)^{1/6}+\left(\frac{nD_n}{j}\right)^{1/6}\right\}\le C_3(nD_n)^{1/6}.$$

CASE 1(b). If $\frac{c_1 n}{\sqrt{D_n}} \le j \le [cn]$, then by (4.1) we obtain that

$$\sum_{k\in A_n}|u_{kn}(x)|\le C\left\{\sum_{k=1}^{\frac{c_1 n}{\sqrt{D_n}}}\frac{1}{k}\left(\frac{k}{nD_n}\right)^{5/6}\left(\frac{n}{j}\right)^{5/2}+\sum_{k=\frac{c_1 n}{\sqrt{D_n}}}^{j/2}\frac{1}{k}\left(\frac{k}{j}\right)^{5/2}+\right.$$

$$\left.+\log(2j)+\sum_{k=2j}^{[cn]}\frac{1}{k}\sqrt{\frac{k}{j}}\right\}\le$$

$$\le C_1\left\{\left(\frac{1}{nD_n}\right)^{5/6}\left(\frac{n}{j}\right)^{5/2}\left(\frac{n}{\sqrt{D_n}}\right)^{5/6}+\frac{1}{j^{5/2}}(j)^{5/2}+\log(2j)+\frac{\sqrt{n}}{\sqrt{j}}\right\}\le$$

$$\le C_2\{1+1+\log(2j)+D_n^{1/4}\}\le C_3(nD_n)^{1/6},$$

where we used the fact that $D_n \le Cn^2$. From these relations it follows that

$$\sum_{k\in A_n}|u_{kn}(x)|\le C(nD_n)^{1/6}\qquad (x\in[\beta_n,v_{0n}],\ n\in\mathbb{N}) \tag{4.27}$$

with a constant $C > 0$ independent of x and n.

CASE 2. Now we estimate the second term of (4.25), i.e. we suppose that $x \approx y_{jn} \in [\beta_n, v_{0n}]$ is a fixed point ($\vartheta_{jn} \in (0, \frac{\pi}{2}]$) and $k \in B_n$, i.e. $y_{kn} \in (a_{-n}, \beta_n]$, $\vartheta_{kn} \in [\frac{\pi}{2}, \pi)$. Then

$$\sin\frac{\vartheta_{jn}+\vartheta_{kn}}{2}\sin\frac{|\vartheta_{jn}-\vartheta_{kn}|}{2}\sim|\vartheta_{jn}^2-\vartheta_{kn}^2|\sim|\vartheta_{jn}-\vartheta_{kn}|.$$

Therefore from (4.18) and (4.4) we get

$$\sum_{k\in B_n}|u_{kn}(x)|\le C\sum_{k\in B_n}\frac{1}{K}\frac{(\pi-\vartheta_{kn})^{5/2}}{\vartheta_{jn}^{1/2}}\frac{1}{|\vartheta_{jn}-\vartheta_{kn}|}. \tag{4.28}$$

CASE 2(a). If $0 < \vartheta_{jn} \le \frac{\pi}{4}$ then $|\vartheta_{kn} - \vartheta_{jn}| \sim 1$. Since $k \in B_n$ thus $cn \le k \le n$ with a constant $c > 0$ independent of n (i.e. $1 \le K = n + 1 -$

$-\, k \le cn$). Consequently by (4.2) we get

$$\sum_{k\in B_n} |u_{kn}(x)| \le C\frac{1}{\vartheta_{jn}^{1/2}}\left\{\sum_{K=1}^{\frac{c_2 n}{\sqrt{D_{-n}}}} \frac{1}{K}\left(\frac{K}{nD_{-n}}\right)^{5/6} + \right.$$

$$\left. + \sum_{K=\frac{c_2 n}{\sqrt{D_{-n}}}}^{[cn]} \frac{1}{K}\left(\frac{K}{n}\right)^{5/2}\right\} \le \frac{C_1}{\vartheta_{jn}^{1/2}}\left\{\frac{1}{D_{-n}^{5/4}} + 1\right\} \le C_2(nD_n)^{1/6}.$$

CASE 2(b). Now, we suppose that $\frac{\pi}{4} \le \vartheta_{jn} \le \frac{\pi}{2}$. If $\vartheta_{kn} \in [\frac{\pi}{2}, \frac{3\pi}{4}]$ then $|\vartheta_{kn} - \vartheta_{jn}| \sim \frac{|k-j|}{n}$ (see (4.4)). If $\vartheta_{kn} \in [\frac{3\pi}{4}, \pi)$ then $|\vartheta_{kn} - \vartheta_{jn}| \sim 1$. Therefore by (4.28) and (4.2) we get

$$\sum_{k\in B_n} |u_{kn}(x)| \le C \sum_{k\in B_n} \frac{1}{K}\frac{(\pi - \vartheta_{kn})^{5/2}}{|\vartheta_{kn} - \vartheta_{jn}|} \le$$

$$\le C\left\{\sum_{\vartheta_{kn}\in[\frac{\pi}{2},\frac{3\pi}{4}]} \frac{1}{K}\frac{n}{|k-j|+1} + \sum_{\vartheta_{kn}\in\left[\frac{3\pi}{4},\pi\right)} \frac{(\pi - \vartheta_{kn})^{5/2}}{K}\right\} \le$$

$$\le C_1\left\{\log n + \sum_{K=1}^{\frac{c_2 n}{\sqrt{D_{-n}}}} \frac{1}{K}\left(\frac{K}{nD_{-n}}\right)^{5/6} + \sum_{K=\frac{c_2 n}{\sqrt{D_{-n}}}}^{[cn]} \frac{1}{K}\left(\frac{K}{n}\right)^{5/2}\right\} \le$$

$$\le C_2 \log n \le C_3(nD_n)^{1/6}.$$

Consequently

$$\sum_{k\in B_n} |u_{kn}(x)| \le C(nD_n)^{1/6} \quad (x \in [\beta_n, \nu_{0n}],\ n \in \mathbb{N}). \tag{4.29}$$

Combining (4.25), (4.27) and (4.29) we get

$$\sum_{k=1}^{n} |u_{kn}(x)| \le C(nD_n)^{1/6} \quad (x \in [\beta_n, \nu_{0n}],\ n \in \mathbb{N}). \tag{4.30}$$

Now let $z_{in} := (y_{in} + y_{i+1,n})/2$ ($\cos\tau_{in} := (z_{in} - \beta_n)/\delta_n$) for $1 \le i \le s$, where s is a fixed index. Then by Theorem 1.19 of [3], (4.1) and (4.15) we have

$$|(p_n w)(z_{in})| \sim \frac{y_{in} - y_{i+1,n}}{\varphi(y_{in})(|y_{in} - a_{-n}||y_{in} - a_n|)^{1/4}} \sim$$

$$\sim (|y_{in} - a_{-n}||y_{in} - a_n|)^{-1/4} \sim \frac{1}{\sqrt{\delta_n \sin\vartheta_{in}}} \sim \frac{1}{\sqrt{\delta_n}} \left(\frac{nD_n}{i}\right)^{1/6}.$$

Let

$$B_{1,n} := \left\{ k \;\middle|\; \frac{\pi}{2} \le \vartheta_{kn} < \pi - \frac{c}{\sqrt{D_{-n}}} \right\} = \left\{ K \;\middle|\; c_2 \frac{n}{\sqrt{D_{-n}}} \le K \le c_3 n \right\}$$

(see (4.2)). Then by (4.2), (4.4) and (4.14)

$$\sum_{k=1}^{n} |u_{kn}(z_{in})| \sim \left(\frac{nD_n}{i}\right)^{1/6} \sum_{k=1}^{n} \frac{(\sin\vartheta_{kn})^{3/2}|\vartheta_{k,n} - \vartheta_{k+1,n}|}{|\tau_{in}^2 - \vartheta_{kn}^2|} \ge$$

$$\ge C(nD_n)^{1/6} \sum_{K \in B_{1,n}} \frac{(\pi - \vartheta_{kn})^{5/2}}{K} \ge C_1 (nD_n)^{1/6} \sum_{K = \frac{c_2 n}{\sqrt{D_{-n}}}}^{[c_3 n]} \frac{1}{K} \left(\frac{K}{n}\right)^{5/2} \ge$$

$$\ge C_2 (nD_n)^{1/6}.$$

This together with (4.30) gives

$$\max_{x \in [\beta_n, v_{0n}]} \sum_{k=1}^{n} |u_{kn}(x)| \sim (nD_n)^{1/6} \quad (n \in \mathbb{N}).$$

For $x \in [v_{n+1,n}, \beta_n]$ similar estimate holds if one replaces D_n by D_{-n}. Thus Theorem 2 is proved. ∎

4.3. Proof of Theorem 2

It is sufficient to prove that

$$\max_{x \in [\beta_n, v_{0n}]} \Lambda_n(w, V_n(w^2), x) \sim \log n \qquad (n \in \mathbb{N}) \tag{4.31}$$

(see Part 4.2).

The weighted Lebesgue function has the form

$$\Lambda_n(w, V_n(w^2), x) = \sum_{k=1}^{n} \left| \frac{(x - v_{n+1,n})(x - v_{0n})}{(y_{kn} - v_{n+1,n})(y_{kn} - v_{0n})} \right| |u_{kn}(x)| +$$

$$+ \left| \frac{(p_n w)(x)}{(p_n w)(v_{0n})} \frac{x - v_{n+1,n}}{v_{0n} - v_{n+1,n}} \right| + \left| \frac{(p_n w)(x)}{(p_n w)(v_{n+1,n})} \frac{x - v_{0n}}{v_{0n} - v_{n+1,n}} \right| =:$$

$$=: \sum\nolimits_1 (x) + S_1(x) + S_2(x).$$

Let $x \approx y_{jn} \in [\beta_n, v_{0n}]$ (i.e. $\vartheta_{jn} \in (0, \frac{\pi}{2}]$). Then by (4.17), (4.14), (4.13) and (4.1) we have

$$|(p_n w)(x)| \sim |(p_n w)'(y_{jn})(x - y_{jn})| \leq C |(p_n w)'(y_{jn})(y_{j+1,n} - y_{jn})| \leq$$

$$\leq C_1 \frac{1}{\sqrt{\delta_n \vartheta_{jn}}} \leq \frac{C_2}{\sqrt{\delta_n}} \left(\frac{n D_n}{j} \right)^{1/6},$$

moreover

$$|(p_n w)(v_{0n})| \geq C |(p_n w)'(y_{1n})(v_{0n} - y_{1n})| \geq$$

$$\leq C_1 \frac{1}{\delta_n^{3/2}} \frac{1}{\vartheta_{1n}^{5/2}} a_n \eta_n \geq \frac{C_2}{\sqrt{\delta_n \vartheta_{1n}}} \geq C_3 \frac{1}{\sqrt{\delta_n}} (n D_n)^{1/6}$$

(see (4.21)). From these it follows that

$$S_1(x) \leq C \qquad (x \in [\beta_n, v_{0n}],\ n \in \mathbb{N}). \tag{4.32}$$

Similarly we get

$$S_2(x) \leq C \qquad (x \in [\beta_n, v_{0n}],\ n \in \mathbb{N}). \tag{4.33}$$

For the estimation of $\sum_1(x)$ we need

$$\begin{aligned} \frac{|x - v_{n+1,n}||x - v_{0n}|}{|y_{kn} - v_{n+1,n}||y_{kn} - v_{0n}|} &\sim \frac{|x - a_{-n}||x - a_n|}{|y_{kn} - a_{-n}||y_{kn} - a_n|} \sim \\ &\sim \frac{|y_{jn} - a_{-n}||y_{jn} - a_n|}{|y_{kn} - a_{-n}||y_{kn} - a_n|} \sim \left(\frac{\sin \vartheta_{jn}}{\sin \vartheta_{kn}} \right)^2 \end{aligned} \tag{4.34}$$

(see (2.8)). Split $\sum_1(x)$ into two parts

$$\sum\nolimits_1 (x) = \sum_{k \in A_n} (x) + \sum_{k \in B_n} (x). \tag{4.35}$$

In order to estimate the first sum, we distinguish two cases.

CASE 1. Let $1 \le j \le c_1 \frac{n}{\sqrt{D_n}}$ and $k \in A_n$, i.e. $\vartheta_{jn}, \vartheta_{kn} \in \left(0, \frac{\pi}{2}\right]$. From (4.26) and (4.34) we obtain that

$$\sum_{k \in A_n} (x) \le C \left\{ \sum_{k=1}^{j/2} \left(\frac{\vartheta_{jn}}{\vartheta_{kn}}\right)^2 \frac{1}{k} \left(\frac{\vartheta_{kn}}{\vartheta_{jn}}\right)^{5/2} + \sum_{k=j/2}^{2j} \frac{1}{|k-j|+1} + \right.$$

$$\left. + \sum_{k=2j}^{[cn]} \left(\frac{\vartheta_{jn}}{\vartheta_{kn}}\right)^2 \frac{1}{k} \sqrt{\frac{\vartheta_{kn}}{\vartheta_{jn}}} \right\} = C \left\{ \sum_{k=1}^{j/2} \frac{1}{k} \sqrt{\frac{\vartheta_{kn}}{\vartheta_{jn}}} + \log(2j) + \sum_{k=2j}^{[cn]} \frac{1}{k} \frac{\vartheta_{jn}^{3/2}}{\vartheta_{kn}^{3/2}} \right\}.$$

(Here we used the Remark in Part 4.1.)

CASE 1(a). If $1 \le j \le c_1 \frac{n}{\sqrt{D_n}}$ then using (4.1) we get

$$\sum_{k \in A_n} (x) \le C \left\{ \sum_{k=1}^{j/2} \frac{1}{k} \left(\frac{k}{j}\right)^{1/6} + \log(2j) + \sum_{k=2j}^{c_1 \frac{n}{\sqrt{D_n}}} \frac{1}{k} \left(\frac{j}{k}\right)^{1/2} + \right.$$

$$\left. + \sum_{k=c_1 \frac{n}{\sqrt{D_n}}}^{[cn]} \frac{1}{k} \left(\frac{j}{nD_n}\right)^{1/2} \left(\frac{n}{k}\right)^{3/2} \right\} \le C_1 \left\{ \frac{1}{j^{1/6}} \sum_{k=1}^{j/2} \frac{1}{k^{5/6}} + \log(2j) + \right.$$

$$\left. + \sum_{k=2j}^{c_1 \frac{n}{\sqrt{D_n}}} \frac{j^{1/2}}{k^{3/2}} + \left(\frac{j}{nD_n}\right)^{1/2} \sum_{k=c_1 \frac{n}{\sqrt{D_n}}}^{[cn]} \frac{n^{3/2}}{k^{5/2}} \right\} \le$$

$$\le C_2 \left\{ 1 + \log(2j) + 1 + \left(\frac{j\sqrt{D_n}}{n}\right)^{1/2} \right\} \le C_3 \log(2j).$$

CASE 1(b). If $c_1 \frac{n}{\sqrt{D_n}} \le j \le [cn]$ then

$$\sum_{k \in A_n} (x) \le C \left\{ \sum_{k=1}^{c_1 \frac{n}{\sqrt{D_n}}} \frac{1}{k} \left(\frac{k}{nD_n}\right)^{1/6} \sqrt{\frac{n}{j}} + \right.$$

$$\left. + \sum_{k=c_1 \frac{n}{\sqrt{D_n}}}^{j/2} \frac{1}{k} \sqrt{\frac{k}{j}} + \log(2j) + \sum_{k=2j}^{[cn]} \frac{1}{k} \left(\frac{j}{k}\right)^{3/2} \right\} \le$$

$$\leq C_1 \left\{ \left(\frac{1}{nD_n}\right)^{1/6} \left(\frac{n}{j}\right)^{1/2} \sum_{k=1}^{c_1 \frac{n}{\sqrt{D_n}}} \frac{1}{k^{5/6}} + \sum_{k=c_1 \frac{n}{\sqrt{D_n}}}^{j/2} \frac{1}{\sqrt{jk}} + \log(2j) + \right.$$

$$\left. + \sum_{k=2j}^{[cn]} \frac{j^{3/2}}{k^{5/2}} \right\} \leq C_2 \left\{ \left(\frac{1}{nD_n}\right)^{1/6} \left(\frac{n}{j}\right)^{1/2} \left(\frac{n}{\sqrt{D_n}}\right)^{1/6} + 1 + \log(2j) + 1 \right\} \leq$$

$$\leq C_3 \left\{ \sqrt{\frac{n}{j\sqrt{D_n}}} + \log(2j) \right\} \leq C_4 \log(2j).$$

Thus we proved that

$$\sum_{k \in A_n} (x) \leq C \log n \qquad (x \in [\beta_n, v_{0n}],\ n \in \mathbb{N}). \tag{4.36}$$

CASE 2. Now we estimate the second term of (4.35), i.e. we suppose that $x \approx y_{jn} \in [\beta_n, v_{0n}]$ is a fixed point ($\vartheta_{jn} \in (0, \frac{\pi}{2}]$) and $k \in B_n$. From (4.34), (4.18) and Lemma 1 we get

$$\begin{aligned} \sum_{k \in B_n} (x) &\leq C \sum_{k \in B_n} \left(\frac{\sin\vartheta_{jn}}{\sin\vartheta_{kn}}\right)^2 \frac{1}{K} \frac{(\pi - \vartheta_{kn})^{5/2}}{\vartheta_{jn}^{1/2}} \frac{1}{|\vartheta_{jn} - \vartheta_{kn}|} \leq \\ &\leq C_1 \sum_{k \in B_n} \frac{\vartheta_{jn}^{3/2}}{K} \frac{(\pi - \vartheta_{kn})^{1/2}}{|\vartheta_{jn} - \vartheta_{kn}|} \end{aligned} \tag{4.37}$$

CASE 2(a). If $0 < \vartheta_{jn} \leq \frac{\pi}{4}$ then $|\vartheta_{kn} - \vartheta_{jn}| \sim 1$. Since $k \in B_n$ thus $cn \leq k \leq n$ with a constant $c > 0$ independent of n (i.e. $1 \leq K = n + 1 - - k \leq cn$). Consequently by (4.2) we get

$$\sum_{k \in B_n} (x) \leq C\vartheta_{jn}^{3/2} \left\{ \sum_{K=1}^{c_2 \frac{n}{\sqrt{D_{-n}}}} \frac{1}{K} \left(\frac{K}{nD_{-n}}\right)^{1/6} + \sum_{K=c_2 \frac{n}{\sqrt{D_{-n}}}}^{[cn]} \frac{1}{K} \left(\frac{K}{n}\right)^{1/2} \right\} \leq$$

$$\leq C_1 \vartheta_{jn}^{3/2} \left\{ \left(\frac{1}{nD_{-n}}\right)^{1/6} \left(\frac{n}{\sqrt{D_{-n}}}\right)^{1/6} + \frac{1}{\sqrt{n}}\sqrt{n} \right\} \leq C_2.$$

CASE 2(b). Now, we suppose that $\frac{\pi}{4} \leq \vartheta_{jn} \leq \frac{\pi}{2}$. If $\vartheta_{kn} \in \left[\frac{\pi}{2}, \frac{3\pi}{4}\right]$ then $|\vartheta_{kn} - \vartheta_{jn}| \sim \frac{|k-j|}{n}$ (see (4.4)). If $\vartheta_{kn} \in [\frac{3\pi}{4}, \pi)$ then $|\vartheta_{kn} - \vartheta_{jn}| \sim 1$.

Therefore by (4.37) and (4.2) we get

$$\sum_{k\in B_n}(x)\le C\sum_{k\in B_n}\frac{1}{K}\frac{(\pi-\vartheta_{kn})^{1/2}}{|\vartheta_{kn}-\vartheta_{jn}|}\le$$
$$\le C\left\{\sum_{\vartheta_{kn}\in[\frac{\pi}{2},\frac{3\pi}{4}]}\frac{1}{K}\frac{n}{|k-j|+1}+\sum_{\vartheta_{kn}\in[\frac{3\pi}{4},\pi)}\frac{(\pi-\vartheta_{kn})^{1/2}}{K}\right\}\le$$
$$\le C_1\left\{\log n+\sum_{K=1}^{\frac{c_2 n}{\sqrt{D_{-n}}}}\frac{1}{K}\left(\frac{K}{nD_{-n}}\right)^{1/6}+\sum_{K=\frac{c_2 n}{\sqrt{D_{-n}}}}^{[cn]}\frac{1}{K}\left(\frac{K}{n}\right)^{1/2}\right\}\le$$
$$\le C_2\{\log n+1+1\}\le C_3\log n.$$

Consequently

$$\sum_{k\in B_n}(x)\le C\log n\quad (x\in[\beta_n,v_{0n}],\ n\in\mathbb{N}).$$

Combining this with (4.35) and (4.36) we get

$$\sum\nolimits_1(x)\le C\log n\quad (x\in[\beta_n,v_{0n}],\ n\in\mathbb{N}).$$

Therefore using (4.32) and (4.33) we obtain that

$$\Lambda_n(w,V_n(w^2),x)\le C\log n\quad (x\in[\beta_n,v_{0n}],\ n\in\mathbb{N}).$$

As in Part 4.2 we can prove that

$$\Lambda_n(w,V_n(w^2),z_{in})\ge C\log n,$$

where $z_{in}:=(y_{in}+y_{i+1,n})/2$ for $1\le s$ (s is a fixed index, independent of n) which proves (4.31). ∎

4.4. Proof of Theorem 3

Here it suffices to show that

$$\sum_{k=1}^{n}|u_{kn}(x)|^2=\sum_{k\in A_n}|u_{kn}(x)|^2+\sum_{k\in B_n}|u_{kn}(x)|^2\le C \tag{4.38}$$
$$(x\in[\beta_n,v_{0n}],\ n\in\mathbb{N}).$$

(see Part **4.2** and (4.24)).

CASE 1. Let $x \approx y_{jn} \in [\beta_n, v_{0n}]$ and $k \in A_n$. By (4.26) we get

$$\sum_{k\in A_n} |u_{kn}(x)|^2 \le C \left\{ \sum_{k=1}^{j/2} \frac{1}{k^2} \left(\frac{\vartheta_{kn}}{\vartheta_{jn}} \right)^5 + \sum_{k=j/2}^{2j} \frac{1}{(|k-j|+1)^2} + \sum_{\substack{k=2j \\ k\in A_n}} \frac{1}{k^2} \frac{\vartheta_{kn}}{\vartheta_{jn}} \right\}.$$

We consider two cases.

CASE 1(a). If $1 \le j \le c_1 \frac{n}{\sqrt{D_n}}$, then by (4.1) we have

$$\sum_{k\in A_n} |u_{kn}(x)|^2 \le C \left\{ \sum_{k=1}^{j/2} \frac{1}{k^2} \left(\frac{k}{j} \right)^{5/3} + 1 + \right.$$

$$\left. + \sum_{k=2j}^{c_1 \frac{n}{\sqrt{D_n}}} \frac{1}{k^2} \left(\frac{k}{j} \right)^{1/3} + \sum_{k=c_1 \frac{n}{\sqrt{D_n}}}^{[cn]} \frac{1}{k^2} \frac{k}{n} \left(\frac{nD_n}{j} \right)^{1/3} \right\} \le$$

$$\le C_1 \left\{ \frac{1}{j^{5/3}} \sum_{k=1}^{j/2} \frac{1}{k^{1/3}} + 1 + \frac{1}{j^{1/3}} \sum_{k=2j}^{c_1 \frac{n}{\sqrt{D_n}}} \frac{1}{k^{5/3}} + \frac{1}{n} \left(\frac{nD_n}{j} \right)^{1/3} \times \right.$$

$$\left. \times \sum_{k=c_1 \frac{n}{\sqrt{D_n}}}^{[cn]} \frac{1}{k} \right\} \le C_2 \left\{ \frac{1}{j} + 1 + \frac{1}{j^{1/3}} + \frac{1}{n} \left(\frac{nD_n}{j} \right)^{1/3} \log n \right\} \le$$

$$\le C_3 \left\{ 1 + 1 + 1 + \left(\frac{D_n}{n^2} \right)^{1/3} \log n \right\} \le C_4,$$

where we used (4.7).

CASE 1(b). If $c_1 \frac{n}{\sqrt{D_n}} \le j \le [cn]$, then (4.1) gives

$$\sum_{k\in A_n} |u_{kn}(x)|^2 \le C \left\{ \sum_{k=1}^{c_1 \frac{n}{\sqrt{D_n}}} \frac{1}{k^2} \left(\frac{k}{nD_n} \right)^{5/3} \left(\frac{n}{j} \right)^5 + \sum_{k=c_1 \frac{n}{\sqrt{D_n}}}^{j/2} \frac{1}{k^2} \left(\frac{k}{j} \right)^5 + \right.$$

$$\left. + \sum_{k=j/2}^{2j} \frac{1}{(|k-j|+1)^2} + \sum_{k=2j}^{[cn]} \frac{1}{k^2} \frac{k}{j} \right\} \le$$

$$\leq C_1 \left\{ \left(\frac{1}{nD_n}\right)^{5/3} \left(\frac{n}{j}\right)^5 \sum_{k=1}^{c_1 \frac{n}{\sqrt{D_n}}} \frac{1}{k^{1/3}} + \frac{1}{j^5} \sum_{k=c_1 \frac{n}{\sqrt{D_n}}}^{j/2} k^3 + 1 + \frac{1}{j} \sum_{k=2j}^{[cn]} \frac{1}{k} \right\} \leq$$

$$\leq C_2 \left\{ \frac{1}{j^5} \left(\frac{n^2}{D_n}\right)^2 + \frac{1}{j} + 1 + \frac{1}{j} \log n \right\} \leq$$

$$\leq C_3 \left\{ \left(\frac{\sqrt{D_n}}{n}\right)^5 \left(\frac{n^2}{D_n}\right)^2 + 1 + 1 + \left(\frac{\sqrt{D_n}}{n}\right) \log n \right\} \leq$$

$$\leq C_4 \left\{ \frac{\sqrt{D_n}}{n} + \frac{\sqrt{D_n}}{n} \log n \right\} \leq C_5,$$

where we used (4.7).

CASE 2. Suppose that $x \approx y_{jn} \in [\beta_n, v_{0n}]$ is a fixed point ($\vartheta_{jn} \in (0, \frac{\pi}{2}]$) and $k \in B_n$. From (4.18) and (4.4) we get

$$\sum_{k \in B_n} |u_{kn}(x)|^2 \leq C \sum_{k \in B_n} \frac{1}{K^2} \frac{(\pi - \vartheta_{kn})^5}{\vartheta_{jn}} \frac{1}{|\vartheta_{jn} - \vartheta_{kn}|^2}. \tag{4.39}$$

CASE 2(a). If $0 < \vartheta_{jn} \leq \frac{\pi}{4}$ then by (4.2) we have

$$\sum_{k \in B_n} |u_{kn}(x)|^2 \leq$$

$$\leq C \frac{1}{\vartheta_{jn}} \left\{ \sum_{K=1}^{c_2 \frac{n}{\sqrt{D_{-n}}}} \frac{1}{K^2} \left(\frac{K}{nD_{-n}}\right)^{5/3} + \sum_{K=c_2 \frac{n}{\sqrt{D_{-n}}}}^{[cn]} \frac{1}{K^2} \left(\frac{K}{n}\right)^5 \right\} \leq$$

$$\leq C_1 \frac{1}{\vartheta_{jn}} \left\{ \left(\frac{1}{nD_{-n}}\right)^{5/3} \sum_{K=1}^{c_2 \frac{n}{\sqrt{D_{-n}}}} \frac{1}{K^{1/3}} + \frac{1}{n^5} \sum_{K=c_2 \frac{n}{\sqrt{D_{-n}}}}^{[cn]} K^3 \right\} \leq$$

$$\leq C_2 \frac{1}{\vartheta_{jn}} \left\{ \left(\frac{1}{nD_{-n}}\right)^{5/3} \left(\frac{n}{\sqrt{D_{-n}}}\right)^{2/3} + \frac{1}{n} \right\} \leq$$

$$\leq C_3 \left(\frac{nD_n}{j}\right)^{1/3} \left\{\frac{1}{nD_{-n}^2} + \frac{1}{n}\right\} \leq C_4 \left(\frac{\sqrt{D_n}}{n}\right)^{2/3} \leq C_5.$$

CASE 2(b). Let $\frac{\pi}{4} \leq \vartheta_{jn} \leq \frac{\pi}{2}$. Since

$$|\vartheta_{jn} - \vartheta_{kn}| \sim \begin{cases} \dfrac{|k-j|}{n}, & \text{if } \vartheta_{kn} \in \left[\frac{\pi}{2}, \frac{3}{4}\pi\right] \\ 1, & \text{if } \vartheta_{kn} \in \left[\frac{3}{4}\pi, \pi\right) \end{cases}$$

thus by (4.39) and (4.2) we obtain that

$$\sum_{k \in B_n} |u_{kn}(x)|^2 \leq C \sum_{k \in B_n} \frac{1}{K^2} \frac{(\pi - \vartheta_{kn})^5}{|\vartheta_{kn} - \vartheta_{jn}|^2} \leq$$

$$\leq C_1 \left\{ \sum_{\vartheta_{kn} \in [\frac{\pi}{2}, \frac{3}{4}\pi]} \frac{1}{K^2} \frac{n^2}{(|k-j|+1)^2} + \sum_{\vartheta_{kn} \in [\frac{3}{4}\pi, \pi)} \frac{(\pi - \vartheta_{kn})^5}{K^2} \right\} \leq$$

$$\leq C_2 \left\{ 1 + \sum_{K=1}^{c_2 \frac{n}{\sqrt{D_{-n}}}} \frac{1}{K^2} \left(\frac{K}{nD_{-n}}\right)^{5/3} + \sum_{K=c_2 \frac{n}{\sqrt{D_{-n}}}}^{[cn]} \frac{1}{K^2} \left(\frac{K}{n}\right)^5 \right\} \leq$$

$$\leq C_3 \left\{ 1 + \left(\frac{1}{nD_{-n}}\right)^{5/3} \left(\frac{n}{\sqrt{D_{-n}}}\right)^{2/3} + \frac{1}{n} \right\} \leq C_4$$

which proves Theorem 3. ∎

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A NOTE ON THE "GOOD" NODES OF WEIGHTED LAGRANGE INTERPOLATION FOR NON-EVEN WEIGHTS

By

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In [4] J. Szabados established the connection between "good" nodes of weighted Lagrange interpolation and the Lebesgue constants. L. Szili [5] extended this result to other even exponential weights. Here we generalize these results to include *non-even exponential weights*. These point systems serve as basis for Erdős type convergence processes in weighted interpolation (see [6]–[8]).

In defining our class of weights, we need the notion of a quasi-increasing function. We say that a function $f : (0, d) \to \mathbb{R}$ is *quasi-increasing* if there exists $C > 0$ such that

$$0 < x \leq y < d \quad \Rightarrow \quad f(x) \leq Cf(y).$$

Obviously, a monotone increasing function is quasi-increasing. Similarly, we may define the notion of a *quasi-decreasing* function. Following is our class of weights (see [3, Chapter 1]).

DEFINITION. Let $I := (c, d)$ $(-\infty \leq c < 0 < d \leq \infty)$ and $w := \exp(-Q)$, where $Q : I \to [0, \infty)$ satisfies the following properties:

(a) Q' is continuous in I and $Q(0) = 0$;

(b) Q'' exists and is positive in $I \setminus \{0\}$;

(c) $\lim_{t\to c+} Q(t) = \infty = \lim_{t\to d-} Q(t)$.

(d) The function

$$T(t) := \frac{tQ'(t)}{Q(t)}, \qquad t \in I \setminus \{0\} \tag{1}$$

is quasi-increasing in $(0, d)$ and quasi-decreasing in $(c, 0)$, with $T(t) \geq \Lambda > 1$, $t \in I \setminus \{0\}$.

(e) There exists $C_1 > 0$ such that

$$\frac{Q''(x)}{|Q'(x)|} \le C_1 |Q'(x)| Q(x), \ a.e.\ x \in I \setminus \{0\}.$$

(f) There exists a compact subinterval J of the open interval I, and $C_2 > 0$ such that

$$\frac{Q''(x)}{|Q'(x)|} \ge C_2 \frac{|Q'(x)|}{Q(x)}, \ a.e.\ x \in I \setminus J.$$

Then we write $w \in \mathcal{F}(C^2+)$.

Let

$$X_n := \{x_{nn} < x_{n-1,n} < \ldots < x_{2n} < x_{1n}\} \subset I \tag{2}$$

be an interpolatory matrix and suppose that $w \in \mathcal{F}(C^2+)$. It is known that the weighted Lebesgue constants $\Lambda_n(w, X_n)$ $(n \in \mathbb{N})$ play an important role in the convergence-divergence behaviour of weighted Lagrange interpolation polynomials. $\Lambda_n(w, X_n)$ is defined as the sup norm on I of the weighted Lebesgue function

$$\lambda_n(w, X_n, x) := \sum_{k=1}^{n} |\ell_{kn}(x)| \frac{w(x)}{w(x_{kn})} = \sum_{k=1}^{n} |q_{kn}(x)|, \tag{3}$$

where

$$\begin{gathered} q_{kn}(x) := q_{kn}(w, X_n, x) := (\pi_n w)(x)(\pi_n' w)(x_{kn})(x - x_{kn}) \\ (1 \le k \le n,\ n \in \mathbb{N}) \end{gathered} \tag{4}$$

are the *fundamental functions of weighted Lagrange interpolation*. Here $\pi_n(x) = c_n \prod_{k=1}^{n} (x - x_{kn})$.

For a fixed weight $w \in \mathcal{F}(C^2+)$ and for all $n \in \mathbb{N}$ we define the *Mhaskar–Rahmanov–Saff numbers* $a_{\pm n} := a_{\pm n}(w)$ to be the roots of the system of equations

$$\frac{1}{\pi} \int_{a_{-n}}^{a_n} \frac{x Q'(x)}{\sqrt{(x - a_{-n})(a_n - x)}} \, dx = n,$$

$$\frac{1}{\pi} \int_{a_{-n}}^{a_n} \frac{Q'(x)}{\sqrt{(x - a_{-n})(a_n - x)}} \, dx = 0.$$

The significance of $a_{\pm n}$ lies partly in the identity

$$\|Pw\| := \max_{x\in I} |(Pw)(x)| = \max_{x\in[a_{-n},a_n]} |(Pw)(x)|$$

valid for all polynomials P of degree at most n (shortly $P \in \mathcal{P}_n$). For more on $a_{\pm n}$, see Chapters 1 and 3 in [3].

For a fixed weight $w \in \mathcal{F}(C^2+)$ and for $n \in \mathbb{N}$ we set

$$\delta_n := \frac{1}{2}(a_n + |a_{-n}|), \quad \eta_{\pm n} := \left(nT(a_{\pm n})\sqrt{\frac{|a_{\pm n}|}{\delta_n}}\right)^{-2/3},$$
(5)
$$D_{\pm n} := T(a_{\pm n})\frac{\delta_n}{|a_{\pm n}|}, \quad D_n^* := \max\{D_{-n}, D_n\}.$$

Our generalization of Proposition 2 in [4] and Theorem in [5] is the following

THEOREM 1. *Let $w \in \mathcal{F}(C^2+)$. If $r_n \in \mathcal{P}_n$ satisfies*

$$\|r_n w\| < e^{C_0\frac{n}{\sqrt{D_n^*}}} \tag{6}$$

with a constant $C_0 > 0$ (independent of n), moreover for a point $y \in I$ we have $(r_n w)(y) = 1$ then

$$a_{-n}\left(1 + C_2\eta_{-n}(\log\|r_n w\|)^{2/3}\right) \le y \le a_n\left(1 + C_1\eta_n(\log\|r_n w\|)^{2/3}\right), \tag{7}$$

where the constants $C_1, C_2 > 0$ depend only on w.

REMARK. For every weight $w \in \mathcal{F}(C^2+)$ there exist $\varepsilon > 0$, $C > 0$ and $n_0 \in \mathbb{N}$ such that

$$D_n = \frac{T(a_n)\delta_n}{a_n} \le Cn^{2-\varepsilon} \qquad (n \ge n_0)$$

(see [3, Lemma 3.7]) and hence

$$\frac{n}{\sqrt{D_n}} \ge Cn^{\varepsilon/2} \to \infty \quad \text{as} \quad n \to \infty.$$

(Similar relations hold for D_{-n}.)

Let's consider a point system X_n for which the weighted fundamental polynomials q_{kn} (see (4)) are uniformly bounded on I, i.e. there exists a constant $A > 0$ such that

$$|q_{kn}(w, X_n, x)| \le A \qquad (x \in I,\ k = 1, 2, \ldots, n,\ n \in \mathbb{N}).$$

This will be called an $E(w)$-*system* (the letter E reminds of Erdős). For many weights and for these type point systems one can construct convergent sequence of weighted Lagrange interpolation polynomials of degree at most $n(1+\varepsilon)$, where $\varepsilon > 0$ is a fixed real number (see [6]–[8]).

From the Theorem 1 and the Remark above immediately follows the following

COROLLARY 2. *Let $w \in \mathcal{F}(C^2+)$ and suppose that the point system X_n ($n \in \mathbb{N}$) is an $E(w)$-system. Then there exist $C_1, C_2 > 0$ such that*

$$a_{-n}(1 + C_2\eta_{-n}) \le x_{kn} \le a_n(1 + C_1\eta_n) \qquad (k = 1, 2, \dots, n,\ n \in \mathbb{N}). \tag{8}$$

REMARK. For even exponential weights L. Szili and P. Vértesi proved (8) using different methods (see [6]–[8]).

Now let's consider the Lebesgue constants $\Lambda_n(w, X_n)$ ($n \in \mathbb{N}$). Let $y_n \in I$ such that

$$\Lambda_n(w, X_n) = \lambda_n(w, X_n, y_n)$$

and consider the weighted polynomial

$$(r_n w)(x) := \sum_{k=1}^{n} \big(\operatorname{sgn} q_{kn}(y_{kn})\big) q_{kn}(x).$$

Then

$$|(r_n w)(x)| \le \lambda_n(w, X_n, x) \le \Lambda_n(w, X_n) = (r_n w)(y_n),$$

that is, $\|r_n w\| = \Lambda_n(w, X_n)$. Since $|(r_n w)(x_{kn})| = 1$, it follows from Theorem 1 that we have

COROLLARY 3. *Let $w \in \mathcal{F}(C^2+)$. Suppose that the point system X_n is such that*

$$\Lambda_n(w, X_n) < e^{C_0 \frac{n}{\sqrt{D_n^*}}} \tag{9}$$

with a constant $C_0 > 0$ independent of n. Then there exist $C_1, C_2 > 0$ such that

$$a_{-n}(1 + C_2\eta_{-n}B_n) \le x_{kn} \le a_n(1 + C_1\eta_n B_n) \qquad (k = 1, 2, \dots, n,\ n \in \mathbb{N}), \tag{10}$$

where

$$B_n := \big(\log \Lambda_n(w, X_n)\big)^{2/3}.$$

For $w \in \mathcal{F}(C^2+)$ it is known that there exists a point system for which $\Lambda_n(w, X_n) \sim \log n$, $n \in \mathbb{N}$. For the construction of these point systems, see [1] and [2]. Moreover P. Vértesi [9] proved that the $\log n$ order is the best possible. Thus the “best” weighted Lagrange interpolation point systems satisfy

$$a_{-n}(1 + C_2\eta_{-n}(\log\log n)^{2/3}) \le x_{kn} \le a_n(1 + C_1\eta_n(\log\log n)^{2/3}) \tag{11}$$
$$(k = 1, 2, \ldots, n,\ n \in \mathbb{N})$$

with some constants $C_1, C_2 > 0$ depending only on w.

Proof of Theorem 1. Let us fix $w \in \mathcal{F}(C^2+)$ and $n \in \mathbb{N}$. For every $r_n \in \mathcal{P}_n$ we have

$$|(r_n w)(x)| \le e^{U_n(x)} \|r_n w\| \qquad (x \in I) \tag{12}$$

where U_n is the “decaying factor” for weighted polynomials, see Lemma 4.4, (1.93), pp. 254 and 473 in [3]. Note that here we have used (4.12) in [3] with $\Omega = t - \frac{2}{p}$.

We shall prove our statement only for the interval $(0, d)$. We can obtain the analogous statement for $(c, 0)$ by replacing x by $-x$.

Let us suppose that $x > 0$. We need the following properties of U_n (see [3, Lemma 4.5 and (1.93)]:

(a) $U_n(x) = 0$ if $x \in [0, a_n]$;

(b) $U_n(x)$ is decreasing and negative for $x \in [a_n, d)$;

(c) for $K > 1$ there exist $C > 0$ and $n_0 \in \mathbb{N}$ such that for $n \ge n_0$ and $a_n \le x \le a_{Kn}$

$$U_n(x) \le -Cn\,T(a_n)\sqrt{\frac{a_n}{\delta_n}}\left(\frac{x}{a_n} - 1\right)^{3/2} = -C\left[\frac{1}{\eta_n}\left(\frac{x}{a_n} - 1\right)\right]^{3/2}, \tag{13}$$

moreover for $x \ge a_{Kn}$

$$U_n(x) \le -Cn\sqrt{\frac{a_n}{T(a_n)\delta_n}} = -C\frac{n}{\sqrt{D_n}}.$$

From Lemma 3.11 in [3], it follows that for $K > 1$, we have

$$\left(\frac{\frac{a_{Kn}}{a_n} - 1}{\eta_n}\right)^{3/2} \sim \left(\frac{1}{T(a_n)\eta_n}\right)^{3/2} = \left[T(a_n)\left(n\,T(a_n)\sqrt{\frac{a_n}{\delta_n}}\right)^{-2/3}\right]^{-3/2} =$$
$$= n\sqrt{\frac{a_n}{T(a_n)\delta_n}} = \frac{n}{\sqrt{D_n}}.$$

From this relation, we obtain that there exists $C_0 > 0$ such that

$$|(r_n w)(x)| \le e^{-C_0 \frac{n}{\sqrt{D_n}}} \|r_n w\| \quad \text{if} \quad x > a_{Kn}.$$

Therefore, if $r_n \in \mathcal{P}_n$ satisfies (6) then $|(r_n w)(x)| < 1$ for $x \ge a_{Kn}$. This means that if for a point $y \in I$ we have $(r_n w)(y) = 1$ then $y < a_{Kn}$ and by (12) and (13) we obtain

$$1 = |(r_n w)(y)| \le e^{-C_1 (\frac{1}{\eta_n} (\frac{y}{a_n} - 1))^{3/2}} \|r_n w\|$$

which gives

$$y \le a_n \left(1 + C_1 \eta_n (\log \|r_n w\|)^{2/3}\right). \tag{14}$$

As we observed earlier, replacing x by $-x$ gives

$$y \ge a_{-n} \left(1 + C_2 \eta_{-n} (\log \|r_n w\|)^{2/3}\right).$$

which together with (14) gives the statement of the theorem. ∎

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A UNIFIED THEORY OF CONTRA-CONTINUITY FOR FUNCTIONS

By

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1. Introduction

Semi-open sets, preopen sets, α-open sets, β-open sets and δ-open sets play an important role in the researches of generalizations of continuity in topological spaces. By using these sets many authors introduced and studied various types of generalizations of continuity. In 1996, DONTCHEV [13] introduced the notion of contra-continuous functions. Recently, new types of contra-continuous functions are introduced and studied: for example, subcontra-continuity [6], contra θ-semi-continuity [9], contra-super-continuity [16], contra-α-continuity [17], contra-semi-continuity [14], contra-precontinuity [18], contra-β-continuity [7]. On the other hand, the present authors introduced and investigated the notions of m-continuous functions [50], almost m-continuous functions [52] and weakly m-continuous functions [53].

In this paper, we introduce the notion of contra m-continuous functions as functions from a set X satisfying some minimal conditions into a topological space and investigate their properties and the relationships between contra m-continuity and other related generalized forms of continuity. It turns out that the contra m-continuity is a unified form of several modifications of contra-continuity due to DONTCHEV [13].

2. Preliminaries

Let (X,τ) be a topological space and A a subset of X. The closure of A and the interior of A are denoted by $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$, respectively. A subset A is said to be *regular closed* (resp. *regular open*) if $\mathrm{Cl}(\mathrm{Int}(A)) = A$ (resp. $\mathrm{Int}(\mathrm{Cl}(A)) = A$). A point $x \in X$ is called a *δ-cluster* (resp. *θ-cluster*) point of A if $\mathrm{Int}(\mathrm{Cl}(V)) \cap A \neq \emptyset$ (resp. $\mathrm{Cl}(V) \cap A \neq \emptyset$) for every open set V containing x. The set of all δ-cluster (resp. θ-cluster) points of A is called the *δ-closure* (resp *θ-closure*) of A and is denoted by $\mathrm{Cl}_\delta(A)$ (resp. $\mathrm{Cl}_\theta(A)$). If $A = \mathrm{Cl}_\delta(A)$ (resp. $A = \mathrm{Cl}_\theta(A)$), then A is said to be δ-closed (resp. θ-closed). The complement of a δ-closed (resp. θ-closed) set is called a δ-open (resp. θ-open) set. The union of all δ-open (resp. θ-open) sets contained in a subset A is called the *δ-interior* (resp. *θ-interior*) of A and is denoted by $\mathrm{Int}_\delta(A)$ (resp. $\mathrm{Int}_\theta(A)$).

DEFINITION 2.1. Let (X, τ) be a topological space. A subset A of X is said to be *semi-open* [22] (resp. *preopen* [29], *α-open* [36], *β-open* [1] or *semi-preopen* [3]) if $A \subset \mathrm{Cl}(\mathrm{Int}(A))$, (resp. $A \subset \mathrm{Int}(\mathrm{Cl}(A))$, $A \subset$ $\subset \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(A)))$, $A \subset \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(A)))$).

The family of all semi-open (resp. preopen, α-open, β-open) sets in X is denoted by $\mathrm{SO}(X)$ (resp. $\mathrm{PO}(X)$, $\alpha(X)$, $\beta(X)$).

DEFINITION 2.2. The complement of a semi-open (resp. preopen, α-open, β-open) set is said to be *semi-closed* [11] (resp. *preclosed* [29], *α-closed* [30], *β-closed* [1] or *semi-preclosed* [3]).

DEFINITION 2.3. The intersection of all semi-closed (resp. preclosed, α-closed, β-closed) sets of X containing A is called the *semi-closure* [11] (resp. *preclosure* [15], *α-closure* [30], *β-closure* [2] or *semi-preclosure* [3]) of A and is denoted by $\mathrm{sCl}(A)$ (resp. $\mathrm{pCl}(A)$, $\alpha\ \mathrm{Cl}(A)$, ${}_\beta\ \mathrm{Cl}(A)$ or $\mathrm{spCl}(A)$).

DEFINITION 2.4. The union of all semi-open (resp. preopen, α-open, β-open) sets of X contained in A is called the *semi-interior* (resp. *preinterior*, *α-interior*, *β-interior* or *semi-preinterior*) of A and is denoted by $\mathrm{sInt}(A)$ (resp. $\mathrm{pInt}(A)$, $\alpha\ \mathrm{Int}(A)$, ${}_\beta\ \mathrm{Int}(A)$ or $\mathrm{spInt}(A)$).

Throughout the present paper, (X, τ) and (Y, σ) denote topological spaces and $f : (X, \tau) \to (Y, \sigma)$ presents a (single valued) function from a topological space (X, τ) into a topological space (Y, σ).

DEFINITION 2.5. A function $f : (X, \tau) \to (Y, \sigma)$ is said to be *contra-continuous* [13] (resp. *contra-super-continuous* [16], *contra-semi-continuous* [14], *contra-precontinuous* [18], *contra α-continuous* [17], *contra β-continuous* [7]) if $f^{-1}(V)$ is closed (resp. δ-closed, semi-closed, preclosed, α-closed, β-closed) for every open set V of Y.

3. Contra m-continuous functions

DEFINITION 3.1. A subfamily m_X of the power set $\mathcal{P}(X)$ of a nonempty set X is called a *minimal structure* (briefly *m-structure*) on X if $\emptyset \in m_X$ and $X \in m_X$. By (X, m_X), we denote a nonempty set X with a minimal structure m_X on X. Each member of m_X is said to be *m_X-open* and the complement of an m_X-open set is said to be *m_X-closed.*

REMARK 3.1. Let (X, τ) be a topological space. Then the families τ, SO(X), PO(X), $\alpha(X)$, $\beta(X)$ are all m-structures on X.

DEFINITION 3.2. Let X be a nonempty set and m_X an m-structure on X. For a subset A of X, the *m_X-closure* of A and the *m_X-interior* of A are defined in [27] as follows:

(1) $m_X\text{-}\mathrm{Cl}(A) = \bigcap\{F : A \subset F, X - F \in m_X\}$,

(2) $m_X\text{-}\mathrm{Int}(A) = \bigcup\{U : U \subset A, U \in m_X\}$.

REMARK 3.2. Let (X, τ) be a topological space and A a subset of X. If $m_X = \tau$ (resp. SO(X), PO(X), $\alpha(X)$, $\beta(X)$), then we have

(1) $m_X\text{-}\mathrm{Cl}(A) = \mathrm{Cl}(A)$ (resp. $\mathrm{sCl}(A)$, $\mathrm{pCl}(A)$, $\alpha\,\mathrm{Cl}(A)$, ${}_\beta\,\mathrm{Cl}(A)$),

(2) $m_X\text{-}\mathrm{Int}(A) = \mathrm{Int}(A)$ (resp. $\mathrm{sInt}(A)$, $\mathrm{pInt}(A)$, $\alpha\,\mathrm{Int}(A)$, ${}_\beta\,\mathrm{Int}(A)$).

LEMMA 3.1. (Maki [27]) *Let X be a nonempty set and m_X a minimal structure on X. For subsets A and B of X, the following properties hold:*

(1) $m_X\text{-}\mathrm{Cl}(X - A) = X - (m_X\text{-}\mathrm{Int}(A))$ *and* $m_X\text{-}\mathrm{Int}(X - A) = X - (m_X\text{-}\mathrm{Cl}(A))$,

(2) *If* $(X - A) \in m_X$, *then* $m_X\text{-}\mathrm{Cl}(A) = A$ *and if* $A \in m_X$, *then* $m_X\text{-}\mathrm{Int}(A) = A$,

(3) $m_X\text{-}\mathrm{Cl}(\emptyset) = \emptyset$, $m_X\text{-}\mathrm{Cl}(X) = X$, $m_X\text{-}\mathrm{Int}(\emptyset) = \emptyset$ *and* $m_X\text{-}\mathrm{Int}(X) = X$,

(4) *If* $A \subset B$, *then* $m_X\text{-}\mathrm{Cl}(A) \subset m_X\text{-}\mathrm{Cl}(B)$ *and* $m_X\text{-}\mathrm{Int}(A) \subset m_X\text{-}\mathrm{Int}(B)$,

(5) $A \subset m_X\text{-}\mathrm{Cl}(A)$ *and* $m_X\text{-}\mathrm{Int}(A) \subset A$,

(6) $m_X\text{-}\mathrm{Cl}(m_X\text{-}\mathrm{Cl}(A)) = m_X\text{-}\mathrm{Cl}(A)$ *and* $m_X\text{-}\mathrm{Int}(m_X\text{-}\mathrm{Int}(A)) = m_X\text{-}\mathrm{Int}(A)$.

LEMMA 3.2. (Popa and Noiri [49]) *Let X be a nonempty set with a minimal structure m_X and A a subset of X. Then $x \in m_X\text{-}\mathrm{Cl}(A)$ if and only if $U \cap A \neq \emptyset$ for every $U \in m_X$ containing x.*

DEFINITION 3.3. A minimal structure m_X on a nonempty set X is said to have the *property* ($\mathscr{B}$) [27] if the union of any family of subsets belonging to m_X belongs to m_X.

LEMMA 3.3. (Popa and Noiri [48]) *For a minimal structure m_X on a nonempty set X, the following properties are equivalent:*

(1) *m_X has the property ($\mathscr{B}$);*

(2) *If m_X-Int$(V) = V$, then $V \in m_X$;*

(3) *If m_X-Cl$(F) = F$, then $X - F \in m_X$.*

LEMMA 3.4. *Let X be a nonempty set and m_X a minimal structure on X satisfying the property ($\mathscr{B}$). For a subset A of X, the following properties hold:*

(1) *$A \in m_X$ if and only if m_X-Int$(A) = A$,*

(2) *A is m_X-closed if and only if m_X-Cl$(A) = A$,*

(3) *m_X-Int$(A) \in m_X$ and m_X-Cl(A) is m_X-closed.*

PROOF. This follows immediately from Lemmas 3.1 and 3.3.

DEFINITION 3.4. A function $f : (X, m_X) \to (Y, \sigma)$, where X is a nonempty set with an m-structure m_X and (Y, σ) is a topological space, is said to be *m-continuous* [50] (resp. *almost m-continuous* [52], *weakly m-continuous* [53]) if for each point $x \in X$ and each open set V of Y containing $f(x)$, there exists $U \in m_X$ containing x such that $f(U) \subset V$ (resp. $f(U) \subset \mathrm{Int}(\mathrm{Cl}(V))$, $f(U) \subset \mathrm{Cl}(V)$).

THEOREM 3.1. *For a function $f : (X, m_X) \to (Y, \sigma)$, where (X, m_X) is a nonempty set with an m-structure m_X and (Y, σ) is a topological space,the following properties are equivalent:*

(1) *f is m-continuous;*

(2) *$f^{-1}(V) = m_X$-Int$(f^{-1}(V))$ for every open set V of (Y, σ);*

(3) *$f^{-1}(K) = m_X$-Cl$(f^{-1}(K))$ for every closed set K of (Y, σ).*

PROOF. This follows from Theorem 3.1 of [50].

COROLLARY 3.1. (Popa and Noiri [48]) *Let X be a nonempty set with an m-structure m_X satisfying the property ($\mathscr{B}$). For a function $f : (X, m_X) \to$ $\to (Y, \sigma)$, the following properties are equivalent:*

(1) *f is m-continuous;*

(2) *$f^{-1}(V) \in m_X$ for every open set V of (Y, σ);*

(3) *$f^{-1}(K)$ is m_X-closed in (X, m_X) for every closed set K of (Y, σ).*

DEFINITION 3.5. A function $f : (X, m_X) \to (Y, \sigma)$ is said to be *contra m-continuous* if $f^{-1}(V) = m_X\text{-Cl}(f^{-1}(V))$ for every open set V of (Y, σ). And also f is said to be *contra m-continuous at* $x \in X$ if for each closed set F containing $f(x)$, there exists $U \in m_X$ containing x such that $f(U) \subset F$.

REMARK 3.3. The notions of m-continuity and contra m-continuity are independent by Examples 2.1 and 2.2 of [7] and Examples 2.1 and 2.2 of [18].

DEFINITION 3.6. Let A be a subset of a topological space (X, τ). The set $\bigcap\{U \in \tau : A \subset U\}$ is called the *kernel* of A [32] and is denoted by Ker(A). In [26], the kernel of A is called a *Λ-set*.

LEMMA 3.5. (Jafari and Noiri [16]) *For subsets A and B of a topological space (X, τ), the following properties hold:*

(1) $x \in \text{Ker}(A)$ *if and only if* $A \cap F \neq \emptyset$ *for any closed set F containing x,*

(2) *If A is open in (X, τ), then* $A = \text{Ker}(A)$,

(3) *If* $A \subset B$, *then* $\text{Ker}(A) \subset \text{Ker}(B)$.

THEOREM 3.2. *For a function $f : (X, m_X) \to (Y, \sigma)$, the following properties are equivalent:*

(1) *f is contra m-continuous;*

(2) $f^{-1}(F) = m_X\text{-Int}(f^{-1}(F))$ *for every closed set F of Y;*

(3) *for each $x \in X$, f is contra m-continuous at x;*

(4) $f(m_X\text{-Cl}(A)) \subset \text{Ker}(f(A))$ *for every subset A of X;*

(5) $m_X\text{-Cl}(f^{-1}(B)) \subset f^{-1}(\text{Ker}(B))$ *for every subset B of Y.*

PROOF. (1) $\Rightarrow$ (2): Let F be any closed set of Y. Then $Y - F$ is open and $f^{-1}(Y - F) = m_X\text{-Cl}(f^{-1}(Y - F))$. By Lemma 3.1, we have $X - f^{-1}(F) = X - [m_X\text{-Int}(f^{-1}(F))]$. Therefore, we have $f^{-1}(F) = m_X\text{-Int}(f^{-1}(F))$.

(2) $\Rightarrow$ (3): Let $x \in X$ and F be a closed set of Y containing $f(x)$. Then $x \in f^{-1}(F)$. By (2), $x \in m_X\text{-Int}(f^{-1}(F))$. There exists $U \in m_X$ containing x such that $x \in U \subset f^{-1}(F)$. Then, $x \in U$ and $f(U) \subset F$.

(3) $\Rightarrow$ (4): Let A be any subset of X. Let $x \in m_X\text{-Cl}(A)$ and F be a closed set of Y containing $f(x)$. Then by (3) there exists $U \in m_X$ containing x such that $f(U) \subset F$; hence $x \in U \subset f^{-1}(F)$. Since $x \in m_X\text{-Cl}(A)$, by

Lemma 3.2 $U \cap A \neq \emptyset$ and hence $\emptyset \neq f(U \cap A) \subset f(U) \cap f(A) \subset F \cap f(A)$. By Lemma 3.5, we have $f(x) \in \mathrm{Ker}(f(A))$ and hence $f(m_X\text{-}\mathrm{Cl}(A)) \subset \mathrm{Ker}(f(A))$.

(4) $\Rightarrow$ (5): Let B be any subset of Y. By (4) and Lemma 3.5,

$$f(m_X\text{-}\mathrm{Cl}(f^{-1}(B))) \subset \mathrm{Ker}(f(f^{-1}(B))) \subset \mathrm{Ker}(B)$$

and hence $m_X\text{-}\mathrm{Cl}(f^{-1}(B)) \subset f^{-1}(\mathrm{Ker}(B))$.

(5) $\Rightarrow$ (1): Let V be any open set of Y. Then by (5) and Lemma 3.5 we have $m_X\text{-}\mathrm{Cl}(f^{-1}(V)) \subset f^{-1}(\mathrm{Ker}(V)) = f^{-1}(V)$. By Lemma 3.1, $m_X\text{-}\mathrm{Cl}(f^{-1}(V)) = f^{-1}(V)$. This shows that f is contra m-continuous.

COROLLARY 3.2. *Let X be a nonempty set with a minimal struture m_X satisfying the property ($\mathscr{B}$) and (Y, σ) a topological space. For a function $f : (X, m_X) \to (Y, \sigma)$, the following properties are equivalent:*

(1) *f is contra m-continuous;*

(2) *$f^{-1}(F) \in m_X$ for every closed set F of Y;*

(3) *$f^{-1}(V)$ is m_X-closed in (X, m_X) for every open set V of Y.*

REMARK 3.4. Let (X, τ) and (Y, σ) be topological spaces. We put $m_X = \tau$ (resp. $\mathrm{SO}(X)$, $\mathrm{PO}(X)$, $\alpha(X)$, $\beta(X)$). Then a contra m-continuous function $f : (X, m_X) \to (Y, \sigma)$ is contra-continuous (resp. contra-semi-continuous, contra-precontinuous, contra-α-continuous, contra-β-continuous). Moreover, Theorem 3.2 and Corollary 3.2 establish their characterizations which are obtained in [13] (resp. [14], [18], [17], [7]).

For contra-β-continuous functions, for example, the following characterizations are known in [7]:

COROLLARY 3.3. *For a function $f : (X, \tau) \to (Y, \sigma)$, the following properties are equivalent:*

(1) *f is contra-β-continuous;*

(2) *$f^{-1}(F) \in \beta(X)$ for every closed set F of Y;*

(3) *for each $x \in X$ and each closed set F containing $f(x)$, there exists $U \in \beta(X)$ containing x such that $f(U) \subset F$;*

(4) *$f({}_{\beta}\mathrm{Cl}(A)) \subset \mathrm{Ker}(f(A))$ for every subset A of X;*

(5) *${}_{\beta}\,\mathrm{Cl}(f^{-1}(B)) \subset f^{-1}(\mathrm{Ker}(B))$ for every subset B of Y.*

4. Almost m-continuity

In this section, we obtain some sufficient conditions for a contra m-continuous function to be almost m-continuous.

DEFINITION 4.1. A function $f : (X, \tau) \to (Y, \sigma)$ is said to be *weakly continuous* [21] (resp. *weakly quasicontinuous* [55] or *weakly semi-continuous* [5], [10], [20]; *almost weakly continuous* [19] or *quasi precontinuous* [43]; *weakly α-continuous* [37]; *weakly β-continuous* [48]) if for each $x \in X$ and each open set V of Y containing $f(x)$, there exists an open (resp. semi-open, preopen, α-open, β-open) set U of X containing x such that $f(U) \subset \mathrm{Cl}(V)$.

REMARK 4.1. Let $f : (X, \tau) \to (Y, \sigma)$ be a function and $m_X = \tau$ (resp. SO(X), PO(X), $\alpha(X)$, $\beta(X)$). Then a weakly m-continuous function $f : (X, m_X) \to (Y, \sigma)$ is weakly continuous (resp. weakly semi-continuous, almost weakly continuous, weakly α-continuous, weakly β-continuous).

It is proved in [18] that every contra-precontinuous function is almost weakly continuous. The following theorem is a generalization of this result.

THEOREM 4.1. *If a function $f : (X, m_X) \to (Y, \sigma)$ is contra m-continuous, then it is weakly m-continuous.*

PROOF. Let $x \in X$ and V be any open set of Y containing $f(x)$. Then $\mathrm{Cl}(V)$ is a closed set containing $f(x)$. Since f is contra m-continuous, by Theorem 3.2 there exists $U \in m_X$ containing x such that $f(U) \subset \mathrm{Cl}(V)$; hence f is weakly m-continuous.

COROLLARY 4.1. *If a function $f : (X, m_X) \to (Y, \sigma)$ is contra-continuous (resp. contra-semi-continuous, contra-precontinuous, contra-α-continuous, contra-β-continuous), it is weakly continuous (resp. weakly semi-continuous, almost weakly continuous, weakly α-continuous, weakly β-continuous).*

DEFINITION 4.2. A function $f : (X, \tau) \to (Y, \sigma)$ is said to be *almost-continuous* [60] (resp. *almost quasicontinuous* [45] or *almost semi-continuous* [28], [33], [38]; *almost precontinuous* [35], [54]; *almost α-continuous* [62] or *almost feebly continuous* [24]; *almost β-continuous* [35]) if for each $x \in X$ and each open set V of Y containing $f(x)$, there exists an open (resp. semi-open, preopen, α-open, β-open) set U of X containing x such that $f(U) \subset \mathrm{Int}(\mathrm{Cl}(V))$.

REMARK 4.2. Let $f : (X, \tau) \to (Y, \sigma)$ be a function and $m_X = \tau$ (resp. SO(X), PO(X), $\alpha(X)$, $\beta(X)$). Then an almost m-continuous function f :

$(X, m_X) \to (Y, \sigma)$ is almost continuous (resp. almost semi-continuous, almost precontinuous, almost α-continuous, almost β-continuous).

DEFINITION 4.3. A function $f : (X, \tau) \to (Y, \sigma)$ is said to be *almost open* [58] (resp. *almost preopen* [47], *almost regular open* [1] or *M-preopen* [31], *α-preopen* [51], *almost β-open* [40]) if for each open (resp. semi-open, preopen, α-open, β-open) set U of X, $f(U) \subset \mathrm{Int}(\mathrm{Cl}(U))$.

DEFINITION 4.4. A function $f : (X, m_X) \to (Y, \sigma)$ is said to be *almost m-open* if $f(U) \subset \mathrm{Int}(\mathrm{Cl}(U))$ for every $U \in m_X$.

It is proved in [18] that every M-preopen contra-precontinuous function is almost precontinuous. The following theorem is a generalization of this result.

THEOREM 4.2. *If $f : (X, m_X) \to (Y, \sigma)$ is almost m-open and contra m-continuous, then f is almost m-continuous.*

PROOF. Let $x \in X$ and V be any open set of Y containing $f(x)$. Then $\mathrm{Cl}(V)$ is a closed set containing $f(x)$. Since f is contra m-continuous, by Theorem 3.2 there exists $U \in m_X$ containing x such that $f(U) \subset \mathrm{Cl}(V)$. Since f is almost m-open, $f(U) \subset \mathrm{Int}(\mathrm{Cl}(f(U))) \subset \mathrm{Int}(\mathrm{Cl}(V))$. Hence f is almost m-continuous.

COROLLARY 4.2. *If $f : (X, \tau) \to (Y, \sigma)$ is contra-continuous (resp. contra-semi-continuous, contra-precontinuous, contra-α-continuous, contra-β-continuous) and almost-open (resp. almost preopen, M-preopen, α-preopen, almost β-open), then it is almost continuous (resp. almost semi-continuous, almost precontinuous, almost α-continuous, almost β-continuous).*

DEFINITION 4.5. A topological space (Y, σ) is said to be *almost-regular* [59] if for any regular closed set F of Y and any $y \notin F$, there exist disjoint open sets U and V such that $y \in U$ and $F \subset V$.

THEOREM 4.3. *If $f : (X, m_X) \to (Y, \sigma)$ is contra m-continuous and (Y, σ) is almost-regular, then f is almost m-continuous.*

PROOF. Let $x \in X$ and V be any open set of Y containing $f(x)$. Since (Y, σ) is almost-regular, by Theorem 2.2 of [59] there exists a regular open set G of Y such that $f(x) \in G \subset \mathrm{Cl}(G) \subset \mathrm{Int}(\mathrm{Cl}(V))$. Since f is contra m-continuous and $\mathrm{Cl}(G)$ is closed in Y, by Theorem 3.2 there exists $U \in m_X$ containing x such that $f(U) \subset \mathrm{Cl}(G) \subset \mathrm{Int}(\mathrm{Cl}(V)$. Hence f is almost m-continuous.

DEFINITION 4.6. A function $f : (X, m_X) \to (Y, \sigma)$ is said to be *faintly m-continuous* if, for each $x \in X$ and each θ-open set V of Y containing $f(x)$, there exists $U \in m_X$ containing x such that $f(U) \subset V$.

REMARK 4.3. Let (X, τ) be a topological space and $m_X = \tau$ (resp. SO(X), PO(X), $\beta(X)$). Then faint m-continuity coincides with faint continuity [23] (resp. faint semi-continuity [39], faint precontinuity [39], faint β-continuity [39]).

LEMMA 4.1. *For a function $f : (X, m_X) \to (Y, \sigma)$, the following properties are equivalent:*

(1) *f is faintly m-continuous;*

(2) *$f^{-1}(V) = m_X$-Int($f^{-1}(V)$) for every θ-open set V of (Y, σ);*

(3) *$f^{-1}(K) = m_X$-Cl($f^{-1}(K)$) for every θ-closed set K of (Y, σ).*

THEOREM 4.4. *If (Y, σ) is a regular space, then the implications* (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ *(4)* $\Rightarrow$ (5) *hold for a function $f : (X, m_X) \to (Y, \sigma)$:*

(1) *f is contra m-continuous;*

(2) *f is almost m-continuous;*

(3) *f is weakly m-continuous;*

(4) *f is faintly m-continuous;*

(5) *f is m-continuous.*

PROOF. (1) $\Rightarrow$ (2): This follows from Theorem 4.3.

(2) $\Rightarrow$ (3): This is obvious.

(3) $\Rightarrow$ (4): Let F be any θ-closed set of Y. It follows from Theorem 3.2 of [53] that m_X-Cl($f^{-1}(F)$) $\subset f^{-1}(\mathrm{Cl}_\theta(F)) = f^{-1}(F)$. By Lemma 4.1, f is faintly m-continuous.

(4) $\Rightarrow$ (5): Let V be any open set of Y. Since Y is regular, V is θ-open. By Lemma 4.1, $f^{-1}(V) = m_X$-Int($f^{-1}(V)$). By Theorem 3.1, f is m-continuous.

COROLLARY 4.3. *If a function $f : (X, \tau) \to (Y, \sigma)$ is contra-β-continuous (resp. contra precontinuous) and (Y, σ) is regular, then f is β-continuous (resp. precontinuous).*

PROOF. This is shown in [7] (resp. [18]).

REMARK 4.4. By Remark 3.1 of [18], every m-continuous function is not always contra m-continuous even if (Y, σ) is regular.

LEMMA 4.2. (Popa and Noiri [52]). *A function $f : (X, m_X) \to (Y, \sigma)$ is almost m-continuous if and only if for any $x \in X$ and any regular open set V containing $f(x)$, there exists $U \in m_X$ containing x such that $f(U) \subset V$.*

We recall that a topological space (X, τ) is said to be *extremally disconnected* (briefly E.D.) if the closure of every open set of X is open in (X, τ).

THEOREM 4.5. *If $f : (X, m_X) \to (Y, \sigma)$ is contra m-continuous and (Y, σ) is E.D., then f is almost m-continuous.*

PROOF. Let $x \in X$ and V be any regular open set of Y containing $f(x)$. Since (Y, σ) is E.D., by Lemma 5.6 of [43] V is clopen. Since f is contra m-continuous, by Theprem 3.2 there exists $U \in m_X$ containing x such that $f(U) \subset V$. Hence, by Lemma 4.2 f is almost m-continuous.

DEFINITION 4.7. A function $f : (X, m_X) \to (Y, \sigma)$ is said to satisfy the *m-interiority condition* if $m_X\text{-}\mathrm{Int}(f^{-1}(\mathrm{Cl}(V))) \subset f^{-1}(V)$ for each open set V of (Y, σ).

THEOREM 4.6. *If $f : (X, m_X) \to (Y, \sigma)$ is contra m-continuous and satisfies the m-interiority condition, then f is m-continuous.*

PROOF. Let V be any open set of Y. Since f is contra m-continuous, by Theorem 3.2 and Lemma 3.1

$$f^{-1}(V) \subset f^{-1}(\mathrm{Cl}(V)) = m_X\text{-}\mathrm{Int}(f^{-1}(\mathrm{Cl}(V))) =$$
$$= m_X\text{-}\mathrm{Int}(m_X\text{-}\mathrm{Int}(f^{-1}(\mathrm{Cl}(V))) \subset m_X\text{-}\mathrm{Int}(f^{-1}(V)) \subset f^{-1}(V).$$

Therefore, we obtain $f^{-1}(V) = m_X\text{-}\mathrm{Int}(f^{-1}(V))$. Hence, by Theorem 3.1 f is m-continuous.

5. Contra m-closed graphs

DEFINITION 5.1. A function $f : (X, m_X) \to (Y, \sigma)$ is said to have a *contra m-closed graph* if for each $(x, y) \in (X \times Y) - \mathrm{G}(f)$, there exist an m_X-open set U containing x and a closed set V of Y containing y such that $(U \times V) \cap \mathrm{G}(f) = \emptyset$.

LEMMA 5.1. *A function $f : (X, m_X) \to (Y, \sigma)$ has a contra m-closed graph if and only if for each $(x, y) \in (X \times Y) - G(f)$, there exist an m_X-open set U containing x and a closed set V of Y containing y such that $f(U) \cap V = \emptyset$.*

THEOREM 5.1. *If $f : (X, m_X) \to (Y, \sigma)$ is a contra m-continuous function and (Y, σ) is Urysohn, then* $\mathrm{G}(f)$ *is contra m-closed.*

PROOF. Suppose that $(x, y) \in (X \times Y) - \mathrm{G}(f)$. Then $y \neq f(x)$. Since Y is Urysohn, there exist open sets V and W in Y containing y and $f(x)$, respectively, such that $\mathrm{Cl}(V) \cap \mathrm{Cl}(W) = \emptyset$. Since f is contra m-continuous, there exists an m_X-open set U containing x such that $f(U) \subset \mathrm{Cl}(W)$. This implies that $f(U) \cap \mathrm{Cl}(V) = \emptyset$ and by Lemma 5.1 $\mathrm{G}(f)$ is contra m-closed.

REMARK 5.1. Let $f : (X, \tau) \to (Y, \sigma)$ be a function. If $m_X = \mathrm{PO}(X)$ (resp. $\alpha(X)$, $\beta(X)$), then by Theorem 5.1 we obtain the results established in Theorem 4.1 of [18] (resp. Theorem 4.1 of [17], Theorem 2.21 of [7]).

THEOREM 5.2. *If $f : (X, m_X) \to (Y, \sigma)$ is an m-continuous function and (Y, σ) is T_1, then* $\mathrm{G}(f)$ *is contra m-closed.*

PROOF. Let $(x, y) \in (X \times Y) - \mathrm{G}(f)$. Then $y \neq f(x)$. Since Y is T_1, there exists an open set V in Y such that $f(x) \in V$ and $y \notin V$. Since f is m-continuous, there exists $U \in m_X$ containing x such that $f(U) \subset V$. Therefore, $f(U) \cap (Y - V) = \emptyset$ and $Y - V$ is a closed set of Y containing y. This shows that $\mathrm{G}(f)$ is contra m-closed.

REMARK 5.2. Let $f : (X, \tau) \to (Y, \sigma)$ be a function. If $m_X = \alpha(X)$, then by Theorem 5.2 we obtain the results established in Theorem 4.2 of [17].

DEFINITION 5.2. A nonempty set X with an m-structure m_X is said to be m-T_2 [50] if for each distinct points $x, y \in X$, there exist $U, V \in m_X$ containing x and y, respectively, such that $U \cap V = \emptyset$.

THEOREM 5.3. *If $f : (X, m_X) \to (Y, \sigma)$ is an injective contra m-continuous function with a contra m-closed graph, then (X, τ) is m-T_2.*

PROOF. Let x and y be any distinct points of X. Then, since f is injective, we have $f(x) \neq f(y)$. Then we have $(x, f(y)) \in (X \times Y) - \mathrm{G}(f)$. Since $\mathrm{G}(f)$ is contra m-closed, by Lemma 5.1 there exist an m_X-open set U of X containing x and a closed set V of Y containing $f(y)$ such that $f(U) \cap V = \emptyset$. Since f is contra m-continuous, there exists $G \in m_X$ containing y such that $f(G) \subset V$. Therefore, we have $f(U) \cap f(G) = \emptyset$. Clearly, we obtain $U \cap G = \emptyset$. This shows that X is m-T_2.

6. Some properties of contra m-continuity

THEOREM 6.1. *Let (X, m_X) be a nonempty set with an m-structure m_X. If for each pair of distinct points x_1 and x_2 in X there exists a function f of (X, m_X) into a Urysohn space (Y, σ) such that $f(x_1) \neq f(x_2)$ and f is contra m-continuous at x_1 and x_2, then X is m-T_2.*

PROOF. Let x and y be any distinct points of X. Then by the hypothesis, there exist an Urysohn space (Y, σ) and a function $f : (X, m_X) \to (Y, \sigma)$ which satisfies the conditions of this theorem. Let $y_i = f(x_i)$ for $i = 1, 2$. Then $y_1 \neq y_2$. Since Y is Urysohn, there exist open sets U_1 and U_2 containing y_1 and y_2, respectively, such that $\mathrm{Cl}(U_1) \cap \mathrm{Cl}(U_2) = \emptyset$. Since f is contra m-continuous at x_i, by Theorem 3.2 there exists $G_{x_i} \in m_X$ containing x_i such that $f(G_{x_i}) \subset \mathrm{Cl}(U_i)$ for $i = 1, 2$. Hence we obtain $G_{x_1} \cap G_{x_2} = \emptyset$. Therefore, X is m-T_2.

COROLLARY 6.1. *If $f : (X, m_X) \to (Y, \sigma)$ is a contra m-continuous injection and (Y, σ) is Urysohn, then (X, m_X) is m-T_2.*

PROOF. For each pair of distinct points x_1 and x_2 in X, f is a contra m-continuous function of (X, m_X) into a Urysohn space (Y, σ) such that $f(x_1) \neq f(x_2)$ because f is injective. Hence, by Theorem 6.1 (X, m_X) is m-T_2.

DEFINITION 6.1. A topological space (Y, σ) is said to be *ultra-Hausdorff* [61] if for each pair of distinct points x and y in Y there exist clopen sets U and V containing x and y, respectively, such that $U \cap V = \emptyset$.

THEOREM 6.2. *If $f : (X, m_X) \to (Y, \sigma)$ is a contra m-continuous injection and (Y, σ) is ultra-Hausdorff, then (X, m_X) is m-T_2.*

PROOF. Let x_1 and x_2 be any distinct points in X. Then, since f is injective, $f(x_1) \neq f(x_2)$. Moreover, since (Y, σ) is ultra-Hausdorff, there exist clopen sets V_1, V_2 such that $f(x_1) \in V_1$, $f(x_2) \in V_2$ and $V_1 \cap V_2 = \emptyset$. By Theorem 3.2, there exists $U_i \in m_X$ containing x_i such that $f(U_i) \subset V_i$ for $i = 1, 2$. Clearly, we obtain $U_1 \cap U_2 = \emptyset$. Thus (X, m_X) is m-T_2.

REMARK 6.1. Let $f : (X, \tau) \to (Y, \sigma)$ be a function. If $m_X = \beta(X)$, then by Theorem 6.1, Corollary 6.1 and Theorem 6.2 we obtain the results established in Theorem 2.14, Corollary 2.1 and Corollary 2.2 of [7].

DEFINITION 6.2. Let (X, m_X) be a nonempty set with an m-structure m_X. A subset A of X is said to be *m-dense* in X if m_X-$\mathrm{Cl}(A) = X$.

THEOREM 6.3. *Let X be a nonempty set with two minimal structures m_X^1 and m_X^2 such that $U \cap V \in m_X^2$ whenever $U \in m_X^1$ and $V \in m_X^2$ and (Y,σ) be a Hausdorff space. If $g : (X, m_X^1) \to (Y,\sigma)$ is almost m-continuous, $f : (X, m_X^2) \to (Y,\sigma)$ is contra m-continuous and $f(x) = g(x)$ on an m-dense set D of (X, m_X^2), then $f(x) = g(x)$ on X.*

PROOF. Let $A = \{x \in X : f(x) = g(x)\}$. Suppose that $x \in X - A$. Then $f(x) \neq g(x)$. Since (Y,σ) is Hausdorff, there exist open sets V and W such that $f(x) \in V$, $g(x) \in W$ and $V \cap W = \emptyset$; hence $\mathrm{Cl}(V) \cap \mathrm{Int}(\mathrm{Cl}(W)) = \emptyset$. Since g is almost m-continuous, there exists $U_1 \in m_X^1$ containing x such that $g(U_1) \subset \mathrm{Int}(\mathrm{Cl}(W))$. Since f is contra m-continuous, by Theorem 3.2 there exists $U_2 \in m_X^2$ containing x such that $f(U_2) \subset \mathrm{Cl}(V)$. Now put $U = U_1 \cap U_2$, then $x \in U$, $U \in m_X^2$ and $U \cap A = \emptyset$. Therefore, by Lemma 3.2 $x \in X - m_X^2\text{-}\mathrm{Cl}(A)$ and hence $A = m_X^2\text{-}\mathrm{Cl}(A)$. On the other hand, $f(x) = g(x)$ on D; hence $D \subset A$. Since D is m-dense in (X, m_X^2), $X = m_X^2\text{-}\mathrm{Cl}(D) \subset m_X^2\text{-}\mathrm{Cl}(A) = A$. Therefore, $X = A$ and $f(x) = g(x)$ for each $x \in X$.

DEFINITION 6.3. A nonempty set X with an m-structure m_X is said to be *m-compact* [50] if every cover of X by m_X-open sets has a finite subcover.

REMARK 6.2. Let (X,τ) be a topological space. If $m_X = \tau$ (resp. $\mathrm{SO}(X)$, $\mathrm{PO}(X)$, $\alpha(X)$) then by Definition 6.3 we obtain the definitions of compact (resp. semi-compact, strongly compact, α-compact) spaces.

DEFINITION 6.4. A topological space (X,τ) is said to be *strongly S-closed* [13] (resp. *semi-compact* [8], *strongly compact* [31], *α-compact* [25]) if every cover of X by closed (resp. semi-open, preopen, α-open) sets of (X,τ) has a finite subcover.

DEFINITION 6.5. A topological space (Y,σ) is said to be *S-closed* [63] (resp. *quasi H-closed* [56]) if for every cover $\{V_\alpha : \alpha \in \Delta\}$ of Y by semi-open (resp. open) sets of (Y,σ), there exists a finite subset Δ_0 of Δ such that $Y = \bigcup\{\mathrm{Cl}(V_i) : \alpha \in \Delta_0\}$.

THEOREM 6.4. *If $f : (X, m_X) \to (Y,\sigma)$ is a contra m-continuous surjection and (X, m_X) is m-compact, then (Y,σ) is strongly S-closed.*

PROOF. Let (X, m_X) be m-compact and $\{V_\alpha : \alpha \in \Delta\}$ any cover of Y by closed sets of (Y,σ). For each $x \in X$, there exists $\alpha(x) \in \Delta$ such that $f(x) \in V_{\alpha(x)}$. Since f is contra m-continuous, by Theorem 3.2 there exists an m_X-open set $U(x)$ containing x such that $f(U(x)) \subset V_{\alpha(x)}$. The family $\{U(x) : x \in X\}$ is a cover of X by m_X-open sets. Since (X, m_X) is m-compact, there exist a finite number of points, say, $x_1, x_2, \ldots, x_n$ in X such that $X = \bigcup\{U(x_k) : x_k \in X, 1 \leq k \leq n\}$. Therefore, we obtain

$$Y = f(X) = \bigcup\{f(U(x_k)) : x_k \in X, 1 \leq k \leq n\} \subset$$
$$\subset \bigcup\{V_{\alpha(x_k)} : x_k \in X, 1 \leq k \leq n\}.$$

This shows that (Y, σ) is strongly S-closed.

COROLLARY 6.2. *If $f : (X, m_X) \to (Y, \sigma)$ is a contra m-continuous surjection and (X, m_X) is m-compact, then (Y, σ) is S-closed and hence quasi H-closed.*

REMARK 6.3. Let (X, τ) be a topological space. If $m_X = \tau$ (resp. SO(X), PO(X), $\alpha(X)$), then by Theorem 6.4 we obtain the results established in Theorem 4.2 of [14] (resp. Theorem 4.2 of [14], Corollary 5.1 of [18], Corollary 5.1 of [17]).

DEFINITION 6.6. A nonempty set (X, m_X) with an m-structure m_X is said to be *m-connected* [50] if X cannot be written as the union of two nonempty sets of m_X.

REMARK 6.4. Let (X, τ) be a topological space. If $m_X = \tau$ (resp. SO(X), PO(X), $\beta(X)$) then by Definition 6.6 we obtain the definitions of connected (resp. semi-connected [44], preconnected [46], β-connected [48]) spaces.

THEOREM 6.5. *Let (X, m_X) be a nonempty set with an m-structure m_X satisfying the property ($\mathscr{B}$) and (Y, σ) a topological space. If $f : (X, m_X) \to$ $\to (Y, \sigma)$ is a contra m-continuous surjection and (X, m_X) is m-connected, then (Y, σ) is connected.*

PROOF. Assume that (Y, σ) is not connected. Then, there exist nonempty open sets V_1, V_2 of (Y, σ) such that $V_1 \cap V_2 = \emptyset$ and $V_1 \cup V_2 = Y$. Hence we have $f^{-1}(V_1) \cap f^{-1}(V_2) = \emptyset$ and $f^{-1}(V_1) \cup f^{-1}(V_2) = X$. Since f is surjective, $f^{-1}(V_1)$ and $f^{-1}(V_2)$ are nonempty sets. Since f is contra m-continuous and V_1, V_2 are clopen sets, by Theorem 3.2, $f^{-1}(V_1) =$ $= m_X\text{-}\mathrm{Int}(f^{-1}(V_1))$ and $f^{-1}(V_2) = m_X\text{-}\mathrm{Int}(f^{-1}(V_2))$. Since m_X has the property ($\mathscr{B}$), by Lemma 3.4, $f^{-1}(V_1)$ and $f^{-1}(V_2)$ are m_X-open sets in (X, m_X). Therefore, (X, m_X) is not m-connected.

REMARK 6.5. Let (X,τ) be a topological space. If $m_X = \mathrm{SO}(X)$ (resp. $\alpha(X)$, $\beta(X)$), then by Theorem 6.5 we obtain the results established in Theorem 5.4 of [14] (resp. Theorem 6.3 of [17], Theorem 3.2 of [7]).

COROLLARY. *If $f : (X,\tau) \to (Y,\sigma)$ is a contra-continuous (resp. contra-precontinuous) surjection and (X,τ) is connected (resp. preconnected), then (Y,σ) is connected.*

PROOF. This is an immediate consequence of Theorem 6.5.

DEFINITION 6.7. Let A be a subset of (X,m_X). The *m_X-frontier* of A, m_X-$\mathrm{Fr}(A)$, is defined by $m_X\text{-}\mathrm{Fr}(A) = m_X\text{-}\mathrm{Cl}(A) \cap m_X\text{-}\mathrm{Cl}(X - A)$.

THEOREM 6.6. *The set of all points $x \in X$ at which a function $f : (X,m_X) \to (Y,\sigma)$ is not contra m-continuous is identical with the union of the m_X-frontiers of the inverse images of closed sets of Y containing $f(x)$.*

PROOF. Suppose that f is not contra m-continuous at $x \in X$. There exists a closed set F of Y containing $f(x)$ such that $f(U) \cap (Y - F) \neq \emptyset$ for every $U \in m_X$ containing x. By Lemma 3.2 we have $x \in m_X\text{-}\mathrm{Cl}(f^{-1}(Y - F)) = m_X\text{-}\mathrm{Cl}(X - f^{-1}(F))$. On the other hand, we have $x \in f^{-1}(F) \subset m_X\text{-}\mathrm{Cl}(f^{-1}(F))$ and hence $x \in m_X\text{-}\mathrm{Fr}(f^{-1}(F))$.

Conversely, suppose that f is contra m-continuous at $x \in X$ and let F be any closed set containing $f(x)$. Then by Theorem 3.2 we have $x \in f^{-1}(F) = m_X\text{-}\mathrm{Int}(f^{-1}(F))$. Therefore, $x \notin m_X\text{-}\mathrm{Fr}(f^{-1}(F))$ for each closed set F containing $f(x)$. This completes the proof.

7. New varieties of contra-continuity

Let A be a subset of a topological space (X,τ). A point x of X is called a *semi-θ-cluster point* of A if $\mathrm{sCl}(U) \cap A \neq \emptyset$ for every $U \in \mathrm{SO}(X)$ containing x. The set of all semi-θ-cluster points of A is called the *semi-θ-closure* [12] of A and is denoted by $\mathrm{sCl}_\theta(A)$. A subset A is said to be *semi-θ-closed* if $A = \mathrm{sCl}_\theta(A)$. The complement of a semi-θ-closed set is said to be *semi-θ-open*. A subset A is said to be *semi-regular* [12] if it is semi-open and semi-closed.

DEFINITION 7.1. Let (X,τ) be a topological space. A subset A of X is said to be

(1) *b-open* [4] if $A \subset \mathrm{Cl}(\mathrm{Int}(A)) \cup \mathrm{Int}(\mathrm{Cl}(A))$,

(2) *δ-preopen* [57] (resp. *δ-semi-open* [42]) if $A \subset \mathrm{Int}(\mathrm{Cl}_\delta(A))$ (resp. $A \subset \mathrm{Cl}(\mathrm{Int}_\delta(A))$).

The family of all b-open (resp. δ-preopen, δ-semi-open, semi-θ-open, θ-open) sets in (X,τ) is denoted by $\mathrm{BO}(X)$ (resp. δ $\mathrm{PO}(X)$, δ $\mathrm{SO}(X)$, $\mathrm{S}\theta\mathrm{O}(X)$, τ_θ).

DEFINITION 7.2. The complement of a b-open (resp. δ-preopen, δ-semi-open) set is said to be *b-closed* [4] (resp. *δ-preclosed* [57], *δ-semi-closed* [42]).

DEFINITION 7.3. The intersection of all b-closed (resp. δ-preclosed, δ-semi-closed) sets of X containing A is called the *b-closure* [4] (resp. *δ-preclosure* [57], *δ-semi-closure* [42]) of A and is denoted by $\mathrm{bCl}(A)$ (resp. $\mathrm{pCl}_\delta(A)$, $\mathrm{sCl}_\delta(A)$).

DEFINITION 7.4. The union of all semi-θ-open (resp. b-open, δ-preopen, δ-semi-open) sets of X contained in A is called the *semi-θ-interior* (resp. *b-interior*, *δ-preinterior*, *δ-semi-interior*) of A and is denoted by $\mathrm{sInt}_\theta(A)$ (resp. $\mathrm{bInt}(A)$, $\mathrm{pInt}_\delta(A)$, $\mathrm{sInt}_\delta(A)$).

LEMMA 7.1. *For a subset A of a topological space (X,τ), the following properties hold:*

(1) *If A is a semi-open set, then* $\mathrm{sCl}(A)$ *is semi-regular,*

(2) *If A is a semi-regular set, then it is semi-θ-open,*

(3) *If A is a semi-regular set, then it is δ-semi-open,*

(4) *If A is a semi-θ-open set, then it is δ-semi-open,*

(5) *If A is a δ-semi-open set, then it is semi-open.*

PROOF. (1) and (2) are shown in Propositions 2.2 and 2.3 of [12].

(3) Let A be a semi-regular set. Then since A is semi-open and semi-closed, we have $\mathrm{Int}(\mathrm{Cl}(A)) \subset A \subset \mathrm{Cl}(\mathrm{Int}(A))$. Since $\mathrm{Int}(\mathrm{Cl}(A))$ is regular open, we obtain $\mathrm{Int}(\mathrm{Cl}(A)) \subset \mathrm{Int}_\delta(A)$ and hence

$$A \subset \mathrm{Cl}(\mathrm{Int}(A)) \subset \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(A))) \subset \mathrm{Cl}(\mathrm{Int}_\delta(A)).$$

This shows that A is δ-semi-open.

(4): Let A be a semi-θ-open set. For each $x \in A$, there exists $U_x \in$ $\in \mathrm{SO}(X)$ such that $x \in U_x \subset \mathrm{sCl}(U_x) \subset A$. By (1), $\mathrm{sCl}(U_x)$ is semi-regular and hence δ-semi-open by (3). Therefore, $A = \bigcup_{x\in A} \mathrm{sCl}(U_x)$ is δ-semi-open by Theorem 3 of [42].

(5) Let A be a δ-semi-open set. Since $\mathrm{Int}_\delta(A) \subset \mathrm{Int}(A)$, $A \subset \mathrm{Cl}(\mathrm{Int}_\delta(A))$ implies $A \subset \mathrm{Cl}(\mathrm{Int}(A))$. This shows that A is semi-open.

By Lemma 7.1, we have the following diagram in which the converses of implications need not be true as shown by the examples stated below.

Diagram I

$$\begin{array}{ccccccccc}
\theta\text{-open} & \rightarrow & \delta\text{-open} & \rightarrow & \text{open} & \rightarrow & \text{preopen} & \rightarrow & \delta\text{-preopen} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & | \\
\text{semi-}\theta\text{-open} & \rightarrow & \delta\text{-semi-open} & \rightarrow & \text{semi-open} & \rightarrow & b\text{-open} & \rightarrow & \text{semi-preopen}
\end{array}$$

EXAMPLE 7.1. Let $X = \{a,b,c\}$ and $\tau = \{X, \emptyset, \{a\}, \{b\}, \{a,b\}\}$. Then $\{a,b\}$ is a δ-open set of (X,τ) which is not θ-open. The subset $\{a,c\}$ is a semi-θ-open set which is not δ-preopen.

EXAMPLE 7.2. (Park et al. [42]) Let $X = \{a,b,c,d\}$ and $\tau = \{X, \emptyset, \{a\}, \{c\}, \{a,b\}, \{a,c\}, \{a,b,c\}, \{a,c,d\}\}$. Then $\{a,c,d\}$ is an open set of (X,τ) which is not δ-semi-open.

EXAMPLE 7.3. Let $X = \{a,b,c,d\}$ and $\tau = \{X, \emptyset, \{c\}, \{a,d\}, \{a,c,d\}\}$. Then $\{a,b,c\}$ is a preopen set of (X,τ) which is not semi-open.

EXAMPLE 7.4. Let $X = \{a,b,c,d\}$ and $\tau = \{X, \emptyset, \{a,b\}, \{a,b,c\}\}$. Then $\{d\}$ is a δ-preopen set of (X,τ) which is not β-open.

LEMMA 7.2. *Let (X,τ) be a topological space and A be a subset of X.*

(1) *If A is open, then* $\mathrm{Cl}_\delta(A) = \mathrm{Cl}(A)$,

(2) *If A is closed, then* $\mathrm{Int}_\delta(A) = \mathrm{Int}(A)$.

PROOF. (1) is known in [64] and (2) follows obviously from (1).

For a topological space (X,τ), the family of all δ-open sets of (X,τ) forms a topology for X, which is weaker than τ. This topology has a base consisting of all regular open sets in (X,τ). It is usually called the *semi-regularization* of τ and is denoted by τ_s. Now, we have the following interesting lemma.

LEMMA 7.3. *Let (X,τ) be a topological space and A be a subset of X.*

(1) *A is δ-semi-open in (X,τ) if and only if A is semi-open in (X,τ_s),*

(2) *A is δ-preopen in (X,τ) if and only if A is preopen in (X,τ_s).*

PROOF. This folllows from Lemma 7.2 and the next facts:

(1) $\mathrm{Cl}(\mathrm{Int}_\delta(A)) = \mathrm{Cl}_\delta(\mathrm{Int}_\delta(A)) = \tau_s\text{-}\mathrm{Cl}(\tau_s\text{-}\mathrm{Int}(A))$,

(2) $\mathrm{Int}(\mathrm{Cl}_\delta(A)) = \mathrm{Int}_\delta(\mathrm{Cl}_\delta(A)) = \tau_s\text{-}\mathrm{Int}(\tau_s\text{-}\mathrm{Cl}(A))$.

DEFINITION 7.5. A function $f : (X,\tau) \to (Y,\sigma)$ is said to be *contra strongly θ-continuous* (resp. *contra strongly semi-θ-continuous*, *contra b-continuous*, *contra δ-precontinuous*, *contra δ-semi-continuous*) if for every open set V of (Y,σ), $f^{-1}(V)$ is θ-closed (resp. semi-θ-closed, b-closed, δ-preclosed, δ-semi-closed) in (X,τ).

REMARK 7.1. Let $f : (X,\tau) \to (Y,\sigma)$ be a function and $m_X = \tau_\theta$ (resp. $S\theta O(X)$, $BO(X)$, $\delta SO(X)$, $\delta PO(X)$). Then a contra m-continuous function $f : (X,m_X) \to (Y,\sigma)$ is contra strongly θ-continuous (resp. contra strongly semi-θ-continuous, contra b-continuous, contra δ-semi-continuous, contra δ-precontinuous).

All the families τ_θ, $S\theta O(X)$, $BO(X)$, $\delta SO(X)$, $\delta PO(X)$ have the property ($\mathscr{B}$). Especially, τ_θ is a topology for X. Therefore, we can apply all results obtained in Sections 3–6 to these new functions. The following theorem is a typical characterization.

THEOREM 7.1. *A function $f : (X,\tau) \to (Y,\sigma)$ is a contra strongly θ-continuous (resp. contra strongly semi-θ-continuous, contra b-continuous, contra δ-semi-continuous, contra δ-precontinuous) function if and only if if for every closed set F of Y, $f^{-1}(F)$ is θ-open (resp. semi-θ-open, b-open, δ-preopen, δ-semi-open) in X.*

PROOF. The proof is obvious from the definition.

By Theorem 7.1 and DIAGRAM I, we obtain the following diagram:

Diagram II.

$$
\begin{tikzcd}
\text{contra st.}\theta\text{-C} \arrow[r] \arrow[d] & \text{contra-super-C} \arrow[r] \arrow[d] & \text{contra-C} \arrow[r] \arrow[d] & \text{contra-}p\text{-C} \arrow[r] \arrow[d] & \text{contra }\delta\text{-}p\text{-C} \arrow[d, dash] \\
\text{contra st. }s\text{-}\theta\text{-C} \arrow[r] & \text{contra }\delta\text{-}s\text{-C} \arrow[r] & \text{contra-}s\text{-C} \arrow[r] & \text{contra }b\text{-C} \arrow[r] & \text{contra-}\beta\text{-C}
\end{tikzcd}
$$

In the diagram above, we abbreviate as follows: C = continuous, st. = strongly, p = pre and s = semi.

THEOREM 7.2. *For a function $f : (X,\tau) \to (Y,\sigma)$, the following properties are equivalent:*

(1) *f is contra strongly θ-continuous;*

(2) *For each point $x \in X$ and each closed set F of Y containing $f(x)$, there exists a θ-open set U of X containing x such that $f(U) \subset F$;*

(3) *For each point $x \in X$ and each closed set F of Y containing $f(x)$, there exists an open set U of X containing x such that $f(\mathrm{Cl}(U)) \subset F$;*

(4) *$f : (X, \tau_\theta) \to (Y, \sigma)$ is contra-continuous.*

PROOF. By Theorem 7.1, (1) is equivalent to (4). We prove only the implication : (3) $\Rightarrow$ (1), the other proofs being obvious. Let F be any closed set of Y and $x \in f^{-1}(F)$. Then $f(x) \in F$ and by (3) there exists an open set U of X containing x such that $f(\mathrm{Cl}(U)) \subset F$. Therefore, we have $x \in U \subset \mathrm{Cl}(U) \subset f^{-1}(F)$ and hence $f^{-1}(F)$ is θ-open in X. It follows from Theorem 7.1 that f is contra strongly θ-continuous.

THEOREM 7.3. *A function $f : (X, \tau) \to (Y, \sigma)$ is contra δ-semi-continuous (resp. contra δ-precontinuous) if and only if $f : (X, \tau_s) \to (Y, \sigma)$ is contra-semi-continuous (resp. contra-precontinuous).*

PROOF. This is an immediate consequence of Lemma 7.3.

THEOREM 7.4. *For a function $f : (X, \tau) \to (Y, \sigma)$, the following properties are equivalent:*

(1) *f is contra strongly semi-θ-continuous;*

(2) *For each point $x \in X$ and each closed set F of Y containing $f(x)$, there exists a semi-θ-open set U of X containing x such that $f(U) \subset F$;*

(3) *For each point $x \in X$ and each closed set F of Y containing $f(x)$, there exists a semi-regular set U of X containing x such that $f(U) \subset F$.*

PROOF. (1) $\Rightarrow$ (2): this is obvious.

(2) $\Rightarrow$ (3): For any semi-θ-open set G and each $x \in G$, there exists a semi-open set H such that $x \in H \subset \mathrm{sCl}(H) \subset G$. By Lemma 7.1, $\mathrm{sCl}(H)$ is a semi-regular set. Set $U = \mathrm{sCl}(H)$, then (3) holds.

(3) $\Rightarrow$ (1): Let F be any closed set of Y and $x \in f^{-1}(F)$. Then there exists a semi-regular set U of X containing x such that $f(U) \subset F$. Then we have $x \in U \subset f^{-1}(F)$. This shows by Lemma 7.1 that $f^{-1}(F)$ is semi-θ-open in X.

Finally, we deal with the preservation theorem of compact-like spaces under new types of contra-continuous surjections.

DEFINITION 7.6. A topological space (X, τ) is said to be

(1) *s-closed* [12] if for every semi-open cover $\{V_\alpha : \alpha \in \Delta\}$ of X, there exists a finite subset Δ_0 of Δ such that $X = \bigcup\{\mathrm{sCl}(V_\alpha) : \alpha \in \Delta_0\}$,

(2) *b-compact* (resp. *δ_s-compact, δ_p-compact*) if every b-open (resp. δ-semi-open, δ-preopen) cover of X has a finite subcover.

THEOREM 7.5. *Let $f : (X,\tau) \to (Y,\sigma)$ be a surjective function. If one of the following conditions holds, then (Y,σ) is strongly S-closed.*

(1) f is contra strongly θ-continuous and (X,τ) is quasi H-closed,

(2) f is contra strongly semi-θ-continuous and (X,τ) is s-closed,

(3) f is contra b-continuous and (X,τ) is b-compact,

(4) f is contra δ-semi-continuous and (X,τ) is δ_s-compact,

(5) f is contra δ-precontinuous) and (X,τ) is δ_p-compact).

PROOF. The proofs for the first and the second statements follow from Theorems 7.2 and 7.4, respectively. The other are obvious.

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ON THE EXTREMAL FROBENIUS PROBLEM IN A NEW ASPECT

By

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1. Introduction and main result

Let $0 < a_1 < a_2 < \ldots < a_n$ be integers with $\gcd(a_1; \ldots; a_n) = 1$. It is well-known that the equation $K = \sum_{i=1}^{n} x_i a_i$ has a solution in non-negative integers x_i provided K is sufficiently large. Define $G(a_1, a_2, \ldots, a_n)$ as the greatest integer K for which the preceding equation has no such solution and $g(n, t)$ by

$$g(n, t) = \max G(a_1, a_2, \ldots, a_n)$$

where the max is taken over all a_i satisfying $1 < a_1 < a_2 < \ldots < a_n \leq t$, $\gcd(a_1; \ldots; a_n) = 1$. The investigation of $G(a_1, a_2, \ldots, a_n)$ and $g(n, t)$ has given rise to many papers, see e.g. [2], [5], [6].

Erdős and Graham asked [3, p.86] "For what choice of n positive integers $1 < a_1 < a_2 < \ldots < a_n \leq t$ is the number of integers not of the form $\sum_i c_i a_i$ maximal, where the c_i range over all non-negative integers? Is the choice $a_i = t - i + 1$, $1 \leq i \leq n$ optimal for this?"

The aim of the present paper is to give a simple proof for this conjecture. In addition we shall give further examples for optimal sets.

Let $N(a_1, a_2, \ldots, a_n)$ be the number of positive integers with no representation by $a_1, a_2, \ldots, a_n$. Analogously to $g(n, t)$ we can define $\nu(n, t)$ as

$$\nu(n, t) = \max N(a_1, a_2, \ldots, a_n)$$

where the max is taken over all a_i satisfying $1 < a_1 < a_2 < \ldots < a_n \leq t$, $\gcd(a_1; \ldots; a_n) = 1$.

The statement of Erdős and Graham can be written in following form with the new notation:

THEOREM 1. *Let n and t be positive integers such that $1 < n \le t$. Then*

$$\nu(n,t) = N(t-n+1, t-n+2, \dots, t). \tag{1}$$

The proof of Theorem 1. uses the following result of DIXMIER [1, Thm.2]: For $a_n \le t$ and any k, at least $\min(t, kn-k+1)$ integers can be represented by $a_1, a_2, \dots, a_n$ in the interval $I_k = [(k-1)t+1, kt]$.

2. Proof of Theorem 1

We prove first the following lemma:

LEMMA. *Let n and t be positive integers such that $1 < n \le t$. Let the integers q and r be defined by $t = q(n-1)+r$, where $1 \le r \le n-1$. Then*

$$N(t-n+1, t-n+2, \dots, t) = \frac{(t-n+r-1)q}{2}.$$

PROOF. Since the numbers $a_i = t-n+i$ are consecutive, all integers in the intervals $J_m = [m(t-n+1), mt]$ are representable $m = 1, 2, \dots$ Hence the integers without a representation are those situated before J_1, between J_1 and J_2, …, between J_{m-1} and J_m as long as these intervals are distinct, i.e. $(m-1)t < m(t-n+1)$, or equivalently $m(n-1) < t$. Hence the last value is $m = q$. So the number of integers without representation is

$$\begin{aligned} &\sum_{m=1}^{q}[m(t-n+1)-(m-1)t-1] = \sum_{m=1}^{q}(t-mn+m-1) = \\ &= qt - \frac{q(q+1)}{2}(n-1) - q = \frac{q}{2}[2t-(q+1)(n-1)-2] = \\ &= \frac{q}{2}[t+q(n-1)+r-(q+1)(n-1)-2] = \frac{(t-n+r-1)q}{2}. \end{aligned} \tag{2}$$

PROOF OF THEOREM 1. Dixmier's theorem mentioned in the introduction claims that the intervals I_k contain at least

$$n;\ 2n-1;\ 3n-2; \dots;\ k(n-1)+1;\ \dots$$

representable elements. So the number of elements in I_k without representation will be at most

$$t-n;\ t-2n+1;\ t-3n+2; \dots;\ t-k(n-1)-1; \dots;\ t-q(n-1)-1 = r-1.$$

This is an arithmetical progression. Hence

$$N(a_1, a_2, \ldots, a_n) \le \sum_{k=1}^{q} (t - k(n-1) - 1)$$

which is the same as the second $\sum$ in (2).

Thus $\nu(n,t) = N(t-n+1, t-n+2, \ldots, t)$ as claimed.

3. Optimal sets

We found that the set $A = \{t-n+1, t-n+2, \ldots, t\}$ is optimal in the sense that $\nu(n,t) = N(A)$. Can we find other optimal sets, as well? In the following theorem we show, that this is possible in many cases.

THEOREM 2. *Let d, n, k be integers such that $2 \le d < n$, $0 \le k < n-d$. If $n-k \equiv 0 \pmod{d+1}$ or $n-k \equiv -1 \pmod{d+1}$ then for $t = dn+k$ there exist at least two optimal sets A, i.e. for which*

$$N(A) = \nu(n, dn+k).$$

PROOF. We have to show that there exists an optimal set different from $\{t-n+1, t-n+2, \ldots, t\}$. We shall use the same sets as in [4] when proving the exact value of $g(n, dn+k)$ for d, k and n satisfying the above conditions.

Case (i). Let $n-k \equiv 0 \pmod{d+1}$. Write $n = l(d+1)+k$, then

$$dn + k = ld(d+1) + dk + k = (d+1)(ld+k).$$

Let $A = \{a_1; a_2; \ldots; a_n\}$ consist of all multiples of $(d+1)$ and the l largest elements of the residue class (-1) modulo $(d+1)$ up to t:

$$A = \{d+1; 2(d+1); \ldots; (ld+k-1)(d+1); (ld+k)(d+1);$$
$$dn+k-1; dn+k-1-(d+1); dn+k-1-2(d+1); \ldots; dn+k-1-(l-1)(d+1)\}.$$

Let $z = dn+k-1-(l-1)(d+1)$ be the smallest element of A, which is not a multiple of $(d+1)$.

It is well-known (see e.g. Sylvester [7]), that $N(b,c) = (b-1)(c-1)/2$, hence

$$N(A) = N(d+1, z) = (z-1)\frac{d}{2} = [dn+k-l(d+1)+d-1]\frac{d}{2} =$$
$$= (dn+k-n+k+d-1)\frac{d}{2}.$$

This coincides with $\nu(n,t) = N(t-n+1, t-n+2, \dots, t)$ since by the notations and result of the Lemma now $t = dn + k = d(n-1) + k + d$, so

$$d = q, r = k + d \text{ and } dn + k - n + k + d - 1 = t - n + r - 1.$$

Case (ii). Suppose $n - k \equiv -1 \pmod{d+1}$. Then $n = l(d+1) + k - 1$ and

$$dn + k = (d+1)dl + dk - d + k = (d+1)(dl + k) - d.$$

So $dn + k - 1 = (d+1)(dl + k - 1)$ is a multiple of $(d+1)$. Let $A = \{a_1; a_2; \dots$ $\dots; a_n\}$ consist of all multiples of $(d+1)$ and the l largest elements of the residue class (1) modulo $(d+1)$ up to t:

$$A = \{d+1; 2(d+1); \dots; (ld + k - 1)(d+1);$$

$$dn + k; dn + k - (d+1); dn + k - 2(d+1); \dots; dn + k - (l-1)(d+1)\}.$$

We obtain $N(A) = \nu(n,t)$ by similar calculations as in Case (i).

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ON TRANSMISSION PROBLEMS FOR NONLINEAR PARABOLIC DIFFERENTIAL EQUATIONS

By

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Introduction

In [3] W. Jäger and N. Kutev considered the following nonlinear transmission (contact) problem for nonlinear elliptic equations:

$$\sum_{i=1}^{n} D_i[a_i(x,u,Du)] + b(x,u,Du) = 0 \text{ in } \Omega \tag{0.1}$$

$$u = g \text{ on } \partial\Omega \tag{0.2}$$

$$\left[\sum_{i=1}^{n} a_i(x,u,Du)\nu_i\right]|_S = 0 \tag{0.3}$$

$$u_1 = \Phi(u_2) \text{ on } S \tag{0.4}$$

where $\Omega \subset R^n$ is a bounded domain with sufficiently smooth boundary $\partial\Omega$ which is divided into two subdomains Ω_1, Ω_2 by means of a smooth surface S which has no intersection point with $\partial\Omega$, the boundary of Ω_1 is S and the boundary of Ω_2 is $S \cup \partial\Omega$. Further, $[f]|_S$ denotes the jump of f on S in the direction of the normal ν, Φ is a smooth strictly increasing function and u_j denotes the restriction of u to Ω_j $(j = 1,2)$. The coefficients of the equation are smooth in $\overline{\Omega_j}$ and satisfy standard conditions but they have jump on the surface S. The problem was motivated e.g. by reaction-diffusion phenomena in porous medium. The authors formulated conditions which

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implied comparison principles, existence and uniqueness of the weak and the classical solution, respectively.

The aim of this paper is to consider similar transmission problems for nonlinear parabolic equations, including nonlocal transmission condition on S and to formulate conditions which imply the existence of weak solutions. In Section 1 we shall consider parabolic equations with a transmission condition which is a bit more general than (0.3), (0.4) and in Section 2 we shall consider equations with nonlocal transmission condition.

1. Nonlinear transmission conditions

Let $\Omega \subset R^n$ be a bounded domain having the uniform C^1 regularity property (see [1]) which is divided into two subdomains Ω_1, Ω_2 by means of a smooth surface S which has no intersection point with $\partial\Omega$, the boundary of Ω_1 is S and the boundary of Ω_2 is $S \cup \partial\Omega$ (such that Ω_1 and Ω_2 have the C^1 regularity property).

In this section we shall consider weak solutions of the following problem:

$$D_t u - \sum_{i=1}^{n} D_i[a_i(t,x,u,Du)] + b(t,x,u,Du) = 0, \quad t \in (0,T), \quad x \in \Omega_1 \cup \Omega_2 \tag{1.5}$$

$$u = 0 \text{ on } \Gamma_T = [0,T] \times \partial\Omega \tag{1.6}$$

$$\sum_{i=1}^{n} a_i(t,x,u_1,Du_1)\nu_i|_{S_T} = \Phi_x'(u_2) \sum_{i=1}^{n} a_i(t,x,u_2,Du_2)\nu_i|_{S_T} \tag{1.7}$$

$$u_1 = \varphi(x,u_2) = \Phi_x(u_2) \text{ on } S_T = [0,T] \times S \tag{1.8}$$

$$u(0,x) = u_0(x), \quad x \in \Omega_1 \cup \Omega_2 \tag{1.9}$$

where $\varphi : \overline{\Omega} \times R \to R$ is a given function with the properties

$$\varphi \in C^2, \quad \Phi_x' > 0, \quad \Phi_x(0) = 0; \lim_{+\infty} \Phi_x = +\infty, \lim_{-\infty} \Phi_x = -\infty,$$

for each fixed $x \in \overline{\Omega}$.

The assumptions on a_i, b are in some sense more general then in [3].

Let $m \geq 2$ be a real number. For any domain $\Omega_0 \subset R^n$ denote by $W^{1,m}(\Omega_0)$ the usual Sobolev space with the norm

$$\| u \| = \left[\int_{\Omega_0} (|Du|^m + |u|^m) \right]^{1/m} .$$

Let V be a closed linear subspace of $W^{1,m}(\Omega_0)$ and denote by $L^m(0, T; V)$ the Banach space of the set of measurable functions $u : (0, T) \to V$ such that $\| u \|^m$ is integrable and define the norm by

$$\| u \|^m_{L^m(0,T;V)} = \int_0^T \| u(t) \|^m_V \, dt.$$

The dual space of $L^m(0, T; V)$ is $L^{\tilde{m}}(0, T; V^\star)$ where $1/m + 1/\tilde{m} = 1$ and $V^\star$ is the dual space of V (see, e.g., [4], [5]).

In order to define the weak solution of (1.5)–(1.9) we define the function U by

$$U = u\chi_{\Omega_1} + \varphi(x, u)\chi_{\Omega_2}$$

where χ_{Ω_j} is the characteristic function of Ω_j. Since for $x \in \Omega_2$

$$U = \Phi_x(u), \quad D_t U = \Phi'_x(u) D_t u, \quad DU = \Phi'_x(u) Du + (D_x \Phi_x)(u),$$

$$u = \Phi_x^{-1}(U), \; D_t u = \frac{1}{\Phi'_x(\Phi_x^{-1}(U))} D_t U,$$

$$Du = \frac{1}{\Phi'_x(\Phi_x^{-1}(U))} [DU - (D_x \Phi_x)(\Phi_x^{-1}(U))]$$

thus u satisfies (1.5) for $x \in \Omega_2$ (in classical sense) if and only if U satisfies

$$D_t U - \Phi'_x(\Phi_x^{-1}(U)) \sum_{i=1}^n D_i \left[a_i \left(t, x, \Phi_x^{-1}(U), \frac{DU - (D_x \Phi_x)(\Phi_x^{-1}(U))}{\Phi'_x(\Phi_x^{-1}(U))} \right) \right] +$$

$$+ \Phi'_x(\Phi_x^{-1}(U)) b \left(t, x, \Phi_x^{-1}(U), \frac{DU - (D_x \Phi_x)(\Phi_x^{-1}(U))}{\Phi'_x(\Phi_x^{-1}(U))} \right) = 0.$$

This equation can be written in the form

$$D_t U - \sum_{i=1}^n D_i \left[\Phi'_x(\Phi_x^{-1}(U)) a_i \left(t, x, \Phi_x^{-1}(U), \frac{DU - (D_x \Phi_x)(\Phi_x^{-1}(U))}{\Phi'_x(\Phi_x^{-1}(U))} \right) \right] +$$

$$+\sum_{i=1}^{n} D_i[\Phi_x'(\Phi_x^{-1}(U))]a_i\left(t,x,\Phi_x^{-1}(U),\frac{DU-(D_x\Phi_x)(\Phi_x^{-1}(U))}{\Phi_x'(\Phi_x^{-1}(U))}\right)+$$

$$+\Phi_x'(\Phi_x^{-1}(U))b\left(t,x,\Phi_x^{-1}(U),\frac{DU-(D_x\Phi_x)(\Phi_x^{-1}(U))}{\Phi_x'(\Phi_x^{-1}(U))}\right)=0$$

where

$$D_i[\Phi_x'(\Phi_x^{-1}(U))]=\Phi_x''(\Phi_x^{-1}(U))(\Phi_x^{-1})'(U)D_iU+(D_{x_i}\Phi_x')(\Phi_x^{-1}(U))=$$

$$=\frac{\Phi_x''}{\Phi_x'}(\Phi_x^{-1}(U))D_iU+(D_{x_i}\Phi_x')(\Phi_x^{-1}(U)).$$

Further, for $x\in S$ we have

$$\sum_{i=1}^{n} a_1(t,x,u_1,Du_1)\nu_i=\Phi_x'(u_2)\sum_{i=1}^{n} a_i(t,x,u_2,Du_2)\nu_i=$$

$$=\sum_{i=1}^{n}\Phi_x'(\Phi_x^{-1}(U))a_i\left(t,x,\Phi_x^{-1}(U),\frac{DU-(D_x\Phi_x)(\Phi_x^{-1}(U))}{\Phi_x'(\Phi_x^{-1}(U))}\right)\nu_i,$$

$$u_1=U.$$

Consequently, u is a classical solution of (1.5)–(1.9) if and only if $U=u\chi_{\Omega_1}+$ $+\varphi(x,u)\chi_{\Omega_2}$ is a classical solution of the problem

$$D_tU-\sum_{i=1}^{n}D_i[A_i(t,x,U,DU)]+B(t,x,U,DU)=0,\qquad t\in(0,T),\quad x\in\Omega_1\cup\Omega_2 \tag{1.10}$$

$$U=0 \text{ on } \Gamma_T=[0,T]\times\partial\Omega \tag{1.11}$$

$$\sum_{i=1}^{n}A_i(t,x,U_1,DU_1)\nu_i|_{S_T}=\sum_{i=1}^{n}A_i(t,x,U_2,DU_2)\nu_i|_{S_T} \tag{1.12}$$

$$U_1=U_2 \text{ on } S_T=[0,T]\times S \tag{1.13}$$

$$U(0,x)=\Phi_x(u_0(x))=U_0(x),\quad x\in\Omega_1\cup\Omega_2 \tag{1.14}$$

where

$$A_i(t,x,z,p)=a_i(t,x,z,p) \text{ for } x\in\Omega_1$$

$$A_i(t,x,z,p)=\Phi_x'(\Phi_x^{-1}(z))a_i\left(t,x,\Phi_x^{-1}(z),\frac{p-(D_x\Phi_x)(\Phi_x^{-1}(z))}{\Phi_x'(\Phi_x^{-1}(z))}\right)\quad\text{for } x\in\Omega_2 \tag{1.15}$$

$$B(t,x,z,p) = b(t,x,z,p) \text{ for } x \in \Omega_1$$

$$(1.16)\quad B(t,x,z,p) = \Phi_x'(\Phi_x^{-1}(z))b\left(t,x,\Phi_x^{-1}(z),\frac{p-(D_x\Phi_x)(\Phi_x^{-1}(z))}{\Phi_x'(\Phi_x^{-1}(z))}\right)+$$

$$+\sum_{i=1}^n\left\{\frac{\Phi_x''}{\Phi_x'}[\Phi_x^{-1}(z)]p_i+(D_{x_i}\Phi_x')[\Phi_x^{-1}(z)]\right\}\cdot$$

$$\cdot a_i\left(t,x,\Phi_x^{-1}(z),\frac{p-(D_x\Phi_x)(\Phi_x^{-1}(z))}{\Phi_x'(\Phi_x^{-1}(z))}\right) \text{ for } x\in\Omega_2.$$

Therefore, it is natural the following

DEFINITION 1.1. We shall say that u is a weak solution of (1.5)–(1.9) if $U = u\chi_{\Omega_1} + \Phi_x(u)\chi_{\Omega_2}$ is a weak solution of (1.10)–(1.14) in the following sense:

$$(1.17)\quad U \in L^m(0,T;V) \text{ with } V = W_0^{1,p}(\Omega), \quad D_t U \in L^{\tilde m}(0,T;V^\star),$$

$$(1.18)\qquad D_t U - \sum_{i=1}^n D_i[A_i(t,x,U,DU)] + B(t,x,U,DU) = 0$$

in usual generalized sense (see below) and

$$(1.19)\qquad U(0,x) = U_0(x), \quad x\in\Omega.$$

Obviously, if U is a classical solution of (1.10)–(1.14) then it satisfies (1.17)–(1.19), i.e. it is a weak solution.

It is well known the following result on the weak solution of (1.17)–(1.19) (which is based on the theory of pseudo-monotone operators, see, e.g. [2]).

Assume that

I. The functions $A_i, B : Q_T \times R^{n+1} \to R$ satisfy the Carathéodory conditions, i.e. $A_i(t,x,z,p)$, $B(t,x,z,p)$ are measurable in $(t,x) \in Q_T = (0,T)\times\Omega$ for each fixed $(z,p)\in R^{n+1}$ and they are continuous in $(z,p) \in R^{n+1}$ for a.e. $(t,x)\in Q_T$.

II. $|A_i(t,x,z,p)| \le c_1[|z|^{m-1}+|p|^{m-1}]+k_1(x)$, for a.e. $(t,x)\in Q_T$, each $(z,p)\in R^{n+1}$ with some constant c_1 and a function $k_1 \in L^{\tilde m}(\Omega)$, $|B(t,x,z,p)| \le c_1[|z|^{m-1}+|p|^{m-1}]+k_1(x)$.

III. $\sum_{i=1}^n[A_i(t,x,z,p)-A_i(t,x,z,p^\star)](p_i-p_i^\star) > 0$ if $p\ne p^\star$.

IV. $\sum_{i=1}^{n} A_i(t,x,z,p)p_i + B(t,x,z,p)z \geq c_2|p|^m - k_2(x)$ with some constant $c_2 > 0$, $k_2 \in L^1(\Omega)$.

Then we may define the operator $G : L^m(0, T; V) \to L^{\tilde{m}}(0, T; V^\star)$ by

$$[G(U), W] = \int_{Q_T} \left[\sum_{i=1}^{n} A_i(t,x, U, DU) D_i W + B(t,x, U, DU) W \right],$$

$$U, W \in L^m(0, T; V)$$

which is bounded (i.e. it maps bounded sets of $L^m(0, T; V)$ into bounded sets of $L^{\tilde{m}}(0, T; V^\star)$) and it is demicontinuous (the strong convergence of a sequence (U_l) in $L^m(0, T; V)$ implies the weak convergence of $(G(U_l))$ in $L^{\tilde{m}}(0, T; V^\star)$. Further, G is pseudomonotone with respect to

$$D(L) = \{W \in L^m(0, T; V) : D_t W \in L^{\tilde{m}}(0, T; V^\star), \quad W(0) = 0\},$$

i.e. if $U_l \in D(L)$, $(U_l) \to U$ weakly in $L^m(0, T; V)$, $(D_t U_l) \to D_t U$ weakly in $L^{\tilde{m}}(0, T; V^\star)$ and $\limsup_{l\to\infty}[G(U_l), U_l - U] \leq 0$ then

$$\lim_{l\to\infty} [G(U_l), U_l - U] = 0 \quad \text{and} \quad (G(U_l)) \to G(U)$$

weakly in $L^{\tilde{m}}(0, T; V^\star)$. Finally, G is coercive:

$$\lim_{\|U\|\to\infty} \frac{[G(U), U]}{\| U \|} = +\infty.$$

By Theorem 4. of [2] we have

THEOREM 1.2. *Assume I–IV. Then for any $U_0 \in V$ there exists*

(1.20) $$U \in L^m(0, T; V) \textit{ such that } D_t U \in L^{\tilde{m}}(0, T; V^\star),$$

(1.21) $$D_t U + G(U) = 0,$$

(1.22) $$U|_{t=0} = U_0.$$

If U satisfies (1.20)–(1.22), we say that U is a weak solution of (1.17)–(1.19).

REMARK 1. If A_i, B satisfy (the monotonicity condition)

(1.23) $$\sum_{i=1}^{n}[A_i(t,x,z,p) - A_i(t,x,z^\star,p^\star)](p_i - p_i^\star) +$$
$$+[B(t,x,z,p) - B(t,x,z^\star,p^\star)](z - z^\star) \geq 0$$

then, obviously, the solution is unique.

Now we formulate conditions on a_i, b which imply I–IV for A_i, B and so the existence of weak solutions of (1.5)–(1.9).

Assume that

I'. The functions a_i, b satisfy the Carathéodory conditions.

II'. For $x \in \Omega_2$ $|a_i(t,x,\tilde{z},\tilde{p})| \leq$

$$\frac{c_1}{\Phi_x'(\tilde{z})}\left[|\Phi_x(\tilde{z})|^{m-1} + [\Phi_x'(\tilde{z})]^{m-1}|\tilde{p}|^{m-1} - |(D_x\Phi_x)(\tilde{z})|^{m-1}\right] + \frac{k_1(x)}{\Phi_x'(\tilde{z})};$$

$$\Bigg| b(t,x,\tilde{z},\tilde{p}) + \frac{1}{\Phi_x'(\tilde{z})}\sum_{i=1}^{n}\left\{\frac{\Phi_x''}{\Phi_x'}(\tilde{z})[\Phi_x'(\tilde{z})\tilde{p}_i + \right.$$

$$\left. + (D_{x_i}\Phi_x)(\tilde{z})] + (D_{x_i}\Phi_x')(\tilde{z})\right\} a_i(t,x,\tilde{z},\tilde{p})\Bigg| \leq$$

$$\leq \frac{c_1}{\Phi_x'(\tilde{z})}\left\{|\Phi_x(\tilde{z})|^{m-1} + [\Phi_x'(\tilde{z})]^{m-1}|\tilde{p}|^{m-1} - |(D_x\Phi_x)(\tilde{z})|^{m-1}\right\} + \frac{k_1(x)}{\Phi_x'(\tilde{z})}.$$

III'. $\sum_{i=1}^{n}[a_i(t,x,\tilde{z},\tilde{p}) - a_i(t,x,\tilde{z},\tilde{p}^\star)](\tilde{p}_i - \tilde{p}_i^\star) > 0$ if $\tilde{p} \neq \tilde{p}^\star$.

IV'. For $x \in \Omega_2$

$$\sum_{i=1}^{n}\left\{1 + \frac{\Phi_x''(\tilde{z})\Phi_x(\tilde{z})}{[\Phi_x'(\tilde{z})]^2}\right\} a_i(t,x,\tilde{z},\tilde{p})\tilde{p}_i + \frac{1}{\Phi_x'(\tilde{z})} b(t,x,\tilde{z},\tilde{p})\Phi_x(\tilde{z}) +$$

$$+ \sum_{i=1}^{n}\left\{\frac{(D_{x_i}\Phi_x)(\tilde{z})}{\Phi_x'(\tilde{z})} + \frac{(D_{x_i}\Phi_x)(\tilde{z})\Phi_x''(\tilde{z})\Phi_x(\tilde{z})}{[\Phi_x'(\tilde{z})]^3} + \frac{(D_{x_i}\Phi_x')(\tilde{z})\Phi_x(\tilde{z})}{[\Phi_x'(\tilde{z})]^2}\right\}\cdot$$

$$\cdot a_i(t,x,\tilde{z},\tilde{p}) \geq c_2\left\{[\Phi_x'(\tilde{z})]^{m-2}|\tilde{p}|^m + \frac{|(D_x\Phi_x)(\tilde{z})|^m}{[\Phi_x'(\tilde{z})]^2}\right\} - \frac{k_2(x)}{[\Phi_x'(\tilde{z})]^2}.$$

V'. For $x \in \Omega_1$ we assume that a_i, b satisfy the same conditions as A_i, B, respectively, in II–IV.

THEOREM 1.3. *Assume I'–V'. Then the problem* (1.5)–(1.9) *has a weak solution for any* u_0 *with the property* $\Phi_x(u_0) \in V$.

PROOF. For $x \in \Omega_2$, the assumption II has the form (according to (1.15), (1.16))

$$\left| \Phi_x'(\Phi_x^{-1}(z)) a_i \left(t, x, \Phi_x^{-1}(z), \frac{p - (D_x\Phi_x)(\Phi_x^{-1}(z))}{\Phi_x'(\Phi_x^{-1}(z))} \right) \right| \leq$$
$$\leq c_1[|z|^{m-1} + |p|^{m-1}] + k_1(x) \text{ and}$$
$$\left| \Phi_x'(\Phi_x^{-1}(z)) b \left(t, x, \Phi_x^{-1}(z), \frac{p - (D_x\Phi_x)(\Phi_x^{-1}(z))}{\Phi_x'(\Phi_x^{-1}(z))} \right) + \right.$$
$$+ \sum_{i=1}^{n} \left[\frac{\Phi_x''}{\Phi_x'}(\Phi_x^{-1}(z)) p_i + (D_{x_i}\Phi_x')(\Phi_x^{-1}(z)) \right] \cdot$$
$$\left. \cdot a_i \left(t, x, \Phi_x^{-1}(z), \frac{p - (D_x\Phi_x)(\Phi_x^{-1}(z))}{\Phi_x'(\Phi_x^{-1}(z))} \right) \right| \leq$$
$$\leq c_1[|z|^{m-1} + |p|^{m-1}] + k_1(x).$$

By using the substitutions

$$\tilde{z} = \Phi_x^{-1}(z), \quad \tilde{p} = \frac{p - (D_x\Phi_x)(\Phi_x^{-1}(z))}{\Phi_x'(\Phi_x^{-1}(z))}, \tag{1.24}$$

one gets: II' implies that A_i, B satisfy assumption II.

Clearly, assumption III' implies III. Finally, by using substitutions (1.24) and (1.15), (1.16) one finds: IV' implies that A_i, B satisfy IV.

Now we consider some special cases when one can check that I'–IV' are fulfilled.

If functions a_i have the form

$$a_i(t, x, \tilde{z}, \tilde{p}) = f(t, x)|\tilde{p}|^{m-1} \operatorname{sign} \tilde{p}_i \text{ for } x \in \Omega_1, \tag{1.25}$$
$$a_i(t, x, \tilde{z}, \tilde{p}) = f(t, x)[\Phi_x'(\tilde{z})]^{m-2}|\tilde{p}|^{m-1} \operatorname{sign} \tilde{p}_i \text{ for } x \in \Omega_2$$

with some measurable function f satisfying $c_0 \leq f(t, x) \leq c_0^\star$ for some positive constants c_0, $c_0^\star$ then a_i satisfies I'–III' and for $x \in \Omega_2$

$$\sum_{i=1}^{n} a_i(t, x, \tilde{z}, \tilde{p})\tilde{p}_i \geq c_2[\Phi_x'(\tilde{z})]^{m-2}|\tilde{p}|^m \text{ with a constant } c_2 > 0. \tag{1.26}$$

We shall formulate conditions which imply I'–IV' in two special cases.

THEOREM 1.4. *Let* $\Phi_x = \Phi$ *(i.e. it is not depending on* x*). Further, assume that* a_i *satisfy I'–III' and* (1.26) *(e.g.* a_i *have the form* (1.25)*); b satisfies I' and*

$$b(t,x,\tilde{z},\tilde{p}) = b^\star(t,x,\tilde{z},\tilde{p}) - \frac{\Phi_x''(\tilde{z})}{\Phi_x'(\tilde{z})}\sum_{i=1}^n a_i(t,x,\tilde{z},\tilde{p})\tilde{p}_i \tag{1.27}$$

where

$$|b^\star(t,x,\tilde{z},\tilde{p})| \le \frac{c_3}{\Phi'(\tilde{z})}|\Phi(\tilde{z})|^{m-1} \text{ and} \tag{1.28}$$

$$b^\star(t,x,\tilde{z},\tilde{p})\tilde{z} \ge 0. \tag{1.29}$$

Then conditions I'–IV' hold.

REMARK 2. (1.28), (1.29) are satisfied e.g. for

$$b^\star(t,x,\tilde{z},\tilde{p}) = \frac{1}{\Phi'(\tilde{z})}|\Phi(\tilde{z})|^{m-1}\operatorname{sign}\tilde{z}.$$

Let

$$\Phi_x(\tilde{z}) = \alpha(x)\tilde{z} \text{ where } \alpha \in C^1(\overline{\Omega_2}), \quad \alpha > 0. \tag{1.30}$$

Since then $\Phi_x'' = 0$, it is not difficult to prove

THEOREM 1.5. *Assume* (1.30)*, I', for* $x \in \Omega_2$

$$|a_i(t,x,\tilde{z},\tilde{p})| \le c_1[|\tilde{z}|^{m-1} + |\tilde{p}|^{m-1}] + k_1(x), \tag{1.31}$$

$$|b(t,x,\tilde{z},\tilde{p})| \le c_1[|\tilde{z}|^{m-1} + |\tilde{p}|^{m-1}] + k_1(x),$$

$$\sum_{i=1}^n a_i(t,x,\tilde{z},\tilde{p})\tilde{p}_i + \tag{1.32}$$

$$+\left[b(t,x,\tilde{z},\tilde{p}) + \frac{2}{\alpha(x)}\sum_{i=1}^n (D_i\alpha)(x)a_i(t,x,\tilde{z},\tilde{p})\right]\tilde{z} \ge$$

$$\ge c_2[|\tilde{p}|^m + |\tilde{z}|^m] \text{ with some constant } c_2 > 0.$$

Then A_i*, B satisfy II'–IV'.*

REMARK 3. Clearly, the followig conditions imply (1.32):

$$\sum_{i=1}^n a_i(t,x,\tilde{z},\tilde{p})\tilde{p}_i \ge \text{const}|\tilde{p}|^m, \quad b(t,x,\tilde{z},\tilde{p})\tilde{z} \ge \text{const}|\tilde{z}|^m,$$

$$\left|\frac{2}{\alpha(x)}\sum_{i=1}^n (D_i\alpha)(x)a_i(t,x,\tilde{z},\tilde{p})\right| \le \frac{1}{2}|b(t,x,\tilde{z},\tilde{p})|.$$

According to Remark 1 it is easy to formulate sufficient conditions for the uniquenss of the weak solution of (1.5)–(1.9). The condition (1.23) (the condition of monotonicity) is satisfied e.g. if the functions a_i are defined by (1.25) and the function b is defined by

$$b(t,x,\tilde{z},\tilde{p}) = \frac{g(t,x)}{\Phi'(\tilde{z})}|\Phi(\tilde{z})|^{m-1}\operatorname{sign}\tilde{z} - \frac{\Phi_x''(\tilde{z})}{\Phi'_x(\tilde{z})}\sum_{i=1}^{n} a_i(t,x,\tilde{z},\tilde{p})\tilde{p}_i$$

with some measurable function g, satisfying $c_0 \le g(t,x) \le c_0^\star$ for some positive constants c_0, $c_0^\star$.

2. Nonlocal transmission conditions

In this section we shall consider weak solutions of the problem

$$D_t u - \sum_{i=1}^{n} D_{x_i}[a_i(t,x,u,Du)] + b(t,x,u,Du) = F(t,x), \tag{2.33}$$

$$t \in (0,T), \quad x \in \Omega_1 \cup \Omega_2,$$

$$u = 0 \text{ on } \Gamma_T = [0,T] \times \partial\Omega, \tag{2.34}$$

$$u_1(t,x) = \alpha u_2(\psi_0(t),\psi(x)) \text{ on } S_T = [0,T] \times S, \tag{2.35}$$

$$\sum_{i=1}^{n} a_i(t,x,u_1(t,x),Du_1(t,x))\nu_i = \tag{2.36}$$

$$= \alpha\psi_0'(t)\sum_{i=1}^{n}\Bigg[a_i(\psi_0(t),\psi(x),u_2(\psi_0(t),\psi(x)),Du_2(\psi_0(t),\psi(x))\cdot$$

$$\cdot\sum_{j=1}^{n}(D_i\psi_j^{-1})(\psi(x))\nu_j\Bigg], \qquad u(0,x) = u_0(x), \quad x \in \Omega_1 \cup \Omega_2 \tag{2.37}$$

where $\psi_0 : [0,T] \to [0,T]$ is a C^1 function, satisfying

$$\psi_0' > 0, \quad 0 \le \psi_0(t) \le t, \quad \psi_0(T) = T;$$

$\psi : \overline{\Omega} \to \overline{\Omega}$ is a C^2 function such that ψ^{-1} exists and $\psi^{-1} \in C^2(\overline{\Omega})$, $\psi(S) = S$, $\psi(\Omega_j) = \Omega_j$; $\alpha > 0$ is a given constant.

In order to define the weak solution of (2.33)–(2.37), we define the function U by

$$U(\tau,\xi) = u(\tau,\xi)\chi_{\Omega_1}(\xi) + \alpha u(\psi_0(\tau),\psi(\xi))\chi_{\Omega_2}(\xi).$$

Since in Ω_2

$$U(\tau,\xi) = \alpha u(\psi_0(\tau), \psi(\xi)) = \alpha u(t,x),$$

with $t = \psi_0(\tau), \quad x = \psi(\xi), \quad \tau = \psi_0^{-1}(t), \quad \xi = \psi^{-1}(x)$, i.e.

$$u(t,x) = \frac{1}{\alpha} U(\tau,\xi) = \frac{1}{\alpha} U(\psi_0^{-1}(t), \psi^{-1}(x)),$$

$$D_t u(t,x) = \frac{1}{\alpha} D_\tau U(\tau, \psi^{-1}(x))(\psi_0^{-1})'(t) = \frac{1}{\alpha \psi_0'(\tau)} D_\tau U(\tau,\xi),$$

$$D_x u(t,x) = \frac{1}{\alpha} D_\xi U(\tau,\xi)(\psi^{-1})'(x)$$

where $(\psi^{-1})'(x) = (\psi')^{-1}(\psi^{-1}(x)) = [\psi'(\xi)]^{-1}$,

thus u satisfies (2.33) in Ω_2 if and only if U satisfies

$$\frac{1}{\alpha\psi_0'(\tau)} D_\tau U(\tau,\xi) -$$

$$-\sum_{i=1}^n D_{x_i} \left\{ a_i(\psi_0(\tau), \psi(\xi), \frac{1}{\alpha} U(\tau,\xi), \frac{1}{\alpha} D_\xi U(\tau,\xi)[\psi'(\xi)]^{-1}) \right\} +$$

$$+ b(\psi_0(\tau), \psi(\xi), \frac{1}{\alpha} U(\tau,\xi), \frac{1}{\alpha} D_\xi U(\tau,\xi)[\psi'(\xi)]^{-1}) = F(\psi_0(\tau), \psi(\xi)), \text{ i.e.}$$

(2.38)
$$\frac{1}{\alpha\psi_0'(\tau)} D_\tau U(\tau,\xi) - \sum_{j=1}^n \left\{ \sum_{i=1}^n \frac{\partial \xi_j}{\partial x_i}(\psi(\xi)) D_{\xi_j} \cdot \right.$$

$$\left. \cdot \left[a_i(\psi_0(\tau), \psi(\xi), \frac{1}{\alpha} U(\tau,\xi), \frac{1}{\alpha} D_\xi U(\tau,\xi)[\psi'(\xi)]^{-1}) \right] \right\} +$$

$$+ b(\psi_0(\tau), \psi(\xi), \frac{1}{\alpha} U(\tau,\xi), \frac{1}{\alpha} D_\xi U(\tau,\xi)[\psi'(\xi)]^{-1}) = F(\psi_0(\tau), \psi(\xi))$$

where we used the notation

$$\frac{\partial \xi_j}{\partial x_i}(\psi(\xi)) = (D_i \psi_j^{-1})(\psi(\xi)).$$

The equation (2.38) can be written in the form

(2.39)
$$D_\tau U(\tau,\xi) - \sum_{j=1}^n D_{\xi_j} \left\{ \alpha \psi_0'(\tau) \cdot \right.$$

$$\cdot \sum_{i=1}^{n} \frac{\partial \xi_j}{\partial x_i}(\psi(\xi)) a_i(\psi_0(\tau), \psi(\xi), \frac{1}{\alpha} U(\tau,\xi), \frac{1}{\alpha} D_\xi U(\tau,\xi)[\psi'(\xi)]^{-1}) \Big\} +$$

$$+ \sum_{j=1}^{n} \sum_{i=1}^{n} \alpha \psi_0'(\tau) D_{\xi_j} \left[\frac{\partial \xi_j}{\partial x_i}(\psi(\xi)) \right] \cdot$$

$$\cdot a_i(\psi_0(\tau), \psi(\xi), \frac{1}{\alpha} U(\tau,\xi), \frac{1}{\alpha} D_\xi U(\tau,\xi)[\psi'(\xi)]^{-1}) +$$

$$+ \alpha \psi_0'(\tau) b(\psi_0(\tau), \psi(\xi), \frac{1}{\alpha} U(\tau,\xi), \frac{1}{\alpha} D_\xi U(\tau,\xi)[\psi'(\xi)]^{-1}) =$$

$$= \alpha \psi_0'(\tau) F(\psi_0(\tau), \psi(\xi)) = F_1(\tau,\xi)$$

where

$$D_{\xi_j} \left[\frac{\partial \xi_j}{\partial x_i}(\psi(\xi)) \right] = \sum_{k=1}^{n} (D_{ik} \psi_j^{-1})(\psi(\xi))(D_j \psi_k)(\xi).$$

Further, for $\xi \in S$ we have

$$\sum_{i=1}^{n} a_i(\tau, \xi, u_1(\tau,\xi), (D_\xi u_1)(\tau,\xi)) \nu_i = \alpha \psi_0'(\tau) \cdot$$

$$\cdot \sum_{i=1}^{n} \left[a_i \left(\psi_0(\tau), \psi(\xi), u_2(\psi_0(\tau), \psi(\xi), Du_2(\psi_0(\tau), \psi(\xi)) \right) \sum_{j=1}^{n} (D_i \psi_j^{-1})(\psi(\xi)) \nu_j \right] =$$

$$= \sum_{j=1}^{n} \left\{ \alpha \psi_0'(\tau) \sum_{i=1}^{n} \left[a_i(\psi_0(\tau), \psi(\xi), \frac{1}{\alpha} U(\tau,\xi), \frac{1}{\alpha} D U(\tau,\xi)[\psi'(\xi)]^{-1}) \cdot \right.\right.$$

$$\left.\left. \cdot (D_i \psi_j^{-1})(\psi(\xi)) \right] \nu_j \right\}.$$

Consequently, u is a classical solution of (2.33)–(2.37) if and only if

$$U(\tau,\xi) = u(\tau,\xi)\chi_{\Omega_1}(\xi) + \alpha u(\psi_0(\tau), \psi(\xi))\chi_{\Omega_2}(\xi).$$

is a solution of the problem

$$D_\tau U - \sum_{j=1}^{n} D_{\xi_j} [A_j(\tau, \xi, U(\tau,\xi), (D_\xi U)(\tau,\xi))] + \tag{2.40}$$

$$B(\tau, \xi, U(\tau,\xi), (D_\xi U)(\tau,\xi)) = F_1(\tau,\xi), \quad \tau \in (0,T), \quad \xi \in \Omega_1 \cup \Omega_2,$$

$$U = 0 \text{ on } \Gamma_T, \tag{2.41}$$

$$U_1 = U_2 \text{ on } S_T, \tag{2.42}$$

$$\sum_{j=1}^{n} A_j(\tau,\xi, U_1, DU_1)\nu_j|_{S_T} = \sum_{j=1}^{n} A_j(\tau,\xi, U_2, DU_2)\nu_j|_{S_T}, \tag{2.43}$$

$$U(0,\xi) = u_0(\psi(\xi)) = U_0(\xi), \quad \xi \in \Omega_1 \cup \Omega_2 \tag{2.44}$$

where

$$A_j(\tau,\xi,z,p) = a_j(\tau,\xi,z,p) \text{ for } \xi \in \Omega_1,$$

$$A_j(\tau,\xi,z,p) = \tag{2.45}$$

$$= \alpha\psi_0'(\tau)\sum_{i=1}^{n} a_i(\psi_0(\tau),\psi(\xi),\frac{1}{\alpha}z,\frac{1}{\alpha}p[\psi'(\xi)]^{-1})(D_i\psi_j^{-1})(\psi(\xi));$$

for $\xi \in \Omega_2$;

$$B(\tau,\xi,z,p) = b(\tau,\xi,z,p) \text{ for } \xi \in \Omega_1,$$

$$B(\tau,\xi,z,p) = \alpha\psi_0'(\tau)b(\psi_0(\tau),\psi(\xi),\frac{1}{\alpha}z,\frac{1}{\alpha}p[\psi'(\xi)]^{-1})+ \tag{2.46}$$

$$+\alpha\psi_0'(\tau)\sum_{i=1}^{n}\left\{a_i(\psi_0(\tau),\psi(\xi),\frac{1}{\alpha}z,\frac{1}{\alpha}p[\psi'(\xi)]^{-1})\cdot\right.$$

$$\left.\cdot\sum_{j,k=1}^{n}(D_{ik}\psi_j^{-1})(\psi(\xi))(D_j\psi_k)(\xi)\right\}$$

for $\xi \in \Omega_2$.

DEFINITION 2.1. We shall say that u is a weak solution of (2.33)–(2.37) if U is a weak solution of (1.17)–(1.19) with F_1 on the right hand side of (1.18) (i.e. U satisfies (1.20)–(1.22) with F_1 on the right hand side of (1.21)).

Now we formulate conditions on a_i, b which imply I–IV for A_j, B (and so the existence of weak solutins to (2.33)–(2.37)).

Assume that

II". For $x \in \Omega_2$

$$|a_i(t,x,\tilde{z},\tilde{p})| \le c_1[|\tilde{z}|^{m-1} + |\tilde{p}|^{m-1}] + k_1(x),$$

$$|b(t,x,\tilde{z},\tilde{p})| \le c_1[|\tilde{z}|^{m-1} + |\tilde{p}|^{m-1}] + k_1(x), \quad k_1 \in L^{\tilde{m}}(\Omega_2).$$

III". For $x \in \Omega_2$ $a_i(t,x,\tilde{z},\tilde{p})$ is continuously differentiable with respect to $\tilde{p}$ and the matrix

$$\left(\frac{\partial a_i}{\partial \tilde{p}_l}(t,x,\tilde{z},\tilde{p})\right)_{i,l=1}^{n}$$

is positive definite for each $(t,x,\tilde{z},\tilde{p})$.

IV".

$$\sum_{i=1}^{n} a_i(t,x,\tilde{z},\tilde{p})\tilde{p}_i \geq c_2|\tilde{p}|^m - k_2(x), \quad k_2 \in L^1(\Omega_2),$$

$$\left\{b(t,x,\tilde{z},\tilde{p}) + \sum_{i=1}^{n}\left[a_i(t,x,\tilde{z},\tilde{p})\sum_{j,k=1}^{n}(D_{ik}\psi_j^{-1})(x)(D_j\psi_k)(\psi^{-1}(x))\right]\right\}\tilde{z} \geq 0.$$

THEOREM 2.2. *Assume I', II"–IV" and V'. Then A_j, B satisfy I–IV, thus the problem* (2.33)–(2.37) *has a weak solution for any $F \in L^{\tilde{m}}(Q_T)$, $u_0 \in V$.*

PROOF. Since $\psi, \psi^{-1} \in C^2(\overline{\Omega})$, from formulas (2.45), (2.46) and II", V' immediately follows that A_j, B satisfy II. Further, it is easy to show that if the matrix

$$\left(\frac{\partial A_j}{\partial \tilde{p}_k}(\tau,\xi,z,p)\right)_{j,k=1}^{n}$$

is positive definite for each (τ,ξ,z,p) then condition III is satisfied. According to (2.45) and $[\psi'(\xi)]^{-1} = (\psi^{-1})'(\psi(\xi))$,

$$\frac{\partial A_j}{\partial p_k}(\tau,\xi,z,p) = \alpha\psi_0'(\tau)\sum_{i=1}^{n}\left[\sum_{l=1}^{n}\frac{\partial a_i}{\partial \tilde{p}_l}\left(\psi_0(\tau),\psi(\xi),\frac{1}{\alpha}z,\frac{1}{\alpha}p(\psi^{-1})'(\psi(\xi))\right)\cdot\right.$$

$$\left.\cdot\frac{1}{\alpha}(D_k\psi_l^{-1})(\psi(\xi))(D_i\psi_j^{-1})(\psi(\xi))\right],$$

$$\text{i.e.}\quad \left(\frac{\partial A_j}{\partial p_k}\right)_{j,k=1}^{n} = \psi_0'(\tau)[(\psi^{-1})'(\psi(\xi))]^T\left(\frac{\partial a_i}{\partial \tilde{p}_l}\right)_{i,l=1}^{n}(\psi^{-1})'(\psi(\xi))$$

which implies by III" that $\left(\frac{\partial A_j}{\partial p_k}\right)_{j,k=1}^{n}$ is positive definite.

Finally, by using the notations

$$x = \psi(\xi), \quad t = \psi_0(\tau), \quad \tilde{z} = \frac{1}{\alpha}z, \quad \tilde{p} = \frac{1}{\alpha}p(\psi^{-1})'(\psi(\xi)),$$

$$\sum_{j=1}^{n} A_j(\tau,\xi,z,p)p_j = \alpha\psi_0'(\tau)\cdot$$

$$\cdot\sum_{j=1}^{n}\left\{\sum_{i=1}^{n} a_i\left(\psi_0(\tau),\psi(\xi),\frac{1}{\alpha}z,\frac{1}{\alpha}p[\psi^{-1}]'(\psi(\xi))\right)(D_i\psi_j^{-1})(\psi(\xi))\right\}p_j =$$

$$= \frac{\alpha}{(\psi_0^{-1})'(t)}\sum_{i=1}^{n} a_i(t,x,\tilde{z},\tilde{p})\sum_{j=1}^{n}(D_i\psi_j^{-1})(x)p_j =$$

$$= \frac{\alpha^2}{(\psi_0^{-1})'(t)}\sum_{i=1}^{n} a_i(t,x,\tilde{z},\tilde{p})\tilde{p}_i \geq \frac{\alpha^2}{(\psi_0^{-1})'(t)}c_2|\tilde{p}|^m - k_2(x) \geq c_3|p|^m - k_2(x)$$

with some constant $c_3 > 0$. Similarly, by (2.46), IV”

$$B(\tau,\xi,z,p)z = \alpha\psi_0'(\tau)b(t,x,\tilde{z},\tilde{p})z+$$

$$+\alpha\psi_0'(\tau)\sum_{i=1}^{n}\left[a_i(t,x,\tilde{z},\tilde{p})\sum_{j,k=1}^{n}(D_i\psi_j^{-1})(x)(D_j\psi_k)(\psi^{-1}(x))\right]z \geq 0.$$

REMARK 4. A simple sufficient condition for the second part of IV” is:

$$b(t,x,\tilde{z},\tilde{p})\tilde{z} \geq 0 \text{ and}$$

$$\left|\sum_{i=1}^{n}\left[a_i(t,x,\tilde{z},\tilde{p})\sum_{j,k=1}^{n}(D_i\psi_j^{-1})(x)(D_j\psi_k)(\psi^{-1}(x))\right]\right| \leq |b(t,x,\tilde{z},\tilde{p})|.$$

By using Remark 1, it is easy to formulate sufficient conditions for the uniqueness of the weak solution. The condition, formulated in Remark 1 is satisfied if the matrix

$$\left(\frac{\partial A_j}{\partial p_k}(\tau,\xi,p_0,p)\right)_{j,k=0}^{n}$$

is positive semidefinite for each fixed (τ,ξ,p_0,p), where we used the notations $A_0 = B$, $p_0 = z$.

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ALMOST SURE FUNCTIONAL LIMIT THEOREMS IN $L^p([0,1]^d)$

By

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1. Introduction

Let $X_{\mathbf{k}}$, $\mathbf{k} \in \mathbb{N}^d$, be a multiindex sequence of independent, identically distributed (i.i.d.) random variables having zero mean and unit variance. Let

$$Y_{\mathbf{n}}(\mathbf{t}) = \frac{1}{\sqrt{|\mathbf{n}|}} \sum_{\mathbf{k} \leq [\mathbf{nt}]} X_{\mathbf{k}}, \qquad \mathbf{t} \in [0,1]^d,$$

$\mathbf{n} \in \mathbb{N}^d$, be the usual random field defined by the partial sums.

Consider also the multidimensional empirical process

$$Z_{\mathbf{n}}(\mathbf{t}) = \frac{1}{\sqrt{|\mathbf{n}|}} \sum_{\mathbf{i} \leq \mathbf{n}} (\chi\{\mathbf{U_i} \leq \mathbf{t}\} - |\mathbf{t}|), \qquad \mathbf{t} \in [0,1]^d,$$

$\mathbf{n} \in \mathbb{N}^k$, where $\mathbf{U_i}$, $\mathbf{i} \in \mathbb{N}^h$, are independent random vectors having uniform distribution on $[0,1]^d$.

The behaviour of the multiindex random fields $Y_{\mathbf{n}}(\mathbf{t})$ and $Z_{\mathbf{n}}(\mathbf{t})$ are usually investigated in the Skorohod space $D([0,1]^d)$. The limit of $Y_{\mathbf{n}}(\mathbf{t})$ is the d-parameter Wiener process $W(\mathbf{t})$, while the limit of $Z_{\mathbf{n}}(\mathbf{t})$ is the d-parameter Brownian bridge $B(\mathbf{t})$. However, to study some statistics, one can consider these random fields as random elements in the space L^p (see OLIVEIRA and SUQUET [11]). Actually, using IVANOV's [8] general theorems, one can easily prove limit theorems for $Y_{\mathbf{n}}(\mathbf{t})$ and $Z_{\mathbf{n}}(\mathbf{t})$ in $L^p([0,1]^d)$.

The main topic of this note is the study of almost sure (a.s.) versions of the above mentioned usual limit theorems (see BERKES [2] for an overview of a.s. limit theorems). There are several methods to prove a.s. (central) limit theorems (see Major [9], Berkes and Csáki [3], Fazekas and Rychlik [6], Móri [10]). Fazekas and Rychlik [7] described a general method to prove a.s. versions of multiindex limit theorems in metric spaces. We shall apply that method to obtain a.s. limit theorems for $Y_{\mathbf{n}}(\mathbf{t})$ and $Z_{\mathbf{n}}(\mathbf{t})$ in $L^p([0,1]^d)$, see Theorem 2.1 and Theorem 3.1, respectively. The proofs are simple, because in L^p we do not need maximal inequalities. The one-dimensional version of the above threorems were presented in Túri [13].

2. The almost sure version of Donsker's theorem in $L^p([0,1]^d)$

Throughout the paper let $1 \le p < \infty$ and $d \in \mathbb{N}$ be fixed. Let $\mathbf{k} = (k_1, \ldots, k_d)$, $\mathbf{n} = (n_1, \ldots, n_d), \ldots \in \mathbb{N}^d$, $\mathbf{1} = (1, \ldots, 1) \in \mathbb{N}^d$. Relations $\le$, min, max, $\to \infty$ are defined coordinatewise. I.e. $\mathbf{n} \to \infty$ means that $n_i \to \infty$, for each $i = 1, \ldots, d$. Let $\log_+ x = \log x$, if $x \ge e$ and $\log_+ x = 1$, if $x < e$. Let $|\mathbf{n}| = \prod_{i=1}^d n_i$ and $|\log \mathbf{n}| = \prod_{i=1}^d \log_+ n_i$, $\mathbf{n} \in \mathbb{N}^d$.

Denote the usual integer part by $[\cdot]$, moreover for $\mathbf{n} \in \mathbb{N}^d$ and $\mathbf{t} \in [0,1]^d$ denote the vector $([n_1 t_1], \ldots, [n_d t_d]) \in \mathbb{N}^d$ also by $[\mathbf{nt}]$.

Denote $\Rightarrow$ the convergence in distribution. $(\Omega, A, \mathbb{P})$ is the underlying probability space, $\omega \in \Omega$ is an elementary event.

Throughout this section $X_{\mathbf{i}}$, $\mathbf{i} \in \mathbb{N}^d$, will be i.i.d. real random variables with $\mathbb{E}X_{\mathbf{1}} = 0$, $\mathbb{D}^2 X_{\mathbf{1}} = 1$ and $\mathbb{E}|X_{\mathbf{1}}|^p < \infty$. Let $S_{\mathbf{k}} = \sum_{\mathbf{i} \le \mathbf{k}} X_{\mathbf{i}}$.

In this part we consider the random field

$$Y_{\mathbf{n}}(\mathbf{t}) = \frac{1}{\sqrt{|\mathbf{n}|}} S_{[\mathbf{nt}]}, \qquad \mathbf{t} \in [0,1]^d. \tag{1}$$

We will use the next result of IVANOV [8].

REMARK 2.1. Let $Y_{\mathbf{n}}(\mathbf{t})$, $\mathbf{n} \in \mathbb{N}^d$, and $Y(\mathbf{t})$ be random elements in $L^p([0,1]^d)$, $p \ge 1$. Assume that

(i) The finite dimensional distributions of $Y_{\mathbf{n}}$ converge weakly to those of Y;

(ii) $\mathbb{E}|Y_{\mathbf{n}}(\mathbf{t})|^p \to \mathbb{E}|Y(\mathbf{t})|^p$, as $\mathbf{n} \to \infty$, for each $\mathbf{t} \in [0,1]^d$;

(iii) $\sup\limits_{\mathbf{n}} \sup\limits_{\mathbf{t} \in [0,1]^d} \mathbb{E}|Y_{\mathbf{n}}(\mathbf{t})|^p < \infty$.

Then $f(Y_{\mathbf{n}}) \Rightarrow f(Y)$, as $\mathbf{n} \to \infty$, for every continuous functional f on $L^p([0,1]^d)$.

We need the next result due the ROSENTHAL (see [1], p. 205).

REMARK 2.2. Let $Y_{\mathbf{i}}$, $\mathbf{i} \le \mathbf{n}$, be independent centered random variables with $\mathbb{E}|Y_{\mathbf{i}}|^p < \infty$, $p \ge 2$. Then there exist a constant $K_p > 0$ depending only on p such that

$$\left(\mathbb{E}\left|\sum_{\mathbf{i}\le\mathbf{n}} Y_{\mathbf{i}}\right|^p\right)^{1/p} \le K_p \max\left\{\left(\sum_{\mathbf{i}\le\mathbf{n}} \mathbb{E}|Y_{\mathbf{i}}|^p\right)^{1/p}, \left(\sum_{\mathbf{i}\le\mathbf{n}} \mathbb{E}|Y_{\mathbf{i}}|^2\right)^{1/2}\right\}.$$

We also need the next result (see [1], p. 136).

REMARK 2.3. Let $X_{\mathbf{i}}$, $\mathbf{i} \in \mathbb{N}^d$, be centered i.i.d. random variables such that $\frac{1}{\sqrt{|\mathbf{n}|}} \sum_{\mathbf{i}\le\mathbf{n}} X_{\mathbf{i}} \Rightarrow \mathcal{N}(0,1)$. If $\mathbb{E}|X_{\mathbf{1}}|^p < \infty$, $p \ge 1$, then

$$\mathbb{E}\left|\frac{1}{\sqrt{|\mathbf{n}|}} \sum_{\mathbf{i}\le\mathbf{n}} X_{\mathbf{i}}\right|^p \to \mathbb{E}|X|^p, \qquad \text{as } \mathbf{n} \to \infty,$$

where X has normal distribution with mean 0 and variance 1.

For the sake of completeness we give a proof for the Donsker theorem in $L^p([0,1]^d)$.

PROPOSITION 2.1. *The multiindex sequence* $Y_{\mathbf{n}}$, $\mathbf{n} \in \mathbb{N}^d$, *of processes defined by* (1) *converges weakly to the* d*-parameter standard Wiener process* W *in* $L^p([0,1]^d)$, *where* $1 \le p < \infty$.

PROOF. We shall prove that the conditions of Remark 2.1 are fulfilled.

Condition (i), that is the convergence of the finite dimensional distributions to those of the Wiener process is an elementary fact.

Apply Remark 2.3 to obtain that condition (ii) is fulfilled.

Now, we will show that condition (iii) of Remark 2.1 is satisfied.

We will distinguish two cases. In the first case $1 \le p \le 2$.

$$\mathbb{E}|\,Y_{\mathbf{n}}(\mathbf{t})|^p = \mathbb{E}\left|\frac{1}{\sqrt{|\mathbf{n}|}}\sum_{\mathbf{i}\le[\mathbf{nt}]} X_{\mathbf{i}}\right|^p \le \frac{1}{(|\mathbf{n}|)^{p/2}}\left(\mathbb{E}\left|\sum_{\mathbf{i}\le[\mathbf{nt}]} X_{\mathbf{i}}\right|^2\right)^{p/2} \le$$

$$\le \frac{(|\mathbf{n}|)^{p/2}}{(|\mathbf{n}|)^{p/2}} = 1 < \infty.$$

In the second case $2 < p < \infty$. Here we use Remark 2.2. If

$$\sum_{\mathbf{i}\le[\mathbf{nt}]} \mathbb{E}|X_{\mathbf{i}}|^p \ge \left(\sum_{\mathbf{i}\le[\mathbf{nt}]} \mathbb{E}|X_{\mathbf{i}}|^2\right)^{p/2},$$

then

$$\mathbb{E}|\,Y_{\mathbf{n}}(\mathbf{t})|^p = \mathbb{E}\left|\frac{1}{\sqrt{|\mathbf{n}|}}\sum_{\mathbf{i}\le[\mathbf{nt}]} X_{\mathbf{i}}\right|^p \le$$

$$\le \frac{K_p}{(|\mathbf{n}|)^{p/2}}\sum_{\mathbf{i}\le[\mathbf{nt}]} \mathbb{E}|X_{\mathbf{i}}|^p \le C\frac{|\mathbf{n}|}{(|\mathbf{n}|)^{p/2}} \le C^* < \infty.$$

If

$$\sum_{\mathbf{i}\le[\mathbf{nt}]} \mathbb{E}|X_{\mathbf{i}}|^p < \left(\sum_{\mathbf{i}\le[\mathbf{nt}]} \mathbb{E}|X_{\mathbf{i}}|^2\right)^{p/2},$$

then

$$\mathbb{E}|\,Y_{\mathbf{n}}(\mathbf{t})|^p = \mathbb{E}\left|\frac{1}{\sqrt{|\mathbf{n}|}}\sum_{\mathbf{i}\le[\mathbf{nt}]} X_{\mathbf{i}}\right|^p \le \frac{K_p}{(|\mathbf{n}|)^{p/2}}\left(\sum_{\mathbf{i}\le[\mathbf{nt}]} \mathbb{E}|X_{\mathbf{i}}|^2\right)^{p/2}$$

$$\le K_p\frac{(|\mathbf{n}|)^{p/2}}{(|\mathbf{n}|)^{p/2}} = K_p < \infty.$$

The proof of Proposition 2.1 is complete. ∎

To prove a.s. Donsker's theorem we shall need the next result due to Fazekas and Rychlik [7].

Let δ_x denote the unit mass at point x. Let μ_X denote the distribution of X.

REMARK 2.4. Let (B,ρ) be a complete separable metric space and $X_{\mathbf{n}}$, $\mathbf{n} \in \mathbb{N}^d$, be a multiindex sequence of random elements in B. Assume that for any pair $\mathbf{h},\mathbf{l} \in \mathbb{N}^d$, $\mathbf{h} \le \mathbf{l}$, there exists a B-valued random element $X_{\mathbf{h},\mathbf{l}}$ with the following properties. $X_{\mathbf{h},\mathbf{l}} = 0$ if $\mathbf{h} = \mathbf{l}$. If $\mathbf{k},\mathbf{l} \in \mathbb{N}^d$, then for $\mathbf{h} = \min\{\mathbf{k},\mathbf{l}\}$ the following random elements are independent: $X_{\mathbf{k}}$ and $X_{\mathbf{h},\mathbf{l}}$; $X_{\mathbf{l}}$ and $X_{\mathbf{h},\mathbf{k}}$; $X_{\mathbf{h},\mathbf{k}}$ and $X_{\mathbf{h},\mathbf{l}}$.

Assume that there exist $C > 0, \beta > 0$, and increasing sequences $\{c_n^{(i)}\}$ of positive numbers with $\lim_{n\to\infty} c_n^{(i)} = \infty$, $c_{n+1}^{(i)}/c_n^{(i)} = O(1)$ for each $i = 1,\dots,d$, such that

$$\mathbb{E}\min\{\rho^2(X_{\mathbf{l}}, X_{\mathbf{h},\mathbf{l}}), 1\} \le C \prod_{\mathbf{i}=1}^{d} \left(\frac{c_{h_i}^{(i)}}{c_{l_i}^{(i)}}\right)^{\beta}$$

for $\mathbf{h} \le \mathbf{l}$. Let $0 \le d_k^{(i)} \le \log(c_{k+1}^{(i)}/c_k^{(i)})$, assume that $\sum_{k=1}^{\infty} d_k^{(i)} = \infty$ for $i = 1,\dots,d$. Let $d_{\mathbf{k}} = \prod_{i=1}^{d} d_{k_i}^{(i)}$ and $D_{\mathbf{n}} = \sum_{\mathbf{k}\le\mathbf{n}} d_{\mathbf{k}}$.

Then for any probability distribution μ on the Borel σ-algebra of B the following two statements are equivalent

$$\frac{1}{D_{\mathbf{n}}} \sum_{\mathbf{k}\le\mathbf{n}} d_{\mathbf{k}} \delta_{X_{\mathbf{k}}(\omega)} \Rightarrow \mu, \qquad \text{as } \mathbf{n} \to \infty, \qquad \text{for almost every } \omega \in \Omega;$$

$$\frac{1}{D_{\mathbf{n}}} \sum_{\mathbf{k}\le\mathbf{n}} d_{\mathbf{k}} \mu_{X_{\mathbf{k}}} \Rightarrow \mu, \qquad \text{as } \mathbf{n} \to \infty.$$

The almost sure version of Donsker's theorem in $L^p([0,1]^d)$ is the following.

THEOREM 2.1. *Let $1 \le p < \infty$. Let the multiindex sequence of fields $Y_{\mathbf{k}}(\mathbf{t},\omega) = Y_{\mathbf{k}}(\mathbf{t})$ be defined in (1). Then*

$$\frac{1}{|\log \mathbf{n}|} \sum_{\mathbf{k}\le\mathbf{n}} \frac{1}{|\mathbf{k}|} \delta_{Y_{\mathbf{k}}(\cdot,\omega)} \Rightarrow \mu_W,$$

in $L^p([0,1]^d)$, as $\mathbf{n} \to \infty$, for almost every $\omega \in \Omega$, where W is the standard d-parameter Wiener process.

PROOF. We shall prove that the conditions of Remark 2.4 are fulfilled. The separability and completeness of space $L^p([0,1]^d)$, $1 \le p < \infty$, are well-known facts.

Let us define the process

$$Y_{\mathbf{k},\mathbf{n}}(\mathbf{t}) = \frac{1}{\sqrt{|\mathbf{n}|}} \left\{ S_{[\mathbf{nt}]} - S_{\min\{\mathbf{k},[\mathbf{nt}]\}} \right\}, \qquad \mathbf{k} \le \mathbf{n},\ \mathbf{t} \in [0,1]^d.$$

Then the independence conditions of Remark 2.4 are satisfied.

We will distinguish two cases.

In the first case $2 \le p < \infty$. Applying Jensen's inequality we get

$$\mathbb{E}(\rho^2(Y_{\mathbf{n}}, Y_{\mathbf{k},\mathbf{n}})) = \mathbb{E}\left(\int\limits_{[0,1]^d} | Y_{\mathbf{n}}(\mathbf{t}) - Y_{\mathbf{k},\mathbf{n}}(\mathbf{t})|^p \, d\mathbf{t} \right)^{2/p} =$$

$$= \mathbb{E}\left(\int\limits_{[0,1]^d} \left| \frac{1}{\sqrt{|\mathbf{n}|}} S_{\min\{\mathbf{k},[\mathbf{nt}]\}} \right|^p d\mathbf{t} \right)^{2/p} \le$$

$$\le \frac{1}{|\mathbf{n}|} \left(\int\limits_{[0,1]^d} \mathbb{E} \left| S_{\min\{\mathbf{k},[\mathbf{nt}]\}} \right|^p d\mathbf{t} \right)^{2/p} = A.$$

We distinguish again two cases and use Remark 2.2. If

$$\sum_{\mathbf{i} \le \min\{\mathbf{k},[\mathbf{nt}]\}} \mathbb{E}|X_{\mathbf{i}}|^p \ge \left(\sum_{\mathbf{i} \le \min\{\mathbf{k},[\mathbf{nt}]\}} \mathbb{E}|X_{\mathbf{i}}|^2 \right)^{p/2}$$

then

$$A \le \frac{1}{|\mathbf{n}|} \left(\int\limits_{[0,1]^d} K_p \sum_{\mathbf{i} \le \min\{\mathbf{k},[\mathbf{nt}]\}} \mathbb{E}|X_{\mathbf{i}}|^p \, d\mathbf{t} \right)^{2/p} \le$$

$$\le \frac{1}{|\mathbf{n}|} \left(\int\limits_{[0,1]^d} K_p \sum_{\mathbf{i} \le \mathbf{k}} \mathbb{E}|X_{\mathbf{i}}|^p \, d\mathbf{t} \right)^{2/p} \le \frac{K_p^{2/p}}{|\mathbf{n}|} (C \cdot |\mathbf{k}|)^{2/p} \le C^* \frac{|\mathbf{k}|}{|\mathbf{n}|}.$$

If

$$\sum_{\mathbf{i} \le \min\{\mathbf{k},[\mathbf{nt}]\}} \mathbb{E}|X_{\mathbf{i}}|^p < \left(\sum_{\mathbf{i} \le \min\{\mathbf{k},[\mathbf{nt}]\}} \mathbb{E}|X_{\mathbf{i}}|^2 \right)^{p/2}$$

then

$$A \le \frac{1}{|\mathbf{n}|}\left(\int\limits_{[0,1]^d} K_p\left(\sum_{\mathbf{i}\le\min\{\mathbf{k},[\mathbf{nt}]\}}\mathbb{E}|X_{\mathbf{i}}|^2\right)^{p/2} d\mathbf{t}\right)^{2/p} \le$$

$$\le \frac{1}{|\mathbf{n}|}\left(\int\limits_{[0,1]^d} K_p\left(\sum_{\mathbf{i}\le\mathbf{k}}\mathbb{E}|X_{\mathbf{i}}|^2\right)^{p/2} d\mathbf{t}\right)^{2/p} = C^*\frac{|\mathbf{k}|}{|\mathbf{n}|}.$$

In the second case $1 \le p < 2$. By Jensen's inequality

$$\mathbb{E}(\rho^2(Y_{\mathbf{n}}, Y_{\mathbf{k},\mathbf{n}})) = \mathbb{E}\left(\int\limits_{[0,1]^d}\left|\frac{1}{\sqrt{|\mathbf{n}|}}S_{\min\{\mathbf{k},[\mathbf{nt}]\}}\right|^p d\mathbf{t}\right)^{2/p} \le$$

$$\le \frac{1}{|\mathbf{n}|}\int\limits_{[0,1]^d}\mathbb{E}\left|S_{\min\{\mathbf{k},[\mathbf{nt}]\}}\right|^2 d\mathbf{t} = \frac{1}{|\mathbf{n}|}\int\limits_{[0,1]^d}\mathbb{E}\left|\sum_{\mathbf{i}\le\min\{\mathbf{k},[\mathbf{nt}]\}}X_{\mathbf{i}}\right|^2 d\mathbf{t} \le C\frac{|\mathbf{k}|}{|\mathbf{n}|}.$$

Therefore we can apply Remark 2.3 with $c_k^{(i)} = k$, $k = 1, 2, \dots$, $i = 1, \dots, d$.

The proof of Theorem 2.1 is complete. ∎

3. The multidimensional empirical process in $L^p([0,1]^d)$

Let $1 \le p < \infty$ and $d, h \in \mathbb{N}$ be fixed. In this section, we consider the multidimensional empirical process

$$Z_{\mathbf{n}}(\mathbf{t}) = \frac{1}{\sqrt{|\mathbf{n}|}}\sum_{\mathbf{i}\le\mathbf{n}}(\chi\{\mathbf{U_i} \le \mathbf{t}\} - |\mathbf{t}|), \qquad \mathbf{t} \in [0,1]^d, \tag{2}$$

where $\mathbf{U_i}$, $\mathbf{i} \in \mathbb{N}^h$, are independent random variables uniformly distributed on $[0,1]^d$.

PROPOSITION 3.1. *The multiindex sequence $Z_{\mathbf{n}}(\mathbf{t})$, $\mathbf{n} \in \mathbb{N}^d$, weakly converges to the d-parameter Brownian bridge B in space $L^p(]0,1[^d)$, where $1 \le p < \infty$.*

PROOF. We shall prove that the conditions of Remark 2.1 are fulfilled.

Condition (i), the convergence of the finite dimensional distributions to those of the Brownian bridge is an elementary fact.

Apply Remark 2.3 to obtain that condition (ii) is fulfilled.

Now, we will show that condition (iii) of Remark 2.1 is satisfied.

We will distinguish two cases. In the first case $1 \le p \le 2$.

$$\mathbb{E}|Z_{\mathbf{n}}(\mathbf{t})|^p \le \frac{1}{(|\mathbf{n}|)^{p/2}} \left(\mathbb{E} \left| \sum_{\mathbf{i} \le \mathbf{n}} (\chi\{\mathbf{U_i} \le \mathbf{t}\} - |\mathbf{t}|) \right|^2 \right)^{p/2} =$$

$$= \frac{1}{(|\mathbf{n}|)^{p/2}} \big(|\mathbf{n}|\, |\mathbf{t}|(1 - |\mathbf{t}|)\big)^{p/2} \le 1 < \infty.$$

In the second case $2 < p < \infty$. Here we distinguish again two cases.

If

$$\sum_{\mathbf{i} \le \mathbf{n}} \mathbb{E}|\chi\{\mathbf{U_i} \le \mathbf{t}\} - |\mathbf{t}||^p \ge \left(\sum_{\mathbf{i} \le \mathbf{n}} \mathbb{E}|\chi\{\mathbf{U_i} \le \mathbf{t}\} - |\mathbf{t}||^2 \right)^{p/2},$$

then by Rosenthal's inequality

$$\mathbb{E}|Z_{\mathbf{n}}(\mathbf{t})|^p \le \frac{K_p}{(|\mathbf{n}|)^{p/2}} \sum_{\mathbf{i} \le \mathbf{n}} \mathbb{E}|\chi\{\mathbf{U_i} \le \mathbf{t}\} - |\mathbf{t}||^p \le \frac{K_p}{(|\mathbf{n}|)^{p/2}} \cdot C|\mathbf{n}| < C^* < \infty.$$

If

$$\sum_{\mathbf{i} \le \mathbf{n}} \mathbb{E}|\chi\{\mathbf{U_i} \le \mathbf{t}\} - |\mathbf{t}||^p < \left(\sum_{\mathbf{i} \le \mathbf{n}} \mathbb{E}|\chi(X_{\mathbf{i}} \le \mathbf{t}) - |\mathbf{t}||^2 \right)^{p/2},$$

then

$$\mathbb{E}|Z_{\mathbf{n}}(\mathbf{t})|^p \le \frac{(K_p)^p}{(|\mathbf{n}|)^{p/2}} (|\mathbf{n}|\, |\mathbf{t}|(1 - |\mathbf{t}|))^{p/2} \le K_p^{p/2} < \infty.$$

The proof of Proposition 3.1 is complete. ∎

The almost sure version of the limit theorem for the empirical process in $L^p([0,1]^d)$ is the following.

THEOREM 3.1. *Let* $1 \le p < \infty$. *Let* $Z_{\mathbf{k}}(\mathbf{t})$ *be the empirical process defined in* (2). *Then*

$$\frac{1}{|\log \mathbf{n}|}\sum_{\mathbf{k}\le\mathbf{n}}\frac{1}{|\mathbf{k}|}\delta_{Z_{\mathbf{k}}(\cdot,\omega)} \Rightarrow \mu_B,$$

in $L^p(]0,1[^d)$, *as* $\mathbf{n}\to\infty$, *for almost every* $\omega\in\Omega$, *where* B *is the* d-*parameter Brownian bridge.*

PROOF. We shall prove that the conditions of Remark 2.4 are fulfilled. Let us define the process

$$Z_{\mathbf{k},\mathbf{n}}(\mathbf{t}) = \frac{1}{\sqrt{|\mathbf{n}|}}\sum_{\mathbf{i}\le\mathbf{n}}(\chi\{\mathbf{U_i}\le\mathbf{t}\}-|\mathbf{t}|) - \frac{1}{\sqrt{|\mathbf{n}|}}\sum_{\mathbf{i}\le\mathbf{k}}(\chi\{\mathbf{U_i}\le\mathbf{t})\}-|\mathbf{t}|),$$

$\mathbf{k}\le\mathbf{n}$, $\mathbf{k},\mathbf{n}\in\mathbb{N}^h$, $\mathbf{t}\in[0,1]^d$.

Then the independence conditions of Remark 2.4 are satisfied.

In the first case let $2 < p < \infty$. Then

$$\mathbb{E}(\rho^2(Z_{\mathbf{n}},Z_{\mathbf{k},\mathbf{n}})) = \mathbb{E}\left(\int_{[0,1]^d}|Z_{\mathbf{n}}-Z_{\mathbf{k},\mathbf{n}}|^p\,d\mathbf{t}\right)^{2/p} =$$

$$= \mathbb{E}\left(\int_{[0,1]^d}\left|\frac{1}{\sqrt{|\mathbf{n}|}}\sum_{\mathbf{i}\le\mathbf{k}}(\chi\{\mathbf{U_i}\le\mathbf{t}\}-|\mathbf{t}|)\right|^p d\mathbf{t}\right)^{2/p} \le$$

$$\le \frac{1}{|\mathbf{n}|}\left(\int_{[0,1]^d}\mathbb{E}\left|\sum_{\mathbf{i}\le\mathbf{k}}(\chi\{\mathbf{U_i}\le\mathbf{t}\}-|\mathbf{t}|)\right|^p d\mathbf{t}\right)^{2/p} = A.$$

We will distinguish again two cases. If

$$\sum_{\mathbf{i}\le\mathbf{k}}\mathbb{E}|\chi\{\mathbf{U_i}\le\mathbf{t}\}-|\mathbf{t}||^p \ge \left(\sum_{\mathbf{i}\le\mathbf{n}}\mathbb{E}|\chi\{\mathbf{U_i}\le\mathbf{t}\}-|\mathbf{t}||^2\right)^{p/2},$$

then by the Rosenthal inequality

$$A \le \frac{1}{|\mathbf{n}|} \left(\int\limits_{[0,1]^d} K_p \sum_{\mathbf{i} \le \mathbf{k}} \mathbb{E} |\chi\{\mathbf{U_i} \le \mathbf{t}\} - |\mathbf{t}||^p \, d\mathbf{t} \right)^{2/p} \le K_p^{2/p} \frac{|\mathbf{k}|^{2/p}}{|\mathbf{n}|} \le C^* \frac{|\mathbf{k}|}{|\mathbf{n}|}.$$

If

$$\sum_{\mathbf{i} \le \mathbf{k}} \mathbb{E} |\chi\{\mathbf{U_i} \le \mathbf{t}\} - |\mathbf{t}||^p < \left(\sum_{\mathbf{i} \le \mathbf{k}} \mathbb{E} |\chi\{\mathbf{U_i} \le \mathbf{t}\} - |\mathbf{t}||^2 \right)^{p/2},$$

then

$$A \le \frac{1}{|\mathbf{n}|} \left(\int\limits_{[0,1]^d} K_p \left(\sum_{\mathbf{i} \le \mathbf{k}} \mathbb{E} |\chi\{\mathbf{U_i} \le \mathbf{t}\} - |\mathbf{t}||^2 \right)^{p/2} d\mathbf{t} \right)^{2/p} =$$

$$= \frac{1}{|\mathbf{n}|} \left(\int\limits_{[0,1]^d} K_p (|\mathbf{k}| \cdot |\mathbf{t}|(1 - |\mathbf{t}|))^{p/2} \, d\mathbf{t} \right)^{2/p} \le K_p \frac{|\mathbf{k}|}{|\mathbf{n}|}.$$

In the second case $1 \le p \le 2$.

$$\mathbb{E}(\rho^2(Z_{\mathbf{n}}, Z_{\mathbf{k},\mathbf{n}})) \le \mathbb{E} \int\limits_{[0,1]^d} |Z_{\mathbf{n}} - Z_{\mathbf{k},\mathbf{n}}|^2 \, d\mathbf{t} =$$

$$= \mathbb{E} \int\limits_{[0,1]^d} \left| \frac{1}{\sqrt{|\mathbf{n}|}} \sum_{\mathbf{i} \le \mathbf{k}} (\chi\{\mathbf{U_i} \le \mathbf{t}\} - |\mathbf{t}|) \right|^2 d\mathbf{t} =$$

$$= \frac{1}{|\mathbf{n}|} \int\limits_{[0,1]^d} \mathbb{E} \left(\sum_{\mathbf{i} \le \mathbf{k}} (\chi\{\mathbf{U_i} \le \mathbf{t}\} - |\mathbf{t}|) \right)^2 d\mathbf{t} =$$

$$= \frac{1}{|\mathbf{n}|} \int\limits_{[0,1]^d} |\mathbf{k}| \, |\mathbf{t}|(1 - |\mathbf{t}|) \, d\mathbf{t} \le \frac{|\mathbf{k}|}{|\mathbf{n}|}.$$

The proof of Theorem 3.1 is complete. ∎

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A NORMAL STRUCTURE OF MCLAIN GROUPS

By

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0. Introduction

Let R be an associative ring with $1 \neq 0$, and I be an infinite linearly ordered set of indices. The $I \times I$ matrix over R is a function $a : I \times I \to R$. All rings considered here are Dedekind-finite, which guarantee that inverse matrix of triangular one is always triangular.

Let $T_f(I, R)$ be a group of all invertible upper triangular matrices with only finite number of entries different from the unit matrix e. By $D_f(I, R)$ and $UT_f(I, R)$ we denote its diagonal and unitriangular subgroups. The group $UT_f(I, R)$ is normal in $T_f(I, R)$ as a kernel of a, homomorphism which sends triangular matrix to a diagonal one with the same main diagonal. $T_f(I, R)$ is generated by diagonal and unitriangular matrices. So we have $T_f(I, R) =$ $= D_f(I, R) \leftthreetimes UT_f(I, R)$.

The group $UT_f(I, R)$ is called (generalized) McLain group. The group $UT_f(\mathbb{Q}, \mathbb{F}_p)$ is infinite locally finite perfect p-group, which is characteristically simple — in contrast to finite p-groups [11]. Its automorphisms were described in [16]. The group $UT_f(\mathbb{N}, \mathbb{F}_p)$ is the simplest example of infinite p-group which does not satisfy a normalizer condition [12]. McLain groups serve as a source of examples and show limitations of many results in group theory: see [13]–[15], [8] for counterexamples to some problems concerning groups with chain condition for subgroups, [18], [19] for examples of large families of characteristically simple groups, [1]–[3] for explicit constructions of some acyclic groups with prescribed properties and some extensions of groups, [5]–[7] for recognizing McLain groups from their automorphism groups and applications to Hahn groups.

In this paper we investigate a normal structure of McLain groups. In our results we use a notion of net subgroups, which was successfully applied to other infinite dimensional groups: for description of parabolic subgroups of Vershik–Kerov's group [10] and subgroups of triangular matrices containing finitary diagonal matrices for a large class of rings [9].

1. Nets of ideals and net, subgroups

A system $\sigma = (\sigma_{ij})$ $(i, j \in I)$ of two sided ideals σ_{ij} of R is called at *net,* if

$$\sigma_{ir} \cdot \sigma_{rj} \subseteq \sigma_{ij} \qquad \text{for all } i, j, r \in I.$$

If the set I is finite we have a finite net.

It is clear that if σ, τ are nets, then a system $\sigma \cap \tau = (\sigma_{ij} \cap \tau_{ij})$ is a net too. The relation $\sigma \leq \tau$ if $\sigma_{ij} \subseteq \tau_{ij}$' defines a partial order on the set of all nets. Here, we consider only *upper nets* σ for which σ_{ij} is trivial for $i \geq j$. We say that σ is nontrivial if $\sigma_{ij} \neq 0$ for at least one pair of indexes (i,j).

Let the set $G(\sigma)$ consist of all matrices $a \in UT_f(I, R)$ such that $a_{ij} \in \sigma_{ij}$ for all $i < j$. Since σ is a net, $G(\sigma)$ is closed under multiplication of matrices.

In fact, we can show more

PROPOSITION 1. *If σ is a net, then $G(\sigma)$ is a subgroup of $UT_f(I, R)$.*

PROOF. It suffices to prove that, $a \in G(\sigma)$ implies $a^{-1} = (a'_{ij}) \in G(\sigma)$. We define the support of $a \in UT_f(I, R)$ as $\sup(a) = \{i \in I : a_{ij} \neq 0 \text{ or } a_{ij} \neq 0$ for some $j \neq i\}$. By use of the homomorphism $UT_f(I, R) \to UT(m, R)$ which forgets all entries out of $\sup(a) \times \sup(a)$ we can restrict our considerations to finite dimensional unitriangular group and a finite net. Since $a^{-1} \cdot a = e$, we have $a'_{12} = a_{12} = 0$ which means that $a'_{12} \in \sigma_{12}$. Now since $a'_{1,j+1} = -a'_{12}a_{2j} - \ldots - a'_{1,j-1}a_{j-1,j} - a_{1j}$ by induction we have $a'_{1,j+1} \in \sigma_{1,j+1}$. Similarly we have $a_{ij} \in \sigma_{ij}$ for all $i < j$, which finishes the proof.

The map $\sigma \to G(\sigma)$ is a bijection and $\sigma \cap \tau \to HG(\sigma \cap \tau) = G(\sigma) \cap G(\tau)$. So we have the following

PROPOSITION 2. *The lattice of nets of ideals of ring R indexed by I is isomorphic to the lattice of net subgroups of $UT_f(I, R)$.*

Not all subgroups of $UT_f(I, R)$ are net subgroups as shows

EXAMPLE 1. Fix $i < k < l$ from I. Let σ be a net with the only nontrivial entries $\sigma_{ik} = \sigma_{il} = \sigma_{kl} = R$. By G we denote a subgroup of $G(\sigma)$ consisting of all matrices for which $a_{ik} = a_{kl}$. It is easy to show that G is not a net subgroup.

It is an interesting question for which σ the net subgroup $G(\sigma)$ is abelian. We characterize below maximal abelian net subgroups.

EXAMPLE 2. Let $i_0 \in L$, then we define nets: σ such that $\sigma_{kl} = R$ if $k \leq i_0$ and $l > i_0$ and $\sigma_{kl} = 0$ otherwise, and τ such that $\tau_{kl} = R$ if $k < i_0$ and $l \geq i_0$ and $\tau_{kl} = 0$ otherwise. It is easy to verify that $G(\sigma)$ and $G(\tau)$ are maximal abelian subgroups [11], [5]. However, in the case of $I = \mathbb{Q}$ we have a net σ such that $s_{kl} = R$ if $k < \sqrt{2}$ and $l > \sqrt{2}$ (and 0 otherwise). For this net $G(\sigma)$ is also maximal abelian subgroup.

These examples show that detailed analysis of net subgroups in McLain groups depends on assumptions on the order properties of I which will appear elsewhere.

2. Normal, subgroups of McLain groups

The net σ is called *normal net* if for all $i < r < j$, $i, j, r \in I$ we have $\sigma_{ir} \subseteq \sigma_{ij}$ and $\sigma_{rj} \subseteq \sigma_{ij}$.

Our main result here is

THEOREM 1. *Let R be an associative ring with 1 additively generated, by invertible elements and such that 1 is a sum of two invertible elements. Let H be a subgroup of $UT_f(I, R)$. The group H is a normal subgroup of $T_f(I, R)$ if and only if $H = G(\sigma)$ for some normal net σ.*

PROOF. If σ is a normal net, then straightforward calculations show that for any $g \in G(\sigma)$ and $v \in T_f(I, R)$ we have $v \cdot g \cdot v^{-1} \in G(\sigma)$.

Now let $H \triangleleft T_f(I, R)$ and $H \subset UT_f(I, R)$. We put $\sigma_{ij} = \{\alpha \in R : t_{ij}(\alpha) \in H\}$ for $i < j$ and $\sigma_{ij} = 0$ otherwise. In view of formulas for conjugations of transvections $d_i(\theta^{-1}) \cdot t_{ij}(\alpha) \cdot d_i(\theta) = t_{ij}(\alpha\theta)$, $d_i(\theta) \cdot t_{ij}(\alpha) \cdot$ $\cdot d_i(\theta^{-1}) = t_{ij}(\alpha\theta)$, and assumptions on R the sets σ_{ij} are two-sided ideals of R. From equalities $[t_{ir}(\alpha), t_{tj}(1)] = t_{ij}(\alpha) \cdot d(\theta) = t_{ij}(\alpha\theta)$, $[t_{ir}(a), t_{rj}(\alpha)] =$ $= t_{ij}(\alpha)$ valid for distinct i, j, r, the net σ is a normal net. Clearly $G(\sigma) \subseteq H$. We prove now that if $a \in H$, then $t_{ij}(a_{ij}) \in H$ for all $i < j$. We have $b = [a^{-1}, d_i(\theta)] \in H$ and $b_{ij} = a_{ii}(\theta - 1)a_{ij}$. If we put $c = [b^{-1}, d_j(\theta^{-1})]$

we will have $[d_i(\theta), c] = t_{ij}((\theta - 1)c_{ij}) \in H$. Since $c_{ij} = b'_{ii}b_{ij}(\theta - 1)$, by conjugations formulas it follows that $t_{ij}(a_{ij}) \in H$. It means that $H \subseteq G(\sigma)$ and Theorem is proved.

In particular case $I = \mathbb{N}$ our result can be deduced also from results of [4] for finite dimensional unitriangular group, because $UT_f(\mathbb{N}, R)$ is a direct limit of finite dimensional groups under natural embeddings.

Since the property of net of ideals 'to be normal' is invariant under lattice operations we have

THEOREM 2. *Under assumptions of Theorem 1, the net subgroups $G(\sigma)$ corresponding to normal nets form a sublattice Λ of the lattice of normal subgroups of McLain groups.*

We note here that if R has an infinite lattice of two-sided ideals, then Λ is uncountable. (In view of the above result it is an interesting question which normal subgroups of $UT_f(I, R)$ are not normal in $T_f(I, R)$.

Now we give examples of large families of net subgroups $G(\sigma)$ which are not normal in McLain group.

EXAMPLE 3. Let $\sim$ be an equivalence relation on I. We define a net $\tilde{\sigma}$ putting $\tilde{\sigma}_{ij} = R$ if $i \sim j$ and $\tilde{\sigma}_{ij} = 0$ otherwise. The net subgroup $G(\tilde{\sigma})$ is called an equi-group (see [12]). We say that equivalence classes C, D form a mini-max pair if $C \neq D$, C has a minimal element, D has a maximal element and $\min C < \max D$. For example, if $I = \mathbb{N}$ then $\sim$ has no mini-max pair if and only if $\sim$ has no finite equivalence class or has exactly one finite class of the form $\{1, 2, \ldots, n\}$. If $\tilde{\sigma}$ has no mini-max pair of classes, then $G(\tilde{\sigma})$ coincides with its normalizer in $UT_f(I, R)$ (Thm. 3 of [12]). It means that McLain group does not satisfy a normalizes condition for subgroups. We note here that if has a mini-max pair of classes, then for all nontrivial net $\tau \subset \tilde{\sigma}$ net subgroup $G(\tau)$ does not coincide with its centralizes in $UT_f(I, R)$ (and so a normalizes) (it is an easy consequence of Thm. 2 of [12]).

3. Monotonic functions and normal subgroups

Let $R = K$ he a field (or simple ring) such that $|K| > 2$. As usual, for two linearly ordered sets A, B we extend the order to disjoint sum $A \sqcup B$ assuming $a < b$ for all $a \in A$ and $b \in B$. We made additional assumption that for linearly ordered set $I \sqcup \{\infty\}$ the following condition holds

($\star$) for all $i \in I$ every subset of interval $[i, \infty]$ has a minimal element.

As examples of a set I satisfying this condition can serve $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{N} \sqcup \mathbb{N}$, $\mathbb{Z} \sqcup \mathbb{N}$ with natural order.

By $MF(I)$ we denote the set of all functions $f : I \to I \cup \{\infty\}$ which are monotonic, i.e. $x < y$ implies $f(x) \leq f(y)$. For $G(\sigma)$ (σ-normal) we define $f_\sigma \in MF(I)$ as follows: $f_\sigma(i) =$ minimal j such that $\sigma_{ij} \neq 0$ and ∞ otherwise. The function f_σ is well defined since normal net has the property: if $\sigma_{ij} = K$ then for all $r > i$, $s > j$ we have $\sigma_{rs} = K$.

The following Lemma is obvious

LEMMA 1. *If σ is a normal net, then $f_\sigma \in MF(I)$.*

The set $MF(I)$ form a lattice under operations

$$(f_\sigma \wedge f_\tau)(i) = \max\{f_\sigma(i), f_\tau(i)\},$$
$$(f_\sigma \vee f_\tau)(i) = \min\{f_\sigma(i), f_\tau(i)\}.$$

The functions $f_{\max}(i) = i + 1$ and $f_{\min} = \infty$ for all i correspond to $UT_f(I, R)$ and $\{e\}$ respectively. Clearly $f_{\min} \leq f_\sigma \leq f_{\max}$ for all f_σ from $MF(I)$. We have also $G(\sigma) \cap G(\tau) \mapsto f_\sigma \wedge f_\tau$ and $G(\sigma \cdot \tau) \mapsto f_\sigma \vee f_\tau$.

So we obtain a generalization of the known result of Weir which states that normal subgroups of $UT_n(K)$ are partition subgroups corresponding to some monotonic functions determined by ‘boundaries’ of these partition subgroups ([17] Thm. 4).

THEOREM 3. *If K is a field, $|K| > 2$, and I satisfies condition ($\star$), then the correspondence $G(\sigma) \mapsto f_\sigma$ defines a lattice isomorphism between the lattice $\Lambda = \{G(\sigma) \in UT_f(I, K) : \sigma\text{-normal net}\}$ and the lattice $MF(I)$.*

4. Subgroups of triangular matrices containing diagonal

The methods and results of previous sections can be used to describe subgroups of $T_f(I, R)$ containing $D_f(I, R)$. By D-net $\sigma = (\sigma_{ij})$ of two-sided ideals of R we mean a net σ such that $\sigma_{kk} = R$ and $\sigma_{ij} = 0$ for all i, j, k ($i > j$). By $G(\sigma)$ we denote the set of all matrices $a \in T_f(I, R)$ such that $a_{ij} \in \sigma_{ij}$ for all $i \leq j$. Similar proof as in Proposition 1 shows that $G(\sigma)$ is a subgroup of $T_f(I, R)$. Small changes in the proof of Theorem 1 give the following

THEOREM 4. *Let R be an associative ring with* 1 *additively generated by invertible elements and such that* 1 *is a sum of two invertible elements. Let H be a subgroup of $T_f(I,R)$ containing $D_f(I,R)$. Then there exists a unique upper D-net $\sigma = (\sigma_{ij})$ of two-sided ideals of R, such that $H = G(\sigma)$.*

This result can he easily generalized to the greater group $T(I,R)$ of triangular matrices with finite number of nonzero elements in every column. For details of the proof in the special case $I = \mathbb{N}$ see [9].

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SOME INVARIANTS IN MIRON'S $Osc^k M$

By

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1. Adapted basis in $T(Osc^k M)$ and $T^*(Osc^k M)$

Here $Osc^k M$ will be defined as a C^∞ manifold in which the transformations of form (1.1) are allowed. It is formed as a tangent space of higher order of the base manifold M.

Let $E = Osc^k M$ be a $(k+1)n$ dimensional C^∞ manifold. In some local chart (U, φ) a point $u \in E$ has coordinates

$$(x^a y^{1a}, y^{2a}, \dots, y^{ka}) = (y^{0a}, y^{1a}, y^{2a}, \dots, y^{ka}) = (y^{\alpha a}),$$

where $x^a = y^{0a}$ and

$$a, b, c, d, e, \dots = 1, 2, \dots, n, \qquad \alpha, \beta, \gamma, \delta, \kappa, \dots = 0, 1, 2, \dots, k.$$

The following abbreviations will be used:

$$\partial_{\alpha a} = \frac{\partial}{\partial y^{\alpha a}}, \qquad \alpha = 1, 2, \dots, k, \qquad \partial_a = \partial_{0a} = \frac{\partial}{\partial x^a} = \frac{\partial}{\partial y^{0a}}.$$

If in some other chart (U', φ') the point $u \in E$ has coordinates $(x^{a'}, y^{1a'}, y^{2a'}, \dots, y^{ka'})$, then in $U \cap U'$ the allowable coordinate transformations are given by:

$$\begin{aligned}
(1.1)\quad x^{a'} &= x^{a'}(x^1, x^2, \dots, x^n)\\
y^{1a'} &= (\partial_a x^{a'}) y^{1a} = (\partial_{0a} y^{0a'}) y^{1a}\\
y^{2a'} &= (\partial_{0a} y^{1a'}) y^{1a} + (\partial_{1a} y^{1a'}) y^{2a}
\end{aligned}$$

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$$y^{3a'} = (\partial_{0a} y^{2a'}) y^{1a} + (\partial_{1a} y^{2a'}) + (\partial_{2a} y^{2a'}) y^{3a}, \qquad \ldots,$$
$$y^{ka'} = (\partial_{0a} y^{(k-1)a}) y^{1a} + (\partial_{1a} y^{(k-1)a}) y^{2a} + \ldots + (\partial_{(k-1)a} y^{(k-1)a}) y^{ka}.$$

THEOREM 1.1. *The transformations of type* (1.1) *on the common domain form a group.*

Some nice example of the space E can be obtained if the points $(x^a) \in \in M$, $\dim M = n$ are considered as the points of the curve $x^a = x^a(t)$, $t \in I$ and $y^{\alpha a}$ $a = 1, 2, \ldots, k$ are determined by

$$y^a = d_t^\alpha x^a, \qquad d_t^\alpha = \frac{d^\alpha}{dt^\alpha}, \qquad d_t = \frac{t}{dt}. \tag{1.2}$$

If in $U \cap U'$ the equation $x^{a'} = x^{a'}(x^1(t), x^2(t), \ldots, x^n(t))$ is valid, then it is easy to see that

$$y^{1a'} = d_t^1 x^{a'}, \quad y^{2a'} = d_t^2 = d_t^2 x^{a'}, \quad \ldots, \quad y^{ka'} = d_t^k x^{a'} \tag{1.3}$$

satisfy (1.1). In [19] $y^{\alpha a} = \frac{1}{\alpha!} d_t^\alpha x^a$ and it results that the structure group is different from (1.1). As from (1.2) and (1.3) it follows

$$\begin{aligned} y^{1a'} &= y^{1a'}(x, y^{1a}), \quad y^{2a'} = y^{2a'}(x, y^{1a}, y^{2a}), \quad \ldots, \\ y^{ka'} &= y^{ka'}(x, y^{1a}, \ldots, y^{ka}) \end{aligned} \tag{1.4}$$

and from the above equation we get (1.1).

Let us introduce the notations:

$${}^{(0)}A_a^{a'} = \partial_a x^{a'}, \quad {}^{(\alpha)}A_a^{a'} = d_t^{\alpha}\, {}^{(0)}A_a^{a'} = \frac{d^{\alpha}\, {}^{(0)}A_a^{a'}}{dt^\alpha}, \qquad \alpha = 1, 2, \ldots, k. \tag{1.5}$$

The natural basis $\bar{B}^*$ of $T^*(E)$ is

$$\bar{B}^* = \{dy^{0a}, dy^{1a}, \ldots, dy^{ka}\}.$$

The elements of $\bar{B}^*$ are not transformed as tensors ([19], [9]).

The adapted basis B^* of $T^*(E)$ is given by

$$B^* = \{\delta y^{0a}, \delta y^{1a}, \delta y^{2a}, \ldots, \delta y^{ka}\}, \tag{1.6}$$

where,

$$\begin{aligned} \delta y^{0a} &= dx^a = dy^{0a}, \\ \delta y^{1a} &= dy^{1a} + M_{0b}^{1a} dy^{0b}, \\ \delta y^{2a} &= dy^{2a} + M_{1b}^{2a} dy^{1b} + M_{0b}^{2a} dy^{0b}, \quad \ldots, \\ \delta y^{ka} &= dy^{ka} + M_{(k-1)b}^{ka} dy^{(k-1)b} + M_{(k-2)b}^{ka} dy^{(k-2)b} + \ldots + M_{0b}^{ka} dy^{0b}. \end{aligned} \tag{1.7}$$

THEOREM 1.2. *The necessary and sufficient conditions that $\delta y^{\alpha a}$ are transformed as d-tensor field, i.e.*

$$\partial y^{\alpha a'} = \frac{\partial x^{a'}}{\partial x^a}\delta y^{\alpha a}, \qquad \alpha = 0, 1, \ldots, k$$

are the following equations:

$$(1.8)\quad M^{(\alpha+\beta)a}_{\alpha b}(\partial_a x^{b'}) = M^{(\alpha+\beta)b'}_{\alpha c'}\partial_{\alpha b}y^{\alpha c'} + M^{(\alpha+\beta)b'}_{(\alpha+1)c'}\partial_{\alpha b}y^{(\alpha+1)c'} + \ldots$$
$$\ldots + M^{(\alpha+\beta)b'}_{(\alpha+\beta-1)c'}\partial_{\alpha b}y^{(\alpha+\beta-1)c'} + \partial_{\alpha b}y^{(\alpha+\beta)b'}$$
$$1 \le \beta, \quad \alpha + \beta \le k.$$

The proof is given in [9].

From (1.8) after some calculation we get

$$M^{(\alpha+\beta)a\,(0)}_{\alpha b}A^{b'}_a = \binom{\alpha}{\alpha}M^{(\alpha+\beta)b'\,(0)}_{\alpha c'}A^{c'}_b + \binom{\alpha+1}{\alpha}M^{(\alpha+\beta)b'\,(1)}_{(\alpha+1)c'}A^{c'}_b + \ldots$$
$$\ldots + \binom{\alpha+\beta-1}{\alpha}M^{(\alpha+\beta)b'}_{(\alpha+\beta-1)c'}{}^{(\beta-1)}A^{c'}_b + \binom{\alpha+\beta}{\alpha}{}^{(\beta)}A^{b'}_b.$$

The natural basis $\bar{B}$ of $T(E)$ is

$$\bar{B} = \{\partial_{0a}, \partial_{1a}, \ldots, \partial_{ka}\}.$$

The transformation law of its elements are given in [19].

Let us denote the adapted basis of $T(E)$ by B, where

$$(1.9)\qquad B = \{\delta_{0a}, \delta_{1a}, \delta_{2a}, \ldots, \delta_{ka}\} = \{\delta_{\alpha a}\}$$

and

$$(1.10)\qquad \begin{aligned} \delta_{0a} &= \partial_{0a} - N^{1b}_{0a}\partial_{1b} - N^{2b}_{0a}\partial_{2b} - \ldots - N^{kb}_{0a}\partial_{kb},\\ \delta_{1a} &= \qquad\quad \partial_{1a} \quad - N^{2b}_{1a}\partial_{2b} \qquad\quad - N^{kb}_{1a}\partial_{kb},\\ &\vdots\\ \delta_{ka} &= \qquad\qquad\qquad\qquad\qquad\qquad\quad \partial_{ka}. \end{aligned}$$

THEOREM 1.3. ([9]) *The necessary and sufficient conditions that B be dual to B^* ((1.6) and (1.10)) when $\bar{B}$ is dual to $\bar{B}^*$ i.e.*

$$\langle \delta_{\alpha a}\delta^{\beta b}\rangle = \delta^\beta_\alpha \delta^b_a$$

are the following relations:

$$N_{\alpha a}^{(\alpha+\beta)b} = M_{\alpha a}^{(\alpha+\beta)b} - M_{(\alpha+1)c}^{(\alpha+\beta)b} N_{\alpha a}^{(\alpha+1)c} - M_{(\alpha+2)c}^{(\alpha+\beta)b} N_{\alpha a}^{(\alpha+2)c} - \dots - M_{(\alpha+\beta-1)c}^{(\alpha+\beta)b} N_{\alpha a}^{(\alpha+\beta-1)c}. \tag{1.11}$$

THEOREM 1.4. ([9]) *The necessary and sufficient conditions that* $\delta_{\alpha a}$ *with respect to* (1.1) *are transformed as* d*-tensors are the following formulae*

$$N_{\alpha a'}^{(\alpha+\beta)b'} (\partial_a x^{a'}) = N_{\alpha a}^{(\alpha+\beta)c} \partial_{(\alpha+\beta)c} y^{(\alpha+\beta)b'} + N_{\alpha a}^{(\alpha+\beta-1)c} \partial_{(\alpha+\beta-1)c} y^{(\alpha+\beta)b'} + \dots + N_{\alpha a}^{(\alpha+1)c} \partial_{(\alpha+1)c} y^{(\alpha+\beta)b'} - \partial_{\alpha a} y^{(\alpha+\beta)b'}. \tag{1.12}$$

2. Liouville vector fields

DEFINITION 2.1. The fields $\Gamma_{(1)}, \Gamma_{(2)}, \dots, \Gamma_{(k)}$, which in the basis $\bar{B}$ $T(Osc^k M)$ have the form:

$$\begin{aligned}
\Gamma_{(1)} &= \binom{k}{0} y^{1a} \partial_{ka}, \\
\Gamma_{(2)} &= \binom{k-1}{0} y^{1a} \partial_{(k-1)a} + \binom{k}{1} y^{2a} \partial_{ka}, \\
&\vdots \\
\Gamma_{(i)} &= \binom{k-(i-1)}{0} y^{1a} \partial_{(k-(i-1))a} + \binom{k-(i-2)}{1} y^{2a} \partial_{(k-(i-2))a} + \\
&\quad + \dots + \binom{k-1}{i-2} y^{(i-1)a} \partial_{(k-1)a} + \binom{k}{i-1} y^{ia} \partial_{ka}, \\
&\vdots \\
\Gamma_{(k)} &= \binom{1}{0} y^{1a} \partial_{1a} + \binom{2}{1} y^{2a} \partial_{2a} + \binom{3}{2} y^{3a} \partial_{3a} + \dots + \binom{k}{k-1} y^{ka} \partial_{ka}
\end{aligned} \tag{2.1}$$

are the Liouville fields in $T(Osc^k M)$.

THEOREM 2.1. *The Liouville fields determined by* (2.1) *are* d*-vector fields of type* $(1, 0)$.

It can be proved that $(k-(i-1))!\Gamma_{(i)}$ (determined by (2.1)) are exactly the Liouville vector fields $\Gamma^{(i)}$ given by R. MIRON and CH. ATANASIU in [16], [17].

For $k = 3$ (2.1) has the form

$$\Gamma_{(1)} = y^{1a}\partial_{3a}, \qquad \Gamma_{(2)} = y^{1a}\partial_{2a} + 3y^{2a}\partial_{3a},$$
$$\Gamma_{(3)} = y^{1a}\partial_{1a} + 2y^{2a}\partial_{2a} + 3y^{3a}\partial_{3a},$$

these vector fields were obtained in [8].

THEOREM 2.2. *The Liouville vector fields in the basis B have the form*

$$\begin{aligned}
(2.2)\quad \Gamma_{(1)} &= z_1^{ka}\delta_{ka},\\
\Gamma_{(2)} &= z_2^{(k-1)a}\delta_{(k-1)a} + z_2^{ka}\delta_{ka},\\
\Gamma_{(3)} &= z_3^{(k-2)a}\delta_{(k-2)a} + z_3^{(k-1)a}\delta_{(k-1)a} + z_3^{ka}\delta_{ka},\\
&\vdots\\
\Gamma_{(i)} &= z_i^{(k-(i-1))a}\delta_{(k-(i-1))a} + z_i^{(k-(i-2))a}\delta_{(k-(i-2))a} + \ldots + z_i^{ka}\delta_{ka},\\
&\vdots\\
\Gamma_{(k)} &= z_k^{1a}\delta_{1a} + z_k^{2a}\delta_{2a} + \ldots + z_k^{ka}\delta_{ka},
\end{aligned}$$

where

$$\begin{aligned}
(2.3)\qquad z_1^{ka} &= z_2^{(k-1)a} = z_3^{(k-2)a} = \ldots = z_i^{(k-(i-1))a} = \ldots = z_k^{1a} = y^{1a}\\
z_2^{ka} &= \binom{k}{1}y^{2a} + \binom{k-1}{0}M_{(k-1)b}^{ka}y^{1b}\\
z_3^{(k-1)a} &= \binom{k-1}{1}y^{2a} + \binom{k-2}{0}M_{(k-2)b}^{(k-1)a}y^{1b}\\
z_3^{ka} &= \binom{k}{2}y^{3a} + \binom{k-1}{1}M_{(k-1)b}^{ka}y^{2b} + \binom{k-2}{0}M_{(k-2)b}^{ka}y^{1b},\\
&\vdots\\
z_i^{(k-j)a} &= \binom{k-j}{i-j-1}y^{(i-j)a} + \binom{k-j-1}{i-j-2}M_{(k-j-1)b}^{(k-j)a}y^{(i-j-1)b} +\\
&\quad + \ldots + \binom{k-i+1}{0}M_{(k-i+1)b}^{(k-j)a}y^{1b}, \qquad (j < i).
\end{aligned}$$

For $k = 3$ in [8] we get

$$z_1^{3a} = z_2^{2a} = z_3^{1a} = y^{1a},$$

$$z_2^{3a} = 3y^{2a} + M_{2b}^{3a} y^{1b}, \qquad z_3^{2a} = 2y^{2a} + M_{1b}^{2a} y^{1b}$$

$$z_3^{3a} = 3y^{3a} + 2M_{2b}^{3a} y^{2b} + M_{1b}^{3a} y^{1b},$$

which coincide with (2.3).

3. Zermello's conditions in $Osc^k M$

DEFINITION 3.1. A differentiable Lagrangian of order k on a C^∞ manifold is a function $L : E \to R$ differentiable on $\tilde{E}$ (where rank $[y^{1a}] = 1$) and continuous in those points of E where y^{1a} are equal to zero.

Let $L : E \to R$ be a differentiable Lagrangian of order k and $c : t \in \in [0,1] \to \{x^a(t)\} \in M$ a smooth parametrized curve, such that $Imc \subset U$, U being the domain of a local chart at the differentiable manifold M. The extension c^* (of c) to k is given by:

$$c^* : t \in [0,1] \to x^a(t)\partial_a + d_t^1 x^a(t)\partial_{1a} + \ldots + d_t^k x^a(t)\partial_{ka}.$$

The integral of action I_{c^*} is

$$I_{c^*} = \int_0^1 L(x, y^1, y^2, \ldots, y^k) dt.$$

The integral of action I_{c^*} does not depend on the parametrization of the curve c^*:

$$x^a = x^a(t) = y^{0a}, \qquad y^{\alpha a} = d_t^\alpha x^a = \frac{d^\alpha x}{dt^\alpha}, \qquad \alpha = 1, 2, \ldots, k, \tag{3.1}$$

if

$$\int_0^1 L(x, y^1, y^2, \ldots, y^k) dt = \int_0^1 L(x, y^{1'}, y^{2'}, \ldots, y^{k'}) ds, \tag{3.2}$$

where $s = s(t)$ is at least C^k function, $s'(t) > 0$ for $t \in [0,1]$, $s(0) = 0$, $s(1) = 1$ and

$$y^{\alpha a'} = d_s^\alpha x^a = \frac{d^\alpha x^a}{ds^\alpha}, \qquad \alpha = 1, 2, \ldots, k. \tag{3.3}$$

The equations which give the conditions when (3.2) is satisfied are called Zermello's conditions.

(3.2) will be satisfied if along the c^*

$$L(x.y^1, y^2, \dots, y^k) = L(x, y^{1'}, y^{2'}, \dots, y^{k'})s', \tag{3.4}$$

where $s' = \frac{ds}{dt}$. We shall use the notation $s^{(\alpha)} = \frac{d^\alpha s}{dt^\alpha}$.

From (3.1) and (3.3) we get (for $s = s(t)$):

$$\begin{aligned} y^{1a} &= \frac{dx^a}{ds}s', \\ y^{2a} &= \frac{\partial y^{1a}}{\partial s}s' + \frac{\partial y^{1a}}{\partial s'}s'', \\ y^{3a} &= \frac{\partial y^{2a}}{\partial s}s' + \frac{\partial y^{2a}}{\partial s'}s'' + \frac{\partial y^{2a}}{\partial s''}s''', \\ &\vdots \\ y^{ka} &= \frac{\partial y^{(k-1)a}}{\partial s}s' + \frac{\partial y^{(k-1)a}}{\partial s'}s'' + \dots + \frac{\partial y^{(k-1)a}}{\partial s^{(k-1)}}s^{(k)}. \end{aligned} \tag{3.5}$$

The above equations follows from the relations:

$y^{1a} = y^{1a}(s, s')$, $y^{2a} = y^{2a}(s, s', s'')$, $\dots$, $y^{(k-1)a} = y^{(k-1)a}(s, s', \dots, s^{(k-1)})$.

Using the notations:

$$A_0^a = \frac{dx^a}{ds}, \qquad A_\alpha^a = d_t^\alpha A_0^a, \qquad \alpha = 1, 2, \dots, k \tag{3.6}$$

and the Leibniz rule we have

THEOREM 3.1. *$y^{\alpha a}$ and $s^{(\alpha)}$, $\alpha = 1, 2, \dots, k$ are connected by formulae:*

$$\begin{aligned} y^{1a} &= A_0^a s' \\ y^{2a} &= \binom{1}{0} A_1^a s' + \binom{1}{1} A_0^a s'', \\ y^{3a} &= \binom{2}{0} A_2^a s' + \binom{2}{1} A_1^a s'' + \binom{2}{2} A_0^a s''', \\ y^{4a} &= \binom{3}{0} A_3^a s' + \binom{3}{1} A_2^a s'' + \binom{3}{2} A_1^2 s''' + \binom{3}{3} A_0^a s^{(4)}, \\ &\vdots \end{aligned} \tag{3.7}$$

$$y^{ka} = \binom{k-1}{0} A^a_{k-1} s' + \binom{k-1}{1} A^a_{k-2} s'' + \ldots + \binom{k-1}{k-1} A^a_0 s^{(k)}.$$

The explicit form of (3.7) is the following:

$$
\begin{aligned}
(3.8)\quad y^{1a} &= y^{1a'} s' \\
y^{2a} &= y^{2a'} (s')^2 + y^{1a'} s'' \\
y^{3a} &= y^{3a'} (s')^3 + y^{2a'} 3s's'' + y^{1a'} s''' \\
y^{4a} &= y^{4a'} (s')^4 + y^{3a'} 6(s')^2 s'' + y^{2a'} (3(s'')^2 + 4s's''') + y^{1a'} s^{(iv)}, \\
&\vdots
\end{aligned}
$$

From (3.7) it follows:

$$
\begin{aligned}
A^a_1 &= \left(\frac{\partial}{\partial s} A^a_0\right) s', \\
(3.9)\qquad A^a_2 &= \left(\frac{\partial}{\partial s} A^a_1\right) s' + \left(\frac{\partial}{\partial s} A^a_0\right) s'', \\
A^a_3 &= \left(\frac{\partial}{\partial s} A^a_2\right) s' + \binom{2}{1} \left(\frac{\partial}{\partial s} A^a_1\right) s'' + \binom{2}{2} \left(\frac{\partial}{\partial s} A^a_0\right) s''', \\
&\vdots
\end{aligned}
$$

THEOREM 3.2. *A^a_α and $s^{(\alpha)}$, $\alpha = 1, 2, \ldots, k$ are connected by the formula:*

$$A^a_\alpha = \binom{\alpha-1}{0} \frac{\partial A^a_{\alpha-1}}{\partial s} s' + \binom{\alpha-1}{1} \frac{\partial A^a_{\alpha-2}}{\partial s} s'' + \ldots + \binom{\alpha-1}{\alpha-1} \frac{\partial A^a_0}{\partial s} s^{(\alpha)}.$$

THEOREM 3.3. *The following relations are valid:*

$$(3.10)\qquad \frac{\partial y^{1a}}{\partial s'} = \frac{\partial y^{2a}}{\partial s''} = \ldots = \frac{\partial y^{ka}}{\partial s^{(k)}} = \frac{dy^{0a}}{ds} = y^{1a'},$$

$$(3.11)\qquad \frac{\partial y^{(\alpha+\beta)a}}{\partial s^{(\alpha)}} = \frac{\alpha+\beta}{\alpha} \frac{\partial y^{(\alpha+\beta-1)a}}{\partial s^{(\alpha-1)}} = \ldots = \binom{\alpha+\beta}{\alpha} \frac{\partial y^{\beta a}}{\partial s},$$

$$0 < \alpha + \beta \leq k.$$

Relations (3.10) and (3.11) are crucial by the determination of Zermello's conditions.

If we take the partial derivatives of (3.4) with respect to s', s'', …, $s^{(k)}$ we get:
(3.12)

$$
\begin{aligned}
(\partial_{1a}L)\frac{\partial y^{1a}}{\partial s'} + (\partial_{2a}L)\frac{\partial y^{2a}}{\partial s'} + \ldots + (\partial_{ka}L)\frac{\partial y^{ka}}{\partial s'} &= L(x, y^{1'}, y^{2'}, \ldots, y^{k'}),\\
(\partial_{2a}L)\frac{\partial y^{2a}}{\partial s''} + \ldots + \ (\partial_{ka})\frac{\partial y^{ka}}{\partial s''} &= 0,\\
&\vdots\\
(\partial_{ka}L)\frac{\partial y^{ka}}{\partial s^{(k)}} &= 0.
\end{aligned}
$$

On the left hand side in (3.12) $L = L(x, y^1, y^2, y^k)$. If we multiply the first equation of (3.12) with s', the second with $2s''$, …, the last with $ks^{(k)}$ and add all these equations and use (3.4) we get:

$$
\begin{aligned}
(3.13)\qquad & (\partial_{1a}L)\frac{\partial y^{1a}}{\partial s'}s' + (\partial_{2a}L)\left(\frac{\partial y^{2a}}{\partial s'}s' + 2\frac{\partial y^{2a}}{\partial s''}s''\right) + \ldots\\
& \ldots + (\partial_{ka}L)\left(\frac{\partial y^{ka}}{\partial s'}s' + 2\frac{\partial y^{ka}}{\partial s''}s'' + \ldots + k\frac{\partial y^{ka}}{\partial s^{(k)}}s^{(k)}\right) =\\
& = L(x, y^{1'}, y^{2'}, \ldots, y^{k'})s' = L(x, y^1, y^2, \ldots, y^k).
\end{aligned}
$$

From (3.11) and (3.5) we obtain:

$$
\begin{aligned}
\frac{\partial y^{1a}}{\partial s'}s' &= \frac{dx^a}{ds}s' = y^{1a},\\
\frac{\partial y^{2a}}{\partial s'}s' + 2\frac{\partial y^{2a}}{\partial s''}s'' &= \frac{2}{1}\frac{\partial y^{1a}}{\partial s}s' + 2\frac{\partial y^{1a}}{\partial s'}s'' = 2y^{2a},\\
&\vdots\\
\frac{\partial y^{ka}}{\partial s'}s' + 2\frac{\partial y^{ka}}{\partial s''}s'' + \ldots + k\frac{\partial y^{ka}}{\partial s^{(k)}} &=\\
= \frac{k}{1}\frac{\partial y^{(k-1)a}}{\partial s}s' + 2\frac{k}{2}\frac{\partial y^{(k-1)a}}{\partial s'}s'' + \ldots + k\frac{k}{k}\frac{\partial y^{(k-1)a}}{\partial s^{(k-1)}}s^{(k)} &= ky^{ka}.
\end{aligned}
$$

The substitution of the above equations into (3.13) results

$$
(3.14)\qquad (\partial_{1a}L)y^{1a} + 2(\partial_{2a}L)y^{2a} + \ldots + k(\partial_{ka}L)y^{ka} = L.
$$

If we multiply the second equation of (3.12) with $\binom{2}{2}s'$ and the following with $\binom{3}{2}s$", $\binom{4}{2}s'''$, …, $\binom{k}{2}s^{(k-1)}$ respectively and add all these equations we get

$$(\partial_{2a}L)\left(\frac{\partial y^{2a}}{\partial s''}s'\right)+(\partial_{3a}L)\left(\frac{\partial y^{3a}}{\partial s''}s'+\binom{3}{2}\frac{\partial y^{3a}}{\partial s'''}s''\right)+\dots$$

$$\dots+(\partial_{ka}L)\left(\frac{\partial y^{ka}}{\partial s''}s'+\binom{3}{2}\frac{\partial y^{ka}}{\partial s'''}s''+\dots+\binom{k}{2}\frac{\partial y^{ka}}{\partial s^{(k)}}s^{(k-1)}\right)=0.$$

If we use (3.11) and (3.5) the above equation takes the form

$$\binom{2}{2}(\partial_{2a}L)y^{1a}+\binom{3}{2}(\partial_{3a}L)y^{2a}+\dots+\binom{k}{2}(\partial_{ka}L)y^{(k-1)a}=0. \tag{3.15}$$

If we multiply the third equation of (3.12) with $\binom{3}{3}s$" and the following with $\binom{4}{3}s'''$, $\binom{5}{3}s^{(iv)}$, …, $\binom{k}{3}s^{(k)}$ respectively we get

$$\binom{3}{3}(\partial_{3a}L)y^{1a}+\binom{4}{3}(\partial_{4a}L)y^{2a}+\dots+\binom{k}{3}(\partial_{ka}L)y^{(k-2)a}=0, \tag{3.16}$$

On the similar way we obtain (for $4\le i\le k$):

$$\binom{i}{i}(\partial_{ia}L)y^{1a}+\binom{i+1}{i}(\partial_{(i+1)a}L)y^{2a}+\dots+\binom{k}{i}(\partial_{ka}L)y^{(k-i+1)a}=0. \tag{3.17}$$

We shall use the notations:

$$\begin{aligned}
I_1&=\binom{k}{k}y^{1a}\partial_{ka},\\
I_2&=\binom{k-1}{k-1}y^{1a}\partial_{(k-1)a}+\binom{k}{k-1}y^{2a}\partial_{ka},\\
I_3&=\binom{k-2}{k-2}y^{1a}\partial_{(k-2)a}+\binom{k-1}{k-2}y^{2a}\partial_{(k-1)a}+\binom{k}{k-2}y^{3a}\partial_{ka},\\
&\vdots\\
I_k&=\binom{1}{1}y^{1a}\partial_{1a}+\binom{2}{1}y^{2a}\partial_{2a}+\dots+\binom{k}{1}y^{ka}\partial_{ka}.
\end{aligned} \tag{3.18}$$

From (3.14)–(3.18) it follows

THEOREM 3.4. *The necessary conditions that the integral of action does not depend on the parametrization of the curve are:*

$$I_1(L) = 0, \quad I_2(L) = 0, \quad \dots, \quad I_{(k-1)}(L) = 0, \quad I_k(L) = L. \tag{3.19}$$

Equations (3.19) are Zermello's conditions. A comparison of (3.18) with (2.1) gives

THEOREM 3.5. *The Liouville vector fields $\Gamma_{(\alpha)}$ and I_α are equal for $\alpha = 1, 2, \dots, k$, i.e.*

$$I_\alpha = \Gamma_{(\alpha)}, \tag{3.20}$$

so I_α are vector fields.

THEOREM 3.6. *The Zermello's conditions can be written in the form*

$$\Gamma_{(1)}(L) = 0, \quad \Gamma_{(2)}(L) = 0, \quad \dots, \quad \Gamma_{(k-1)}(L) = 0, \quad \Gamma_{(k)}(L) = L. \tag{3.21}$$

4. Energies of higher order

DEFINITION 4.1. We call $\varepsilon_\alpha(L)$ energies of order α, $\alpha = 1, 2, \dots, k$ of the Lagrangian $L(x, y^1, \dots, y^k)$. They are defined along a curve c by the invariants I_α in the following form:

$$\begin{aligned}
\varepsilon_k(L) &= [I_k - d_t^1 I_{k-1} + d_t^2 I_{k-2} - \dots + (-1)^{k-1} d_t^{k-1} I_1]\,(L) - L,\\
\varepsilon_{k-1}(L) &= [-I_{k-1} + d_t^1 I_{k-2} - \dots + (-1)^{k-1} d_t^{k-2} I_1]\,(L),\\
\varepsilon_{k-2}(L) &= [I_{k-2} - \dots + (-1)^{k-1} d_t^{k-3} I_1]\,(L),\\
&\vdots\\
\varepsilon_2(L) &= [(-1)^{k-2} I_2 + (-1)^{k-1} d_t^1 I_1]\,(L),\\
\varepsilon_1(L) &= [(-1)^{k-1} I_1]\,(L).
\end{aligned} \tag{4.1}$$

PROPOSITION 4.1. *The following identities hold:*

$$\begin{aligned}
(\varepsilon_k - d_t^1 \varepsilon_{k-1})(L) &= I_k(L) - L,\\
(\varepsilon_{k-1} - d_t^1 \varepsilon_{k-2})(L) &= -I_{k-1}(L),\\
&\vdots\\
(\varepsilon_2 - d_t^1 \varepsilon_1)(L) &= (-1)^{k-2} I_2(L).
\end{aligned} \tag{4.2}$$

From (3.19), (4.1) and (4.2) it follows

THEOREM 4.1. *For the Lagrangian $L(x, y^1, \ldots, y^k)$, for which the Zermello's conditions are satisfied, the higher order energies are equal to zero, i.e.*

$$\varepsilon_1(L) = 0, \quad \varepsilon_2(L) = 0, \quad \ldots, \quad \varepsilon_k(L) = 0.$$

PROPOSITION 4.2. *For any differentiable Lagrangian $L(x, y^1, y^2, \ldots, y^k)$ and any differentiable function $F = F(t)$ defined along the curve $c : [0,1] \to$ $\to M$ we have*

$$(4.3)\quad (d_t^1 F)L - [(d_t^1 F)I_k + (d_t^2 F)I_{k-1} + \ldots + (d_t^k F)I_1](L) = F(d_t^1 \varepsilon_k)L+$$
$$+d_t^1[-F\varepsilon_k + (d_t^1 F)\varepsilon_{k-1} - (d_t^2 F)\varepsilon_{k-2} + \ldots + (-1)^k (d_t^{k-1} F)\varepsilon_1](L).$$

PROOF. The right hand side of the above equation can be written in the form

$$[-(d_t^1 F)(\varepsilon_k - d_t^1 \varepsilon_{k-1}) + (d_t^2 F)(\varepsilon_{k-1} - d_t^1 \varepsilon_{k-2}) - \ldots$$
$$\ldots + (-1)^{k-1}(d_t^{k-1} F)(\varepsilon_2 - d_t^2 \varepsilon_1) + (-1)^k (d_t^k F)\varepsilon_1](L) =$$
$$= -(d_t^1 F)(I_k(L) - L) - [(d_t^2 F)I_{k-1} + (d_t^3 F)I_{k-2} + \ldots + (d_t^{k-2} F)I_2 + (d_t^k F)I_1](L) =$$
$$= (d_t^1 F)L - [(d_t^1 F)I_k + (d_t^2 F)I_{k-1} + \ldots + (d_t^k F)I_1](L).$$

The above equation is involved in the Noether theory of symmetries of the higher order Lagrangians [19].

THEOREM 4.2. *For any differentiable function $F(x(t), y^1(t), \ldots, y^k(t))$ the operators d_t^1, E_a^0 and I_1, I_2, $\ldots$, I_k are connected by formula:*

$$(4.4)\quad d_t^1(F) = [y^{1a}\partial_{0a} + y^{2a}\partial_{1a} + y^{3a}\partial_{2a} + \ldots + d_t^1 y^{ka}\partial_{ka}]F =$$
$$= [y^{1a}E_a^0 + d_t^1 I_k - d_t^2 I_{k-1} + d_t^3 I_{k-2} + \ldots + (-1)^k d_t^k I_1]F.$$

THEOREM 4.3. *For any differentiable Lagrangian along the smooth curve $c : [0,1] \to x^a(t) \in M$ we have* [19]

$$(4.5)\qquad d_t^1(\varepsilon_k(L)) = -y^{1a}E_a^0(L),$$

where

$$E_a^0 = \partial_{0a} - d_t^1 \partial_{1a} + d_t^2 \partial_{2a} + \ldots + (-1)d_t^k \partial_{ka}.$$

PROOF. Let us introduce the notation

$$(4.6)\qquad B = I_k - d_t^1 I_{k-1} + d_t^2 I_{k-1} - \ldots + (-1)^k d_t^{k-1} I_1.$$

From (4.1) we have

$$\varepsilon_k(L) = B(L) - L,$$

$$d_t^1(\varepsilon_k(L)) = (d_t^1 B)L - d_t^1 L. \tag{4.7}$$

From (4.4) and (4.6) we have

$$d_t^1 L = y^{1a} E_a^0(L) + (d_t^1 B)L. \tag{4.8}$$

If we substitute $(d_t^1 B)L$ from (4.8) into (4.7) we obtain (4.5).

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THE RESULT OF EVEN ALLOCATION OF FUNDS FOR POSTGRADUATE TRAINING

By

M. FARKAS

(April 9, 2003)

Introduction

A simple competitive system of ODEs is constructed and analysed along with its implications modeling the dynamics of staff in a large institute which trains its own would be staff in postgraduate courses. A certain field of science is considered with competing branches. The model shows that more popular branches increase their number related to less popular ones faster than linear.

1. The model

Suppose that in a large institute or university in a certain field of science, mathematics say, there is a fixed amount of funds available in unit time (a year or three years) for postgraduate training and the funds are distributed according to the respective numbers of postgraduate, say PhD, students among the different branches of the given field. Students are admitted by their merits, by an entrance examination, say, and then they choose the branch freely. The training of a student in unit time (tuition and scholarship provided by the institute or the state) costs a certain amount of money. We use this amount as the unit of funds and assume that the *total amount of funds* in unit time is K units. Different branches have different popularities depending on the hardships, on how fast may one get to the point of working on some research problem and being able to publish, on the quality and personality of

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the possible supervisors etc. We denote the *popularity* of the i-th branch by a_i. This may be measured by the average number of postgraduate students of a researcher in the i-th field. The number of staff (possible supervisors, PhD title holders, say) in branch i at time t will be denoted by $N_i(t)$. We assume also that successful students at the end of their training enter the staff of the given branch at the same institute and that the institute does not recruit staff from outside. This condition may be relaxed assuming that the students having obtained their degree enter other institutes in the country as well, provided that popularities and funds have the same values at the different institutes. Under these assumptions if branch i were the only existing branch then the simplest assumption is that N_i follows the logistic dynamics, i.e.

$$\frac{dN_i}{dt} = r_i N_i(1 - a_i N_i/K)$$

where $r_i = a_i - m$ is the *intrinsic growth rate of the branch*, m being the rate of reaching pension age of the staff, considered to be independent of the branch and small compared to a_i.

If there are n branches then the dynamics is governed by the system

$$\frac{dN_i}{dt} = r_i N_i \left(1 - \sum_{j=1}^{n} a_j N_j/K\right), \qquad i = 1, 2, \ldots, n. \tag{1.1}$$

2. Implications of the model

System (1.1) is a simple degenerate Lotka-Volterra system (see e.g. Farkas[2001]). According to our assumptions all the parameters are positive. The equilibria, apart from those on the coordinate hyperplanes, i.e. those that represent the absence of some of the branches, are the points of the open simplex

$$S = \left\{ N \in \Re^n \;\middle|\; \sum_{j=1}^{n} a_j N_j/K = 1, N_i > 0,\ i = 1, 2, \ldots, n \right\}.$$

According as $\sum_{j=1}^{n} a_j N_j/K$ is larger or less than one, the quantity of staff is decreasing (by lay off, say), resp. increasing in all the branches. Denoting an arbitrary point of S by $\overline{N} = (\overline{N}_1, \overline{N}_2, \ldots, \overline{N}_n)$ the characteristic polynomial at this point is

$$D(\lambda) = \begin{vmatrix} -\frac{r_1\overline{N}_1 a_1}{K} - \lambda & -\frac{r_1\overline{N}_1 a_2}{K} & \ldots & -\frac{r_1\overline{N}_1 a_n}{K} \\ -\frac{r_2\overline{N}_2 a_1}{K} & -\frac{r_2\overline{N}_2 a_2}{K} - \lambda & \ldots & -\frac{r_2\overline{N}_2 a_n}{K} \\ \cdot & \cdot & \ldots & \cdot \\ -\frac{r_n\overline{N}_n a_1}{K} & -\frac{r_n\overline{N}_n a_2}{K} & \ldots & -\frac{r_n\overline{N}_n a_n}{K} - \lambda \end{vmatrix}.$$

Simple row and column operations yield

$$D(\lambda) = \lambda^{n-1}\left(\lambda + \sum_{i=1}^{n} a_i r_i \overline{N}_i / K\right),$$

i.e. 0 is an $(n-1)$-tuple root and the n-th root is

$$\lambda_n = -\sum_{i=1}^{n} a_i r_i \overline{N}_i / K) < 0.$$

This means that the simplex S is the center manifold of each equilibrium $\overline{N}$ on it and each equilibrium has a one dimensional stable manifold. As a consequence, the simplex is a global attractor of the system with respect to the open positive orthant of $\Re^n$, i.e. every solution with positive initial conditions tends towards a point on S as t tends to infinity. This means that in the long run the distribution of the staff by branches will settle at a point of the simplex. Which point will it be, depends on the initial conditions. It is also easy to determine the equation of the trajectories. We may divide each equation of system (1.1) by the first one obtaining

$$\frac{dN_i}{dN_1} = \frac{r_i N_i}{r_1 N_1}, \ i = 1, 2, \ldots, n.$$

Thus the equation of the trajectory corresponding to the initial point $(N_{10}, N_{20}, \ldots, N_{n0})$ in the interior of the positive orthant parametrized by the first coordinate is

$$N_i = \frac{N_{i0}}{N_{10}^{r_i/r_1}} N_1^{r_i/r_1}, \ i = 1, 2, \ldots, n.$$

These results may be used several ways.

If the initial distribution of staff among the branches, the popularity of the branches and the funds available for postgraduate training are known we may forecast the long run distribution of staff.

If we want to fix the long run overall quantity of staff we may control the funding accordingly.

If we want to achieve a certain relative distribution among the branches we may try to find methods that increase the popularity of some branches and decrease that of some others.

What is important, we don't have to fix the aimed situation at the start. We may fix the parameters and the initial values appropriately and leave it to the dynamics to sort out.

3. The case $n = 2$

We illustrate the results of the previous Section in case there are only two branches. In this case the simplex of equilibria is the straight line segment

$$a_1 N_1/K + a_2 N_2/K = 1,$$

(in the positive quadrant of the plane N_1, N_2) and the equation of the trajectories is

$$N_2 = \frac{N_{20}}{N_{10}^{r_2/r_1}} N_1^{r_2/r_1},$$

i.e. they are parts of parabolae that pass through the origin. We may assume without loss of generality that $r_1 < r_2$ implying that these are parabolae convex down. The Figure (produced by Maple-V) shows the phase portrait when $a_1 = 0.31$, $a_2 = 0.61$, $r_1 = 0.3$, $r_2 = 0.6$, $K = 10$ and the initial conditions are given in the Table.

Table. Initial values with corresponding equilibria

(N_{10}, N_{20})	$(\overline{N_1}, \overline{N_2})$
$(1, 2)$	$(2.85, 14.47)$
$(2, 2)$	$(5.47, 13.13)$
$(3, 2)$	$(7.87, 11.91)$
$(4, 2)$	$(10.05, 10.81)$
$(5, 2)$	$(12.03, 9.82)$
$(6, 2)$	$(13.83, 8.92)$

As we see, trajectories end up at points of the straight line segment

$$N_1/32.26 + N_2/16.39 = 1.$$

The second branch is twice as popular as the first one. While the number of staff in the first branch grows 2-3 fold in the long run, in the second branch

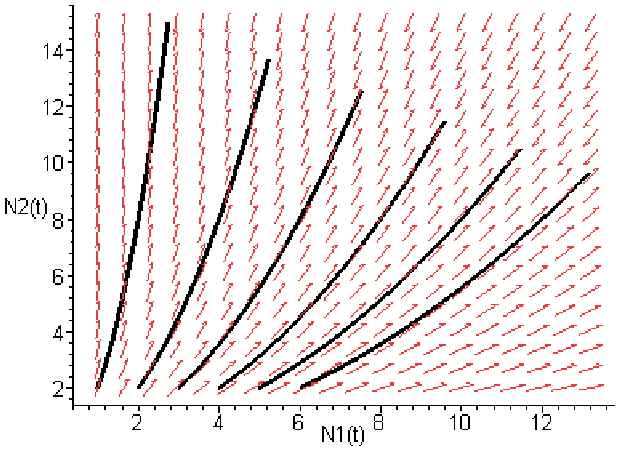

Fig. 1

it grows 7-5 fold. If we want to have 10 staff in each branch in the long run, say, then we have to start with 4 in the first one and 2 in the second one.

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